Compact cardinals and fine filters

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1989

Introduction

In these days large cardinals play important roles in set theory. Not only their properties are interesting, but also their existence is indispensable to show the consistency of many statements concerening small cardinals like the least uncountable cardinal ω_1 and the second ω_2.

What are large cardinals? We consider a cardinal κ large, if it is a limit point of the set of cardinals with a property which κ itself also has, or if it has a combinatorial property which small cardinals never have. The development of large cardinals in recent years is dramatic and the theory of them has a long history. Hausdorff [H] had already presented the notion of weakly inaccessible cardinals before Zermelo introduced his axiomatization of set theory, where κ is weakly inaccessible if it is a regular limit cardinal. Sierpinski and Tarski [S-T] introduced the notion of strongly inaccessible cardinals, where κ is strongly inaccessible if it is weakly inaccessible and the cardinality of power set of ν is less than κ for any cardinal $\nu < \kappa$. In this case every operation of set theory can be carried out inside V_κ, which is the set of the sets with their rank less than κ, and V_κ becomes a model of set theory. More important developments began after Ulam [U] considered the abstract measure problem: Is there an infinite set X with a measure $\mu : P(X) \to [0,1]$ (where $P(X)$ is the set of subsets of X) such that

(i) $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for any $x \in X$; and

(ii) μ is countably additive, i.e. if $X_n \subset X$ $(n < \omega)$ are pairwise disjoint, then $\mu(\cup_n X_n) = \Sigma_n \mu(X_n)$.

Of course, the Lebesgue measure on the unit interval satisfies (i) and (ii), but is not defined for all subsets of the unit interval. Suppose that such a set X exists, then there exists a cardinal κ with a measure μ on κ such that μ not only satisfies (i) and (ii) but also is κ-additive, i.e. if $X_\alpha \subset X$ $(\alpha < \nu < \kappa)$ are pairwise disjoint, then $\mu(\cup_{\alpha<\nu} X_\alpha) = \Sigma_{\alpha<\nu} \mu(X_\alpha)$. In case μ has no atom, such a κ is called a real-valued measurable cardinal nowdays. Otherwise, we conclude that there is a two-valued κ-additive measure on κ. Such a cardinal κ is called a measurable cardinal and $U = \{X \subset \kappa : \mu(X) = 1\}$ becomes a non-principal κ-complete ultrafilter on κ. The technical invention of taking the ultrapower of the universe by such an ultrafilter due to Scott [S] is very important. He proved that the existence of a measurable cardinal implies $V \neq L$, i.e. the universe is not equal to Godel's constructible universe, and he also showed that large cardinals introduce some structure into the universe of set theory. Treating large cardinals by elementary embeddings or ultrafilters has been popular after his work and many strengthenings of measurability have been considered. One of them is the notion of strongly compact cardinals. It first appeared in a problem of the Compactness Theorem for infinitary language. But now we have another definition of it using elementary embeddings or ultrafilters, which are main subjects of this paper.

In Chapter I, we characterize the cardinals which are fixed points of elementary embeddings induced by fine measures on $P_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$. This was first done by Barbanel [B] for normal measures. Since every normal measure is fine by definition, our result improves his one. Normal measures define supercompact cardinals and fine measures do strongly compact cardinals. As we mention below, a strongly compact cardinal is not always supercompact. Therefore, our improvement is essential. Our proof is more complicated for this reason.

It is a prominent open question whether strong compactness and supercompactness are equiconsistent. At early stage they looked the same concept, but Magidor [Ma] proved that the first strongly compact cardinal may be either the first measurable or the first supercompact. (There exist many measurable cardinals below a supercompact cardinal [S-R-K].) Then, Apter [A] provided a model of set theory where the first strongly compact is the least ordinal with certain degree of supercompactness. Though he started with so many supercompact cardinals, we get the same result assuming only one supercompact in chapter II. The second part of the chapter is used for the observation about the normality and the weak normality of fine measures investigated by Menas [Me1].

In chapter III, we extend certain properties of filters on κ to those of fine filters on $P_\kappa\lambda$ where κ is not necessarily large. We show that any fine filter of the Menas type cannot be an extension of the (strongly) closed unbounded filter, and give full consideration to its weak normality, improving the results in the last chapter. We also present two distinct isomorphic fine measures containing the closed unbounded filter on $P_\kappa\lambda$ in case λ is inaccessible. Menas [Me2] has proved the corresponding statement for λ strong limit with cofinality less than κ.

The author is grateful to Nobuyoshi Motohashi and Moto-o

Takahashi for their helpful advice and encouragement, and to Katsuya Eda for initiating the author into the theory of large cardinals.

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Chapter I

THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 3, Sept. 1984

STRONGLY COMPACT CARDINALS, ELEMENTARY EMBEDDINGS AND FIXED POINTS

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§0. J. Barbanel [1] characterized the class of cardinals fixed by an elementary embedding induced by a normal ultrafilter on $P_\kappa\lambda$ assuming that κ is supercompact. In this paper we shall prove the same results from the weaker hypothesis that κ is strongly compact and the ultrafilter is fine.

§1. Preliminaries. We work in ZFC throughout. Our set-theoretic notation is quite standard. In particular, if X is a set, $|X|$ denotes the cardinality of X and $P(X)$ denotes the power set of X. Greek letters will denote ordinals. In particular γ, κ, η and λ will denote cardinals. If κ and λ are cardinals, then $\lambda^{<\kappa}$ is defined to be $\sup_{\gamma<\kappa}\lambda^\gamma$. Cardinal exponentiation is always associated from the top. Thus, for example, $2^{\lambda^{<\kappa}}$ means $2^{(\lambda^{<\kappa})}$. V denotes the universe of all sets. If M is an inner model of ZFC, $|X|^M$ and $P(X)^M$ denote the cardinality of X in M and the power set of X in M respectively.

We review the basic facts on fine ultrafilters and the corresponding elementary embeddings. (For detail, see [2].)

DEFINITION. Assume κ and λ are cardinals with $\kappa \le \lambda$. Then, $P_\kappa\lambda = \{X \subset \lambda \mid |X| < \kappa\}$.

It is important to note that $|P_\kappa\lambda| = \lambda^{<\kappa}$.

DEFINITION. Assume that U is a collection of subsets of $P_\kappa\lambda$. U is a *fine ultrafilter* on $P_\kappa\lambda$ iff the following conditions hold:

(i) For each $Z \in P_\kappa\lambda$, $\{Z\} \notin U$ (U is nonprincipal).

(ii) If $X \in U$ and $X \subset Y \subset P_\kappa\lambda$, then $Y \in U$.

(iii) For each $X \subset P_\kappa\lambda$, exactly one of X, $P_\kappa\lambda\backslash X$ is in U.

(iv) If $\{X_\alpha \mid \alpha < \gamma\}$ is a collection of elements of U where $\gamma < \kappa$, then $\bigcap_{\alpha<\gamma} X_\alpha \in U$ (U is κ-complete).

(v) For each ordinal $\alpha < \lambda$, $\{Z \in P_\kappa\lambda \mid \alpha \in Z\} \in U$ (U is fine).

DEFINITION. For cardinals κ and λ, κ is *λ-compact* iff there is a fine ultrafilter on $P_\kappa\lambda$. κ is *strongly compact* iff κ is λ-compact for all $\lambda \ge \kappa$.

Suppose now that U is a fine ultrafilter on $P_\kappa\lambda$. Let $V^{P_\kappa\lambda}/U$ denote the ultrapower of V with respect to U, and let $e\colon V \to V^{P_\kappa\lambda}/U$ denote the usual elementary embedding by constant functions. It is straightforward to show, using κ-completeness of

Received December 26, 1982.

0022-4812/84/4903-0007/$01.50

U, that $V^{P_\kappa\lambda}/U$ is well-founded. Thus $V^{P_\kappa\lambda}/U$ is isomorphic to a transitive class M. Let $\pi: V^{P_\kappa\lambda}/U \to M$ be this isomorphism. Then M is an inner model and $j: V \to M$ defined by $j = \pi \circ e$ is an elementary embedding. It is well known that κ is the least ordinal moved by j, $j(\kappa) > \lambda$; and, $X \subset M$ and $|X| \leq \lambda$ implies there is a $Y \in M$ such that $X \subset Y$ and $|Y|^M < j(\kappa)$. Our object is to describe the action of j on cardinals. Only a familiarity with ultrapower technique is expected, and we use Solovay's famous result on powers of cardinals, which was already used by Barbanel. That is:

If κ is strongly compact, $\eta > \kappa$, and $2^\gamma < \eta$, then

$$\eta^\gamma = \eta \quad \textit{if } \mathrm{cf}(\eta) > \gamma,$$
$$\eta^\gamma = \eta^+. \quad \textit{if } \mathrm{cf}(\eta) \leq \gamma.$$

In particular, for $\lambda > \kappa$,

$$\lambda^{<\kappa} = \lambda \quad \text{if } \mathrm{cf}(\lambda) \geq \kappa,$$
$$\lambda^{<\kappa} = \lambda^+ \quad \text{if } \mathrm{cf}(\lambda) < \kappa.$$

In the following, κ is a fixed strongly compact cardinal and λ is a cardinal greater than or equal to κ. U is a fine ultrafilter on $P_\kappa\lambda$ and j is the canonical elementary embedding; $j: V \to M \simeq V^{P_\kappa\lambda}/U$. η always denotes a cardinal and $[f]_U$ is the equivalence class of f in $V^{P_\kappa\lambda}/U$. We often drop the subscript U.

§2. The action of j on small cardinals. We begin by considering what happens to small cardinals under j. If $\eta < \kappa$, then $j(\eta) = \eta$.

LEMMA 1. $j(2^\lambda) \geq 2^{\lambda^{<\kappa}}$.

PROOF. We inject $P(P_\kappa\lambda)$ into $F = \{f: P_\kappa\lambda \to P_\kappa(P\lambda)\}/U$. For $A \subset P_\kappa\lambda$, $f_A: P_\kappa\lambda \to P_\kappa(P\lambda)$ is defined by $f_A(x) = \{z \in A \mid z \subset x\}$. We have to show the map g which sends A to $[f_A]_U$ is one-to-one. Let $A, B \subset P_\kappa\lambda$, $A \neq B$. We may assume $\exists y \in A(y \notin B)$. Since U is fine, $\{x \in P_\kappa\lambda \mid y \subset x\} \in U$. (We use the fact that $|y| < \kappa$ and U is κ-complete.) Then $\{x \in P_\kappa\lambda \mid y \in f_A(x)\} \in U$. But for every $x \in P_\kappa\lambda$, $y \notin f_B(x)$. Thus $[f_A]_U \neq [f_B]_U$. Hence g is one-to-one. By the way, F represents $P_{j(\kappa)}(P(j(\lambda))$ in M.

$M \models$ "$j(\kappa)$ is strongly compact and $\mathrm{cf}(2^{j(\lambda)}) > j(\kappa)$." This implies $M \models$ "$|F| = 2^{j(\lambda)^{<j(\kappa)}} = 2^{j(\lambda)} = j(2^\lambda)$." Recall that g is one-to-one. Hence $2^{\lambda^{<\kappa}} = |P(P_\kappa\lambda)| \leq |F| \leq |F|^M = j(2^\lambda)$. □

THEOREM 1. *If $\kappa \leq \eta \leq 2^{\lambda^{<\kappa}}$, then $j(\eta) > \eta$.*

PROOF. We know that $j(\kappa) > 2^\lambda$ (see [3]) and $2^{\lambda^{<\kappa}} = 2^\lambda$ or $2^{(\lambda^+)}$. Hence there is no trouble in the case that $2^{\lambda^{<\kappa}} = 2^\lambda$. In fact, $\eta < 2^{\lambda^{<\kappa}} = 2^\lambda < j(\kappa) \leq j(\eta)$. Suppose $2^{\lambda^{<\kappa}} = 2^{(\lambda^+)} > 2^\lambda$. If $\kappa \leq \eta \leq 2^\lambda$, then $\eta \leq 2^\lambda < j(\kappa) \leq j(\eta)$. If $2^\lambda < \eta \leq 2^{\lambda^{<\kappa}}$, then $j(2^\lambda) < j(\eta)$. By the previous lemma, we have $j(2^\lambda) \geq 2^{\lambda^{<\kappa}} \geq \eta$. Hence $j(\eta) > \eta$. □

§3. The action of j on larger cardinals. We now consider cardinals above $2^{\lambda^{<\kappa}}$. Theorem 2 is due to [1]. The proof in [1] can be carried out also in our case since the assumption that ${}^\lambda M \subset M$ is not used there.

THEOREM 2. *Let η be a cardinal greater than $2^{\lambda^{<\kappa}}$. If η satisfies one of the conditions below, then $j(\eta) = \eta$.*

(1) $\eta = (2^{\lambda^{<\kappa}})^+$.

(2) $\eta = \gamma^{++}$ *for some γ.*

(3) *η is a limit cardinal and* $\mathrm{cf}(\eta) < \kappa$ *or* $\mathrm{cf}(\eta) > \lambda^{<\kappa}$.

(4) $\eta = \gamma^+$ *for some γ which satisfies* (3).

Next we consider the case that η is a limit cardinal and $\kappa \leq \mathrm{cf}(\eta) \leq \lambda^{<\kappa}$.

LEMMA 2. *If η is a limit cardinal strictly greater than $2^{\lambda^{<\kappa}}$ and $\kappa \leq \mathrm{cf}(\eta) \leq \lambda$, then $j(\eta) > \eta$.*

PROOF. Suppose $j(\eta) = \eta$. Let $\gamma = \mathrm{cf}(\eta)$ and $f: \gamma \to \eta$ be a cofinal function such that

$$\forall \delta < \gamma \exists \delta' (2^{\lambda^{<\kappa}} < f(\delta) = \delta'^{++}).$$

For each $\delta < \gamma$, let $[\langle \delta_x \mid x \in P_\kappa \lambda \rangle]_U = \delta$ and $\forall x(\delta_x < \kappa)$. (Since $\delta < \gamma \leq \lambda < j(\kappa)$, this can be done.) Define k: $P_\kappa\lambda \to V$ by

$$k(x) = \left\{ \left\langle \delta_x, \bigcup_{\substack{\xi_x = \delta_x \\ \xi \in x}} f(\xi) \right\rangle \middle| \delta \in x \text{ and } \delta < \gamma \right\}.$$

(1) Clearly,

$$M \models \text{"}[k] \text{ is a function and } |\mathrm{dom}[k]| < j(\kappa).\text{"}$$

Since $\forall x \in P_\kappa\lambda(|x| < \kappa \leq \mathrm{cf}(\eta))$ and $\forall \xi < \gamma(f(\xi) < \eta)$, $\forall x \in P_\kappa\lambda(\mathrm{ran}(k(x)) \subset \eta)$. Hence,

(2) $$M \models \text{"}\mathrm{ran}[k] \subset \eta = j(\eta).\text{"}$$

By the definition of f, $\forall \alpha < \eta \exists \delta_\alpha < \gamma(\alpha < f(\delta_\alpha))$. Since U is fine, $\{x \in P_\kappa\lambda \mid \delta_\alpha \in x\} \in U$. Hence $\delta_\alpha \in \mathrm{dom}[k]$ and $\alpha < f(\delta_\alpha) = j(f(\delta_\alpha)) \leq [k](\delta_\alpha)$. Thus,

(3) $$M \models \text{"}[k] \text{ is cofinal in } \eta = j(\eta).\text{"}$$

By (1)–(3), $M \models$ "$\mathrm{cf}(\eta) < j(\kappa)$." But $V \models$ "$\mathrm{cf}(\eta) \geq \kappa$" implies $M \models$ "$\mathrm{cf}(\eta) = \mathrm{cf}(j(\eta)) \geq j(\kappa)$." Contradiction. Therefore $j(\eta) > \eta$. □

DEFINITION. A κ-complete ultrafilter W on $P_\kappa\lambda$, is *uniform* iff $\forall A \in W(|A| = |P_\kappa\lambda|)$.

REMARK. A fine ultrafilter is uniform.

PROOF. Let W be a fine ultrafilter on $P_\kappa\lambda$ and $A \in W$. For each $y \in A$, $A_y = \{x \in P_\kappa\lambda \mid x \subset y\}$. Since W is fine, $P_\kappa\lambda = \bigcup_{y \in A} A_y$. $|A_y| \leq 2^{|y|} < \kappa$. Hence $|A| = \lambda^{<\kappa} = |P_\kappa\lambda|$. □

In proving the next lemma, we need only the fact that U is uniform.

LEMMA 3. *If $\mathrm{cf}(\lambda) < \kappa$ and η is a limit cardinal greater than $2^{\lambda^{<\kappa}}$ with cofinality λ^+, then $j(\eta) > \eta$.*

PROOF. Note that $|P_\kappa\lambda| = \lambda^{<\kappa} = \lambda^+$. Let $\{x_\xi \mid \xi < \lambda^+\}$ be an enumeration of $P_\kappa\lambda$ and $f: \lambda^+ \to \eta$ a cofinal function such that $\forall \delta < \lambda^+ \exists \delta' (2^{\lambda^{<\kappa}} < f(\delta) = \delta'^{++})$. Suppose $j(\eta) = \eta$. Define $k: P_\kappa\lambda \to V$ by $k(x_\xi) = \{\langle \delta, f(\delta) \rangle \mid \delta < \xi\}$. It is easily verified that

(1) $$M \models \text{"}[k] \text{ is a function, } |\mathrm{dom}[k]| \leq j(\lambda) \text{ and } \mathrm{ran}[k] \subset \eta.\text{"}$$

Since U is uniform, $\{x_\xi \mid \delta < \gamma\} \in U$ for every $\delta < \lambda^+$. Hence

$$\{x \in P_\kappa\lambda \mid f(\delta) \in \mathrm{ran}(k(x))\} \in U \quad \text{and} \quad j(f(\delta)) = f(\delta) \in \mathrm{ran}[k].$$

Thus, $M \models$ "$[k]$ is cofinal in η and $|\mathrm{dom}[k]| \leq j(\lambda)$." Hence $M \models$ "$\mathrm{cf}(\eta) \leq j(\lambda)$." But $V \models$ "$\mathrm{cf}(\eta) = \lambda^+$" implies $M \models$ "$\mathrm{cf}(j(\eta)) = \mathrm{cf}(\eta) = j(\lambda^+) > j(\lambda)$." Contradiction. Hence $j(\eta) > \eta$. □

THEOREM 3. *If* η *is a limit cardinal greater than* $2^{\lambda^{<\kappa}}$ *and* $\kappa \leq \mathrm{cf}(\eta) \leq \lambda^{<\kappa}$, *then* $j(\eta) > \eta$.

PROOF. If $\mathrm{cf}(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \lambda$. Hence the conclusion is right by Lemma 2. If $\mathrm{cf}(\lambda) < \kappa$, then $\lambda^{<\kappa} = \lambda^+$. In the case that $\mathrm{cf}(\eta) \leq \lambda$, we use Lemma 2. If $\mathrm{cf}(\eta) = \lambda^+$, then we use Lemma 3. $\square$

The arguments for Lemmas 4 and 5 are analogous to those for Lemmas 2 and 3.

LEMMA 4. *If* η *is a limit cardinal greater than* $2^{\lambda^{<\kappa}}$ *and* $\kappa \leq \mathrm{cf}(\eta) \leq \lambda$, *then* $j(\eta^+) > \eta^+$.

PROOF. By Theorem 3, $j(\eta) > \eta$. Suppose $j(\eta^+) = \eta^+$. Since $\eta < j(\eta) < j(\eta^+) = \eta^+$, $V \vDash$ "$j(\eta)$ is not cardinal and $|j(\eta)| = \eta$." Hence there is a function g from η onto $j(\eta)$. Let $\{\alpha_\delta \mid \delta < \gamma = \mathrm{cf}(\eta)\}$ be a cofinal increasing sequence in η and, for each $\beta < \eta$, $\delta_\beta =$ the least δ such that $\beta < \alpha_\delta$. Define $f: P_\kappa\lambda \to V$ by

$$f(x) = \{\langle \beta, g(\beta)_x \rangle \mid \delta_\beta \in x\}, \quad \text{where } [\langle g(\beta)_x \mid x \in P_\kappa\lambda \rangle] = g(\beta).$$

Obviously,

(1) $$M \vDash \text{"}[f] \text{ is a function."}$$

Also, $\forall x \in P_\kappa\lambda(\mathrm{dom}(f(x)) = \bigcup_{\delta\in x}\{\beta < \eta \mid \delta_\beta = \delta\})$. Since $\{\beta < \eta \mid \delta_\beta = \delta\} \subset \alpha_\delta < \eta$ and $|x| < \kappa \leq \mathrm{cf}(\eta)$, $|\mathrm{dom}(f(x))| < \eta$. Hence

(2) $$M \vDash \text{"}|\mathrm{dom}[f]| < j(\eta)\text{."}$$

From our assumption on g, $\forall\xi < j(\eta)\exists\beta < \eta(\xi = g(\beta))$. Since U is fine and $\delta_\beta < \gamma \leq \lambda$, $\forall\beta < \eta(\{x \in P_\kappa\lambda \mid \delta_\beta \in x\} \in U)$. Thus $\forall\beta < \eta(\{x \in P_\kappa\lambda \mid g(\beta)_x \in \mathrm{ran}(f(x))\} \in U$, i.e. $g(\beta) \in \mathrm{ran}[f]$. Hence

(3) $$M \vDash \text{"}j(\eta) \subset \mathrm{ran}[f]\text{."}$$

By (1)–(3), $M \vDash$ "$j(\eta)$ is not a cardinal." Contradiction. $\square$

The next lemma holds whenever U is uniform.

LEMMA 5. *If* η *is a limit cardinal greater than* $2^{\lambda^{<\kappa}}$ *and* $\mathrm{cf}(\eta) = \lambda^+$, $\mathrm{cf}(\lambda) < \kappa$, *then* $j(\eta^+) > \eta^+$.

PROOF. Suppose $j(\eta^+) = \eta^+$. As in the previous lemma, there is a function g from η onto $j(\eta)$. Let $\{x_\xi \mid \xi < \lambda^+\}$ be an enumeration of $P_\kappa\lambda$ and $\{\alpha_\delta \mid \delta < \lambda^+\}$ be a cofinal increasing sequence in η. (Note that $|P_\kappa\lambda| = \lambda^{<\kappa} = \lambda^+$ since $\mathrm{cf}(\lambda) < \kappa$.) For each $\beta < \eta$, $\delta_\beta =$ the least δ such that $\alpha_\delta > \beta$. Define $f: P_\kappa\lambda \to V$ by $f(x_\xi) = \{\langle \beta, g(\beta)_x \rangle \mid \delta_\beta < \xi\}$. Obviously

(1) $$M \vDash \text{"}[f] \text{ is a function."}$$

Also, $\forall x_\xi \in P_\kappa\lambda(\mathrm{dom}(f(x_\xi)) = \bigcup_{\delta<\xi}\{\beta < \eta \mid \delta_\beta = \delta\})$. Since $\{\beta < \eta \mid \delta_\beta = \delta\} \subset \alpha_\delta < \eta$ and $\mathrm{cf}(\eta) = \lambda^+ > |\xi|$, $|\mathrm{dom}(f(x_\xi))| < \eta$. We get

(2) $$M \vDash \text{"}|\mathrm{dom}[f]| < j(\eta)\text{."}$$

Since U is uniform, $\{x_\xi \in P_\kappa\lambda \mid \delta_\beta < \xi\} \in U$ for every $\beta < \eta$. Then, $\{x \in P_\kappa\lambda \mid g(\beta)_x \in \mathrm{ran}(f(x))\} \in U$. This means $g(\beta) \in \mathrm{ran}[f]$. Hence,

(3) $$M \vDash \text{"}j(\eta) \subset \mathrm{ran}[f]\text{."}$$

By (1)–(3), $M \vDash$ "$j(\eta)$ is not a cardinal." Contradiction. $\square$

THEOREM 4. *If* η *is a limit cardinal greater than* $2^{\lambda^{<\kappa}}$ *and* $\kappa \leq \mathrm{cf}(\eta) \leq \lambda^{<\kappa}$, *then* $j(\eta^{+}) > \eta^{+}$.

Theorem 4 follows from Lemmas 4 and 5 just as Theorem 3 did from Lemmas 2 and 3.

We have characterized the class of cardinals fixed by j and improved Barbanel's results by weakening the assumption of supercompactness to that of strong compactness.

The proofs of Theorem 2 and Lemmas 3 and 5 needed only uniformity (apparently weaker than fineness) of U. It is not known whether uniformity suffices for the other results.

The author wishes to express his gratitude to the referee for his very thorough reading of the manuscript and correcting many errors. His suggestions were very helpful in preparing this paper.

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Chapter II

THE JOURNAL OF SYMBOLIC LOGIC
Volume 50, Number 4, Dec. 1985

SOME RESULTS CONCERNING STRONGLY COMPACT CARDINALS

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§0. This paper consists of two parts. In §1 we mention the first strongly compact cardinal. Magidor proved in [6] that it can be the first measurable and it can be also the first supercompact. In [2], Apter proved that Con(ZFC + there is a supercompact limit of supercompact cardinals) implies Con(ZFC + the first strongly compact cardinal κ is $\phi(\kappa)$-supercompact + no $\alpha < \kappa$ is $\phi(\alpha)$-supercompact) for a formula ϕ which satisfies certain conditions.

We shall get almost the same conclusion as Apter's theorem assuming only one supercompact cardinal. Our notion of forcing is the same as in [2] and a trick makes it possible.

In §2 we study a kind of fine ultrafilter on $P_\kappa\lambda$ investigated by Menas in [7], where κ is a measurable limit of strongly compact cardinals. He showed that such an ultrafilter is not normal in some case (Theorems 2.21 and 2.22 in [7]). We shall show that it is not normal in any case (even if κ is supercompact). We also prove that it is weakly normal in some case.

We work in ZFC and much of our notation is standard. But we mention the following: the letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, whereas $\kappa, \lambda, \mu, \ldots$ are reserved for cardinals. $R(\alpha)$ is the collection of sets rank $<\alpha$. ϕ^M denotes the realization of a formula ϕ to a class M. Except when it is necessary, we drop "M". For example, $M \models$ "κ is $\phi(\kappa)$-supercompact" means "κ is $\phi^M(\kappa)$-supercompact in M". If x is a set, $|x|$ is its cardinality, Px is its power set, and $P_\kappa x = \{y \in Px \mid |y| < \kappa\}$. If also $x \subseteq$ OR, $\bar{x}$ denotes its order type in the natural ordering. The identity function with the domain appropriate to the context is denoted by id. For the notation concerning ultrapowers and elementary embeddings, see [11]. When we talk about forcing, "$\Vdash$" will mean "weakly forces" and "$p \leq q$" means "p is stronger than q".

§1. On the first strongly compact cardinal. We state Apter's theorem precisely.

THEOREM (APTER). *Assume $V \models$ "$ZFC + \delta$ is a supercompact limit of supercompact cardinals". Let ϕ be a formula which defines an increasing Σ_2 function from OR to OR and, in addition, has the following properties.*

(1) *For G V-generic on P, $|P| = \kappa$, P a cardinal preserving partial ordering, if $\alpha > \kappa$, then $V \models$ "$\beta = \phi(\alpha)$" iff $V[G] \models$ "$\beta = \phi(\alpha)$".*

Received June 8, 1984.

0022-4812/85/5004-0002/$01.70

(2) *If* $\alpha < \beta$, α *is* $\phi(\alpha)$*-supercompact and* β *is* $\phi(\beta)$*-supercompact, then* $\phi(\alpha) < \beta$.

Then it is consistent for the least strongly compact cardinal μ *to be at least* $\phi(\mu)$*-supercompact yet not to be fully supercompact. In fact there is no* α *below* μ *that is* $\phi(\alpha)$*-supercompact.*

We, in some sense, improve this to the following.

THEOREM 1. *Assume* $V \models$ "$ZFC + \kappa$ *is supercompact*". *Let* ϕ *be a formula which defines an increasing function from OR to OR and, in addition, has the following properties.*

(A) *For every partial ordering* Q, *if* $\alpha > |Q|$, *then* $V \models$ "$\beta = \phi(\alpha)$" *iff* $V[G] \models$ "$\beta = \phi(\alpha)$".

(B) *There is a formula* ψ *such that*

(1) $\phi(\alpha) < \psi(\alpha)$ *for every* α,

(2) *if* $\alpha < \beta$ *and* β *is* $\phi(\beta)$*-supercompact, then* $\psi(\alpha) < \beta$, *and*

(3) *if* M, N *are models of* ZFC *and* $R(\psi^M(\alpha)) \cap M = R(\psi^M(\alpha)) \cap N$, *then* $\phi^M(\alpha) = \phi^N(\alpha)$.

Then, for some generic filter G,

$$V[G] \models \text{"}\kappa \text{ is the first strongly compact} + \kappa \text{ is } \phi(\kappa)\text{-supercompact} + \text{no } \alpha < \kappa \text{ is } \phi(\alpha)\text{-supercompact"}.$$

We note that ϕ may imply additional hypothesis. For example, if ϕ says "Send α to the least measurable $>\alpha$", then a measurable cardinal $>\kappa$ is assumed to exist. Also note that many ϕ's satisfy our assumption. If $\phi(\alpha) = \alpha^+$, 2^α, the least inaccessible $>\alpha$, then $\psi(\alpha) = \phi(\alpha) + 1$. If $\phi(\alpha) =$ the least measurable $>\alpha$, then $\psi(\alpha) = \phi(\alpha) + 2$.

Our forcing notion is the same as in [2], that is, Magidor's iterated Prikry forcing that destroys measurability of each element of a given set A of measurable cardinals. (We call this forcing notion "the iterated Prikry forcing on A".) We assume the reader is familiar with this forcing. See [6] for details. Most of our notation is the same as in [6]. But we use $\|\cdot\|$ for the distant function instead of $|\cdot|$ that is used in [6].

PROOF OF THEOREM 1. Let $V \models$ "$ZFC + \kappa$ is supercompact". We may assume that $2^\alpha = \alpha^{++}$ for α inaccessible and $2^\alpha = \alpha^+$ otherwise. (See [8].) Following Apter [2], we inductively define the sequences $\{\lambda_\alpha; \alpha \leq \kappa\}$ and $\{P_\alpha; \alpha \leq \kappa\}$.

$P_0 = 0$ and $\lambda_0 =$ the least cardinal λ that is $\phi(\lambda)$-supercompact.

$P_\alpha =$ the iterated Prikry ordering on $\{\lambda_\beta \mid \beta < \alpha\}$.

$\lambda_\alpha =$ the least ordinal λ such that $\exists p \in P_\alpha(p \Vdash$ "λ is $\phi(\lambda)$-supercompact"). Let $P = P_\kappa$.

LEMMA 1.1. $\alpha < \beta \rightarrow \lambda_\alpha < \lambda_\beta$.

PROOF. Suppose not. Let α be the least ordinal such that $\exists \beta < \alpha(\lambda_\alpha \leq \lambda_\beta)$. Let H be V-generic on P_α such that

(1) $$V[H] \models \text{"}\lambda_\alpha \text{ is } \phi(\lambda_\alpha)\text{-supercompact"}$$

$V[H] \models$ "$\mathrm{cf}(\lambda_\beta) = \omega$" by the definition of P_α. Thus $\lambda_\alpha < \lambda_\beta$. Let γ be the least ordinal such that $\lambda_\alpha < \lambda_\gamma$. λ_γ is not a limit point of $\{\lambda_\xi \mid \xi < \alpha\}$. Hence, by Lemma 2.3 in [6],

(2) $$V[H \restriction \lambda_\gamma] \text{ and } V[H] \text{ have the same bounded subsets of } \lambda_\gamma.$$

$|P_\gamma| \leq 2^{\lambda}\alpha < \lambda_\gamma$ since λ_γ is measurable in V. (No new measurable cardinal is created by an iterated Prikry forcing—Theorem 3.1 in [6].) Hence by property (A) of ϕ, (1), and the Lévy-Solovay results (see [5]),

$$\phi^V(\lambda_\gamma) = \phi^{V[H\restriction\lambda_\gamma]}(\lambda_\gamma) \text{ and } V \models \text{“}\lambda_\gamma \text{ is } \phi(\lambda_\gamma)\text{-supercompact”}.$$

Again by the Lévy-Solovay results,

$$(3) \qquad V[H\restriction\lambda_\gamma] \models \text{“}\lambda_\gamma \text{ is } \phi(\lambda_\gamma)\text{-supercompact”}.$$

By (3) and property (B)(2) of ϕ, $\psi^{V[H\restriction\lambda_\gamma]}(\lambda_\alpha) < \lambda_\gamma$. Then by (2) and (3),

$$R(\psi^{V[H\restriction\lambda_\gamma]}(\lambda_\alpha)) \cap V[H\restriction\lambda_\gamma] = R(\psi^{V[H\restriction\lambda_\gamma]}(\lambda_\alpha)) \cap V[H].$$

By property (B)(3) of ϕ,

$$\phi^{V[H\restriction\lambda_\gamma]}(\lambda_\alpha) = \phi^{V[H]}(\lambda_\alpha).$$

Using (1), (2), (3) and the fact that $\phi^{V[H\restriction\lambda_\gamma]}(\lambda_\alpha) < \lambda_\gamma$,

$$V[H\restriction\lambda_\gamma] \models \text{“}\lambda_\alpha \text{ is } \phi(\lambda_\alpha)\text{-supercompact”}.$$

But $\lambda_\alpha < \lambda_\gamma$. This contradicts the definition of λ_γ. □

LEMMA 1.2. *$\{\lambda_\alpha \mid \alpha < \kappa\}$ is an unbounded subset of κ.*

PROOF. Since $\{\alpha < \kappa \mid V \models \text{“}\alpha \text{ is } \phi(\alpha)\text{-supercompact”}\}$ is unbounded in κ and $|P_\alpha| < \kappa$ for every $\alpha < \kappa$ (easily shown by induction), our claim is obvious. □

By the previous two lemmas, we get

LEMMA 1.3. $\lambda_\kappa \geq \kappa$.

LEMMA 1.4. *If G is V-generic on P, then*

$$V[G] \models \text{“no } \alpha < \kappa \text{ is } \phi(\alpha)\text{-supercompact} + \kappa \text{ is the first strongly compact”}.$$

PROOF. By Lemma 1.3, the first assertion is clear. For $\alpha < \kappa$,

$$V[G] \models \text{“cf}(\lambda_\alpha) = \omega + 2^{\lambda_\alpha} \geq \lambda_\alpha^{++} + \lambda_\alpha \text{ is strong limit”}.$$

Hence the second part is true by Solovay's theorem [10] and the fact that an iterated Prikry forcing below κ preserves the strong compactness of κ [6, Theorem 3.4]. □

Now what is left to show is that for some generic filter G, $V[G] \models \text{“}\kappa \text{ is } \phi(\kappa)\text{-supercompact”}$.

Let $A = \{\alpha \in \text{OR} \mid \exists p \in P(p \Vdash \text{“}\alpha = \psi(\kappa)\text{”})\}$. We have that $|A| \leq 2^\kappa$, since $|P| \leq 2^\kappa$. Let $\delta = \bigcup A + 1$. Then every condition $p \in P$ forces “$\delta > \psi(\kappa)$”.

Let $j: V \to M$ be an elementary embedding such that

(1) κ is the first ordinal moved by j, and

(2) ${}^{|R(\delta)|^+}M \subset M$.

Note that $j(P)_\kappa = P$ and $|R(\delta)| > 2^\kappa$.

LEMMA 1.5. *If G is V-generic on P, $x \in V[G]$, $x \subset M[G]$, and $V[G] \models$ “$|x| \leq |R(\delta)|$”, then $x \in M[G]$.*

PROOF. It will suffice to assume that x is a set of ordinals since both $V[G]$ and $M[G]$ are models of AC. Let **x** be a term denoting x in $V[G]$. Since P satisfies κ^+-c.c. [6, Lemma 4.4], there is a set $D \in V$ such that $1_P \Vdash \text{“}\mathbf{x} \subset D\text{”}$ and $|D| \leq |R(\delta)|$.

For $\alpha \in D$, let A_α be a maximal disjoint subset of $\{p \in P \mid p \Vdash \text{“}\alpha \in \mathbf{x}\text{”}\}$. Since $|A_\alpha| \leq \kappa$, $|D| \leq |R(\delta)|$, and ${}^{|R(\delta)|}M \subset M$, $\langle A_\alpha \mid \alpha \in D\rangle \in M$. But $x = \{\alpha \in D \mid A_\alpha \cap G \neq 0\}$. Hence $x \in M[G]$. □

LEMMA 1.6. *Let* $G \subset P$.

(i) *G is V-generic iff G is M-generic.*

(ii) $\phi^{V[G]}(\kappa) = \phi^{M[G]}(\kappa)$.

PROOF. (i) is clear by the two facts that $|P| \leq 2^\kappa$ and ${}^{2^\kappa}M \subset M$. By Lemma 1.5, and an easy induction, $R(\delta) \cap V[G] = R(\delta) \cap M[G]$. Since $\psi^{V[G]}(\kappa) < \delta$,

$$R(\psi^{V[G]}(\kappa)) \cap V[G] = R(\psi^{V[G]}(\kappa)) \cap M[G].$$

Hence $\phi^{V[G]}(\kappa) = \phi^{M[G]}(\kappa)$ by property (B)(3) of ϕ. □

LEMMA 1.7. *For some V-generic filter G on P,*

$$V[G] \models \text{“}\kappa \text{ is } \phi(\kappa)\text{-supercompact”}.$$

PROOF. *Case* 1. $\lambda_\kappa^M = \kappa$. There is an M generic filter G on P such that

$$M[G] \models \text{“}\kappa \text{ is } \phi(\kappa)\text{-supercompact”}.$$

Let U be a normal ultrafilter on $P_\kappa\phi^{M[G]}(\kappa)$ in $M[G]$. By Lemma 1.6, G is also V-generic and $\phi^{V[G]}(\kappa) = \phi^{M[G]}(\kappa)$.

Since $V[G] \models \text{“}|P_\kappa\phi(\kappa)| \leq 2^{\phi(\kappa)} \leq 2^{\psi(\kappa)} \leq |R(\delta)|\text{”}$, we have that

$$PP_\kappa\phi^{V[G]}(\kappa) \cap V[G] = PP_\kappa\phi^{M[G]}(\kappa) \cap M[G]$$

by Lemma 1.5. Hence U is also a normal ultrafilter on $P_\kappa\phi^{V[G]}(\kappa)$ in $V[G]$. Thus

$$V[G] \models \text{“}\kappa \text{ is } \phi(\kappa)\text{-supercompact”}.$$

Case 2. $\lambda_\kappa^M > \kappa$. Let G be a V-generic filter on P. There is an M-generic filter H such that

$$M[H] \models \text{“}\lambda_\kappa^M \text{ is } \phi(\lambda_\kappa^M)\text{-supercompact”}.$$

Since $M \models \text{“}\lambda_\kappa^M \text{ is measurable} + |P| \leq 2^\kappa\text{”}$, $|P|^M < \lambda_\kappa^M$. Hence $\phi^M(\lambda_\kappa^M) = \phi^{M[H]}(\lambda_\kappa^M)$ by property (A) of ϕ. Moreover, by the Lévy-Solovay results,

$$M \models \text{“}\lambda_\kappa^M \text{ is } \phi(\lambda_\kappa^M)\text{-supercompact”}.$$

G is also M-generic. Again by the Lévy-Solovay results and property (A) of ϕ,

$$M[G] \models \text{“}\lambda_\kappa^M \text{ is } \phi(\lambda_\kappa^M)\text{-supercompact”}.$$

By property (B)(1), (2) of ϕ, $\phi^{M[G]}(\kappa) < \psi^{M[G]}(\kappa) < \lambda_\kappa^M$. By Lemma 1.6, $\phi^{V[G]}(\kappa) < \lambda_\kappa^M$.

We define a term $\mathbf{U}$ by

$p \Vdash \text{“}\tau \in \mathbf{U}\text{”}$ iff $p \Vdash \text{“}\tau \subseteq P_\kappa(\phi(\kappa))\text{”}$
and $\exists q \leq j(p)(\|q - j(p)\| = 0, q \restriction \kappa = j(p) \restriction \kappa = p$, and $q \Vdash \text{“}j''\phi(\kappa) \in j(\tau)\text{”})$.

Note that $j''\phi^{V[G]}(\kappa) = j''\phi^{M[G]}(\kappa) \in M[G]$. Using the fact that $\phi^{V[G]}(\kappa) < \lambda_\kappa^M$ and λ_κ^M is not a limit point of $j(P)$, we can show that

$$1_P \Vdash \text{“}\mathbf{U} \text{ is a normal ultrafilter on } P_\kappa\phi(\kappa)\text{”}.$$

For details, see [2] and [6]. □

By Lemmas 1.4 and 1.7, the proof of Theorem 1 is complete.

§2. **A kind of fine ultrafilter and its normality.** In this section, κ is a measurable limit of strongly compact cardinals and λ is a cardinal $\geq \kappa$, $\mathcal{U}_\alpha$ is a fine ultrafilter on $P_\alpha\lambda$ for $\alpha < \kappa$ strongly compact, and U is a κ-complete nonprincipal ultrafilter on κ such that $\{\alpha < \kappa \mid \alpha \text{ is strongly compact}\} \in U$. $\mathcal{U}$ is defined by

$$X \in \mathcal{U} \quad \text{iff} \quad X \subset P_\kappa\lambda \text{ and } \{\alpha < \kappa \mid X \cap P_\alpha\lambda \in \mathcal{U}_\alpha\} \in U.$$

Menas proved in [7] that $\mathcal{U}$ is a fine ultrafilter on $P_\kappa\lambda$.

We are interested in the next fact that he also proved in [7].

THEOREM (MENAS). (i) *If λ is regular and $\{\alpha < \kappa \mid \mathcal{U}_\alpha$ is minimal$\} \in U$, then $\mathcal{U}$ is not normal.*

(ii) *If κ is the least measurable limit of strongly compact cardinals and $\lambda \geq 2^\kappa$, then $\mathcal{U}$ is not normal.*

REMARK. (1) $\mathcal{U}_\alpha$ is minimal iff for all functions q from $P_\alpha\lambda$ into $P_\alpha\lambda$ such that $\{X \subset P_\alpha\lambda \mid q^{-1}[X] \in \mathcal{U}_\alpha\}$ is a fine ultrafilter on $P_\alpha\lambda$, q is injective on a set of measure one. ($q^{-1}[X] = \{x \mid q(x) \in X\}$.)

(2) If λ is regular or $\mathrm{cf}(\lambda) < \alpha$, then every normal ultrafilter on $P_\alpha\lambda$ is minimal.

We extend his theorem to the following.

THEOREM 2. *Let κ, U, $\langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$, and $\mathcal{U}$ be as above. Then $\mathcal{U}$ is not normal.*

DEFINITION. (i) For $x \in P_\kappa\lambda$, let α_x be the least strongly compact cardinal $> |x|$.

(ii) Define $f: P_\kappa\lambda \to V$ by $f(x) = x \cap \alpha_x$.

LEMMA 2.1. *If $\mathcal{U}$ is normal, then $[f]_\mathcal{U} = \kappa$.*

PROOF. Suppose that $\mathcal{U}$ is normal and let $j: V \to M \simeq V^{P_\kappa\lambda}/\mathcal{U}$ be the cannonical elementary embedding. Since $\mathcal{U}$ is normal, $j''\lambda = [\mathrm{id}]_\mathcal{U}$ and $[\langle |x| \mid x \in P_\kappa\lambda \rangle]_\mathcal{U} = |j''\lambda|^M = \lambda$. Hence $M \models$ "$j(\kappa)$ is a strongly compact cardinal $>\lambda$" and $M \models$ "$[\langle \alpha_x \mid x \in P_\kappa\lambda \rangle]_\mathcal{U}$ is the least strongly compact $>\lambda$". Thus

$$[\langle \alpha_x \mid x \in P_\kappa\lambda \rangle]_\mathcal{U} \leq j(\kappa).$$

Hence

$$\kappa = j''\kappa \cap \kappa \subset [\langle x \cap \alpha_x \mid x \in P_\kappa\lambda \rangle]_\mathcal{U} \subset j''\lambda \cap j(\kappa) = j''\kappa = \kappa.$$

Thus $[f]_\mathcal{U} = \kappa$. $\square$

PROOF OF THEOREM 2. Suppose that $\mathcal{U}$ is normal. By Lemma 2.1, $[f]_\mathcal{U} = \kappa$. Let α be strongly compact and $\alpha < \kappa$. Note that $\alpha_x \leq \alpha$ for $x \in P_\alpha\lambda$. Let $\alpha < \beta < \kappa$. Since $\mathcal{U}_\alpha$ is fine and $\beta < \kappa \leq \lambda$,

$$A = \{x \in P_\alpha\lambda \mid \beta \in x\} \in \mathcal{U}_\alpha.$$

Since $\alpha < \beta < \kappa$, $x \cap \alpha_x \subset x \cap \alpha \subsetneq x \cap \kappa$ for $x \in A$. Hence

$$\{x \in P_\alpha\lambda \mid x \cap \alpha_x \subseteq x \cap \kappa\} \in \mathcal{U}_\alpha.$$

Now we get

$$\{\alpha < \kappa \mid \{x \in P_\alpha\lambda \mid x \cap \alpha_x \subseteq x \cap \kappa\} \in \mathcal{U}_\alpha\} \in U,$$

i.e. $\{x \in P_\kappa\lambda \mid f(x) \subsetneq x \cap \kappa\} \in \mathcal{U}$. Since $\mathcal{U}$ is normal,

$$[\langle x \cap \kappa \mid x \in P_\kappa\lambda \rangle]_\mathcal{U} = \kappa.$$

So $\kappa = [f]_\mathcal{U} \subsetneq \kappa$. Contradiction. Hence $\mathcal{U}$ is not normal. $\square$

One may feel strange in comparing Theorem 2 to the following. "Let λ be a measurable cardinal $>\kappa$ and $\mathscr{V}_\alpha$ be a normal ultrafilter on $P_\kappa\alpha$ for α between κ and λ. If D is a normal ultrafilter on λ and W is defined by $X \in W$ iff $X \subseteq P_\kappa\lambda$ and $\{\alpha < \lambda \mid X \cap P_\kappa\alpha \in \mathscr{V}_\alpha\} \in D$, then W is a normal ultrafilter on $P_\kappa\lambda$."

Though $\mathscr{U}$ is not normal, it has weak normality in some case. To show this we first state two lemmas.

LEMMA 2.2 (SEE [1] AND [3]). *Let ν be strongly compact and W be a fine ultrafilter on $P_\nu\eta$. If $\zeta > 2^{\eta^{<\nu}}$ satisfies one of* (i)–(iv) *below, then $j_W(\zeta) = \zeta$.*

(i) $\zeta = \xi^{++}$ *for some* ξ.

(ii) ζ *is a limit cardinal and* $\mathrm{cf}(\zeta) < \nu$ *or* $\mathrm{cf}(\zeta) > \eta^{<\nu}$.

(iii) $\zeta = \xi^+$ *for some ξ that satisfies* (ii).

(iv) $\zeta = (2^{\eta^{<\nu}})^+$.

LEMMA 2.3 (i) *Let W be a fine ultrafilter on $P_\kappa\kappa$ and define D by $X \in D$ iff $X \subset \kappa$ and $X = Y \cap \kappa$ for some $Y \in W$. Then D is a κ-complete nonprincipal ultrafilter on κ.*

(ii) *Conversely, let D be a κ-complete nonprincipal ultrafilter on κ and define W by $X \in W$ iff $X \subset P_\kappa\kappa$ and $X \cap \kappa \in D$. Then W is a fine ultrafilter on $P_\kappa\kappa$.*

PROPOSITION 2.4. *Assume $\{\alpha < \kappa \mid \mathscr{U}_\alpha$ is weakly normal$\} \in U$ and $\lambda > 2^\kappa$ satisfies one of the conditions below.*

(i) $\lambda = \xi^{++}$ *for some* ξ.

(ii) λ *is a limit cardinal and* $\mathrm{cf}(\lambda) \neq \kappa$.

(iii) $\lambda = \xi^+$ *for some ξ that satisfies* (ii).

(iv) $\lambda = (2^\kappa)^+$.

If f is a function from $P_\kappa\lambda$ into λ such that $\{x \in P_\kappa\lambda \mid f(x) \in x\} \in \mathscr{U}$, then $\{x \in P_\kappa\lambda \mid f(x) \le \delta\} \in \mathscr{U}$ for some $\delta < \lambda$.

PROOF. Since $\{x \in P_\kappa\lambda \mid f(x) \in x\} \in \mathscr{U}$, $\{\alpha < \kappa \mid \{x \in P_\alpha\lambda \mid f(x) \in x\} \in \mathscr{U}_\alpha\} \in U$. By the weak-normality of $\mathscr{U}_\alpha$,

$$\{\alpha < \kappa \mid \{x \in P_\alpha\lambda \mid f(x) \le \gamma_\alpha\} \in \mathscr{U}_\alpha \text{ for some } \gamma_\alpha < \lambda\} \in U.$$

Let $[\langle\gamma_\alpha \mid \alpha < \kappa\rangle]_{\mathscr{U}} = \delta$. Clearly $\delta < j_U(\lambda)$. By Lemmas 2.2 and 2.3, $j_U(\lambda) = \lambda$. (Note that $\kappa^{<\kappa} = \kappa$. Replace η and ν by κ.) Hence $\delta < \lambda$.

Since $\delta \le j_U(\delta)$, $\{\alpha < \kappa \mid \gamma_\alpha \le \delta\} \in U$. Thus

$$\{\alpha < \kappa \mid \{x \in P_\alpha\lambda \mid f(x) \le \delta\} \in \mathscr{U}_\alpha\} \in U.$$

This means that $\{x \in P_\kappa\lambda \mid f(x) \le \delta\} \in \mathscr{U}$. □

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Chapter III

Weakly normal filters and the closed unbounded filter on $P_\kappa\lambda$

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In the theory of κ-ultrafilters on a measurable cardinal κ, the closed unbounded filter (the club filter) plays an important role. For instance, Ketonen showed that any two distinct κ-ultrafilters containing the club filter are not isomorphic.

Weakly normal filters on a regular cardinal are also important. A filter is weakly normal iff it is a p-point containing the club filter. Every countably complete ultrafilter is minimal in the RK-ordering iff it is isomorphic to a weakly normal ultrafilter.

Jech is the first to introduce some combinatorial principles into $P_\kappa\lambda$ from the usual fields of κ. At first $P_\kappa\lambda$ seemed the same as κ. But it turned out to be more complicated. Menas proved that every normal measure on $P_\kappa\lambda$ where λ is a strong limit with the cofinality less than κ is isomorphic to a fine measure containing the club filter on $P_\kappa\lambda$. (See Proposition 12 in [9].) In [4], Gitik constructed a model in which there is a stationary subset of $P_\kappa\kappa^+$ that can not be split into κ^+ disjointed stationary sets.

Applying Menas' result, we present two isomorphic fine measures on $P_\kappa\lambda$ both of which contain the club filter under the hypothesis that κ is supercompact and λ is strongly inaccessible.

In [1], a kind of fine measure on $P_\kappa\lambda$ investigated by Menas, was studied. By the embedding argument, it was pointed out that such a measure is not normal and can be weakly normal in suitable conditions. We take a combinatorial approach and show that filters of the same type do not contain a standard club set, indeed strongly closed unbounded. We extend the results in [1] on the weak normality of such a filter.

At last, some remarks on the relation between the RK-order and weakly normal fine measures, the strongly club filter and the partition property are added.

§0. Definitions and notations. κ is a regular uncountable cardinal and λ is a cardinal>κ throughout. $P_\kappa\lambda = \{x \subset \lambda: |x| < \kappa\}$. When we speak of a filter on $P_\kappa\lambda$ it is assumed to be κ-complete and fine, where U is fine iff $\{x: \alpha \in x\} \in U$ for all $\alpha < \lambda$.

Definition 0.1. U is normal if every regressive function is constant on a set of positive measure. (We write $X \in U^+$ if X is positive measure.) U is weakly normal if every regressive function is bounded by some $\gamma < \lambda$ on a set in U. We call U a fine measure if it is an ultrafilter.

A subset C of $P_\kappa\lambda$ is said to be unbounded if for each $a \in P_\kappa\lambda$ there is an $x \in C$ so that $a \subset x$. $\hat{a}$ denotes the set $\{x \in P_\kappa\lambda: a \subset x\}$. Thus C is unbounded if $\hat{a} \cap C \neq 0$ for all $a \in P_\kappa\lambda$. C is closed if $\bigcup A \in C$ whenever A is a $\subset$-increasing chain of length<κ in C. C is strongly closed if $\bigcup A \in C$ for all $A \subset C$ with $|A| < \kappa$. The club filter $CF_{\kappa\lambda}$ is the filter generated by the closed unbounded sets. The strongly club filter $SCF_{\kappa\lambda}$ is the filter generated by the strongly closed unbounded sets.

Let U be a fine measure on $P_\kappa\lambda$ and $f: P_\kappa\lambda \longrightarrow P_\kappa\lambda$. The ultrafilter $f_*(U)$ defined by " $X \in f_*(U)$ if $f^{-1}(X) \in U$" is a fine measure provided that $\{x: \alpha \in f(x)\} \in U$ for all $\alpha < \lambda$.

Definition 0.2. Suppose that U and D are fine measures on $P_\kappa\lambda$. We write $U \leq D$ if $U = f_*(D)$ for some $f: P_\kappa\lambda \longrightarrow P_\kappa\lambda$. U and D are isomorphic ($U \cong D$) if $U = f_*(D)$ and f is one-to-one on a set $X \in D$. D is minimal in the RK-order if D is isomorphic to all $U \leq D$.

Definition 0.3. Suppose that f is an ordinal valued function with domain $P_\kappa\lambda$. f is the first function of U if $\{x: f(x) > \gamma\} \in U$ for any $\gamma < \lambda$, and $\{x: g(x) < \gamma\} \in U$ for some $\gamma < \lambda$ whenever $\{x: g(x) < f(x)\} \in U$.

The first function tells us whether a fine measure is minimal or not under the certain assumption on λ.

Definition 0.4. A fine measure U has the partition property if every $F: [P_\kappa\lambda]^2 = \{\{x,y\}: x,y \in P_\kappa\lambda \text{ and } x \subsetneq y\} \longrightarrow 2$ has a homogeneous set in U. (A is homogeneous for F if there is a $k<2$ so that for all $x,y \in A$ with $x \subsetneq y$, $F(\{x,y\}) = k$.)

§1. Isomorphic fine measures. In this section, λ is a fixed inaccessible cardinal greater than κ a supercompact. We shall present two isomorphic fine measures including $CF_{\kappa\lambda}$. Though we extend the result of Menas, we have to start from it.

Lemma 1.1. (Menas [9]) Let δ be a strong limit cardinal with the cofinality less than κ. Then every normal measure on $P_\kappa\delta$ is isomorphic to a nonnormal fine measure containing $CF_{\kappa\delta}$.

Let $A = \{\delta: \kappa < \delta < \lambda, \delta \text{ is strong limit, } cf(\delta) < \kappa\}$. For each $\delta \in A$, there is a function $q^\delta: P_\kappa\delta \longrightarrow P_\kappa\delta$ so that $CF_{\kappa\delta} \subset q^\delta{}_*(U_\delta) \cong U_\delta$ where U_δ is a normal measure on $P_\kappa\delta$. We shall sum up these U_δ's and $q^\delta{}_*(U_\delta)$'s with a suitable ultrafilter on λ.

Lemma 1.2. There exists a κ-complete ultrafilter on λ including $\{A\} \cup CF_\lambda$. (CF_λ is the club filter on λ.)

Proof. Since λ is inaccessible, A is stationary. Hence we have a λ-complete filter $E = \{X \subset \lambda$: A-X is not stationary$\}$. It is easily seen that $\{A\} \cup CF_\lambda \subset E$. Then the strong compactness of κ gives us a κ-complete ultrafilter D extending E.

We use the above D. Define F_1 and F_2 by

$$X \in F_1 \quad \text{if} \quad X \subset P_\kappa\lambda \text{ and } \{\delta \in A: X \cap P_\kappa\delta \in U_\delta\} \in D.$$
$$X \in F_2 \quad \text{if} \quad X \subset P_\kappa\lambda \text{ and } \{\delta \in A: X \cap P_\kappa\delta \in q^\delta{}_*(U_\delta)\} \in D.$$

F_1 and F_2 are fine measures on $P_\kappa\lambda$. We want to show that they are isomorphic and contain $CF_{\kappa\lambda}$. The next is an easy but key lemma.

Lemma 1.3. Assume that $cf(\eta)<\kappa$ and U is a fine measure on $P_\kappa\eta$. Then $\{x \in P_\kappa\eta$: $\sup(x) = \eta\} \in U$.

Proof. Let $\{\eta_\alpha: \alpha<cf(\eta)\}$ be a cofinal subset of η. Since U is fine, $\{x: \eta_\alpha \in x\} \in U$ for each $\alpha<cf(\eta)$. Using the κ-completeness of U and the fact that $cf(\eta)<\kappa$, we get $\{x: \eta_\alpha \in x$ for every $\alpha<cf(\eta)\} \in U$.

Corollary 1.4. For every $\delta \in A$, $\{x \in P_\kappa\delta$: $\sup(x) = \delta\} \in U_\delta$ and $\{x \in P_\kappa\delta$: $\sup(q^\delta(x)) = \delta\} \in U_\delta$.

Proof. Since $q^\delta{}_*(U_\delta)$ is also a fine measure on $P_\kappa\delta$ and $cf(\delta)<\kappa$, $\{x$: $\sup(x) = \delta\} \in q^\delta{}_*(U_\delta)$. This is equivalent to $\{x$: $\sup(q^\delta(x)) = \delta\} \in U_\delta$.

For $x \in P_\kappa\lambda$, let δ_x = the least member of A such that $x \in P_\kappa\delta$. And $q: P_\kappa\lambda \longrightarrow P_\kappa\lambda$ is defined by;

$$q(x) = q^{\delta_x}(x).$$

By our construction,

Lemma 1.5. For every $\delta \in A$, $\{x \in P_\kappa\delta : \delta_x = \delta\} \in U_\delta$ hence $\{x : q(x) = q^\delta(x)\} \in U_\delta$.

We can see that F_1 and F_2 are isomorphic.

Lemma 1.6. q is one-to-one on a set in F_1.

Proof. Let $B_\delta \in U_\delta$ be such that q^δ is one-to-one on B_δ. We have already known that $C_\delta = \{x \in B_\delta : q(x) = q^\delta(x),\ \sup(q^\delta(x)) = \sup(x) = \delta\} \in U_\delta$. Hence $C = \bigcup \{C_\delta : \delta \in A\}$ is a member of F_1.

Suppose that $x, y \in C$ and $q(x) = q(y)$. There is a $\delta \in A$ such that $\delta = \sup(x) = \sup(q(x)) = \sup(q(y)) = \sup(y)$. Since x and y are in the same C_δ and $q \restriction C_\delta = q^\delta \restriction C_\delta$ is one-to-one, we have $x = y$. Thus q is one-to-one on $C \in F_1$.

Lemma 1.7. $F_2 = q_*(F_1)$.

Proof. Recall that $X \in F_2$ iff $\{\delta \in A : X \cap P_\kappa\delta \in q^\delta{}_*(U_\delta)\} \in D$, and that $X \cap P_\kappa\delta \in q^\delta{}_*(U_\delta)$ is equivalent to $\{x \in P_\kappa\delta : q^\delta(x) \in X \cap P_\kappa\delta\} \in U_\delta$. By 1.5, the last paraphrase is the same as $\{x \in P_\kappa\delta : q(x) \in X\} \in U_\delta$.

Let $Y = \{x \in P_\kappa\lambda : q(x) \in X\}$. We have shown that $X \in F_2$ is equivalent to $\{\delta \in A : Y \cap P_\kappa\delta \in U_\delta\} \in D$. The latter says that $Y \in F_1$ and $X \in q_*(F_1)$. Hence $X \in F_2$ iff $X \in q_*(F_1)$.

What is left to show is that both F_1 and F_2 contain $CF_{\kappa\lambda}$. Note that $\{\delta < \lambda : X \cap P_\kappa\delta \in CF_{\kappa\delta}\} \in CF_\lambda$ for every $X \in CF_{\kappa\lambda}$.

Lemma 1.8. $CF_{\kappa\lambda} \subset F_1 \cap F_2$.

Proof. Suppose that $X \in CF_{\kappa\lambda}$. Then $X' = \{\delta<\lambda: X \cap P_\kappa\delta \in CF_{\kappa\delta}\} \in CF_\lambda \subset D$. Since U_δ and $q^\delta{}_*(U_\delta)$ contain $CF_{\kappa\delta}$, $X \cap P_\kappa\delta$ belongs to both U_δ and $q^\delta{}_*(U_\delta)$ for all $\delta \in X'$. Hence $X \in F_1 \cap F_2$.

Now we have done.

Theorem 1.9. If λ is a strongly inaccessible cardinal greater than κ a supercompact, there are two distinct isomorphic fine measures on $P_\kappa\lambda$ containing the club filter.

The author does not know whether a normal measure on $P_\kappa\lambda$ is isomorphic to a fine measure containing $CF_{\kappa\lambda}$ under the same assumption. It is also still open whether two fine measures can be isomorphic for a successor cardinal λ. The case that λ is not strong limit is also open.

§2. $SCF_{\kappa\lambda}$, prestationary sets and the partition property. For the subsets of regular uncountable cardinals, the situation is simple. That is, $S \subset \kappa$ is stationary iff for any regressive function f on S, there is an unbounded set $T \subset S$ on which f is constant. But this does not hold for the subsets of $P_\kappa\lambda$.

In this section, κ is a regular uncountable cardinal and $\lambda > \kappa$. We begin by Menas' invention again.

Proposition 2.1. (Menas [8]) There is a nonstationary subset S of $P_\kappa\lambda$ such that every regressive function is constant on an unbounded subset of S.

Definition 2.2. We call such a set S " prestationary".

Menas characterized S "stationary" as follows:

Proposition 2.3. (Menas [8]) $S \subset P_\kappa\lambda$ is stationary iff any function $f:S \longrightarrow \lambda\times\lambda$ so that $f(y) \in y\times y$ for all y in S, is constant on some unbounded $T \subset S$.

In the spirit of proposition 2.3, we can express stationarity using prestationarity.

Proposition 2.4. If $S \subset P_\kappa\lambda$ is prestationary and every regressive function is constant on a prestationary $T \subset S$, then S is stationary.
Proof. Let $f:S \longrightarrow \lambda\times\lambda$, f_1, $f_1:S \longrightarrow \lambda$ so that $f(y) \in y\times y$ for all $y \in S$ and $f(y) = (f_1(y), f_2(y))$. Since $f_1(y) \in y$ for all $y \in S$, there is a prestationary $T_1 \subset S$ on which f_1 is constant. Again by the fact that $f_2(y) \in y$ for every $y \in T_1$ that is prestationary, there is an unbounded $T_2 \subset T_1$ so that $f_2 \upharpoonright T_2$ is constant. Then $f \upharpoonright T_2$ is constant.

The stationary subsets are the sets which have nonempty inter--section with every closed unbounded set. Now we characterize the prestationary sets with $SCF_{\kappa\lambda}$. First recall the theorem for $SCF_{\kappa\lambda}$ in Carr [3].

Lemma 2.5. (Carr) $C \in SCF_{\kappa\lambda}$ iff there is a sequence of sets in $P_\kappa\lambda$, $\langle x_\alpha \mid \alpha<\lambda\rangle$ so that $\Delta\langle \hat{x}_\alpha \mid \alpha<\lambda\rangle := \{y: x_\alpha \subset y \text{ for all } \alpha \in y\} \subset C$.

Proposition 2.6. $S \subset P_\kappa\lambda$ is prestationary iff $S \cap C \neq 0$ for all $C \in SCF_{\kappa\lambda}$.
Proof. Suppose that S is prestationary and $S \cap C = 0$ for some $C \in SCF_{\kappa\lambda}$. By 2.5, there is a sequence $\langle x_\alpha \mid \alpha<\lambda\rangle$ so that $\Delta\langle \hat{x}_\alpha \mid \alpha<\lambda\rangle \subset C$. For every

$x \in S$, there exists an $\alpha \in x$ such that $x_\alpha \not\subset x$. Since S is prestationary, there is an ordinal γ so that $\{x \in S: x_\gamma \not\subset x\}$ is unbounded. Contradiction.

For the converse, assume that $S \cap C \neq 0$ for all $C \in SCF_{\kappa\lambda}$ and S is not prestationary. There is a regressive function f such that for every $\alpha<\lambda$ there is an $a_\alpha \in P_\kappa\lambda$ so that $\{x \in S: f(x) = \alpha\} \cap \hat{a}_\alpha = 0$. Let $C = \Delta\{\hat{a}_\alpha | \alpha<\lambda\}$, then $C \in SCF_{\kappa\lambda}$. Pick an $x \in C \cap S$ and suppose that $f(x) = \alpha$. Since $\alpha \in x$ and $x \in C$, $a_\alpha \subset x$. Then $f(x) \neq \alpha$ by the definition of a_α. This is absurd.

We connect the above fact to the partition property of fine measures.

Corollary 2.7. If U is a fine measure with the partition property assigning measure one to the strongly club sets, then U is normal.

This is really proposition 11 in Menas [9], where he proved it for the club sets version. Menas' proof is applicable in our case as well.

§3. Weakly normal filters on $P_\kappa\lambda$. For weakly normal filters on κ regular, see Kanamori [7]. We briefly review the basic facts.

Proposition 3.1. For any filter on κ, the following are equivalent.

(i) U is weakly normal.

(ii) Every filter extension of U is weakly normal.

(iii) If $\{X_\alpha: \alpha<\kappa\}$ are sets of positive measure such that $X_\beta \subset X_\alpha$ whenever $\alpha<\beta$, then $\Delta\{X_\alpha: \alpha<\kappa\} = \{\alpha<\kappa: \alpha \in X_\beta \text{ for all } \beta<\alpha\}$ has a positive measure.

(iv) U is a p-point filter extending CF_κ. (U is a p-point if every function $f:\kappa\longrightarrow\kappa$ such that $\kappa - f^{-1}(\{\alpha\})\in U$ for all $\alpha<\kappa$ is $<\kappa$ to one on some $X\in U$.)

It is natural to ask whether the same thing happens to filters on $P_\kappa\lambda$. We easily get that (i)~(iii) are also equivalent for any filter on $P_\kappa\lambda$. (Note that $\Delta\{X_\alpha : \alpha<\lambda\} = \{x\in P_\kappa\lambda : x\in X_\alpha \text{ for all } \alpha\in x\}$.)

But for (iv), the author only knows the following.

Proposition 3.2. (i) Suppose that U is weakly normal. If f is a function with the domain $P_\kappa\lambda$ and $\{x: f(x)>\alpha\}\in U^+$ for all $\alpha<\lambda$, then there is a set X of positive measure so that $X\cap f^{-1}(\{\alpha\})\subset P_\kappa\alpha$ for all $\alpha<\lambda$.

(ii) Suppose that U extends $SCF_{\kappa\lambda}$ and for any $\alpha<\lambda$ there is an $X\in U^+$ such that $X\cap f^{-1}(\{\alpha\})\subset P_\kappa\beta$ for some $\beta<\lambda$ whenever f satisfies $\{x: f(x)>\gamma\}\in U^+$ for all $\gamma<\lambda$. Then U is weakly normal.

Proof. (i) Let $X_\xi = \{x: f(x)>\xi\}$ for each $\xi<\lambda$. Then $X_\xi\in U^+$ and $X_\eta\subset X_\xi$ if $\xi<\eta$. Now $\Delta\{X_\xi : \xi<\lambda\}\in U^+$ by (iii). If $x\in\Delta\{X_\xi : \xi<\lambda\}$ and $f(x) = \alpha$, then $\xi<\alpha$ for all $\xi\in x$. Hence $x\subset\alpha$.

(ii) Suppose that f is a regressive function on $P_\kappa\lambda$. Since U extends $SCF_{\kappa\lambda}$, every X of positive measure is prestationary. Hence there is an $\alpha<\lambda$ so that $X\cap f^{-1}(\{\alpha\})$ is unbounded. By our hypothesis, $\{x: f(x)<\gamma\}\in U$ for some $\gamma<\lambda$.

The question left is whether every weakly normal filter extends $CF_{\kappa\lambda}$ or $SCF_{\kappa\lambda}$. In [1], the fine measure investigated by Menas was revisited and shown to be nonnormal. We again observe it and get more information, which gives a negative answer to the question. The author wishes to express his gratitude to A. Blass whose advice led to a simplified proof. We concentrate on a filter defined below. We assume that κ is a regular limit cardinal.

Let $\langle U_\alpha \mid \alpha<\kappa\rangle$ be a sequence of fine filters on $P_\alpha\lambda$ and D be a κ-complete uniform filter on κ. Then a fine measure U is defined by

$$X \in U \quad \text{if} \quad X \subset P_\kappa\lambda \text{ and } \{\alpha<\kappa: X \cap P_\alpha\lambda \in U_\alpha\} \in D.$$

Theorem 3.3. (Inspired by Blass.) U does not extend $SCF_{\kappa\lambda}$ hence is nonnormal.

Proof. Let $C = \{x \in P_\kappa\lambda: \ x \cap \kappa \text{ is an ordinal}\}$. Then C is strongly closed unbounded. We shall show that $C \cap P_\alpha\lambda$ is not unbounded for all $\alpha<\kappa$. If $x \in X \cap P_\alpha\lambda$ and $\alpha^+ \in x$, then $\alpha^+ \subset x$. But this contradicts to $|x|<\alpha$. Hence $\alpha^+ \notin x$ for all $x \in C \cap P_\alpha\lambda$ and $C \cap P_\alpha\lambda \notin U_\alpha$. Thus $C \notin U$. Note that $\alpha^+<\kappa<\lambda$ since κ is a limit cardinal.

For certain $A \subset \kappa$ we have a strongly club set which is not unbounded for any $\alpha \in A$. More precisely;

Proposition 3.4. Suppose that $\lambda^{<\kappa} = \lambda$ and $A \subset \kappa$. There is a $C \in SCF_{\kappa\lambda}$ so that if $\alpha \in A$ and $\sup(A \cap \alpha) \neq \alpha$, then $C \cap P_\alpha\lambda$ is not unbounded.

Proof. Let $\{x_\xi: \xi<\lambda\}$ be an enumeration of $P_\kappa\lambda$ and α_ξ = the least member of $A>|x|$. Then, we pick a $y_\xi \supset x$ with $|y_\xi| \geq \alpha_\xi^+$. Finally, $C = \Delta\langle \hat{y}_\xi \mid \xi<\lambda\rangle$.

Suppose that $\alpha \in A$ and $\sup(A \cap \alpha) \neq \alpha$. Then $\alpha = \alpha_\xi$ for some x_ξ. Assume that there exists an $x \in C \cap P_\alpha\lambda$ with $\xi \in x$. By our definition of C, $x \supset y_\xi$. This implies $|x| \geq |y_\xi| \geq \alpha_\xi^+ > \alpha$ contradicting $x \in P_\alpha\lambda$. Hence $(C \cap P_\alpha\lambda) \cap \widehat{\{\xi\}} = 0$.

Now we turn to the weak normality of U under the assumption that U_α is weakly normal for all $\alpha<\kappa$, and improve proposition 2.4 in [1] by a simple argument. In the next theorem, κ is not necessarily a limit cardinal in (i) and (iii).

Theorem 3.5. (i) If $cf(\lambda)>\kappa$, then U is weakly normal.
(ii) If $cf(\lambda) = \kappa$, then U is not weakly normal.
(iii) If $cf(\lambda)<\kappa$ and (a) or (b) is satisfied, then, U is weakly normal.

(a) U is an ultrafilter.

(b) D is $cf(\lambda)$-decendingly complete. That is; if $\langle X_\xi \mid \xi<cf(\lambda)\rangle$ is a sequence of positive measure such that $X_\eta \subset X_\xi$ whenever $\xi<\eta$, then $\bigcap\{X_\xi: \xi\ cf(\lambda)\} \neq 0$. (Note that D is not required to be an ultrafilter.)

Proof. Suppose that $f(x)\in x$ for every $x\in P_\kappa\lambda$.
(i) For $\alpha<\kappa$, δ_α is an ordinal$<\lambda$ such that $\{x\in P_\alpha\lambda: f(x)<\delta_\alpha\}\in U_\alpha$. Since $cf(\lambda)>\kappa$, $\delta = \sup(\{\delta_\alpha: \alpha<\kappa\})<\lambda$. Obviously $\{x\in P_\kappa\lambda: f(x)<\delta\}\in U$.
(ii) Let $\{\lambda_\alpha: \alpha<\kappa\}$ be a cofinal subset of λ and $\lambda_\alpha<\lambda_\beta$ if $\alpha<\beta$. For each $\alpha<\kappa$, $\{x\in P_\alpha\lambda: \lambda_\alpha\in x \text{ and } \lambda_{|x|}<\lambda_\alpha\}\in U_\alpha$. Hence we have $\{x\in P_\kappa\lambda: x-\lambda_{|x|} \neq 0\}\in U$.

So, there is a function $g:P_\kappa\lambda\longrightarrow\lambda$ such that $g(x)\in x$ and $g(x)>\lambda_{|x|}$ for almost all x (mod. U). For any $\alpha<\kappa$, we know that $\{x\in P_\alpha\lambda: x\supset\alpha^+\}\in U$ and then $\{x: \lambda_{|x|}>\lambda_\alpha\}\in U$. Hence $\{x\in P_\kappa\lambda: g(x)>\lambda_\alpha\}\in U$ for every $\alpha<\kappa$. We are done because g is an unbounded regressive function.
(iii) Suppose that (a) holds. We already showed in Lemma 1.3 that every fine measure on $P_\kappa\lambda$ is weakly normal if $cf(\lambda)<\kappa$. In fact,

Fact 3.6. A fine measure is weakly normal iff its first function maps x to sup(x). (We denote such function by Sup.)

When (b) holds, let $\{\lambda_\alpha: \alpha<\delta\}$ be a cofinal subset of λ with $\delta = cf(\lambda)$ so that $\lambda_\alpha<\lambda_\beta$ if $\alpha<\beta$. Suppose that $\{x\in P_\kappa\lambda: f(x)<\lambda_\alpha\}\notin U'$ for all $\alpha<\delta$. Then $\{\xi<\kappa: \{x\in P_\xi\lambda: f(x)<\lambda_\alpha\}\in U_\xi\}\notin D$ for any $\alpha<\delta$. Hence

$$C_\alpha = \{\xi<\kappa: \{x\in P_\xi\lambda: f(x)<\lambda_\alpha\}\notin U_\xi\}\in D^+.$$

If $\alpha<\beta$, then $\{x\in P_\xi\lambda: f(x)<\lambda_\beta\}\notin U_\xi$ implies $\{x\in P_\xi\lambda: f(x)<\lambda_\alpha\}\notin U_\xi$ since $\lambda_\alpha<\lambda_\beta$. So, $C_\beta\subset C_\alpha$. Then $C = \bigcap\{C_\alpha: \alpha<\delta\} \neq 0$.

Pick a $\xi\in C$. $\{x\in P_\xi\lambda: f(x)<\lambda_\alpha\}\notin U_\xi$ for any $\alpha<\delta$. This contradicts the hypothesis that U_ξ is weakly normal.

Note that a filter F on $P_\kappa\lambda$ is weakly normal if it is $cf(\lambda)$--decendingly complete.

Combining Theorems 3.3 and 3.5, we have;

Corollary 3.7. There is a weakly normal filter which does not extend $SCF_{\kappa\lambda}$.

Jech [5] and Carr [3] showed that $CF_{\kappa\lambda}$ is the minimal normal filter. Is there a nice analogue for weakly normal filter? Or, what is the consistency of weakly normal filters? (Note here we assume that any filter is fine and κ-complete.)

§4. Weakly normal fine measures and the RK-ordering. In this section, κ is a fixed strongly compact cardinal. We observe the weak normality in view of the RK-ordering. First we review the fact established by Menas in [8].

Theorem 4.1. (Menas) (i) If $cf(\lambda)<\kappa$ or λ is regular, then every normal measure on $P_\kappa\lambda$ is minimal.

(ii) If λ is regular and the first function of U is one-to-one on a set of measure one, then U is minimal.

We hope that every weakly normal measure is minimal as in the theory of uniform ultrafilters on a regular cardinal. In fact any minimal fine measure is isomorphic to a weakly normal measure.

Proposition 4.2. Every fine measure has a weakly normal measure below it.

Proof. Let U be a fine measure and g its first function. Define $f: P_\kappa\lambda \longrightarrow P_\kappa\lambda$ by $f(x) = x \cap g(x)$.

By an easy observation, $\{x: \alpha \in f(x)\} \in U$ for all $\alpha < \lambda$ and $f_*(U)$ is a fine measure.

Suppose that $\{x: f(x) \in x\} \in f_*(U)$. It means that $\{x: h \circ f(x) \in x \cap g(x)\} \in U$. Since g is the first function of U, we have $\{x: h \circ f(x) < \gamma\} \in U$ for some $\gamma < \lambda$. Hence $\{x: h(x) < \gamma\} \in f_*(U)$.

The next fact appeared already in [8] implicitly.

Proposition 4.3. Let λ be regular and U a fine measure on $P_\kappa\lambda$. U is minimal iff its first function is one-to-one on a set $X \in U$.

Proof. Let $\{A_\lambda(\alpha): \alpha < \lambda\}$ be a partition of $\{\alpha < \lambda: cf(\alpha) = \omega\}$ into disjointed stationary subsets. Let f be the first function and define q by $q(x) = \{\alpha < f(x): A_\lambda(\alpha) \cap f(x)$ is stationary in $f(x)\}$. Then $q_*(U)$ is a minimal fine measure. (Theorem 2.14 in [8])

Suppose that U is minimal. $q \restriction X$ is one-to-one for some $X \in U$. But $q(x) = q(y)$ if $f(x) = f(y)$. Hence $f \restriction X$ is one-to-one.

Corollary 4.4. A weakly normal measure on $P_\kappa\lambda$ with λ regular is minimal iff Sup is one-to-one on a set of measure one.

A filter F on a regular cardinal ρ is called a q-point if every $<\rho$ to one function from ρ to ρ is one-to-one on a set $X \in F$. It is known that any filter extending CF_ρ is a q-point. $SCF_{\kappa\lambda}$ also plays a role on the minimality of weakly normal measures.

Proposition 4.5. Let λ be regular. If U is a minimal fine measure on $P_\kappa\lambda$ that is not weakly normal, then $SCF_{\kappa\lambda} \not\subset U$.

Proof. Let f be the first function. By our assumption, there is a set $X \in U$ so that $f \upharpoonright X$ is one-to-one and $f(x) < \sup(x)$ for all $x \in X$.

Suppose that $SCF_{\kappa\lambda} \subset U$. Then X is prestationary. For $x \in X$, set $g(x)$ = the least member of x greater than $f(x)$. There is an unbounded set $Y \subset X$ such that $g''Y = \{\gamma\}$ for some $\gamma < \lambda$. Thus, $f''Y \subset \gamma$ and $|Y| = \lambda^{<\kappa} > \gamma$, which contradicts the fact that $f \upharpoonright Y$ is one-to-one.

Corollary 4.6. Let λ be regular. If U is normal and $f_*(U) \subset SCF_{\kappa\lambda}$, then $f_*(U)$ is weakly normal and $\{x: \sup(f(x)) = \sup(x)\} \in U$.

Corollary 4.7. For any regular $\lambda > \kappa$, there is a non-minimal fine measure extending $CF_{\kappa\lambda}$.

Proof. Let $A = \{\alpha < \lambda: cf(\alpha) < \kappa\}$ which is stationary in λ. We repeat the construction in §1.

There is a κ-complete ultrafilter on λ, $D \supset CF_\lambda \cup \{A\}$. For each $\alpha \in A$, fix a fine measure U_α on $P_\kappa\alpha$ extending $CF_{\kappa\alpha}$, and define U by

$$X \in U \quad \text{iff } \{\alpha < \lambda: X \cap P_\kappa\alpha \in U_\alpha\} \in D.$$

Then U is a fine measure extending $CF_{\kappa\lambda}$.

We shall see that U is not weakly normal, hence non-minimal by Proposition 4.5.

Since $A \in D$, D is not normal. Thus there is a function g so that $[g]_D = \lambda$ and $\{\alpha<\lambda: g(\alpha)<\alpha\} \in D$.

For $x \in P_\kappa\lambda$, let α_x = the least α such that $x \in P_\kappa\alpha$ and $f(x) = g(\alpha_x)$. For every $\alpha \in A$, $\{x \in P_\kappa\alpha: f(x)<\sup(x)\} \in U_\alpha$ since $\{x: \alpha_x = \alpha = \sup(x)\} \in U_\alpha$. Let h(x) = the least member of x greater than f(x). h is a regressive function on a set in U.

Pick a $\gamma<\lambda$. Then $B = \{\alpha \in A: \gamma<g(\alpha)\} \in D$. For all $\alpha \in B$, $\{x \in P_\kappa\alpha: f(x) = g(\alpha_x) = g(\alpha)\} \in U_\alpha$ and $\{x \in P_\kappa\alpha: f(x)>\gamma\} \in U_\alpha$. Hence $\{x \in P_\kappa\lambda: \gamma<f(x)\} \in U$. It shows that Sup is not the least function.

On the other hand, we have a minimal fine measure which is weakly normal and does not extend $SCF_{\kappa\lambda}$. We recall the fine measure in §3. Suppose that λ is regular and $\langle U_\alpha | \alpha<\kappa\rangle$ is a sequence of normal measures on $P_\alpha\lambda$ and D is a normal measure on κ. Define U by

$$X \in U \quad \text{iff} \quad \{\alpha<\kappa: X \cap P_\alpha\lambda \in U_\alpha\} \in D.$$

Following the argument of 3.1. 3, 4 in [10], we get;

Lemma 4.8. (i) $\{x$: the order type of x is regular$\} \in U$.
(ii) Let G be a ω-Jonsson function over λ. (G is ω-Jonsson over y if $G: {}^\omega y \rightarrow y$ and $G''z = y$ whenever $z \subset y$ and $|z| = |y|$.) Then we have $\{x: G\restriction{}^\omega x$ is ω-Jonsson over $x\} \in U$.
(iii) There is an $X \in U$ so that $\mathrm{Sup}\restriction X$ is one-to-one.

Note that normality of U_α's is necessary in the above. Using the results proved in §3, we can show;

Theorem 4.9. For every regular $\lambda>\kappa$, there is a weakly normal minimal fine measure which does not extend $SCF_{\kappa\lambda}$.
Proof. It is clear that every normal measure is weakly normal.

Hence our U is weakly normal by Theorem 3.5-(i). Theorem 3.3 asserts that U does not extend $SCF_{\kappa\lambda}$. At last U is minimal by Fact 3.6, Theorem 4.1-(ii), and Lemma 4.8-(iii).

It is not known whether U can be isomorphic to some fine measure extending $SCF_{\kappa\lambda}$. We also do not know whether non-minimal weakly normal measures exist.

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