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SOME ASPECTS OF CONVEXITY IN GEOMETRY OF GEODESICS

By

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THESIS

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PREFACE

The spaces with geodesics would be (complete) Riemannian manifolds, (complete) Finsler manifolds and G-spaces created by H. Busemann. The difference between a Riemannian manifold and a Finsler manifold is that the former behaves locally like a Euclidean, the latter locally like a Minkowski space. One of the important properties of a Riemannian manifold M is that Euclidean spaces as tangent spaces are developed in a single Euclidean space along a curve in M. G-spaces have no differentiability, so that it is impossible to approximate the space locally by a linear space. In this reason the theory of G-spaces is said as a direct method of geometry.

In this thesis I will deal with convexity in Riemannian manifolds and in G-spaces. Functions and sets with convexity are naturally defined on these spaces, because geodesics play the same role as lines in linear spaces. Since the behavior of geodesics depends on the metric structure of the space, the existence of definite convexity are possibly influenced by its topological and metric structure. I would like to show some aspects of influences on convexity in Riemannian manifolds and G-spaces.

A fundamental property of a 3-simplex in an affine space is that it is obtained by connecting points of the six segments which join the four vertices. Also it is trivial that a triangle Δ(abc) is decomposed into two triangles Δ(aba') and Δ(aca') where a' is an arbitrary point in the segment T(b,c) joining b and c. In Chapter I these properties together with the axiom of n-planes (which is also called Beltrami's theorem) will furnish a necessary and sufficient condition for a Riemannian metric to be of constant curvature.

A function F on a G-space R is said to be convex (or concave, affine) if $F \circ x$ is a one-variable convex (or concave, affine) function where $x(t)$, $-\infty < t < \infty$, is a representation of an arbitrary geodesic in R. The Busemann function of a ray (see Chapter III) often becomes a function with convexity. For example, the Busemann function of every ray in a complete noncompact Riemannian manifold of nonnegative sectional curvature is convex, and the Busemann function of every ray on a simply connected G-space with nonpositive curvature is concave, and the Busemann function of a ray $\{x\} \times [0,\infty)$ on a complete Riemannian product $N \times \mathbb{R}$ is affine. Greene-Shiohama ([15] and [16]) have determined the topological and differentiable structures of complete Riemannian manifolds with locally nonconstant convex functions. They use essentially the properties which hold on Riemannian manifolds, for instance, the existence of strongly convex balls and the first and second variation formulas. In contrust to the Riemannian case where convex functions are automatically Lipschitz continuous, it is not true in general that convex functions on a G-space is continuous. But if the dimension of a G-space is two, then the space becomes topologically a surface and it turns out that every convex function on it is continuous. This situation makes it possible to investigate the topology of

2-dimensional G-spaces which possess convex functions. Without using existence of strongly convex balls and variation formulas I will show in Chapter II that if a G-surface R admits a locally nonconstant convex function, then R is topologically either a plane, a cylinder $S^1 \times \mathbb{R}$ or an open Moebius strip.

I would like to emphasize in Chapter IV that the existence of an affine function on a complete Riemannian manifold M gives a strong restriction to the Riemannian structure. Namely, a complete Riemannian manifold M admits a nontrivial affine function if and only if M is isometric to a Riemannian product $N \times \mathbb{R}$. Thus a characterization of a Euclidean space is obtained via affine functions. This particular property of affine functions is applied to Busemann functions on complete and non-compact Riemannian manifolds of nonnegative (or nonpositive) sectional curvature to give new splitting theorems.

Due to convexity of Busemann functions the study of these functions is important. In fact, it plays an essential role in the study of complete Riemannian manifolds of nonnegative (or nonpositive) sectional curvature. In Chapter III, I shall discuss the "differentiability" of a Busemann function on a G-space R. Here it will be proved that if the set of all "non-differentiable" points of a Busemann function on R is bounded, then a neighborhood of the point at infinity of R is topologically a cylinder and R has exactly one end. Concerning the metric structure of R it follows that if dim R = 2 and the total excess of R (which corresponds to the total curvature in Riemannian case) exists and if the set of all "non-differentiable" points of a Busemann function is bounded, then the total excess is $2\pi\chi(R)$. It should be noted that this value is the possible maximum value by the well-known theorem due to Cohn-Vossen in [12] which has been extended to G-surfaces by H. Busemann in [4].

I am deeply imdebted to Professor Katsuhiro Shiohama for introducing me to G-spaces and convexity in geometry and for his continuing and stimulating interest in the present work. Also I would like to express my thanks to Professor Hisao Nakagawa for his advice and encourgement to carry out this plan.

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CHAPTER I

The axiom of n-planes and convex hulls

1. Introduction.

A characterization of a space of constant curvature is an interesting problem in Riemannian geometry. It has been done by the various methods as seen in [4], [8] and [27]. In particular we are interested in the axiom of n-planes which is stated as follows; a Riemannian manifold M of dimension $m \geqq 3$ is said to satisfy the axiom of n-planes if for each p in M and any n-dimensional subspace T_p' of the tangent space T_pM, there is an n-dimensional totally geodesic submanifold N containing p such that the tangent space of N at p is T_p', where n is a fixed integer $2 \leqq n < m$. É. Cartan proved in [6] that if M satisfies the axiom of n-planes for some n, then M is a space of constant curvature.

Historicaly, E. Beltrami proved in [2] that a space of constant curvature M satisfies the axiom of 2-planes, and the converse was proved by F. Schur [37]. É. Cartan also indicated in [7] that Schur's theorem had been proved by L. Schlaefli [36] in combination with F. Klein [26].

The purpose of this chapter is to exhibit this axiom in terms of convex analysis, i.e., convex combinations and convex hulls.

In this chapter let M be a Riemannian manifold without boundary. For a point p in M let $B(p,\rho)$ denote the <u>strongly convex open ball</u> with center p and radius ρ, i.e., every ball which is contained in $B(p,\rho)$ is convex where the term "convex" is used in the following sense: A set $D \subset M$ is <u>convex</u> iff q, r in D implies that there is a unique (distance minimizing geodesic) segment $T(q,r)$ and it is contained in D. From [44] (and also see [17], [19]) we know that for each p in M there is an $\rho > 0$ such that $B(p,\rho)$ is strongly convex. Since the constancy of curvature is a local property, we may restrict our attention to the interior of a strongly convex ball.

If U is a subset of $B(p,\rho)$, then we consider the smallest convex set which contains U. We call it the <u>convex hull</u> of U and denote it by <u>HU</u>. Clearly $HU \subset B(p,\rho)$.

For a set U in M, <u>CU</u> is by definition the set of all points each of which belongs to some segment which joints two points of U, and we put $C^kU := C(C^{k-1}U)$ inductively, $k = 1,2,\ldots$, $C^0U := U$. Clearly $HU = \bigcup_{k=0}^{\infty} C^kU$ holds for any $U \subset B(p,\rho)$. We may think that C^k corresponds to convex combinations in the linear space. In fact, Valentine stated in [43] that C^k was observed by Brunn [3], Sierpiński [40], Abe-Kubota-Yoneguchi [1] in Euclidean space.

It is the nature of a space of constant curvature that the convex hull of sufficiently close n+1 points $q_0, q_1, \ldots, q_n$ can be obtained by $C^k\{q_0,q_1,\ldots,q_n\}$, where the integer k satisfies $2^{k-1} \leqq n < 2^k$. And if M satisfies the axiom of n-planes with $2 \leqq n < \dim M$, then the set of n+1 points $q_0, q_1, \ldots, q_n$, which are sufficiently close to each other, has the property that $C^k\{q_0,q_1,\ldots,q_n\} = H\{q_0,q_1,\ldots,q_n\}$, where the integer k satisfies $2^{k-1} \leqq n < 2^k$.

However it is not easy to verify the converse. This is because $C^k\{\ \}$ does not in general carry the structure of a smooth submanifold, and because the dimension of $H\{\ \}$ is possibly greater than n.

Thus our main result of this chapter is

Theorem 1. Let dim M be greater than 3. If for each point p in M there exists a convex neighborhood V of p in M such that $H\{q_0,q_1,q_2,q_3\} = C^2\{q_0,q_1,q_2,q_3\}$ for any points q_0, q_1, q_2, q_3 in V, then M is a space of constant curvature.

The author does not know whether the above theorem for convex combinations of three points is true. On this problem the following holds.

Theorem 2. Let dim M be greater than 2. If for each point p in M there exists a convex neighborhood V of p in M such that $H\{q_0,q_1,q_2\} = C^2\{q_0,q_1,m(q_1,q_2)\} \cup C^2\{q_0,q_2,m(q_1,q_2)\}$ for any points q_0, q_1, q_2 in V, where $m(q_1,q_2)$ is the midpoint of the segment $T(q_1,q_2)$ which joins q_1 and q_2, then M is a space of constant curvature.

In the proofs of our theorems we shall need to estimate the dimensions (defined in [20], p.24) of convex hulls. For this purpose we will often use the <u>a-measure</u> $m_a(X)$, $0 \leqq a < \infty$, of a (separable) metric space X which is defined in [20], p.102 as follows: Given $\varepsilon > 0$, let $m_a^{\varepsilon}(X) := \inf \sum_{i=1}^{\infty} [\delta(A_i)]^a$, where $X = \bigcup_{i=1}^{\infty} A_i$ is any decomposition of X in a countable number of subsets such that for every i the diameter $\delta(A_i)$ of A_i is less than ε, and the superscript a denotes the exponentiation. Let $m_a(X) := \sup_{\varepsilon > 0} m_a^{\varepsilon}(X)$.

Concerning this measure it is well known ([20], p.104) that if X is a metric space such that $m_{n+1}(X) = 0$, $0 \leqq n < \infty$, then $\dim X \leqq n$, and this fact is used in the proof of Lemma 4 in Section 2.

In Section 2 we shall give lemmas which are used in the proofs of our theorems and we will prove theorems in Section 3. In Section 4 we give remarks of the theorems.

2. Convex hull and Hausdorff measure.

In [11] Cheeger-Gromoll showed that if S is a connected locally convex set in M, then there is a smooth totally geodesic imbedded submanifold N of M such that $N \subset S \subset N^c$, where N^c is the closure of N.

This fact and the axiom of 2-planes funish the following.

Lemma 3. Let m:= dim M be greater than 2. If for each point p in M and for some n, $2 \leqq n < m$, there is a convex neighborhood V of p in M such that $\dim H\{q_0,q_1,\ldots,q_n\} \leqq n$ for any points $q_0, q_1, \ldots, q_n$ in V, then M is a space of constant curvature.

Proof. We first claim that $\dim H\{q_0,q_1,\ldots,q_k\} \leqq k$ holds for every k, $2 \leqq k \leqq n$, and for any points $q_0, q_1, \ldots, q_k$ in V. Suppose that $\dim H\{q_0,q_1,\ldots,q_{n-1}\} > n-1$ for some points $q_0, q_1, \ldots, q_{n-1}$ in V, i.e., $\dim H\{q_0,q_1,\ldots,q_{n-1}\} = n$. Then there exists a smooth totally geodesic n-dimensional imbedded submanifold N such that $N \subset \mathring{H}\{q_0,q_1,\ldots,q_{n-1}\} \subset N^c$. Take a point q in N and a normal vector v of N at q such that $\exp_N v \in V$. Then $\dim H\{q_0,q_1,\ldots,q_{n-1},\exp_N v\} > n$, a contradiction. Thus we obtain $\dim H\{q_0,q_1,\ldots,q_{n-1}\} \leqq n-1$ for any points $q_0, q_1, \ldots, q_{n-1}$ in V. By iterating the same argument we have our claim. In particular, $\dim \mathring{H}\{q_0,q_1,q_2\} \leqq 2$ for any points q_0, q_1, q_2 in V.

Now we show that M satisfies the axiom of 2-planes. Let T_p' be an arbitrary 2-dimensional subspace of T_pM and let v_1

and v_2 be vectors in T_p' such that $q_1 := \exp_p v_1$ and $q_2 := \exp_p v_2$ belong to V, and p, q_1, q_2 are non-colinear. Take q in the interior of the segment $T(q_1, q_2)$ joining q_1 and q_2, and take q_0 in V on the other side of q with respect to p on the extension of T(p,q). Let N_0 be the set of all points which belong to segments from q_0 to points on $T(q_1, q_2)$. Then N_0 is a smooth surface except at q_0. We need to prove that $T_p N_0 = T_p$' and $N_0 - \{q_0\}$ is totally geodesic in M. Let N be a smooth totally geodesic submanifold in M such that $N \subset \dot{H}\{q_0, q_1, q_2\} \subset N^c$. Since $p \in N_0 \subset H\{q_0, q_1, q_2\}$ and $\dim \dot{H}\{q_0, q_1, q_2\} \leqq 2$, it follows that $N_0 \subset N^c$ and $\dim N_0 = \dim N = 2$. Thus $T_p N_0 = T_p$' and $N_0 - \{q_0\}$ is totally geodesic in M.

We know from this lemma that in order to prove our theorems we have only to estimate the dimension of H{ }. We then need the following lemma.

<u>Lemma 4</u>. Let T_1 and T_2 be two segments contained in a convex set V in M. Let A be the set of all points each of which belongs to some segment joining a point of T_1 and a point of T_2. Then $\dim A \leqq 3$ and A is closed.

<u>Proof</u>. Let $x(\tau)$, $0 \leqq \tau \leqq \alpha$, and $y(\nu)$, $0 \leqq \nu \leqq \beta$, represent segments T_1 and T_2 respectively, and for each q in V let W_q be a subset of $T_q M$ such that $\exp_q | W_q : W_q \longrightarrow V$ is a diffeomorphism. Define a map G of $[0,1] \times [0,\alpha] \times [0,\beta]$ into $T_p M$ by $G(\mu, \tau, \nu) :=$

$(\exp_p|W_p)^{-1}[\exp_{x(\tau)}\{\mu(\exp_{x(\tau)}|W_{x(\tau)})^{-1}(y(\nu))\}]$ for $(\mu,\tau,\nu) \in [0,1] \times [0,\alpha] \times [0,\beta]$, where p is a fixed point in V. Then G is differentiable, and hence G is Lipschitz continuous. Therefore it follows from the definition of 4-measure that the 4-measure of the image of G is zero since the 4-measure of $[0,1] \times [0,\alpha] \times [0,\beta]$ is zero. Note that A is the image of $\exp_p \circ G$ and that the property of having at most dimension n is topologically invariant. Thus we conclude dim $A \leqq 3$ by the fact in Section 1.

Closedness of A is evident.

Lemma 5. Let p be a fixed point in M. For an arbitrary $\alpha > 0$, there exists a $\rho > 0$ such that for any points q, r and s in $B(p,\rho)$,

$$\mu(1-\alpha)d(r,s) \leqq d(w_r(\beta\mu),w_s(\gamma\mu)) \leqq \mu(1+\alpha)d(r,s)$$

for any $\mu \in [0,1]$, where $w_r(\tau)$, $0 \leqq \tau \leqq \beta$, and $w_s(\nu)$, $0 \leqq \nu \leqq \gamma$, represent segments $T(q,r)$ and $T(q,s)$ respectively.

Proof. By a straightforward generalization of Proposition 9.10 in [19], p.54 we obtain that for given $0 < \varepsilon < 1$ there is a $\rho > 0$ such that for any non-colinear points q, r and s in $B(p,\rho)$,

$$1-\varepsilon < \|(\exp_q|B_\rho)^{-1}(r)-(\exp_q|B_\rho)^{-1}(s)\|_q/d(r,s) < 1+\varepsilon,$$

where $\| \; \|_q$ is the norm in T_qM, and B_ρ is the ρ-ball in T_qM centered at the origin.

We then have

$1-\varepsilon <$

$$\|(\exp_q|B_\rho)^{-1}(w_r(\beta\mu))-(\exp_q|B_\rho)^{-1}(w_s(\gamma\mu)\|_q/d(w_r(\beta\mu),w_s(\gamma\mu))$$
$$< 1+\varepsilon,$$

for $\mu \neq 0$. Therefore

$$(1-\varepsilon)/(1+\varepsilon) < (1/\mu)(d(w_r(\beta\mu),w_s(\gamma\mu))/d(r,s)) < (1+\varepsilon)/(1-\varepsilon).$$

If we choose an $\varepsilon > 0$ which satisfies

$$1-\alpha < (1-\varepsilon)/(1+\varepsilon) < (1+\varepsilon)/(1-\varepsilon) < 1+\alpha,$$

then it follows that $\mu(1-\alpha)d(r,s) < d(w_r(\beta\mu),w_s(\gamma\mu)) < \mu(1+\alpha)\times d(r,s)$ for any $\mu \in [0,1]$

3. Convex hull and dimension

In this section we show by estimating dimension that the convex hulls of points satisfy the axiom of n-planes under the assumptions in Theorems 1 and 2.

3.1. Proof of Theorem 1.

We denote six segments each of which joins q_i and q_j, $0 \leqq i < j \leqq 3$, by T_k, $k = 1, 2, \ldots, 6$. Then from the assumption $H\{q_0,q_1,q_2,q_3\} = \bigcup_{1 \leqq i<j \leqq 6}\{q \in V;$ q belongs to some segment which connects a point of T_i and a point of $T_j\}$. Therefore it follows from Lemma 4 and the sum theorem ([20], p.30), i.e., a separable metric space which is the countable sum of closed subsets of dimension $\leqq$ n has dimension $\leqq$ n, that dim $H\{q_0,q_1,q_2,q_3\} \leqq 3$. Hence we obtain our theorem by Lemma 3.

3.2. Proof of Theorem 2.

Let $\alpha > 0$ satisfy that $6((1+\alpha)/2)^3 < 1$. And for this α we choose a $\rho > 0$ such that $B(p,\rho) \subset V$ satisfies the conclusion of Lemma 5 and the 4ρ-ball with center p is strongly convex.

By Lemma 3 it suffices to show that dim $H\{q_0,q_1,q_2\} \leqq 2$. If dim $C^2\{r_0,r_1,r_2\} \leqq 2$ for any points r_0, r_1 and r_2 in $B(p,\rho)$, then dim $H\{q_0,q_1,q_2\} \leqq 2$ because of the sum theorem.

In fact, dim $C^2\{r_0,r_1,r_2\} \leqq 2$ is established as follows. From the definition of $C^2\{r_0,r_1,r_2\}$ the diameter of $C^2\{r_0,r_1,r_2\}$ is not greater than $d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0)$, because $C^2\{r_0,r_1,r_2\}$ is contained in $B(r_0, (d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0))$

/2). Since $H\{r_0,r_1,r_2\} \supset C^2\{r_0,r_1,r_2\}$, it holds that

$$C^2\{r_0,r_1,r_2\} \subset C^2\{r_0,r_1,m(r_1,r_2)\} \cup C^2\{r_0,r_2,m(r_1,r_2)\}.$$

Hence if we put $r_0' := m(r_1,r_2)$, $r_1' := m(r_0,r_2)$, $r_2' := m(r_0,r_1)$ and $r' := m(r_0,r_0')$, then

$$C^2\{r_0,r_1,r_2\} \subset C^2\{r_0,r_1',r'\} \cup C^2\{r_0,r',r_2'\} \cup C^2\{r_1',r_2,r_0'\}$$
$$\cup C^2\{r_1',r_0',r'\} \cup C^2\{r',r_0',r_2'\} \cup C^2\{r_2',r_0',r_1\},$$

and the diameter of each $C^2\{\ \}$ on the right side are not greater than $((1+\alpha)/2)(d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0))$ (by Lemma 5). If we repeat this (n-1) times for each $C^2\{\ \}$ of the right hand side, then we obtain 6^n $C^2\{\ \}$'s and their diameters are not greater than $((1+\alpha)/2)^n(d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0))$. Hence for given $\varepsilon > 0$ there is an n_0 such that $n \geqq n_0$ implies $((1+\alpha)/2)^n \times (d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0)) < \varepsilon$. Since $m_3^{\varepsilon}(C^2\{r_0,r_1,r_2\}) \leqq \sum[\delta(C^2\{\ \})]^3 \leqq 6^n[((1+\alpha)/2)^n(d(r_0,r_1)+d(r_1,r_2)+d(r_2,r_0)]^3$ for $n \geqq n_0$, we get $m_3(C^2\{r_0,r_1,r_2\}) = 0$. By the fact introduced in Section 1, dim $C^2\{r_0,r_1,r_2\} \leqq 2$. This completes the proof of Theorem 2.

4. Applications and remarks

If we try to describe Theorem 1 with only convex combinations we have Corollary 6. This is because $C^{k+1}U = C^kU$ for every subset U of $B(p,\rho)$ in M means $HU = C^kU$.

Corollary 6. Let dim M be greater than 3. If for each point p in M there is a convex neighborhood V of p in M such that $C^2\{q_0,q_1,q_2,q_3\} = C^2\{q_0,q_1,q_2,q_3\}$ for any points q_0, q_1, q_2, q_3 in V, then M is a space of constant curvature.

The following corollary is evident by the fact that $H\{r_0,r_1,r_2\} \supset C^2\{r_0,r_1,r_2\}$ for any r_0, r_1, r_2 in $B(p,\rho) \subset M$. Moreover it is directly proved by the same way as in the proof of Theorem 2.

Corollary 7. Let dim M be greater than 2. If for each point p in M there is a convex neighborhood V of p in M such that $H\{q_0,q_1,q_2\} = H\{q_0,q_1,m(q_1,q_2)\} \cup H\{q_0,q_2,m(q_1,q_2)\}$ for any points q_0, q_1 and q_2 in V, then M is a space of constant curvature.

It is natural to ask whether $H\{q_0,q_1,q_2\}$ in the assumption of Theorem 2 could be replaced by $C^2\{q_0,q_1,q_2\}$. On this question we show the following.

Theorem 8. Let dim M be greater than 2. If for each p in M there exists a convex neighborhood V of p in M such that $C^2\{q_0,q_1,q_2\} = C^2\{q_0,q_1,q\} \cup C^2\{q_0,q_2,q\}$ for any points q_0, q_1 and q_2 in V and for any point q in the segment $T(q_1,q_2)$, then M is a space of constant curvature.

If it is possible to replace q in the assumption with $m(q_1,q_2)$, then this theorem is stronger than Theorem 2. However the author does not know the possibility.

We prepare a lemma for the proof of Theorem 8.

Lemma 9. Let M satisfy the assumption in Theorem 8. Let $x(\tau)$, $0 \leqq \tau \leqq \alpha$, and $y(\nu)$, $0 \leqq \nu \leqq \beta$, represent segments $T_1 := T(r_0,r_1)$ and $T_2 := T(r_0,r_2)$ respectively in $B(p,\rho) \subset V$ and let $t := \mathrm{Max}\{d(x(\alpha\mu),y(\beta\mu));\ \mu \in [0,1]\}$. If the 3ρ-ball with center p is strongly convex, then $C^2\{r_0,r_1,r_2\}$ is contained in the union of the closed t-neighborhood of T_1 and the closed t-neighborhood of T_2 in M.

Proof of Lemma 9. Choose a partition $0 = \mu_0 < \mu_1 < \dots < \mu_n = 1$ of $[0,1]$ such that $\alpha(\mu_i-\mu_{i-1}) < t$ and $\beta(\mu_i-\mu_{i-1}) < t$ for $1 \leqq i \leqq n$. Then $C^2\{x(\alpha\mu_{i-1}),x(\alpha\mu_i),y(\beta\mu_{i-1})\} \subset H\{x(\alpha\mu_{i-1}),x(\alpha\mu_i),y(\beta\mu_{i-1})\} \subset B(x(\alpha\mu_{i-1}),t)^c$ and $C^2\{x(\alpha\mu_i),y(\beta\mu_{i-1}),y(\beta\mu_i)\} \subset B(y(\beta\mu_i),t)^c$ for every $1 \leqq i \leqq n$, because $B(x(\alpha\mu_{i-1}),t)^c$ and $B(y(\beta\mu_i),t)^c$ are contained in $B(p,3\rho)$ for every $1 \leqq i \leqq n$, and hence are convex. Since $C^2\{r_0,r_1,r_2\} \subset \bigcup_{i=1}^{n} C^2\{x(\alpha\mu_{i-1}),x(\alpha\mu_i),y(\beta\mu_{i-1})\} \cup C^2\{x(\alpha\mu_i),y(\beta\mu_{i-1}),y(\beta\mu_i)\}$, $C^2\{r_0,r_1,r_2\}$ is contained in the union of the closed t-neighborhood of T_1 and the closed t-neighborhood of T_2 in M.

Proof of Theorem 8. Let q_0, q_1 and q_2 be any points in $B(p,\rho) \subset V$ where ρ is a positive such that the 3ρ-ball with center p is strongly convex. Let S be the set of all points each of which belongs to the segment $T(q_0,q)$ for some q in $T(q_1,q_2)$. $S \subset C^2\{q_0,q_1,q_2\}$ is clear. We claim $S = C^2\{q_0,q_1,q_2\}$. In fact, suppose there exists a point $s \in C^2\{q_0,q_1,q_2\} - S$. Let η be the distance between s and S. Since S is closed, we have $\eta > 0$. Choose a partition $q_1 = s_1, s_2, \ldots, s_n = q_2$ of $T(q_1,q_2)$ in this order such that if $z_i(\tau)$, $0 \leqq \tau \leqq \alpha_i$, represents the segment $T(q_0,s_i)$ for each $1 \leqq i \leqq n$, and if we put $t_i := \mathrm{Max}\,\{d(z_i(\alpha_i\mu), z_{i+1}(\alpha_{i+1}\mu));\ \mu \in [0,1]\}$ for each $1 \leqq i \leqq n-1$, then $t_i < \eta$ for all $1 \leqq i \leqq n-1$. By Lemma 9 and the assumption, $C^2\{q_0,q_1,q_2\}$ is contained in the open η-neighborhood of S in M, a contradiction.

Next we assert $H\{q_0,q_1,q_2\} = C^2\{q_0,q_1,q_2\}$. Let r and s be any points of $C^2\{q_0,q_1,q_2\}$. Then by the above argument there are points r' and s' in $T(q_1,q_2)$ such that $r \in T(q_0,r')$ and $s \in T(q_0,s')$. Since $C^2\{q_0,q_1,q_2\} = C^2\{q_0,q_1,r'\} \cup C^2\{q_0,r',s'\} \cup C^2\{q_0,q_2,s'\}$, where we assume without loss of generality that q_1, r', s' and q_2 are in this order on $T(q_1,q_2)$, and since $T(r,s)$ is contained in $C^2\{q_0,r',s'\}$, $T(r,s)$ is contained in $C^2\{q_0,q_1,q_2\}$, which implies convexity of $C^2\{q_0,q_1,q_2\}$.

Thus $H\{q_0,q_1,q_2\} = C^2\{q_0,q_1,q_2\} = S$. Then we conclude dim $H\{q_0,q_1,q_2\} \leqq 2$, and hence we obtain our theorem by Lemma 3.

CHAPTER II

Convex functions on G-surfaces

1. Introduction

A function F defined on a complete Riemannian manifold M without boundary is said to be convex if F is a one-variable convex function on each arc-length parametrized geodesic. F is locally Lipschitz continuous, and hence continuous on M. It is a natural question to ask to what extent the existence of a convex function on M implies restrictions to the topology of M. In the recent work [16], the topology of M with locally nonconstant convex functions has been studied in detail. One of their results gives a classification theorem of 2-dimensional complete Riemannian manifolds which admit locally nonconstant convex functions; they are diffeomorphic to either a plane, a cylinder, or an open Moebius strip.

A classical result of Cohn-Vossen [12] states that a complete noncompact 2-dimensional Riemannian manifolds with nonnegative Gaussian curvature is homeomorphic to a plane, a cylinder, or an open Moebius strip. Moreover, Cheeger-Gromoll have proved in [11] that if a complete noncompact Riemannian manifold has nonnegative sectional curvature, then every Busemann function on it is convex (and locally nonconstant).

H. Busemann generalized Cohn-Vossen's result in [4], proving that a noncompact G-surfaces with finite connectivity and zero excesses whose angular measure is uniform at π is topologically a plane, a cylinder, or an open Moebius strip.

The purpose of this chapter is to prove the following:

Theorem. Let R be a noncompact 2-dimensional G-space. If R admits a locally nonconstant convex function, then R is homeomorphic to either a plane, a cylinder $S^1 \times \mathbb{R}$, or an open Moebius strip.

It should be noted that in the proof of the above result, there is no analogy with the Riemannian case. This is because every point of a G-surface R does not in general have convex balls around it. Hence, for every closed convex set C of a G-space R and for every point $q \in R - C$ which is sufficiently close to C, we cannot conclude the uniqueness of a segment which connects q to a point on C, and whose length realizes the distance between q and C. It should be also noted that a convex function on a G-space R is in general not necessarily continuous. But in the case where dim R = 2, R is a topological manifold and every convex function on it is locally Lipschitz continuous.

In Section 2 we shall give the definition and some basic notions for G-spaces which are used later. They are found in the book of H. Busemann [4]. In Section 3 we shall discuss G-surfaces which possess locally nonconstant convex functions, directing our attention to the levels of the functions. In Section 4 we have a classification of G-surfaces which admit locally nonconstant nearly peakless functions without continuity.

This result will be introduced in [6]. The method is different from that the convex case. We do not give our attention to the levels but study only the existence of intersecting closed geodesics according to the idea of G. Thorbergsson [41].

2. G-spaces

Let R be a metric space, and let $d(p,q)$ denote the distance between points p and q on R. Let (pqr) denote that p, q and r are mutually distinct and $d(p,q)+d(q,r) = d(p,r)$; let $B(p,\rho)$ denote the set $\{q;\ d(p,q) < \rho\}$, which is called the (open) ball with center p and radius ρ. The axioms for a G-space R are:

1. The space is a symmetric metric space with distance $d(p,q) = d(q,p)$.

2. The space is finitely compact, i.e., a bounded infinite set has at least one accumulation point.

3. The space is (Menger) convex, i.e., for given two distinct points p and r, a point q with (pqr) exists.

4. To every point s of the space there corresponds $\rho_s > 0$ such that for any two distinct points p and q in $B(s,\rho_s)$ a point r with (pqr) exists. (Axiom of local prolongation)

5. If (pqr_1), (pqr_2) and $d(q,r_1) = d(q,r_2)$, then $r_1 = r_2$. (Axiom of uniqueness of prolongation)

The axioms ensure the existence of a (continuous) curve which connects given two points p and q and whose length is equal to the distance between them, and this curve is called a segment and denoted by $T(p,q)$. If an r with (pqr) exists, then the segment $T(p,q)$ is unique. If $p,\ q \in B(r,\rho)$ for some r in R, then $T(p,q) \subset B(r,2\rho)$. Let $\rho(p)$ be the least upper bound of those ρ_p which satisfy Axiom 4. Then either $\rho(p) = \infty$ for all p or

$0 < \rho(p) < \infty$ and $|\rho(p)-\rho(p)| \leqq d(p,q)$, which implies continuity of the function $\rho(.)$ on R.

A geodesic g is a certain class of mappings of the real line into R which is locally a segment, i.e., g has a representation $x(\tau)$, $-\infty < \tau < \infty$ such that for every τ_0 there exists an $\varepsilon(\tau_0) > 0$ such that $d(x(\tau_1),x(\tau_2)) = |\tau_1 - \tau_2|$ for $|\tau_0 - \tau_i| \leqq \varepsilon(\tau_0)$, i = 1, 2, and for another representation $y(\tau)$, $-\infty < \tau < \infty$, there exist $\alpha = \pm 1$, $\beta \in \mathbb{R}$ which satisfy that $x(\tau) = y(\alpha\tau + \beta)$ for all τ. If a representation of a geodesic is a globally isometric map of $\mathbb{R}$ into R, or a plane circle into R, then we call it a straight line or a great circle, respectively.

Fact 1. If $y(\tau)$, $\alpha \leqq \tau \leqq \beta$, $\alpha < \beta$, represents a segment in a G-space, then there is a unique representation $x(\tau)$, $-\infty < \tau < \infty$, of a geodesic such that $x(\tau) = y(\tau)$ for $\alpha \leqq \tau \leqq \beta$.

Fact 2. If $x_n(\tau)$, $-\infty < \tau < \infty$, represents a geodesic n = 1, 2,... and the sequence $\{x_n(\tau_0)\}$ is bounded, then $\{x_n(\tau)\}$ contains a subsequence $\{x_k(\tau)\}$ which converges (uniformly in every bounded set of $\mathbb{R}$) to a representation $x(\tau)$, $-\infty < \tau < \infty$, of a geodesic.

Fact 3. A class of homotopic closed curves through p which is not contractible contains a geodesic loop (a piece of a geodesic) with endpoint p.

We use the notion of dimension in the sense defined by Menger and Urysohn

Fact 4. A G-space of dimension 2 is a topological manifold. $B(p,\rho(p))$ is a homeomorphic to an open ball in a plane $\mathbb{R}^2$.

Fact 5. Every point of a 2-dimensional G-space is an interior point of a closed and of an open convex set whose boundary consists of three segments, where a convex set C means that p, q in C implies that $T(p,q)$ exists uniquely and is contained in C. We call such a convex set a triangle.

3. Locally nonconstant convex functions

Let R be a 2-dimensional G-space and F be a <u>convex function</u> on R. This means that for each geodesic with a representation $x(\tau)$, $-\infty < \tau < \infty$, F satisfies the inequality:

$$F(x(\lambda\tau_1 + (1 - \lambda)\tau_2)) \leqq \lambda F(x(\tau_1)) + (1 - \lambda)F(x(\tau_2)),$$

for any $\lambda \in [0,1]$, and for any $\tau_1, \tau_2 \in \mathbb{R}$.

The plane, cylinder and open Moebius strip with canonical metric evidently possess (locally nonconstant) convex functions. As we are interested in the topological structure of R, we may assume that a convex function F is <u>locally nonconstant</u>, i.e., nonconstant on each open set of R. If a non-trivial convex function F is constant on an open set $U \subset R$, then we can construct from R a topologically distinct R' on which a non-trivial convex function is defined. This is done as follows: There is a disk $D \subset U$. R' is obtained via the connected sum $(R - D)\#V$, where V is an arbitrary G-space with boundary S^1 which is identified with the boundary ∂D. The convex function on R' is equal to F on R' - V and is constant on V, and agree with F on ∂D. Thus the existence of a non-trivial convex function does not imply a topological restriction on the G-space except a trivial one, namely, noncompactness (see [15]). Throughout this section let F be locally nonconstant on R. And let $[F = a]$ and $[a \leqq F \leqq b]$ denote the sets $\{q \in R;\ F(q) = a\}$ and $\{q \in R;\ a \leqq F(q) \leqq b\}$, respectively.

<u>Lemma 6</u>. F is locally Lipschitz continuous on R.

Proof. We first show that F is locally bounded above. Let q_1, q_2 and q_3 be the vertices of the convex triangle C mentioned in Fact 5, and let $p' \in \text{Int } C$. Choose q on $T(q_2,q_3)$ such that $p' \in T(q_1,q)$. Then by convexity of F, we have

$$F(p') \leqq (d(p',q)/d(q_1,q))F(q_1) + (d(q_1,p')/d(q_1,q))F(q)$$
$$\leqq (d(p',q)/d(q_1,q))F(q_1) + (d(q_1,p')/d(q_1,q)) \times ((d(q,q_3)/d(q_2,q_3))F(q_2) + (d(q_2,q)/d(q_2,q_3))F(q_3)).$$

Therefore $F(p') \leqq \text{Max}\{F(q_1), F(q_2), F(q_3)\}$.

Secondly, we show that F is locally bounded. Let $B(p,\rho) \subset \text{Int } C$, $\rho \leqq \rho(p)$, and let $q \in B(p,\rho)$. Then by convexity of F we have

$$F(p) \leqq (F(q) + F(q'))/2,$$

where q' satisfies that (qpq') and $d(p,q) = d(p,q')$. Hence

$$F(q) \geqq 2F(p) - \text{Max}\{F(q_1), F(q_2), F(q_3)\},$$

for $q \in B(p,\rho)$. Thus F is locally bounbed.

In order to prove local Lipschitz continuity of F, we work in the above $B(p,\rho)$. Let $u, v \in B(p,\rho/3)$. Extend $T(u,v)$ in both directions until its endpoints arrive at $\partial B(p,\rho)$. Take points u_1, u_2, v_2, v_1 in this extension of $T(u,v)$ such that u_1, u_2, u, v, v_2 and v_1 are in this order and $d(u_1,u_2) = d(v_2,v_1) = \rho/3$ and u_1, $v_1 \in \partial B(p,\rho)$. Then by convexity of F,

$$(F(u_2) - F(u_1))/d(u_1,u_2) \leqq (F(v) - F(u))/d(u,v) \leqq (F(v_1) - F(v_2))/d(v_2,v_1).$$

Hence there is an $L > 0$ such that $|F(v) - F(u)| \leqq Ld(u,v)$ for $u, v \in B(p,\rho/3)$. Thus F is locally Lipschitz continuous, which is our goal.

Lemma 7. $[F = a]$, $a > \inf F(R)$, has the structure of an embedded 1-dimensional topological submanifold without boundary.

Proof. Let $p \in [F = a]$. There is a point q such that $q \in B(p,\rho(p))$ and $F(q) < F(p)$. Take an r on an extension of $T(p,q)$ such that $F(r) > F(p)$ and $r \in B(p,\rho(p))$. Let T' be a segment through r and contained in $B(p,\rho(p))$ and which intersects the extension of $T(p,q)$ at exactly r and on which $F > F(p)$. Then $T(q,q') \cap [F = a]$ is exactly one point for every $q' \in T'$, because F is strictly monotone increasing along $T(q,q') \cap (R - [F \leq F(q)])$, and the totality of those points is homeomorphic to T'. Hence this set is a neighborhood of p in $[F = a]$, and it has no selfintersections. This completes the proof.

We conclude from Lemma 7 that $[F = a]$, $a > \inf F(R)$, is homeomorphic to either a real line $\mathbb{R}$ or a circle S^1.

Lemma 8. R is noncompact.

Proof. A bounded convex function is constant.

Concerning the number of components of a level $[F = a]$, $a > \inf F(R)$, of F the following holds.

Proposition 9. Let p and q be distinct points of $[F = a]$,

$a > \inf F(R)$. If there is a geodesic curve from p to q such that F does not assume $\inf F(R)$ on it, then p and q are contained in the same component of $[F = a]$.

Proof. Let $x(\tau)$, $\alpha \leqq \tau \leqq \beta$, represent the geodesic curve in the assumption. If $\min F(x([\alpha,\beta])) = a$, then $F(x(\tau)) = a$ for every τ, $\alpha \leqq \tau \leqq \beta$, by convexity of F. So p and q are contained in the same component of $[F = a]$. Thus we may assume without loss of generality that there exists $\tau_0 \in [\alpha,\beta]$ such that $F(x(\tau_0)) = \min F(x([\alpha,\beta])) < a$. Since $F(x(\tau_0)) > \inf F(R)$, we can choose an r such that $r \in B(x(\tau_0),\rho(x(\tau_0))/3)$, and $F(r) < F(x(\tau_0))$. Put $\alpha' := \mathrm{Max}\{\alpha, \tau_0 - \rho(x(\tau_0))/3\}$, $\beta' := \min\{\beta, \tau_0 + \rho(x(\tau_0))/3\}$, and $m := \min\{(F(x(\tau)) - F(r))/d(r,x(\tau)); \alpha' \leqq \tau \leqq \beta'\}$, The choice of r implies $m > 0$. For each τ, $\alpha' \leqq \tau \leqq \beta'$, there is exactly one representation $y_\tau'(\nu)$, $-\infty < \nu < \infty$, of a geodesic by Fact 1 which satisfies that $y_\tau'(0) = r$, and $y_\tau'(d(r,x(\tau))) = x(\tau)$. Then we have

$$\begin{aligned} & m(\nu - d(r,x(\tau))) + F(x(\tau)) \\ & \leqq ((F(x(\tau)) - F(r))/d(r,x(\tau)))(\nu - d(r,x(\tau))) + F(x(\tau)) \\ & \leqq F(y_\tau'(\nu)), \end{aligned}$$

for every $\nu \geqq d(r,x(\tau))$. Since $y_\tau'(\nu)$, $d(r,x(\tau)) \leqq \nu < \rho(x(\tau_0))/3 + d(r,x(\tau))$, is contained in $B(x(\tau_0),2\rho(x(\tau_0))/3)$ for each τ, $\alpha' \leqq \tau \leqq \beta'$, there is a ν such that $y_\tau'(\nu) \in B(x(\tau_0),\rho(x(\tau_0)))$, and $m\rho(x(\tau_0))/3 + F(x(\tau)) \leqq F(y_\tau'(\nu))$.

Let $\varepsilon(\tau)$, $\alpha' \leqq \tau \leqq \beta'$, be a continuous function which satisfies that $\varepsilon(\alpha') = \varepsilon(\beta') = 0$, and $0 < \varepsilon(\tau) < m\rho(x(\tau_0))/3$ for any τ, $\alpha' < \tau < \beta'$. Convexity of F implies that the geodesic curve with a representation $y_\tau'(\nu)$, $\nu > 0$, intersects $[F = F(x(\tau)) + \varepsilon(\tau)]$ at exactly one point, which is denoted by $y(\tau)$. We are going to see that $y(\tau)$, $\alpha' \leqq \tau \leqq \beta'$, is a continuous curve such that $y(\alpha') = x(\alpha')$ and $y(\beta') = x(\beta')$ and $y(\tau) \in B(x(\tau_0),\rho(x(\tau_0)))$. Let a sequence $\{\tau_i\}$ tends to τ, $\alpha' \leqq \tau \leqq \beta'$. Since $\{y(\tau_i)\}$ is bounded, the sequence $\{y(\tau_i)\}$ contains a subsequence $\{y(\tau_k)\}$ which converges to a point y_0. Then, since the equality $F(y_0) = F(x(\tau)) + \varepsilon(\tau)$ follows from continuity of ε and F, y_0 belongs to $[F = F(x(\tau)) + \varepsilon(\tau)]$. On the other hand, y_0 is on the geodesic curve with a representation $y_\tau'(\nu)$, $-\infty < \nu < \infty$, since the sequence $\{y_{\tau_i}'(\nu)\}$ of representations of geodesics converges to $y_\tau'(\nu)$, $-\infty < \nu < \infty$, by Fact 2 when $\{\tau_i\}$ tends to τ. Therefore we have from the definition of $y(\tau)$ that $y_0 = y(\tau)$.

Next, for each τ, $\alpha' \leqq \tau \leqq \beta'$, let $z_\tau'(\nu)$, $-\infty < \nu < \infty$, be a representation of a geodesic which satisfies that $z'(0) = x(\tau_0)$ and $z_\tau'(d(x(\tau_0),y(\tau))) = y(\tau)$. This is well-defined because $y(\tau) \in B(x(\tau_0),\rho(x(\tau_0)))$. From the construction of $z_\tau'(\nu)$ we see that $z_{\alpha'}'(\nu) = x(\tau_0 - \nu)$, $z_{\beta'}'(\nu) = x(\tau_0 + \nu)$, and hence each of them has a unique intersection with $[F = a]$. The desired curve from p to q in $[F = a]$ is obtained as follows: From the construction of $z_\tau'(\nu)$, $\alpha' \leqq \tau \leqq \beta'$, $\nu > 0$, we see that $F(z_{\alpha'}'(\nu))$ and $F(z_{\beta'}'(\nu))$ are monotone nondecreasing for $\nu > 0$, and moreover $F(z_\tau'(\nu))$, $\alpha' < \tau < \beta'$, is strictly monotone increasing for $\nu > d(x(\tau_0),y(\tau))$. Thus, for each τ, $\alpha' \leqq \tau \leqq \beta'$, $z_\tau'(\nu)$, $\nu > 0$, has a unique intersection with $[F = a]$, which is denoted by $z(\tau)$, and the intersection is continuous with τ. In fact, to prove it around $\tau = \alpha'$, fix $p' = z_{\alpha}{}'(\tau_0 - \alpha + 1)$. Then convexity of F along $z_{\alpha}{}'(\nu)$, $\nu > 0$, implies that $F(p') > a$. There is a neighborhood of p' on which $F > a$. Therefore we find a $\delta_1 > 0$ such that $z_{\tau}{}'(\tau_0 - \alpha + 1)$ is in the neighborhood if $|\tau - \alpha'| < \delta_1$. Then continuity of $z(\tau)$, $\alpha' \leqq \tau < \alpha' + \delta_1$, is obvious. In the same way we find a $\delta_2 > 0$ such that $z(\tau)$, $\beta' - \delta_2 < \tau \leqq \beta'$, is continuous. To prove continuity of $z(\tau)$, $\alpha' + \delta_1 \leqq \tau \leqq \beta' - \delta_2$, put $m_1 := \inf\{[F(y(\tau)) - F(x(\tau_0))]/d(x(\tau_0),y(\tau));\ \alpha' + \delta_1 \leqq \tau \leqq \beta' - \delta_2\}$. Then we can see that $m_1 > 0$ and that for each τ, $\alpha' + \delta_1 \leqq \tau \leqq \beta' - \delta_2$,

$$
\begin{aligned}
&m_1(\nu - d(x(\tau_0),y(\tau))) + F(y(\tau)) \\
&\leqq [\{F(y(\tau)) - F(x(\tau_0))\}/d(x(\tau_0),y(\tau))](\nu - d(x(\tau_0),y(\tau))) \\
&\qquad\qquad + F(y(\tau)) \\
&\leqq F(z_{\tau}{}'(\nu)),
\end{aligned}
$$

for every $\tau > d(x(\tau_0),y(\tau))$. Thus the set $\{z(\tau);\ \alpha' + \delta_1 \leqq \tau \leqq \beta' - \delta_2\}$ is bounded. Continuity of $z(\tau)$, $\alpha' \leqq \tau \leqq \beta'$, holds by means of the same argument as continuity of $y(\tau)$. This completes the proof.

As a direct consequence of the proof of Proposition 9, we have

<u>Proposition 10</u>. If p and q are taken from different components of $[F = a]$, then F assumes $\inf F(R)$ on every geodesic curve which joins p and q and $\inf F(R)$ is assumed at exactly

one point on it.

Theorem 11. If there is a value a such that $[F = a]$ is not connected, then the following hold.

(1) F assumes inf F(R) at some point.

(2) $[F = \min F(R)]$ is totally convex and it is either a straight line or a great circle.

(3) $R - [F = \min F(R)]$ consists of two components. If $b > \inf F(R)$, then $[F = b]$ has exactly two components.

Proof. (1) is a consequence of Proposition 10. In (2), total convexity of $[F = \min F(R)]$ is trivial. If $\partial[F = \min F(R)] = \emptyset$ as a 1-dimensional manifold, then it follows from total convexity of $[F = \min F(R)]$ and (9.6) in [2],p.46 that $[F = \min F(R)]$ is either a straight line or a great circle. If $\partial[F = \min F(R)] \neq \emptyset$ or $[F = \min F(R)]$ consists of only one point, then we can see that $[F = a]$ is connected, a contradiction. In fact, we can prove this as follows. Take points p and q in $[F = a]$ and r in $\partial[F = \min F(R)]$ and join from r to p, and from r to q by segments $T(r,p)$ and $T(r,q)$. Since $B(r,\rho(r))$ is not separated by $[F = \min F(R)]$, we get a continuous curve $y(\tau)$, $\alpha \leqq \tau \leqq \beta$, joining $\partial B(r,\rho(r)/2) \cap T(r,p)$ and $\partial B(r,\rho(r)/2) \cap T(r,q)$ which is contained in $B(r,\rho(r))$ and does not intersect $[F = \min F(R)]$. If for each τ, $\alpha \leqq \tau \leqq \beta$, $z(\tau)$ is defined by the intersection of a geodesic curve in the direction from r to $y(\tau)$ and $[F = a]$ as in the proof of Proposition 9, then $z(\tau)$,

$\alpha \leqq \tau \leqq \beta$, is a continuous curve joining p and q.

To prove (3), fix a point p of $[F = \min F(R)]$. Then $B(p, \rho(p)/2) - [F = \min F(R)]$ has exactly two components by (2). We denote its components by V_1 and V_2. For each $q \in R - [F = \min F(R)]$, let $x(\tau)$, $0 \leqq \tau \leqq \alpha$, $x(0) = p$, $x(\alpha) = q$ be a representation of a geodesic curve from p to q. Then $x(\tau)$, $0 < \tau < \rho(p)/2$, is contained in only one of V_1 and V_2. Put $A :=$ $\{q \in R - [F = \min F(R)]$; all geodesic curves from p to q on which sufficiently small parts near p intersect $V_1\}$, $B := \{q \in R - [F = \min F(R)]$; all geodesic curves from p to q on which sufficiently small parts near p intersect $V_2\}$ and $C := \{q \in R - [F = \min F(R)]$; there are geodesic curves from p to q such that one of their representations $y'(\tau)$, $0 \leqq \tau \leqq \beta$, $y'(0) = p$, $y'(\beta) = q$, intersects V_1 and another $z'(\tau)$, $0 \leqq \tau \leqq \gamma$, $z'(0) = p$, $z'(\gamma) = q$, intersects $V_2\}$.

Both A and B are open and connected in $R - [F = \min F(R)]$ if they are nonempty. If we show that $A \cup B = R - [F = \min F(R)]$, i.e., $C = \emptyset$, then the first part of (3) will be proved. This is because if $C = \emptyset$, then $V_1 \subset A$ and $V_2 \subset B$ will follow from the argument stated below.

Suppose $q \in C$ exists. Then we have a contradiction from the following considerations. Fix a point $q_0 = y(\tau_0)$ such that $F(q_0) = a$, where $y(\tau)$, $-\infty < \tau < \infty$, is a representation of a geodesic determined by $y'(\tau)$, $0 \leqq \tau \leqq \beta$, in the definition of C. And let $z''(\tau)$, $0 \leqq \tau \leqq |\tau_0 - \beta| + \gamma$, be a continuous curve from p to q_0 such that if $\tau_0 \geqq \beta$, then $z''(\tau) = z'(\tau)$ for $0 \leqq$

$\tau \leqq \gamma$ and $z''(\tau) = y(\tau + \beta - \gamma)$ for $\gamma \leqq \tau \leqq \tau_0 - \beta + \gamma$ and if $\tau_0 < \beta$ then $z''(\tau) = z'(\tau)$ for $0 \leqq \tau \leqq \gamma$ and $z''(\tau) = y(\beta - \tau + \gamma)$ for $\gamma \leqq \tau \leqq \beta - \tau_0 + \gamma$, where $z'(\tau)$, $0 \leqq \tau \leqq \gamma$, is in the definition of C. We consider the class of all curves from p to q_0 whose interiors are in R - [F = min F(R)] and such that sufficiently small parts near p meet V_2 but not meet V_1. This class is nonempty. Since [F = min F(R)] is totally convex and is either a great circle or a straight line, we find (5.18) in [2],p.25, a geodesic curve from p to q_0, in the class, whose interior is contained in R - [F = min F(R)] and which is different from a geodesic curve with a representation $y(\tau)$, $0 \leqq \tau \leqq \beta$. Let $z(\tau)$, $0 \leqq \tau \leqq \delta$, $z(0) = p$, $z(\delta) = q_0$, represent this geodesic curve. Using these representation $y(\tau)$, $0 \leqq \tau \leqq \beta$, and $z(\tau)$, $0 \leqq \tau \leqq \delta$, we connect any two points q_1 and q_2 in [F = a] by a continuous curve in [F = a], a contradiction. This is done as follows: If $y_1(\tau)$, $0 \leqq \tau \leqq \beta_1$, and $y_2(\tau)$, $0 \leqq \tau \leqq \beta_2$, represents geodesic curves such that $y_1(0) = y_2(0) = p$ and $y_1(\beta_1) = q_1$, $y_2(\beta_2) = q_2$ and if $y_i(\tau)$, $0 < \tau < \rho(p)/2$, $i = 1, 2$, are contained in V_1, then we can connect $y_1(\rho(p)/3)$ and $y_2(\rho(p)/3)$ by a continuous curve in V_1, and hence as in Proposition 9 we find a curve in [F = a] which joins q_1 and q_2. Thus, without loss of generality, we may consider that $y_i(\tau)$, $0 < \tau < \rho(p)/2$, are contained in V_i for $i = 1, 2$. By the same idea as in Proposition 9, we can find two curves in [F = a] such that one of them joins q_1 to q_0 and the other joins q_2 to q_0. Thus [F = a] is connected, a contradiction. Hence $C = \emptyset$ is proved.

The above arguments show that $V_1 \subset A$ and $V_2 \subset B$, and hence they are not empty. Therefore $R - [F = \min F(R)]$ is the disjoint union of A and B.

To prove the second part of (3) it is enough to see that both $[F = b] \cap A$ and $[F = b] \cap B$ are connected for any $b > \inf F(R)$. This is evident by the above argument.

To continue our investigations, we need the notion of an <u>end</u> ε which is by definition an assigment to each compact set K in R a component $\varepsilon(K)$ of $R - K$ in such a way that $\varepsilon(K_1) \supset \varepsilon(K_2)$ if $K_1 \subset K_2$.

<u>Theorem 12</u>. If there is a compact component of a level of F, then all levels are compact.

<u>Proof</u>. Theorem 11 implies that every level consists of one or two components. So we first consider the case where all levels are connected.

Let $[F = a]$ be compact. And suppose that $[F = b]$ is noncompact for some b with $a < b$. We fix a point p in $[F = a]$ and choose an unbounded sequence $\{q_i\}$, $q_i \in [F = b]$. Let $x_i(\tau)$, $\tau \geqq 0$, be a representation of a geodesic curve such that $x_i(0) = p$ and $x_i(d(p,q_i)) = q_i$, $i = 1, 2, \ldots$. Then we have a subsequence $\{x_k(\tau)\}$ of $\{x_i(\tau)\}$ which converges to a representation $x(\tau)$, $\tau \geqq 0$, of a geodesic curve. If we see that $F(x(\tau)) = a$ for any $\tau \geqq 0$, then $[F = a]$ is noncompact since $x(\tau)$, $\tau \geqq 0$, represents a ray, i.e., a half-straight line. This is a contradiction. Thus, if [F = a] is compact, then [F = b] compact for all $b > a$. It remains to prove that $F(x(\tau)) = a$ for every $\tau \geqq 0$. For each $\tau \geqq$ the diameter of [F = a], there is a number k_0, such that $d(p,q_k) > \tau$ for $k \geqq k_0$. For this k_0, it follows from convexity of F that $k \geqq k_0$ implies $F(x_k(\tau)) \leqq b$. Therefore we have

$$F(x(\tau)) = F(\lim x_k(\tau)) = \lim F(x_k(\tau)) \leqq b.$$

On the other hand, since $\tau \geqq$ the diameter of [F = a],

$$F(x(\tau)) = F(\lim x_k(\tau)) = \lim F(x_k(\tau)) \geqq a.$$

$F(x(\tau))$, the diameter of [F = a] $\leqq \tau < \infty$, is bounded and monotone nondecreasing, so F is constant on it. Therefore it follows from convexity of F that $F(x(\tau)) = a$ for $\tau \geqq 0$.

Suppose that [F = a] is noncompact and [F = b] compact for some $a < b$. In this case, there are at least two ends of R because of the existence of a straight line intersecting [F = b] along which F is nonconstant and monotone. In particular, R is not simply connected. Under the assumption above, we claim that F does not assume inf F(R). In fact, suppose F assumes inf F(R). Then the minimum set is noncompact, otherwise all levels are compact because of the above argument. Thus the minimum set consists of either a ray or a straight line. Since R is not simply connected, there is a non-null homotopy class of closed curves with any fixed point $p \in$ [F = min F(R)] as a base point. Then we get a geodesic loop at p by Fact 3. Along this geodesic loop, F is constant because $F(x(\tau)) \leqq F(p)$, where $x(\tau)$, $0 \leqq \tau$

$\leqq \alpha$, is a representation of this geodesic loop. Since this geodesic loop is contained in neither that ray nor that straight line, there is an open set U in the neighborhood of p such that F is constant on U. This contradicts local non-constancy of F. Thus we can suppose that F does not assume inf F(R).

Now, we are going to obtain the final contradiction in this case. Put $t_0 := \inf\{t \in \mathbb{R}\,;\ [F = t] \text{ is compact}\}$. If we prove that $[F \leqq t_0]$ is homeomorphic to the closed half-plane, which is proved in Proposition 13, then as R is not simply connected we have a geodesic loop with endpoint p where $F(p) < t_0$. This geodesic loop must intersect $[F = t_0]$. Thus this contradicts convexity of F.

Next we consider the case where there is a level which is not connected. In this case it follows from Theorem 11 that F assumes inf F(R) and the minimum set is either a straight line or a great circle.

If the minimum set is a great circle, all levels are compact by the same reason as we have already shown that [F = b] is compact if so is [F = a] for $a \leqq b$.

In the case where the minimum set is a straight line, each component of any level is noncompact. In fact, suppose that there is a compact component of some level. Then R has at least two ends and hence it is not simply connected. Then for every $p \in [F = \min F(R)]$ there is a geodesic loop at p which is not homotopic to a point curve. Thus the geodesic loop at p must lie in the minimum set which is a straight line, a contradiction. This completes the proof for all cases.

The following proposition is directly used in the proof of Theorem 12. Once we establish Theorem 12, we may find by the same reasoning as Proposition 13 that $[a \leqq F \leqq b]$, $b > a > \inf F(R)$, is topologically a part of a cylinder $S^1 \times [a,b]$ if $[F = b]$ is compact.

Proposition 13. If there is a value $b > \inf F(R)$ such that $[F = b]$ is noncompact, then there exists a homeomorphism h of $\mathbb{R} \times [a,b]$, $a > \inf F(R)$, onto each component of $[a \leqq F \leqq b]$ such that $F \circ h(u,v) = v$ for every $(u,v) \in \mathbb{R} \times [a,b]$.

It should be noted that each level $[F = c]$, $a \leqq c \leqq b$, has a neighborhood which is homeomorphic to the union of triangles of $\mathbb{R}^2$, since from the same idea as in Lemma 7, $[F = c]$ is covered by triangles whose interiors are mutually disjoint. Hence our aim is to extend this homeomorphism to globally satisfy the condition.

Proof. We know as in the first part of the proof of Theorem 12 that every c, $c \leqq b$, $[F = c]$ is noncompact and hence homeomorphic to $\mathbb{R}$. Fix a value d, $\inf F(R) < d < a$, and choose an unbounded sequence $\{p_i\}_{-\infty < i < \infty}$ in this order of an orientation of $[F = b]$ in both directions of $[F = b]$. For each

i, $-\infty < i < \infty$, let f_i be a point in $[F = d]$ such that $d(p_i, f_i) = d(p_i, [F \leqq d])$. Then for each c, $a \leqq c \leqq b$, $T(p_i, f_i) \cap [F = c]$ is exactly one point which we denote by $p_i(c)$. Clearly $p_i(b) = p_i$ for every i, $-\infty < i < \infty$. And for every c, $a \leqq c \leqq b$, the sequence $\{p_i(c)\}$ is unbounded in both directions of $[F = c]$. Otherwise there exists a ray starting at an accumulation point of $\{f_i\}$ and passing through an accumulation point q of $\{p_i(c)\}$ on which F is bounded above by b and the right derivative at q is positive, a contradiction. Moreover, the sequence $\{p_i(c)\}$ is in this order. This is proved as follows: Let W be a neighborhood of $T(p_0, f_0)$. $W \cap [a \leqq F \leqq b]$ is separated by $T(p_0, f_0)$ into two components W_{-1} and W_1. Let G be a function of $[a \leqq F \leqq b]$ to $\{-1, 0, 1\}$ which satisfies that if $q \in T(p_0(b), p_0(a))$, then $G(q) = 0$, if $q \in [a \leqq F \leqq b] - T(p_0(b), p_0(a))$, $F(q) = c$ and the subarc of $[F = c]$ from $p_0(c)$ to q meet W_{-1}, then $G(q) = -1$, and otherwise $G(q) = 1$. By the remark above the proof, G is continuous except on $T(p_0(b), p_0(a))$, and hence for every c, $a \leqq c \leqq b$, and for every integer $k \geqq 1$, $p_k(c)$ is in only one of two sides of $T(p_0(b), p_0(a))$, and for $k \leqq -1$, is in the other side. Since this fact is true for each i, $-\infty < i < \infty$, the sequence $\{p_i(c)\}$ is in this order on $[F = c]$ for every c, $a \leqq c \leqq b$.

Let M_i denote the set which is surrounded by $[F = b]$, $[F = a]$, $T(p_i(b), p_i(a))$ and $T(p_{i+1}(b), p_{i+1}(a))$, more precisely, the totality of the subarcs of $[F = c]$'s from $p_i(c)$ to $p_{i+1}(c)$, for all c, $a \leqq c \leqq b$.

It suffices to construct a homeomorphism h_i of the domain

$\{(u,v) \in \mathbb{R}^2 ; i \leqq u \leqq i + 1, a \leqq v \leqq b\}$ onto M_i such that $F \circ h_i(u,v) = v$. Because if we connect h_i's we get h. This is done as follows: Put $\rho := \min\{\rho(q)/2 ; q \in M_i \cap [F = b]\}$. If we consider a neighborhood $U := \{q ; d(q,[F = b]) < \rho\}$ of $M_i \cap [F = b]$, then there is an $\varepsilon > 0$ and $b' < b$ such that $M_i \cap [b' - \varepsilon \leqq F \leqq b] \subset U$. Choose $s_j \in [F = b] \cap U$ and $r_j \in [F = b' - \varepsilon] \cap U$, $j = 0, 1, \ldots, n$, in this order, such that $s_0 = p_i(b)$, $s_n = p_{i+1}(b)$, $r_0 = p_i(b'-\varepsilon)$ and $r_n = p_{i+1}(b'-\varepsilon)$ and $s_j, s_{j+1}, r_j, r_{j+1} \in B(t_j,\rho/2)$ for some $t_j \in [F = b]$, $j = 0, 1, \ldots, n - 1$. For each j, $0 \leqq j \leqq n - 1$, let M_{ij} be the domain which is surrounded by $[F = b]$, $[F = b']$, $T(s_j,r_j)$ and $T(s_{j+1},r_j)$ and let M_{ij}' be the domain which is surrounded by $T(s_{j+1},r_j)$, $T(s_{j+1}, r_{j+1})$ and $[F = b']$. Then we can construct, by the same techniques as in Lemma 7 and Proposition 9, a homeomorphism of M_{ij} onto the trapezoid in $\mathbb{R}^2$ whose vertices are $(i + j/n,b)$, $(i + j/n,b')$, $(i + (j + 1)/n,b)$ and $(i + j/n + \varepsilon/n(b - b' + \varepsilon),b')$, and a homeomorphism of M_{ij}' onto the triangle in $\mathbb{R}^2$ whose vertices are $(i + (j + 1)/n,b)$, $(i + j/n + \varepsilon/n(b - b' + \varepsilon),b')$ and $(i + (j + 1)/n,b')$. The homeomorphisms agree on the segment $T(r_j,s_{j+1}) \cap [b' \leqq F \leqq b]$. If we connect these homeomorphisms we get a homeomorphism $h_i'^{-1}$ of $M_i \cap [b' \leqq F \leqq b]$ to $\{(u,v) \in \mathbb{R}^2 ; i \leqq u \leqq i + 1, b' \leqq v \leqq b\}$ which satisfies $F \circ h_i'((u,v)) = v$ for all $u \in [i,i + 1]$. We do not know whether M_i is compact, so the desired homeomorphism is obtained as follows: Let b_0 be the greatest lower bound of $\{b' ; [b' \leqq F \leqq b] \cap M_i$ has a homeomorphism $h_i'^{-1}$ which satisfies the condition$\}$. Then $b_0 = a$.

Otherwise we can construct by the same way as above a homeomorphism $h_i'^{-1}$ of $[b' \leqq F \leqq b] \cap M_i$, for some $b' < b_0$, which satisfies the condition, a contradiction to the choice of b_0.

Clearly h_i^{-1} and h_{i+1}^{-1} agree on the segment $T(p_{i+1}(b), p_{i+1}(a))$, so we obtain the desired homeomorphism h^{-1} of $[a \leqq F \leqq b]$ onto $\{(u,v) \in \mathbb{R}^2 ; -\infty < u < \infty, a \leqq v \leqq b\}$ after connecting h_i^{-1} and h_{i+1}^{-1} for all i, $-\infty < i < \infty$.

We shall observe how the existence of a locally nonconstant convex function on R will restrict the number of ends of R.

Lemma 14. If there is a compact level of F, then R has at most two ends.

Proof. Suppose that R has more than two ends. Then there is a compact set K such that R - K consists exactly three unbounded components U_1, U_2 and U_3. We will prove that F is bounded above on two of the U_1, U_2 and U_3. Therefore F is bounded above on exactly two of them, since F is not bounded above. In order to see this, we may suppose that $\sup F(U_1) = \sup F(U_2) = \infty$. Then we can find such a high level that it does not intersect K but intersects U_1 and U_2. This implies that this level is not connected. Therefore F is bounded above on U_3, since Theorem 11 says that all levels except the minimum set consist of exactly two components. Choose a point in the minimum set and an unbounded sequence $\{q_i\}$ in U_3. Let $x_i(\tau)$, $\tau \geqq 0$, represents a geodesic curve, $i = 1, 2, \ldots$, which satisfies that $x_i(0) = p$ and $x_i(d(p,q_i)) = q_i$. Because of Fact 2 there is a subsequence $\{x_k(\tau)\}$ of $\{x_i(\tau)\}$ which converges to a representation $x(\tau)$, $\tau \geqq 0$, of a geodesic curve. In the same way as in the proof of Theorem 12, we see that $\{x(\tau) \; ; \; \tau \geqq 0\}$ belongs to the minimum set. Since $x(\tau)$, $\tau \geqq 0$, represents a ray, this contradicts compactness of levels.

Thus we may suppose without loss of generality that F is bounded above on U_2 and U_3. If we put $m := \min F(K)$, then $m = \inf F(R)$. In fact, if we suppose that there exists a point q with $F(q) < m$, then we can find a ray on which F is constant equal to $F(q)$ or nonincreasing in the same way as above argument after taking an unbounded sequence $\{q_i\}$ contained in U_j, $j = 2$ or 3, which does not contain q. However, since this ray intersects K, this is impossible.

Let $p \in K$ satisfy that $F(p) = m$. Then there is a ray emanating from p, and an unbounded subarc of which lines in $K \cup U_2$. But F is constant $= m$ on the ray. Thus $[F = m]$ is noncompact, contradicting Theorem 12. This completes the proof.

Lemma 15. Suppose there exists a noncompact level. If R has more than one end, then F assumes $\inf F(R)$ and the minimum set intersects every $\varepsilon(K)$, where ε is an arbitrary end and K is any compact set of R.

It turns out from Proposition 17 that if F has a noncompact level, then R has exactly one end. But this lemma gives a step of the proof of Proposition 17.

Proof. From the assumption there exist two ends ε_1 and ε_2 and a compact set K which satisfies that $\varepsilon_1(K)$ and $\varepsilon_2(K)$ are distinct unbounded components of R - K. Put a:= min F(K), b:= Max F(K). We will find that a:= inf F(K). Otherwise there is a point p such that $F(p) < a$. We may suppose without loss of generality that $p \notin \varepsilon_1(K)$. If F is bounded on $\varepsilon_1(K)$, then making use of p and an unbounded sequence in $\varepsilon_1(K)$, in the same way as before, we obtain a representation $x(\tau)$, $\tau \geqq 0$, of a ray intersecting K and on which F is constant. This contradicts the choice of p and a.

The above argument also shows that every $\varepsilon(K)$ intersects the minimum set if R - K has at least two unbounded components.

Let K_1 be any compact set. Then there is a compact set $K \supset K_1$ such that every $\varepsilon(K)$ intersects the minimum set, and hence so does $\varepsilon(K_1)$. Thus the proof of the final statement is complete.

Theorem 16. If R is a noncompact G-surface which admits a locally nonconstant convex function, then the number of ends is at most two.

Proof. Suppose that the number of ends of R is not less than three. Then Theorem 12 says that all levels are noncompact and Lemma 15 concludes that F assumes inf F(R) and the minimum set intersects every $\varepsilon(K)$. Since the noncompact minimum set is a ray or a straight line, it cannot intersect more than two $\varepsilon(K)$'s. This is a contradiction.

Now we classify G-surfaces which admit locally nonconstant convex functions. First we consider the case where R has two ends.

Proposition 17. If R has two ends, then each component of every level is homeomorphic to a circle S^1 and R is homeomorphic to a cylinder $S^1 \times \mathbb{R}$.

Proof. In the first step we will see that all levels are compact. Suppose that there is a noncompact level. Lemma 15 and two ends of R imply that F assumes inf F(R) and the minimum set is a straight line. Since R has two ends, there is a geodesic loop whose endpoint is contained in the minimum set and which is not homotopic to a point curve, a contradiction. We know the existence of a homeomorphism of R to a cylinder from the remark above Proposition 13.

In the case that F does not assume inf F(R), then all levels are connected. Therefore each level intersects a straight line at one point which connects two ends. Thus R is topologically a cylinder by the remark above Proposition 13.

Next we claim that if F assumes inf F(R), then the minimum set is a great circle. Otherwise, since it is a point or a segment, all levels are connected. The existence of two ends implies that there is a compact set K such that R - K consists of exactly two unbounded components, so F is bounded above on one of components of R - K. Thus, in the same way as in the proof of Theorem 12, we can derive a contradiction, namely the minimum set contains a ray. It turns out at the same time from this consideration that there exist levels which are not connected. Therefore each component of a level intersects a straight line at one point which passes through the minimum set and connects two ends.

Now we can conclude the following;

Theorem 18. If R is a noncompact G-surface which admits a locally nonconstant convex function, then R is homeomorphic to either a plane, a cylinder $S^1 \times \mathbb{R}$, or an open Moebius strip.

Proof. The case where R has two ends has already been treated in Proposition 17. We may suppose by Theorem 16 that R has one end. First we will prove that if there is a compact level, then F assumes inf F(R). Suppose that F does not assumes inf F(R). Then we produce a straight line through a certain compact level by choosing two sequences $\{q_i\}$ and $\{q_i'\}$ which satisfy that lim $F(q_i)$ = inf F(R), and lim $F(q_i') = +\infty$, and connecting q_i and q_i' by a segment. Hence R has at least two ends, a contradiction. Therefore, the minimum set of F is either a point, a segment, or a great circle. Since the desired construction of homeomorphisms is the same as in Proposition 13, we need only to see how to map a level.

When the minimum set is a point, we map it to the origin of canonical plane $\mathbb{R}^2$ and $[F = a]$, $a > \min F(R)$, onto a circle in $\mathbb{R}^2$ with center (0,0) and radius $a - \min F(R)$.

When the minimum set is a segment, we map it to a segment T in canonical plane $\mathbb{R}^2$ and $[F = a]$, $a > \min F(R)$, onto the set $\{w \in \mathbb{R}^2;\ d(w,T) = a - \min F(R)\}$ in $\mathbb{R}^2$.

If the minimum set is a great circle, we map it to the shortest great circle T in canonical open Moebius strip M and $[F = a]$, $a > \min F(R)$, onto the set $\{w \in M\ ;\ d(w,T) = a - \min F(R)\}$ in M.

Next we consider the case where all levels are noncompact. If F does not assume $\inf F(R)$, Proposition 13 implies that $[F = a]$, $a > \inf F(R)$, is mapped onto the set $\{(u,v) \in \mathbb{R}^2;\ -\infty < u < \infty,\ v = a\}$ in canonical plane $\mathbb{R}^2$ and R is topologically a plane. Hence we examine the case where F assumes $\inf F(R)$.

When the minimum set is a ray (or a straight line), we map it to a half-line T (or a straight line T') in canonical plane $\mathbb{R}^2$ and $[F = a]$, $a > \min F(R)$ onto the set $\{w \in \mathbb{R}^2;\ d(w,T) = a - \min F(R)\}$ ($\{w \in \mathbb{R}^2;\ d(w,T') = a - \min F(R)\}$) in $\mathbb{R}^2$.

From these considerations we conclude that if the minimum set is a great circle, then R is topologically an open Moebius strip and that otherwise R is a plane $\mathbb{R}^2$ topologically.

4. Locally nonconstant nearly peakless functions

In this section we extend the classification theorem in Section 3 to the case of the existence of functions more general than convex ones. A convex function F on a G-space R has the following properties:

(1) $F(x(\tau_2)) \leqq \mathrm{Max}\{F(x(\tau_1)), F(x(\tau_3))\}$ for $\tau_1 \leqq \tau_2 \leqq \tau_3$, where $x(\tau)$, $-\infty < \tau < \infty$, is a representation of a geodesic.

(2) Whenever the equality holds in (1), $F(x(\tau_1)) = F(x(\tau_2)) = F(x(\tau_3))$.

(3) If $F(x(\tau_1)) < F(x(\tau_2))$ for $\tau_1 < \tau_2$, then $F(x(\tau)) \longrightarrow \sup F(R)$, as $\tau \longrightarrow \infty$, and if $F(x(\tau_1)) > F(x(\tau_2))$ for $\tau_1 > \tau_2$, then $F(x(\tau)) \longrightarrow \sup F(R)$ as $\tau \longrightarrow -\infty$.

A function which satisfies (1) (or (1) and (2)) is said to be nearly peakless (or peakless). Those properties have just played crucial roles in our reseach of Section 3. H. Busemann actually pointed out that all the results which we have obtained in Section 3 are true under the existence of a continuous peakless function with (3) and we need no changes in materials for the proofs. He is right. However if we are not interested in the behavior of the levels, then we will have the same classification under the existence of a nearly peakless function which is locally nonconstant.

Theorem 19. If a G-surface R admits a locally nonconstant, nearly peakless function F on R, then R is topologically either a plane, an open cylinder or an open Moebius strip.

The author emphasize that continuity of the function F are not assumed. In fact, nearly peakless functions are in general not continuous. For the proof we need two lemmas under the assumption of Theorem 19.

Lemma 20. F does not assume sup F(R).

Proof. Note that if $\Delta(q_1q_2q_3)$ is a convex triangle whose vertices are q_1, q_2 and q_3 and if $q \in \Delta(q_1q_2q_3)$, then $F(q) \leqq \mathrm{Max}\{F(q_1), F(q_2), F(q_3)\}$. Suppose that there exists a point p such that F(p) = sup F(R). From local nonconstancy of F we can have non-colinear three points q_1, q_2 and q_3 near p such that $F(q_i) < F(p)$, i = 1, 2, 3. Then it follows from the note above and near peaklessness of F that $F(r) \geqq F(p)$ for those r which satisfies (qpr) for some $q \in \Delta(q_1q_2q_3)$, and hence F(r) = F(p). Thus if q runs on $\Delta(q_1q_2q_3)$, then r runs on the set with non-empty interior. This contradicts local nonconstancy of F.

Lemma 21. R is noncompact.

Proof. Suppose that R is compact. Let a sequence $\{p_i\}$ satisfy that $F(p_i) \longrightarrow \sup F(R)$ and $\{p_i\}$ tends to a point in R, say p. There exist three non-colinear points q_1, q_2 and q_3 near p such that $p \in \Delta(q_1q_2q_3)$. By Lemma 20, $F(p_i) > \mathrm{Max}\{F(q_1), F(q_2), F(q_3)\}$ for a sufficiently large i, contradicting near peaklessness of F.

We return to the proof of Theorem 19.

Proof of Theorem 19. It is sufficient to prove that if R is orientable, then R is topologically a plane or an open cylinder, because otherwise the two-fold covering of R is orientable and the composition of F and the covering projection is nearly peakless and locally nonconstant. Suppose that R is orientable. Then R is topologically a sphere from which points are removed and to which hundles are attached. If R is homeomorphic to neither a plane nor an open cylinder, then there exists a self-intersecting closed geodesic in R or a pair of closed geodesics in R which intersect each other (see [41]). Since F is constant on every closed geodesic, in the same technique as in Lemma 20, R contains an open set U near the intersection point such that F is constant on U (also see $(9)_i$ in [6]), a contradiction. This completes the proof.

CHAPTER III

Busemann functions and total excess

1. Introduction

A G-space is defined by Busemann [4] as a metric space in which any two points can be joined by a segment and in which any segment may be prolonged uniquely to a geodesic. All complete Riemannian manifolds are, of course, G-spaces.

If A is a ray with origin $q \in R$ in a noncompact G-space R, then we can define a function on R by $f_A(p) := \lim\{d(q,z) - d(p,z)\}$ where $d(q,z) \longrightarrow \infty$ with $z \in A$, because $d(q,z) - d(p,z) \leq d(q,z') - d(p,z') \leq d(p,q)$ for $z, z' \in A$ with $d(q,z) \leq d(q,z')$. The function f_A is called the Busemann function of A

It is recently known that Busemann functions play important roles in the study of complete manifolds. For example, Cheeger-Gromoll [11], Gromoll-Meyer [18] and Wu [45] have obtained the topological and differentiable structures of manifolds of non-negative or positive sectional curvature by using convexity of Busemann functions on such spaces. Eberlein-O'Neill [13] used them as a useful implement in determining their axial and parabolic manifolds. Shiohama [38] and [39] discovered that the existence of exhaustion or nonexhaustion Busemann functions was controlled by the total curvature in noncompact complete 2-dimensional manifolds. Neverthless we have not yet known what points spoil differentiability of a Busemann function in non-straight spaces. In straight Riemannian G-spaces, i.e., complete simply connected Riemannian manifolds without conjugate points, all Busemann functions are differentiable. Eschenburg [14] has proved this.

The purposes of this chapter are to exhibit points which spoil differentiability of a Busemann function and to investigate the set of such all points.

We are, of course, interested in Riemann and Finsler G-spaces. However in order to clarify the properties which ensure our conclusions we make the arguments start from general G-spaces and put the requisite assumptions on the spaces case by case.

In order to discuss differentiability of functions on a G-space we need the following notions. A G-space R is in general not a differentiable manifold, so differentiability of a function on R at a point $p \in R$ is defined along a geodesic curve through p. Let F be a function on an open set U of R. F is by definition <u>differentiable</u> at $p \in U$ if F restricted to any geodesic curve through p is differentiable at p for a parameter of a representation of the geodesic curve. Also F is <u>differentiable</u> on U if F is differentiable at each point $p \in U$. The distance on R is by definition <u>differentiable around p</u> if there exists an open neighborhood V_p of p such that the distance function from p is differentiable on $V_p - \{p\}$. We say that if for each point p of R the distance on R is <u>differentiable around p</u> and if there exists a positive η_p such that for each $r \in B(p,\eta_p) := \{r \; ; \; d(p,r) < \eta_p\}$ we can choose an open neighborhood V_r of r with the property as stated above and

$V_r \ni p$, then the distance of R is locally differentiable on R.

In a G-space R we need to assume another important property concerning the distance function which holds in differentiable manifolds. For any point $p \in R$ the distance function from p restricted to a segment from p is differentiable except at p for a parameter of a representation of the segment and its derivative is equal to 1 or -1. We say that the distance from $p \in R$ is regular at $q \in R$ if the differential coefficient of the distance function from p restricted to any geodesic curve through q exists at q and it is equal neither to 1 nor to -1 whenever the geodesic curve through q is not contained in any segment from p. And also we say that the distance of R is regular if for each point $p \in R$ the distance from p is regular at every point $q \in R$ at which it is differentiable.

We denote by B(A) the set of all points at each of which the Busemann function of a ray A is not differentiable. Our first aim is to explain B(A) in very geometrical words. To do this, we need the notion of co-points to a ray A which are defined as follows. Let A be a ray with origin $q \in R$ in a non-compact G-space R. A co-ray B from p to A is by definition the limit of a converging sequence of segments $T(p_n, z_n)$ where $p_n \longrightarrow p$ and $z_n \in A$ with $d(q, z_n) \longrightarrow \infty$. The union of all co-rays which contain a co-ray B to A is either a straight line or a ray. In the first case, we call the line, with the orientation for which B is a positive subray, an asymptote to A; in the second a maximal co-ray to A and its origin a co-point to A. We denote by C(A) the set of all co-points to A and by $C_2(A)$ the subset of C(A) whose points are origins of two or more co-rays to A. C(A) and $C_2(A)$ are in general not closed in R (see [31]). Lewis [28] and Nasu [31], [32] and [33] developed the theory of C(A) and $C_2(A)$ (also see Busemann [5]).

Under these notations we prove

Theorem 2. Let R be a noncompact G-space with locally differentiable and regular distance and let A be a ray in R. Then $C_2(A) \subset B(A) \subset C(A)$. Further, if all sphere $S(p,\rho) := \{r \; ; d(p,r) = \rho\}$ with $0 < \rho < \rho(p)$ are not contractible, then B(A) is dense in C(A). (see Section 2 of Chapter II for the definition of $\rho(p)$)

From this fact we understand the study of C(A) to be very interesting. We have in the case where C(A) is bounded

Corollary 10. Let R be a noncompact G-space with domain invariance. If there exists a ray A in R such that C(A) is non-empty and bounded, then R has exactly one end, no asymptote to A exists, and therefore f_A is an exhaustion function.

These are main contents of Section 2.

In Section 3 we clarify the relation between the boundedness of C(A) and the total excess of a G-surface (for the implication of G-surface see Section 2 of Chapter II). According Corollary 10, this is a detail of our investigation. To do this, we need the notion of angular measure which is defined by Busemann [4] as follows. Let p be a point of a G-surface R. A direction from, or with origin p, is an oriented segment $T^+(p,q)$ of length $\sigma(p) := \min\{\rho(p)/4, 1\}$. Then for each $q \in S := S(p,\sigma(p))$ there exists a unique direction from p to q. We conceive of an angle A with vertex p as a set of directions with origin p passing through the points of a sub-arc of S. A measure for the angles at p is a nonnegative function $|A|$ which is defined for all angles A with vertex p and has the following properties: 1) $|A| = \pi$ if and only if A contains exactly one segment through p and connecting points of S, which is called a straight angle, 2) If $A_1 \cap A_2$ consists of exactly one direction, then $|A_1 \cup A_2| = |A_1| + |A_2|$. We speak of an angular measure on a G-surface R if a definite angular measure has been defined at each of its points.

Using the angular measure above we can define the excesses of triangles and the total excesses of certain regions on a G-surface (see Section 3) which is, of course, in the case of Riemannian G-surfaces with the Riemannian angular measure, identified with the total curvatures of triangles and the total curvatures of regions by the Gauss-Bonnet Theorem.

The main result of Section 3 is

Theorem 14. Let R be a noncompact G-surface with a continuous angular measure and let the total excess $\varepsilon(R)$ of R exist. If there is a ray A in R such that C(A) is nonempty and bounded, then $\varepsilon(R) \geqq 2\pi\chi(R)$ where $\chi(R)$ is the Euler characteristic of R.

If the angular measure has the property which insures that, in a uniform way, an angle cannot be nearly straight without having a measure close to π, then in the inequality above $\varepsilon(R) = 2\pi\chi(R)$ holds. Such an angular measure is said to be "uniform at π" on R (see Section 3). Of course, the Riemannian angular measure on Riemannian G-surfaces are uniform at π.

In Section 4 we investigate the union of the angles composed of the directions of co-rays from each point to a ray. In a certain G-plane R the totality of the measures of these angles at each point is at most the total excess (see Theorem 21). As an application, we show, in combination with Maeda [29] and [30], the existence of infinitely many rays from each point $p \in R$ which are not co-rays from p to a ray (see Corollary 24).

2. Busemann functions and co-points

Let R be a noncompact G-space and let A be a ray in R. If B is a co-ray to A, then B is also a co-ray to any ray contained in or containing A as a sub-ray. Furthermore, the limit of a converging sequence of co-rays to a ray A is likewise a co-ray to A. The existence of co-rays to A from a point $p \in R$ is in general not unique. However the co-ray to A from any point of a co-ray B other than the origin p of B is unique and a sub-ray of B (see [4], p.136). For co-rays B_1 and B_2 to A, unless one is a sub-ray of the other B_1 and B_2 do not intersect each other except at the origins.

Our first observation is

Proposition 1. Let R be a noncompact G-space with locally differentiable distance and let A be a ray from q. If $p \in R$ is not a co-point to A, then the Busemann function f_A is differentiable at p.

From this we have

Theorem 2. Let R be a noncompact G-space with locally differentiable and regular distance. Let A be a ray. Then $C_2(A) \subset B(A) \subset C(A)$. Further, if all spheres $S(p,\rho)$ with $0 < \rho < \rho(p)$ are contractible, then B(A) is dense in C(A).

It is known that if a G-space R is finite-dimensional, then all spheres $S(p,\rho)$ with $0 < \rho < \rho(p)$ are not contractible (see [5], p.16).

From Theorem 2 any noncompact G-space in which no sraight lines exist does not admit any differentiable Busemann function.

We say that a G-space R is <u>straight</u> if all geodesics are straight lines. In a straight G-space C(A) is empty for every ray A (see [4], p.138).

<u>Corollary 3</u>. Let R be as in Proposition 3. If R is straight, then all Busemann functions are differentiable on R.

This is the version of the result of Eschenburg [14].

<u>Corollary 4</u>. Let R and $S(p,\rho)$ be as in Theorem 2. If B(A) is empty for a ray A in R, then there is a homeomorphism h of R onto a product $R_1 \times \mathbb{R}$ such that R_1 is homeomorphic to $[f_A = 0]$ by h^{-1} and $\{p\} \times \mathbb{R}$ is the image of an asymptote to A by h.

<u>Proof</u>. The homeomorphism h is given as follows. If for each $q \in R$ $h_1(q)$ is the point at which the asymptote through q to A intersects $[f_A = 0]$, then we have only to put $h(q) := (h_1(q), f_A(q))$, since generally $f_A(y(s)) = s + f_A(y(0))$ holds where y(s) is a representation of a co-ray to A (see [4], p.134) and since the asymptote through $p \in R$ to A continuously depends on p.

From the definition of ends (see Section 3 of Chapter II) a product $R_1 \times \mathbb{R}$ has one end if R_1 is noncompact and otherwise two ends. Hence we have

Corollary 5. Let R and $S(p,\rho)$ be as in Theorem 2. If R has at least three ends, then there are no differentiable Busemann functions on R.

Nasu [33] stated that in a noncompact G-space if C(A) is nonempty and compact for a ray A, then no asymptote to A exists (also see [5], p.89). Since $f_A(p) \geqq d(p,C(A)) + \inf f_A(C(A))$ for every $p \in R$, we have

Corollary 6. Let R and $S(p,\rho)$ be as above. If B(A) is nonempty and compact, then f_A is exhaustive, i.e., $[-\infty < f_A \leqq a]$ is compact in R for every $a \in \mathbb{R}$.

In combination with Nasu's structure theorem [31] of a G-space having an isolated co-point to a ray, we have

Corollary 7. Let R and $S(p,\rho)$ be as in Theorem 2. If B(A) contains an isolated point p, then $B(A) = \{p\}$ and the space is the union of all co-rays to A with origin p and R is contractible to p. The levels of f_A are the ordinary spheres $S(p,\rho)$ with center p.

We return to the proofs of Proposition 1 and Theorem 2.

Proof of Proposition 1. Let B be a co-ray through p to A. By the definitions of co-rays to A and of co-points to A there is a sequence of points w_n and $z_n \in A$ such that $w_n \longrightarrow w$ ($\in B$), p follows w on B, $d(p,z_n) \longrightarrow \infty$, and segments $T(w_n,z_n) \longrightarrow B$. Further, from local differentiability of the distance, choose points u_n and v_n on $T(w_n,z_n)$ such that w_n, u_n, p_n and v_n are in this order on $T(w_n,z_n)$ and $u_n \longrightarrow u \in B$, $v_n \longrightarrow v \in B$ with $p \in V_u \cap V_v$, where V_u and V_v are the domains as in the definition of local differentiability of the distance. Let $x(s)$, $-\varepsilon \leqq s \leqq \varepsilon$, be a representation of any geodesic curve with $x(0) = p$ in $V_u \cap V_v$. We must prove that $f_A(x(s))$ is differentiable at $s = 0$.

By the triangle inequalities, $d(x(s),z_n) \leqq d(x(s),v_n) + d(p_n,z_n) - d(p_n,v_n)$ for every n and for every s, and hence $d(p_n,v_n) - d(x(s),v_n) \leqq d(p_n,z_n) - d(x(s),z_n)$. Also $d(u_n,p_n) + d(p_n,z_n) \leqq d(x(s),u_n) + d(x(s),z_n)$ for every n and for every s, and therefore $d(p_n,z_n) - d(x(s),z_n) \leqq d(x(s),u_n) - d(p_n,u_n)$. Hence, because $d(p_n,z_n) - d(x(s),z_n) = \{d(q,z_n) - d(x(s),z_n)\} - \{d(q,z_n) - d(p_n,z_n)\}$ for every s, we have, as $n \longrightarrow \infty$, $-(d(x(s),v) - d(p,v)) \leqq f_A(x(s)) - f_A(x(0)) \leqq d(x(s),u) - d(p,u)$ for every s. Thus we have only to show that $\lim (d(x(s),u) - d(p,u))/s = -\lim (d(x(s),v) - d(p,v))/s$. This is proved as follows.

Since the function $d(x(s),u) + d(x(s),v)$ for s assumes a minimum at $s = 0$ and since $d(x(s),u)$ and $d(x(s),v)$ are differentiable at $s = 0$, $0 = \lim (d(x(s),u) + d(x(s),v) - d(u,v))$

/s = lim (d(x(s),u) - d(p,u))/s + lim (d(x(s),v) - d(p,v))/s. This completes the proof.

It should be noted that, under the notations of the proof, even if p is a co-point to A, the inequality $-(d(x(s),v) - d(p, v)) \leqq f_A(x(s)) - f_A(p)$ holds for every s and the equality sign holds at s = 0, so that if f_A is differentiable at p, then the differential coefficient of f_A at p along x(s) is equal to the one of the negative distance function from v at p along x(s).

<u>Proof of Theorem 2</u>. From Proposition 1 it follows that B(A) $\subset$ C(A). Suppose there exists a point $p \in C_2(A) - B(A)$. Let A_1 and A_2 be two distinct co-rays from p to A. Let y(s), $0 \leqq s < \infty$, be a representation of a ray A_2 with y(0) = p and let v be a point on A_1 such that it follows p and $p \in V_v$. From the fact that $f_A(y(s)) = s + f_A(p)$ the differential coefficient of $f_A(y(s))$ at s = 0 is equal to 1. However, by regularity of the distance, the differential coefficient of d(y(s),v) is not equal to -1, a contradiction to the above remark. Thus $C_2(A) \subset B(A)$.

The denseness of B(A) in C(A) follows from a result in [5], p.89. This completes the proof.

We notice that if R is a Riemannian G-space, then the gradient of f_A is continuous on R - B(A).

In the following we will discuss and partially answer the question; in what spaces there exists a ray A such that C(A) is bounded.

We say that a Hausdorff space R has the property of <u>domain invariance</u> if a subset of R which is homeomorphic to an open set of R is open in R. According to Busemann [5], p.16, domain invariance holds in finite-dimensional G-spaces.

<u>Theorem 8</u>. Let R be a noncompact G-space with domain invariance and let A be a ray in R. If C(A) is nonempty, then $f_A(C(A))$ is dense in the interval $[\inf f_A(R), \sup f_A(C(A))]$.

To give corollary of Theorem 8 we need a short remark. If a noncompact G-space R has at least two ends, then for every ray A there exists an asymptote to A, and f_A, in particular, assumes no minimums. In fact, from the assumption, there exists a compact set $K \subset R$ such that R - K contains at least two unbounded components and a sub-ray of A is contained in one component W of R - K. Take an unbounded sequence of points p_j in R - K each of which is contained in distinct components from W. Since for each j a co-ray A_j from p_j to A intersects K, there exists a subsequence $\{A_k\}$ of the sequence $\{A_j\}$ converging to an asymptote to A.

<u>Corollary 9</u>. Let R be a noncompact G-space with domain invariance and with at least two ends. If C(A) is nonempty for a ray A, then C(A) is unbounded. In particular, if R has at least three ends, then C(A) is nonempty and unbounded for every ray A.

Corollary 10. Let R be a noncompact G-space with domain invariance. If there exists a ray A in R such that C(A) is nonempty and bounded, then R always has exactly one end, no asymptote to A exists, and hence f_A is exhaustive.

This corollary should be compared with Corollary 6.

To the proof of Theorem 8 we need the following lemma.

Lemma 11. Let R be a noncompact G-space and let A be a ray in R. Then, $[a \leqq f_A < \infty]$ is pathwise connected for every $a \in f_A(R)$.

Proof. We first prove that we connect a point p with $f_A(p) > a$ and a point q with $f_A(q) > a$ by a curve. To show this we will obtain two curves in $[a \leqq f_A < \infty]$ from p to some point $z \in A$ and from q to the same point $z \in A$. If for each $z \in A$ a segment T(p,z) from p to z (or T(q,z)) intersects $[f_A = a]$, then we denote by p(z) (or q(z)) a point in the intersection. Thus $d(r,z) - d(p,z) = d(r,z) - d(p(z),z) - d(p,p(z))$ or $d(r,z) - d(q,z) = d(r,z) - d(q(z),z) - d(q,q(z))$ for every $z \in A$, respectively, where r is the origin of A. In general if z' follows z on A, then $d(r,z) - d(s,z) \leqq d(r,z') - d(s,z')$ for every point $s \in R$. Hence $d(r,z) - d(p,z) \leqq a - d(p,p(z))$ or $d(r,z) - d(q,z) \leqq a - d(q,q(z))$ for every $z \in A$. From this, for every $z \in A$ such that d(r,z) is sufficiently large, d(p,p(z)) and d(q,q(z)) is negative, a contradiction. Therefore using such a point $z \in A$ we can obtain a curve $T(p,z) \cup T(q,z)$ from p to q.

For a point p with $f_A(p) = a$ and for a point q with $f_A(q) > a$, for example, we pick out a point $p_1 \neq p$ on a co-ray from p to A and a curve T from p_1 to q in $[a \leqq f_A < \infty]$ as above. Then we get a curve $T(p,p_1) \cup T$ from p to q in $[a \leqq f_A < \infty]$.

Proof of Theorem 8. Suppose that there exist values a and b such that for every t, $a < t < b \leqq \sup f_A(C(A))$, $[f_A = t]$ contains no co-points to A. Fix t_0, $a < t_0 < b$. From Lemma 11, $[t_0 \leqq f_A < \infty]$ is connected. Now let X be the set of all points each of which is contained in the co-ray to A from some point $p \in [f_A = t_0]$. Clearly $X \subset [t_0 \leqq f_A < \infty]$. We show that X is closed and open in $[t_0 \leqq f_A < \infty]$.

Assume that $\{p_j\} \subset X$ converges to a point p_0 in $[t_0 \leqq f_A < \infty]$. By the construction of X, for each p_j there exists a unique point $q_j \in [f_A = t_0]$ the co-ray to A from which contains p_j. For each j the distance between p_j and q_j is at most $\sup_j\{f_A(p_j)\} - t_0$ and hence $\{q_j\}$ is bounded. Since $[f_A = t_0]$ is closed, there exists a subsequence $\{q_k\}$ of $\{q_j\}$ converges to a point q_0 in $[f_A = t_0]$. Because $[f_A = t_0]$ contains no co-points to A and therefore the co-ray from a point q in $[f_A = t_0]$ to A continuously depends on q, $\{p_k\}$ converges to p_0, which is contained in a unique co-ray from q_0 to A, namely $p_0 \in X$. Hence X is closed in $[t_0 \leqq f_A < \infty]$.

To prove that X is open in $[t_0 \leqq f_A < \infty]$ we need the property of domain invariance of R. For any point q' in X there is a unique point q in $[f_A = t_0]$ the co-ray to A from which contains q'. If $f_A(q') = t_0$, then we have nothing to prove because $[t_0 \leqq f_A < b]$ is a neighborhood of q' in X. So we may assume that $f_A(q') > t_0$. Since $[t_0 < f_A < b]$ is open in R, for given point $p \in T(q,q') \cap [t_0 < f_A < b]$ there exists an open neighborhood V of p in R which is contained in $[t_0 < f_A < b]$. Define a map of V into X by sending $r \in V$ to a point r' such that r' is contained in the co-ray from r to A and the distance between r and r' is d(p,q') (q' = p' under our new notation). Obviously this map h is a homeomorphism of V onto its image containing p. From domain invariance of R the image $h(V) \subset X$ is open in $[t_0 \leqq f_A < \infty]$. Hence X is open in $[t_0 \leqq f_A < \infty]$.

Therefore $X = [t_0 \leqq f_A < \infty]$. However this is impossible, because X contains no co-points to A from the construction of X and because $[t_0 \leqq f_A < \infty]$ contains co-points to A. This completes the proof.

In the proof above we have just shown that there are no values a and $b \leqq \sup f_A(C(A))$ such that no components of $[a < f_A < b]$ exist such that it does not contain any co-points to A. Hence we have

<u>Proposition 12</u>. Let R be a noncompact G-space with domain invariance and with at least two ends. Let A be a ray in R. If C(A) is nonempty, then for each compact set K there are co-points

to A in all unbounded components of R - K, except in only one which contains a sub-ray of A.

For the investigations in Section 3 it is convenient to notice here the case of G-surfaces in more details.

Proposition 13. Let R be a noncompact G-surface and let A be a ray in R. If C(A) is nonempty and bounded, then no asymptote to A exists, f_A is exhaustive, and moreover there is a t_0 such that $[f_A = t]$ is a one-dimensional embedded submanifold of R which is homeomorphic to a circle S^1 for every $t \geqq t_0$ and $[t_0 \leqq f_A < \infty]$ is homeomorphic to a product $[f_A = t_0] \times [0,\infty)$.

Proof. The first two assertions are contained in Corollary 10. since C(A) is bounded, there is a t_0 such that $[t_0-1 \leqq f_A < \infty] \cap C(A)$ is empty. By the same reasoning as Corollary 4, $[t_0 \leqq f_A < \infty]$ is homeomorphic to a product $[f_A = t_0] \times [t_0,\infty)$. The proof of the remainder is as follows.

For given $p \in [f_A = t]$, $t \geqq t_0$, there is a point q such that $t_0 > f_A(q) > t_0 - 1$ and the co-ray B from q to A contains p. Let T be a segment through q such that T is contained in $[t_0 -1 < f_A < t_0]$ and the intersection $T \cap B$ consists of only one point q. Note that for every co-ray B' from $r \in T$ to A the intersection $T \cap B'$ consists only one point, since if $T \cap B'$ contains two points, then q must be a co-point to A. Let D be the set of all points each of which is contained in the co-ray from some point $r \in T$ to A. Then, D is homeomorphic to a half-strip $[0,1] \times [0,\infty)$, since the co-ray from $r \in T$ to A continuously depends on r. In particular, D is a neighborhood of p. Obviously the intersection $D \cap [f_A = t]$ is homeomorphic to T, so that $[f_A = t]$ is a one-dimensional embedded submanifold of R without boundary. Since f_A is exhaustive, $[f_A = t]$ is compact, and hence $[f_A = t]$ is topologically a circle S^1. This completes the proof.

3. Co-points and total excesses

In this section we see to what extent the total excess of a G-surface influences the set of all co-points to a ray. We confirm the notions which are used in this section. They are found in the book of Busemann [4], pp.273-305.

We have already defined the angles and the angular measure in the introduction. We say that the angles A_n with vertex p_n tends to the angle A with vertex p if $p_n \longrightarrow p$ and the point set carring the directions in A_n tends to the point set carring the directions in A. An angular measure is by definition continuous if $A_n \longrightarrow A$ implies $|A_n| \longrightarrow |A|$. The excess of a triangle $\Delta(abc)$ in $B(p,\rho(p)/8)$ is the value $\varepsilon(abc) = |abc| + |bca| + |cab| - \pi$, where $|abc|$, for example, implies the measure of the angle composed of the directions with origin b and through points of the segment T(a,c). Let D be a compact polygonal domain, i.e., its boundary, if exists, consists of finitely many simple closed geodesic polygons. We put $\varepsilon(D) := \sum_{i=1}^{n} \varepsilon(\Delta(a_i b_i c_i))$, where $\Delta(a_i b_i c_i)$ is the decomposition of D into triangles each of which is contained in $B(p,\rho(p)/8)$ for some p. $\varepsilon(D)$ is independent of the choice of the decomposition and is called the total excess of D. Busemann has stated in [4], p.283, that $\varepsilon(D) = 2\pi\chi(D) - \sum_{j=1}^{m} (\pi - \beta_j)$, where β_j is the measure of the inner angle of D with vertex p_j on the boundary of D and $\chi(D)$ is the Euler characteristic of D. An angular measure is said to be "uniform at π" in the subset G of R if a nondecreasing positive function $\delta(\varepsilon) < 1$ and a positive function $\omega(p,\varepsilon)$

$\leq \rho(p)/4$ defined for $0 < \varepsilon < \pi$ and $p \in G$ exist, such that the relations $0 < d(a_1,p) = d(a_2,p) < \omega(p,\varepsilon)$ and $d(a_1,a_2)/(d(a_1,p) + d(p,a_2)) > 1 - \delta(\varepsilon)$ imply $|a_1pa_2| > \pi - \varepsilon$. This property insures that, in a uniform way, an angle cannot be nearly straight without having a measure close to π. Of course, the Riemannian angular measure on a Riemannian G-surface R is uniform at π on R (see [4], p.293). Finally we need the notion of the total excess of a noncompact region G. Let G be a subset of R which is the union of an increasing sequence $\{D_n\}$ of compact polygonal domains $G = \cup D_n$ such that $p_i \in D_{n_i+1} - D_{n_i}$ implies that $\{p_i\}$ has no accumulation point for any sequence $\{n_i\}$ going to infinity. If $\varepsilon(G) := \lim \varepsilon(D_n)$ exists for each such sequence $\{D_n\}$, $\pm\infty$ admitted, then we call it the <u>total excess</u> of G. Busemann [4], p.300, has proved that if the G-surface R of finite connectivity possesses the total excess with respect to the angular measure which is uniform at π on R, then $\varepsilon(R) \leq 2\pi\chi(R)$. This is a generalization of a result of Cohn-Vossen [12].

Under the notions above we establish

<u>Theorem 14</u>. Let R be a noncompact G-surface with a continuous angular measure and let the total excess $\varepsilon(R)$ of R exist. If there exists a ray A in R such that C(A) is nonempty and bounded, then $\varepsilon(R) \geq 2\pi\chi(R)$.

According to the remark preceding Theorem 14 we have

Corollary 15. Let R and C(A) be as above. Further, if the angular measure is uniform at π on R, then $\varepsilon(R) = 2\pi\chi(R)$. In particular, if R is a Riemannian G-surface with the Riemannian angular measure, then the total curvature of R, $\varepsilon(R)$, is equal to $2\pi\chi(R)$.

If for any point p outside a bounded set in a G-surface there is only one ray starting from p, then the assumption concerning C(A) is automatically satisfies. Hence we obtain

Corollary 16. Let R be as in Theorem 14. If there is a bounded set K of R such that for every $p \notin K$ only one ray from p exists, then $\varepsilon(R) \geqq 2\pi\chi(R)$.

Before proving Theorem 14 we provide the notations and three lemmas under the assumption of Theorem 14.

From Proposition 13 there is a $t_0 > 0$ such that C(A) is contained in $[-\infty < f_A \leqq t_0-2]$ and $[t_0-1 \leqq f_A < \infty]$ is topologically a cylinder $S^1 \times [t_0,\infty)$.

Lemma 17. For every $t \geqq t_0$ there is a point $z(t) \in A$ such that for every $z \in A$ which follows z(t) on A and for every $q \in [f_A = t]$ all segments from q to z does not intersect $[-\infty < f_A \leqq t-1]$.

Proof. Suppose that this is false. Then there exists a sequence $\{q_j\} \subset [f_A = t]$ and a sequence $\{z_j\} \subset A$ such that $d(q_j, z_j) \longrightarrow \infty$ and a segment from q_j to z_j, for each j, intersects $[-\infty < f_A \leqq t-1]$. The seque-nce of these segments containes a subsequence converging to a co-ray B from a point in $[f_A = t]$ to A. f_A is monotone increasing on B, so B cannot intersect $[-\infty < f_A \leqq t-1]$, a contradiction.

Lemma 18. Let $t \geqq t_0$. Then for every $z \in A$ which follows $z(t) \in A$ as above there exists a point $q(z) \in [f_A = t]$ such that at least two segments from q(z) to z exist and the union of two of these segments, if chosen, surrounds a compact polygonal region D(z) which contains $[-\infty < f_A \leqq t-1]$ in its interior.

Proof. Let p be a point on A such that $f_A(p) = t_0$ and let $c(s)$, $0 \leqq s \leqq \alpha$, be a representation of $[f_A = t]$ with $c(0) = c(\alpha) = p$. Fix $z \in A$ which follows z(t) on A. For each s, $0 \leqq s \leqq \alpha$, we denote by X(s) (or Y(s)) the set of all curves from c(s) to z in $R' := R - [-\infty < f_A < t-1]$ each of which is homotopic to the curve $c([0,s]) \cup T(p,z)$ (or $c([s,1]) \cup T(p,z)$) in R'. We note that for every $\tilde{x} \in X(s)$ and for every $\tilde{y} \in Y(s)$ the union $\tilde{x} \cup \tilde{y}$ is homotopic to the union $T(z,p) \cup [f_A = t] \cup T(p,z)$ in R', so that $\tilde{x} \cup \tilde{y}$ surrounds $[-\infty < f_A \leqq t-1]$. For each s, $0 \leqq s \leqq \alpha$, let $\tilde{x}(s)$ be an element of X(s) such that for every element $\tilde{x}$ of X(s) the length of $\tilde{x}(s)$ is not greater than the one of $\tilde{x}$. Similarly let $\tilde{y}(s)$ be an element of Y(s) with minimum length in Y(s). We denote by $\lambda(s)$ and $\mu(s)$ the lengths of $\tilde{x}(s)$

and $\tilde{y}(s)$ respectively. Obviously $\tilde{x}(0) = T(p,z)$ and $\tilde{y}(\alpha) = T(p,a)$, so $\lambda(0) < \mu(0)$ and $\lambda(\alpha) > \mu(\alpha)$, because of the uniqueness of a segment $T(p,z)$. By semicontinuity of the lengths of curves (see [4], p.20) $\lambda(s)$ and $\mu(s)$, $0 \leqq s \leqq \alpha$, are continuous, so that there is an s_0 with $0 < s_0 < \alpha$ such that $\lambda(s_0) = \mu(s_0)$. Then both $\tilde{x}(s_0)$ and $\tilde{y}(s_0)$ are segments from $c(s_0)$ to z in R. In fact, for every s, $0 \leqq s \leqq \alpha$, Lemma 17 implies that $\tilde{x}(s)$, $\tilde{y}(s)$ or both are segments in R, because R' is topologically a half-cylinder. Thus we have only to put $q(z) := c(s_0)$, and then $\tilde{x}(s_0) \cup \tilde{y}(s_0)$ surrounds a desired compact domain $D(z)$. This completes the proof.

Note that $\chi(R) = \chi([-\infty < f_A \leqq t-1]) = \chi(D(z))$ for every $t \geqq t_0$.

Lemma 19. For any $\varepsilon > 0$ and for every $t \geqq t_0$ there is a $w(t) \in A$ such that $w(t)$ follows $z(t)$ on A and $\varepsilon(D(w(t))) > 2\times\pi\chi(R) - \varepsilon$ where $D(w(t))$ is as in Lemma 18.

Proof. Since $[f_A = t]$ is compact, there exists a sequence $\{z_i\}$ on A such that $d(p,z_i) \longrightarrow \infty$ and $q(z_i)$ converges to some point in $[f_A = t]$. If $\alpha(z_i)$ is the measure of inner angle of $D(z_i)$ at $q(z_i)$ and so is $\beta(z_i)$ at z_i, then $\alpha(z_i) \longrightarrow 2\pi$ as $d(p,z_i) \longrightarrow \infty$, since the angular measure is continuous and since for every point in $[f_A = t]$ the existence of a co-ray from it to A is unique. Hence there exists a $w(t) := z_{i_0} \in A$ which follows

$z(t)$ on A such that $\varepsilon(D(w(t))) = 2\pi\chi(R) - (\pi - \alpha(w(t))) - (\pi - \beta(w(t))) > 2\pi\chi(R) - \varepsilon$.

Now we will prove Theorem 14.

Proof of Theorem 14. We want to prove that $\varepsilon(R) > 2\pi\chi(R) - \varepsilon$ for any $\varepsilon > 0$. Put $D_0 := D(w(t_0))$ as above. We assume by induction hypothesis that there exists a sequence $D_0, D_1, \ldots, D_n$ of compact polygonal domains such that $D_i \subset D_{i+1}$ for $i = 0, 1, \ldots, n-1$ and $\varepsilon(D_i) > 2\pi\chi(R) - \varepsilon$ for $i = 0, 1, \ldots, n$. If $a :=$ Max $f_A(D_n)$, then we can put $D_{n+1} := D(w(a + 2))$ from Lemma 19. Obviously the sequence $\{D_i\}$ satisfies the conditions which are needed for the definition of the total excess of R. Thus $\varepsilon(R) \geqq 2\pi\chi(R)$. This completes the proof.

4. Remarks on the existence of co-rays

In this section we treat the case that a Busemann function f_A on a G-plane, i.e., topologically a plane, is convex. It is known that every Busemann function on a Riemannian manifold of nonnegative sectional curvature is convex (see [11], [45]). In a G-plane if a convex function assumes a minimum, then the minimum set is either a point, a segment, a ray, or a straight line (see Section 3 of Chapter II). However the last does not occur in the case of convex Busemann functions as in the following.

Proposition 20. Let R be a noncompact G-space. The minimum set of a Busemann function does not separate the space R.

For we have already proved this in the proof of Lemma 11.

It is proved in Section 3 of Chapter II that unless a convex function on a G-surface R has nonempty minimum set which is a one-dimensional manifold without boundary, then all levels are connected and levels except the minimum set are one-dimensional embedded submanifolds without boundary, and hence all levels are topologically either circles or real lines.

Let R be a G-plane and let a Busemann function f_A be convex and nonexhaustive. Then from the fact above every level except the minimum set is homeomorphic to a real line. If A_1 and A_2 are distinct co-rays from $p \in C_2(A)$ to A, then they intersect $[f_A = t]$ at distinct points for all $t > f_A(p)$, and therefore they cut off the sub-arc $\tilde{c}(t)$ of $[f_A = t]$ for all $t > f_A(p)$.

Theorem 21. Let R be a G-plane which possesses an continuous angular measure with nonnegative excess and let a Busemann function f_A be convex and nonexhaustive. Then for each point $p \in C_2(A)$ the measure of the inner angle at p of the domain $D := \bigcup_{t > f_A(p)} \tilde{c}(t)$ as above is not greater than the total excess $\varepsilon(R)$ of R.

We say that an angular measure on a G-surface R has non-negative excess if every non-degenerate triangle in $B(p,\rho(p)/8)$ R has nonnegative excess. We notice that since the excess of every triangle is nonnegative, $\varepsilon(R)$ always exists (see [4], p. 299).

Proof. We have only to prove the statement for two co-rays A_1 and A_2 from p to A such that any co-ray from p to A passes through the sub-arcs cut off from levels by A_1 and A_2. Let p_1 and p_2 be points in A_1 and A_2 respectively with $f_A(p_1) = f_A(p_2) > f_A(p)$. First we show that there exists a $z_0 \in A$ such that if $z \in A$ follows z_0 on A, then the union $T(p,p_1) \cup T(p_1,z) \cup T(z,p_2) \cup T(p_2,p)$ of four segments is a simple quadrilateral, and such that the compact domain D(z) which is surrounded by the quadrilateral tends to the domain D in the statement. If q is the origin of A, then $d(q,z) - d(p_i,z) \leqq f_A(p_i) - d(p_i,r_i(z))$, $i = 1, 2$, as in the proof of Lemma 11, where $z \in A$ and $r_i(z)$ is, if exists, a point of the intersection of a segment $T(p_i,z)$ and $[f_A = f_A(p_i)]$. Hence $d(p_i,r_i(z))$, $i = 1, 2$, tends to zero. Thus there is a $z_0 \in A$ such that $d(p_i,r_i(z)) < d(p_i,T(p_j,p))$, $i \neq j$, for every z following z_0 on A. Then our quadrilateral is simple, because, for example, a segment $T(p_1,z)$ intersects the union $T(p,p_1) \cup T(p,p_2) \cup T(p_2,z)$ of segments at only two points p_1 and z. Since each of components of $R - [f_A = f_A(p_1)]$ is homeomorphic to a plane and since segments $T(p_1,z)$ and $T(p_2,z)$ tends to the sub-rays of A_1 and A_2 respectively as $d(p,z) \longrightarrow \infty$ with $z \in A$, the domain $D(z)$ surrounded by the quadrilateral tends to the domain D in the statement.

Because the excesses are nonnegative, $\varepsilon(D(z)) = 2\pi - (\pi - \alpha(z)) - (\pi - \beta(z)) - (\pi - \gamma(z)) - (\pi - \delta(z)) \leqq \varepsilon(R)$ where $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$ are the measures of the inner angles of $D(z)$ at p, p_1, p_2 and z respectively. Hence $\alpha(z) + \beta(z) + \gamma(z) - 2\pi \leqq \varepsilon(R)$. Since the angular measure is continuous, $\beta(z)$ and $\gamma(z)$ tends to π as $d(p,z) \longrightarrow \infty$ with $z \in A$. Therefore $\alpha(z) \leqq \varepsilon(R)$, which is our goal.

Let D be the region which is composed of all co-rays from p to a ray A. In general $D - \{p\}$ is not connected and each component may contain many co-rays from p to A. For each component D_λ of $D - \{p\}$ if we apply Theorem 21 to the boundary of $D_\lambda \cup \{p\}$, then the measure of the inner angle of $D_\lambda \cup \{p\}$ at p is at most $\varepsilon(R)$. Nevertheless the proof of Theorem 21 yields a stronger result than this.

Corollary 22. Let R be a G-plane which possesses a continuous angular measure with nonnegative excess and let a Busemann function f_A be convex and nonexhaustive. Then for each point p the totality of the measures of the inner angles of $D_\lambda \cup \{p\}$'s at p is at most the total excess $\varepsilon(R)$ of R.

In a Riemannian G-plane M^2 with the Riemannian angular measure K. Shiohama has proved in [38] that if M^2 is of nonnegative Gaussian curvature and if the total curvature is not greater than π, then all Busemann functions on M^2 are nonexhaustive. Hence we have

Corollary 23. Let M be a Riemannian G-plane with the Riemannian angular measure and of nonnegarive Gaussian curvature. If the total curvature $\varepsilon(M)$ is not greater than π, then for each ray A and for every point p the totality of the measures of the inner angles of $D_\lambda \cup \{p\}$'s at p is at most $\varepsilon(M)$.

As another application we prove the existence of rays from each point p which are not co-rays from p to a given ray A. To do this, we carefully observe Maeda's result in [29] and [30], i.e., if M is a Riemannian G-plane with the Riemannian angular measure and of nonnegative Gaussian curvature, then the totality of the measures of the inner angles at each point p of the domains which are composed of all rays from p is at least $2\pi - \varepsilon(M)$ where $\varepsilon(M)$ is the total curvature of M. Using this we have the following.

Corollary 24. Let M be a Riemannian G-plane with the Riemannian angular measure and of nonnegative Gaussian curvature. If the total curvature $\varepsilon(M)$ is smaller than π, then for each ray A and for every point $p \in M$ there are infinitely many rays from p which are not co-rays from p to A, more precisely, the totality of the measures of the angles at p of the domains which are composed of rays from p that are not co-rays to A is at least $2\pi - 2\varepsilon(M)$.

Our estimate is, of course, valid at only points of $C_2(A)$.

CHAPTER IV

Affine functions and splitting theorems

1. Introduction

A function F defined on a complete Riemannian manifold M without boundary is said to be <u>convex</u> iff on each geodesic F is a one-variable convex function. In [15] and [16], such functions are studied in details. For example, if F assumes no minimum, then M is diffeomorphic to $N \times \mathbb{R}$, where N is homeomorphic to a level of F, and if F has a compact level, then all levels are compact and the diameter function of levels of F is monotone nondecreasing as a function for values of F. Moreover M with a locally nonconstant convex function F has at most two ends, and M has one end if F has a noncompact level. These facts will be used in Section 4.

In the present chapter we study functions on M which are affine functions on geodesics, and we apply them to prove splitting theorems of Riemannian manifolds.

Let M be a complete Riemannian manifold without boundary. A function F on M is by definition <u>affine</u> if for a representation $x(t)$, $-\infty < t < \infty$, of any geodesic in M, $F\circ x(st_1 + (1 - s)t_2) = sF\circ x(t_1) + (1 - s)F\circ x(t_2)$ for every $s \in (0,1)$ and for every $t_1, t_2 \in (-\infty,\infty)$. A function F on a Riemannian product manifold $M := N \times \mathbb{R}$ is clearly affine on M if $F(p,t) = t$ for each $(p,t) \in N \times \mathbb{R}$.

The main theorem of our investigation is

Main Theorem. A complete Riemannian manifold M without boundary admits a non-trivial affine function if and only if M is isometric to a Riemannian product $N \times \mathbb{R}$.

More precisely, we prove

Theorem 1. Let M be a complete Riemannian manifold without boundary. If M admits a non-trivial affine function F, then $[F = a]$, for every $a \in \mathbb{R}$, is a totally geodesic submanifold of M without boundary, and furthermore there exist an isometric map I of $[F = a] \times \mathbb{R}$ onto M and a constant b such that $F \circ I(p, t) = bt + a$ for every $p \in [F = a]$ and for every $t \in \mathbb{R}$.

Examples and applications of this theorem are as follows.

Let V be the totality of all affine functions on M. Then V is evidently a vector space containing all constant functions on M and hence dim V is at least one. If M is the n-dimensional Euclidean space, then V is an $(n + 1)$-dimensional vector space. Conversely, from the fact that grad F of an affine function F is parallel on M and by iterating Theorem 1, we have

Theorem 2. Let M be an n-dimensional complete noncompact Riemannian manifold without boundary. Then $1 \leqq \dim V \leqq n + 1$. If $\dim V = k + 1$, then M is isometric to the Riemannian product $N \times \mathbb{R}^k$, where N admits no non-trivial affine functions. In particular M is the Euclidean space if and only if $\dim V = n + 1$.

Next we discuss when an affine function on M exists.

A geodesic with a representation $x(t)$, $-\infty < t < \infty$, (or $0 \leqq t < \infty$) is by definition a <u>straight line</u> (or a <u>ray</u>) if $d(x(t_1), x(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in (-\infty, \infty)$ (or $t_1, t_2 \in [0, \infty)$).

<u>Theorem 3</u>. Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature and without boundary. If there exist two rays A_1 and A_2 with representations $x_1(t)$, $0 \leqq t < \infty$, and $x_2(t)$, $-0 \leqq t < \infty$, and a positive constant a such that for every $t \in [0, \infty)$, $2t - d(x_1(t), x_2(t)) < a$, then the Busemann functions $f_{A_i}(.) := \lim\{t - d(., x_1(t)\}$, $i = 1, 2$, of A_i are non-trivial affine functions. In particular M is isometrically a Riemannian product $N \times \mathbb{R}$.

Using the Toponogov comparison theorem (see [42]), we know that the existence of A_1 and A_2 in the assumption is equivalent to the existence of a straight line. Thus we obtain a restatement of the Toponogov splitting theorem (see [10], [11] and [42]).

However if M is of nonpositive sectional curvature, the existence of a straight line does not imply the existence of a non-trivial affine function on M (see Example 6 in Section 4). But the following holds in this case.

Theorem 4. Let M be a complete noncompact Riemannian manifold of nonpositive sectional curvature and without boundary. Suppose there exists an isometry δ of M such that it translates a straight line A with a representation $x(t)$, $-\infty < t < \infty$, i.e., there exists a constant $a \neq 0$ such that $\delta \circ x(t) = x(t + a)$ for every $t \in (-\infty,\infty)$, which connects different ends of M and it leaves all free homotopy classes of closed curves in M invariant. Then the Busemann functions $f_{A_\pm}(.) := \lim\{d(.,x_\pm(t)) - t\}$ are non-trivial affine functions and hence M is isometrically a Riemannian product $N \times \mathbb{R}$, where A_+ (or A_-) is a positive (or negative) sub-ray of A with representations $x_+(t) = x(t)$ (or $x_-(t) = x(-t)$) for all $t \in [0,\infty)$.

In Section 2 we give the proof of Theorem 1. In Section 3 we prove Theorem 3. The proof is very short and the idea is useful to that of Theorem 4. In Section 4 we deal with Theorem 4 and we shall see there another statement (Proposition 5) which explains satisfactorily the meaning of splitting.

Note: Busemann and Phadke (see [6]) have independently proved the analogous result to Theorem 2 in G-spaces.

Main Theorem is a contrast to a result of T. Yamaguchi [46] who has studied the isometry groups of a complete Riemannian manifold admitting strictly convex functions.

2. Affine functions

In this section we give the proof of Theorem 1. Throughout this section the space we treat is a complete noncompact Riemannian manifold without boundary.

Clearly for each geodesic with a representation $x(t)$, $-\infty < t < \infty$, there are constants m and $n \in \mathbb{R}$ such that $F \circ x(t) = mt + n$ for all $t \in (-\infty, \infty)$ and hence F has no minimum on M. It follows from this formula that all levels of F are connected totally convex set (for definition see [11]) and hence totally geodesic embedded hypersurface without boundary (see [9], [11]). Because if the geodesic passes through two points in a level of F, then $m = 0$.

We are going to show that the exponential map of the normal bundle of each level onto M yields the desired isometric map. For a subset Q of M and for a point q in M, we call a point f $\in Q$ a foot of q on Q if $d(q,f) = d(q,Q)$. We shall often use this notion. We take the following steps to complete the proof.

Assertion 1. For given $a \in \mathbb{R}$ and for each $q \notin [F = a]$, let f be a foot of q on $[F = a]$. Then a segment $T(q,f)$ from q to f satisfies the following properties.

(1) For every $c \in [a, F(q)]$ (or $[F(q), a]$ if $F(q) < a$), $T(q,f)$ intersects $[F = c]$ at exactly one point, say f_c.

(2) f_c is a foot of q on $[F = c]$ and f is a foot of f_c on $[F = a]$.

Proof. (1) is evident, since F is a non-trivial affine function along $T(q,f)$. Concerning the second part of (2), see (20.6) in [4], p.120.

Suppose that there is a point f_c' in $[F = c]$ such that $d(q,f_c) > d(q,f_c') = d(q,[F = c])$. If $F(q) > c$ (or if $F(q) < c$), then $(F(q) - c)/d(q,f_c) < (F(q) - c)/d(q,f_c')$ (or $(c - F(q))/d(q,f_c) < (c - F(q))/d(q,f_c')$). Let f' be a point in $[F = a]$ at which the extension of $T(q,f_c')$ meets $[F = a]$. Then the length of $T(q,f)$ is greater than the length of the extension of $T(q,f_c')$ up to the point f'. This contradicts the choice of f. Hence the first part of (2) is proved.

Assertion 2. Each point $q \notin [F = a]$ has a unique foot on $[F = a]$ and moreover there is a unique segment $T(q,f)$ from q to f.

Proof. From the existence of a strongly convex ball around q (see [44]) and the total convexity of $[F = a]$ it follows that for sufficiently small positive ε there is a unique foot of q on $[F = F(q)-\varepsilon]$ (or $[F = F(q)+\varepsilon]$). If there are distinct feet f and f' of q on $[F = a]$, then Assertion 1 implies that there are at least two feet of q on $[F = F(q)-\varepsilon]$ (or $[F = F(q)+\varepsilon]$) through which $T(q,f)$ and $T(q,f')$ pass respectively, a contradiction.

The argument above is useful to prove the second part. If uniqueness of the existence of the segment from q to f is false, then there exist at least two feet of q on a level of F between [F = a] and [F = F(q)], a contradiction.

Let f be a foot of q on [F = a] and let x(t), $-\infty < t < \infty$, be a representation of the geodesic A determined by x(0) = f and x(d(q,f)) = q.

Assertion 3. $d(x(t),f) = d(x(t),[F = a]) = |t|$ for all $t \in (-\infty,\infty)$.

Proof. Let t_0 be the least upper bound of those t such that x(t) has the foot f on [F = a]. We must prove that $t_0 = \infty$. Suppose $t_0 < \infty$. Then $x(t_0 - \varepsilon)$ is evidently a foot of $x(t_0+\varepsilon)$ on $[F = F(x(t_0-\varepsilon))]$ for sufficiently small $\varepsilon > 0$, since Assertion 1 implies that A is perpendicular to $[F = F(x(t_0-\varepsilon))]$ at $x(t_0-\varepsilon)$. However since $x(t_0+\varepsilon)$ has a foot on [F = a] different from f, $x(t_0-\varepsilon)$ cannot be a foot of $x(t_0+\varepsilon)$ on $[F = F(x(t_0-\varepsilon))]$, a contradiction.

We can prove similarly on the nonpositive part of A.

By Assertions 1 to 3 we obtain the following.

Assertion 4. The exponential map of the normal bundle of [F = a] onto M is a diffeomorphism.

It turns out that this diffeomorphism is an isometric map by the following assertion and thus the proof of Theorem 1 is complete.

Assertion 5. Let q and q' be any points in [F = c] and let f and f' be the feet of q and q' on [F = a] respectively. Then $d(q,q') = d(f,f')$ and $d(q,f) = d(q',f')$.

The first part implies that all levels of F are isometric to each others and the second part implies that there exists an isometric map I of the Riemannian product $[F = a] \times \mathbb{R}$ onto M such that $F \circ I(p,t) = bt + a$ for all $p \in [F = a]$ and for all $t \in \mathbb{R}$.

Proof. First we consider the case where q' is close to q, more precisely, the least upper bound of the set of all the lengths of $T(f_d,f_d')$, $a \leqq d \leqq c$ (or $c \leqq d \leqq a$ if $c < a$), is smaller than the greatest lower bound of the set of all the convex radii of points in $T(q,f)$, where f_d and f_d' are points at which $T(q,f)$ and $T(q',f')$ intersect $[F = d]$ respectively. Since $T(q,f)$ and $T(q',f')$ are perpendicular to each level through which they pass, it follows from the first variation formula (see [17]) that $d(q,q') = d(f,f')$.

In general case, take a partition of a segment $T(q,q')$, $q = q_0, q_1, \ldots, q_n = q'$, in this order such that every pair of q_i and q_{i+1}, $i = 0, 1, \ldots, n-1$, satisfies the condition above. And let f_i, $i = 0, 1, \ldots, n$, be a foot of q_i on [F = a]. Then

$$d(q,q') = \sum_{i=0}^{n-1} d(q_i,q_{i+1}) = \sum_{i=0}^{n-1} d(f_i,f_{i+1}) \geqq d(f,f').$$

From Assertion 1, (2), it follows that f (or f') is the foot of q (or q') on [F = a] if and only if q (or q') is the foot of f (or f') on [F = c]. Thus we obtain $d(f,f') \geqq d(q,q')$. This proves that $d(q,q') = d(f,f')$.

On the second part, we have only to consider the variation made of the totality of segments each of which joins a point of $T(q,q')$ and its foot on [F = a]. This completes the proof of Assertion 5.

3. Direct applications of affine functions

In this section we give the proof of Theorem 3.

Let M be a complete noncompact Riemannian manifold of non-negative sectional curvature and without boundary. It is well known (see [11], [45]) that all Busemann functions are convex on M. Hence $f_{A_1} + f_{A_2}$ is convex on M where f_{A_1} and f_{A_2} are functions in the assumption of Theorem 3. Moreover $f_{A_1} + f_{A_2}$ is bounded above by a on M, since $f_{A_1}(p) + f_{A_2}(p) = \lim\{t - d(p,x_1(t))\} + \lim\{t - d(p,x_2(t))\} = \lim\{2t - (d(p,x_1(t)) + d(p,x_2(t)))\} \leqq \lim\{2t - d(x_1(t),x_2(t))\} \leqq a$ for all $p \in M$. Thus $f_{A_1} + f_{A_2}$ is constant on M, say c, and therefore f_{A_i}, $i = 1, 2$, is affine. In fact, for every geodesic with a representation $x(t)$, $-\infty < t < \infty$,

$$\begin{aligned} f_{A_1}\circ x(st_1 + (1-s)t_2) &\leqq sf_{A_1}\circ x(t_1) + (1-s)f_{A_1}\circ x(t_2) \\ &= s(c - f_{A_2}\circ x(t_1)) + (1-s)(c - f_{A_2}\circ x(t_2)) \\ &= c - (sf_{A_2}\circ x(t_1) + (1-s)f_{A_2}\circ x(t_2)) \\ &\leqq c - f_{A_2}\circ x(st_1 + (1-s)t_2) \\ &= f_{A_1}\circ x(st_1 + (1-s)t_2) \end{aligned}$$

for every $s \in (0,1)$ and for every t_1, $t_2 \in (-\infty,\infty)$. Hence f_{A_1} is affine and also f_{A_2} is. This completes the proof of Theorem 3.

4. Splitting theorems in the case of nonpositive curvature

Notions in this section are due to P. Eberlein and B. O'Neill [13].

A Hadamard manifold H is a complete, simply connected Riemannian manifold of dimension $n \geqq 2$ having nonpositive sectional curvature. In this space all geodesics are straight lines. Oriented geodesics A and B in H with representations $x(t)$, $-\infty < t < \infty$, and $y(t)$, $-\infty < t < \infty$, are asymptotic provided there exists a positive c such that $d(x(t),y(t)) \leqq c$ for all $t \geqq 0$. This definition is equivalent to that in Chapter III on H. The asymptote relation is an equivalence relation on the set of all oriented geodesics in H. Let $H(\infty)$ be the set of all asymptote classes of oriented geodesics of H and let $\bar{H} := H \cup H(\infty)$. $\bar{H}$ with the cone topology (see [13]) is homeomorphic to the closed unit n-ball. If A is an oriented geodesic in H, then $A^{\cdot}$ is by definition the asymptote class of A and $A_{-}^{\cdot}$ is the asymptote class of the reverse geodesic A_{-}. If ϕ is an isometry of H and X is a point in $H(\infty)$ we set $\phi(X) := (\phi \circ A)^{\cdot}$, where A is any oriented geodesic representing X. Thus we obtain a well-defined map $\phi : \bar{H} \longrightarrow \bar{H}$ which is bijective and carries $H(\infty)$ into itself.

A complete manifold M of dimension $n \geqq 2$ and of nonpositive sectional curvature is precisely the quotient manifold H/D where D is a properly discontinuous group of isometries of H. A continuous curve $c(t)$, $0 \leqq t < \infty$, is by definition divergent if for any compact set K in M there exists $t = t_K$ such that for $s \geqq t$, $c(s) \in M - K$. Divergent curves c and d in M will be called cofinal, if given any compact set K in M some final sub-arcs $c([s,\infty))$ and $d([t,\infty))$ lie in the same component of M - K. This is clearly an equivalence relation on the set of all divergent curves in M, and the resulting equivalence classes are the ends of M. This is obviously equivalent to the definition of ends given in Section 2 of Chapter II . A geodesic A with a representation $x(t)$, $0 \leqq t < \infty$, in M is by definition almost minimizing if there is a positive c such that $d(x(0),x(t)) \geqq t - c$ for all $t \geqq 0$. Further, $X \in H(\infty)$ is by definition almost D-minimizing if for any geodesic A representing X, $\pi(A)$ is almost minimizing, where π is the covering projection of H onto M = H/D. P. Eberlein and B. O'Neill have proved in [13] that if there exists an $X \in H(\infty)$ such that it is almost D-minimizing and a common fixed point of D, then a Busemann function of any $A \in X$ is invariant under D. Hence the function f on M is induced from f_A and it is convex, since every Busemann function on H is convex, and hence M is homeomorphic to a product manifold $N \times \mathbb{R}$ where N is a level of f. Furthermore it follows that $f = f_{\pi(A)}$ on M (see [25]).

Now we consider the case that there are two points in $H(\infty)$ which are common fixed points of D and almost D-minimizing.

Proposition 5. Let M be a complete noncompact Riemannian manifold without boundary and of nonpositive sectional curvature and let M = H/D. If there exist distinct points X and Y in $H(\infty)$ such that (1) they are common fixed points of D, (2) they are almost D-minimizing and (3) $\pi(X)$ and $\pi(Y)$ are in different ends of M, then M is isometric to a Riemannian product $N \times \mathbb{R}$.

<u>Proof</u>. (3) in the assumption implies in combination with (2) the existence of a straight line A with a representation $x(t)$, $-\infty < t < \infty$, such that it connects $\pi(X)$ and $\pi(Y)$, more precisely, $A \in \pi(X)$ and the reverse geodesic $A_- \in \pi(Y)$. In fact, since geodesics $B \in \pi(X)$ and $C \in \pi(Y)$ in M are almost minimizing, and hence divergent, a sequence of segments from $y(t)$ to $z(t)$ for $t \geqq 0$ contains a subsequence which converges to the desired straight line, where $y(t)$, $-\infty < t < \infty$, and $z(t)$, $-\infty < t < \infty$, are representations of B and C respectively.

Let $x_\pm(t)$, $0 \leqq t < \infty$, be representations of two rays such that $x_+(t) = x(t)$ and $x_-(t) = x(-t)$ for $t \geqq 0$. The Busemann functions $f_{A_\pm}(.) := \lim\{d(.,x_\pm(t)) - t\}$ are locally nonconstant convex functions without minimum by the preceding remark to Proposition 5. Moreover, from the facts at the beginning of Section 1, M is topologically a cylinder $N_+ \times \mathbb{R}$ (or $N_- \times \mathbb{R}$), where N_+ (or N_-) is a level of f_{A_+} (or f_{A_-}), and all levels are compact.

There exists a compact set K of M such that $[f_{A_\pm} = 0] \subset K$ and $p \in M - K$ implies that $f_{A_+}(p) < 0$ or $f_{A_-}(p) < 0$. In fact, otherwise there exists an unbounded sequence $\{p_i\}$ of points in M such that $f_{A_\pm}(p_i) \geqq 0$ and hence there exists an i_0 such that $f_{A_+}(p_{i_0}) > f_{A_+}(p_1) \geqq 0$ and $f_{A_-}(p_{i_0}) > f_{A_-}(p_1) \geqq 0$. Let $x_0(t)$, $-\infty < t < \infty$, be a representation of a geodesic A_0 such that

$x_0(0) = p_1$ and $x_0(d(p_1,p_{i_0})) = p_{i_0}$. The $f_{A_\pm} \circ x_0(t)$, $t \geqq d(p_1,p_{i_0})$, are positive and monotone increasing from convexity of $f_{A_\pm}$. Clearly A_0 is divergent. We assert that A_0 is not contained in the ends of M containing A_+ and A_-, contradicting the fact that M has at most two ends.

Suppose A_0 and A_+ (or A_-) are contained in the same end of M. If K' is a compact set containing $[f_{A_+} = 0]$ and $[f_{A_-} = 0]$, then by the definition of ends there is an $s = t_K$, such that $x_0([s,\infty))$ and $x_+([s,\infty))$ (or $x_-([s,\infty))$) are in the same component of M - K'. If $t' > s$ and if $\varepsilon(\tau)$, $0 \leqq \tau \leqq 1$, is a curve in the component of M - K' joining $x_0(t')$ and $x_+(t')$ (or $x_-(t')$), then continuity of $f_{A_\pm}$ implies that $\varepsilon([0,1])$ must meet K', since $f_{A_+} \circ x_0|[s,\infty) > 0$ (or $f_{A_-} \circ x_0|[s,\infty) > 0$) and $f_{A_+} \circ x_+|[s,\infty) < 0$ (or $f_{A_+} \circ x_+|[s,\infty) < 0$ (or $f_{A_-} \circ x_-|[s,\infty) < 0$), a contradiction.

Next we assert that there is a positive c such that $d(p, x(\mathbb{R})) < c$ for any $p \in M$. In fact, if $f_{A_+}(p) < 0$ (or $f_{A_-}(p) < 0$), then by the fact stated at the beginning of Section 1, $d(p,x(\mathbb{R}))$ $\leqq$ the diameter of $[f_{A_+} = f_{A_+}(p)]$ (or $[f_{A_-} = f_{A_-}(p)]$) $\leqq$ the diameter of $[f_{A_+} = 0]$ (or $[f_{A_-} = 0]$) $\leqq$ the diameter of K.

If we prove that $f_{A_+} + f_{A_-}$ is bounded above on M, then from convexity of $f_{A_+} + f_{A_-}$, it is constant and hence f_{A_+} and f_{A_-} are affine. For any point $p \in M$ let t_1 be such that $d(p,x(\mathbb{R})) = d(p,x(t_1))$. Then

$$f_{A_+}(p) + f_{A_-}(p) = \lim\{d(p,x_+(t)) - t + d(p,x_-(t)) - t\}$$

$= \lim\{d(p,x(t)) - t + d(p,x(-t)) - t\}$

$= \lim[\{d(p,x(t)) - (t - t_1)\} + \{d(p,x(-t)) - (t_1 + t)\}]$

$\leqq 2d(p,x(\mathbb{R})) \leqq 2c.$

This completes the proof of Proposition 5.

Now we can proceed to the proof of Theorem 4. It turns out that the assumption of Theorem 4 would be equivalent to that of Proposition 5, and the proof of Theorem 4 will be achived by Proposition 5.

Proof of Theorem 4. Let $\tilde{x}(t)$, $-\infty < t < \infty$, be any lift of $x(t)$, $-\infty < t < \infty$, to H. First we will construct an isometry $\tilde{\delta}$ over δ such that $\tilde{\delta}\circ\phi = \phi\circ\tilde{\delta}$ for any $\phi \in D$ and $\tilde{\delta}\circ\tilde{x}(t) = \tilde{x}(t + a)$ for all $t \in (-\infty,\infty)$.

For points q and r in H let $T(q,r)$ be a geodesic from q to r in H. For every point p in H, define $\tilde{\delta}p$ by a point in H which is the endpoint of the lift of $\delta\circ\pi(T(p,\tilde{x}(0)))$ to H starting at $\tilde{x}(a)$. We know from (28.7) in [4], p.177, that $\tilde{\delta}$ is well-defined and an isometry to H over δ. From the construction of $\tilde{\delta}$, $\tilde{\delta}\circ\tilde{x}(t) = \tilde{x}(t + a)$ for all $t \in (-\infty,\infty)$.

Now we prove that $\tilde{\delta}\circ\phi = \phi\circ\tilde{\delta}$ for any $\phi \in D$. Since all free homotopy classes of closed curves in M are invariant under δ, $\delta\circ\pi(T(p,\phi p))$ corresponds to ϕ for every point p in H. And $\delta\circ\pi(T(p,\phi p)) = \pi\circ\tilde{\delta}(T(p,\phi p)) = \pi(T(\tilde{\delta}p,\tilde{\delta}\circ\phi p))$. Thus $\tilde{\delta}\circ\phi = \phi\circ\tilde{\delta}$.

Let X and Y be points in $H(\infty)$ which contain $\tilde{x}$ and the reverse geodesic $\tilde{x}_-(t) := \tilde{x}(-t)$, $-\infty < t < \infty$, respectively. Since

$d(\tilde{x}(na),\phi\circ\tilde{x}(na)) = d(\tilde{\delta}^n\circ\tilde{x}(0),\phi\circ\tilde{\delta}^n\circ\tilde{x}(0)) = d(\tilde{\delta}^n\circ\tilde{x}(0),\tilde{\delta}^n\circ\phi\circ\tilde{x}(0))$ $= d(\tilde{x}(0),\phi\circ\tilde{x}(0))$ for all integer n, X (or Y) contains $\phi\circ\tilde{x}$ (or $\phi\circ\tilde{x}_-$) and therefore X and Y are common fixed points of D. Since A is a straight line, X and Y are almost D-minimizing and hence the assumption of Proposition 5 is satisfied. This completes the proof.

It is necessary for δ to leave all free homotopy classes of closed curves in M invariant. In fact, there is an example of a surface S of nonpositive Gaussian curvature in the Euclidean 4-space E^4 which is not isometric to a flat cylinder and has two ends and on which a non-trivial isometry δ exists and translates a straight line on S along itself but δ does not leave all free homotopy classes of closed curves invariant.

Example 6. The S is constructed as a union of countably many congruent flat tori in E^4 with two plane disks removed and countably many congruent topological cylinders $S^1 \times [0,1]$ which are joined along their boundary circles.

The construction is done by putting congruent flat tori, to which cylinders are attached, into a deliberate order in such a way that each tours contains 2 parallel plane squares along some line from which disks are removed, and the plane squares on the tori and the boundary circles of cylinders are all parallel along the line in E^4.

Let $(\alpha_1(t),\alpha_2(t))$, $0 \leqq t \leqq \lambda$, be an arc-length parametrized C^∞-curve in E^2 with $(\alpha_1(0),\alpha_2(0)) = (\alpha_1(\lambda),\alpha_2(\lambda)) = (0,-1)$ such that (1) it contains two segments, $\{(x_1,1) ; -1 \leqq x_1 \leqq 1\}$ and $\{(x_1,-1) ; -1 \leqq x_1 \leqq 1\}$, (2) it is contained in the strip $\{(x_1,x_2) ; -1 \leqq x_2 \leqq 1\}$ and (3) it is symmetric with respect to the origin of E^2. And let X denote its image in E^2. Then we can consider canonically $Y' := X \times X$ as the figure in E^4.

Y' has clearly 4 disjoint flat squared faces and we denote two of them, $\{(x_1,1,x_3,1) ; -1 \leqq x_1, x_3 \leqq 1\}$ and $\{(x_1,-1,x_3,1) ; -1 \leqq x_1, x_3 \leqq 1\}$, by A_1 and A_2 respectively. We remove the disk D_1 (or D_2) from A_1 (or A_2) with center $(0,1,0,1)$ (or $(0,-1,0,1)$) and radius 1/2. And we denote the resulting figure by Y. It should be noted that $x_0(t) := (\alpha_1(t),\alpha_2(t),0,1)$, $1/2 \leqq t \leqq (\lambda - 1)/2$, is a shortest join from ∂D_2 to ∂D_1 in Y. Because in the universal covering space H of Y' any lift of x_0 to H is a distance minimizing segment from the lift of ∂D_2 to the lift of ∂D_1.

Y is joined with a certain topological cylinder $S^1 \times [0,\mu]$ along their boundary circles, and this is done in the affine $(x_1,x_2,x_3,1)$-subspace as follows. Let $c(t) := (\beta_1(t),\beta_2(t),0,1)$, $0 \leqq t \leqq \mu$, be an arc-length parametrized C^∞-convex curve in the $(x_1,x_2,0,1)$-space with $c(\mu) = (1/2,2,0,1)$ such that (1) c does not intersect the x_2-axis, (2) c starts at $b := (1/2,1,0,1) = (\alpha_1((\lambda - 1)/2),\alpha_2((\lambda - 1)/2),0,1)$, (3) c contains the segment such that $c([0,1/4]) = \{(x_1,1,0,1) ; 1/4 \leqq x_1 \leqq 1/2\}$ and (4) c is symmetric with respect to the line $\{(x_1,3/2,0,1) ; -\infty < x_1 < \infty\}$ in the $(x_1,x_2,0,1)$-space. Revolving c about the x_2-axis in the $(x_1,x_2,x_3,1)$-space produces a surface C with boundary ∂D_1 and $\partial D_1'$ which is congruent to ∂D_1 and hence ∂D_2. Attaching C to Y along ∂D_1 we obtain a surface with boundary $\partial D_1' \subset C$ and $\partial D_2 \subset Y$, and they are on the parallel planes normal to the x_2-axis in the $(x_1,x_2,x_3,1)$-space. We denote this surface by W.

For each $i = 0, \pm 1, \pm 2, \ldots$, let ϕ_i be a translation along the x_2-axis in E^4 such that $\phi_i(x_1,x_2,x_3,x_4) = (x_1,x_2+3i,x_3,x_4)$. Thus $S := \bigcup \phi_i W$ is the desired surface, namely, S is of non-positive curvature and has an isometry δ which satisfies the assumption of Theorem 4 except for invariancy of all free homotopy classes of closed curves under δ.

The desired δ is obtained by putting $\delta = \phi_1$. Clearly δ does not leave all free homotopy classes of closed curves in S invariant. Now we find a straight line A with a representation $x(t)$, $-\infty < t < \infty$, which is translated by δ. Let $x_0(t)$, $1/2 \leqq t \leqq (\lambda - 1)/2$, be the distance minimizing geodesic segment in W which is already realized, i.e., the endpoints are $(\alpha_1(1/2), \alpha_2(1/2),0,1) = (1/2,-1,0,1) \in \partial D_2$ and $(\alpha_1((\lambda - 1)/2),\alpha_2((\lambda - 1)/2),0,1) = (1/2,1,0,1) \in \partial D_1$. Then ∂D_1 and ∂D_2 are plane circles, so $x_0([1/2,(\lambda - 1)/2])$ is perpendicular to both ∂D_1 and ∂D_2, and therefore the extention of $x_0(t)$, $1/2 \leqq t \leqq (\lambda - 1)/2$, into C in W is nothing but a profile curve, and the endpoint $(1/2, 2,0,1)$ is identified with $\phi_1(1/2,-1,0,1) = (1/2,2,0,1) \in \phi_1 \partial D_2$. Hence by iterating this step $x_0(t)$, $1/2 \leqq t \leqq (\lambda - 1)/2$, is smoothly extended in S and the resulting geodesic is identified with $\bigcup \phi_i x_0([1/2,(\lambda - 1)/2]) =: A$. Because $x_0(t)$, $1/2 \leqq t \leqq$

$(\lambda - 1)/2$, is distance minimizing from ∂D_2 to ∂D_1 and hence so is $\phi_i x_0(t)$, $1/2 \leqq t \leqq (\lambda - 1)/2$, from $\phi_i \partial D_2$ to $\phi_i \partial D_1$, for every i, and because so is a profile curve of $\phi_i C$, for each i, A is a straight line. And δ obviously translates A along itself.

References

[1]. Abe, Y., Kubota, T. and Yoneguchi, H., Some properties of a set of points in Euclidean space, Kôdai Math. Sem. Rept., (1950) 117-119.

[2]. Beltrami, E., Teoria Fondamentale dogli spazii di curvatura costante, Annali di matem. (2) 2, (1868-69) 232 -255.

[3]. Brunn, H., Ueber das durch eine beliebige endliche Figur bestimmte Eigebilde, Boltzmann-Festschrift, Leipzig, (1904) 94-104.

[4]. Busemann, H., The geometry of geodesics, Academic Press, New York, 1955.

[5]. Busemann, H., Recent synthetic differential geometry, Springer, Berlin-Heidelberg-New York, 1970.

[6]. Busemann, H. and Phadke, B. B., Peakless and monotone functions on G-spaces, to appear in Tsukuba J. Math.

[7]. Cartan, É., L'axiome du plan et la géométrie différentielle metrique, Œuvres complètes, Partie III, 2, Gauthier-Villars, Paris, 1955.

[8]. Cartan, É., Leçons sur la géométrie des espaces de Riemann, Paris, 1928.

[9]. Cheeger, J. and Ebin, D. G., Comparison theorems in Riemannian geometry, North Holland, Amer. Elsevier, 1975.

[10]. Cheeger, J. and Gromoll, D., The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geometry 6 (1971) 119-129.

[11]. Cheeger, J. and Gromoll, D., On the structure of complete manifolds of nonnegative curvature, Ann. of Math. 96 (1972) 415-443.

[12]. Cohn-Vossen, S., Kürzeste Wege und Totalkrummung auf Flächen, Compositio Math. 2 (1935) 63-133.

[13]. Eberlein, P. and O'Neill, B., Visibility manifolds, Pacific J. of Math. 46 (1973) 45-109.

[14]. Eschenburg, J.-H., Horospheres and the stable part of the geodesic flow, Math. Z. 153 (1977) 237-251.

[15]. Greene, R. E. and Shiohama, K., Convex functions on complete noncompact manifolds: Topological structure, Inventions Math. 63 (1981) 129-157.

[16]. Greene, R. E. and Shiohama, K., Convex functions on complete noncompact manifolds: Differentiable structure, Ann. Sci. Ecole Norm. Sup., 14 (1981) 357-367.

[17]. Gromoll, D., Klingenberg, W. and Meyer, W., Riemannsche geometrie im Grossen, Lecture Notes in Math. 55, Springer, Berlin-Heidelberg-New York, 1968.

[18]. Gromoll, D. and Meyer, W., On complete open manifolds of positive curvature, Ann. of Math., 90 (1969) 75-90.

[19]. Helgason, S., Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, 1978.

[20]. Hurewicz, W. and Wallmann, H., Dimension theory, Princeton Univ. Press, Princeton, New Jersey, 1948.

[21]. Innami, N., A classification of Busemann G-surfaces which possess convex functions, Acta Math., 148 (1982) 15-29.

[22]. Innami, N., The axiom of n-planes and convexity in Riemannian manifolds, to appear in J. Math. Soc. Japan.

[23]. Innami, N., Splitting theorems of Riemannian manifolds, to appear in Compositio Math.

[24]. Innami, N., Differentiability of Busemann functions and total excess, Math. Z., 180 (1982) 235-247.

[25]. Innami, N., Convexity in Riemannian manifolds without focal points, Preprint.

[26]. Klein, F., Ueber die sogenannte nicht-euklidische Geometrie, Math. Annalen, 6 (1873).

[27]. Leung, D. S. and Nomizu, K., The axiom of spheres in Riemannian geometry, J. Differential Geometry, 5 (1971) 487-489.

[28]. Lewis, G. M., Cut loci of points at infinity, Pacific J. of Math., 43 (1972) 675-690.

[29]. Maeda, M., On the existence of rays, Sci. Rep. Yokohama Nat. Univ. Sect. I 26 (1979) 1-4.

[30]. Maeda, M., Remarks on the distribution of rays, Preprint.

[31]. Nasu, Y., On asymptotic conjugate points, Tôhoku Math. J. Ser. 2, 7 (1955) 159-165.

[32]. Nasu, Y., On asymptotes in metric spaces with non-positive curvature, Tôhoku Math. J. Ser. 2, 9 (1957) 68-95.

[33]. Nasu, Y., On asymptotes in two-dimensional metric space, Tensor New Ser., 7 (1957) 173-184.

[34]. Roberts-Verberg, Convex functions, Academic Press,

New York, 1973.

[35]. Rockafellar, R. T., Convex analysis, Princeton Univ. Press, Princeton, New Jersey, 1972.

[36]. Schlaefli, L., Nota alla Memoria del sig. Beltrami, (Sugli spazii di curvatura costante), Annali di matem. (2) 5, (1871-73).

[37]. Schur, F., Ueber den Zusammenhang der Räume constanten Riemannschen Krümmungsmasses mit den projectiven Räumen, Math. Ann., t 27 (1886) 537-567.

[38]. Shiohama, K., Busemann function and total curvature, Inventiones Math., 53 (1979) 281-297.

[39]. Shiohama, K., A role of total curvature on complete non-compact Riemannian 2-manifolds, Preprint.

[40]. Sierpiński, W., Ueber den kleinsten konvexen Bereich, der eine gegebene Menge enthält, (in Polish), Wektor (Math. Phys. Z.), 3 (1914) 268-307.

[41]. Thorbergsson, G., Closed geodesics on non-compact Riemannian manifolds, Math. Z., 159 (1978) 245-258.

[42]. Toponogov, V. A., Spaces with straight lines, Amer. Math. Soc. Transl., 37 (1964) 278-280.

[43]. Valentine, F. A., Convex sets, Robert E. Krieger Publishing company, New York, 1976.

[44]. Whitehead, J. H. C., Convex regions in the geometry of paths, Quart. J. Math., 3 (1932) 33-42.

[45]. Wu, H., An elementary method in the study of nonnegative curvature, Acta Math., 142 (1979) 57-78.

[46]. Yamaguchi, T., The isometry groups of Riemannian manifolds admitting strictly convex functions, to appear in Ann. Sci. Ecole Norm. Sup.