FOURIER ANALYSIS AND LATTICE POINT PROBLEMS

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Preface

Fourier analysis and lattice point problems are classical fields with a long history. The study of lattice point problems flourished in the beginning of the 20th century. The outstanding mathematicians in lattice point problems are Hardy, Littlewood, Landau, Walfisz, Szegö, Jarnik and Cramér. Recently, B. Novák and W. G. Nowak have been very active.

On the other hand, Fourier analysis, in particular, convergence problems (almost everywhere convergence, norm convergence, and summability) has been studied by Hardy, Littlewood, Paley, Zygmund, Marcinkiewicz, M. Riesz and Kolmogorov, and others. In regards to pointwise convergence problems of Fourier series, Kolmogorov [1925, 1926] proved that the Fourier series of a certain L^1- function diverges everywhere. On norm convergence M. Riesz [1927] proved that the Fourier series of L^p-functions converges in norm if $p > 1$. On the other hand, the problem that the Fourier series of $L^2(T^1)$-functions converges almost everywhere (i.e., Lusin's conjecture) continued to attract the attention of many mathematicians. About 40 yeas after M. Riesz's proof the long story was finally closed in 1966, when Carleson proved the Lusin conjecture.

While the interest of Fourier analysis researchers was focused on one variable cases, the two fields couldn't readily intesect. Since 1966, the central concern of Fourier analysis researchers has been *several variables*. In the 1930's, Bochner solved the localization property of multiple Fourier series at *critical index*. This shows that there is a delicate difference between Fourier integrals and series. In the late 1950's Stein started an extensive study with respect to Fourier analysis of several variables and he obtained many striking results. One particular sentence in one of Stein's papers [1961] was unforgetable for me. He wrote "*We let S_R^δ be the Riesz means of the Fourier expansion*

$$S_R^\delta(x,f) = \sum_{|n|\le R}\left(1-\frac{|n|^2}{R^2}\right)^\delta a_n e^{inx}.$$

One of the main difficulties with this problem is due to our incomplete knowledge of the behavior of kernels

$$(1,1) \qquad \sum_{|n|\le R}\left(1-\frac{|n|^2}{R^2}\right)^\delta e^{inx}, \qquad 0\le\delta\le\frac{(k-1)}{2}$$

In the previous paper we found that, neverthless, one can obtain a variety of results by by-passing a direct estimation of (1.1). It might be worthwhile to observe that a rather complete knowledge of the behavior of the kernels (1.1) would bring with it a solution to some outstanding problems in the theory of numbers. (E.g., take $x = 0$, $\delta = 0$, $k = 2$, then the sum (1,1) becomes the number of lattice points in a circle of radius R)". This sentence was the first to impress upon us the necessity of studying not only Fourier analysis but also lattice point problems. In the 1970's C. Fefferman made several very important contributions on multiple Fourier series. He won the Fields Prize at the International Congress of Mathematicians 1978 at Helsinki. He became the first medalist in the field of Fourier analysis.

In the late 1960's, fortunately, a Czechoslovakian mathematician, B. Novák, started the study of lattice point problems with weight which is perfectly related to *the knowledge of the behavior of kernels.* We think that the two fields of multiple Fourier series and lattice point problems do not have a great deal in common. In this paper we will consider some of the mutual points of contact between them.

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Chapter I. Fourier analysis

Section 1. A simple summary of Fourier series of one variable

Let $f(x)$ be an element of $L^1[0,1]$ (i.e., a function integrable in $[0,1]$ and periodic with the period 1). We put

$$\hat{f}(m) = \int_0^1 f(x)e^{-2\pi imx}dx \quad (m \in Z),$$

and call $\hat{f}(m)$ the Fourier coefficients of f. The formal series

$$\sum_{m\in Z} \hat{f}(m)e^{2\pi imx}$$

is called the Fourier series of f. The Nth partial sums $S_N(f;x)$ of the Fourier series of f can be written in the form

$$S_N(f;x) = S_N(f)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+t)D_N(t)dt,$$

where

$$D_N(t) = \frac{sin(2\pi(N+\frac{1}{2})t)}{sin(\pi t)}$$

and the function $D_N(t)$ is called the Dirichlet kernel.

The convergence problem of Fourier series is: *When does $S_N(x)$ converge to $f(x)$?.*

The following are the important results in the history of the convergence problem of Fourier series.

THEOREM (P. de Bois-Reymond [1876]). *There exists a continuous function such that its Fourier series diverges at a point.*

THEOREM (A. N. Kolmogorov [1926]). *There exists an integrable function such that its Fourier series diverges everywhere.*

THEOREM (M. Riesz [1927]). *If* $1 < p < +\infty$, *then* $\|S_N(f) - f\|_p$ *converges to* 0 *as* $N \to +\infty$ *for every* $f \in L^p$.

(This result fails to hold for $p = 1$ *or* $p = +\infty$*)*

THEOREM (J. P. Kahane and Y. Katznelson [1966]). *For any given null set* E, *there exists a continuous function such that its Fourier series diverges on every point in* E.

THEOREM (L. Carleson [1966] and R. A. Hunt [1967]). *If* $p > 1$, *then Fourier series of* f *belonging to* L^p *converges almost everywhere.*

Section 2. Multiple Fourier series

We consider the n dimensional Euclidean space R^n, whose elements are denoted by $x = (x_1, \ldots, x_n)$, and for $x, y \in R^n$ we put

$$xy = (x, y) = x_1 y_1 + \ldots + x_n y_n \ , \ |x| = (x, x)^{\frac{1}{2}}.$$

The set T^n of all vectors x with $-1/2 < x_\nu \le 1/2$ for each $\nu = 1, \ldots, n$, that is, $T^n = (-1/2, 1/2]^n$ is called the n dimensional torus. The set $Z^n \subset R^n$ of all vectors with integral coordinates is called the integer lattice in R^n, and the elements of Z^n are denoted by $m = (m_1, \ldots, m_n)$, $mx = m_1 x_1 + \ldots + m_n x_n$, and $|m|^2 = m_1^2 + \ldots + m_n^2$. We call the element of Z^n *the lattice point.*

Let $f(x)$ be an element of $L(T^n)$ (i.e., a function integrable in T^n and periodic in each variable with the period 1). We put

$$\hat{f}(m) = \int_{T^n} f(x) e^{-2\pi i m x} dx, \quad m \in Z^n$$

and call $\hat{f}(m)$ *the Fourier coefficients* of f. The formal series

$$S[f] = \sum_m \hat{f}(m) e^{2\pi i m x}$$

is called *the Fourier series* of f. Thus Fourier coefficients and Fourier series are formally the same as in one variable case. But in the case of several variables, how the concept of partial sums of the series should be taken is an essential problem. The concepts which are studied most extensively are the square partial sum and the spherical partial sum.

a) *The square partial sum* $S_k(f;x)$, specified by a natural number k, has the form

$$S_k(f;x) = \sum_{m_1=-k}^{k} \cdots \sum_{m_n=-k}^{k} \hat{f}(m) e^{2\pi i m x}.$$

We consider that $S[f]$ converges squarely to A if $S_k(f;x)$ converges to A as $k \to +\infty$.

b) *The spherical partial sum* $S_t(f;x)$ specified by a positive number t, has the form

$$S_t(f;x) = \sum_{|m|^2<t} \hat{f}(m)e^{2\pi imx}.$$

We say that $S[f]$ converges spherically to A if $S_t(f;x)$ converges to A as $t \to +\infty$.

The square and spherical partial sums of an $S[f]$ behave in many respects quite differently. Each type requires a different technique to treat and appeals to different properties of f. For example, C. Fefferman proved the almost everywhere convergence problem for the square partial sum by applying the Carleson-Hunt theorem, but the corresponding problem for the spherical case has not been proved at the present moment.

THEOREM (C. Fefferman [1971]). *For $p > 1$, $S[f]$ converges squarely to $f(x)$ almost everywhere for $f \in L^p(T^n)$.*

(Fefferman also proved this theorem in regards to *" polygonal")*

From now on we will consider exclusively the spherical partial sum, because our interest is in intersections between Fourier analysis and lattice point problems. Moreover, the spherical partial sum is natural from the point of view of eigenvalue problems for Laplacian on T^n. S. Bochner refers to the following:

The elementary exponentials

$$u(x) = e^{i(n_1x_1+\ldots+n_kx_k)}$$

(all $n_1, \ldots, n_k$ integers) are a complete set of regular solutions of the characteristic value problem

$$\triangle u(x) = -\lambda u(x),$$

if this equation is being considered on the (closed) torus

$$0 \leq x_1 < 2\pi, \ldots, 0 \leq x_k < 2\pi,$$

and $\triangle u(x)$ is the Laplace operator with respect to the Euclidean metric on the torus, *namely,*

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_k^2}.$$

Since $\lambda = n_1^2 + \ldots + n_k^2$, *our way of writing series* (1)* *in the the form* (5)** *satisfies the very natural principle of ordering the terms in series (1) according to the magnitude of the characteristic values* λ.

(N. B.,* $f(x) \sim \sum a_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$,

** $\sum_{j=0}^{+\infty} \sum_{\nu=j} a_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$; $\nu^2 = n_1^2 + \ldots + n_k^2$)

(S. Bochner [1936], p. 179)

Section 3. The Bochner-Riesz means of multiple Fourier series

We introduce the Bochner-Riesz means of $S[f]$, i.e., $\alpha > -1$, $t > 0$

$$S_t^{\alpha}(f;x) = \sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} \hat{f}(m) e^{2\pi i m x}.$$

We say that $S[f]$ is Bochner-Riesz summable of order α to A if $S_t^{\alpha}(f;x)$ converges to A as $t \to +\infty$. The Bochner-Riesz means have the following property.

a) For $\alpha > -1$,

$$S_t^{\alpha}(f;x) = \int_{T^n} f(x-y) K_{\alpha}(t;y)\,dy,$$

where

$$K_{\alpha}(t;y) = \sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} e^{2\pi i m y}.$$

(We call this kernel *the Bochner-Riesz kernel of order* α)

b) For $\alpha > -1$,

$$S_t^{\alpha}(f;x) = \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha} dS_s^0(f;x),$$

where the right integral means the Riemann-Stieltjes integral on variable s.

c) $For \alpha > 0$,

$$S_t^{\alpha}(f;x) = \frac{\alpha}{t} \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} S_s^0(f;x)\,ds.$$

We set $p(\alpha) = \frac{2n}{(n+1+2\alpha)}$, and also $p'(\alpha) = \frac{2n}{(n-1-2\alpha)}$ which is the dual exponent of $p(\alpha)$ if $\alpha \le \frac{n-1}{2}$. Let the sets S, U, H be the following:

$S = \{(\alpha, p) : 1 \le p \le \infty$ and $\alpha > (n-1)|\frac{1}{p} - \frac{1}{2}|\}$ (Stein's domain),

$U = \{(\alpha, p) : 0 \le \alpha \le \frac{n-1}{2}$ and $p(\alpha) < p < p'(\alpha)\}$ (Unknown domain),and

$H = \{(\alpha, p) : \alpha \ge 0$ and $p(\alpha) \ge p$ or $p'(\alpha) \le p\}$ (Herz's domain).

THEOREM (S. Bochner [1936]). There exists a function $f \in L^1(T^n)$ vanishing in a neighborhood of the origin such that

$$\limsup_{t \to +\infty} \left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^{\frac{n-1}{2}} \hat{f}(m) \right| = +\infty$$

(In Fourier integrals case, it is well known that this *localization principle* holds true for *critical index* $\alpha = (n-1)/2$. See Alimov-Ashurov-Pulatov [1990], p. 34)

THEOREM (S. Bochner [1936] and E. M. Stein [1958]).

(1) If $(\alpha, p) \in S$, then for each function $f \in L^p(T^n)$, $S_t^\alpha(f; x)$ converges in norm to $f(x)$.

and

(2) If $1 \le p \le 2$ and $(\alpha, p) \in S$, then for each function $f \in L^p(T^n)$, $S_t^\alpha(f; x)$ converges almost everywhere to $f(x)$.

THEOREM (C. S. Herz [1954] and K. I. Babenko [1973]).

(1) If $(\alpha, p) \in H$, then for some function $f \in L^p(T^n)$, $S_t^\alpha(f; x)$ diverges in norm.

and

(2) If $1 \le p \le 2$ and $(\alpha, p) \in H$, then for some function $f \in L^p(T^n)$, $S_t^\alpha(f, x)$ diverges on a set of positive measure.

In Chapter 3 some functions act as a go-between of Fourier analysis and lattice point problems. The following lemma is seen in Stein-Weiss's text ([1971], p. 256) and Wainger's memoirs ([1965], Theorem 7).

LEMMA. Let $0 < \sigma < n$ *and* $\tau > 0$. *Then the series* $\sum_{m \neq 0} \left(|m|^\sigma \log^\tau |m| \right)^{-1} e^{2\pi i m x}$ *is the Fourier series of an integrable function on* T^n *which is of class* C^∞ *on* $T^n - \{0\}$. *At the origin this function has the same singularity as the function whose values are* $\gamma_{\sigma\tau} |x|^{\sigma - n} \log^{-\tau} |x|^{-1}$. *That is,*

$$\gamma_{\sigma\tau} |x|^{\sigma-n} \log^{-\tau} |x|^{-1} + b(x) \sim \sum_{m \neq 0} \frac{1}{|m|^\sigma \log^\tau |m|} e^{2\pi i m x} \qquad (x \in T^n),$$

where $b \in C^\infty(T^n)$ *and* $\gamma_{\sigma\tau}$ *is a constant number.*

Throughout this thesis we write the function as $\zeta_{\sigma\tau}$, that is,

$$\zeta_{\sigma\tau}(x) \sim \sum_{m \neq 0} \frac{1}{|m|^\sigma \log^\tau |m|} e^{2\pi i m x}.$$

Therefore $\zeta_{\sigma\tau} \in L^p(T^n)$ is an integrable function if and only if

$$\int_0^1 r^{p(\sigma - \frac{n}{p'}) - 1} \log^{-p\tau} dr < +\infty \text{ , that is, } \sigma > n/p', \text{ or } \sigma = n/p' \text{ and } \tau > 1/p.$$

Section 4. Multiple Fourier integrals

Let $f(x)$ be an element of $L(R^n)$ (i.e., a function integrable on R^n). We put

$$\hat{f}(y) = \int\limits_{R^n} f(x) e^{-2\pi i y x} dx,$$

and call it the Fourier integral of f. Further we put the Bochner-Riesz means of order α:

$$R_t^\alpha(f;x) = \int\limits_{|y|<t} \left(1 - \frac{|y|^2}{t}\right)^\alpha \hat{f}(y) e^{2\pi i x y} dy = \int\limits_{R^n} f(x-y) R_\alpha(t;y) dy,$$

where

$$R_\alpha(t;x) = \int\limits_{|y|^2<t} \left(1 - \frac{|y|^2}{t}\right)^\alpha e^{-2\pi i x y} dy = \frac{\Gamma(\alpha+1)}{\pi^\alpha} t^{\frac{1}{2}(\frac{n}{2}-\alpha)} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|x|)}{|x|^{\frac{n}{2}+\alpha}}.$$

In multiple Fourier integrals the convergence problems are also very important.

When does $R_t^\alpha(f;x)$ converge in norm (or almost everywhere) to $f(x)$?

We have the following:

THEOREM (C. S. Herz [1954]). If $(\alpha,p) \in H$, then there exists a function $f \in L^p(R^n)$ such that $R_t^\alpha(f;x)$ diverges in norm.

THEOREM (C. Fefferman, 1970). If $\alpha > \frac{n-1}{4}$ and $(\alpha,p) \in U$, then for each function $f \in L^p(R^n)$, $R_t^\alpha(f;x)$ converges in norm to $f(x)$.

THEOREM (L. Carleson and P. Sjölin [1972]). *If* $\alpha > 0$ *and* $(\alpha, p) \in U$, *then for each function* $f \in L^p(R^2)$, $R_t^\alpha(f; x)$ *converges in norm to* $f(x)$.

THEOREM (C. Fefferman [1972]). $R_t^0(f; x)$ *diverges in norm unless* $n = 1$ *or* $p = 2$.

Thus the research of norm convergence in $L^p(R^n)$ is further developed than of that in $L^p(T^n)$. But the problem of almost everywhere convergence of $R_t^\alpha(f; x)$ has not seen much progress although there are some partial results (Christ [1985], Kanjin [1988] and Prestini [1988], et al.). Of course the case of $S_t^\alpha(f; x)$ has not yet been developed either.

Section 5. Unsolved problems in multiple Fourier series

When α and p satisfy the condition of $(\alpha, p) \in U$ i.e.,

$$1 \le p \le 2 \text{ and } n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} < \alpha \le (n-1)\left(\frac{1}{p} - \frac{1}{2}\right),$$

the problems on norm or almost everywhere convergence in $L^p(T^n)$ remain unsolved. We refer to the most important problems in multiple Fourier series as *Problems 1* and *2.*

PROBLEM 1.

When $\alpha > 0, n \ge 2$ and $p(\alpha) < p < p'(\alpha)$, does $S_t^\alpha(f;x)$ converge in norm?

PROBLEM 2.

When $\alpha > 0$, $n \ge 2$ and $2 \ge p > p(\alpha)$ (in other words, $1 \le p \le 2$ and $\alpha > \frac{n-1}{2} - \frac{n}{p'}$), does $S_t^\alpha(f;x)$ converge almost everywhere?

Stein proposed problems of the same type in the case of Fourier integrals ([1979], Problem 1 and Problem 2). The convergence problem of Fourier series is more difficult than of Fourier integrals. Further, the almost everywhere convergence problem is more difficult than the norm convergence problem. Actually, in the case of one variable, M. Riesz solved the norm convergence problem in 1927, then, 40 years later Carleson solved the big problem. I think that, Problem 2 especially is extremely difficult.

Chapter II. Lattice point problems

Section 1. Gauss's circle problem and Hardy's conjecture

The arithmetical function $r_2(k)$ counts the number of representations of a nonnegative integer k as a sum of two integral squares: in other words, the number of solutions of eqation $x^2+y^2=k$, in integers x, y.

We denote by $A(t)$ the number of lattice points on a circle whose center is the origin and whose radius is $\sqrt{t}$, i.e., $A(t)=\sum_{0\le k<t} r_2(k)$. The magnitude of $A(t)$ is approximately equal to πt, the area of the circle. We put $P(t)=A(t)-\pi t$. The problem of estimating $P(t)$ is called Gauss's circle problem. The following are well known:

$$P(t)=O(t^{\frac{13}{40}+\varepsilon}),$$

$$P(t)=\Omega(t^{\frac{1}{4}}(\log t)^{\frac{1}{4}}),$$

($f(t)=\Omega(g(t))$ means $f(t)\neq o(g(t))$)

and

$$\left(\frac{1}{t}\int_0^t |P(s)|^2 ds\right)^{\frac{1}{2}} = Ct^{\frac{1}{4}} + O\left(\frac{(\log t)^2}{t^{\frac{1}{4}}}\right).$$

(cf. Encyclopedic Dictionary of Mathematics (2nd ed.) [1987], p. 892)

We will use ϑ as the infimum of the set $\{\alpha : P(t)=O(t^\alpha)\}$. G. H. Hardy conjectured that $\vartheta=\frac{1}{4}$. The best result up to now is $35/108$ $(=0.324074\ldots)$ by W. G. Nowak [1984].

Section 2. Lattice point problems with weight

As an extension of Gauss's circle problem, it is natural to consider the number of lattice points $m = (m_1, \ldots, m_n)$ satisfying $m_1^2 + \ldots + m_n^2 < t$.

This problem is extended to a case in which the weight $e^{2\pi i(m_1x_1+\ldots+m_nx_n)}$ is placed at each lattice point. We call this problem *lattice point problem with weight in n dimensional sphere.* Then our interest is to estimate the following :

$$A(t; x, y) = \sum_{|m+y|^2<t} e((m+y)x),$$

$$P(t; x, y) = A(t; x, y) - \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} e(xy)\delta(x),$$

where $e(s) = e^{2\pi is}$, and $\delta(x) = 1$ or 0 according to whether $x \in Z^n$ or not.

For exmple, in the case of $n = 1$, the following equalities are used often:

$$\sum_{|m|\le N} e^{2\pi imx} - \int_{-N}^{N} e^{2\pi iyx} dy = \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin \pi x} - \frac{\sin(2\pi Nx)}{\pi x}$$

$$= \sin(2\pi Nx)\left(\frac{\pi x - \tan \pi x}{\pi x \tan \pi x}\right) + \cos(2\pi Nx) = O(1).$$

The following Landau's result is important in the sense that it is true for general dimension n .

THEOREM (E. Landau [1915]).

$$P(t; 0, 0) = O(t^{\frac{n}{2} - \frac{n}{n+1}}).$$

In particular,

$$A(t,0,0) = \sum_{|m|^2<t} 1 = \sum_{0\le k<t} r_n(k),$$

where $r_n(k)$ = number of lattice points $m = (m_1, \ldots, m_n)$ satisfying

$$m_1^2 + \ldots + m_n^2 = k.$$

Vinogradov (1960) obtained deeper results for a special case $n = 3$:

$$P(t;0,0) = O(t^{\frac{19}{28}+\varepsilon}).$$

We will use estimations on $r_n(k)$ by Siegel and Malyšev in CHAPTER 3. Then we will introduce the Bochner -Riesz means of multiple trigonometric series:

$$\sum_{m\in Z^n} e((m+y)x).$$

For $\alpha > -1$,$t \ge 0$, $m \in Z^n$ and x, $y \in R^n$,

$$A_\alpha(t;x,y) = \frac{1}{\Gamma(\alpha+1)} \sum_{|m+y|^2<t} (t - |m+y|^2)^\alpha e((m+y)x),$$

$$P_\alpha(t;x,y) = A_\alpha(t;x,y) - \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n}{2}+\alpha+1)} e(xy)\delta(x),$$

where

$$\frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n}{2}+\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \int_{|y|^2<t} (t-|y|^2)^\alpha dy.$$

We point out the relation between the kernel of the Bochner-Riesz means for order α, and $A_\alpha(t;x,0)$:

$$K_\alpha(t;x) = \frac{\Gamma(\alpha+1)}{t^\alpha} A_\alpha(t;x,0).$$

We call the problem of estimating $P_\alpha(t;x,y)$ *"lattice points problem"* throughout our Thesis.

THEOREM (E. Landau [1915] and V. Jarnik [1968]).

$$P_\alpha(t;0,0) = \begin{cases} O\left(t^{\frac{n-1}{4}+\frac{\alpha}{2}}\right) & if \alpha > \frac{n-1}{2}, \\ O\left(t^{\frac{n-1}{4}+\frac{\alpha}{2}}\log t\right) & if \alpha = \frac{n-1}{2}, \\ O\left(t^{\frac{n}{2}+\alpha-\frac{n}{n+1-2\alpha}}\right) & if 0 \leq \alpha < \frac{n-1}{2}. \end{cases}$$

THEOREM (G. Szegö [1926], B. C. Berndt [1971] and J. L. Hafner [1982]).

$$P_\alpha(t;0,0) = \begin{cases} \Omega\left(t^{\frac{n-1}{4}+\frac{\alpha}{2}}\right) & if \alpha > \frac{n-1}{2}, \\ \Omega\left(t^{\frac{n-1}{4}+\frac{\alpha}{2}}\log\log t\right) & if \alpha = \frac{n-1}{2} \text{ and } n \text{ is even }, \\ \Omega\left(t^{\frac{n-1}{4}+\frac{\alpha}{2}}(\log t)^{\frac{n-1-2\alpha}{2n}}\right) & if 0 \leq \alpha < \frac{n-1}{2}. \end{cases}$$

Berndt and Hafner extended Szegö's theorem to more general lattice point problems. Their method is based on ideas used by Szegö to obtain this theorem. This Szegö's idea can be used in cases with weight. We carry it out in Section 6 of Chapter 3.

A merit of consideration of two variables x and y can be found in the following property. This proposition is proven in TYPICAL MEANS [1952] (p. 126) by K. Chandrasekharan & S. Minakshisundaram in the case of $x = 0$ and Stein-Weiss' text [1971] (p. 255) in the case of $y = 0$.

Proposition. *$P_\alpha(t,x,y)$ is a perodic function with the period 1 about variable y, and its Fourier expansion is the following :*

$$P_\alpha(t;x,y) \sim \frac{t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)}}{\pi^\alpha} \sum_{m-x\neq 0} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}} e^{2\pi i m y},$$

where function J_β is the Bessel function for order β.

Proof. We have

$$A_\alpha(t;x,y) = \frac{1}{\Gamma(\alpha+1)} \sum_{m\in Z^n} 1_{B(0,t)}(m+y)(t-|m+y|^2)^\alpha e^{2\pi i(m+y)x}$$

$$= \sum_{m\in Z^n} \varphi(m+y),$$

where $\varphi(y) = \frac{1}{\Gamma(\alpha+1)} 1_{B(0,\sqrt{t})}(y)(t-|y|^2)^\alpha e^{2\pi i y x}$. We well know:

$$\hat{\varphi}(\omega) = \frac{t^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\pi^\alpha} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|\omega - x|)}{|\omega - x|^{\frac{n}{2}+\alpha}}.$$

$$\int_{T^n} A_\alpha(t;x,y) e^{-2\pi i k y} dy = \int_{T^n} \{ \sum_{m\in Z^n} \varphi(m+y)\} e^{-2\pi i k y} dy$$

$$= \sum_{m\in Z^n} \int_{T^n - m} \varphi(y) e^{-2\pi i k(y-m)} dy$$

$$= \sum_{m\in Z^n} \int_{T^n - m} \varphi(y) e^{-2\pi i k y} dy$$

$$= \int_{R^n} \varphi(y) e^{-2\pi i k y} dy = \hat{\varphi}(k),$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_{|y|^2<t} (t-|y|^2)^\alpha dy = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+\alpha+1)} t^{\frac{n}{2}+\alpha}.$$

Therefore we have

$$\int_{T^n} A_\alpha(t;x,y) e^{-2\pi i k y} dy = \begin{cases} \frac{t^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\pi^\alpha} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|k-x|)}{|k-x|^{\frac{n}{2}+\alpha}} & if k - x \neq 0, \\ \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+\alpha+1)} t^{\frac{n}{2}+\alpha} & if k - x = 0. \end{cases}$$

$$A_\alpha(t;x,y) \sim \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n}{2}+\alpha+1)} e^{2\pi i x y} \delta(x) + \sum_{m-x\neq 0} \frac{t^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\pi^\alpha} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}} e^{2\pi i m y}.$$

Therefore we have

$$P_\alpha(t;x,y) \sim \sum_{m-x\neq 0} \frac{t^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\pi^\alpha} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}} e^{2\pi i m y}.$$

This completes the proof of the Proposition.

By well known estimations on the Bessel function: $J_\beta(s) \ll \max\{s^\beta, s^{-1/2}\}$. We have the following two Corollaries.

Corollary 1. If $\alpha > (n-1)/2$, the Fourier series of $A_\alpha(t;x,y)$ is absolutely convergent. Especially, in the case of $y = 0$, we have the formula for the Bochner-Riesz kernel of order α:

$$K_\alpha(t;x) = \sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha e^{2\pi i m x}$$

$$= \frac{\Gamma(\alpha+1)t^{\frac{1}{2}(\frac{n}{2}-\alpha)}}{\pi^\alpha} \sum_{m\in Z^n} \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}},$$

where

$$\left.\frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}}\right|_{m-x=0} = \frac{\pi^{\frac{n}{2}+\alpha}t^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\Gamma(\frac{n}{2}+\alpha+1)}.$$

Stein and Weiss obtained this expansion from the Poisson's summation formula ([1971], p. 253). Our method can give a meaning also for the case of $\alpha \le \frac{n-1}{2}$ that each term of expansion is a kind of Fourier coefficient.

Corollary 2. For $\alpha > (n-1)/2$, there exists a positive constant C_α such that

$$\left| \sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha e^{2\pi i m x} - \int_{|y|^2<t} \left(1 - \frac{|y|^2}{t}\right)^\alpha e^{2\pi i y x} dy \right| \le C_\alpha t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}$$

uniformly in $x \in [-1/2, 1/2]^n$.

Proof. It is obvious from the next inequality:

$$\frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|m-x|)}{|m-x|^{\frac{n}{2}+\alpha}} \leq C\frac{t^{-\frac{1}{4}}}{|m|^{\frac{n+1}{2}+\alpha}}$$

for $m \neq 0$.

From corollary 1 we can obtain also the well known fundamental result.

Corollary 3. *For $\alpha > \frac{n-1}{2}$, we have*

$$A_\alpha(t;x) = \begin{cases} O\left(t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)}\right) \\ \Omega\left(t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)}\right) & \textit{for all } x. \end{cases}$$

(cf. Jarnik [1969], the proof of *Satz 4*)

Section 3. On the contributions of B. Novák

Let

$$Q(u) = Q(u_1, \dots, u_n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_i u_j$$

be a positive definite quadratic form with determinant D. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be elements of n dimensional Euclidean space R^n, $M = (M_1, \dots, M_n)$ be $M_j > 0 (j = 1, \dots, n)$ and $m = (m_1, \dots, m_n)$ be n dimensional integral point, i.e., $n \in Z^n$. For $t > 0$, let us consider the function

$$A_Q(t; x) = A_Q(t; x, y, M) \stackrel{\text{def}}{=} \sum_{Q(m \circ M + y) < t} e(x(m \circ M + y)),$$

where $m \circ M = (m_1 M_1, \dots, m_n M_n)$, $x(m \circ M + y) = x_1(m_1 M_1 + y_1) + \dots + x_n(m_n M_n + y_n)$ and $e(x(m \circ M + y)) = e^{2\pi i x(m \circ M + y)}$.

When we put

$$V_Q(t; x) = V_Q(t; x, y, M) \stackrel{\text{def}}{=} \frac{e(xy)}{\sqrt{D} \Pi_{j=1}^n M_j} \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \delta(x \circ M),$$

where $\delta(x \circ M) = 1$ or 0 according to whether $x \circ M \in Z^n$ or not. Then

$$P_Q(t; x) = P_Q(t; x, y, M) \stackrel{\text{def}}{=} A_Q(t; x) - V_Q(t; x)$$

is known as *"lattice rest"*. Further we introduce some notations.

$$A_\alpha(t; x) = A_{Q,\alpha}(t; x, y, M) \stackrel{\text{def}}{=} \sum_{Q(m \circ M + y) < t} \left(t - \frac{Q(m \circ M + y)}{t} \right)^\alpha e(x(m \circ M + y)),$$

$$V_\alpha(t; x) = V_{Q,\alpha}(t; x, y, M) \stackrel{\text{def}}{=} \frac{e(xy)}{\sqrt{D} \Pi_{j=1}^n M_j} \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2} + \alpha}}{\Gamma(\frac{n}{2} + \alpha + 1)} \delta(x \circ M),$$

$$P_\alpha(t;x) = P_{Q,\alpha}(t;x,y,M) \overset{\text{def}}{=} A_{Q,\alpha}(t;x,y,M) - V_{Q,\alpha}(t;x,y,M),$$

and

$$T_\alpha(t;x) = T_{Q,\alpha}(t;x,y,M) \overset{\text{def}}{=} \left(\frac{1}{t}\int_0^t |P_\alpha(s;x)|^2\, ds\right)^{\frac{1}{2}}.$$

Interesting results in *lattice rest* were derived in a series of papers by the Czechoslovakian mathematician B. Novák. He studied in detail, and thoroughly, how the estimations of $P_{Q,\alpha}(t,x,y,M)$ or $T_{Q,\alpha}(t;x,y,M)$ are influenced by the coefficients of Q, that is, u_{ij}, x or y. In this section, we state the part of his study related to our interest. That is, the case of $Q(u) = u_1^2 + \ldots + u_n^2$, $y = 0$ and $M = (1,\ldots,1)$.

I. For $0 \le \alpha < \frac{n}{2} - 2$,

$$P_\alpha(t;x) = \begin{cases} O\left(t^{\frac{n}{2}-1}\right),\ \Omega\left(t^{\frac{n}{2}-1}\right) & \text{for} \quad x \in Q^n, \\ o\left(t^{\frac{n}{2}-1}\right) & \text{for} \quad x \notin Q^n, \\ O\left(t^{\frac{n}{4}+\frac{\alpha}{2}}\log^\tau t\right) & \text{for} \quad \text{almost all } x, \end{cases}$$

where $\tau = 3n - 1$ for $\alpha > 0$ and $\tau = 3n$ for $\alpha = 0$.

([1972] Satz 4, Satz 5, Satz 7)

II. For $x \in Q^n$,

$$T_\alpha(t;x) = \begin{cases} K(x)t^{\frac{n}{2}-1} + O(t^\kappa) & \text{for} \quad 0 \le \alpha < \frac{n-3}{2}, \\ K(x)t^{\frac{n}{2}-1}\log^{\frac{1}{2}} t + O\left(t^{\frac{n}{2}-1}\log^\beta t\right) & \text{for} \quad \alpha = \frac{n-3}{2}, \\ K(x)t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} + O\left(t^{\frac{n-3}{4}+\frac{\alpha}{2}}\right) & \text{for} \quad \alpha > \frac{n-2}{2}, \end{cases}$$

where $K(x)$ is the positive constant number depending only on x, and κ is any number less than $\frac{n}{2} - 1$ and β is any number less than $1/2$.

Consequently,

$$P_\alpha(t;x) = \begin{cases} \Omega\left(t^{\frac{n}{2}-1}\right) & \text{for} \quad 0 \le \alpha < \frac{n-3}{2}, \\ \Omega\left(t^{\frac{n}{2}-1}\sqrt{\log t}\right) & \text{for} \quad \alpha = \frac{n-3}{2}, \\ \Omega\left(t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)}\right) & \text{for} \quad \alpha > \frac{n-2}{2}. \end{cases}$$

([1969] Hauptsatz 1, [1971] Hauptsatz 1, Hauptsatz 2 and [1973] Hauptsatz)

and for all x,

$$t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} \ll T_\alpha(t;x) \ll \begin{cases} t^{\frac{n}{2}-1} & \text{for} \quad 0 \le \alpha < \frac{n-3}{2}, \\ t^{\frac{n}{2}-1}\sqrt{\log t} & \text{for} \quad \alpha = \frac{n-3}{2}, \\ t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} & \text{for} \quad \alpha > \max\{\frac{n-3}{2}, 0\}, \end{cases}$$

and for $n = 2$, $t^{\frac{1}{4}} \ll T_0(t;x) \ll t^{\frac{1}{4}}$.

([1967] Satz 1, [1970] Theorem 3 and [1969] Hauptsatz 3)

$$T_\alpha(t;x) \ll t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} \log^{(3n-1)/2} t \quad \text{for almost all } x.$$

([1970] Theorem 5)

The following results are well known in the Diopantine Approximation Theory:

i) *There are infinitly many integer solutions of* $q^{1/n}P_q(x) < n/(n+1)$, *where* $P_q(x) \stackrel{\text{def}}{=} \max\{\langle qx_\nu\rangle : 1 \le \nu \le n\}$. (Cassels [1965], p. 14, Theorem VII)

ii) *Let* $\psi(q)$ *be a monotonely decreasing function of the integer variable* $q > 0$ *with* $0 \le \psi \le 1/2$. *Then the set of inequality* $P_q(x) < \psi(q)$ *has infinitly many integer solutions* $q > 0$ *for almost no or for almost all sets of* $x = (x_1, \dots, x_n)$ *according as* $\sum(\psi(q))^n$ *converges or diverges.* (Cassels [1965], p. 120, Theorem I)

Let $\gamma(x) \stackrel{\text{def}}{=} \sup\{\tau > 0 : P_k(x) \ll k^{-\tau}$ has infinitely many solutions$\}$. Then, consequently, we have

(1) $\gamma(x) \geq \frac{1}{n}$ for all x,

(2) $\gamma(x) = \frac{1}{n}$ for almost all x, and

(3) $\gamma(x) = +\infty$ for any $x \in Q^n$.

III. (1) If $x_1 = \ldots = x_n = s$ and $0 \leq \alpha < \frac{n-6}{2}$, then

$$\limsup_{t \to +\infty} \frac{\log |P_\alpha(t;x)|}{\log t} = \frac{n}{2} - 1 - \frac{1}{2(\gamma(x)+1)}\left(\frac{n}{2} - 1 - \alpha\right).$$

([1972] Hauptsatz)

(2) For $0 \leq \alpha \leq (n-3)/2 - 1/\gamma(x)$, we have

$$\limsup_{t \to +\infty} \frac{\log |P_\alpha(t;x)|}{\log t} = \frac{n}{2} - 1 - \frac{1}{2(\gamma(x)+1)}\left(\frac{n}{2} - 1 - \alpha\right).$$

([1974] Theorem 3)

(3) For $\alpha > (n-3)/2 - 1/2\gamma(x)$ and $\alpha \geq 0$,

$$t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} \ll T_\alpha(t;x) \ll t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)} \log^\kappa t,$$

where $\kappa = 1$ for $\alpha = (n-3)/2$, else $\kappa = 0$. And for $0 \leq \alpha < (n-3)/2 - 1/2\gamma(x)$,

$$\limsup_{t \to +\infty} \frac{\log T_\alpha(t;x)}{\log t} = \frac{n}{2} - 1 - \frac{1}{2(\gamma(x)+1)}\left(\frac{n}{2} - 1 - \alpha\right).$$

([1974] Theorem 4)

Chapter III. Fourier analysis and lattice point problems

Section 1. Some points of contact between Fourier analysis and lattice point problem

The special trigonometric series: $\sum_{m\neq 0}\frac{1}{|m|^\sigma}e^{2\pi imx}$ forms a strong connection between the two fields. The first device is the following simple idea:

(1) This multiple trigonometric series is the Fourier series of the function $\zeta_\sigma(x)$ and it is necessarily and sufficiently for the function ζ_σ to belong to $L^p(T^n)$ that $\sigma > n/p'$.

(2) If $\sum_{m\neq 0}|m|^{-\sigma}e^{2\pi imx}$ converges for some point, then $\sum_{0<|m|^2<t}|m|^{-\sigma}e^{2\pi imx} = O(1)$. Therefore, by Lemma2 of Section 3,

$$\sum_{0<|m|^2<t} e^{2\pi imx} = \sum_{0<k<t} k^{\frac{\sigma}{2}}\Big(\sum_{|m|^2=k}|m|^{-\sigma}e^{2\pi imx}\Big) = O(t^{\frac{\sigma}{2}})$$

at the point.

(3) If $\sum_{|m|^2<t} e^{2\pi imx} = O(t^\alpha)$ for some point x and $2\alpha < \sigma$, then

$$\sum_{s\le|m|^2<t}\frac{1}{|m|^\sigma}e^{2\pi imx} = \int_s^t \frac{1}{u^{\frac{\sigma}{2}}}dD(u;x)$$

$$= \left[\frac{1}{u^{\frac{\sigma}{2}}}D(u;x)\right]_s^t - \frac{\sigma}{2}\int_s^t D(u;x)\frac{du}{u^{\frac{\sigma}{2}+1}} = O(s^{\alpha-\sigma/2}) = o(1).$$

That is, $\sum_{m\neq 0}|m|^{-\sigma}e^{2\pi imx}$ coverges at the point.

Therefore we have the following equalities for $x\notin Z^n$: $\sigma(x) = 2\alpha(x)$, where

$$\sigma(x) \overset{\text{def}}{=} \sup\left\{\sigma : \sum_{m\neq 0} \frac{e^{2\pi imx}}{|m|^\sigma} \text{ is divergent}\right\} = \inf\left\{\sigma : \sum_{m\neq 0} \frac{e^{2\pi imx}}{|m|^\sigma} \text{ is convergent}\right\}$$

and

$$\alpha(x) \overset{\text{def}}{=} \sup\left\{\alpha : \sum_{|m|^2<t} e^{2\pi imx} = \Omega(t^\alpha)\right\} = \inf\left\{\alpha : \sum_{|m|^2<t} e^{2\pi imx} = O(t^\alpha)\right\}.$$

(because $\sigma(x) \geq 2\alpha(x)$ by (2) and $2\alpha(x) \geq \sigma(x)$ by (3).)

The second device is that the lines $\{(1/p, \alpha) : \alpha = \frac{n-1}{2} - \frac{n}{p'}\}$ and $\{(\sigma, \alpha) : \alpha + \sigma = \frac{n-1}{2}\}$ become the same one by linear transformation $\sigma = \frac{n}{p'}$. The former line is discovered in Section 3 of Chapter 1. On the other hand, the latter line is the boundary on convergence of the series:

$$\sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi imx}}{|m|^\sigma}.$$

For example, If $\alpha + \sigma < \frac{n-1}{2}$, $\alpha > 0$ and $\sigma > 0$, then this series diverges for every x, but if $\alpha + \sigma > \frac{n-1}{2}$, then it is not always so. This fact is stated in Section 4 of this Chapter.

Section 2. On the relation between Fourier coefficients and $r_n(k)$

We know that if $f \in L^1(T^n)$, $\hat{f}(m)$ converges to 0 as $|m| \to +\infty$ by the Riemann-Lebesgue's lemma, and if $f \in L^2(T^n)$,

$$\sum_{|m|^2=k} \left|\hat{f}(m)\right|^2 \leq \|f\|_2 \quad \text{for every } k \in N$$

by Parseval's equality, and since the right hand side of the inequality can be made arbitrarily small by subtracting from f a polynomial, $\lim_{k\to+\infty} \sum_{|m|^2=k} \left|\hat{f}(m)\right|^2 = 0$.

According to Zygmund [1974] (p. 189), Charles Fefferman proposed some time ago: "*Does there exist a positive number p strictly less than* 2 *such that*

$$\left(\sum_{|m|^2=k} \left|\hat{f}(m)\right|^2\right)^{\frac{1}{2}} \leq A\|f\|_p \quad \textit{for all } f \in L^p(T^2),$$

where A is independent of $k \in N$."

Zygmund gives an answer to the problem.

THEOREM (A. Zygmund [1974]). For any k, we have

$$\left(\sum_{|m|^2=k} \left|\hat{f}(m)\right|^2\right)^{\frac{1}{2}} \leq A\|f\|_{4/3},$$

where $A = 5^1/4$.

We formulate the following problem: *Determine* ϑ_n, *where* $\vartheta_n \overset{\text{def}}{=} \inf\Bigg\{p \in [1, +\infty)$:

$$\Bigg(\sum_{|m|^2=k} \left|\hat{f}(m)\right|^2\Bigg)^{\frac{1}{2}} \leq A\|f\|_p \textit{ for every } f \in L^p(T^n) \textit{ and every } k\Bigg\}.$$

(A depends only on p and n.)

Zygmund's result implies $\vartheta_2 \leq 4/3$. We obtained the following and it implies $\vartheta_n \geq 2n/(n+2)$ if $n \geq 3$.

Theorem (Kuratsubo [1979]). If $n \geq 3$ *and* $1 \leq p < 2n/(n+2)$*, there exists a function* $f \in L^p(T^n)$ *such that*

$$\limsup_{k\to+\infty} \sum_{|m|=k} \left|\hat{f}(m)\right|^2 = +\infty.$$

We use some estimations for $r_n(k)$ to prove our theorem.

For every positive number ε,

(1) $\qquad r_3(k) \geq C_\varepsilon k^{\frac{1}{2}-\varepsilon}, \qquad$ for $m \equiv 1$, 2(mod 4) and $m \equiv 3$(mod 8).

If $n \geq 4$,

(2) $\qquad r_n(k) \geq C_\varepsilon k^{\frac{n}{2}-1-\varepsilon} \qquad$ for any k.

((1) by C. L. Siegel [1935] and (2) by A. V. Malyšev [1962]. cf. Linnik [1968])

Proof of Theorem. From the given conditions, we have the inequality; $n(p-1)/p < (n-2)/2$. We use for σ a number which satisfies the inequality: $n(p-1)/p < \sigma < (n-2)/2$.

Then

(1) $\qquad n/2 - 1 - \sigma > 0$ and

(2) $\qquad (n-\sigma)p < n.$

Now, put $f(x) = \zeta_{\sigma 0}(x)$. Then $f \in L^p(T^n)$ by (2) and

$$\sum_{|m|^2=k} \left| \hat{f}(m) \right|^2 = \sum_{|m|^2=k} \left(\frac{1}{|m|^\sigma} \right)^2 = \frac{1}{k^\sigma} r_n(k).$$

Therefore

$$\limsup_{k \to +\infty} \sum_{|m|^2=k} \left| \hat{f}(m) \right|^2 = +\infty. \qquad \text{(by (1))}.$$

This completes the proof of the Theorem.

Section 3. On an inequality of Il'in & Alimov's inequality and its applications

The following two lemmas were proved by Il'in and Alimov for $b(t) = t^\beta$ $(\beta > 0)$ and $s > 0$.

LEMMA 1 (V. A. Il'in and Sh. A. Alimov [1970]). *Suppose $s > -1$, $\beta > 0$ and $s = r + \kappa$, where r is an integer and κ satisfies $0 < \kappa \le 1$, and suppose $b(t) \in C^{r+2}(0, +\infty)$. Further, for any numerical series $\sum_{j\ge 1} a_j$ and $\lambda > 0$, let*

$$\sigma_\lambda^s = \sum_{j\le\lambda}\left(1 - \frac{j}{\lambda}\right)^s a_j$$

and

$$\overline{\sigma_{j\le\lambda}^s} = \sum_{j\le\lambda}\left(1 - \frac{j}{\lambda}\right)^s b(j)a_j.$$

Then we have

$$\overline{\sigma_\lambda^s} = b(\lambda)\sigma_\lambda^s + (-1)^{r+1}\int_0^1 \frac{t^{r+1}}{(r+1)!}\sigma_{t\lambda}^{r+1}\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda) - b(\lambda))(1-t)^s]dt.$$

Proof. We prove first the case of $-1 < s \le 0$.

$$\overline{\sigma_\lambda^s} = \sum_{j<\lambda} a_j b(j)\left(1 - \frac{j}{\lambda}\right)^s = b(\lambda)\sigma_\lambda^s + \sum_{j<\lambda} a_j(b(j) - b(\lambda))\left(1 - \frac{j}{\lambda}\right)^s$$

$$= b(\lambda)\sigma_\lambda^s + \frac{1}{\lambda^s}\sum_{j<\lambda} a_j(b(j) - b(\lambda))(\lambda - j)^s$$

$$= b(\lambda)\sigma_\lambda^s + \frac{1}{\lambda^s}\int_0^\lambda (\lambda - t)^s(b(t) - b(\lambda))d\sigma_t^0,$$

where, by integration by parts,

$$\int_0^1 (\lambda - t)^s (b(t) - b(\lambda)) d\sigma_t^0 = [(\lambda - t)^s (b(t) - b(\lambda))\sigma_t^0]_o^\lambda - \int_0^\lambda \frac{d}{dt}[(\lambda - t)^s (b(t) - b(\lambda))]\sigma_t^0 dt$$

$$= \int_0^\lambda \frac{d}{dt}[(\lambda - t)^s (b(t) - b(\lambda))]\sigma_t^0 dt.$$

Therefore

$$\overline{\sigma_\lambda^s} = b(\lambda)\sigma_\lambda^s - \frac{1}{\lambda^s}\int_0^\lambda \frac{d}{dt}[(\lambda - t)^s (b(t) - b(\lambda))]\sigma_t^0 dt.$$

Next let us consider the case of $s > 0$. Put $A_0(t) = \sum_{j<t} a_j = \sigma_t^0$ and $A_{k+1}(t) = \int_0^t A_k(u)du = \Gamma(k+2)^{-1}\sum_{j<t}(t-j)^{k+1}a_j = \Gamma(k+2)^{-1}t^{k+1}\sigma_t^{k+1}$, inductively. Using $r+1$ times Integration by parts,we complete our proof.

To prove Lemma 2 and 3 we need the following well known relation of σ_t^α for various orders α. Let $s > -1$ and $\delta > 0$. Then we have

$$(*) \qquad \sigma_\lambda^{s+\delta} = B(\delta, s+1)^{-1}\int_0^1 (1-t)^{\delta-1} t^s \sigma_{t\lambda}^s dt.$$

(cf. Stein and Weiss [1971], p. 269)

Lemma 2. Under the same notation as in Lemma 1, when $b(t) = t^\beta \log^\tau t$, 0 according to $\lambda \le e$, $\lambda < e$ respectively. for some positive constant C, we have

$$\int_0^1 \frac{t^{r+1}}{(r+1)!}\left|\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda) - b(\lambda))(1-t)^s]\right| dt \le Cb(\lambda)\ ,\ for\ any\ \lambda \ge 0,$$

and

$$|\overline{\sigma_\lambda^s}| \le Cb(\lambda)\sup_{0<t\le\lambda}|\sigma_t^s|.$$

Proof. It follows from the next.

$$\left|\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda)-b(\lambda))(1-t)^s]\right|$$

$$\leq Cb(\lambda)\{(1-t^\beta)(1-t)^{s-r-2}+\sum_{j=1}^{r+2}t^{\beta-j}(1-t)^{s-r+j-2}\},$$

and

$$\int_0^1 t^{r+1}(1-t^\beta)(1-t)^{s-r-2}dt<+\infty,$$
$$\int_0^1 t^{r+1+\beta-j}(1-t)^{s-r+j-2}dt<+\infty \qquad (1\leq j\leq r+2).$$

Therefore

$$\left|\overline{\sigma_\lambda^s}\right|\leq b(\lambda)\left|\sigma_\lambda^s\right|+Cb(\lambda)\sup_{o<t\leq\lambda}\left|\sigma_t^{r+1}\right|.$$

Here applying $(*)$ for $r<s<r+1$ and δ which $r+1=s+\delta$,we have

$$\sigma_\lambda^{r+1}=B(\delta,s+1)^{-1}\int_0^1(1-t)^{\delta-1}t^s\sigma_{t\lambda}^s dt.$$

Therefore we have $\left|\overline{\sigma_\lambda^s}\right|\leq Cb(\lambda)\sup_{0<t\leq\lambda}\left|\sigma_t^s\right|.$

Lemma 3 (Kuratsbo [1986]). Under the same notation as before there exists a positive constant C such that

$$\frac{1}{T}\int_0^T\left|\overline{\sigma_\lambda^s}\right|^2 d\lambda\leq Cb(T)^2\sup_{0<t\leq T}\left(\frac{1}{t}\int_0^t\left|\sigma_\lambda^s\right|^2 d\lambda\right),\ \textit{for every } T>0.$$

Proof. By Lemma 2 we have the next inequality.

$$\int_0^T\left|\overline{\sigma_\lambda^s}\right|^2 d\lambda\leq 2\Big\{\int_0^T b(\lambda)^2\left|\sigma_\lambda^s\right|^2 d\lambda$$

$$+\int_0^T \left| \int_0^1 \frac{t^{r+1}}{(r+1)!} \sigma_{t\lambda}^{r+1} (\frac{d}{dt}[(b(t\lambda) - b(\lambda))(1-t)^s] dt \right|^2 d\lambda \Bigg\}$$

$$= 2\{I_1 + I_2\}.$$

Now, by monotone increase of $b(\lambda)$ we have $I_1 \leq b(T)^2 \int_0^T |\sigma_\lambda^s|^2 d\lambda$. Next, by Schwarz's inequality, Fubini's theorem, Lemma 2 and its proof we have

$$I_2 \leq \int_0^T \Bigg(\int_0^1 \frac{t^{r+1}}{(r+1)!} \left| (\frac{d}{dt})^{r+2} [(b(t\lambda) - b(\lambda))(1-t)^s] \right| dt \Bigg)$$

$$\times \Bigg(\int_0^1 \frac{t^{r+1}}{(r+1)!} \left|\sigma_{t\lambda}^{r+1}\right|^2 \left| (\frac{d}{dt})^{r+2} [(b(t\lambda) - b(\lambda))(1-t)^s] \right| dt \Bigg) d\lambda$$

$$\leq C \int_0^T b(\lambda)^2 \Bigg(\int_0^1 t^{r+1} \Big\{ (1-t^\beta) 1 - t)^{s-r-2}$$

$$+ \sum_{j=1}^{r+2} t^{\beta - j} (1-t)^{s-r+j-2} \Big\} \left|\sigma_{t\lambda}^{r+1}\right|^2 dt \Bigg) d\lambda$$

$$\leq CTb(T)^2 \int_0^1 t^{r+1} \Big\{ (1-t^\beta)(1-t)^{s-r-2}$$

$$+ \sum_{j=1}^{r+2} t^{\beta - j} (1-t)^{s-r+j-2} \Big\} \frac{1}{Tt} \int_0^{Tt} \left|\sigma_{\lambda}^{r+1}\right|^2 d\lambda dt$$

$$\leq CTb(T)^2 \sup_{0<t\leq T} \Bigg(\frac{1}{t} \int_0^t \left|\sigma_{\lambda}^{r+1}\right|^2 d\lambda \Bigg).$$

In the same way as in the proof of Lemma 2, we have

$$\frac{1}{t} \int_0^t \left|\sigma_{\lambda}^{r+1}\right|^2 d\lambda \leq B(\delta, s+1)^{-1} \int_0^1 (1-u)^{\delta-1} u^s \Bigg(\int_0^t |\sigma_{u\lambda}^s|^2 d\lambda \Bigg) du$$

$$\leq \sup_{0<u\leq t}\left(\frac{1}{u}\int_0^u |\sigma_\lambda^s|^2\, d\lambda\right).$$

These complete the proof of Lemma 3.

Lemma 4. *If, in Lemma 2, each term a_j of the numerical sequence is $L^p(T^n)$-integrable, there exists a positive constant C such that*

$$\|\overline{\sigma_\lambda^s}\|_p \leq Cb(\lambda) \sup_{0<t\leq\lambda} \|\sigma_t^s\|_p.$$

Proof. By Lemma 1, we have

$$\left|\overline{\sigma_\lambda^s(x)}\right| \leq b(\lambda)\left|\sigma_\lambda^s(x)\right|$$

$$+\int_0^1 \frac{t^{r+1}}{(r+1)!}\left|\left(\frac{d}{dt}\right)^{r+2}[b(t\lambda)-b(\lambda))(1-t)^s]\right|\left|\sigma_{t\lambda}^{r+1}(x)\right| dt.$$

By *the generalized Minkovski's inequality*, that is,

$$\|\int f(t,\cdot)dt\|_p \leq \int \|f(t,\cdot)\|_p dt,$$

we have the next inequality:

$$\|\overline{\sigma_\lambda^s}\|_p \leq b(\lambda)\|\sigma_\lambda^s\|_p +$$

$$+\int_0^1 \frac{t^{r+1}}{(r+1)!}\left|\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda)-b(\lambda))(1-t)^s]\right| \|\sigma_{t\lambda}^{r+1}\|_p dt.$$

Further, by the relation $(*)$, we have

$$\|\sigma_{t\lambda}^{r+1}\|_p \leq B(\delta, s+1)^{-1}\int_0^1 (1-u)^{\delta-1}u^s\|\sigma_{ut\lambda}^s\|_p du,$$

where $r < s < r+1$ and δ which $r+1 = s+\delta$.

Therefore, using the first half of Lemma 2, we complete the proof of Lemma 4.

Section 4. On the summability of Fourier series by the Bochner-Riesz means

E. M. Stein [1958] has shown that

$$\text{if } f \in L^p(T^n) \text{ and } \alpha > (n-1)\left(\frac{1}{p}-\frac{1}{2}\right), \ \lim_{t\to+\infty} S^\alpha_{t^2}(f;x) = f(x) \text{ for almost all } x.$$

In the same paper he has solved the problem of the strong summability. The problem of the strong summability of the Bochner-Riesz means of multiple Fourier series is one of dealing with the validity of the following:

$$(\star) \qquad \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left|S^\alpha_{t^2}(f;x) - f(x)\right|^2 dt = 0.$$

E. M. Stein has shown the following theorem:

THEOREM (E. M. Stein [1958]). *If $f \in L^p(T^n)$ and $\alpha > \alpha_p \stackrel{\text{def}}{=} n(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2} = (n-1)(\frac{1}{p}-\frac{1}{2}) - \frac{1}{p'} = \frac{n-1}{2} - \frac{n}{p'}$, then $(\star)$ holds true almost everywhere in x.*

But the case $\alpha \le \alpha_p$ has been unknown except for the case $p = 1$ and $\alpha = \alpha_1$ $(= \frac{n-1}{2})$. (In [1958] Stein stated the affirmative result without the proof.)

The purpose of this section is to prove a negative result for the case $\alpha < \alpha_p$ and to prove some corollarys. The method consists of joining multiple Fourier series and lattice point problems by means of a special function ζ_σ or $\zeta_{\sigma\tau}$ whose Fourier coefficients is given by $\hat{\zeta}_\sigma(m) = 1/|m|^\sigma$ or $\hat{\zeta}_{\sigma\tau}(m) = 1/|m|^\sigma \log^\tau |m|$ (See Section 3 of Chapter 1). Our main results in this section are the following.

Theorem (Kuratsubo [1986]). *Suppose $1 \le p \le 2$, $-1 < \alpha < \alpha_p$ and $\tau > \frac{1}{p}$. Then* *there exists a function* $f \in L^p(T^n)$ *such that*

$$\frac{1}{T}\int_0^T |S_t^\alpha(f;x)|^2\,dt = \Omega\Big(T^{\alpha_p-\alpha}\log^{-2\tau}T\Big) \text{ as } T \to +\infty$$

for every x*, in particular,*

$$\limsup_{T\to+\infty}\frac{1}{T}\int_0^T |S_t^\alpha(f;x)|^2\,dt = +\infty$$

for every x*, where we can take* $\zeta_{\sigma\tau}(x)$ *for such a function* $f(x)$*.*

(In the above statement, $g(T) = \Omega(h(T))$ implies $g(T) \neq o(h(T))$)

We must comment on a small gap between Stein's and our definition of the strong summability and we do so at the end of this section.

From this theorem we have directly the following corollary.

Corollary (Kuratsubo [1981] and [1986]). *Suppose* $1 \le p \le 2$, $-1 < \alpha < \alpha_p$ *and* $\tau > \frac{1}{p}$*. Then there exists a function* $f \in L^p(T^n)$ *such that*

$$S_t^\alpha(f;x) = \Omega\left(t^{\frac{1}{2}(\alpha_p-\alpha)}\log^{-\tau}t\right) \qquad \text{as } t \to +\infty$$

for every x*, in particular,*

$$\limsup_{t\to+\infty}|S_t^\alpha(f;x)| = +\infty$$

for every x*, where we can take* $\zeta_{\sigma\tau}(x)$ *for such a function* $f(x)$*.*

The existence of $f \in L^p(T^n)$ such that $\limsup_{t\to+\infty}|S_t^\alpha(f;x)| = +\infty$ for almost all x and the fact that we can take ζ_σ for such a function f have been shown for $\alpha = 0$ by Stein and Weiss [1971] and for $0 \le \alpha < (n-1)/2$ by Babenko [1973] (see also Alimov, Il'in and Nikishin [1976]). Especially, Babenko's result becomes the following form in the special case that the positive elliptic operator with constant coefficients is Laplacian :

THEOREM. *Suppose $1 \le p \le 2n/(n+1)$, $\alpha < \alpha_p$ and $n/p' < \sigma < (n-1)/2 - \alpha$. Then $\zeta_\sigma \in L^p(T^n)$ and $S_t^\alpha(\zeta_\sigma; x)$ are unboundedly divergent almost everywhere on T^n.*

In connection with this theorem he proposed the problem of how to characterize sets on which the Fourier series of ζ_σ is summable by the Bochner-Riesz means of order $\alpha < \alpha_p$ and stated the following: "*(according to THEOREM) such a set must be of Lebesgue measure zero. But in R^n for $n > 1$ such a characterization is too broad, because among the sets of measure zero there are sets of various dimensions. It is more profitable to characterize such sets in terms of their Hausdorff measure. The problem arises of finding a more redifined characterization of the exceptional set in THEOREM.*

Is this set empty in general, and does the series $\sum_{m \in Z^n} |m|^{-s} e^{2\pi imx}$ fail to be summable by the Riesz method of order $\alpha < \alpha_p$ for all $x \in T^n$? It would be very interesting to resolve this question for the Laplace operator." (Babenko [1973], p. 194)

Our Corollary gives the complete answer for the Laplace operator.

THEOREM (B. Novák [1970]). *Suppose $\alpha > -1$. Then we have*

$$\frac{1}{T}\int_0^T \left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^\alpha e^{2\pi imx} \right|^2 dt \gg T^{(n-1)/2-\alpha}$$

for every x.

("$f(T) \gg g(T)$" implies $f(T) \ge Cg(T)$ for some constant number C which may depend on x in general.)

Indeed,

$$\left(T_\alpha(t;x)\right)^2 = \frac{1}{t}\int_0^t |P_\alpha(s;x)|^2\, ds \le \frac{1}{t}\int_0^t \left| \sum_{|m|^2 < s} (s - |m|^2)^\alpha e^{2\pi imx} \right|^2 ds$$

$$= \frac{1}{t}\int_0^t s^{2\alpha} \left| \sum_{|m|^2 < s} \left(1 - \frac{|m|^2}{s}\right)^\alpha e^{2\pi imx} \right|^2 ds \le t^{2\alpha}\frac{1}{t}\int_0^t \left| \sum_{|m|^2 < s} \left(1 - \frac{|m|^2}{s}\right)^\alpha e^{2\pi imx} \right|^2 ds.$$

On the other hand, by Novák's result (Chapter 2, Section 3, II), $t^{1/2\{(n-1)/2+\alpha\}} \ll T_\alpha(t;x)$ for all x. Therefore, we obtain the proof of theorem.

(Novák's result was proved in the case of $\alpha \geq 0$, but an examination of the proof shows that it can apply without significant change to the present situation.)

Lemma (Kuratsubo [1986]). Suppose $\alpha > -1$, $\sigma \geq 0$, $\alpha + \sigma < (n-1)/2$ and τ is a nonnegative number. Then we have

$$\frac{1}{T}\int_0^T \left| \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} \frac{e^{2\pi i m x}}{|m|^{\sigma} \log^{\tau} |m|} \right|^2 dt = \Omega\Big(T^{(n-1)/2-\alpha-\sigma} \log^{-2\tau} T\Big)$$

for every x.

Proof. Applying Lemma 3 to the case

$$a_j = \sum_{|m|^2=j} \frac{e^{2\pi i m x}}{|m|^{\sigma} \log^{\tau} |m|}, \qquad \beta = \frac{\sigma}{2}, \qquad \text{and } s = \alpha.$$

Then

$$\sigma_t^{\alpha} = \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} \frac{e^{2\pi i m x}}{|m|^{\sigma} \log^{\tau} |m|} \quad \text{and} \quad \overline{\sigma_t^{\alpha}} = \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{2} e^{2\pi i m x}.$$

Suppose, for some x,

$$\frac{1}{T}\int_0^T \left| \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} \frac{e^{2\pi i m x}}{|m|^{\sigma} \log^{\tau} |m|} \right|^2 dt = o\Big(T^{(n-1)/2-\alpha-\sigma} \log^{-2\tau} T\Big).$$

Then by Lemma 3 of Section 3, we have

$$\frac{1}{T}\int_0^T |\overline{\sigma_t^{s}}|^2 \, dt \ll T^{2\sigma} \log^{2\tau} T \sup_{0<t\leq T} \left(\frac{1}{t}\int_0^t |\sigma_{\lambda}^{s}|^2 \, d\lambda\right) = o\Big(T^{(n-1)/2-\alpha}\Big).$$

This contradicts our hypothesis.

Proof of Theorem. Now let σ be equal to n/p', τ larger than $1/p$ and $\zeta_{\sigma\tau}$ the function such that

$$\hat{\zeta}_{\sigma\tau}(m) = \frac{1}{|m|^\sigma \log^\tau |m|}, \qquad \text{for } |m| > 1.$$

Then, from the Lemma of Section 3 of Chapter 2, the function $\zeta_{\sigma\tau}$ belongs to $L^p(T^n)$ and from Kuratsubo's Lemma we have

$$\frac{1}{T}\int_0^T |S_t^\alpha(\zeta;x)|^2\,dt = \frac{1}{T}\int_0^T \left| \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|} \right|^2 dt$$

$$= \Omega\left(T^{(n-1)/2-\alpha-n/p'} \log^{-2\tau} T\right)$$

$$= \Omega\left(T^{\alpha_p - \alpha} \log^{-2\tau} T\right)$$

for every x. This completes the proof of the Theorem.

Remarks. Stein's and our definitions of the strong summability are equivalent. That is,

Stein:

$$\Psi_S(T) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T |S_{t^2}^\alpha(f;x) - f(x)|^2\,dt = \frac{1}{T}\int_0^{T^2} |S_t^\alpha(f;x) - f(x)|^2 \frac{dt}{2\sqrt{t}}.$$

Kuratsubo:

$$\Psi_K(T) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T |S_t^\alpha(f;x) - f(x)|^2\,dt.$$

Proposition. $\Psi_S(T) \to 0$ as $T \to +\infty$ if and only if $\Psi_K(T) \to 0$ as $T \to +\infty$.

Proof. Let $\phi(t) \geq 0$ and $\Phi(T)$ be $\int_0^T \phi(t)dt$. Then it is sufficient to show that

$$\int_0^T \phi(t)dt = o(T) \quad \text{if and only if} \quad \int_0^{T^2} \phi(t)\frac{dt}{2\sqrt{t}} = o(T).$$

In the first place,

$$\frac{1}{T}\int_0^{T^2} \phi(t)\frac{dt}{2\sqrt{t}} = \frac{1}{T^2}\int_0^{T^2} \phi(t)\frac{1}{2}(\frac{T}{t})^{\frac{1}{2}}dt \geq \frac{1}{2T^2}\int_0^{T^2} \phi(t)dt,$$

because $T/t \geq 1$. Therefore if $\int_0^{T^2} \phi(t)\frac{dt}{2\sqrt{t}} = o(T)$, then $\int_0^T \phi(t)dt = o(T)$.

On the other hand,

$$\int_1^{T^2} \phi(t)\frac{dt}{2\sqrt{t}} = \int_1^{T^2} \frac{1}{2\sqrt{t}}d\Phi(t) = \left[\Phi(t)\frac{1}{2\sqrt{t}}\right]_1^{T^2} + \frac{1}{4}\int_1^{T^2} \Phi(t)\frac{dt}{t^{\frac{3}{2}}}.$$

Therefore if $\Phi(T) = o(T)$,

$$\int_1^{T^2} \phi(t)\frac{dt}{2\sqrt{t}} = \frac{1}{T}o(T^2) + O(1) + \int_1^{T^2} o(t)\frac{dt}{t^{\frac{3}{2}}} = o(T).$$

This completes the proof of the Proposition.

In this section we obtained the divergence problem of the strongly Bochner-Riesz summability of Fourier series from a result in lattice point problems of Novák. In the latter part of this section, we show conversely that we can obtain a estimation of $T_\alpha(t;x)$ from a result of Fourier series.

Theorem. *For any $\alpha > 0$ and any $\varepsilon > 0$, we have*

$$T_\alpha(t;x) = O\left(t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)+\varepsilon}\right) \textit{ for almost all } x.$$

Proof. Let p be a number such that $1 < p < 2$ and $(n-1)/2 - n/p' + \varepsilon > \alpha > (n-1)/2 - n/p'$. Then $(n-1)/2 - \alpha + \varepsilon > n/p' > (n-1)/2 - \alpha$. Next, let σ be a number such that $(n-1)/2 - \alpha + \varepsilon > \sigma > n/p'$. By the inequality $\sigma > n/p'$, $\zeta_\sigma(x) \in L^p(T^n)$ and by the inequality $\alpha > \alpha_p = (n-1)/2 - n/p'$, the Bochner-Riesz means of $S_t^\alpha(\zeta_\sigma; x)$ is strongly summable to $\zeta_\sigma(x)$ (by Stein [1958]). That is,

$$\frac{1}{t}\int_0^t \left| \sum_{|m|^2<s} \left(1 - \frac{|m|^2}{s}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma} \right|^2 ds = O(1).$$

By lemma 3 of section 3, we have

$$\frac{1}{t}\int_0^t \left| \sum_{|m|^2<s} \left(1 - \frac{|m|^2}{s}\right)^\alpha e^{2\pi i m x} \right|^2 ds = O(t^\sigma).$$

Therefore, we have the following inequality:

$$\frac{1}{t}\int_0^t \left| \sum_{|m|^2<s} (s - |m|^2)^\alpha e^{2\pi i m x} \right|^2 ds \leq t^{2\alpha} \frac{1}{t}\int_0^t \left| \sum_{|m|^2<s} \left(1 - \frac{|m|^2}{s}\right)^\alpha e^{2\pi i m x} \right|^2 ds$$

$$= O(t^{2\alpha+\sigma}).$$

Here, since $2\alpha + \sigma = (\alpha + \sigma) + \alpha < (n-1)/2 + \alpha + \varepsilon$, we complete the proof of the Theorem.

Section 5. On a majorant for the partial sums of a multiple trigonometric series

For $n = 1$ trigonometric series $\sum_{m \neq 0} |m|^{-\sigma} e^{2\pi imx}$ equals to $2\sum_{m>0} m^{-\sigma}\cos(2\pi mx)$. This series in the case of $0 < \sigma < 1$ is very important and the following facts are known:

(1) $C|x|^{\sigma-1} + g_\sigma(x) \sim \sum_{m=1}^{+\infty} m^{-\sigma}\cos(2\pi mx)$, where C is a constant number and $g_\sigma(x)$ is a infinitly differrentiable function on $(-\infty, +\infty)$.

(2) This series converges for every $x \neq 0$.

and

(3) There exists a constant number C such that $\left|\sum_{m<t}\cos(2\pi mx)m^{-\sigma}\right| \leq C|x|^{\sigma-1}$.

(cf. Zygmund ([1968], Chapter V, p. 191)

Kohn extended the inequality of (3) to the following form in 1972.

THEOREM (M. J. Kohn [1972]). If $n-1 < \sigma < n$, there exists a constant number C_σ such that

$$\left|\sum_{|m|^2<t} \frac{e^{2\pi imx}}{|m|^\sigma}\right| \leq C_\sigma \frac{1}{|x|^{n-\sigma}}$$

for any $t > 0$ and $x \in T^n$.

We can prove, in the case $n \geq 4$, that Kohn's restriction "$n-1$" can be extended to "$n-2$" and this is the best.

Theorem (Kuratsubo [1984]). Let $n \geq 4$. If $n-2 < \sigma < n$, then there exists a constant number C_σ such that

$$\left|\sum_{0<|m|^2<t} \frac{e^{2\pi mx}}{|m|^\sigma}\right| \leq C_\sigma \frac{1}{|x|^{n-\sigma}}$$

for every $x \in T^n$ and every $t > 0$. On the other hand, if $0 < \sigma < n-2$, then there does not exist any such constant number.

Throughout this section, the letter C denotes (in general different) positive constants, depending only on n and σ , and the notation $A \ll B$ implies $|A| \leq CB$ for some C.

Let $D(t;x)$ and $K(t;x)$ denote the Dirichlet kernels for T^n and R^n respectively, that is,

$$D(t;x) = \sum_{|m|^2<t} e^{2\pi imx}$$

and

$$K(t;x) = \int_{|y|^2<t} e^{2\pi xy}dy = \begin{cases} \pi^{\frac{n}{2}} t^{\frac{n}{2}} \frac{J_{\frac{n}{2}}(2\pi\sqrt{t}|x|)}{(\pi\sqrt{t}|x|)^{\frac{n}{2}}} & (x \neq 0), \\ \pi^{\frac{n}{2}} t^{\frac{n}{2}} \frac{1}{\Gamma(\frac{n}{2}+1)} & (x = 0), \end{cases}$$

where $J_{\frac{n}{2}}(s)$ is the Bessel function of order $\frac{n}{2}$.

The following theorems are essential for the proof of our theorem.

THEOREM 1 (K. I. Babenko [1978]). Let $n \geq 4$. Then we have

$$D(t;x) - K(t;x) \ll \begin{cases} t^{\frac{n}{2}-1} & (n > 4), \\ t\log^2 t & (n = 4) \end{cases}$$

for every $x \in T^n$ and every $t > 0$.

THEOREM (B. Novák [1971]). Let $n \geq 4$. Then we have

$$\limsup_{t\to+\infty} \frac{|D(t;x)|}{t^{\frac{n}{2}-1}} > 0$$

for every $x \in Q^n$, where Q^n is the set $\{(x_1, \ldots, x_n) :$ each x_i is rational$\}$.

Further, recalling the following estimation:

$$J_{\frac{n}{2}}(s) \ll \max\{|s|^{-\frac{1}{2}}, |s|^{\frac{n}{2}}\},$$

we have

$$K(t;x) \ll \begin{cases} t^{\frac{n}{2}} & \text{if } \sqrt{t}|x| \leq 1, \\ \frac{t^{(n-1)/4}}{|x|^{(n+1)/2}} & \text{if } \sqrt{t}|x| > 1. \end{cases}$$

Proof of Theorem. For simplicity of description only the case $n > 4$ is treated. The case $n = 4$ is also similar. By the formula of summation by parts, we have

$$\sum_{0<|m|^2<t} \frac{e^{2\pi i m x}}{|m|^\sigma} = \frac{1}{t^{\frac{\sigma}{2}}} D(t;x) + \frac{\sigma}{2} \int_{1/2}^{t} D(u;x) \frac{du}{u^{\frac{\sigma}{2}+1}}$$

$$= \frac{1}{t^{\frac{\sigma}{2}}}(D(t;x) - K(t;x)) + \frac{\sigma}{2} \int_{\frac{1}{2}}^{t} (D(u;x) - K(u;x)) \frac{du}{u^{\frac{\sigma}{2}+1}}$$

$$+ \frac{1}{t^{\frac{\sigma}{2}}} K(t;x) + \frac{\sigma}{2} \int_{1/2}^{t} K(u;x) \frac{du}{u^{\frac{\sigma}{2}+1}}$$

$$= I_1 + I_2 + I_3 + I_4$$

According to (3), we have

$$I_1 \ll \frac{t^{n/2-1}}{t^{\sigma/2}} = t^{\frac{1}{2}(n-2-\sigma)} \ll 1 \ll \frac{1}{|x|^{n-\sigma}}$$

and

$$I_2 \ll \int_{\frac{1}{2}}^{t} \frac{u^{n/2-1}}{u^{\sigma/2+1}} du = \int_{\frac{1}{2}}^{t} u^{\frac{1}{2}(n-2-\sigma)-1} du \ll 1 \ll \frac{1}{|x|^{n-\sigma}}.$$

Next, acording to (3) if $\sqrt{t}|x| \leq 1$, then we have

$$I_3 \ll t^{\frac{1}{2}(n-\sigma)} \ll \frac{1}{|x|^{n-\sigma}}$$

and if $\sqrt{t}|x| \geq 1$, then we have

$$I_3 \ll \frac{t^{\frac{n-1}{4}-\frac{\sigma}{2}}}{|x|^{\frac{n+1}{2}}} \ll |x|^{-(\frac{n+1}{2}+(\frac{n-1}{2}-\sigma))} = \frac{1}{|x|^{n-\sigma}},$$

because $(n-1)/2 - \sigma < (n-1)/2 - (n-2) < -(n-3)/2 < 0$. Similarly, acording to (3) we have

$$I_4 \ll \int_{\frac{1}{2}}^{1/|x|^2} \frac{u^{\frac{n}{2}}}{u^{\frac{\sigma}{2}+1}} + \int_{1/|x|^2}^{+\infty} \frac{u^{(n-1)/4}}{|x|^{\frac{n+1}{2}}} \frac{du}{u^{\sigma/2+1}}$$

$$\ll (\frac{1}{|x|^2})^{\frac{n-\sigma}{2}} + \frac{1}{|x|^{\frac{n+1}{2}}} \int_{1/|x|^2}^{+\infty} u^{\frac{1}{2}(\frac{n-1}{2}-\sigma)-1} du \ll \frac{1}{|x|^{n-\sigma}}.$$

Consequently, these estimates complete the first half of the proof of the theorem. Lastly, we shall prove the second half. If we put

$$T_\sigma(t;x) = \sum_{0<|m|^2<t} \frac{e^{2\pi i m x}}{|m|^\sigma},$$

then by the formula of summation by parts, we have

$$D(t;x) = t^{\frac{\sigma}{2}} T_\sigma(t;x) - \frac{\sigma}{2} \int_{\frac{1}{2}}^{t} T_\sigma(u;x) u^{\frac{\sigma}{2}-1} du.$$

From this relation it follows that if $\limsup_{t\to+\infty} |T_\sigma(t;x_0)| < +\infty$ for some x_0, then we have $\limsup_{t\to+\infty} \frac{|D(t;x_0)|}{t^{\frac{\sigma}{2}}} < +\infty$. Therefore this shows that

$$\limsup_{t\to+\infty} |T_\sigma(t;x)| = +\infty, \text{ whenever } \limsup_{t\to+\infty} \frac{|D(t;x)|}{t^{\frac{\sigma}{2}}} = +\infty.$$

Thus, when $0 < \sigma < n-2$, (2) implies that $\limsup_{t\to+\infty} |T_\sigma(t;x)| = +\infty$ for every $x \in T^n$. This completes the proof of the second half.

Section 6. On lattice point problems with weight in ellipsoid

For any positive definite quardrtic form:

$$Q(u) = Q(u_1, \ldots, u_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_iu_j$$

with determinant D, we consider the most general *"lattice rest"*,

$$P_Q(t;x) = P_Q(t;x,y,M) = A_Q(t;x,y,M) - V_Q(t;x,y)$$

The notations of this section follow the ones of Section 3 of Chapter 2.

The first interesting result of the *lattice rest* was derived by B. Novák. In it the influence of *"weight"* x has been discussed. For example,

THEOREM I (B. Novák [1968]). If Q, M and y are rational, for every x, we have

$$P_Q(t;x) = \begin{cases} O(t^{\frac{n}{2}-1}) & \text{for } n \geq 5, \\ O(t\log^2 t) & \text{for } n = 4. \end{cases}$$

He further proved the following.

THEOREM II (B. Novák [1968]). If Q, M and y are rational and $n \geq 5$, we have the following:

1) If x is not rational, $P_Q(t;x) = o(t^{\frac{n}{2}-1})$.

2) the estimation of I can't improve generally. That is, for each monotone decreasing function $\psi(t)$ which converges to 0 as $t \to +\infty$, there exists x such that

$$P_Q(t;x) = \Omega(t^{\frac{n}{2}-1}\psi(t)).$$

3) For almost everywhere x, we have

$$P_Q(t;x) = O(t^{\frac{n}{4}} \log^{3n} t).$$

On the other hand, Landau proved the result which is true for every case.

THEOREM III (E. Landau [1915]). *For $n \geq 2$ and all Q, x, y and M, we have*

$$P_Q(t;x,y,M) = O(t^{\frac{n}{2}-1+\frac{1}{n+1}}).$$

(Fricker's text [1982] comments in detail on these results and the main results up to 1981.)

Our purpose in this section is to prove the next Theorem which is a generalization and an improvement of THEOREM II. 3). Our method is quite different. Our theorem is better for case $n \geq 4$ than Landau's and Novák's , but for the case of $n = 2, 3$ Landau's result is better than ours.

Theorem (Kuratsubo [1982]). *For $n \geq 2$ and all Q, y and M ,and for $\kappa > \frac{3}{2}$. we have*

$$P_Q(t:x,y,M) = O(t^{\frac{n}{4}} \log^{\kappa} t) \text{ for almost everywhere } x \in R^n.$$

In the following the letter C denotes (in general, different) positive constants, depending only on Q, y and M. The next three propositions are trivial.

Proposition 1. *There exist positive constants C_1, C_2 such that*

$$C_1|u|^2 \leq Q(u) \leq C_2|u|^2$$

for all $u \in R^n$.

Proposition 2. *There exist constants C, C_1 and C_2 such that if $|m| > C$, then*

$$C_1|m| \leq |m \circ M + y| \leq C_2|m|.$$

We denote by $\{\phi_\nu\}_{\nu=1}^{+\infty}$ the sequence $\{e(x(m \circ M + y)) : m \in Z^n\}$ rearranged in order of ascending of magnitude of $Q(m \circ M + y)$, then we obtain the following proposition from proposition 1 and 2.

Proposition 3. *There exists a positive constant C such that*
if $\phi_\nu(x) = e(x(m \circ M + y))$, then $\nu \leq C|m|^n$.

Further, let ν_0 be a natural number such that if $\phi_{\nu_0}(x) = e(x(m \circ M + y))$, then $Q(m \circ M + y) \geq 2$.

Lemma. *Let $\Omega_0 = \Pi_{j=1}^n [0, M_j^{-1}]$, then $\{\phi_\nu\}_{\nu=1}^{+\infty}$ is a orthonormal system on Ω_0 with respect to the measure $(\Pi_{j=1}^n M_j)dx$ where dx is the n dimensional Lebesgue measure. Further , let $c_\nu = \left(Q(m \circ M + y)^{\frac{n}{4}} \log^\kappa Q(m \circ M + y)\right)^{-1}$ for each $\nu \geq \nu_0$. then*

$$\sum_{\nu=\nu_0}^{+\infty} |c_\nu|^2 \log^2 \nu < +\infty.$$

Proof. The first half is trivial. The second half follows from the following inequality.

$$\sum_{\nu \geq \nu_0} |c_\nu|^2 \log^2 \nu$$

$$\leq C \sum_{Q(m\circ M+y)\geq 2} \left(Q(m\circ M+y)^{\frac{n}{4}} \log^{\kappa} Q(m\circ M+y) \right)^{-2} \log^2 |m|$$

$$\leq C \sum_{|m|\geq 2} \frac{1}{|m|^n \log^{2\kappa-2}|m|} < +\infty.$$

In the proof of the theorem , the theorem of Menšov-Rademacher is essential and this theorem is stated in the following form.

THEOREM OF MENŠOV-RADEMACHER. *Let (Ω, A, μ) be a measure space and let $\{\phi_\nu\}_{\nu=1}^{+\infty}$ be a orthonormal system on Ω. If a sequence $\{c_\nu\}_{\nu=1}^{+\infty}$ satisfies the condition:*

$$\sum_{\nu=1}^{+\infty} |c_\nu|^2 \log^2 \nu < +\infty,$$

then $\sum_{\nu=1}^{+\infty} c_\nu \phi_\nu$ converges almost everywhere in Ω.

(See Zygmund [1968], vol. II, p. 193 (10, 21))

By the lemma and the theorem of Menšov-Rademacher, we have

$$\sum_{Q(m\circ M+y)\leq t} \left(Q(m\circ M+y)^{\frac{n}{4}} \log^{\kappa} Q(m\circ M+y) \right)^{-1} e(x(m\circ M+y)) = O(1)$$

for almost everywhere $x \in R^n$. We denote the left hand by $S_Q(t;x) = S_Q(t;x,y,M)$. Then we have

$$\sum_{Q(m\circ M+y)\leq t} e(x(m\circ M+y)) = \int_2^t u^{\frac{n}{4}} \log^{\kappa} u dS_Q(u;x) + O(1)$$

$$= t^{\frac{n}{4}} \log^{\kappa} t S_Q(t;x) - \int_2^t S_Q(u;x) \frac{d}{du}(u^{\frac{n}{4}} \log^{\kappa} t) du + O(1)$$

$$= O(t^{\frac{n}{4}} \log^{\kappa} t) + O\left(\int_2^t \left|\frac{d}{du}(u^{\frac{n}{4}} \log^{\kappa} t)\right| dt\right) + O(1)$$

$$= O(t^{\frac{n}{4}} \log^{\kappa} t).$$

Therefore

$$\sum_{Q(m \circ M + y) \leq t} e(x(m \circ M + y)) = O(t^{\frac{n}{4}} \log^{\kappa} t)$$

almost everywhere on R^n. This completes the proof of the Theorem.

Section 7. On some special multiple trigonometric series

We cosider in this section the Bochner-Riesz means of some special multiple trigonometric series, that is,

$$\sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^{\alpha}\frac{1}{|m|^{\sigma}}e^{2\pi imx}.$$

Firstly, we present already-known results in Fourier analysis by Stein, Stein-Weiss, Babenko, Il'in, Shapiro, Alimov-Il'n-Nikishin.

Stein:

[S] $$\|\sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^{\frac{n-1}{2}}e^{2\pi imx}\|_1 = C\log t + O(1).$$

(cf. Stein [1961], THEOREM 4)

E.M.Stein and *G.Weiss*:

[S − W*] $$\limsup_{t\to+\infty}\left|\sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^{\frac{n-1}{2}}e^{2\pi imx}\right| = +\infty \text{ for } x\in S,$$

and

[S − W**] $$\limsup_{t\to+\infty}\left|\sum_{0<|m|^2<t}\frac{1}{|m|^{\frac{n-1}{2}}}e^{2\pi imx}\right| = +\infty \text{ for } x\in S,$$

where S is the set of points x where the denumerable collection of real numbers $\{|x-m| : m\in Z^n\}$ is linearly independent over the rational and it is known that the complement of S has measure 0 when $n>1$. (Stein and Weiss [1971], LEMMA. 4. 11 and THEOREM 4. 3)

K.I.Babenko:

(1) Suppose $1 \le p \le 2n/(n+1)$, $0 \le \alpha < \alpha_p$ and $\sigma < n/p'$. Then

$$[B^*] \qquad t^{\frac{1}{2}(\frac{n-1}{2}-\alpha-\sigma)} \ll \| \sum_{0<|m|^2<t} \left(1-\frac{|m|^2}{t}\right)^{\alpha} \frac{1}{|m|^{\sigma}} e^{2\pi i m x} \|_p \ll t^{\frac{1}{2}(\frac{n-1}{2}-\alpha-\sigma)}.$$

(2) Suppose $1 \le p \le 2n/(n+1)$, $\alpha = \alpha_p$ and $\sigma < n/p'$. Then

$$[B^{**}] \qquad t^{\frac{n}{p'}-\sigma}(\log t)^{\frac{1}{p}} \ll \| \sum_{0<|m|^2<t} \left(1-\frac{|m|^2}{t}\right)^{\alpha_p} \frac{1}{|m|^{\sigma}} e^{2\pi i m x} \|_p \ll t^{\frac{n}{p'}-\sigma}(\log t)^{\frac{1}{p}}.$$

(3) Suppose $1 \le p \le 2n/(n+1)$, $\alpha < \alpha_p$ and $n/p' < \sigma < (n-1)/2-\alpha$. Then $\zeta_\sigma \in L^p(T^n)$ and its Bochner-Riesz means of order α are unboundedly divergent almost everywhere on T^n.

(Babenko [1973], THEOREM 3.3 and THEOREM 3.7)

V.A.Il'in and *H.S.Shapiro*:

$$[I-S] \qquad t^{\frac{n-1}{4}} \ll \| \sum_{|m|^2<t} e^{2\pi i m x} \|_1 \ll t^{\frac{n-1}{4}}.$$

(The left inequality by Il'in ([1968], (23)) and the right inequality by Shapiro [1975])

Sh.A.Alimov, *V.A.Il'in* and *E.M.Nikishin*:

$$[A-I-N^*] \qquad \| \sum_{0<|m|^2<t} \frac{e^{2\pi i m x}}{|m|^{\frac{n-1}{2}}} \|_p = O(1) \qquad (1 \le p < \frac{2n}{(n+1)}).$$

$$[A-I-N^{**}] \qquad \| \sum_{0<|m|^2<t} \frac{e^{2\pi i m x}}{|m|^{\sigma}} \|_p \ne O(1) \qquad (1 \le p\ , \text{ and } \sigma < \frac{n-1}{2}).$$

(Alimov, Il'n and Nikishin [1977], COROLLARY 3, COROLLARY 5)

I°. On the results of Stein-Weiss

Stein-Weiss show first $[S - W^*]$, and then show $[S - W^{**}]$ by using the Riesz's convexity theorem. We can obtain the following:

Theorem. *Suppose* $0 \leq \alpha < \frac{n-1}{2}$, $\sigma \geq 0$ *and* $\alpha + \sigma = \frac{n-1}{2}$. *Then we have*

$$\limsup_{t \to +\infty} \left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha} \frac{e^{2\pi i m x}}{|m|^{\sigma}} \right| = +\infty \text{ for } x \in S.$$

The Covexity Theorem of Riesz means (M. Riesz [1923]). *Consider a numerical series* $\sum_{k \geq 0} a_k$ *and its Riesz means* $\sigma_t^{\alpha} = \sum_{0 \leq k < t} (1 - k/t)^{\alpha} a_k$, $\alpha \geq 0$. *Suppose that* $\sigma_t^{\alpha_j} = O(t^{\beta_j})$ *as* $t \to +\infty$, *for* $j = 0, 1$. *Then, if* $0 < \theta < 1$, $\alpha = \alpha_0(1 - \theta) + \alpha_1 \theta$, *and* $\beta = \beta_0(1 - \theta) + \beta_1 \theta$, *we can conclude that* $\sigma_t^{\alpha} = O(t^{\beta})$ *as* $t \to +\infty$.

Proof. Suppose, for some α_0 which $0 \leq \alpha_0 < \frac{n-1}{2}$,

$$\limsup_{t \to +\infty} \left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha_0} \frac{e^{2\pi i m x_0}}{|m|^{\frac{n-1}{2} - \alpha_0}} \right| < +\infty, \text{ at } x_0 \in S.$$

Then, by Lemma 2 of Section 3, we have

$$\left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha_0} e^{2\pi i m x_0} \right| \ll t^{\frac{1}{2}(\frac{n-1}{2} - \alpha_0)} \sup_{0 \leq s \leq t} \left| \sum_{|m|^2 < s} \left(1 - \frac{|m|^2}{s}\right)^{\alpha_0} \frac{e^{2\pi i m x_0}}{|m|^{\frac{n-1}{2} - \alpha_0}} \right|$$

$$= O\left(t^{\frac{1}{2}(\frac{n-1}{2} - \alpha_0)}\right).$$

On the other hand, if $\alpha_1 > \frac{n-1}{2}$, we have

$$\sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^{\alpha_1} e^{2\pi i m x_0} = O\left(t^{\frac{1}{2}(\frac{n-1}{2} - \alpha_1)}\right).$$

Now, let θ be a number such that $0 < \theta < 1$ and $\frac{n-1}{2} = \theta \alpha_1 + (1 - \theta)\alpha_0$. Then, since

$\theta(\frac{n-1}{4}-\frac{\alpha_1}{2})+(1-\theta)(\frac{n-1}{4}-\frac{\alpha_0}{2})=0$, by Riesz's Convexity Theorem, we can get

$$\sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^{\frac{n-1}{2}}e^{2\pi i m x_0}=O(1)$$

This contradicts the $[S-W^*]$.

This theorem can be improved to a more precise form by using the theorem [Szegö-Berndt-Hafner],

THEOREM (G. Szegö [1926], B. C. Berndt [1971] and J. L. Hafner [1982]). For $0\leq\alpha<\frac{n-1}{2}$,

$$\sum_{|m|^2<t}(t-|m|^2)^{\alpha}e^{2\pi i m x}=\Omega\left(t^{\frac{1}{2}(\frac{n-1}{2}+\alpha)}(\log t)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)}\right),$$

and for $\alpha=\frac{n-1}{2}$ *and even* n,

$$\sum_{|m|^2<t}(t-|m|^2)^{\frac{n-1}{2}}e^{2\pi i m x}=\Omega(t^{\frac{n-1}{2}}\log\log t),$$

everywhere.

This proof is based on the following equality

$$\frac{1}{k!}\int_0^{+\infty}e^{-t}t^kP_\alpha(\xi t;x,y)dt=\frac{\xi^{\frac{1}{2}(\frac{n}{2}+\alpha)}}{\pi^\alpha}\sum_{m-x\neq 0}\frac{H_k^{(\frac{n}{2}+\alpha)}(\pi^2|m-x|^2\xi)}{|m-x|^{\frac{n}{2}+\alpha}}e^{2\pi i m x},$$

where $H_k^{(\beta)}(u)=u^{\frac{\beta}{2}}e^{-u}L_k^{(\beta)}(u)$ and $L_k^{\beta}(u)$ is the Laguerre polynomial of order β. In the case of $\alpha=0$ and $x=0$, this was obtained by Szegö [1926], by Berndt [1971] in the case of $\alpha=0$ and general x, and by Hafner [1982] in the case of $\alpha\geq 0$ and geneal x.

Theorem (Kuratsubo [1988]). Suppose $\tau \geq 0$, $0 \leq \alpha < \frac{n-1}{2}$, $\sigma \geq 0$ and $\alpha + \sigma \leq \frac{n-1}{2}$. Then we have

$$\sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|} = \Omega\left(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha-\sigma)} (\log t)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)-\tau}\right)$$

everywhere.

Proof. Assume the existence of x such that

$$\sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|} = o\left(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha-\sigma)} (\log t)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)-\tau}\right).$$

Next, applyning Lemma 2 of Section 3 to the case $s = \alpha$, $\beta = \sigma$, $\tau = \tau$ and

$$a_k = \sum_{|m|^2=k} \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|},$$

then we have

$$\sigma_\lambda^s = \sum_{1<|m|^2<\lambda} \left(1 - \frac{|m|^2}{\lambda}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|},$$

and

$$\overline{\sigma_\lambda^s} = \sum_{1<|m|^2<\lambda} \left(1 - \frac{|m|^2}{\lambda}\right)^\alpha e^{2\pi i m x}.$$

Therefore, we have

$$\left|\sum_{1<|m|^2<\lambda} \left(1 - \frac{|m|^2}{\lambda}\right)^\alpha e^{2\pi i m x}\right| \ll \lambda^{\frac{\sigma}{2}} \log^\tau \lambda \, o\left(\lambda^{\frac{1}{2}(\frac{n-1}{2}-\alpha-\sigma)} (\log \lambda)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)-\tau}\right)$$

$$= o\left(\lambda^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} (\log \lambda)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)}\right).$$

This is inconsistent with [S-B-H].

Corollary 1 (Kuratusbo [1988]). If $0 \le \alpha < \alpha_p$ and $1 \le p \le \frac{2n}{n+1}$, then, for any $\tau > \frac{1}{p}$, there exists a function $f(x) \in L^p(T^n)$ such that

$$S_t^\alpha(f;x) = \Omega\left(t^{\frac{1}{2}(\alpha_p-\alpha)}(\log t)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)-\tau}\right) \text{ everywhere,}$$

where we can take for such a function $f(x)$, the function $\zeta_{\sigma,\tau}(x)$.

Corollary 2. For $0 \le \alpha < \frac{n-1}{2}$, we have a precis form of the previous theorem:

$$\sum_{0<|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi imx}}{|m|^{\frac{n-1}{2}-\alpha}} = \Omega\left((\log t)^{\frac{1}{n}(\frac{n-1}{2}-\alpha)}\right) \text{ everywhere .}$$

Especially,

$$\limsup_{t\to+\infty}\left|\sum_{0<|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi imx}}{|m|^{\frac{n-1}{2}-\alpha}}\right| = +\infty \text{ everywhere .}$$

2°. On estimations of L^p-norm for the Bochner-Riesz kernel

For $0 < \xi < 1/2$, defining the function $V_t^\alpha(x)$ as $R_\alpha(t;x)$ or 0 according to whether $|x| \le \xi$,or $|x| > \xi$ and $x \in T^n$, where

$$R_\alpha(t;x) = \int_{|y|^2<t}\left(1-\frac{|y|^2}{t}\right)^\alpha e^{2\pi ixy}dy = \frac{\Gamma(\alpha+1)}{\pi^\alpha}t^{\frac{1}{2}(\frac{n}{2}-\alpha)}\frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|x|)}{|x|^{\frac{n}{2}+\alpha}}.$$

Then

$$\widehat{V_t^\alpha}(m) = \int_{|x|\le\xi} R_\alpha(t;x)e^{-2\pi imx}dx$$

$$= 2\pi\frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{t^{\frac{1}{2}(\frac{n}{2}-\alpha)}}{|m|^{\frac{n}{2}-1}}\int_0^\xi J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}r)J_{\frac{n}{2}-1}(2\pi|m|r)\frac{dr}{r^\alpha}.$$

By the well known formula (Watson [1966], p. 411): *For* $0 \le a$ *and* b,

$$\int_0^{+\infty} \frac{J_\mu(at)J_\nu(bt)}{b^\nu t^{\mu-\nu-1}}dt = \begin{cases} 0 & \textit{if } a < b, \\ \frac{1}{2}\frac{1}{2^{\mu-\nu-1}a^{\nu+1}\Gamma(\mu-\nu)} & \textit{if } a = b, \\ \frac{(a^2-b^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^{\mu}\Gamma(\mu-\nu)} & \textit{if } a > b. \end{cases}$$

we have

$$2\pi\frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{t^{\frac{1}{2}(\frac{n}{2}-\alpha)}}{|m|^{\frac{n}{2}-1}}\int_0^{+\infty} J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}r)J_{\frac{n}{2}-1}(2\pi|m|r)\frac{dr}{r^\alpha}$$

$$= \begin{cases} 0 & \text{if } t < |m|^2, \\ \frac{1}{2} & \text{if } t = |m|^2 \text{ and } \alpha = 0, \\ (1-\frac{|m|^2}{t})^\alpha & \text{if } t > |m|^2. \end{cases}$$

Therefore we have

$$\widehat{V_t^\alpha}(m) = \left(1-\frac{|m|^2}{t}\right)^\alpha \delta_t^\alpha(m) - c_\alpha \frac{t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}}{|m|^{\frac{n-1}{2}}} I_t^\alpha(m),$$

where $\delta_t^\alpha(m) = 1$, $1/2$ and 0 according to whether $|m|^2 < t$, $|m|^2 = t$ and $\alpha = 0$, or $|m|^2 > t$, $c_\alpha = 2\pi\Gamma(\alpha+1)/\pi^\alpha$, and

$$I_t^\alpha(m) = t^{\frac{1}{4}}|m|^{\frac{1}{2}}\int_\xi^{+\infty} J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}r)J_{\frac{n}{2}-1}(2\pi|m|r)\frac{dr}{r^\alpha}.$$

Therefor, for $T \ge t$, we have

$$K_\alpha(t;x) = \sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^\alpha e^{2\pi imx}$$

$$= \sum_{|m|^2<T}\widehat{V_t^\alpha}(m)e^{2\pi mx} + c_\alpha \sum_{0<|m|^2<T}\frac{t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}}{|m|^{\frac{n-1}{2}}}I_t^\alpha(m)e^{2\pi imx}.$$

Since $V_t^\alpha(x) \in L^2(T^n)$, $\sum_{|m|^2<T}\widehat{V_t^\alpha}(m)e^{2\pi imx} \to V_t^\alpha(x)$ in L^2-*norm* as $T \to +\infty$.

Therefore we have the following Lemma.

Lemma 1. *The expansion* $\sum_{m\neq 0} \frac{I_t^\alpha(m)}{|m|^{\frac{n-1}{2}}} e^{2\pi i m x}$ *is the Fourier series of a function* $\theta_t^\alpha(x) \in L^2(T^n)$ *and* $K_\alpha(t;x) = V_t^\alpha(x) + c_\alpha t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}\theta_t^\alpha(x)$.

Lemma 2 (V. A. Il'in and Sh. A. Alimov [1973], p. 525).

$$(1) \qquad |I_t^\alpha(m)| \leq C_1 \xi^{-\alpha},$$

and

$$(2) \qquad |I_t^\alpha(m)| \leq C_2 \frac{\xi^{-\alpha-1}}{|t-|m|^2|}.$$

where C_1 *and* C_2 *are depend only on* α *and* n.

Using this Lemma, we have the following lemma.

Lemma 3 (V. A. Il'in and Sh. A. Alimov [1973], p. 526). $\|\theta_t^\alpha\|_2 \leq C\xi^{-\alpha-1}$.

Lemma 4. *For* $p \geq 1$, *we have*

$$\|V_t^\alpha\|_p = c_{\alpha,p} t^{\frac{n}{2p'}} \left(\int_0^{2\pi\sqrt{t}\xi} s^{(n-1)-p(\frac{n}{2}+\alpha)} \left| J_{\frac{n}{2}+\alpha}(s) \right|^p ds \right)^{\frac{1}{p}},$$

where $c_{\alpha,p} = 2^{\frac{n}{2}+\alpha-\frac{n-1}{p}} \pi^{-\frac{n}{2p}+\frac{n}{2}} \Gamma(\alpha+1)\Gamma(\frac{n}{2})^{-\frac{1}{p}}$.

Proof.

$$\|V_t^\alpha\|_p = \frac{\Gamma(\alpha+1)}{\pi^\alpha} t^{\frac{1}{2}(\frac{n}{2}-\alpha)} \Bigg(\int\limits_{|x|\le\xi} \left| \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}|x|)}{|x|^{\frac{n}{2}+\alpha}} \right|^p dx \Bigg)^{\frac{1}{p}}$$

$$= \frac{\Gamma(\alpha+1)}{\pi^\alpha} t^{\frac{1}{2}(\frac{n}{2}-\alpha)} \Bigg(\int\limits_0^\xi \left| \frac{J_{\frac{n}{2}+\alpha}(2\pi\sqrt{t}r)}{r^{\frac{n}{2}+\alpha}} \right|^p r^{n-1} |\Sigma_{n-1}| \, dr \Bigg)^{\frac{1}{p}}$$

$$= c_{\alpha,p} t^{\frac{n}{2p'}} \Bigg(\int\limits_0^{2\pi\sqrt{t}\xi} s^{n-1-p(\frac{n}{2}+\alpha)} \left| J_{\frac{n}{2}+\alpha}(s) \right|^p ds \Bigg)^{\frac{1}{p}}.$$

Theorem 1. *If $\alpha > \alpha_p$ and $1 \le p \le 2$, then we have*

$$\|K_\alpha(t;\cdot)\|_p = c^*_{\alpha,p} t^{\frac{n}{2p'}} + O(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}),$$

where

$$c^*_{\alpha,p} = c_{\alpha,p} \Bigg(\int\limits_0^{+\infty} s^{n-1-(\frac{n}{2}+\alpha)p} \left| J_{\frac{n}{2}+\alpha}(s) \right|^p ds \Bigg)^{\frac{1}{p}}.$$

$(\alpha > \alpha_p \iff \alpha > (n-1)/2 - n/p' \iff n/p' > (n-1)/2 - \alpha)$

Proof. First,

$$\|V_t^\alpha\|_p - \|\widetilde{\theta_t^\alpha}\|_p \le \|K_\alpha(t;\cdot)\|_p \le \|V_t^\alpha\|_p + \|\widetilde{\theta_t^\alpha}\| \text{ , where } \widetilde{\theta_t^\alpha}(x) = t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \theta_t^\alpha(x).$$

and $\|\widetilde{\theta_t^\alpha}\|_p \le \|\widetilde{\theta_t^\alpha}\|_2 = t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \|\theta_t^\alpha\|_2 = O(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)})$ (by lemma 3.).

Next,

$$t^{\frac{n}{2p'}} \Bigg(\int\limits_{2\pi\sqrt{t}\xi}^{+\infty} s^{n-1-p(\frac{n}{2}+\alpha)} \left| J_{\frac{n}{2}+\alpha}(s) \right|^p ds \Bigg)^{\frac{1}{p}}$$

$$= O\Bigg(t^{\frac{n}{2p'}} \Bigg(\int\limits_{2\pi\sqrt{t}\xi}^{+\infty} s^{n-1-p(\frac{n+1}{2}+\alpha)} ds \Bigg)^{\frac{1}{p}} \Bigg) = O(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}).$$

These complete the proof of the theorem.

Lemma 5.

(1) $$\|V_t^\alpha\|_p = c^{**}_{\alpha_p,p} t^{\frac{n}{2p'}} (\log t)^{\frac{1}{p}} + O(t^{\frac{n}{2p'}}) \ , \ \textit{if } \alpha = \alpha_p,$$

and

(2) $$\|V_t^\alpha\|_p = c^{\star}_{\alpha,p} t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \xi^{\frac{n}{p}-(\frac{n+1}{2}+\alpha)} + O\left(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)-1}\right) \ , \ \textit{if } \alpha < \alpha_p,$$

where $$c^{**}_{\alpha,p} = c_{\alpha,p}\sqrt{\frac{2}{\pi}}\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos s|^p ds\right)^{\frac{1}{p}},$$

and $$c^{\star}_{\alpha,p} = \left(\frac{2^{\frac{1}{p}}}{\pi^{1-\frac{n}{2p}}}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{n}{2})^{\frac{1}{p}}}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos s|^p ds\right)^{\frac{1}{p}}\left\{n - p\left(\frac{n+1}{2}+\alpha\right)\right\}^{-\frac{1}{p}}.$$

Proof. By well known formula: $J_\mu(s) = \sqrt{2/\pi s}\cos(s-\pi\mu/2-\pi/4)+O(s^{-3/2})$, we have

$$\left(\int_0^{2\pi\sqrt{t}\xi} s^{n-1-p(\frac{n}{2}+\alpha)} \left|J_{\frac{n}{2}+\alpha}(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$= O(1) + \sqrt{\frac{2}{\pi}}\left(\int_1^{2\pi\sqrt{t}\xi} s^{n-1-p(\frac{n+1}{2}+\alpha)} \left|\cos\left(s - \frac{n+1+2\alpha}{4}\pi\right)\right|^p ds\right)^{\frac{1}{p}}$$

$$+O\left(\left(\int_1^{2\pi\sqrt{t}\xi} s^{n-1-p(\frac{n+3}{2}+\alpha)} ds\right)^{\frac{1}{p}}\right)$$

Here,

$$\int_1^u s^{n-1-p(\frac{n+1}{2}+\alpha)} |\cos(s+A)|^p \, ds = \int_1^u s^{n-1-p(\frac{n+1}{2}+\alpha)} \left(\int_1^s |\cos(r+A)|^p \, dr\right)' ds$$

$$= \left[s^{n-1-p(\frac{n+1}{2}+\alpha)} \int_1^s |\cos(r+A)|^p \, dr \right]_1^u$$

$$-\left\{ n-1-p\left(\frac{n+1}{2}+\alpha\right)\right\} \int_1^u s^{n-2-p(\frac{n+1}{2}+\alpha)} \left(\int_1^s |\cos(r+A)|^p \, dr \right) ds = I_1 + I_2.$$

Since $\int_1^s |\cos(r+A)|^p dr = s/2\pi \int_0^{2\pi} |\cos r|^p dr + O(1)$, we have

$$I_1 = \left(\frac{1}{2\pi} \int_1^{2\pi} |\cos r|^p dr \right) u^{n-p(\frac{n+1}{2}+\alpha)} + O\left(u^{n-1-p(\frac{n+1}{2}+\alpha)} \right).$$

In the case of $\alpha = \alpha_p$,

$$I_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |cosr|^p dr \right) \int_1^u \frac{1}{s} ds + O\left(\int_1^u s^{n-2-p(\frac{n+1}{2}+\alpha)} ds \right)$$

$$= \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos r|^p dr \right) \log u + O(1).$$

Therefore, in this case,

$$\int_1^u s^{n-1-p(\frac{n+1}{2}+\alpha)} |cos(r+A)|^p ds = \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos r|^p dr \right) \log u + O(1).$$

In the case of $\alpha < \alpha_p$ ($\iff p(\frac{n+1}{2}+\alpha) < n$), $I_2 =$

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |\cos r|^p dr \right) \left\{ n - p\left(\frac{n+1}{2}+\alpha\right) \right\}^{-1} u^{n-p(\frac{n+1}{2}+\alpha)} + O(1) + O\left(u^{n-1-p(\frac{n+1}{2}+\alpha)} \right).$$

Therefore, we have

$$\int_1^u s^{n-1-p(\frac{n+1}{2}+\alpha)} |\cos(r+A)|^p ds$$

$$= \left(\frac{1}{2\pi}\int_0^{2\pi} |\cos r|^p dr\right)\left\{n - p\left(\frac{n+1}{2}+\alpha\right)\right\}^{-1} u^{n-p(\frac{n+1}{2}+\alpha)}$$

$$+O\left(u^{n-1-p(\frac{n+1}{2}+\alpha)}\right) + O(1).$$

These complete the proof of the lemma.

Theorem. *For the case of $\alpha = \alpha_p$, we have*

$$\|K_\alpha(t;\cdot)\|_p = c^{**}_{\alpha_p,p} t^{\frac{n}{2p'}} (\log t)^{\frac{1}{p}} + O(t^{\frac{n}{p'}}).$$

Especially, when $p = 1$,

$$\| \sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{(n-1)/2} e^{2\pi i m x}\|_1 = c_0 \log t + o(\log t),$$

where $c_0 = c^{**}_{\frac{n-1}{2},1} = 4\pi^{-3/2}\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}$.

Remark. When $n = 1$, $p = 1$, the constant c_0 becomes $4/\pi^2$. This is consistent with the so-called *Lebesgue number* L_k. That is,

$$L_k \stackrel{\text{def}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{\nu=-k}^{k} e^{2\pi i \nu x}\right| dx = \frac{4}{\pi^2}\log k + O(1).$$

(See Zygmund [1968], p. 67)

Theorem. *In the case of $\alpha < \alpha_p$ ($\iff 1 \le p < 2n/(n+1+2\alpha)$), for some positive constants C_1, C_2,*

$$C_1 t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \le \|K_\alpha(t;\cdot)\|_p \le C_2 t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}.$$

Especially, for $\alpha = 0$ and $p = 1$, we have the result [I-S].

Proof. First, by lemma 3 and Lemm 5, we obtain the right inequality. Next, for each $\rho \geq 1$, we have

$$\|K_\alpha(t;\cdot)\|_p \geq \left(\int_{|x|<\xi^\rho} |K_\alpha(t;x)|^p dx \right)^{\frac{1}{p}}$$

$$\geq \left(\int_{|x|<\xi^\rho} |V_t^\alpha(x)|^p dx \right)^{\frac{1}{p}} - \left(\int_{|x|<\xi^\rho} |\widetilde{\theta_t^\alpha}(x)|^p dx \right)^{\frac{1}{p}}.$$

where $\widetilde{\theta_t^\alpha}(x) = c_\alpha t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \theta_t^\alpha(x)$. By the definition of the function $V_t^\alpha(x)$, lemma 4 and lemma 5,

$$\left(\int_{|x|<\xi^\rho} |V_t^\alpha(x)|^p \, dx \right)^{\frac{1}{p}} = c_{\alpha,p} t^{\frac{n}{2p'}} \left(\int_0^{2\pi\sqrt{t}\xi^\rho} s^{(n-1)-p(\frac{n}{2}+\alpha)} \left| J_{\frac{n}{2}+\alpha}(s) \right|^p ds \right)^{\frac{1}{p}}$$

$$= c_{\alpha,p}^{**} t^{\frac{1}{2}(\frac{n}{2}-\alpha)} \xi^{\rho(\frac{n-1}{2}-\alpha-\frac{n}{p'})} + O(t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)-1}).$$

On the other hand, by the Schwarz inequality and Lemma 3, we have

$$\left(\int_{|x|<\xi^\rho} \left|\widetilde{\theta_t^\alpha}(x)\right|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{T^n} \left|\widetilde{\theta_t^\alpha}(x)\right|^2 dx \right)^{\frac{1}{2}} \left(\int_{|x|<\xi^\rho} 1 dx \right)^{(1-\frac{p}{2})\frac{1}{p}}$$

$$\ll \|\widetilde{\theta_t^\alpha}\|_2 \xi^{\rho n(\frac{1}{p}-\frac{1}{2})} \ll t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)} \xi^{-\alpha-1+\rho n(\frac{1}{p}-\frac{1}{2})}.$$

Therefore, let k be a number such that $\rho(\frac{n-1}{2} - \alpha - \frac{n}{p'}) < -\alpha - 1 + \rho n(\frac{1}{p} - \frac{1}{2})(\iff \rho > \frac{2(\alpha+1)}{(2\alpha+1)})$, and let ξ be sufficiently small. Then we have the following inequality:

$$\|K_\alpha(t;\cdot)\|_p \gg t^{\frac{1}{2}(\frac{n-1}{2}-\alpha)}.$$

Section 8. A remark on Babenko's theorem

Babenko proved the theorem which gives an estimation of $D(t;x) - K(t:x)$. I also prove it in the following form and my method is the same as B. Novák's in which he proved a pointwise estimation. This work was announced at a seminar on the study of Hardy space by the real analytic method and multiple Fourier analysis at RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY (1980. 2/12∼ 2/14) [1980]:

Theorem. *There exists a constant C independing on x such that*

$$\left| \sum_{|m|^2<t} e^{2\pi imx} - \int_{|y|^2<t} e^{2\pi iyx} dy \right| \leq \begin{cases} Ct^{\frac{n}{2}-1} \log t & \text{for } n > 4, \\ Ct \log^3 t & \text{for } n = 4. \end{cases}$$

for all $x \in T^n$.

The main part of our proof of this theorem is to confirm that each inequality of Satz 3 of Novák [1968] is valid uniformly in x.

Notations: $t > 0$, $\langle t \rangle = \min\{|j-t| : j \in Z\}$, $x \in R^n$, $a_k(x) = \sum_{|m|^2=k} e^{2\pi imx}$, $A(t;x) = \sum_{|m|^2<t} e^{2\pi imx} = \sum_{0\leq k<t} a_k(x)$, $A'(t;x) = A(t;x)$, $A(t;x) + 1/2a_k(x)$ according to whether t is not integer or integer, $K(t;x) = \int_{|y|^2<t} e^{2\pi iyx} dy$ and

$$\Theta(s;x) = \sum_{k=0}^{+\infty} a_k(x) e^{-ks} = \sum_{m \in Z^n} e^{2\pi imx - |m|^2 s} \text{ (Theta-function).}$$

Lemma 1. *For $b > 0$,*

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{e^{us}}{s}ds = \begin{cases} 1+\mu\frac{e^{ub}}{Tu} & \text{for } u > o, \\ 1/2+\mu\frac{b}{T} & \text{for } u = 0, \\ \mu\frac{e^{ub}}{T|u|} & \text{for } u < 0. \end{cases}$$

where μ denotes positive constants depending on T, u and b such that $|\mu| \leq 1$.

Lemma 2. Let $T \gg t^n$, and $\langle t\rangle \geq t^{-n/2}$ or $\langle t\rangle = 0$. Then we have

$$A'(t;x) = \frac{1}{2\pi i}\int_{1/t-iT}^{1/t+iT}\frac{e^{ts}\Theta(s;x)}{s}ds + O(1),$$

where $O(1)$ is uniformly in x.

Proof.

$$\frac{1}{2\pi i}\int_{1/t-iT}^{1/t+iT}\frac{e^{ts}\Theta(s;x)}{s}ds = \sum_{k=0}^{+\infty}a_k(x)\frac{1}{2\pi i}\int_{1/t-iT}^{1/t+iT}\frac{e^{(t-k)s}}{s}ds$$

$$= \sum_{0\leq k<t}a_k(x)\left(1+\mu\frac{e^{(t-k)/t}}{T(t-k)}\right) + \sum_{k=t}a_k(x)\left(1/2+\mu\frac{b}{T}\right) + \sum_{k>t}a_k(x)\left(\mu\frac{e^{(t-k)/t}}{T|t-k|}\right)$$

$$= A'(t;x) + \frac{1}{T}O\left(\sum_{k=0}^{+\infty}|a_k(x)|\alpha_k\right),$$

where $\alpha_k = e^{(t-k)/t}/T|t-k|$, $1/(Tt)$ according to whether $k \neq t$ or $k = t$.

Put $S_1 = \sum_{k<t/2 \text{ or } k>2t}|a_k(x)|\alpha_k$ and $S_2 = \sum_{t/2\leq k\leq 2t}|a_k(x)|\alpha_k$.

First, we estimate S_1. Since $|a_k(x)| \ll k^{(n-1)/2}$ and $|k-t| > t/2$ if $k < t/2$ or $k > 2t$,

$$S_1 \ll \frac{1}{t}\sum_{k=0}^{+\infty}|a_k(x)|e^{(t-k)/t} \ll \frac{1}{t}\sum_{k=0}^{+\infty}k^{(n-1)/2}e^{(t-k)/t}$$

$$\ll \frac{1}{t}t^{\frac{n+1}{2}} = O(T) \text{ , since } \sum_{k=0}^{+\infty}k^{\frac{n-1}{2}}e^{-k/t} \ll \int_0^{+\infty}r^{\frac{n-1}{2}}e^{-r/t}dr = \Gamma(\frac{n+1}{2})t^{\frac{n+1}{2}}.$$

Next, we estimate S_2 in two cases separately, i.e., according as t is integral or not, respectively. In the *"not integer"* case, let N be $\langle t\rangle = |t-N|$. Then

$$S_2 = \left\{ \sum_{\frac{t}{2}\le k\le N-1} + \sum_{k=N} + \sum_{N+1\le k\le 2t} \right\} |a_k(x)| \frac{e^{(t-k)/t}}{|t-k|}$$

$$\ll t^{\frac{n-1}{2}} \left(\sum_{j=1}^{[t]} \frac{1}{j} + \frac{1}{\langle t\rangle} \right) \ll t^{\frac{n-1}{2}} (\log t + t^{\frac{n}{2}}) \ll T.$$

In the *"integer"* case, let t be N. Then

$$S_2 = \left\{ \sum_{\frac{t}{2}\le k\le N-1} + \sum_{N+1\le k\le 2t} \right\} |a_k(x)| \frac{e^{(t-k)/t}}{|t-k|} + \sum_{n=N} |a_k(x)| \frac{1}{t}$$

$$\ll t^{\frac{n-1}{2}} (\log t + \frac{1}{t}) \ll T.$$

Therefore we complete the proof of the Lemma.

Definition. (The generalized Gaussian sums)

$$S_{h,k,(m)} = S_{h,k,(m_1,\ldots,m_n)} \stackrel{\text{def}}{=} \sum_{a_1=1}^{k} \sum_{a_2=1}^{k} \cdots \sum_{a_n=1}^{k} \exp\left(-2\pi i \frac{h}{k} |a|^2 + \frac{2\pi i}{k} ma \right).$$

Lemma 3. Let s be a complex number, $Re(s) > 0$. Then

$$\Theta(s;x) = \frac{\pi^{\frac{n}{2}}}{k^n (s - \frac{2\pi i h}{k})^{\frac{n}{2}}} \sum_{m\in Z^n} S_{h,k,(m)} \exp\left(-\frac{\pi^2 |m-kx|^2}{k^2 (s - \frac{2\pi i h}{k})} \right),$$

where by s^β, $\beta > 0$, we mean a branch of the function s^β which is positive for positive values of s. Further, we have

$$\left| S_{h,k,(m)} \right| \le (2\sqrt{k})^n \text{ and, especially, } \left| S_{h,k,(m)} \right| = k^{\frac{n}{2}} \text{, if } (k,h) = 1.$$

Proof.

$$\sum_{j=-\infty}^{+\infty} e^{2\pi i j x - j^2 s} = \sum_{j=-\infty}^{+\infty} \left(\sum_{a=1}^{k} e^{2\pi i (jk+a)x - (jk+a)^2 s} \right)$$

$$= \sum_{a=1}^{k} e^{-2\pi i \frac{h}{k} a^2} \sum_{j=-\infty}^{+\infty} e^{-(s-2\pi i \frac{h}{k})|a+jk|^2 + 2\pi i (a+jk)x}$$

$$= \sum_{a=1}^{k} e^{-2\pi i \frac{h}{k} a^2} \left\{ \frac{\pi^{\frac{1}{2}}}{k(s-2\pi i \frac{h}{k})^{\frac{1}{2}}} \sum_{j=-\infty}^{+\infty} \exp\left(2\pi i \frac{j}{k} a - \frac{\pi^2 (j-xk)^2}{k^2 (s - 2\pi i \frac{h}{k})} \right) \right\},$$

where we used the well known Theta-transformation formula:

$$\sum_{j=-\infty}^{+\infty} e^{-\pi\omega|a+kj|^2 + 2\pi(a+kj)x} = \frac{1}{k\omega^{\frac{1}{2}}} \sum_{j=-\infty}^{+\infty} e^{-\pi(j-kx)^2 \frac{1}{k^2\omega} + 2\pi i \frac{j}{k} a}$$

$$= \frac{\pi^2}{k(s-2\pi i \frac{h}{k})^{\frac{1}{2}}} \sum_{j=-\infty}^{+\infty} e^{-\frac{\pi^2 (j-kx)^2}{k^2(s-2\pi i \frac{h}{k})}} \sum_{a=1}^{k} e^{-2\pi i \frac{h}{k} a^2 + 2\pi i \frac{j}{k} a}.$$

Therefore, we have

$$\Theta(s;x) = \Pi_{\nu=1}^{n} \left\{ \sum_{m_\nu=-\infty}^{+\infty} e^{2\pi i m_\nu x_\nu - m_\nu{}^2 s} \right\}$$

$$= \frac{\pi^{\frac{n}{2}}}{k^n (s-2\pi i \frac{h}{k})^{\frac{n}{2}}} \Pi_{\nu=1}^{n} \left\{ \sum_{m_\nu=-\infty}^{+\infty} e^{-\frac{\pi^2 (m_\nu - kx)^2}{k^2(s-2\pi i \frac{h}{k})}} \sum_{a_\nu=1}^{k} e^{-2\pi i \frac{h}{k} a_\nu{}^2 + 2\pi i \frac{m_\nu}{k} a_\nu} \right\}$$

$$= \frac{\pi^{\frac{n}{2}}}{k^n (s-2\pi i \frac{h}{k})^{\frac{n}{2}}} \sum_{m \in Z^n} S_{h,k,(m)} e^{-\frac{\pi^2 |m-kx|^2}{k^2(s-2\pi i \frac{h}{k})}}.$$

Therefore this completes the proof of the Lemma.

Definition. $P_k(x) \overset{\text{def}}{=} \max\{\langle kx_\nu \rangle : 1 \le \nu \le n\}$, $R_k(x) \overset{\text{def}}{=} \min\{|m-kx|^2 : m \in Z^n\}$ and $S_{h,k}(x) \overset{\text{def}}{=} S_{h,k,(m)}$ if m satisfies $R_k(x) = |m-kx|^2$. (It is easy to see that $1/nP_k{}^2(x) \le R_k(x) \le nP_k{}^2(x)$.) Further, the Farey fractions corresponding to $t^{1/2}$ means fractions of the form h/k where $(h,k) = 1$ and $0 < k \le t^{1/2}$. If $h_1/k_1 < h/k < h_2/k_2$ are three consecutive fractions of this type (i.e., between h_1/k_1 and h_2/k_2 lies just one Farey fraction coresponding to $t^{1/2}$, namely, h/k), then we denote by $V_{h,k}$ the interval

$$\left[2\pi\frac{h+h_1}{k+k_1}\ ,\ 2\pi\frac{h+h_2}{k+k_2}\right].$$

Then we have

$$V_{h,k}=\left[2\pi\frac{h}{k}-\frac{\theta_1}{kt^{1/2}}\ ,\ 2\pi\frac{h}{k}+\frac{\theta_2}{kt^{1/2}}\right],$$

where $\pi\le\theta_1,\ \theta_2<2\pi$. Thus, for $t\in V_{h,k}$, we have

$$|t-\frac{2\pi h}{k}|\ll 1/kt^{1/2}.$$

The union of all these intervals is the whole real axis. In particular, we have $V_{0,1}=[-w\ ,\ w]$, where $w=2\pi/(1+[t^{1/2}])\ll 1/t^{1/2}$.

Lemma 4. If $\sigma\in V_{h,k}$, then we have

$$\Theta(s;x)=\frac{\pi^{\frac{n}{2}}S_{h,k}(x)}{k^n(s-2\pi i\frac{h}{k})^{\frac{n}{2}}}e^{-\frac{\pi^2R_k(x)}{k^2(s-2\pi i\frac{h}{k})}}+C\mu\frac{t^{\frac{n}{2}}e^{-\frac{ct}{k^2(1+t^2\left|\sigma-2\pi\frac{h}{k}\right|^2)}}}{k^{\frac{n}{2}}(1+t^2\left|\sigma-2\pi\frac{h}{k}\right|^2)^{\frac{n}{4}}},$$

where $s=1/t+i\sigma$ and μ is complex numbers depending on x which $|\mu|\le 1$.

Proof. Let m' be the integer for which $R_k(x)=|m'-kx|^2$. Then we have

$$\Theta(s;x)=\frac{\pi^{\frac{n}{2}}}{k^n(s-2\pi\frac{h}{k})^{\frac{n}{2}}}S_{h,k}(x)e^{-\frac{\pi^2R_k(x)}{k^2(s-2\pi i\frac{h}{k})}}+\frac{\pi^{\frac{n}{2}}}{k^n(s-2\pi i\frac{h}{k})^{\frac{n}{2}}}\sum_{m\neq m'}\{\ldots\}$$

$=$ (main term)+(error term).

$$(\text{error term})\ll\frac{1}{k^{\frac{n}{2}}\left|s-2\pi i\frac{h}{k}\right|^{\frac{n}{2}}}\sum_{m\neq m'}k^{\frac{n}{2}}\exp\left(-\pi^2|m-kx|^2\ \mathrm{Re}\frac{1}{k^2(s-2\pi i\frac{h}{k})}\right)$$

$$\ll\frac{t^{\frac{n}{2}}}{k^{\frac{n}{2}}(1+t^2(\sigma-2\pi\frac{h}{k})^2)^{\frac{n}{4}}}\left\{\sum_{j\neq 0}e^{-\frac{Cj^2t}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}}\right\}\left\{\sum_{j=-\infty}^{+\infty}e^{-\frac{cj^2t}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}}\right\}^{n-1}$$

$$\ll \frac{t^{\frac{n}{2}}}{k^{\frac{n}{2}}(1+t^2(\sigma-2\pi\frac{h}{k})^2)^{\frac{n}{4}}} e^{-\frac{Ct}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}} \left\{1-e^{-\frac{ct}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}}\right\}^{-n}$$

$$\ll \frac{t^{\frac{n}{2}}}{k^{\frac{n}{2}}(1+t^2(\sigma-2\pi\frac{h}{k})^2)^{\frac{n}{4}}} e^{-\frac{Ct}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}},$$

because

$$\frac{t}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)} \gg \frac{t}{k^2+t} \geq \frac{1}{2} \quad \text{,when } \sigma \in V_{h,k} \text{ , i.e., } \left|\sigma - 2\pi\frac{h}{k}\right| \ll \frac{1}{kt^{\frac{1}{2}}}.$$

Lemma 5. For every t such that $\langle t\rangle \geq t^{-\frac{1}{2}}$ *or* $\langle t\rangle = 0$,

$$A'(t;x) = \pi^{\frac{n}{2}} \sum_{k\leq\sqrt{t}} \sum_{|h|\leq kT_1} \frac{S_{h,k}(x)}{k^n} \frac{1}{2\pi} \int_{V_{h,k}} \frac{\exp\left(ts - \frac{\pi^2 R_k(x)}{k^2(s-2\pi i\frac{h}{k})}\right)}{(s-2\pi i\frac{h}{k})^{\frac{n}{2}} s} d\sigma + O(t^{\frac{n}{4}}\log t),$$

where $T_1 = [t^n]$, $s = 1/t + i\sigma$, *and h and k satisfy* $(h,k) = 1$ *always.*

Proof. We cover the interval $[-2\pi T_1, 2\pi T_1]$ by $\{V_{h,k} : h/k$ are the Farey fractions corresponding to $t^{1/2}\}$. The length of the most efficient covering is $2T$ which satisfies the relation: $T = 2\pi T_1 + O(1/t^{1/2})$.

By Lemma 2, we have

$$A'(t;x) = \sum_{k\leq\sqrt{t}} \sum_{|h|\leq kT_1} \frac{1}{2\pi} \int_{V_{h,k}} \frac{e^{ts}\Theta(s;x)}{s} d\sigma + O(1),$$

and by Lemma 4,

$$\frac{1}{2\pi}\int_{V_{h,k}} \frac{e^{ts}\Theta(s;x)}{s} d\sigma = \left(\frac{\pi}{k}\right)^{\frac{n}{2}} S_{h,k}(x) \frac{1}{2\pi} \int_{V_{h,k}} \frac{\exp\left(ts - \frac{\pi^2 R_k(x)}{k^2(s-2\pi i\frac{h}{k})}\right)}{(s-2\pi i\frac{h}{k})^{\frac{n}{2}} s} d\sigma$$

$$+C\mu\left(\frac{t}{k}\right)^{\frac{n}{2}} \int_{V_{h,k}} \frac{\exp\left(-\frac{ct}{k^2(1+t^2(\sigma-2\pi\frac{h}{k})^2)}\right)}{(1+t^2(\sigma-2\pi\frac{h}{k})^2)^{\frac{n}{4}}|s|} d\sigma.$$

Denoting the second term by $T_{h,k}$,

$$T_{h,k} = C\mu\left(\frac{t}{k}\right)^{\frac{n}{2}} \int_{-c/k\sqrt{t}}^{c/k\sqrt{t}} \frac{\exp(-\frac{ct}{k^2(1+t^2u^2)})}{(1+t^2u^2)^{\frac{n}{4}}\left|1/t+i(u+2\pi\frac{h}{k})\right|}du.$$

In the case of $(h,k)=(0,1)$,

$$T_{0,1} \ll t^{\frac{n}{4}+\frac{1}{2}} \int_0^{c/\sqrt{t}} \left(\frac{t}{1+t^2u^2}\right)^{\frac{n}{4}+\frac{1}{2}} e^{-\frac{ct}{1+t^2u^2}}du \ll t^{\frac{n}{4}},$$

because the function $\sigma^d e^{-c\sigma}$ is bounded on $[0,+\infty)$ for each c, $d>0$. Next, since, in the case of $h \neq 0$ (we assume $h>0$ after this), $\left|1/t+i(u+2\pi\frac{h}{k})\right| \gg h/k$,

$$T_{h,k} \ll \left(\frac{t}{k}\right)^{\frac{n}{2}} \frac{k}{h} \int_0^{c/k\sqrt{t}} \frac{e^{-\frac{ct}{k^2(1+t^2u^2)}}}{(1+t^2u^2)^{\frac{n}{4}}}du$$

$$= t^{\frac{n}{4}}\frac{k}{h} \int_0^{c/k\sqrt{t}} \left(\frac{t}{k^2(1+t^2u^2)}\right)^{\frac{n}{4}} e^{-\frac{ct}{k^2(1+t^2u^2)}}du \ll t^{\frac{n}{4}-\frac{1}{2}}\frac{1}{h}.$$

Therefore we have

$$\sum_{k\le\sqrt{t}}\sum_{|h|\le kT_1} T_{h,k} \ll t^{\frac{n}{4}} + t^{\frac{n}{4}-\frac{1}{2}} \sum_{k\le\sqrt{t}}\sum_{0<|h|\le kT_1} \frac{1}{|h|} \ll t^{\frac{n}{4}}\log t.$$

This completes the proof of the Lemma 5.

Lemma 6. *For $a \in R$, $b \ge 0$,*

$$F(t;a,b) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{1/t-i\infty}^{1/t+\infty} \frac{e^{ts-b/(s-ia)}}{(s-ia)^{\frac{n}{2}}s}ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ts-b/(s-ia)}}{(s-ia)^{\frac{n}{2}}s}d\sigma,$$

where $s=1/t+i\sigma$. Then, If $\langle t\rangle \ge t^{-\frac{n}{2}}$ or $\langle t\rangle = 0$, we have

$$A'(t;x) = \pi^{\frac{n}{2}} \sum_{k\le\sqrt{t}}\sum_{|h|\le kT_1} \frac{S_{h,k}(x)}{k^n}F(t:2\pi\frac{h}{k},\frac{\pi^2R_k(x)}{k^2}) + O(t^{\frac{n}{4}}\log^{\kappa} t),$$

where $\kappa = 1$ or 2 according to whether $n>4$ or $n=4$.

Proof.

$$F(t;0,\pi^2R_k(x)) - \frac{1}{2\pi}\int\limits_{V_{0,1}} \frac{e^{ts-\frac{\pi^2 R_1(x)}{s}}}{s^{n/2+1}}d\sigma = \frac{1}{2\pi}\int\limits_{|\sigma|\geq c/\sqrt{t}} \frac{e^{ts-\frac{\pi^2 R_1(x)}{s}}}{s^{n/2+1}}d\sigma$$

$$\ll \int\limits_{|\sigma|\geq c/\sqrt{t}} \frac{d\sigma}{|1/t+i\sigma|^{n/2+1}} \ll \int\limits_{|\sigma|\geq c/\sqrt{t}} \frac{d\sigma}{\sigma^{n/2+1}} \ll t^{\frac{n}{4}},$$

because $Re\{ts-\pi^2 R_1(x)/s\} \leq 1$.

Next, in the case of $h \neq 0$(we assume $h>0$.),

$$\frac{S_{h,k}(x)}{k^n}\left(F(t;2\pi\frac{h}{k},\frac{\pi^2R_k(x)}{k^2}) - \frac{1}{2\pi}\int\limits_{V_{h,k}} \frac{\exp(ts-\frac{\pi^2R_k(x)}{k^2(s-2\pi\frac{h}{k})})}{(s-2\pi\frac{h}{k})^{\frac{n}{2}}s}d\sigma\right)$$

$$\ll \frac{t^{n/2+1}}{k^{n/2}}\int\limits_{c/k\sqrt{t}} \frac{du}{(1+t^2u^2)^{\frac{n}{4}}(1+t^2(u-2\pi\frac{h}{k})^2)^{1/2}} \ll \frac{t^{n/2+1}}{k^{n/2}}\left\{\int\limits_{c/k\sqrt{t}}^{\pi h/k} + \int\limits_{\pi h/k}^{+\infty}\right\}$$

$= I_{h,k}$.

In the case of $\frac{c}{k\sqrt{t}} \leq u \leq \pi h/k$, (denominator of integrand)$\gg$ $(t^2u^2)^{n/4}(t^2\frac{h^2}{k^2})^{1/2}$ $= t^{n/2+1}u^{n/2}h/k$.

On the other hand, in the case of $\pi h/k \leq u < +\infty$, (denominator of integrand)$\gg$ $(t^2u^2)^{n/4} = (tu)^{n/2}$. Therefore,

$$I_{h,k} \ll \frac{t^{n/2+1}}{k^{n/2}}\left\{\int\limits_{\frac{c}{k\sqrt{t}}}^{\pi h/k} \frac{du}{t^{n/2+1}u^{n/2}h/k} + \int\limits_{\pi h/k}^{+\infty} \frac{du}{t^{n/2}u^{n/2}}\right\}$$

$$\ll \frac{1}{hk^{n/2-1}}\int\limits_{c/k\sqrt{t}}^{+\infty} \frac{du}{u^{n/2}} + \frac{t}{k^{n/2}}\int\limits_{\pi h/k}^{+\infty} \frac{du}{u^{n/2}} \ll \frac{t^{n/4-1/2}}{h} + \frac{t}{kh^{n/2-1}}.$$

Therefore, we have

$$\sum_{k\leq\sqrt{t}}\sum_{|h|\leq kT_1} I_{h,k} \ll t^{n/4} + \sum_{k\leq\sqrt{t}}\sum_{0<|h|\leq kT_1}\left\{\frac{t^{n/4-1/2}}{h} + \frac{t}{kh^{n/2-1}}\right\}$$

$$\ll t^{n/4} + t^{n/4-1/2} \sum_{k \le \sqrt{t}} \sum_{0<|h|\le kT_1} \frac{1}{h} + t \sum_{k\le\sqrt{t}} \frac{1}{k} \sum_{0<|h|\le kT_1} \frac{1}{h^{n/2-1}}$$

$$\ll t^{n/4} + t^{n/4}\log t + \begin{cases} t\log t & \text{for } n>4, \\ t\log^2 t & \text{for } n=4. \end{cases}$$

This completes the proof of Lemma 6.

Lemma 7 (The Hankel's formula). *For $d>0$, $A>0$, $B>0$ and $\mu>0$, we have*

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{e^{As-\frac{B}{s}}}{s^{\mu+1}}ds = \left(\frac{A}{B}\right)^{\frac{\mu}{2}} J_\mu(2\sqrt{AB}).$$

(See Landau [1962] (p. 262))

Lemma 8. *Let ν be the smallest integer such that $n/4-1\le\nu$. Then we have*

$$A'(t;x) - \pi^{\frac{n}{2}} t^{\frac{n}{2}} j_{\frac{n}{2}}(2\pi\sqrt{t}|x|)$$

$$= \pi^{\frac{n}{2}} \sum_{0\le r<\frac{n}{4}-1} \frac{(-1)^r}{(2\pi i)^{r+1}} \sum_{k\le\sqrt{t}} \frac{t^{\frac{n}{2}-r-1}}{k^{n-r-1}} j_{\frac{n}{2}-r-1}\left(2\pi\sqrt{\frac{R_k(x)t}{k^2}}\right) \sum_{0<|h|\le kT_1} \frac{S_{h,k}(x)e^{2\pi i\frac{h}{k}t}}{h^{r+1}}$$

$+O(t^{\frac{n}{4}}\log^\kappa t)$, *where* $j_\mu(z) = J_\mu(z)/(\frac{1}{2}z)^\mu$.

Proof. First,

$$F(t;0,\pi^2R_1(x)) = \frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+\infty} \frac{e^{ts-\frac{\pi^2R_1(x)}{s}}}{s^{\frac{n}{2}+1}}ds = t^{n/2} j_{\frac{n}{2}}(2\pi\sqrt{tR_1(x)}).$$

Since $1/(s+ia) = \sum_{r=0}^{\nu-1}(-1)^r s^r/(ia)^{r+1} + 1/(1+ia)(-1)^\nu s^\nu/(ia)^\nu$, we have

$$F(t;a,b) = \frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty} \frac{e^{ts-\frac{b}{s-ia}}}{(s-ia)^{\frac{n}{2}}s}ds = \frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty} \frac{e^{t(s+ia)-b/s}}{s^{\frac{n}{2}}(s+ia)}ds$$

$$= e^{iat}\sum_{r=0}^{\nu-1}\frac{(-1)^r}{(ia)^{r+1}}\frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-b/s}}{s^{\frac{n}{2}-r}}ds + e^{iat}\frac{(-1)^\nu}{(ia)^\nu}\frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-b/s}}{s^{\frac{n}{2}-\nu}(s+ia)}ds$$

$$= e^{iat}\sum_{r=0}^{\nu-1}\frac{(-1)^r}{(ia)^{r+1}}\left(\frac{t}{b}\right)^{\frac{1}{2}(\frac{n}{2}-r-1)}J_{\frac{n}{2}-r-1}(2\sqrt{tb}) + e^{iat}\frac{(-1)^\nu}{(ia)^\nu}\frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-b/s}}{s^{\frac{n}{2}-\nu}(s+ia)}ds$$

$$= e^{iat}\sum_{r=0}^{\nu-1}\frac{(-1)^r}{(ia)^{r+1}}t^{\frac{n}{2}-r-1}j_{n/2-r-1}(2\sqrt{tb}) + e^{iat}\frac{(-1)^\nu}{(ia)^\nu}\frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-b/s}}{s^{n/2-\nu}(s+ia)}ds,$$

and

$$\left|\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-b/s}}{s^{\frac{n}{2}-\nu}(s+ia)}ds\right| \ll \int_{-\infty}^{+\infty}\frac{Re\{e^{t(1/t+i\sigma)-b/(1/t+i\sigma)}\}}{|1/t+i\sigma|^{\frac{n}{2}-\nu}\,|1/t+i(\sigma+a)|}d\sigma$$

$$\ll t^{\frac{n}{2}-\nu}\int_{-\infty}^{+\infty}\frac{1}{(1+|\sigma|)^{\frac{n}{2}-\nu}(1+|\sigma+at|)}d\sigma \ll t^{\frac{n}{2}-\nu}\int_{-\infty}^{+\infty}\frac{1}{(1+|\sigma|)(1+|\sigma+at|)}d\sigma$$

$$\ll t^{n/2-\nu}\frac{\log(1+|a|t)}{|a|t}\text{ , because }\frac{n}{2}-\nu > \frac{n}{4}\geq 1.$$

Therefore, we have the following:

$$A'(t;x) - (\pi t)^{\frac{n}{2}}j_{\frac{n}{2}}(2\pi\sqrt{tR_1(x)})$$

$$= \pi^{\frac{n}{2}}\sum_{r=0}^{\nu-1}\frac{(-1)^r}{(2\pi i)^{r+1}}\sum_{k\leq\sqrt{t}}\frac{t^{n/2-r-1}}{k^{n-r-1}}j_{\frac{n}{2}-r-1}\left(2\pi\sqrt{t\frac{R_k(x)}{k^2}}\right)\sum_{0<|h|\leq kT_1}\frac{S_{h,k}(x)}{h^{r+1}}e^{2\pi i\frac{h}{k}t}$$

$$+\frac{(-1)^\nu\pi^{\frac{n}{2}}}{(2\pi i)^\nu}\sum_{k\leq\sqrt{t}}\frac{1}{k^{n-\nu}}\sum_{0<|h|\leq kT_1}\frac{S_{h,k}(x)}{h^\nu}e^{2\pi i\frac{h}{k}t}\frac{1}{2\pi i}\int_{1/t-i\infty}^{1/t+i\infty}\frac{e^{ts-\frac{\pi^2R_k(x)}{sk^2}}}{s^{n/2-\nu}(s+2\pi i\frac{h}{k})}ds$$

$+O(t^{\frac{n}{4}}\log^\kappa t)$ =(main term)+(error term)+$O(t^{\frac{n}{4}}\log^\kappa t)$.

$$(\text{error term}) \ll t^{n/2-\nu-1}\log t \sum_{k\le\sqrt{t}} \frac{1}{k^{n/2-\nu-1}} \sum_{0<|h|\le kT_1} \frac{1}{h^{\nu+1}} \ll \begin{cases} t^{\frac{n}{4}}\log t & \text{for } n>4, \\ t^{\frac{n}{4}}\log^3 t & \text{for } n=4. \end{cases}$$

This completes the proof of Lemma 8.

Lemma 9. Let $T_1=[t^n]$, and $\langle t\rangle \ge t^{-n/2}$ or $\langle t\rangle = 0$. Then we have

$$A'(t;x)-(\pi t)^{n/2} j_{\frac{n}{2}}(2\pi\sqrt{t}|x|) \ll t^{\frac{n}{2}-1}\log^{\kappa} t,$$

where $\kappa=1$ or 3 according to whether $n>4$ or $n=4$.

Proof. We consider only the case of $n>4$. By the boundedness of the function $j_\mu(z)$ and the estimation of the generalized Gaussian sum: $S_{h,k,(m)} \ll k^{n/2}$, we have

$$A'(t;x)-(\pi t)^{\frac{n}{2}} j_{\frac{n}{2}}(2\pi\sqrt{t}|x|) \ll \sum_{0\le r<\frac{n}{4}-1} \sum_{k\le\sqrt{t}} \frac{t^{n/2-r-1}}{k^{n-r-1}} \sum_{0<|h|\le kT_1} \frac{k^{\frac{n}{2}}}{|h|^{r+1}} + t^{\frac{n}{4}}\log^{\kappa} t$$

$$\ll \sum_{0<r<\frac{n}{4}-1} t^{\frac{n}{2}-r-1} \sum_{k\le\sqrt{t}} \frac{1}{k^{n/2-r-1}} \sum_{o<h\le kT_1} \frac{1}{h^{r+1}} + t^{\frac{n}{2}-1} \sum_{k\le\sqrt{t}} \frac{1}{k^{\frac{n}{2}-1}} \sum_{0<h\le kT_1} \frac{1}{h} + t^{\frac{n}{4}}\log^{\kappa} t$$

$$\ll \sum_{0<r<\frac{n}{4}-1} t^{\frac{n}{2}-r-1} + t^{\frac{n}{2}-1}\log t + t^{\frac{n}{4}}\log t \ll t^{\frac{n}{2}-2} + t^{\frac{n}{2}-1}\log t + t^{\frac{n}{4}}\log^{\kappa} t \ll t^{\frac{n}{2}-1}\log t.$$

Therefore we complete the proof of Lemma 9.

Proof of theorem. Suppose $0<\langle t\rangle < t^{-\frac{n}{2}}$. We consider t_1 such that $\langle t_1\rangle = t_1^{-\frac{n}{2}}$ and $|t_1-t| \le t_1^{-\frac{n}{2}}$. Then

$$A'(t;x)=A'(t_1;x)=A(t;x) \text{ and } t_1^{\frac{n}{2}-1}-t^{\frac{n}{2}-1} \ll 1 \text{ by “mean value theorem”}$$

and

$$(\pi t_1)^{\frac{n}{2}} j_{\frac{n}{2}}(2\pi\sqrt{t_1 R_1(x)}) - (\pi t)^{\frac{n}{2}} j_{\frac{n}{2}}(2\pi\sqrt{t R_1(x)}) = \int\limits_{|y|<t_1} e^{2\pi i y x} dy - \int\limits_{|y|<t} e^{2\pi i y x} dy$$

$$= \int\limits_{t \le |y| < t_1} e^{2\pi i y x} dy \ll \int\limits_{t \le |y| < t_1} 1 dy = t_1^{\frac{n}{2}} - t^{\frac{n}{2}} \ll 1.$$

Therefore for all $t > 1$ we have

$$A'(t;x) - (\pi t)^{\frac{n}{2}} j_{\frac{n}{2}}(2\pi\sqrt{t R_1(x)}) \ll t^{\frac{n}{2}-1} \log^{\kappa} t.$$

For not integral t, we obtain the wanted inequality. On the other hand, for integer $t = k$, we also obtain it by the fact that $\lim_{t \to k-0} A'(t;x) = A(k;x)$. This completes the proof of the Theorem.

Section 9. On an unsolved problem of multiple Fourier series

In Chapter I (Section 5), we point out the existence of a big unsolved problem in multiple Fourier series, that is, *When* $n \geq 2$, $1 \leq p \leq 2$ *and* $\alpha > \max\{\frac{n-1}{2} - \frac{n}{p'}, 0\}$, *does* $S_t^\alpha(f;x)$ *converge almost everywhere? (Problem II)*

This problem is closely related to lattice point problems with weight in the following sense.

Theorem. *If Problem II can be solved affirmatively, then we have the following:*

For $0 \leq \alpha < \frac{n-1}{2}$

$$\sum_{|m|^2<t}\left(1-\frac{|m|^2}{t}\right)^\alpha e^{2\pi imx} = O\left(t^{\frac{n-1+2\alpha}{4}+\varepsilon}\right) \text{ almost everywhere in } x \in R^n.$$

Lemma 1. *For* $\alpha \geq 0$ *and any numerical sequence* $\{a_k\}_{k=0}^{+\infty}$, *suppose*

$$\sum_{k<t}(t-k)^\alpha a_k = At^\alpha + o(t^\alpha).$$

Then, we have the following estimation:

$$A_k - A = o(k^\alpha) \qquad \text{where } A_k = \sum_{j<k} a_j.$$

(See Chandrasekharan and Minakshisundaram [1952] (p. 13, COROLLARY 1.61.))

Lemma 2. *If* $\sum_{k<t}(1-\frac{k}{t})^\alpha a_k$ *converges to* A *for some* $\alpha \geq 0$, *then* $\sum_{k<t} a_k b_k$ *converges for each sequence* $\{b_k\}_{k=0}^{+\infty}$ *which* $b_k = (k^\beta)^{-1}$ *for some* $\beta > \alpha$ *or* $b_k = (k^\alpha \log^\tau k)^{-1}$ *for some* $\tau > 1$.

Proof. We consider only the case of $b_k = (k^\alpha \log^\tau k)^{-1}$. First, considering $a_0 - A$ as a_0 renewedly, it is enough to consider for the case $A = 0$. Then, by Lemma 1. we have: $A_k = o(k^\alpha)$. Therefore

$$\sum_{k=\mu}^{\nu} a_k b_k = \sum_{k=\mu}^{\nu} (A_{k+1} - A_k) b_k = \sum_{\mu+1}^{\nu} A_k (b_{k-1} - b_k) + A_{\nu+1} b_\nu - A_\mu b_\mu$$

$$= o\left(\sum_{k=\mu+1}^{\nu} \frac{1}{k \log^\tau k} \right) + o\left(\frac{1}{\log^\tau \nu} \right) + o\left(\frac{1}{\log^\tau \mu} \right) = o(1).$$

Proof of theorem. (1) The case of $\alpha > 0$: For any $\varepsilon > 0$, let σ be $\frac{n-1}{2} - \alpha + \varepsilon$ and let p be a number such that $1 \le p \le 2$ and $\frac{n-1}{2} - \alpha < \frac{n}{p'} < \sigma$. Then

$$\zeta_{\sigma 0} \in L^p(T^n) \text{ and } \alpha > \frac{n-1}{2} - \frac{n}{p'}.$$

From our hypothesis, we have:

$$S_t^\alpha(\zeta_{\sigma 0}; x) = \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t} \right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma} \to \zeta_{\sigma 0}(x) \text{ almost everywhere in } x \text{ as } t \to +\infty.$$

Now, let $a_k(x)$ be $\sum_{|m|^2 = k} e^{2\pi i m x}/|m|^\sigma$. Then $\sum_{k<t} a_k(x)(1 - \frac{k}{t})^\alpha = O(1)$ almost everywhere in x.

Therefore, by Lemma 2 of Section 3, we have

$$\sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t} \right)^\alpha e^{2\pi i m x} = \sum_{k<t} \left(1 - \frac{k}{t} \right)^\alpha k^{\frac{\sigma}{2}} a_k(x)$$

$$= O(t^{\frac{\sigma}{2}}) = O(t^{\frac{n-1-2\alpha+2\varepsilon}{4}}) \text{ almost everywhere in } x.$$

This completes the proof of Case (1).

(2) The case of $\alpha = 0$: Let σ be $\frac{n-1}{2}$ and, for any $\varepsilon > 0$, let p be a number such that $\frac{n-1}{2} > \frac{n}{p'} > \frac{n-1}{2} - \varepsilon$. Then $\varepsilon > \frac{n-1}{2} - \frac{n}{p'} > 0$ and $\zeta_{\sigma 0} \in L^p(T^n)$. Therefore, from our hypothesis, we have the following.

$$\sum_{|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^{\varepsilon} \frac{e^{2\pi imx}}{|m|^{\sigma}} \to \zeta_{\sigma 0}(x) \text{ a.e. as } t \to +\infty.$$

By Lemma2,

$$\sum_{k\geq 0} \frac{a_k(x)}{k^{2\varepsilon}} \text{ converges almost everywhere, where } a_k(x) = \sum_{|m|^2=k} \frac{e^{2\pi imx}}{|m|^{\sigma}}.$$

Therefore, by Lemma 2 of Section 3, we have

$$\sum_{|m|^2<t} e^{2\pi imx} = \sum_{k<t} k^{2\varepsilon+\frac{\sigma}{2}} \sum_{|m|^2<t} \frac{e^{2\pi imx}}{|m|^{\sigma+4\varepsilon}}$$

$$= O(t^{2\varepsilon+\frac{\sigma}{2}}) = O(t^{\frac{n-1}{4}+2\varepsilon}) \text{ almost everywhere in } x.$$

This completes the proof of Case (2).

Remark. Especially, in $n = 2$ and $\alpha = 0$, this theorem maintains that

$$\sum_{|m|^2<t} e^{2\pi imx} = O(t^{\frac{1}{4}+\varepsilon}) \text{ for almost all } x.$$

This corresponds to *"Hardy's conjecture"*.

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