Behavior of geodesics in foliated manifolds with bundle-like metrics

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THESIS

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1. Introduction.

Foliated manifolds are studied by C. Ehresmann, A. Haefliger, G. Reeb and many people. Many of works are topological (non-riemannian) cases. The early study of riemannian case was done by B. L. Reinhart[24], that is, he defined foliated manifolds with "bundle-like" metrics with respect to the foliations and tried to prove so-called Reeb stability theorem for this case. The foliated manifolds with bundle-like metrics are studied by R. Hermann[4], A. M. Naveira[19], J. S. Pasternack[22, 23], B. L. Reinhart[24, 25], R. Sacksteder [26], I. Vaisman[28, 29] and others.

The typical examples of foliated manifolds with bundle-like metrics are followings; (i) each fiber space under a suitable choice of metric, (ii) the foliation of a riemannian manifold by the orbits of a group of isometries having all its orbits of the same dimension.

In this paper we discuss the behavior of geodesics in foliated manifolds with bundle-like metrics. As a well-known and fundamental result in this direction, we may state:

Theorem(B. L. Reinhart[24]). *A geodesic of a bundle-*

like metric is orthogonal to the leaf at one point if and only if it is orthogonal to the leaf at every point.

We discuss geodesics making constant angle with leaves, and these are generalizations of [14]. We also discuss focal points of leaves along transversal geodesics, and, in the case of codimension 1 , we have non-existence of focal points of leaves along transversal geodesics. The relations between the Levi-Civita connection and the second connection defined by I. Vaisman[28] are discussed. And we have the definitions of geodesic and Jacobi field with respect to the second connection, and we discuss the properties of them.

The topological obstructions for the existence of the foliation with a bundle-like metric were studied by H. Kitahara and S. Yorozu[12], J. S. Pasternack[22] and R. Sacksteder [26]. The existence of the complete bundle-like metric was discussed by H. Kitahara[8, 9].

In this paper, we shall be in C^{∞}-category and manifolds are paracompact, connected Hausdorff spaces. Latin indices run from 1 to p , and Greek indices run from $p+1$ to $p+q$. We use the Einstein's summation convention unless otherwise stated.

2. Foliated manifolds.

Let M and N be manifolds of dimension $p+q$ and q, respectively, and let f be a map $M \longrightarrow N$. For each point $m \in M$, we have the linear map

$$(f_*)_m : T_mM \longrightarrow T_{f(m)}N ,$$

where T_mM (resp. $T_{f(m)}N$) denotes the tangent space of M at m (resp. the tangent space of N at $f(m)$). Then we have

Definition 2.1. A map $f : M \longrightarrow N$ is called an <u>immersion</u> if $(f_*)_m$ is injective for any point $m \in M$, and f is called a <u>submersion</u> if $(f_*)_m$ is surjective for any point $m \in M$.

Let TM denote the tangent bundle (or, its total space) over M. For a sub-bundle E of TM, $\Gamma(E)$ (resp. $\Gamma(E|_U)$) is a set of all sections of E (resp. E over $U \subset M$).

Definition 2.2. A sub-bundle E of TM is called <u>integrable</u> if locally E is the kernel of the differential of a submersion. An integrable sub-bundle E of TM is called a <u>foliation</u> E, and the pair (M,E) or, simply M, is called a <u>foliated manifold with a foliation</u> E. The maximal connected integral manifolds of E are called <u>leaves</u>.

Precisely, a foliation on M is defined by the following data:

(i) An open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M.

(ii) A pseudogroup Γ of local diffeomorphisms of $\mathbb{R}^q$.

(iii) Submersions $f_\lambda : U_\lambda \longrightarrow \mathbb{R}^q$.

(iv) For $U_\lambda \cap U_\mu \neq \phi$, there exists $\gamma_{\lambda\mu} \in \Gamma$ such that f_λ

$= \gamma_{\lambda\mu} f_\mu$.

Definition 2.3. In the above foliation on M , q is called the <u>codimension</u> of the foliation.

We may choose a local coordinates (x^i, x^α) on each $\{U_\lambda, f_\lambda\}$ such that

(i) $|x^i| < 1$, $|x^\alpha| < 1$.

(ii) The integral manifolds of E are given locally by $x^\alpha = c^\alpha$ for constants c^α satisfying $|c^\alpha| < 1$.

Definition 2.4. The above local coordinates (x^i, x^α) on $\{U_\lambda, f_\lambda\}$ is called <u>flat coordinates</u>, and a triple $\{U_\lambda, f_\lambda, (x^i, x^\alpha)\}$ or, simply, $U(x^i, x^\alpha)$ a <u>flat coordinate chart</u>. In a flat coordinate chart $U(x^i, x^\alpha)$, each of slices given by equations $x^\alpha = c^\alpha$ is called a <u>plaque</u>.

Definition 2.5. A foliated manifold M is called <u>regular at point</u> $m \in M$ if there exists a flat coordinate chart of m which meets each leaf in at most one plaque. A leaf in M is called a <u>regular leaf</u> if M is regular at any point of the leaf.

If $U(x^i, x^\alpha)$ and $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$ are flat coordinate charts such that $U \cap \bar{U} \neq \phi$, then $\partial/\partial x^i$ transforms by coordinate change into a combination of $\partial/\partial \bar{x}^1, \cdots, \partial/\partial \bar{x}^p$, since the tangent space to a leaf goes into the tangent space to the leaf. Thus the coordinate transformation is of form

$$\bar{x}^i = \bar{x}^i(x^k, x^\tau) \quad \text{and} \quad \bar{x}^\alpha = \bar{x}^\alpha(x^\tau) \quad .$$

In each flat coordinate chart $U(x^i, x^\alpha)$, we may choose

1-forms $w^1, \cdots, w^p$ such that $\{ w^1, \cdots, w^p, dx^{p+1}, \cdots, dx^{p+q} \}$ is a basis for the cotangent space at each point in U , and vectors $v_{p+1}, \cdots, v_{p+q}$ such that $\{ \partial/\partial x^1, \cdots, \partial/\partial x^p, v_{p+1}, \cdots, v_{p+q} \}$ is the dual base for the tangent space. We have

$$w^i = dx^i + A^i_\alpha \, dx^\alpha$$

$$v_\alpha = \partial/\partial x^\alpha - A^i_\alpha \, \partial/\partial x^i$$

for any functions $A^i_\alpha = A^i_\alpha(x^k, x^\tau)$ on U . If we transform the flat coordinate chart $U(x^i, x^\alpha)$ into $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$ and choose $\bar{w}^i$ and $\bar{v}_\alpha$ in $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$, then $\bar{w}^i$ transforms into a combination of the w^j and $\bar{v}_\alpha$ into a combination of the v_β .

Notice that not every sub-bundle E of TM is integrable. The following theorems, usually referred to jointly as the "Frobenius Theorem", give the integrability of a sub-bundle E of TM .

Theorem 2.1. *Let E be a sub-bundle of TM described locally as the simutaneous kernel of everywhere linearly independent 1-forms $\varphi_{p+1}, \cdots, \varphi_{p+q}$, that is, for m in the domain of φ_α's,*

$$E_m = \{ u \in T_m M \mid \varphi_{p+1}(u) = \cdots = \varphi_{p+q}(u) = 0 \} .$$

Then E is integrable if and only if each $d\varphi_\alpha$ locally may be written of the form

$$d\varphi_\alpha = \sum_\beta \varphi_\beta \wedge \theta_{\alpha\beta}$$

for certain 1-forms $\theta_{\alpha\beta}$.

Theorem 2.2. *A sub-bundle* E *of* TM *is integrable if and only if, for any* $X, Y \in \Gamma(E)$, $[X, Y] \in \Gamma(E)$ *where* $[\ ,\ ]$ *denotes the bracket operator*.

3. Bundle-like metric and examples.

Let Q be the quotient bundle TM/E . The natural projection $\pi : TM \longrightarrow Q$ induces a map $\pi : \Gamma(TM) \longrightarrow \Gamma(Q)$. For each $Z \in \Gamma(Q)$, choose $\widehat{Z} \in \Gamma(TM)$ such that $Z = \pi(\widehat{Z})$. Then we may define a "partial" connection $\hat{\nabla} : \Gamma(E) \times \Gamma(Q) \longrightarrow \Gamma(Q)$ by

$$\hat{\nabla}_X Z = \pi([X, \tilde{Z}]) \tag{3.1}$$

for $X \in \Gamma(E)$, $Z \in \Gamma(Q)$. $\hat{\nabla}$ is well-defined. $\hat{\nabla}$ is a $\mathbb{R}$-bilinear map and satisfies the conditions:

$$\hat{\nabla}_X (fZ) = X(f)Z + f \hat{\nabla}_X Z \quad ,$$

$$\hat{\nabla}_{fX} Z = f \hat{\nabla}_X Z \quad .$$

Definition 3.1. $\hat{\nabla}$ is called the <u>Bott partial connection</u> on Q .

Next we have

Definition 3.2. In each flat coordinate chart $U(x^i, x^\alpha)$, a frame $\{ X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q} \}$ is an <u>adapted frame to the foliation</u> E if $\{ X_1, \cdots, X_p \}$ and $\{ \pi(X_{p+1}), \cdots, \pi(X_{p+q}) \}$ span $\Gamma(E|_U)$ and $\Gamma(Q|_U)$ respectively.

In each $U(x^i, x^\alpha)$, frames $\{ \partial/\partial x^i, \partial/\partial x^\alpha \}$ and $\{ \partial/\partial x^i, v_\alpha \}$ are adapted frames to E (See [14], [22], [23], [24]).

Definition 3.3. The adapted frame $\{ \partial/\partial x^i, v_\alpha \}$ is called the <u>basic adapted frame to the foliation</u> E .

It holds that

(3.2) $\pi([X, v_\alpha]) = 0$ for any $X \in \Gamma(E|_U)$.

Choosing a suitable riemannian metric on TM , we may identify the quatient bundle Q with the orthogonal complement bundle $E^\perp$ to E in TM . We have

(3.3) $TM \cong E \oplus Q \cong E \oplus E^\perp$

where $\oplus$ denotes the Whitney sum.

For two riemannian manifolds M and N , a riemannian submersion $f : M \longrightarrow N$ is a map satisfying the following axioms:

(i) f is a submersion.

(ii) f_* preserves lengths of vectors which are orthogonal to $f^{-1}(f(m))$ for each $m \in M$.

Let M be an n dimensional riemannian manifold with a riemannian metric $\langle\ ,\ \rangle$. A foliation E of codimension q ($= n - p$, $0 < p < n$) on M is a <u>riemannian</u> <u>foliation</u> if following data are given:

(i) A open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M .

(ii) An auxiliary q dimensional riemannian manifold N and a pseudogroup Γ of local isometries of N .

(iii) Submersions $f_\lambda : U_\lambda \longrightarrow N$.

(iv) For $U_\lambda \cap U_\mu \neq \phi$, there exists $\gamma_{\lambda\mu} \in \Gamma$ such that $f_\lambda = \gamma_{\lambda\mu} f_\mu$.

Then the metric $\langle\ ,\ \rangle$ has a local expression

(3.4) $\langle\ ,\ \rangle|_U = g_{ij}(x^k, x^\tau)\, w^i \cdot w^j + g_{\alpha\beta}(x^\tau)\, dx^\alpha \cdot dx^\beta$

in each flat coordinate chart $U(x^i, x^\alpha)$. We notice that, for each flat coordinate chart $U(x^i, x^\alpha)$, the submersion $f : U \longrightarrow N$ is a riemannian submersion such that $f^{-1}(f(m))$ is a plaque for any $m \in U$. (3.4) means that $(\partial/\partial x^i) g_{\alpha\beta} = 0$ where $g_{\alpha\beta} = \langle v_\alpha , v_\beta \rangle$.

Definition 3.4. Let M be a foliated manifold with a foliation E and a riemannian metric $\langle\ ,\ \rangle$. The metric $\langle\ ,\ \rangle$ is called a <u>bundle-like metric with respect to the foliation</u> E if it has a local expression (3.4).

A curve $\sigma(t)$ is called to be transverse to E if $\dot{\sigma}(t) \in \Gamma(E^\perp|_{\sigma(t)})$ where $\dot{\sigma}(t)$ denotes the tangent vector of $\sigma(t)$. Also, vector fields are called similar as curves. Then we have

Lemma 3.1 (See [24]). <u>The following conditions are equivalent</u>:

(i) $\langle\ ,\ \rangle$ <u>is a bundle-like metric with respect to</u> E .

(ii) <u>Any two transverse curves with the same projection on</u> N <u>have the same length</u>.

(iii) <u>Any two transverse vectors with the same projection on</u> N <u>have the same length</u>.

Now, we have following theorem which will play an important role in the next section.

Theorem 3.1 (See [14]). <u>The riemannian metric</u> $\langle\ ,\ \rangle$ <u>on a foliated manifold</u> M <u>with a foliation</u> E <u>of codimension</u> q <u>is a bundle-like metric with respect to</u> E <u>if and only if, for each flat coordinate chart</u> $U(x^i, x^\alpha)$, <u>there exists an</u>

orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that

$$< \nabla_{X_\alpha} X_i, X_\beta > + < \nabla_{X_\beta} X_i, X_\alpha > = 0$$

for $1 \leqq {}^\forall i \leqq p$ and $p+1 \leqq {}^\forall\alpha, {}^\forall\beta \leqq p+q$, where ∇ denotes the Levi-Civita connection with respect to the metric $< \ , \ >$.

First we prove the following lemma.

Lemma 3.2. *The metric $< \ , \ >$ is a bundle-like metric with respect to E if and only if, for each flat coordinate chart $U(x^i, x^\alpha)$, there exists an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E such that*

$$\hat{\nabla}_X \pi(X_\alpha) = 0 \qquad \text{for any } X \in \Gamma(E|_U) ,$$

where $\hat{\nabla}$ denotes the Bott partial connection.

Proof. Suppose that the metric $< \ , \ >$ is a bundle-like metric with respect to E. We have an orthonormal frame $\{X_i, X_\alpha\}$ from the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E by the Schmidt's orthogonalization process to $\{\partial/\partial x^i\}$ and $\{v_\alpha\}$ respectively. Then $\{X_i, X_\alpha\}$ is an orthonormal adapted frame to E. By (3.2), $< v_\alpha, v_\beta > = g_{\alpha\beta}(x^\tau)$ implies $\pi([X_i, X_\alpha]) = 0$. Therefore, we have $\hat{\nabla}_X \pi(X_\alpha) = 0$ for any $X \in \Gamma(E|_U)$.

Conversely, let $\{X_i, X_\alpha\}$ be an orthonormal adapted frame to E such that $\hat{\nabla}_X \pi(X_\alpha) = 0$ for any $X \in \Gamma(E|_U)$. Since the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E satisfies $< \partial/\partial x^i, v_\alpha > = 0$, we may write v_α of form $v_\alpha = h_\alpha^\beta X_\beta$ where h_α^β are functions on U. Then we have

$$\hat{\nabla}_{\partial/\partial x^i} \pi(v_\alpha) = \pi([\partial/\partial x^i, v_\alpha])$$

$$= \pi([\partial/\partial x^i, h_\alpha^\beta X_\beta])$$

$$= h_\alpha^\beta \pi([\partial/\partial x^i, X_\beta]) + (\partial/\partial x^i)(h_\alpha^\beta) \pi(X_\beta)$$

$$= h_\alpha^\beta \hat{\nabla}_{\partial/\partial x^i} \pi(X_\beta) + (\partial/\partial x^i)(h_\alpha^\beta) \pi(X_\beta)$$

$$= (\partial/\partial x^i)(h_\alpha^\beta) \pi(X_\beta) .$$

by (3.2), $\hat{\nabla}_{\partial/\partial x^i} \pi(v_\alpha) = \pi([\partial/\partial x^i, v_\alpha]) = 0$. Then
ıave

$$(\partial/\partial x^i)(h_\alpha^\beta) = 0$$

:he linearly independence of $\pi(X_\beta)$. Thus we have

$$(\partial/\partial x^i) \langle v_\alpha , v_\beta \rangle = (\partial/\partial x^i) \langle h_\alpha^\gamma X_\gamma , h_\beta^\tau X_\tau \rangle$$

$$= (\partial/\partial x^i) (h_\alpha^\gamma h_\beta^\tau \delta_{\gamma\tau})$$

$$= 0 .$$

:efore, the metric $\langle\ ,\ \rangle$ is a bundle-like metric.

Q.E.D.

Proof of Theorem 3.1. Suppose that the metric $\langle\ ,\ \rangle$ is
ındle-like metric. By Lemma 3.2, there exists an orthonormal
ɔted frame $\{ X_i, X_\alpha \}$ to E such that $\hat{\nabla}_X \pi(X_\alpha) = 0$ for
$X \in \Gamma(E|_U)$. Then we have

$$\langle \nabla_{X_\alpha} X_i, X_\beta \rangle = \langle \nabla_{X_i} X_\alpha, X_\beta \rangle - \langle [X_i, X_\alpha], X_\beta \rangle$$

$$= \langle \nabla_{X_i} X_\alpha, X_\beta \rangle ,$$

since $\hat{\nabla}_{X_i} \pi(X_\alpha) = 0$ implies $[\, X_i, X_\alpha \,] \in \Gamma(E|_U)$. Therefore, we have

$$< \nabla_{X_\alpha} X_i, X_\beta > + < \nabla_{X_\beta} X_i, X_\alpha >$$

$$= < \nabla_{X_i} X_\alpha, X_\beta > + < \nabla_{X_i} X_\beta, X_\alpha >$$

$$= X_i < X_\alpha, X_\beta >$$

$$= 0 \quad .$$

Conversely, suppose that there exists an orthonomal adapted frame $\{ X_i, X_\alpha \}$ to E such that $< \nabla_{X_\alpha} X_i, X_\beta > + < \nabla_{X_\beta} X_i, X_\alpha > = 0$. Then we may write the basic adapted frame $\{ \partial/\partial x^i, v_\alpha \}$ to E of form $\partial/\partial x^i = h_i^j X_j$ and $v_\alpha = h_\alpha^\gamma X_\gamma$ where h_i^j and h_α^γ are functions on U . Thus we have

$$(\partial/\partial x^i) < v_\alpha, v_\beta >$$

$$= < \nabla_{\partial/\partial x^i} v_\alpha, v_\beta > + < v_\alpha, \nabla_{\partial/\partial x^i} v_\beta >$$

$$= h_i^j h_\beta^\tau (X_j (h_\alpha^\gamma) \delta_{\gamma\tau} + h_\alpha^\gamma < [\, X_j, X_\gamma \,], X_\tau >)$$

$$+ h_i^j h_\alpha^\gamma h_\beta^\tau < \nabla_{X_\gamma} X_j, X_\tau >$$

$$+ h_i^j h_\alpha^\gamma (X_j (h_\beta^\tau) \delta_{\gamma\tau} + h_\beta^\tau < X_\gamma, [\, X_j, X_\tau \,] >)$$

$$+ h_i^j h_\alpha^\gamma h_\beta^\tau < X_\gamma, \nabla_{X_\tau} X_j > \quad .$$

Since we have

$$X_j (h_\alpha^\gamma) \delta_{\gamma\tau} + h_\alpha^\gamma < [\, X_j, X_\gamma \,], X_\tau >$$

$$= X_j(h_\alpha^\gamma)\delta_{\gamma\tau} + \langle [\ X_j,\ h_\alpha^\gamma X_\gamma\ ],\ X_\tau \rangle$$

$$- \langle X_j(h_\alpha^\gamma)\ X_\gamma,\ X_\tau \rangle$$

$$= \langle [\ X_j,\ v_\alpha\ ],\ X_\tau \rangle$$

$$= 0\ ,$$

we have

$$(\partial/\partial x^i) \langle v_\alpha,\ v_\beta \rangle$$

$$= h_i^j\ h_\alpha^\gamma\ h_\beta^\tau\ (\langle \nabla_{X_\gamma} X_j,\ X_\tau \rangle + \langle \nabla_{X_\tau} X_j,\ X_\gamma \rangle)$$

$$= 0\ .$$

Therefore, the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E . Q.E.D

Definition 3.5. A leaf L in a foliated manifold M with a foliation E and the Levi-Civita connection ∇ is called <u>totally geodesic</u> if $\nabla_X Y|_m \in T_m L$ for each point $m \in L$, any flat coordinate chart U ($m \in U$) and any $X, Y \in \Gamma(E|_U)$, where $T_m L$ denotes the tangent space of L at m .

We remark that an immersed sub-manifold N of a manifold M with the Levi-Civita connection ∇ is totally geodesic (= the second fundamental form of N identically vanishes) if and only if $\nabla_X Y \in \Gamma(TN)$ for any $X, Y \in \Gamma(TN)$ (See [15]).

The foliated manifolds all leaves of which are totally geodesic are studied by many pepole (See [2], [3], [5], [16]).

We are often able to find out the foliated manifolds with bundle-like metrics in the study of differential geometry:

(i) Let M be a riemannian manifold acted on by a group of isometries such that all orbits are of same dimension. M is a foliated manifold with orbits as its leaves, and the riemannian metric on M is a bundle-like metric with respect to the foliation (See [5], [7], [22], [23], [24]).

(ii) Let M be the tangent bundle TN over a q dimensional riemannian manifold N . Then M is a foliated manifold with fibers as leaves, and the Sasaki metric (See [27]) on TN is a bundle-like metric with respect to the foliation.

(iii) Let $\varphi : M \longrightarrow N$ be a riemannian submersion (See [6], [20]). M is a foliated manifold with fibers $\varphi^{-1}(b)$ ($b \in N$) as leaves, and the riemannian metric on M is a bundle-like metric with respect to the foliation.

We may also introduce examples.

Example 3.1. Let M be a flat torus which is identified with $\{(x,y) \in \mathbb{R}^2 \mid 0 \leqq x \leqq 1 , 0 \leqq y \leqq 1\}$. M is considered a foliated manifold by a family of straight lines with constant slope α . If α is a rational number then all leaves are regular, but if α is an irrational number then no leaves are regular. The flat metric on M is a bundle-like metric with respect to the foliation.

Example 3.2. Let M be a flat Moebius strip which is identified with $\{(x,y) \in \mathbb{R}^2 \mid 0 \leqq x \leqq 1 , -1 < y < 1 , (0,y)$ is identified with $(1,-y)\}$. M is considered a foliated manifold whose leaves are given by straight lines $y = \text{constant}$.

A leaf through a point $(0,0)$ is not regular, and another all leaves are regular. The flat metric on M is a bundle-like metric with respect to the foliation.

We remark that not all foliated manifolds have bundle-like metrics with respect to the foliations. For example, the canonical metric on a 3 dimensional sphere S^3 is not bundle-like metric with respect to the Reeb foliation.

4. Geodesic making constant angle with leaves.

Let $\gamma(s)$ (or γ) be a geodesic in M parametrized by arc-length s, that is, $\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = 0$ where $\dot{\gamma}(s)$ denotes the tangent vector of γ at $\gamma(s)$.

For any point $\gamma(s)$, we may choose a flat coordinate chart $U(x^i, x^\alpha)$ such that $\gamma(s) \in U$ and an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E in U. Let $\{\theta^i, \theta^\alpha\}$ be its dual adapted frame. Then we define $f = \{f_U\}$ by

$$f_U(s) = f_U(\gamma(s)) = \Sigma_{i=1}^{p} (\theta^i (\dot{\gamma}(s)))^2 \quad . \tag{4.1}$$

Lemma 4.1. <u>The function</u> $f = \{f_U\}$ <u>defined by</u> (4.1) <u>is independent of the choice of</u> U. f <u>is a differentiable function on</u> I_γ <u>which is a range of parameter</u> s <u>of</u> γ.

The geometric meaning of $f(s)$ is a square of the length of orthographic vector in $E_{\gamma(s)}$ of a vector $\dot{\gamma}(s)$ in $T_{\gamma(s)}M$. Let $\alpha(s)$ be an angle between the orthographic vector of $\dot{\gamma}(s)$ and $\dot{\gamma}(s)$. Then we have that

$$f(s) = (\cos \alpha(s))^2 \quad .$$

Definition 4.1. A geodesic $\gamma(s)$ parametrized by arc-length s is called a <u>geodesic making constant angle with leaves</u> if the function f is a constant, that is, $df(s)/ds = 0$ for any $s \in I_\gamma$.

Theorem 4.1. <u>Let</u> M <u>be a foliated manifold with a foliation</u> E <u>of codimension</u> q ($= n - p$) <u>and with a riemannian metric</u> $\langle \ , \ \rangle$. <u>Suppose that all leaves are totally geodesic</u>.

(i) If the metric $<\ ,\ >$ is a bundle-like metric with respect to E, then any geodesic in M is a geodesic making constant angle with leaves.

(ii) If all geodesics in M are of making constant angle with leaves, then the metric $<\ ,\ >$ is a bundle-like metric with respect to E.

Proof. (i) Let $\gamma(s)$ be a geodesic parametrized by arc-length s. In a flat coordinate chart $U(x^i, x^\alpha)$ such that $\gamma(s) \in U$ for any fixed $s \in I_\gamma$, by Theorem 3.1, we have an orthonormal adapted frame $\{ X_i, X_\alpha \}$ to E satisfying $< \nabla_{X_\alpha} X_i, X_\beta > + < \nabla_{X_\beta} X_i, X_\alpha > = 0$, that is,

$$\hat{\Gamma}^\beta_{\alpha i} + \hat{\Gamma}^\alpha_{\beta i} = 0 \tag{4.2}$$

where $\nabla_{X_A} X_B = \hat{\Gamma}^C_{AB} X_C$ ($A, B, C = 1, 2, \cdots, p, p+1, \cdots, p+q$).
And, by the orthonormality of the frame, we have

$$\hat{\Gamma}^C_{AB} + \hat{\Gamma}^B_{AC} = 0 \quad . \tag{4.3}$$

Then we have

$$df(s)/ds = \frac{d}{ds}\left(\Sigma^p_{i=1} \left(\theta^i(\dot{\gamma}(s)) \right)^2 \right)$$

$$= 2 \Sigma^p_{i=1} \left(\theta^i(\dot{\gamma}(s)) \right) \frac{d}{ds}\left(\theta^i(\dot{\gamma}(s)) \right)$$

and

$$0 = \theta^i(\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s))$$

$$= \frac{d}{ds}(\theta^i(\dot{\gamma}(s)))$$

$$+ \hat{\Gamma}^i_{jk} \theta^j(\dot{\gamma}(s)) \theta^k(\dot{\gamma}(s)) + \hat{\Gamma}^i_{j\beta} \theta^j(\dot{\gamma}(s)) \theta^\beta(\dot{\gamma}(s))$$

$$+ \hat{\Gamma}^{i}_{\beta j} \; \theta^{\beta}(\dot{\gamma}(s)) \, \theta^{j}(\dot{\gamma}(s)) + \hat{\Gamma}^{i}_{\alpha\beta} \; \theta^{\alpha}(\dot{\gamma}(s)) \, \theta^{\beta}(\dot{\gamma}(s))$$

where $\{ \theta^{i}, \theta^{\alpha} \}$ denotes the dual frame of $\{ X_i, X_\alpha \}$. Thus we have, by (4.2) and (4.3),

$$df(s)/ds = 2 \sum_{i,j,\beta} \hat{\Gamma}^{\beta}_{ji} \, \theta^{i}(\dot{\gamma}(s)) \, \theta^{j}(\dot{\gamma}(s)) \, \theta^{\beta}(\dot{\gamma}(s)) \quad .$$

Since all leaves are totally geodesic, we have $\hat{\Gamma}^{\beta}_{ji} = 0$. Therefore we have $df(s)/ds = 0$ for any $s \in I_\gamma$.

(ii) For any point $m \in M$, we take a flat coordinate chart $U(x^{i}, x^{\alpha})$ at m and any geodesic $\gamma(s)$ through m making constant angle with leaves. By the method of Schmidt's orthonormalization, we may make the basic adapted frame $\{ \partial/\partial x^{i}, v_\alpha \}$ to E into an adapted frame $\{ \widetilde{X}_i, \widetilde{X}_\alpha \}$ to E such that $\{ \widetilde{X}_i \}$ is mutually orthonormal and $\widetilde{X}_\alpha = v_\alpha$. We set $\nabla_{\widetilde{X}_A} \widetilde{X}_B = \widetilde{\Gamma}^{C}_{AB} \widetilde{X}_C$, we have

$$\widetilde{\Gamma}^{i}_{Ak} + \widetilde{\Gamma}^{k}_{Ai} = 0 \quad , \quad \widetilde{\Gamma}^{i}_{j\beta} + \widetilde{\Gamma}^{\tau}_{ji} \, g_{\beta\tau} = 0 \quad , \quad \widetilde{\Gamma}^{\beta}_{ji} = 0 \quad .$$

Thus we have

$$0 = df(s)/ds$$

$$= - 2 \sum_{i,\alpha,\beta} \widetilde{\Gamma}^{i}_{\alpha\beta} \, \widetilde{\theta}^{i}(\dot{\gamma}(s)) \, \widetilde{\theta}^{\alpha}(\dot{\gamma}(s)) \, \widetilde{\theta}^{\beta}(\dot{\gamma}(s))$$

for any $s \in I_\gamma$, where $\{ \widetilde{\theta}^{i}, \widetilde{\theta}^{\alpha} \}$ denotes the dual frame of $\{ \widetilde{X}_i, \widetilde{X}_\alpha \}$. As the choice of a geodesic γ is arbitrary, we have, for each i,

$$\widetilde{\Gamma}^{i}_{\alpha\beta} \, \widetilde{\theta}^{\alpha}(\dot{\gamma}(s)) \, \widetilde{\theta}^{\beta}(\dot{\gamma}(s)) = 0 \quad .$$

We set $\dot{\gamma}(s) = f^{i} \widetilde{X}_i + f^{\alpha} \widetilde{X}_\alpha = f^{i} \widetilde{X}_i + f^{\alpha} v_\alpha$. Then we have

$$(4.4) \qquad \widetilde{\Gamma}^{i}_{\alpha\beta}\, f^{\alpha} f^{\beta} = 0 \qquad (\widetilde{\theta}^{\alpha}(\dot{\gamma}(s)) = f^{\alpha}) .$$

Thus

$$\widetilde{X}_i < f^{\alpha} \widetilde{X}_{\alpha},\ f^{\beta} \widetilde{X}_{\beta} >$$

$$= < \nabla_{f^{\alpha} \widetilde{X}_{\alpha}} \widetilde{X}_i,\ f^{\beta} \widetilde{X}_{\beta} > + < [\ \widetilde{X}_i,\ f^{\alpha} \widetilde{X}_{\alpha}\ ],\ f^{\beta} \widetilde{X}_{\beta} >$$

$$+ < f^{\alpha} \widetilde{X}_{\alpha},\ \nabla_{f^{\beta} \widetilde{X}_{\beta}} \widetilde{X}_i > + < f^{\alpha} \widetilde{X}_{\alpha},\ [\ \widetilde{X}_i,\ f^{\beta} \widetilde{X}_{\beta}\ ] >$$

$$= 2\ f^{\alpha} f^{\beta}\, \widetilde{\Gamma}^{\tau}_{\alpha i}\, g_{\tau\beta} + 2\ f^{\alpha}\, \widetilde{X}_i(f^{\beta})\, g_{\alpha\beta} .$$

Here we note that $[\ \widetilde{X}_i,\ \widetilde{X}_{\alpha}\ ] \in \Gamma(E|_U)$.

On the other hand, we have

$$\widetilde{X}_i < f^{\alpha} \widetilde{X}_{\alpha},\ f^{\beta} \widetilde{X}_{\beta} > = \widetilde{X}_i(\ f^{\alpha} f^{\beta} g_{\alpha\beta}\ )$$

$$= 2\ f^{\alpha}\, \widetilde{X}_i(f^{\beta})\, g_{\alpha\beta} + f^{\alpha} f^{\beta}\, \widetilde{X}_i(g_{\alpha\beta}) .$$

Thus we have

$$2\ f^{\alpha} f^{\beta}\, \widetilde{\Gamma}^{\tau}_{\alpha i}\, g_{\tau\beta} = f^{\alpha} f^{\beta}\, \widetilde{X}_i(g_{\alpha\beta}) .$$

Since $< \nabla_{\widetilde{X}_{\alpha}} \widetilde{X}_{\beta},\ X_i > + < \widetilde{X}_{\beta},\ \nabla_{\widetilde{X}_{\alpha}} \widetilde{X}_i > = 0$, that is, $\widetilde{\Gamma}^{i}_{\alpha\beta} + \widetilde{\Gamma}^{\tau}_{\alpha i}\, g_{\tau\beta} = 0$, we have

$$f^{\alpha} f^{\beta}\, \widetilde{X}_i(g_{\alpha\beta}) = -\ 2\ f^{\alpha} f^{\beta}\, \widetilde{\Gamma}^{i}_{\alpha\beta} = 0 \quad (\text{ from } (4.4)\ ) .$$

As the choice of the geodesic γ is arbitrary, we have

$$\widetilde{X}_i(g_{\alpha\beta}) = 0 .$$

By the construction of $\widetilde{X}_i$, we have $\widehat{X}_i = \sum_{k=1}^{i} h^k_i\, \partial/\partial x^k$ ($1 \leqq {}^{\forall} i \leqq p$, h^k_i are functions on U), and thus we have

$$(\partial/\partial x^i)\ g_{\alpha\beta} = 0$$

for $1 \leqq {}^{\forall}i \leqq p$ and $p+1 \leqq {}^{\forall}\alpha, {}^{\forall}\beta \leqq p+q$. Therefore the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E .

Q.E.D.

The condition that all leaves are totally geodesic is necessary:

Example 4.1. Let $\mathbb{R}^2$ be a x-y plane with the flat metric. We set $M = \mathbb{R}^2 - \{$ the origin point $\}$, then M is considered a foliated manifold whose leaves are $L_r = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \}$ for any $r > 0$ and a metric $\langle\ ,\ \rangle$ on M is induced from the flat metric on $\mathbb{R}^2$. All leaves are not totally geodesic. A geodesic given by $y = \text{constant} = c$ is to be tangent to L_c at $(0,c)$ and make an angle of $\pi/3$ with the leaf L_{2c} at $(\sqrt{3}\,c,c)$.

For the geodesics orthogonal to the leaves, we may omit the condition that all leaves are totally geodesic.

Theorem 4.2. <u>Let</u> M <u>be a foliated manifold with a foliation</u> E <u>of codimension</u> $q\ (= n - p)$ <u>and with a riemannian metric</u> $\langle\ ,\ \rangle$.

(i) (B. L. Reinhart[24]) <u>If the riemannian metric</u> $\langle\ ,\ \rangle$ <u>is a bundle-like metric with respect to</u> E , <u>then any geodesic orthogonal to the leaf at some point on the geodesic is to be orthogonal to the leaves at all points on the geodesic</u>.

(ii) <u>If, for any point</u> $m \in M$, <u>all geodesics that are to be orthogonal to the leaf at</u> m <u>are to be orthogonal to the leaves at all points on the geodesics, then the metric</u> $\langle\ ,\ \rangle$ <u>is a bundle-like metric with respect to</u> E .

Theorem 4.2 (i) is a generalization of the corresponding results of Y. Muto[17], B. O'Neil[21] and S. Sasaki[27]. Our proof of (i) differs from Reinhart's one.

Proof. (i) Let $\gamma(s)$ be a geodesic orthogonal to a leaf at some point $\gamma(s_0)$. We take a flat coordinate chart $U(x^i, x^\alpha)$ ($|x^i|$, $|x^\alpha| < 1$) of $\gamma(s_0)$, where we use two adapted frames $\{ \partial/\partial x^i, \partial/\partial x^\alpha \}$ and $\{ \partial/\partial x^i, v_\alpha \}$.

The bundle-like metric $\langle \ , \ \rangle$ is expressed locally by

$$\langle \ , \ \rangle|_U = h_{ij}\, dx^i \cdot dx^j + 2h_{i\beta}\, dx^i \cdot dx^\beta + h_{\alpha\beta}\, dx^\alpha \cdot dx^\beta$$

$$= g_{ij}(x^k, x^\tau)\, w^i \cdot w^j + g_{\alpha\beta}(x^\tau)\, dx^\alpha \cdot dx^\beta$$

where $g_{ij} = h_{ij}$ and $g_{\alpha\beta} = h_{\alpha\beta} - h_{ij} A^i_\alpha A^j_\beta$.

In the fixed chart $U(x^i, x^\alpha)$, we define a metric $\langle\!\langle \ , \ \rangle\!\rangle_U$ on U by

$$\langle\!\langle \ , \ \rangle\!\rangle_U = g_{ij}\, dx^i \cdot dx^j + g_{\alpha\beta}\, dx^\alpha \cdot dx^\beta$$

and let $\hat{\nabla}_U$ denote the Levi-Civita connection on U with respect to the metric $\langle\!\langle \ , \ \rangle\!\rangle_U$.

Now the geodesic $\gamma(s)$ in U is given by

$$\gamma(s) = (\ \gamma^i(s),\ \gamma^\alpha(s)\ )$$

and

$$\gamma(s_0) = (\ \gamma^i(s_0),\ \gamma^\alpha(s_0)\ )$$

$$\dot{\gamma}(s_0) = a^\alpha\, v_\alpha \big|_{\gamma(s_0)}$$

where a^α are constants. In U , we may define a $\hat{\nabla}_U$-geodesic $\tilde{\gamma}$, $\tilde{\gamma}(t) = (\ \gamma^i(s_0),\ \tilde{\gamma}^\alpha(t)\ )$, by

$$\hat{\nabla}_{U\ \tilde{\gamma}'(t)}\ \tilde{\gamma}'(t) = 0 \quad \text{for any } t \in (-\varepsilon,\varepsilon) \quad (\ \varepsilon > 0\ )$$

and

$$\tilde{\gamma}^{\alpha}(0) = \gamma^{\alpha}(s_{\circ})$$

$$\tilde{\gamma}'^{i}(0) = 0 \quad , \quad \tilde{\gamma}'^{\alpha}(0) = a^{\alpha}$$

where $'$ denotes the derivative with respect to t . Then, in U , we may define a curve σ by

$$\sigma(t) = (\ \varphi^{i}(t),\ \tilde{\gamma}^{\alpha}(t)\ )$$

where

$$\varphi^{i}(0) = \gamma^{i}(s_{\circ}) \quad , \quad \varphi'^{i}(t) = -A^{i}_{\alpha}\ \tilde{\gamma}'^{\alpha}(t)$$

for any $t \in (-\varepsilon,\varepsilon)$.

Therefore we have

$$(4.5) \qquad \tilde{\gamma}'(t) = \tilde{\gamma}'^{\alpha}(t)\ (\ \partial/\partial x^{\alpha}\ )\big|_{\tilde{\gamma}(t)}$$

$$(4.6) \qquad \sigma'(t) = \tilde{\gamma}'^{\alpha}(t)\ v_{\alpha}\big|_{\sigma(t)}$$

and

$$\sigma(0) = \gamma(s_{\circ})$$

$$\sigma'(0) = a^{\alpha}\ v_{\alpha}\big|_{\sigma(0)} = \dot{\gamma}(s_{\circ}) \quad .$$

Let $\{ w^{i},\ dx^{\alpha} \}$ (resp. $\{ \hat{\theta}^{i},\ \hat{\theta}^{\alpha} \}$) be a dual adapted frame of $\{ \partial/\partial x^{i},\ v_{\alpha} \}$ (resp. $\{ \partial/\partial x^{i},\ \partial/\partial x^{\alpha} \}$). We have

$$w^{i}(\ \nabla_{\sigma'(t)}\ \sigma'(t)\ )$$

$$= \frac{d}{dt}(\ w^{i}(\ \sigma'(t)\ )\ )$$

$$+ \Gamma^{i}_{jk}\ w^{j}(\ \sigma'(t)\ )\ w^{k}(\ \sigma'(t)\ )$$

$$+ \Gamma^{i}_{j\beta}\, w^{j}(\ \sigma'(t)\ )\ dx^{\beta}(\ \sigma'(t)\ )$$

$$+ \Gamma^{i}_{\beta j}\, dx^{\beta}(\ \sigma'(t)\ )\ w^{j}(\ \sigma'(t)\ )$$

$$+ \Gamma^{i}_{\alpha\beta}\, dx^{\alpha}(\ \sigma'(t)\ )\ dx^{\beta}(\ \sigma'(t)\ )$$

$$= \Gamma^{i}_{\alpha\beta}\, dx^{\alpha}(\ \sigma'(t)\ )\ dx^{\beta}(\ \sigma'(t)\ ) \qquad (\text{ from } w^{i}(\ \sigma'(t)\ ) = 0\ )$$

By $\Gamma^{i}_{\alpha\beta} + \Gamma^{i}_{\beta\alpha} = 0$, we have

$$w^{i}(\ \nabla_{\sigma'(t)}\ \sigma'(t)\ ) = 0 \tag{4.7}$$

for $1 \leqq {}^{\forall}i \leqq p$.

On the other hand, we have

$$dx^{\alpha}(\ \nabla_{\sigma'(t)}\ \sigma'(t)\ )$$

$$= \frac{d}{dt}(\ dx^{\alpha}(\ \sigma'(t)\ )\ )$$

$$+ \Gamma^{\alpha}_{ij}\, w^{i}(\ \sigma'(t)\ )\ w^{j}(\ \sigma'(t)\ )$$

$$+ \Gamma^{\alpha}_{j\beta}\, w^{j}(\ \sigma'(t)\ )\ dx^{\beta}(\ \sigma'(t)\ )$$

$$+ \Gamma^{\alpha}_{\beta j}\, dx^{\beta}(\ \sigma'(t)\ )\ w^{j}(\ \sigma'(t)\ )$$

$$+ \Gamma^{\alpha}_{\beta\tau}\, dx^{\beta}(\ \sigma'(t)\ )\ dx^{\tau}(\ \sigma'(t)\ )$$

$$= \frac{d}{dt}(\ dx^{\alpha}(\ \sigma'(t)\ ) + \Gamma^{\alpha}_{\beta\tau}\, dx^{\beta}(\ \sigma'(t)\ )\ dx^{\tau}(\ \sigma'(t)\ )$$

$$= \frac{d}{dt}(\ \widetilde{\gamma}'^{\alpha}(t)\ ) + \Gamma^{\alpha}_{\beta\tau}\, \widetilde{\gamma}'^{\beta}(t)\ \widetilde{\gamma}'^{\tau}(t) \qquad (\text{ from } (4.6)\ ) .$$

Then we have

(4.8) $\quad dx^{\alpha}(\nabla_{\sigma'(t)} \sigma'(t))$

$$= \frac{d}{dt}(\hat{\theta}^{\alpha}(\tilde{\gamma}'(t))) + \Gamma^{\alpha}_{\beta\tau} \hat{\theta}^{\beta}(\sigma'(t)) \hat{\theta}^{\tau}(\tilde{\gamma}'(t))$$

by (4.5). We note that

$$\Gamma^{\alpha}_{\beta\tau} = \frac{1}{2} g^{\alpha\varepsilon}(v_{\beta}(g_{\varepsilon\tau}) + v_{\tau}(g_{\beta\varepsilon}) - v_{\varepsilon}(g_{\beta\tau}))$$

$$= \frac{1}{2} g^{\alpha\varepsilon}(\partial g_{\varepsilon\tau}/\partial x^{\beta} + \partial g_{\beta\varepsilon}/\partial x^{\tau} - \partial g_{\beta\tau}/\partial x^{\varepsilon})$$

since $\partial g_{\beta\tau}/\partial x^{i} = 0$.

Now, we set

$$\hat{\nabla}_{U\, \partial/\partial x^{\beta}} \partial/\partial x^{\tau} = \hat{\Gamma}^{k}_{\beta\tau} \partial/\partial x^{k} + \hat{\Gamma}^{\alpha}_{\beta\tau} \partial/\partial x^{\alpha} ,$$

then we have

$$\hat{\Gamma}^{k}_{\beta\tau} = 0$$

$$\hat{\Gamma}^{\alpha}_{\beta\tau} = \frac{1}{2} g^{\alpha\varepsilon}(\partial g_{\varepsilon\tau}/\partial x^{\beta} + \partial g_{\beta\varepsilon}/\partial x^{\tau} - \partial g_{\beta\tau}/\partial x^{\varepsilon}) .$$

Thus we have

(4.9) $$\Gamma^{\alpha}_{\beta\tau} = \hat{\Gamma}^{\alpha}_{\beta\tau}$$

and, from (4.8) and (4.9), we have

(4.10) $\quad dx^{\alpha}(\nabla_{\sigma'(t)} \sigma'(t))$

$$= \frac{d}{dt}(\hat{\theta}^{\alpha}(\tilde{\gamma}'(t))) + \hat{\Gamma}^{\alpha}_{\beta\tau} \hat{\theta}^{\beta}(\tilde{\gamma}'(t)) \hat{\theta}^{\tau}(\tilde{\gamma}'(t))$$

$$= \hat{\theta}^{\alpha}(\hat{\nabla}_{U\, \tilde{\gamma}'(t)} \tilde{\gamma}'(t))$$

$$= 0 ,$$

since $\tilde{\gamma}$ is a $\hat{\nabla}_U$-geodesic. From (4.7) and (4.10), we have

$$\nabla_{\sigma'(t)} \sigma'(t) = 0 ,$$

that is, a curve $\sigma(t)$ is a geodesic orthogonal to the leaves at all points $\sigma(t)$ ($-\varepsilon < t < \varepsilon$). The geodesic $\sigma(t)$ satisfies

$$\sigma(0) = \gamma(s_o) \qquad \sigma'(0) = \dot{\gamma}(s_o) .$$

Let $V_o(x^i, x^\alpha)$ be the smallest flat coodinate chart of $\gamma(s_o)$ in $U(x^i, x^\alpha)$ containing the geodesic $\sigma(t)$ ($t \in (-\varepsilon, \varepsilon)$). Thus, by the uniqueness of solution of differential equation system, we have

$$\gamma(s) = \sigma(s - s_o)$$

in V_o . Therefore the geodesic γ is a geodesic orthogonal to the leaves at all points on γ in V_o .

Next, we take a point $\gamma(s_1)$ in V_o which differs from the point $\gamma(s_o)$. We may assume that $s_o < s_1$. The geodesic γ is orthogonal to the leaf at $\gamma(s_1)$. We take a flat coordinate chart $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$ ($|\bar{x}^i|$, $|\bar{x}^\alpha| < 1$) of $\gamma(s_1)$, and, as above way, we may take a flat coordinate chart $V_1(\bar{x}^i, \bar{x}^\alpha)$ in which the geodesic γ is a geodesic orthogonal to the leaves at all points on γ . And this way goes on. Thus we have sequences $\{\gamma(s_N)\}$ and $\{V_N\}$, where $s_N < s_{N+1}$ and $N = 1, 2, 3, \cdots$. Let I_γ denote the range of the parameter s of the geodesic γ . Then we have

$$\sup \{ s_N \mid N = 1, 2, 3, \cdots \} = \sup I_\gamma .$$

In fact, if there exists $s^* \in I_\gamma$ such that $\lim_{N \to \infty} \gamma(s_N) = \gamma(s^*)$ and $s^* < \sup I_\gamma$, then γ is orthogonal to the leaf at $\gamma(s^*)$.

Then the above argument may be applied at $\gamma(s^*)$.

And if we take a sequence $\{\gamma(s_N)\}$ such that $s_N > s_{N+1}$ ($N = 1, 2, 3, \cdots$), then we have

$$\inf \{ s_N \mid N = 1, 2, 3, \cdots \} = \inf I_\gamma \ .$$

Therefore the geodesic γ is a geodesic orthogonal to the leaves at all points on γ .

(ii) For any point $m \in M$, we take a flat coordinate chart $U(x^i, x^\alpha)$ of the point m . Let $\gamma(s)$ be any geodesic through m orthogonal to the leaves. We take an adapted frame $\{\bar{X}_i, \bar{X}_\alpha\}$ to E such that $\{\bar{X}_i\}$ are mutually orthogonal and are given by the method of Schmidt's orthogonalization from $\{\partial/\partial x^i\}$, and $\bar{X}_\alpha = v_\alpha$. Let $\{\bar{\theta}^i, \bar{\theta}^\alpha\}$ denote the dual frame of $\{\bar{X}_i, \bar{X}_\alpha\}$. For each i , we have

$$\begin{aligned}
0 &= \bar{\theta}^i (\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s)) \\
&= \frac{d}{ds} (\bar{\theta}^i (\dot{\gamma}(s))) \\
&\quad + \Gamma^i_{jk} \ \bar{\theta}^j (\dot{\gamma}(s)) \ \bar{\theta}^k (\dot{\gamma}(s)) \\
&\quad + \Gamma^i_{j\beta} \ \bar{\theta}^j (\dot{\gamma}(s)) \ \bar{\theta}^\beta (\dot{\gamma}(s)) \\
&\quad + \Gamma^i_{\beta j} \ \bar{\theta}^\beta (\dot{\gamma}(s)) \ \bar{\theta}^j (\dot{\gamma}(s)) \\
&\quad + \Gamma^i_{\alpha\beta} \ \bar{\theta}^\alpha (\dot{\gamma}(s)) \ \bar{\theta}^\beta (\dot{\gamma}(s)) \\
&= \Gamma^i_{\alpha\beta} \ \bar{\theta}^\alpha (\dot{\gamma}(s)) \ \bar{\theta}^\beta (\dot{\gamma}(s)) \ .
\end{aligned}$$

By the same way as the proof of Theorem 4.1 (ii), we have that the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E .

Q.E.D.

Theorem 4.1 and 4.2 are generalizations of [14].

Definition 4.2. A geodesic γ on M is called a <u>transversal geodesic</u> if γ is to be orthogonal the the leaves at all points on γ .

Even if M admits only one transversal geodesic, then the metric $\langle\ ,\ \rangle$ on M is not necessarilly a bundle-like metric with respect to the foliation:

Example 4.2. Let $\mathbb{R}^2$ be a u-v plane with the flat metric $\langle\ ,\ \rangle$. $\mathbb{R}^2$ is a foliated manifold whose leaves are given by $\{ (u,v) \in \mathbb{R}^2 \mid v = u^2 - a \}$ for any $a \in \mathbb{R}$. A geodesic given by $u = 0$ is only one transversal geodesic. We set

$$f(u) = \frac{1}{2}\left(2u(4u^2 + 1)^{1/2} + \log(2u + (4u^2 + 1)^{1/2}) \right)$$

$$x = f(u) \quad , \qquad y = v - u^2 \quad .$$

Setting $w = dx + 2u(4u^2 + 1)^{-1/2} dy$, we have

$$\begin{aligned} \langle\ ,\ \rangle &= du\cdot du + dv\cdot dv \\ &= w\cdot w + (4u^2 + 1)^{-1} dy\cdot dy \\ &= w\cdot w + (4(f^{-1}(x))^2 + 1)^{-1} dy\cdot dy \quad . \end{aligned}$$

Thus the metric $\langle\ ,\ \rangle$ is not a bundle-like metric with respect to the foliation.

5. Focal point of a leaf.

We recall that the bundle-like metric $\langle\ ,\ \rangle$ on M is locally expressed by

$$\langle\ ,\ \rangle|_U = g_{ij}(x^k, x^\tau)\, w^i \cdot w^j + g_{\alpha\beta}(x^\tau)\, dx^\alpha \cdot dx^\beta$$

in each flat coordinate chart $U(x^i, x^\alpha)$. Here and hereafter, vector fields, forms, tensor fields and etc. are locally expressed by the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E and its dual $\{w^i, dx^\alpha\}$, where

$$w^i = dx^i + A^i_\alpha\, dx^\alpha \quad \text{and} \quad v_\alpha = \partial/\partial x^\alpha - A^i_\alpha\, \partial/\partial x^i \ .$$

We set, in U ,

$$\nabla_{\partial/\partial x^i}\, \partial/\partial x^j = \Gamma^k_{ij}\, \partial/\partial x^k + \Gamma^\tau_{ij}\, v_\tau$$

$$\nabla_{\partial/\partial x^i}\, v_\beta = \Gamma^k_{i\beta}\, \partial/\partial x^k + \Gamma^\tau_{i\beta}\, v_\tau$$

$$\nabla_{v_\alpha}\, \partial/\partial x^j = \Gamma^k_{\alpha j}\, \partial/\partial x^k + \Gamma^\tau_{\alpha j}\, v_\tau$$

$$\nabla_{v_\alpha}\, v_\beta = \Gamma^k_{\alpha\beta}\, \partial/\partial x^k + \Gamma^\tau_{\alpha\beta}\, v_\tau$$

and

$$\begin{aligned}[v_\alpha , v_\beta] &= (\partial A^i_\alpha /\partial x^\beta - \partial A^i_\beta /\partial x^\alpha + A^j_\alpha\, \partial A^i_\beta /\partial x^j \\ &\qquad - A^j_\beta\, \partial A^i_\alpha /\partial x^j)\, \partial/\partial x^i \\ &= B^i_{\alpha\beta}\, \partial/\partial x^i \ .\end{aligned}$$

Lemma 5.1. <u>Suppose that the metric</u> $\langle\ ,\ \rangle$ <u>is a bundle-like metric with respect to</u> E , <u>then</u>

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} (\partial g_{hj} /\partial x^i + \partial g_{ih} /\partial x^j - \partial g_{ij} /\partial x^h)$$

$$\Gamma^\tau_{ij} = \frac{1}{2} g^{\tau\varepsilon} (g_{hj} \partial A^h_\varepsilon /\partial x^i + g_{ih} \partial A^h_\varepsilon /\partial x^j - v_\varepsilon (g_{ij}))$$

$$\Gamma^k_{\alpha j} = \frac{1}{2} g^{kh} (v_\alpha (g_{hj}) + g_{hf} \partial A^f_\alpha /\partial x^j - g_{jf} \partial A^f_\alpha /\partial x^h)$$

$$\Gamma^k_{j\alpha} = \Gamma^k_{\alpha j} - \partial A^k_\alpha /\partial x^j$$

$$\Gamma^k_{\alpha\beta} = - \Gamma^k_{\beta\alpha} = \frac{1}{2} B^k_{\alpha\beta}$$

$$\Gamma^\tau_{\alpha j} = \Gamma^\tau_{j\alpha} = - \frac{1}{2} g^{\tau\varepsilon} B^h_{\alpha\varepsilon} g_{hj}$$

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2} g^{\tau\varepsilon} (\partial g_{\varepsilon\beta} /\partial x^\alpha + \partial g_{\alpha\varepsilon} /\partial x^\beta - \partial g_{\alpha\beta} /\partial x^\varepsilon) \quad .$$

By the decomposition (3.3), $TM \cong E \oplus E^\perp$, any $Y \in \Gamma(TM)$ is decomposed as

$$Y = Y_E + Y_{E^\perp} \quad ,$$

where Y_E (resp. $Y_{E^\perp}$) denotes a $\Gamma(E)$- (resp. $\Gamma(E^\perp)$-) component of Y . In a flat coordinate chart $U(x^i, x^\alpha)$, Y_E and $Y_{E^\perp}$ are locally expressed by

$$Y_E = Y^i \partial/\partial x^i \quad , \quad Y_{E^\perp} = Y^\alpha v_\alpha \quad .$$

Let $\gamma(t)$ be a transversal geodesic in M parametrized proportionally to arc-length, then, setting $\dot\gamma(t) = X^\alpha v_\alpha$ in U , we have

(5.1) $$X^\alpha v_\alpha (X^\tau) + X^\alpha X^\beta \Gamma^\tau_{\alpha\beta} = 0 \qquad (p+1 \leqq {}^\forall\tau \leqq p+q) \ .$$

According to B. O'Neill[21], we have

Definition 5.1. If $Y(t) = Y = Y_E + Y_{E^\perp}$ is a vector field along a transversal geodesic $\gamma(t)$ in M , then

$$\hat{Y}(t) = \hat{Y} = (\nabla_{\dot\gamma(t)} Y_E)_E - (\nabla_{Y_E} \dot\gamma(t))_E + 2(\nabla_{\dot\gamma(t)} Y_{E^\perp})_E$$

is called the <u>derived vector field of</u> Y , and $\hat{Y}(t) \in \Gamma(E|_{\gamma(t)})$.

Hereafter, we assume that M has a bundle-like metric $\langle\ ,\ \rangle$ with respect to E .

Proposition 5.1. <u>For a vector field</u> Y <u>along a transversal geodesic</u> $\gamma(t)$ <u>in</u> M , <u>it holds that</u>

$$(5.2) \quad (\nabla_{\dot\gamma(t)} \nabla_{\dot\gamma(t)} Y + R(Y, \dot\gamma(t)) \dot\gamma(t))_E$$

$$= (\nabla_{\dot\gamma(t)} Y)_E + (\nabla_{\hat{Y}} \dot\gamma(t))_{E^\perp}$$

$$(5.3) \quad (\nabla_{\dot\gamma(t)} \nabla_{\dot\gamma(t)} Y + R(Y, \dot\gamma(t)) \dot\gamma(t))_{E^\perp}$$

$$= (\nabla_{\dot\gamma(t)} (\nabla_{\dot\gamma(t)} Y_{E^\perp})_{E^\perp})_{E^\perp}$$

$$- (\nabla_{\dot\gamma(t)} (\nabla_{Y_{E^\perp}} \dot\gamma(t))_{E^\perp})_{E^\perp}$$

$$- (\nabla_{([Y_{E^\perp}, \dot\gamma(t)])_{E^\perp}} \dot\gamma(t))_{E^\perp}$$

$$+ 2(\nabla_{\dot\gamma(t)} \hat{Y})_{E^\perp}$$

<u>where</u> R <u>denotes the curvature tensor of</u> ∇ , <u>that is</u>,
$R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V, W]} Z$.

This is proved by the direct calculation, taking notice of Lemma 5.1 and (5.1).

Let $\gamma : [0,1] \longrightarrow M$ be a transversal geodesic in M parametrized proportionally to arc-length. Let $L_{\gamma(t)}$ denote the leaf through a point $\gamma(t)$ and $T_{\gamma(t)}L$ the tangent space to $L_{\gamma(t)}$ at $\gamma(t)$.

A linear space $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ (resp. $\mathcal{E}(L_{\gamma(0)}, \gamma(1))$) consists of piece-wise differentiable vector fields $Y(t)$ along $\gamma(t)$ orthogonal to $\dot{\gamma}(t)$ satisfying $Y(0) \in T_{\gamma(0)}L$ and $Y(1) \in T_{\gamma(1)}L$ (resp. $Y(1) = 0$). Then the index form I on $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ is given by

$$
\begin{aligned}
I(Y,Z) & \\
&= \frac{1}{L(\gamma)} \Big[- \int_0^1 < \nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Z > dt \\
&\quad + < \nabla_{\dot{\gamma}(t)} Y - S_{\dot{\gamma}(t)} Y, Z > \Big|_0^1 \\
&\quad + \sum_{i=1}^{k-1} < (\nabla_{\dot{\gamma}(t)} Y)(t_i^-) - (\nabla_{\dot{\gamma}(t)} Y)(t_i^+), Z(t_i) > \Big] .
\end{aligned}
$$

where $L(\gamma)$ denotes the length of γ , S denotes the second fundamental form:

$$< S_{\dot{\gamma}(t)} Y, Z > = - < \nabla_Y Z, \dot{\gamma}(t) > ,$$

$0 < t_1 < t_2 < \cdots < t_{k-1} < 1$ are points where Y is not differentiable, and $(\nabla_{\dot{\gamma}(t)} Y)(t_i^-)$ (resp. $(\nabla_{\dot{\gamma}(t)} Y)(t_i^+)$) denotes the left (resp. right) limit of $\nabla_{\dot{\gamma}(t)} Y$ at t_i (See [15], [18], [21]).

Lemma 5.2. _Let_ Y _be a vector field along a transversal geodesic_ $\gamma(t)$ _in_ M . _If_ $Y(0) \in T_{\gamma(0)}L$, _then_

$$\hat{Y}(0) = (\nabla_{\dot{\gamma}(t)} Y_E)_E(0) - (\nabla_{Y_E} \dot{\gamma}(t))_E(0)$$

$$(\nabla_{\dot{\gamma}(t)} Y)_E(0) = (\nabla_{\dot{\gamma}(t)} Y_E)_E(0)$$

$$S_{\dot{\gamma}(t)} Y (0) = (\nabla_{Y_E} \dot{\gamma}(t))_E(0) \quad .$$

This lemma is easily proved.

Lemma 5.3. <u>Let</u> Y <u>be a piece-wise differentiable vector field along a transversal geodesic</u> $\gamma(t)$ <u>in</u> M <u>and</u> $0 < t_1 < t_2 < \cdots < t_{k-1} < 1$ <u>broken points of</u> Y . <u>Then, for each</u> i $(1 \leqq i \leqq k-1)$,

$$(\nabla_{\dot{\gamma}(t)} Y)(t_i^-) - (\nabla_{\dot{\gamma}(t)} Y)(t_i^+)$$

$$= \hat{Y}(t_i^-) - \hat{Y}(t_i^+) + (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_{E^\perp}(t_i^-) - (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_{E^\perp}(t_i^+) \quad .$$

Proof. For any $t \in (t_i, t_{i+1})$ (or (t_{i-1}, t_i)),

$$\nabla_{\dot{\gamma}(t)} Y(t) = (\nabla_{\dot{\gamma}(t)} Y_E)_E(t) + (\nabla_{\dot{\gamma}(t)} Y_E)_{E^\perp}(t)$$

$$+ (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_E(t) + (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_{E^\perp}(t)$$

$$= \hat{Y}(t) + (\nabla_{\dot{\gamma}(t)} Y_E)_{E^\perp}(t) + (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_{E^\perp}(t)$$

$$+ (\nabla_{Y_E} \dot{\gamma}(t))_E(t) - (\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_E(t) \quad .$$

We may set $Y_E = Y^i \, \partial/\partial x^i$, $Y_{E^\perp} = Y^\alpha v_\alpha$ and $\dot{\gamma}(t) = X^\alpha v_\alpha$, thus we have

$$(\nabla_{\dot{\gamma}(t)} Y_E)_{E^\perp} = X^\alpha Y^i \Gamma^\tau_{\alpha i} v_\tau$$

$$(\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_E = X^\alpha Y^\beta \Gamma^k_{\alpha\beta} \, \partial/\partial x^k$$

$$(\nabla_{Y_E} \dot{\gamma}(t))_E = Y^i X^\beta \Gamma^k_{i\beta} \partial/\partial x^k \quad .$$

Since $Y_E(t)$ and $Y_E{}^\perp(t)$ are continuous on $[0,1]$, we have

$$(\nabla_{\dot{\gamma}(t)} Y_E)_E{}^\perp(t_i^-) = (\nabla_{\dot{\gamma}(t)} Y_E)_E{}^\perp(t_i^+)$$

$$(\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E(t_i^-) = (\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E(t_i^+)$$

$$(\nabla_{Y_E} \dot{\gamma}(t))_E(t_i^-) = (\nabla_{Y_E} \dot{\gamma}(t))_E(t_i^+) \quad .$$

Therefore we have the assertion. Q.E.D.

From above two lemmas, the index form I on $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ is rewritten:

(5.4) $I(Y,Z)$

$$= \frac{1}{L(\gamma)} \Big[- \int_0^1 \langle \nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Z \rangle \, dt$$

$$+ \langle \hat{Y}, Z \rangle \Big|_0^1 + \textstyle\sum_{i=1}^{k-1} \langle \Delta \hat{Y}(t_i), Z(t_i) \rangle$$

$$+ \textstyle\sum_{i=1}^{k-1} \langle \Delta(\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E{}^\perp(t_i), Z_E{}^\perp(t_i) \rangle \Big] \quad ,$$

where $\Delta \hat{Y}(t_i) = \hat{Y}(t_i^-) - \hat{Y}(t_i^+)$ and $\Delta(\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E{}^\perp(t_i)$ $= (\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E{}^\perp(t_i^-) - (\nabla_{\dot{\gamma}(t)} Y_E{}^\perp)_E{}^\perp(t_i^+)$.

Definition 5.2. A vector field Y along a geodesic $\gamma(t)$ is called a <u>Jacobi field along</u> γ if Y satisfies the Jacobi equation: $\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t)) \dot{\gamma}(t) = 0$.

Let γ be a transversal geodesic in M parametrized proportionally to arc-length and $\mathcal{J}(\gamma)$ the linear space of all

Jacobi fields along γ orthogonal to γ. Then we consider the following subspaces of $\mathcal{J}(\gamma)$:

$$\mathcal{J}_L(\gamma) = \{ Y \in \mathcal{J}(\gamma) \ ; \ \hat{Y} = 0 \}$$

$$\mathcal{J}(\gamma ; L) = \{ Y \in \mathcal{J}(\gamma) \ ; \ Y(t) \in T_{\gamma(t)}L \text{ for any } t \in [0,1] \}$$

$$\mathcal{J}(\gamma ; L_{\gamma(0)}, L_{\gamma(1)}) = \{ Y \in \mathcal{J}(\gamma) \ ; \ Y(0) \in T_{\gamma(0)}L \text{ and } Y(1) \in T_{\gamma(1)}L \}$$

$$\mathcal{J}(\gamma ; L_{\gamma(0)}, \gamma(1)) = \{ Y \in \mathcal{J}(\gamma) \ ; \ Y(0) \in T_{\gamma(0)}L \text{ and } Y(0) = 0 \}$$

$$\mathcal{J}_L(\gamma ; L) = \mathcal{J}_L(\gamma) \cap \mathcal{J}(\gamma ; L)$$

$$\mathcal{J}_L(\gamma ; L_{\gamma(0)}, L_{\gamma(1)}) = \mathcal{J}_L(\gamma) \cap \mathcal{J}(\gamma ; L_{\gamma(0)}, L_{\gamma(1)})$$

$$\mathcal{J}_L(\gamma ; L_{\gamma(0)}, \gamma(1)) = \mathcal{J}_L(\gamma) \cap \mathcal{J}(\gamma ; L_{\gamma(0)}, \gamma(1)) \ .$$

Lemma 5.4. The space $\mathcal{J}_L(\gamma ; L)$ consists of all solutions Y of

(5.5) $$\nabla_{\dot{\gamma}(t)} Y = (\nabla_{\dot{\gamma}(t)} Y_E)_{E^{\perp}} + (\nabla_{\dot{\gamma}(t)} Y_{E^{\perp}})_E + (\nabla_{Y_E} \dot{\gamma}(t))_E$$

on γ such that $Y(0) \in T_{\gamma(0)}L$. Moreover $\dim \mathcal{J}_L(\gamma ; L) = p = n - q$.

Proof. If $Y = Y_E + Y_{E^{\perp}}$ satisfies (5.5) and

(5.6) $$Y_{E^{\perp}}(0) = 0 \ ,$$

then we have

(5.7) $$(\nabla_{\dot\gamma(t)} Y_{E^\perp})_{E^\perp} = 0 \quad ,$$

since $(\nabla_{\dot\gamma(T)} Y)_{E^\perp} = (\nabla_{\dot\gamma(t)} Y_E)_{E^\perp} + (\nabla_{\dot\gamma(t)} Y_{E^\perp})_{E^\perp}$. By (5.6) and (5.7), we have a differential equation system

$$\begin{cases} \dfrac{d\, Y^\tau}{dt} + \mathfrak{A}^\tau_\beta Y^\beta = 0 \\ \\ Y^\tau(0) = 0 \end{cases} \qquad (p+1 \leqq {}^\forall\tau \leqq p+q)$$

where $Y_{E^\perp} = Y^\tau v_\tau$, $\dot\gamma(t) = X^\alpha v_\alpha$ and $\mathfrak{A}^\tau_\beta = X^\alpha \Gamma^\tau_{\alpha\beta}$. Thus we have, for each τ , $Y^\tau = 0$, that is,

$$Y = Y_E \quad .$$

Since $(\nabla_{\dot\gamma(t)} Y_E)_E = (\nabla_{Y_E} \dot\gamma(t))_E$ and $\hat{Y}_E = (\nabla_{\dot\gamma(t)} Y_E)_E - (\nabla_{Y_E} \dot\gamma(t))_E$, we have that $\hat{Y}_E = 0$, that is,

$$\hat{Y} = 0 \quad .$$

By Proposition 5.1, we have

$$\nabla_{\dot\gamma(t)} \nabla_{\dot\gamma(t)} Y + R(Y, \dot\gamma(t)) \dot\gamma(t) = 0 \quad .$$

And we have $\langle Y, \dot\gamma(t) \rangle = \langle Y_E, \dot\gamma(t) \rangle = 0$. Therefore we have $Y \in \mathcal{J}_L(\gamma ; L)$.

Conversely, if $Y \in \mathcal{J}_L(\gamma ; L)$, then it is trivial that Y satisfies (5.5).

And we easily have $\dim \mathcal{J}_L(\gamma ; L) = p$. Q.E.D.

Lemma 5.5. <u>Let</u> Y <u>be a Jacobi field along a transversal geodesic</u> γ . <u>If, for some</u> $t_1 \in [0,1]$, $Y(t_1) = 0$, <u>then</u> $Y \in \mathcal{J}_L(\gamma)$.

Proof. From (5.2), Y satisfies $(\nabla_{\dot\gamma(t)} \hat{Y})_E + (\nabla_{\hat{Y}} \dot\gamma(t))_E = 0$. Thus we have

$$\nabla_{\dot{\gamma}(t)} \hat{Y} = (\nabla_{\dot{\gamma}(t)} \hat{Y})_{E^{\perp}} - (\nabla_{\hat{Y}} \dot{\gamma}(t))_{E}$$

$$\hat{Y}(t_1) = 0 \quad ,$$

that is, we have a differential equation system

$$\begin{cases} \dfrac{d \hat{Y}^i}{dt} + \mathcal{B}^i_j \hat{Y}^j = 0 \\ \hat{Y}^i(t_1) = 0 \end{cases} \qquad (1 \leqq {}^{\forall} i \leqq p)$$

where $\hat{Y} = \hat{Y}^i \partial/\partial x^i$, $\dot{\gamma}(t) = X^{\alpha} v_{\alpha}$ and $\mathcal{B}^i_j = X^{\alpha} \partial A^i_{\alpha} /\partial x^j$. Thus we have $\hat{Y}^i = 0$, that is, $\hat{Y} = 0$. Therefore we have $Y \in \mathcal{J}_L(\gamma)$. Q.E.D.

Proposition 5.2. <u>A vector field</u> $Y \in \mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ <u>belongs to</u> $\mathcal{J}_L(\gamma\, ;\, L_{\gamma(0)}, L_{\gamma(1)})$ <u>if and only if</u> $I(Y,Z) = 0$ <u>for any</u> $Z \in \mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$.

Proof. If Y belongs to $\mathcal{J}_L(\gamma\, ;\, L_{\gamma(0)}, L_{\gamma(1)})$, we have

$$Y \in \mathcal{J}(\gamma) \quad , \quad \hat{Y}(0) = 0 \quad ,$$

$$Y(0) \in T_{\gamma(0)}L \quad , \quad Y(1) \in T_{\gamma(1)}L \quad ,$$

that is,

$$\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t)) \dot{\gamma}(t) = 0 \quad ,$$

$$\hat{Y} = 0 \quad ,$$

$$\Delta \hat{Y} = 0 \quad , \quad \Delta(\nabla_{\dot{\gamma}(t)} Y_{E^{\perp}})_{E^{\perp}} = 0 \quad .$$

By (5.4), we have $I(Y,Z) = 0$ for any $Z \in \mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$.

Conversely, suppose that $I(Y,Z) = 0$ for any $Z \in \mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$. Then Y is a Jacobi field along γ (See

[15], [18]). For sufficiently small $t' > 0$, we take an arbitrary $Z \in \mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ such that $Z = 0$ on $\gamma|_{[t',1]}$. Then, by the assumption, we have

$$0 = I(Y,Z)$$

$$= -\frac{1}{L(\gamma)} \langle \hat{Y}(0), Z(0) \rangle \quad .$$

By the arbitrarity of the choice of $Z(0) \in T_{\gamma(0)}L$, we have that $\hat{Y}(0) = 0$. From Lemma 5.5, we have $\hat{Y} = 0$. Therefore $Y \in \mathcal{J}_L(\gamma ; L_{\gamma(0)}, L_{\gamma(1)})$. Q.E.D.

By the same ways, the nullspace of the index form I on $\mathcal{E}(L_{\gamma(0)}, \gamma(1))$ is $\mathcal{J}_L(\gamma ; L_{\gamma(0)}, \gamma(1))$. Thus we have

Definition 5.3. Let $\gamma(t)$ be a transversal geodesic in M parametrized proportionally to arc-length. A point $\gamma(1)$ is a <u>focal point of the leaf</u> $L_{\gamma(0)}$ <u>along</u> γ if there exists a non-zero vector field Y belonging to $\mathcal{J}_L(\gamma ; L_{\gamma(0)}, \gamma(1))$.

Proposition 5.3. <u>Let</u> $Y = Y_E + Y_E^{\perp}$ <u>be a vector field along a transversal geodesic</u> γ <u>in</u> M . <u>If</u> $Y_E \in \mathcal{J}_L(\gamma ; L_{\gamma(0)}, \gamma(1))$, <u>then</u> $Y_E = 0$.

Proof. We set $Y_E = Y^i \partial/\partial x^i$ and $\dot{\gamma}(t) = X^{\alpha} v_{\alpha}$. From $Y_E \in \mathcal{J}_L(\gamma ; L_{\gamma(0)}, \gamma(1))$, we have

$$\hat{Y}_E = 0 \quad , \quad Y_E(1) = 0 \; ,$$

that is,

$$\begin{cases} \dfrac{d\, Y^i}{dt} + \mathcal{C}^i_j\, Y^j = 0 \\ \\ Y^i(1) = 0 \end{cases} \qquad (1 \leqq {}^{\forall} i \leqq p)$$

where $\mathcal{C}^i_j = X^\alpha \partial A^i_\alpha / \partial x^j$. Then we have $Y_E = 0$.

Q.E.D.

Definition 5.4. Let $\gamma(t)$ ($t \in [0,1]$) be a transversal geodesic in M and $\alpha : [0,1] \times (-\varepsilon,\varepsilon) \longrightarrow M$ ($\varepsilon > 0$) a variation of γ , that is, $\alpha(t,0) = \gamma(t)$. The variation α of γ is a $(L_{\gamma(0)}, L_{\gamma(1)})$-<u>geodesic variation of</u> γ if it satisfies the following:

(i) For each $u \in (-\varepsilon,\varepsilon)$, a curve $\alpha_u(t)$ ($= \alpha(t,u)$) is a geodesic.

(ii) Two curves $\alpha^0(u) = \alpha(0,u)$ and $\alpha^1(u) = \alpha(1,u)$ are in $L_{\gamma(0)}$ and $L_{\gamma(1)}$ respectively.

Proposition 5.4. <u>Let</u> $\gamma(t)$ ($t \in [0,1]$) <u>be a transversal geodesic in</u> M <u>parametrized proportionally to arc-length and</u> $\alpha : [0,1] \times (-\varepsilon,\varepsilon) \longrightarrow M$ ($\varepsilon > 0$) <u>a</u> $(L_{\gamma(0)}, L_{\gamma(1)})$-<u>geodesic variation of</u> γ . <u>Then the variational vector field</u> $Y(t) = \alpha_*(\partial/\partial u)(t,0)$ <u>along</u> γ <u>belongs to</u> $\mathcal{T}_L(\gamma; L_{\gamma(0)}, L_{\gamma(1)})$.

Proof. We have easily that Y is a Jacobi field along γ and $\langle Y(t), \dot\gamma(t) \rangle = 0$ for any $t \in [0,1]$. By Lemma 5.2 and $[\, Y, \dot\gamma(t) \,]\big|_{t=0} = 0$, we have

$$
\begin{aligned}
\hat{Y}(0) &= (\nabla_{\dot\gamma(t)} Y_E)_E(0) - (\nabla_{Y_E} \dot\gamma(t))_E(0) \\
&= ([\, \dot\gamma(t), Y_E \,])_E(0) \\
&= ([\, \dot\gamma(t), Y \,])_E(0) \\
&= 0 \quad .
\end{aligned}
$$

By Lemma 5.5, $Y \in \mathcal{T}_L(\gamma;\ L_{\gamma(0)}, L_{\gamma(1)})$. Q.E.D.

Proposition 5.5.

$$\mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)) \oplus \mathcal{T}_L(\gamma;\ L) \subset \mathcal{T}_L(\gamma;\ L_{\gamma(0)}, L_{\gamma(1)})$$

and $\dim \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)) \leqq q-1$, where $\oplus$ denotes the direct sum.

Proof. We take an arbitrary vector field $Y \in \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)) \cap \mathcal{T}_L(\gamma;\ L)$. Then we have

$$Y = Y_E$$

$$\nabla_{\dot\gamma(t)} Y = (\nabla_{\dot\gamma(t)} Y_E)_{E^\perp} + (\nabla_{Y_E} \dot\gamma(t))_E$$

$$Y(1) = 0 \quad .$$

If we set $\sigma(t) = \gamma(1-t)$ and $Z(t) = Y(1-t)$, then we have

$$Z = Z_E$$

$$\nabla_{\dot\sigma(t)} Z = (\nabla_{\dot\sigma(t)} Z_E)_{E^\perp} + (\nabla_{Z_E} \dot\sigma(t))_E$$

$$Z(0) = 0 \quad ,$$

that is, $Z = Z_E = Z^i\, \partial/\partial x^i$ satisfies

$$\left\{\begin{array}{l} \dfrac{d\,Z^i}{dt} + \mathcal{D}^i_j Z^j = 0 \\ \\ Z^i(0) = 0 \end{array}\right. \qquad (1 \leqq {}^\forall i \leqq p)$$

where $\dot\sigma(t) = V^\alpha v_\alpha$ and $\mathcal{D}^i_j = V^\alpha \partial A^i_\alpha / \partial x^j$. Then we have $Z = 0$, that is, $Y = 0$. Therefore we have

(5.8) $\qquad \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ ) \cap \mathcal{T}_L(\gamma;\ L\ ) = \{0\}$.

We take an arbitrary vector field $Y \in \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ ) + \mathcal{T}_L(\gamma;\ L\ )$, $Y = Z + W$ where $Z \in \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ )$ and $W \in \mathcal{T}_L(\gamma;\ L\ )$. Then, from $W_{E^\perp} = 0$, we have

$$(Z + W)_E = Z_E + W_E$$

$$(Z + W)_{E^\perp} = Z_{E^\perp}$$

and

$$\begin{aligned} \hat{Y} &= \widehat{(Z + W)} \\ &= \hat{Z} + \hat{W} \\ &= 0 \quad . \end{aligned}$$

It is obvious that $Y(0) \in T_{\gamma(0)}L$ and $Y(1) \in T_{\gamma(1)}L$. Thus we have that $Y \in \mathcal{T}_L(\gamma;\ L_{\gamma(0)}, L_{\gamma(1)}\ )$. Therefore we have

(5.9) $\quad \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ ) + \mathcal{T}_L(\gamma;\ L\ ) \subset \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ )$.

By (5.8) and (5.9), we have

$$\mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ ) \oplus \mathcal{T}_L(\gamma;\ L\ ) \subset \mathcal{T}_L(\gamma;\ L_{\gamma(0)}, L_{\gamma(1)}\ ) .$$

Since $\dim \mathcal{T}_L(\gamma;\ L_{\gamma(0)}, L_{\gamma(1)}\ ) = n - 1$ and $\dim \mathcal{T}_L(\gamma;\ L\ ) = n - q$, we have

$$\begin{aligned} \dim \mathcal{T}_L(\gamma;\ L_{\gamma(0)},\ \gamma(1)\ ) &\leqq (n - 1) - (n - q) \\ &= q - 1 . \end{aligned}$$

Q.E.D.

Theorem 5.1. Let M be a foliated manifold with a foliation E of codimension 1 and with a bundle-like metric with respect to E. For any point $m \in M$, there is not a focal point of the leaf L_m through m along every transversal geodesic γ starting from m.

Proof. By Proposition 5.5, we have

$\dim \mathcal{T}_L(\gamma ; L_{\gamma(0)}, \gamma(1)) = 0$. Q.E.D.

Example 5.1. Let $\mathbb{R}^3$ be the set of triple (x,y,z) of real numbers. $\mathbb{R}^3$ is considered a riemannian manifold with a riemannian metric $\langle\ ,\ \rangle = dx\cdot dx - 2z\, dx\cdot dy + (1 + z^2)\, dy\cdot dy + dz\cdot dz$. Then $\mathbb{R}^3$ is considered a foliated manifold whose leaves are orbits of a vector field $\partial/\partial x$, and the metric is a bundle-like metric with respect to the foliation, that is, $\langle\ ,\ \rangle = w\cdot w + dy\cdot dy + dz\cdot dz$ where $w = dx - z\, dy$. For any point $(x_0, y_0, z_0) \in \mathbb{R}^3$, let γ be an arbitrary transversal geodesic starting from (x_0, y_0, z_0) and $L_{(x_0,y_0,z_0)}$ the leaf through the point (x_0, y_0, z_0). Then there is no focal point of the leaf $L_{(x_0,y_0,z_0)}$ along γ.

Example 5.2. Let $\mathbb{R}^4$ be identified with the quaternion number field $\mathbb{Q}$, and let $S^3 \subset \mathbb{R}^4$ be a set $\{ a \in \mathbb{Q} \mid \|a\| = 1 \}$ where $\|a\|^2 = a\cdot\bar{a}$ and $\bar{a}$ denotes the conjugate of a. For any $a \in S^3$, L_a denotes a set given by $\{ (\cos\theta)\cdot a + (\sin\theta)\cdot(i\cdot a) \mid 0 \leqq \theta \leqq 2\pi \}$. Then S^3 is a foliated manifold by a family of the set L_a. The metric on S^3 induced from the flat metric on $\mathbb{R}^4$ is a bundle-like metric with respect to the foliation (See [3], [12], [14]). For any $a \in S^3$, let L_a be the leaf through a and $\gamma(s)$ a transversal geodesic parametrized by arc-length such that $\gamma(0) = a$. Then a point $\gamma(\pi/2)$ is a focal point of L_a along γ.

6. Clairaut's foliation.

The following Clairaut's theorem is a basic tool for studying geodesics on a surface of revolution.

Clairaut's Theorem. *Let* $\mathbb{r}$ *be the distance to the axis of revolution, and let* α *be the angle between a geodesic and the meridians, viewed as a function on the parameter of the geodesic. Then* $\mathbb{r} \sin \alpha$ = constant.

Then we have the following definition:

Definition 6.1. Let M be a foliated manifold with a foliation E of codimension q and with a riemannian metric $\langle \ , \ \rangle$. The foliation E is called a *Clairaut's foliation* if there exists a positive valued function $\mathbb{r} : M \longrightarrow \mathbb{R}$ such that, for any geodesic $\gamma(t)$ parametrized proportionally to arc-length,

$$\mathbb{r} \sin \alpha = \text{constant} ,$$

where $\alpha = \alpha(t)$ is defined by

$$\cos^2 \alpha(t) = \| X_E{}^{\perp}(t) \|^2 / \| X(t) \|^2$$

$(0 \leqq \alpha(t) \leqq \pi/2)$, $\dot{\gamma}(t) = X(t) = X_E(t) + X_E{}^{\perp}(t)$ and $\| X(t) \|^2 = \langle X(t), X(t) \rangle$. The function $\mathbb{r}$ is called the *girth of* E (See [1], [10]).

Let $\gamma(t)$ be a geodesic in M parametrized proportionally to arc-length and $\dot{\gamma}(t) = X(t) = X_E + X_E{}^{\perp}$. Setting $\rho^2 = \| X(t) \|^2$ = constant , we have

$$\langle X_E, X_E \rangle = \rho^2 \sin^2 \alpha$$

$$< X_E{}^{\perp}, X_E{}^{\perp} > = \rho^2 \cos^2\alpha \quad .$$

R. L. Bishop[1] defined and studied Clairaut submersions, and H. Kitahara[10] discussed the Clairaut's foliations of codimension 1 . We discuss the foliated manifold M with a Clairaut's foliation E of codimension q and with a bundle-like metric with respect to E .

Definition 6.2. A function f on M is called a foliated function if f is constant on each leaf of M .

Proposition 6.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $< \ , \ >$ with respect to E . If E is a Clairaut's foliation with the girth $\mathfrak{r} = e^f$, where f is a function on M , then f is a foliated function on M .*

Proof. Let $\gamma(t)$ be a geodesic parametrized proportionally to arc-length. By assumption, $\mathfrak{r} \sin\alpha$ = constant , thus we have

$$0 = \frac{d}{dt}(\mathfrak{r} \sin\alpha)$$

$$= \mathfrak{r} \frac{df}{dt} \sin\alpha + \mathfrak{r} \cos\alpha \frac{d\alpha}{dt} \quad .$$

Then we have

$$0 = (\frac{d}{dt}(\mathfrak{r} \sin\alpha)) \rho^2 \sin\alpha$$

$$= \mathfrak{r} \frac{df}{dt} < X_E, X_E > + \mathfrak{r} < \nabla_{\dot\gamma(t)} X_E, X_E >$$

$$= \mathfrak{r} < \dot\gamma(t), < X_E, X_E > \text{grad } f > + \mathfrak{r} < \nabla_{\dot\gamma(t)} X_E, X_E > \quad ,$$

since $df/dt = \langle \dot{\gamma}(t), \operatorname{grad} f \rangle$ and $\dot{\gamma}(t) = X = X_E + X_E{}^{\perp}$.
Thus we have

$$\langle \dot{\gamma}(t), \langle X_E, X_E \rangle \operatorname{grad} f \rangle = - \langle \nabla_{\dot{\gamma}(t)} X_E, X_E \rangle \quad .$$

We have

$$\frac{d}{dt} \langle X_E + X_E{}^{\perp}, X_E \rangle = \frac{d}{dt} \langle X_E, X_E \rangle$$

$$= 2 \langle \nabla_{\dot{\gamma}(t)} X_E, X_E \rangle \quad .$$

On the other hand, we have

$$\frac{d}{dt} \langle X_E + X_E{}^{\perp}, X_E \rangle = \frac{d}{dt} \langle \dot{\gamma}(t), X_E \rangle$$

$$= \langle \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t), X_E \rangle + \langle \dot{\gamma}(t), \nabla_{\dot{\gamma}(t)} X_E \rangle$$

$$= \langle \dot{\gamma}(t), \nabla_{\dot{\gamma}(t)} X_E \rangle$$

$$= \langle X_E, \nabla_{\dot{\gamma}(t)} X_E \rangle + \langle X_E{}^{\perp}, \nabla_{\dot{\gamma}(t)} X_E \rangle \quad .$$

Thus we have

$$\text{(6.1)} \qquad \langle \nabla_{\dot{\gamma}(t)} X_E, X_E \rangle = \langle X_E{}^{\perp}, \nabla_{\dot{\gamma}(t)} X_E \rangle \quad .$$

Therefore

$$\langle \dot{\gamma}(t), \langle X_E, X_E \rangle \operatorname{grad} f \rangle = - \langle X_E{}^{\perp}, \nabla_{\dot{\gamma}(t)} X_E \rangle \quad .$$

By Lemma 5.1, we have

$$\langle X_E{}^{\perp}, \nabla_{X_E{}^{\perp}} X_E \rangle = \langle X^{\alpha} v_{\alpha}, \nabla_{X^{\beta} v_{\beta}} (X^i \partial/\partial x^i) \rangle$$

$$= X^{\alpha} X^{\beta} X^i \Gamma^{\tau}_{\beta i} g_{\alpha\tau}$$

$$= - \frac{1}{2} X^{\alpha} X^{\beta} X^i B^k_{\beta\alpha} g_{ki}$$

$$= 0 \quad ,$$

thus we have

(6.2) $$< X_E{}^{\perp}, \nabla_{\dot{\gamma}(t)} X_E > \; = \; < X_E{}^{\perp}, \nabla_{X_E} X_E > \quad .$$

Therefore we have

(6.3) $$< \dot{\gamma}(t), < X_E, X_E > \operatorname{grad} f > = - < X_E{}^{\perp}, \nabla_{X_E} X_E > \quad .$$

Now, for any point $m \in M$ and any non-zero vector $Y \in T_m L$, we take a geodesic $\gamma(t)$ such that

$$\gamma(0) = m \qquad \dot{\gamma}(0) = Y$$

and then we have, by (6.3) at $t = 0$,

$$< Y, < Y, Y > \operatorname{grad} f|_m > = 0 \quad ,$$

thus we have

$$< Y, \operatorname{grad} f|_m > = 0 \quad .$$

Therefore, grad f is orthogonal to a leaf at each point, and f is a constant on each leaf. Thus f is a foliated function on M. Q.E.D.

Definition 6.3. Let $\{ X_i, X_\alpha \}$ be an orthonormal adapted frame to E. The <u>mean curvature vector</u> H_m <u>at</u> $m \in M$ <u>of the leaf</u> L_m is defined by

$$H_m = \frac{1}{n-q} \sum_{i,\alpha} < \nabla_{X_i} X_i|_m , X_\alpha|_m > X_\alpha|_m \quad .$$

Definition 6.4. A leaf is called <u>totally umbilic</u> if, for each point m of the leaf, it holds

$$< X|_m , Y|_m > H_m = (\nabla_X Y)_E{}^{\perp} |_m$$

for any $X, Y \in \Gamma(E|_U)$ where U is a flat coordinate chart at m .

Proposition 6.2. *Let M be a foliated manifold with a foliation E and with a bundle-like metric $<\ ,\ >$ with respect to E. If E is a Clairaut's foliation with the girth $r = e^f$ where f is a function on M. Then the mean curvature vector H of each leaf is* $-\operatorname{grad} f$.

Proof. For a geodesic $\gamma(t)$, $\dot{\gamma}(t) = X_E + X_E^{\perp}$, we have

$$(6.4)\qquad < X_E^{\perp},\ < X_E,\ X_E > \operatorname{grad} f > = -< X_E^{\perp},\ \nabla_{X_E} X_E > \ ,$$

since grad f is orthogonal to each leaf and (6.3) holds.

For any fixed point $m \in M$ and any non-zero vector $Y^{\alpha} X_{\alpha}|_m$ at m , we may take geodesics $\gamma_i(t)$ ($i = 1, 2, \cdots, p$) such that

$$\gamma_i(0) = m \qquad \dot{\gamma}_i(0) = X_i|_m + Y^{\alpha} X_{\alpha}|_m \ ,$$

where $\{X_i, X_{\alpha}\}$ is an orthonormal adapted frame to E . By (6.4), we have, for each i ,

$$< Y^{\alpha} X_{\alpha}|_m\ ,\ \operatorname{grad} f|_m > = -< Y^{\alpha} X_{\alpha}|_m\ ,\ \nabla_{X_i} X_i|_m > \ .$$

And, for each i and α ,

$$< X_{\alpha}|_m\ ,\ \nabla_{X_i} X_i|_m > = -< X_{\alpha}|_m\ ,\ \operatorname{grad} f|_m > \ .$$

Thus we have

$$\sum_{i,\alpha} < X_{\alpha}|_m\ ,\ \nabla_{X_i} X_i|_m > X_{\alpha}|_m$$

$$= -\ (n-q) \sum_{\alpha} < X_{\alpha}|_m\ ,\ \operatorname{grad} f|_m > X_{\alpha}|_m$$

$$= -\ (n-q)\ \operatorname{grad} f|_m \ .$$

Therefore, by the choice of m, we have $H = -\operatorname{grad} f$.

Q.E.D.

Theorem 6.1. Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. Suppose that all leaves are totally umbilic and the mean curvature vector H of each leaf is $-\operatorname{grad} f$, where f is a function on M. Then E is a Clairaut's foliation with the girth $\mathfrak{r} = e^f$.

Proof. Let $\gamma(t)$ be an arbitrary geodesic parametrized proportionally to arc-length and $\dot\gamma(t) = X = X_E + X_{E^\perp}$. We set

$$\rho = \|\dot\gamma(t)\| \quad (= \text{constant})$$

$$\cos^2\alpha = \|X_{E^\perp}\|^2 / \|X\|^2$$

$$\mathfrak{r} = e^f \quad .$$

We have

$$\langle \dot\gamma(t), \langle X_E, X_E\rangle \operatorname{grad} f \rangle$$

$$= \langle X_{E^\perp}, \langle X_E, X_E\rangle \operatorname{grad} f \rangle$$

$$= -\langle X_{E^\perp}, \langle X_E, X_E\rangle H \rangle \qquad (\text{from that } H = -\operatorname{grad} f)$$

$$= -\langle X_{E^\perp}, \nabla_{X_E} X_E \rangle \qquad (\text{from that all leaves are totally umbilic})$$

$$= -\langle X_{E^\perp}, \nabla_{\dot\gamma(t)} X_E \rangle \qquad (\text{from } (6.2))$$

$$= -\langle \nabla_{\dot\gamma(t)} X_E, X_E \rangle \qquad (\text{from } (6.1)) \quad .$$

Thus

$$< X_E, X_E > < \dot{\gamma}(t), \text{grad } f > + < \nabla_{\dot{\gamma}(t)} X_E, X_E > = 0 ,$$

that is,

$$2 \frac{df}{dt} \rho^2 \sin^2\alpha + \frac{d}{dt}(\rho^2 \sin^2\alpha) = 0 .$$

Then we have

$$2 e^f \frac{df}{dt} \rho^2 \sin^2\alpha + e^f \frac{d}{dt}(\rho^2 \sin^2\alpha) = 0 ,$$

that is,

$$2 \rho^2 \sin\alpha \left(\frac{d\mathbb{r}}{dt} \sin\alpha + \mathbb{r} \frac{d}{dt}(\sin\alpha) \right) = 0 .$$

By the choice of γ , we have $d(\mathbb{r} \sin\alpha)/dt = 0$. Therefore, E is a Clairaut's foliation with the girth $\mathbb{r} = e^f$.

Q.E.D.

Example 6.1. Let $\mathbb{R}^2$ be a x-y plane with the flat metric $< , >$. We consider $\mathbb{R}^2 - \{ (0,0) \}$ a foliated manifold whose leaves are sets $L_r = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \}$ ($r > 0$). The metric $< , > |_{\mathbb{R}^2 - \{(0,0)\}}$ is a bundle-like metric with respect to the foliation. Then the foliation is a Clairaut's foliation with the girth $\mathbb{r} = (x^2 + y^2)^{1/2}$.

7. Second connection.

I. Vaisman proved the following theorem:

Theorem(I. Vaisman[28, 29]). Let M be a foliated manifold with a foliation E of codimension q and with a riemannian metric $\langle\ ,\ \rangle$. Then there exists a connection D uniquely defined by the following conditions:

(i) If $Y \in \Gamma(E)$ (resp. $\Gamma(E^{\perp})$), then $D_X Y \in \Gamma(E)$ (resp. $\Gamma(E^{\perp})$) for any $X \in \Gamma(TM)$.

(ii) If $X, Y, Z \in \Gamma(E)$ (or $\Gamma(E^{\perp})$), then $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$.

(iii) $(T(X,Y))_E = 0$ if at least one of the arguments is in $\Gamma(E)$, and $(T(X,Y))_{E^{\perp}} = 0$ if at least one of the arguments is in $\Gamma(E^{\perp})$. Here T denotes the torsion tensor of D , that is, $T(X,Y) = D_X Y - D_Y X - [X,Y]$.

This is proved by similar way to prove the existence and uniqueness of the Levi-Civita connection on a manifold with a riemannian metric.

Definition 7.1. The connection D of above theorem is called the second connection on a foliated manifold.

The second connection is not metrical with respect to the riemannian metric and has non-zero torsion in general. The foliated manifolds with second connections are studied by H. Kitahara[11], H. Kitahara and S. Yorozu[13], I. Vaisman[28] and others.

Now, we have expressions of the second connection D and its torsion tensor T by using the basic adapted frame .

$\{\partial/\partial x^i, v_\alpha\}$ to E in a flat coordinate chart $U(x^i, x^\alpha)$. We first recall that the metric $\langle\ ,\ \rangle$ is locally expressed by

$$\langle\ ,\ \rangle|_U = g_{ij}(x^k, x^\tau)\, w^i \cdot w^j + g_{\alpha\beta}(x^k, x^\tau)\, dx^\alpha \cdot dx^\beta$$

where $w^i = dx^i + A^i_\alpha\, dx^\alpha$.

Lemma 7.1. *It holds that*

$$D_{\partial/\partial x^i}\, \partial/\partial x^j = \Gamma^k_{ij}\, \partial/\partial x^k \qquad D_{v_\alpha}\, \partial/\partial x^j = \Gamma^k_{\alpha j}\, \partial/\partial x^k$$

$$D_{\partial/\partial x^i}\, v_\beta = 0 \qquad D_{v_\alpha}\, v_\beta = \Gamma^\tau_{\alpha\beta}\, v_\tau ,$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} \left(\partial g_{hj}/\partial x^i + \partial g_{ih}/\partial x^j - \partial g_{ij}/\partial x^h \right)$$

$$\Gamma^k_{\alpha j} = \partial A^k_\alpha/\partial x^j$$

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2} g^{\tau\varepsilon} \left(v_\alpha(g_{\varepsilon\beta}) + v_\beta(g_{\alpha\varepsilon}) - v_\varepsilon(g_{\alpha\beta}) \right) .$$

Moreover

$$T(\partial/\partial x^i, \partial/\partial x^j) = 0 \qquad T(\partial/\partial x^i, v_\beta) = 0$$

$$T(v_\alpha, v_\beta) = (\partial A^k_\alpha/\partial x^\beta - \partial A^k_\beta/\partial x^\alpha + A^h_\alpha\, \partial A^k_\beta/\partial x^h - A^h_\beta\, \partial A^k_\alpha/\partial x^h)\, \partial/\partial x^k .$$

Lemma 7.2. *It holds that*

$$(\partial/\partial x^i)\langle \partial/\partial x^j, \partial/\partial x^k \rangle = \langle D_{\partial/\partial x^i}\, \partial/\partial x^j, \partial/\partial x^k \rangle + \langle \partial/\partial x^j, D_{\partial/\partial x^i}\, \partial/\partial x^k \rangle$$

$$v_\alpha \langle v_\beta , v_\tau \rangle = \langle D_{v_\alpha} v_\beta , v_\tau \rangle + \langle v_\beta , D_{v_\alpha} v_\tau \rangle .$$

Moreover, if the metric $\langle \ , \ \rangle$ _is a bundle-like metric with respect to_ E _, then_

$$(\partial/\partial x^i) \langle v_\alpha , v_\beta \rangle = \langle D_{\partial/\partial x^i} v_\alpha , v_\beta \rangle + \langle v_\alpha , D_{\partial/\partial x^i} v_\beta \rangle$$

$$= 0 .$$

We omit proofs of above two lemmas.

We discuss the ralation between the second connection D and the Levi-Civita connection ∇ .

Proposition 7.1. _Let_ M _be a foliated manifold with a foliation_ E _of codimension_ q _and with a bundle-like metric_ $\langle \ , \ \rangle$ _with respect to_ E . _If all leaves are totally geodesic and_ $E^\perp$ _is integrable, then_ $\nabla = D$.

Proof. By the integrability of $E^\perp$, we have

$$[v_\alpha , v_\beta] = B^i_{\alpha\beta} \, \partial/\partial x^i$$

$$= 0 ,$$

that is, $B^i_{\alpha\beta} = 0$ for every i , α, β . Then we have

$$\Gamma^k_{\alpha\beta} = \Gamma^k_{\beta\alpha} = \Gamma^\tau_{\alpha j} = \Gamma^\tau_{j\alpha} = 0$$

$$\text{for every } k, j, \alpha, \beta, \tau$$

by Lemma 5.1.

Since all leaves are totally geodesic, we have $\Gamma^\tau_{ij} = 0$ and, by Lemma 5.1,

$$v_\varepsilon (g_{ij}) = g_{kj} \, \partial A^k_\varepsilon / \partial x^i + g_{ik} \, \partial A^k_\varepsilon / \partial x^j . \tag{7.1}$$

Substituting above equality to the right side of the third equality in Lemma 5.1, we have

$$\Gamma^{k}_{\cdot\alpha j} = \partial A^{k}_{\alpha} / \partial x^{j} \quad .$$

Thus we have

$$\Gamma^{k}_{j\alpha} = 0 \quad .$$

Therefore, by Lemma 5.1 and 7.1, we have

$$\Gamma^{k}_{ij} = \mathbb{\Gamma}^{k}_{ij} \quad , \quad \Gamma^{k}_{\alpha j} = \mathbb{\Gamma}^{k}_{\alpha j} \quad , \quad \Gamma^{\tau}_{\alpha\beta} = \mathbb{\Gamma}^{\tau}_{\alpha\beta}$$

and others vanish. Q.E.D.

8. Geodesic with respect to the second connection.

Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. Let $\gamma(t)$ be a curve in M. Locally, $\gamma(t)$ is expressed by $\gamma(t) = (\gamma^i(t), \gamma^\alpha(t))$ in a flat coordinate chart $U(x^i, x^\alpha)$, and

$$\dot\gamma(t) = \dot\gamma^i(t)\,\partial/\partial x^i + \dot\gamma^\alpha(t)\,\partial/\partial x^\alpha$$

$$= (\dot\gamma^i(t) + A^i_\alpha\,\dot\gamma^\alpha(t))\,\partial/\partial x^i + \dot\gamma^\alpha(t)\,V_\alpha \quad .$$

Definition 8.1. A curve $\gamma(t)$ in M is called a D-<u>geodesic</u> if

$$D_{\dot\gamma(t)}\,\dot\gamma(t) = 0 \quad .$$

Such a parameter t is called a D-<u>affine</u> <u>parameter</u>.

Remark. To distinguish a geodesic with respect to the Levi-Civita connection ∇ from a D-geodesic, we will use "∇-geodesic" instead of " geodesic with respect to ∇ ".

A curve $\gamma(t)$ in M is a D-geodesic if and only if $\gamma(t)$ satisfies a differential equation system:

$$\left\{\begin{array}{l} \dfrac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\dfrac{d\gamma^i}{dt}\dfrac{d\gamma^j}{dt} \\[2ex] \qquad + 2(\Gamma^k_{ij}A^i_\alpha + \partial A^k_\alpha/\partial x^j)\dfrac{d\gamma^j}{dt}\dfrac{d\gamma^\alpha}{dt} \\[2ex] \qquad + (\partial A^k_\alpha/\partial x^\beta + A^i_\beta\,\partial A^k_\alpha/\partial x^i + \Gamma^k_{ij}A^i_\alpha A^j_\beta \\[2ex] \qquad\qquad\qquad - \Gamma^\tau_{\alpha\beta}A^k_\tau)\dfrac{d\gamma^\alpha}{dt}\dfrac{d\gamma^\beta}{dt} = 0 \\[2ex] \dfrac{d^2\gamma^\tau}{dt^2} + \Gamma^\tau_{\alpha\beta}\dfrac{d\gamma^\alpha}{dt}\dfrac{d\gamma^\beta}{dt} = 0 \quad . \end{array}\right.$$

On the other hand, a curve $\gamma(s)$ in M is a ∇-geodesic if and only if $\gamma(s)$ satisfies a differential equation system:

$$
(8.1)\quad \left\{
\begin{aligned}
&\frac{d^2\gamma^k}{ds^2} + \Gamma^k_{ij}\frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} \\
&\quad + 2(\Gamma^k_{\alpha j} + \Gamma^k_{ij}A^i_\alpha)\frac{d\gamma^j}{ds}\frac{d\gamma^\alpha}{ds} \\
&\quad + (\partial A^k_\alpha/\partial x^\beta + 2\Gamma^k_{\alpha j}A^j_\beta - A^j_\alpha \partial A^k_\beta/\partial x^j \\
&\qquad\qquad + \Gamma^k_{ij}A^i_\alpha A^j_\beta)\frac{d\gamma^\alpha}{ds}\frac{d\gamma^\beta}{ds} \\
&\quad + A^k_\beta\frac{d^2\gamma^\beta}{ds^2} = 0 \\
&\frac{d^2\gamma^\tau}{ds^2} + (\Gamma^\tau_{\alpha\beta} + \Gamma^\tau_{ij}A^i_\alpha A^j_\beta + 2\Gamma^\tau_{\alpha i}A^i_\beta)\frac{d\gamma^\alpha}{ds}\frac{d\gamma^\beta}{ds} \\
&\quad + \Gamma^\tau_{ij}\frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} \\
&\quad + 2(\Gamma^\tau_{\alpha j} + \Gamma^\tau_{ij}A^i_\alpha)\frac{d\gamma^j}{ds}\frac{d\gamma^\alpha}{ds} = 0
\end{aligned}
\right.
$$

Let $\gamma(u)$ be a D-geodesic in M parametrized by a parameter $u = u(t)$ where t is a D-affine parameter. Then we have

$$(8.2)\qquad D_{\gamma'(u)}\gamma'(u) = -\left((d^2u/dt^2)/(du/dt)^2 \right)\gamma'(u)$$

where $\gamma'(u) = (d\gamma^i/du + A^i_\alpha\, d\gamma^\alpha/du)\,\partial/\partial x^i + (d\gamma^\alpha/du)\, v_\alpha$.

Now, let $\gamma(t)$ be a D-geodesic parametrized by a D-affine parameter t and $\dot{\gamma}(t) = X = X_E + X_E{}^\perp$. Let s be the arc-length along γ . Then we have

$$\frac{ds}{dt} = \left(\langle X_E, X_E\rangle + \langle X_E{}^\perp, X_E{}^\perp\rangle \right)^{1/2}$$

$$(8.3) \qquad \frac{d^2 s}{dt^2} = \frac{1}{2}(\langle X_E, X_E\rangle + \langle X_E{}^{\perp}, X_E{}^{\perp}\rangle)^{-1/2}\{X(\langle X_E, X_E\rangle + \langle X_E{}^{\perp}, X_E{}^{\perp}\rangle)\} .$$

By Lemma 7.2, we have

$$X(\langle X_E, X_E\rangle + \langle X_E{}^{\perp}, X_E{}^{\perp}\rangle)$$

$$= 2\langle D_{X_E} X_E, X_E\rangle + X_E{}^{\perp}\langle X_E, X_E\rangle$$

$$+ 2\langle D_{X_E} X_E{}^{\perp}, X_E{}^{\perp}\rangle + 2\langle D_{X_E{}^{\perp}} X_E{}^{\perp}, X_E{}^{\perp}\rangle$$

$$= 2\langle D_X X, X_E\rangle + 2\langle D_X X, X_E{}^{\perp}\rangle + X_E{}^{\perp}\langle X_E, X_E\rangle$$

$$- 2\langle D_{X_E{}^{\perp}} X_E, X_E\rangle$$

$$= X_E{}^{\perp}\langle X_E, X_E\rangle - 2\langle D_{X_E{}^{\perp}} X_E, X_E\rangle .$$

Thus, setting $X_E = x^i \partial/\partial x^i$ and $X_E{}^{\perp} = x^{\alpha} v_{\alpha}$, we have

$$(8.4) \qquad X(\langle X_E, X_E\rangle + \langle X_E{}^{\perp}, X_E{}^{\perp}\rangle)$$

$$= x^{\alpha} x^j (x^i v_{\alpha}(g_{ij}) - 2 x^k g_{ij}\partial A^i_{\alpha}/\partial x^k) .$$

Proposition 8.1. <u>Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. Suppose that all leaves are totally geodesic. Then the arc-length parameter is a D-affine parameter.</u>

Proof. By the assumption, we have that $v_{\alpha}(g_{ij}) = g_{kj}\partial A^k_{\alpha}/\partial x^i + g_{ik}\partial A^k_{\alpha}/\partial x^j$ (See (7.1)). By (8.4), we have

$$X(\langle X_E, X_E \rangle + \langle X_E{}^{\perp}, X_E{}^{\perp} \rangle)$$

$$= X^{\alpha} X^{j} (X^{i} g_{kj} \partial A^{k}_{\alpha} / \partial x^{i} + X^{i} g_{ik} \partial A^{k}_{\alpha} / \partial x^{j} - 2 X^{k} g_{ij} \partial A^{i}_{\alpha} / \partial x^{k})$$

$$= X^{\alpha} X^{j} (X^{i} g_{ik} \partial A^{k}_{\alpha} / \partial x^{j} - X^{k} g_{ij} \partial A^{i}_{\alpha} / \partial x^{k})$$

$$= X^{\alpha} X^{j} X^{i} g_{ik} \partial A^{k}_{\alpha} / \partial x^{j} - X^{\alpha} X^{j} X^{k} g_{ij} \partial A^{i}_{\alpha} / \partial x^{k}$$

$$= 0 \quad .$$

Thus, by (8.3), $d^2s/dt^2 = 0$. Therefore we have $D_{\gamma'(s)}\, \gamma'(s) = 0$ where $'$ denotes the derivative with respect to the arc-length parameter s . Q.E.D.

Definition 8.2. A D-geodesic $\gamma(t)$ in M is called a <u>transversal</u> D-<u>geodesic</u> if $\dot{\gamma}(t) \in \Gamma(E^{\perp}|_{\gamma(t)})$ for every t .

Theorem 8.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric with respect to E . A curve $\gamma(t)$ in M is a transversal D-geodesic if and only if $\gamma(t)$ is a transversal ∇-geodesic.*

Proof. In a flat coordinate chart $U(x^i, x^{\alpha})$, setting $\gamma(t) = (\gamma^{i}(t), \gamma^{\alpha}(t))$, we have

$$\dot{\gamma}(t) = \left(\frac{d\,\gamma^{i}}{dt} + A^{i}_{\alpha} \frac{d\,\gamma^{\alpha}}{dt} \right) \partial/\partial x^{i} + \frac{d\,\gamma^{\alpha}}{dt}\, v_{\alpha} \quad .$$

If $\gamma(t)$ is a transversal D-geodesic, that is, $D_{\dot{\gamma}(t)}\, \dot{\gamma}(t) = 0$ and $d\,\gamma^{i}/dt = - A^{i}_{\alpha}\, d\,\gamma^{\alpha}/dt$. Then we have

$$(8.5)\quad \begin{cases} \dfrac{d\,\gamma^k}{dt} = -A^k_\alpha \dfrac{d\,\gamma^\alpha}{dt} \\[2ex] \dfrac{d^2\,\gamma^\tau}{dt^2} + \Gamma^\tau_{\alpha\beta} \dfrac{d\,\gamma^\alpha}{dt} \dfrac{d\,\gamma^\beta}{dt} = 0 \end{cases}$$

By Lemma 5.1 and Lemma 7.1, (8.5) makes into

$$\begin{cases} \dfrac{d\,\gamma^k}{dt} = -A^k_\alpha \dfrac{d\,\gamma^\alpha}{dt} \\[2ex] \dfrac{d^2\,\gamma^\tau}{dt^2} + \Gamma^\tau_{\alpha\beta} \dfrac{d\,\gamma^\alpha}{dt} \dfrac{d\,\gamma^\beta}{dt} = 0 \end{cases} .$$

Thus $\gamma^k(t)$ and $\gamma^\alpha(t)$ satisfy the equation (8.1). Therefore $\gamma(t)$ is a transversal ∇-geodesic.

The converse is obvious. Q.E.D.

9. Jacobi field with respect to the second connection.

Let M be in above section, and we define a D-Jacobi field along a D-geodesic in M.

Definition 9.1. Let $\gamma(t)$ be a D-geodesic in M. A vector field $Y = Y(t)$ along $\gamma(t)$ is called a D-Jacobi field along $\gamma(t)$ if Y satisfies the Jacobi equation:

$$D_{\dot{\gamma}(t)} D_{\dot{\gamma}(t)} Y + R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t)$$

$$+ D_{\dot{\gamma}(t)} (T(Y, \dot{\gamma}(t))) = 0$$

where R_D denotes the curvature tensor of D and T denotes the torsion tensor of D (See [15]).

We notice that

$$(9.1) \qquad (D_{\dot{\gamma}(t)} (T(Y, \dot{\gamma}(t))))_{E^{\perp}} = 0$$

by Lemma 7.1.

Remark. We will use "∇-Jacobi field" and "∇-focal point" instead of "Jacobi field" and "focal point" in section 5, respectively.

Definition 9.2. A vector field Y on M is called transversal if $Y \in \Gamma(E^{\perp})$.

By Lemma 7.1 and (9.1), we have

Lemma 9.1. If Y is a transversal D-Jacobi field along a transversal D-geodesic $\gamma(t)$ in M, then

$$D_{\dot{\gamma}(t)} D_{\dot{\gamma}(t)} Y + R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t) = 0 .$$

Every transversal D-geodesic $\gamma(t)$ admits two D-Jacobi fields in a natural way. One is given by $\dot{\gamma}(t)$ and the other is given by $t\,\dot{\gamma}(t)$.

Proposition 9.1. Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E . Then every D-Jacobi field $Y = Y(t)$ along a transversal D-geodesic $\gamma(t)$ in M is uniquely decomposed in the following form:

$$Y(t) = (at + b)\, \dot{\gamma}(t) + V(t) \quad ,$$

where a and b are real constants, and $V(t)$ is a D-Jacobi field along $\gamma(t)$ orthogonal to $\gamma(t)$.

Proof. We set

$$a = \langle \dot{\gamma}(t_0), \ (D_{\dot{\gamma}(t)}\ Y)(t_0) \rangle \,/\, \| \dot{\gamma}(t_0) \|^2$$

$$b = \langle \dot{\gamma}(t_0), \ Y(t_0) \rangle \,/\, \| \dot{\gamma}(t_0) \|^2$$

for some t_0 , and

$$V(t) = Y(t) - (at + b)\, \dot{\gamma}(t) \quad .$$

Since $Y(t)$ and $(at + b)\, \dot{\gamma}(t)$ are D-Jacobi field along $\gamma(t)$, so does $V(t)$. We have, by (9.1),

$$(9.2) \qquad 0 = \langle D_{\dot{\gamma}(t)}\, D_{\dot{\gamma}(t)}\, V(t), \ \dot{\gamma}(t) \rangle + \langle R_D(V(t), \dot{\gamma}(t))\, \dot{\gamma}(t), \ \dot{\gamma}(t) \rangle$$

$$= \langle D_{\dot{\gamma}(t)}\, D_{\dot{\gamma}(t)}\, V(t), \ \dot{\gamma}(t) \rangle \quad .$$

By the properties of D (See Theorem(Vaisman) in section 7),

$$\dot{\gamma}(t) < V(t), \dot{\gamma}(t) > = \dot{\gamma}(t) < V(t)_E + V(t)_{E^\perp}, \dot{\gamma}(t) >$$

$$= \dot{\gamma}(t) < V(t)_E, \dot{\gamma}(t) >$$

$$+ \dot{\gamma}(t) < V(t)_{E^\perp}, \dot{\gamma}(t) >$$

$$= \dot{\gamma}(t) < V(t)_{E^\perp}, \dot{\gamma}(t) >$$

$$= < D_{\dot{\gamma}(t)} V(t)_{E^\perp}, \dot{\gamma}(t) > ,$$

and $< D_{\dot{\gamma}(t)} V(t)_E, \dot{\gamma}(t) > = 0$. Thus we have

(9.3) $\dot{\gamma}(t) < V(t), \dot{\gamma}(t) > = < D_{\dot{\gamma}(t)} V(t), \dot{\gamma}(t) >$.

By (9.2) and (9.3), we have

$$\frac{d^2}{dt^2} < V(t), \dot{\gamma}(t) > = 0 ,$$

thus $< V(t), \dot{\gamma}(t) > = ct + d$ where c and d are constants.

We have

$$\begin{aligned} d &= < V(t_0), \dot{\gamma}(t_0) > \\ &= < Y(t_0) - b\,\dot{\gamma}(t_0), \dot{\gamma}(t_0) > \\ &= < Y(t_0), \dot{\gamma}(t_0) > - b < \dot{\gamma}(t_0), \dot{\gamma}(t_0) > \\ &= b \| \dot{\gamma}(t_0) \|^2 - b \| \dot{\gamma}(t_0) \|^2 \\ &= 0 . \end{aligned}$$

As $D_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$, we have $D_{\dot{\gamma}(t)} V(t) = D_{\dot{\gamma}(t)} Y(t) - a\,\dot{\gamma}(t)$.
Thus we have

$$c = \frac{d}{dt} < V(t), \dot{\gamma}(t) > \Big|_{t=t_0}$$

$$= \langle D_{\dot{\gamma}(t)} V(t), \dot{\gamma}(t) \rangle |_{t=t_\circ}$$

$$= \langle (D_{\dot{\gamma}(t)} Y)(t_\circ) - a \dot{\gamma}(t_\circ), \dot{\gamma}(t_\circ) \rangle$$

$$= \langle (D_{\dot{\gamma}(t)} Y)(t_\circ), \dot{\gamma}(t_\circ) \rangle - a \langle \dot{\gamma}(t_\circ), \dot{\gamma}(t_\circ) \rangle$$

$$= a \| \dot{\gamma}(t_\circ) \|^2 - a \| \dot{\gamma}(t_\circ) \|^2$$

$$= 0 \quad .$$

Therefore we have

$$\langle V(t), \dot{\gamma}(t) \rangle = 0 \quad .$$

The uniqueness of decomposition of $Y(t)$ is easily proved.

Q.E.D.

Proposition 9.2. Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle \ , \ \rangle$ with respect to E. Let $\gamma(t)$ ($t \in [0,1]$) be a transversal D-geodesic in M and Y a transversal D-Jacobi field along $\gamma(t)$. If $\langle R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Y \rangle \leqq 0$ and Y vanishes at two points $\gamma(0)$ and $\gamma(1)$, then Y vanishes identically.

Proof. We have

$$\frac{d}{dt} \langle D_{\dot{\gamma}(t)} Y, Y \rangle$$

$$= \langle D_{\dot{\gamma}(t)} D_{\dot{\gamma}(t)} Y, Y \rangle + \langle D_{\dot{\gamma}(t)} Y, D_{\dot{\gamma}(t)} Y \rangle$$

$$= - \langle R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Y \rangle + \langle D_{\dot{\gamma}(t)} Y, D_{\dot{\gamma}(t)} Y \rangle \quad ,$$

thus we have

$$\int_0^1 \left\{ < D_{\dot{\gamma}(t)} Y, D_{\dot{\gamma}(t)} Y > - < R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Y > \right\} dt$$

$$= < (D_{\dot{\gamma}(t)} Y)(0), Y(0) > - < (D_{\dot{\gamma}(t)} Y)(1), Y(1) >$$

$$= 0 .$$

Since $< R_D(Y, \dot{\gamma}(t)) \dot{\gamma}(t), Y > \leqq 0$, we have

$$< D_{\dot{\gamma}(t)} Y, D_{\dot{\gamma}(t)} Y > = 0$$

for any $t \in [0,1]$. Since Y vanishes at $\gamma(0)$, $D_{\dot{\gamma}(t)} Y = 0$ implies $Y(t) = 0$ for any $t \in [0,1]$. Q.E.D.

Now we have the non-existence of ∇-focal points of each leaf under a certain condition of R_D .

For a point $m \in M$, a plane π in the tangent space $T_m M$ is called a transversal plane if π is spaned by linearly independent vectors X_m, Y_m such that X_m, $Y_m \in E^{\perp}_m$ (that is, X_m and Y_m are transversal vectors). For each point $m \in M$ and each transversal plane π in $T_m M$, the transversal D-sectional curvature $K(m,\pi)$ is defined by

$$K(m,\pi) = \frac{< R_D(X_m, Y_m) Y_m, X_m >}{< X_m, X_m >< Y_m, Y_m > - < X_m, Y_m >^2}$$

where X_m and Y_m are linearly independent vectors and span a transversal plane π . If $K(m,\pi) \leqq 0$ for each point $m \in M$ and for all transversal plane π in $T_m M$, then M is called to have non-positive transversal D-sectional curvature.

Then we have

Theorem 9.1. Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $<\ ,\ >$ with respect to E. Suppose that M has non-positive transversal D-sectional curvature. Then, for any point $m \in M$, there is not a ∇-focal point of the leaf through m along every transversal ∇-geodesic starting from m.

Proof. Let $\gamma(t)$ ($t \in [0,1]$) be a transversal ∇-geodesic starting from m. We assume that a point $\gamma(1)$ is a ∇-focal point of the leaf L_m through m along γ. That is, we assume that there exists a non-zero ∇-Jacobi field $Y \in \mathcal{J}_L(\gamma;\, L_{\gamma(0)},\, \gamma(1))$. Then we have $Y_E{}^{\perp} \neq 0$ by Proposition 5.3. Thus we have

$$\hat{Y} = 0 \quad , \quad Y(0) \in T_{\gamma(0)}L \quad , \quad Y(1) = 0$$

and, by Proposition 5.1,

$$(9.4) \qquad 0 = (\nabla_{\dot{\gamma}(t)} (\nabla_{\dot{\gamma}(t)} Y_E{}^{\perp})_E{}^{\perp})_E{}^{\perp} - (\nabla_{\dot{\gamma}(t)} (\nabla_{Y_E{}^{\perp}} \dot{\gamma}(t))_E{}^{\perp})_E{}^{\perp} - (\nabla_{([Y_E{}^{\perp},\, \dot{\gamma}(t)])_E{}^{\perp}} \dot{\gamma}(t))_E{}^{\perp} .$$

The transversal ∇-geodesic γ is also a transversal D-geodesic by Theorem 8.1. By Lemma 5.1 and Lemma 7.1, (9.4) implies

$$0 = D_{\dot{\gamma}(t)} D_{\dot{\gamma}(t)} Y_E{}^{\perp} - D_{\dot{\gamma}(t)} D_{Y_E{}^{\perp}} \dot{\gamma}(t) - D_{([Y_E{}^{\perp},\, \dot{\gamma}(t)])_E{}^{\perp}} \dot{\gamma}(t)$$

$$= D_{\dot{\gamma}(t)} D_{\dot{\gamma}(t)} Y_E{}^{\perp} + R_D(Y_E{}^{\perp}, \dot{\gamma}(t)) \dot{\gamma}(t) \quad .$$

Thus $Y_E{}^{\perp}$ is a transversal D-Jacobi field along γ and satisfies $Y_E{}^{\perp}(0) = Y_E{}^{\perp}(1) = 0$. By $\langle R_D(Y_E{}^{\perp}, \dot{\gamma}(t)) \dot{\gamma}(t), Y_E{}^{\perp} \rangle \leqq 0$ and Proposition 9.2, we have $Y_E{}^{\perp} = 0$. This is a contradiction. Q.E.D.

See Example 5.1.

Appendix

Let G be a $p+q$ dimensional connected Lie group, and let $\mathfrak{g}$ be the associated Lie algebra consisting of all left invariant vector fields on G . If we take a Lie sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$, then G is regarded as a foliated manifold. In fact, a sub-bundle $E = \bigcup_{a \in G} E_a$, $E_a = \{ X_a \mid X \in \mathfrak{h} \}$ of the tangent bündle TG of G is integrable. And we assume that $\dim E_a = p$ for every $a \in G$. Then we have

Definition. The above cople $(G; \mathfrak{h})$ is called a _foliated Lie group with a foliation_ $\mathfrak{h}$ _of codimension_ q .

We notice that if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then the leaf space $G/\mathfrak{h}$ has a Lie group structure.

We assume that G admits a left invariant riemannian metric $\langle \ , \ \rangle$. Then we take an orthonormal adapted frame $\{ X_i, X_\alpha \}$ to the foliation $\mathfrak{h}$ with respect to the metric $\langle \ , \ \rangle$. The Levi-Civita cinnection ∇ is defined by

$$\langle \nabla_X Y, Z \rangle$$

$$= \frac{1}{2} \Big\{ \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \Big\}$$

for any $X, Y, Z \in \mathfrak{g}$. Setting

$$C_{ABC} = \langle [X_A, X_B], X_C \rangle \quad ,$$

it follows that

$$\nabla_{X_A} X_B = \sum_C \frac{1}{2} \{ c_{ABC} - c_{BCA} + c_{CAB} \} X_C$$

where $A, B, C = 1, 2, \cdots, p, p+1, \cdots, p+q$. (See, J. Milnor, Curvature of left invariant metrics on Lie group, Advances in Math. 21 (1976), 293-329).

Let $\gamma(s)$ be a geodesic in G parametrized by arc-length s and $\gamma(0) = e$ (the unit of G). We set

$$(*) \qquad \dot{\gamma}(s) = \sum_{i=1}^{p} \dot{\gamma}^i(s) \; X_i|_{\gamma(s)} + \sum_{\alpha=p+1}^{p+q} \dot{\gamma}^\alpha(s) \; X_\alpha|_{\gamma(s)}$$

for an orthonormal adapted frame $\{ X_i, X_\alpha \}$ to η . From Theorem 4.1, if the riemannian metric $\langle \;,\; \rangle$ is a bundle-like metric with respect to η and all leaves are totally geodesic, then

$$\sum_{i=1}^{p} \{ \dot{\gamma}^i(s) \}^2 = \text{constant}$$

for any geodesic $\gamma(s)$ with (*). But each $\dot{\gamma}^i(s)$ is not constant in general.

For each $X \in \mathfrak{g}$, a linear transformation $ad(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by

$$ad(X)\, Y = [\, X, Y \,]$$

for any $Y \in \mathfrak{g}$, and it is called skew-adjoint if it satisfies

$$\langle ad(X)\, Y, Z \rangle + \langle Y, ad(X)\, Z \rangle = 0$$

for any $Y, Z \in \mathfrak{g}$.

Theorem. Let $(G; \eta)$ be a $p+q$ dimensional foliated Lie group with a foliation η of codimension q . Suppose that G has a left invariant riemannian metric $\langle \;,\; \rangle$ and that

$ad(X)$ *is skew-adjoint for every* $X \in \eta$. *Then, for every geodesic* $\gamma(s)$ *in* G *parametrized by arc-length* s ,

$$\dot{\gamma}^i(s) = \text{constant} \qquad 1 \leqq {}^\forall i \leqq p$$

where $\dot{\gamma}(s) = \sum_{i=1}^{p} \dot{\gamma}^i(s)\, X_i\big|_{\gamma(s)} + \sum_{\alpha=p+1}^{p+q} \dot{\gamma}^\alpha(s)\, X_\alpha\big|_{\gamma(s)}$, *and* $\{ X_i, X_\alpha \}$ *denotes an orthonormal adapted frame to* η .

Remark. Under the above assumptions, we have that the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to η and all leaves are totally geodesic. If η is the non-trivial center of $\mathfrak{g}$, then $ad(X)$ is skew-adjoint for every $X \in \eta$. If the metric $\langle\ ,\ \rangle$ is a bi-invariant metric on G , then each $\dot{\gamma}^A(s) = \text{constant}$ ($A = 1, 2, \cdots, p, p+1, \cdots, p+q$).

Proof. Let $\gamma(s)$ be a geodesic in G . Then we have

$$\begin{aligned} 0 &= \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) \\ &= \nabla_{\dot{\gamma}^A(s)\, X_A} \dot{\gamma}^B(s)\, X_B \\ &= \sum_C \Big\{ \frac{d\,\dot{\gamma}^C(s)}{ds} + \frac{1}{2} \dot{\gamma}^A(s)\, \dot{\gamma}^B(s) (C_{ABC} - C_{BCA} \\ &\qquad\qquad + C_{CAB}) \Big\} X_C\big|_{\gamma(s)} \end{aligned}$$

Thus we have, for each i ,

$$\frac{d\,\dot{\gamma}^i(s)}{ds} = -\frac{1}{2} \dot{\gamma}^A(s)\, \dot{\gamma}^B(s) (C_{ABi} - C_{BiA} + C_{iAB}) .$$

Now we have

$$C_{ABC} + C_{BAC} = 0$$

and, from that $ad(X_i)$ is skew-adjoint, we have

$$C_{iBC} + C_{iCB} = 0 \quad .$$

Thus we have

$$\frac{d\,\dot{\gamma}^i(s)}{ds} = -\frac{1}{2}\ \dot{\gamma}^A(s)\ \dot{\gamma}^B(s)\,(\ C_{ABi} - C_{iBA} + C_{iAB}\ )$$

$$= -\frac{1}{2}\ \dot{\gamma}^A(s)\ \dot{\gamma}^B(s)\ C_{ABi}$$

$$= 0 \quad .$$

Therefore, for each i, $\dot{\gamma}^i(s) = \text{constant}$. Q.E.D.

We give a example of G satisfying the assumptions in the above theorem.

Example. We set

$$G = \left\{ \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \;\middle|\; \begin{matrix} A_1 \in SO(n) \ , & A_2 \in M(n,m:\mathbb{R}) \quad A_2 \neq 0 \\ A_3 \in GL(m;\mathbb{R}) & \det A_3 > 0 \end{matrix} \right\}$$

$$\mathfrak{g} = \left\{ \begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix} \;\middle|\; \begin{matrix} X_1 \in \mathfrak{so}(n) \ , & X_2 \in \mathfrak{m}(n,m;\mathbb{R}) \\ X_3 \in \mathfrak{gl}(m;\mathbb{R}) & \end{matrix} \right\}$$

$$\mathfrak{h} = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \;\middle|\; X \in \mathfrak{so}(n) \right\}$$

Then $\mathfrak{g}$ is the associated Lie algebra of G , and $\mathfrak{h}$ is a Lie sub-algebra of $\mathfrak{g}$ and is not an ideal of $\mathfrak{g}$.

For each $a \in G$, the left translation $\ell_a : G \longrightarrow G$ is defined by $\ell_a(b) = ab$ for any $b \in G$. The left translation ℓ_a induces a map $(\ell_{a*})_e : \mathfrak{g} = T_eG \longrightarrow T_aG$. Thus we may take that $T_aG = (\ell_{a*})_e(\mathfrak{g})$ for every $a \in G$. For any $X, Y \in \mathfrak{g}$, we may define $\langle X, Y \rangle$ by

$$\langle X, Y \rangle = \mathrm{Trace}(\ {}^tXY\ )$$

where ${}^{t}X$ denotes the transposed matrix of X . Then G has a left invariant riemannian metric $\langle\ ,\ \rangle$. In fact, for any $\widetilde{X}, \widetilde{Y} \in T_aG$ ($a \in G$), we may define $\langle \widetilde{X}, \widetilde{Y} \rangle_a$ by

$$\langle \widetilde{X}, \widetilde{Y} \rangle_a = \langle X, Y \rangle$$

where $\widetilde{X} = (\ell_{a*})_e(X)$ and $\widetilde{Y} = (\ell_{a*})_e(Y)$.

Next, we show that $ad(X)$ is skew-adjoint for every $X \in \mathfrak{n}$. We take

$$\forall X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{n}$$

$$\forall Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_3 \end{bmatrix} , \quad \forall Z = \begin{bmatrix} Z_1 & Z_2 \\ 0 & Z_3 \end{bmatrix} \in \mathfrak{g} .$$

Then we have

$$\langle ad(X)\ Y,\ Z \rangle$$

$$= \mathrm{Trace}(\ {}^{t}(\ XY - YX\ )Z\ )$$

$$= \mathrm{Trace}(\ {}^{t}(\ X_1Y_1 - Y_1X_1\ )Z_1 + {}^{t}(\ X_1Y_2\ )Z_2\ )$$

$$\langle Y,\ ad(X)\ Z \rangle$$

$$= \mathrm{Trace}(\ {}^{t}Y_1(\ X_1Z_1 - Z_1X_1\ ) + {}^{t}Y_2(\ X_1Z_2\ )\ ) .$$

We have

$$\langle ad(X)\ Y,\ Z \rangle + \langle Y,\ ad(X)\ Z \rangle$$

$$= \mathrm{Trace}(\ {}^{t}(\ X_1Y_1 - Y_1X_1\ )Z_1 + {}^{t}Y_1(\ X_1Z_1 - Z_1X_1\ )\ )$$

$$+ \operatorname{Trace}({}^t(X_1Y_2)Z_2 + {}^tY_2(X_1Z_2)) \quad .$$

Since $X_1, Y_1, Z_1 \in \mathfrak{so}(n)$, we have

$$\operatorname{Trace}({}^t(X_1Y_1 - Y_1X_1)Z_1 + {}^tY_1(X_1Z_1 - Z_1X_1))$$

$$= \operatorname{Trace}(- {}^tX_1{}^tY_1Z_1 - {}^tY_1Z_1X_1)$$

$$= \operatorname{Trace}(- X_1Y_1Z_1 - {}^t(X_1Z_1Y_1))$$

$$= \operatorname{Trace}(- X_1Y_1Z_1 - X_1Z_1Y_1)$$

$$= \operatorname{Trace}(- X_1(Y_1Z_1 - Z_1Y_1))$$

$$= \operatorname{Trace}(- X_1((Y_1Z_1) + {}^t(Y_1Z_1)))$$

$$= 0 \quad ,$$

and

$$\operatorname{Trace}({}^t(X_1Y_2)Z_2 + {}^tY_2(X_1Z_2))$$

$$= \operatorname{Trace}({}^tY_2{}^tX_1Z_2 + {}^tY_2X_1Z_2)$$

$$= \operatorname{Trace}(- {}^tY_2X_1Z_2 + {}^tY_2X_1Z_2)$$

$$= 0 \quad .$$

Thus we have

$$\langle \operatorname{ad}(X)\, Y, Z \rangle + \langle Y, \operatorname{ad}(X)\, Z \rangle = 0$$

for any $X \in \mathfrak{h}$, $Y, Z \in \mathfrak{g}$.

We set

$$E_{ij} = \begin{array}{cc} & \qquad j \\ i) & \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \end{array} \in \mathfrak{gl}(m;\mathbb{R}) \qquad 1 \leqq {}^{\forall}i,\ {}^{\forall}j \leqq m$$

$$X_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & E_{ij} \end{bmatrix} \in \mathfrak{g} .$$

Then we notice that $ad(X_{1m})$ is not skew-adjoint on $\mathfrak{g}$. In fact, we have

$$< ad(X_{1m})\ X_{mm},\ X_{1m} > = 1$$

$$< X_{mm},\ ad(X_{1m})\ X_{1m} > = 0 \quad .$$

Therefore $(G; \mathfrak{J})$ is a foliated Lie group with a foliation $\mathfrak{J}$ satisfying the assumptions in the above theorem.

References

[1] R. L. Bishop, Clairaut submersions, Differential Geometry, in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 21-31.

[2] P. Dombrowski, Jacobi fields, totally geodesic foliations and geodesic differential forms, Resultate der Math. 1 (1980), 156-194.

[3] R. H. Escobales, Jr., Riemannian submersions with totally geodesic fibers, J. Differential Geometry, 10 (1975), 253-276.

[4] R. Hermann, On the differential geometry of foliations, Ann. of Math., 72 (1960), 445-457.

[5] R. Hermann, Totally geodesic orbits of groups of isometries, Indag. Math., 24 (1962), 291-298.

[6] S. Ishihara and M. Konishi, Differential geometry of fibred spaces, Publications of the Study Group of Geometry 8, Tokyo, 1973.

[7] S. Kashiwabara, The structure of a Riemannian manifold admitting a parallel field of one-dimensional tangent vector subspaces, Tohoku Math. J., 11 (1959), 327-350.

[8] H. Kitahara, The existence of complete bundle-like metrics, Ann. Sci. Kanazawa Univ., 9 (1972), 37-40.

[9] H. Kitahara, The existence of complete bundle-like metrics II, Ann. Sci. Kanazawa Univ., 10 (1973), 51-54.

[10] H. Kitahara, The completeness of a Clairaut's foliation, Ann. Sci. Kanazawa Univ., 11 (1974), 37-40.

[11] H. Kitahara, On a parametrix form in a certain V-submersion, Lecture Notes in Math., 792, Springer-Verlag, 1980, 264-298.

[12] H. Kitahara and S. Yorozu, On the cohomology groups of a manifold with a nonintegrable subbundle, Proc. Amer. Math. Soc., 56 (1976), 201-204.

[13] H. Kitahara and S. Yorozu, A formula for the normal part of the Laplace-Beltrami operator on the foliated manifold, Pacific J. Math., 69 (1977), 425-432.

[14] H. Kitahara and S. Yorozu, On some differential geometric characterizations of a bundle-like metrics, Kodai Math. J., 2 (1979), 130-138.

[15] S. Kobayashi and K. Nomizu, Foundations of differential geometry, II, Interscience Pub., New York, 1969.

[16] R. Maltz, The nullity spaces of the curvature operator, Cahiers Topologie Géom. Différentielle, 8 (1966), 1-20.

[17] Y. Muto, On some properties of a fibred Riemannian manifold, Sci. Rep. Yokohama Nat. Univ. Sect. I, 1 (1952), 1-14.

[18] H. Nakagawa, Riemannian geometry in the large (in Japanese), Kaigai-boeki, Tokyo, 1977.

[19] A. M. Naveira, Variedades foliadas con metrica casifibrada, Collect. Math., 21 (1970), 41-97.

[20] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J., 13 (1966), 459-469.

[21] B. O'Neill, Submersions and geodesics, Duke Math. J., 34 (1967), 363-373.

[22] J. S. Pasternack, Topological obstructions to integrability and the Riemannian geometry of smooth foliations, Thesis, Princeton Univ., 1970.

[23] J. S. Pasternack, Foliations and compact Lie group actions, Comment. Math. Helv., 46 (1971), 467-477.

[24] B. L. Reinhart, Foliated manifolds with bundle-like metrics, Ann. of Math., 69 (1959), 119-132.

[25] B. L. Reinhart, Closed metric foliations, Michigan Math. J., 8 (1961), 7-9.

[26] R. Sacksteder, Foliations and pseudogroups, Amer. J. Math., 87 (1965), 79-102.

[27] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J., 10 (1958), 338-354.

[28] I. Vaisman, Variétés riemanniennes feuilletées, Czechoslovak Math. J., 21 (1971), 46-75.

[29] I. Vaisman, Cohomology and differentiable forms, Marcel Dekker, INC., New York, 1973.

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