Cyclic Quotients of 2-dimensional Quasi-Homogeneous Hypersurface Singularities.

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THESIS

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PREFACE

Normal surface singularities have been researched by many mathematicians from several points of view (i.e., complex analysis, algebraic geometry, commutative algebra, differential geometry and topology). There, normal surface singularities with $\mathbb{C}^*$-action have important roles. Those singularities are defined by an ideal which is generated by quasi-homogeneous polynomials with same weight (the converse is also true). Then the affine ring of such singularity has a graded structure and there are many investigations in the commutative ring theory. On the other hand the resolution space of any normal surface singularity $\mathbb{C}^*$-action has a geometric structure which is similar to a line bundle and there are many investigations about them in the topology.

Normal surface singularities with $\mathbb{C}^*$-action have appeared often in the classifications from the point of view of complex analysis or algebraic geometry. For example, M. Artin [1] gave the definition of rational singurality (i.e., the geometric genus is 0) and exhausted every rational surface singularity whose multiplicity is two. We usually call them rational double poits. Eevery rational double point is a hypersurface singularity whose defining polynomial is quasi homogeneous. Also, we can find many quasi-homogeneous hypersurface singularities in Arnold's classification of function germs.

In [4], P.Orlik and Ph.Wagreich investigated the algebraic and geometric structure of normal surface singularities with $\mathbb{C}^*$-action. They showed that the weighted dual graphs of such singularities are star-shaped. Moreover, for quasi-homogeneous hypersurface singularities, they showed how we can compute the numerical data of the resolution (the genus and the self-intersection number of any irreducible component of exceptional set) from the weight and the degree of the defining equation. In [5], H. Pinkham reformulated Orlik and Wagreich's result above. He described the associated affine ring is constructed as the graded ring which is associated with a $\mathbb{Q}$- divisor on an algebraic curve (i.e., compact Riemann surface). Furthermore Orlik and Wagreich's result has important meanings for other researches of surface singularity theory, because many examples have been computed by using their method in the latter research.

In Chapter 1 of present paper, we investigate normal surface singularities with $\mathbb{C}^*$-action which are obtained as diagonal cyclics quotient of 2-dimensional quasi-homogeneous hypersurface singularities. Although they are very special normal surface singularities with $\mathbb{C}^*$-action, it has very important meaning to investigate them. Because, for the class we can obtain a numerical formula to compute the several invariants (for example, geometric genus) and a numerical criterion for the singularities to be Gorenstein when the cyclic group G does not contain any reflection. Further this class contains many examples which are well known.

In section 2 of Chapter 1, we give the method to resolve normal surface singularities with $\mathbb{C}^*$-action which are obtained as diagonal cyclic quotient of 2-

dimensional quasi-homogeneous hypersurface singularities. Our method is based on the techniques of Orlik and Wagreich [4] and A. Fujiki [2]. By using this method, we can find many examples which have meaning in the research of normal surface singularities([10]).

In section 4 of Chapter 1, we classify diagonal cyclic quotients of the simple elliptic singularity $\tilde{E}_8$, which are not simple elliptic singularities or quotient singularities. These singularities had been already found by other mathematicians from several points of view. Our result gives a concrete representation for those singularities.

In Chapter 2, we investigate a class of normal surface singularities with $\mathbb{C}^*$-action which are determined by a Weierstrass point on a algebraic curve (i.e., compact Riemann surface). Let C be an algebraic curve and $P \in C$ a Weierstrass point. We consider the graded ring $R(d,e) = \bigoplus_{k=0}^{\infty} H^0(C, \mathcal{O}_C([\frac{ke}{d}] \cdot P) \cdot t^k$ and the associated singularity $X(d,e)$, where $1 \leqq e \leqq d$. In the case of d=e=1, the ring $R(1,1)$ has been already investigated in the field of commtative ring theory.

In section 1 and 2 of Chapter 2, we consider the embedding dimension of $X(d,e)$ and the generator system of the defining ideal. For a normal surface singularity, if it is a rational or minimally elliptic singularity, the embedding dimension of it is written by numerical data (i.e., fundamental cycle) on the exceptional set of the minimal resolution. In general, the embedding dimension not only depends on the numerical data but also depends on the moduli of the singularity. However, F. VanDyke [11] proved that if (X, x) is a normal surface singularity with $\mathbb{C}^*$- action and the self-intersection number is high enough, then the embedding dimension of the singularity is determined by numerical data of the exceptional set. The singularity $X(d,e)$ does not adapt to the situation of VanDyke, because the self-intersection number of the central curve of $X(d,e)$ is -1. About the embedding dimension of $X(d,e)$ we can find a few phenomena which are different from the VanDyke's observations.

In section 3 of Chapter 2, we consider the case in which the semi-group for a Weierstrass point has two generators. Then the singularity $X(d,e)$ is written as a cyclic quotient of a quasi-homogeneous hypersurface singularity. In this case we obtain a necessary and sufficient condition for the singularities to be complete intersection. We prove this by using K-i. Watanabe's results [13]. Moreover, in section 4 we give a formula of the Poincare series of $R(d,e)$.

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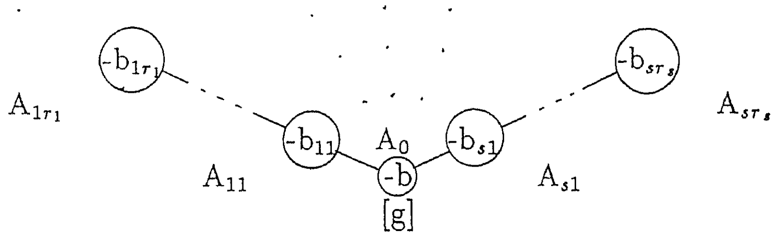
CONTENTS

Chapter I

Cyclic quotients of 2-dimensional quasi-homogeneous hypersurface singularities.

0. Introduction. Let (X, x) be a normal surface singularity with $\boldsymbol{C}^*$-action and $\sigma : (\tilde{X}, E) \longrightarrow (X, x)$ be the minimal good resolution, so the exceptional set E is a normal crossing divisor and the any irreducible component is smooth. Then, by P. Orlik and P. Wagreich [12], the w.d. graph (= weighted dual graph) of E has the star-shaped form as follows :



(0.1),

where $A_{ij} \simeq \boldsymbol{P}^1$ for $i = 1, \cdots, s : j = 1, \cdots, r_i$, and A_0 is a curve of genus g which is called the central curve. Let (Y_i, y_i) be the singularity which is obtained by blowing-down of the i-th branch $(-b_{i1}) - \cdots - (-b_{ir_i})$ $(i = 1, \cdots, s)$. It is isomorphic to the cyclic quotient singularity $C_{\alpha_i;1,\beta_i} = \boldsymbol{C}^2 / < \begin{pmatrix} e_{\alpha_i} & 0 \\ 0 & e_{\alpha_i}^{\beta_i} \end{pmatrix} >$ (see [17]), where $\alpha_i/\beta_i = b_{i1} - \frac{1}{b_{i2} - \cdots - \frac{1}{b_{ir_i}}}$ (the continued fractional expantion). We call (α, β) (resp. α) the type (resp. the cyclic order) of $C_{\alpha;1,\beta}$. In [16], H. Pinkham wrote the affine graded ring R_X of (X, x) in terms of the above numerical data. Let D be $\boldsymbol{Q}$-coefficient divisor $D_0 - \sum_{i=1}^{s} \beta_i/\alpha_i \cdot P_i$ on A_0, and let $D^{(k)}$ be $kD_0 - \sum_{i=1}^{s} \{k\beta_i/\alpha_i\} \cdot P_i$, where D_0 is a divisor on A_0 such that $\mathcal{O}_{A_0}(D_0)$ is the conormal sheaf of A_0 in $\tilde{X}$, and for any $a \in \boldsymbol{R}$, $\{a\}$ is the least integer greater than, or equal to a. Then $R_X = R(A_0, D_0) = \bigoplus_{k=0}^{\infty} H^0(A_0, \mathcal{O}_{A_0}(D^{(k)})) \cdot t^k$. We call this representation Pinkham's construction. Also, he gave a formula of the geometric genus $p_g(X, x)$ in terms of D_0. Moreover, K-i. Watanabe [24] gave a condition for (X, x) to be Gorenstein in terms of D_0. However, in general cases, when we give a curve C with genus $g(C) > 0$ and $\boldsymbol{Q}$-divisor D_0 and consider the normal surface singularity with $\boldsymbol{C}^*$-action (X, x) associated to $R(A_0, D_0)$, it is not easy to obtain the concrete value of $p_g(X, x)$ and to decide if it is Gorenstein.

In this paper we study normal surface singularities with $\boldsymbol{C}^*$-action which are obtained as diagonal cyclic quotients of 2-dimensional quasi-homogeneous hypersurface singularities. Although they are very special normal surface singularities with $\boldsymbol{C}^*$-action, it is meaningful to study them. Because, for the class we can obtain a numerical formula to compute the pluri-genera δ_m (so $\delta_1 = p_g$) and a numerical criterion to be Gorenstein when G does not contain any reflection. If the order of G is small, then we can also compute the embedding dimension. Moreover, this class contains many examples which are well known (for example, every quotient singularity, every simplly elliptic singularity and rational singularity as in section 4 of this paper) and many others.

In section 1, we prepare some facts about finite group actions on weighted projective space or $\boldsymbol{C}^3$. In section 2, we shall give a numerical algorithm to compute the weighted dual graph associated to $(X/G, \pi(0))$ from numerical data. For the case where G is trivial (i.e., the case of hypersurface), Orlik and Wagreich [14] obtained it by topological methods. Further, A. Fujiki [6] gave a method to resoluve 3-dimensional cyclic quotient singularities by lifting the group action through blowing-ups. Our algorithm will be shown by using their techniques. Although the algorithm is not so simple, it is easy to program for the computer. In section 3, we shall give a numerical criterion for $(X/G, \pi(0))$ to be Gorenstein and a formula for the pluri-genera δ_m of $(X/G, \pi(0))$. In section 4, we shall consider the rational singularities which are obtained as cyclic quotients of the simple elliptic singularity $\tilde{E}_8$ by small actions (i.e., actions which do not contain any reflection.).

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1. Preliminaries. Let $\boldsymbol{P}(q_0, \cdots, q_k)$ be the weighted projective space which is defined as the quotient space of $\boldsymbol{C}^{k+1} - \{0\}$ by the $\boldsymbol{C}^*$-action $t \cdot (x_0, \cdots, x_k) = (t^{q_0} x_0, \cdots, t^{q_k} x_k)$ for $t \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ (see [5]). Suppose G is a finite subgroup of $GL(k+1, \boldsymbol{C})$ which acts naturally on $\boldsymbol{P}(q_0, \cdots, q_k)$. If $g = (a_{ij})_{0 \le i,j \le k} \in G$, then from the well-definedness we have $\sum_{j=0}^{k} (t^{q_i} - t^{q_j}) x_j a_{ij} = 0$ for any $t \in \boldsymbol{C}^*$ and $(x_0, \cdots, x_k) \in \boldsymbol{C}^{k+1}$ $(i = 0, \cdots, k)$. From this we have

$$a_{ij} = 0 \quad \text{if} \quad q_i \neq q_j \ (\ i, j = 0, \cdots, k \ ) \qquad (1.1).$$

Therefore if $q_0 = \cdots = q_{s_1-1} < q_{s_1} = \cdots = q_{s_1+s_2-1} < \cdots < q_{s_1+\cdots+s_{m-1}} = \cdots = q_{s_1+\cdots+s_m-1}$ $(k + 1 = s_1 + \cdots + s_m)$, then g $\in G$ has a form $g = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$, where $A_j \in GL(s_j, \boldsymbol{C})$ for $j = 1, 2, \cdots, m$. Hence G has the decomposition $G = G_1 \oplus \cdots \oplus G_m$, where $G_j \subset GL(s_j, \boldsymbol{C})$. In particular, if $k_i \neq k_j$ for any $i \neq j$, then any $g \in G$ is a diagonal matrix and so G is an abelian group. In this case, since g has a finite order, $a_{ii} = e_\ell^{p_i}$ for some p_i and $\ell \in \boldsymbol{N}$ ($i = 0, \cdots, k$).

Definition 1.1. In the above situation, we denote the *Veronese subgroup* $V(G)$ *of* G

as follows : $V(G) = \{\, g \in G \,|\, g$ acts trivially on $\boldsymbol{P}(q_0, \cdots, q_k)\}$.

We can easily see that $V(G)$ is a cyclic group generated by an element $(e_N^{q_0}, \cdots, e_N^{q_k})$ for some $N \in \boldsymbol{N}$. We call the order $|V(G)|$ the Veronese order of G. Also if $G = V(G)$, we call the action a Veronese action. The invariant subring by the Veronese action is the Veronese subring of the polynomial ring $\boldsymbol{C}[x_0, \cdots, x_k]$ with weight $(q_0, \cdots, q_k)$ (see [7]).

Now let $f \in \boldsymbol{C}[x_0, \cdots, x_k]$ be a quasi-homogeneous polynomial of type $(d; q_0, \cdots, q_k)$ (i.e., $f(t^{q_0} \cdot x_0, \cdots, t^{q_k} \cdot x_k) = t^d \cdot f(x_0, \cdots, x_k)$ for any $t \in \boldsymbol{C}^*$) such that $(X, \{0\}) = \{f = 0\}$ is an isolated singularity. Let G be a finite cyclic group which is generated by a diagonal element $g = (e_n^{i_0}, \cdots, e_n^{i_k})(:= \begin{pmatrix} e_n^{i_0} & & 0 \\ & \ddots & \\ 0 & & e_n^{i_2} \end{pmatrix})$. Assume G acts $\boldsymbol{C}^*$-equivariantly on $(X, \{0\})$. Hence f is a semi-invariant polynomial of G (i.e., $g^* f = e_n^s f$ for some integer s). We put $Y = X - \{0\}/\boldsymbol{C}^* \subset \boldsymbol{P}(q_0, \cdots, q_k)$ and put $V_X(G) = \{g \in G | g$ acts trivially on Y$\}$. Since Y is not contained in any coordinate hyperplane $H_i = \{x_i = 0\} \subset \boldsymbol{P}(q_0, \cdots, q_k)\,(i = 0, \cdots, k), V_X(G) = V(G)$. Thus we always abbreviate $V_X(G)$ as $V(G)$ in this paper.

Let $G =< g >=< (e_n^{i_0}, \cdots, e_n^{i_k}) >$ be a cyclic group acting diagonally on $\boldsymbol{P}(q_0, \cdots, q_k)$, then we obtain the following numerical formula of $|V(G)|$.

Lemma 1.2. $\quad |G/V(G)|=\min \left\{ s \in \boldsymbol{N} \; ; \; \begin{array}{c} q_j x \equiv i_j \mod n/s \;\; (j = 0, \cdots, k) \\ \text{for some } x \in \boldsymbol{Z} \end{array} \right\}.$

Proof. $|G/V(G)|$ is the minimal positive integer s satisfying $g^s \in V(G)$. If $g^s \in V(G)$, then n is divided by s. Thus $t^{q_j} = e_{n/s}^{i_j}$ for some $t \in \boldsymbol{C}^*$ $(j = 0, \cdots, k$). Hence if we put $t = e_{n/s}^x$, then $q_j x \equiv i_j \mod n/s$. q.e.d.

Now let G be an abelian group generated by two elements g_1 and g_2, where $g_j = (e_n^{a_j}, e_n^{b_j}, e_n^{c_j})$ ($j = 1, 2$). It acts naturally on $\boldsymbol{C}^3$. We want to compute the generators of the stabilizer G_p at a point $p = (\omega, 0, 0)$, where $\omega \neq 0$. We denote the following integers :

$$d_6 = (a_1, a_2, n), d_5 = (a_1, a_2)/d_6, d_4 = (a_1, n)/d_6, d_3 = (a_2, n)/d_6,$$

$$d_2 = n/d_3 d_4 d_6, d_1 = a_2/d_3 d_5 d_6, d_0 = a_1/d_4 d_5 d_6 \ .$$

Furthermore we denote an integer λ by the equation $d_1 \lambda + d_0 \equiv 0 \mod d_2$ ($0 \leq \lambda < d_2$). Since $(d_1, d_2) = 1$, such λ always exists.

Lemma 1.3. G_p is generated by $f_1 = g_1^{d_3} \cdot g_2^{d_4 \lambda}$ and $f_2 = g_2^{d_2 d_4}$.

Proof. Suppose $g_1^i g_2^j$ is an element of G_p, so we have $a_1 i + a_2 j \equiv 0$ mod n. Thus $d_0 d_4 i + d_1 d_3 j \equiv 0 \mod d_2 d_3 d_4$. Since d_3 (resp. d_4) is relatively prime to $d_0 d_4$ (resp. $d_1 d_3$), we can put $i = d_3 I$ and $j = d_4 J$ for some integers I and J. Then $d_0 I + d_1 J \equiv 0 \mod d_2$. From the definition of λ, we have $d_0 I + d_1 \lambda I \equiv 0 \mod d_2$, so $d_1 J \equiv -d_0 I \equiv d_1 I \lambda \mod d_2$.

Since d_1 is relatively prime to d_2, $J \equiv \lambda I \mod d_2$. Then there exists an integer K satisfying $J = \lambda I + d_2 K$, so we have $g_1^i g_2^j = f_1^I f_2^K$. q.e.d.

Remark 1.4. Let $H =< g_1, g_2 >=< (e_m^{a_1}, e_n^{b_1}), (e_m^{a_2}, e_n^{b_2}) >$ be a group acting on $\boldsymbol{C}^2$ and assume H doesn't contain any reflection of the form $(1, e_n^{\lambda})$, where λ is an integer with $0 < \lambda < n$. Put $k = a_1 b_2 - a_2 b_1, s = g.c.d.(a_1, a_2), a_1 = c_1 s$ and $a_2 = c_2 s$. Let p and q be integers satisfying $pc_1 + qc_2 = 1$ and let $g = g_1^p g_2^q = (e_m^s, e_n^{pb_1 + qb_2})$. Then $g^{c_1} g_1^{-1} = (1, e_n^{qk/s})$ and $g^{c_2} g_2^{-1} = (1, e_n^{pk/s})$. By the assumption, $g^{c_1} = g_1$ and $g^{c_2} = g_2$, so g is a generator of H. This will arise when we resolve $(X/G, \pi(0))$ in the next step.

2. Weighted dual graph associated to $(X/G, \pi(0))$. Suppose $f \in \boldsymbol{C}[x_0, x_1, x_2]$ is a quasi-homogeneous polynomial such that $(X, \{0\}) = \{f = 0\}$ is an isolated singularity. Then the singularity is diffeomorphic to a singularity which is defined by a polynomial belonging to one of the following eight classes (p 61 in [15]) :

$$
\begin{array}{llll}
(I) & x_0^{a_0} + x_1^{a_1} + x_2^{a_2}, & (II) & x_0^{a_0} + x_1^{a_1} + x_1 x_2^{a_2} \ (a_2 > 1), \\
(III) & x_0^{a_0} + x_1^{a_1} x_2 + x_2^{a_2} x_1 \ (a_1, a_2 > 1), & (IV) & x_0^{a_0} + x_0 x_1^{a_1} + x_1 x_2^{a_2} \ (a_0 > 0), \\
(V) & x_0^{a_0} x_1 + x_1^{a_1} x_2 + x_2^{a_2} x_0, & (VI) & x_0^{a_0} + x_1 x_2, \\
(VII) & \multicolumn{3}{l}{x_0^{a_0} + x_0 x_1^{a_1} + x_0 x_2^{a_2} + x_1^{b_1} x_2^{b_2} \ (\ (a_0 - 1)(a_1 b_2 + a_2 b_1)/a_0 a_1 a_2 = 1),} \\
(VIII) & \multicolumn{3}{l}{x_0^{a_0} x_1 + x_0 x_1^{a_1} + x_0 x_2^{a_2} + x_1^{b_1} x_2^{b_2} \ (\ (a_0 - 1)(a_1 b_2 + a_2 b_1)/a_2(a_0 a_1 - 1) = 1),}
\end{array}
\tag{2.1}
$$

Let $G =< (e_n^{i_0}, e_n^{i_1}, e_n^{i_2}) >$ be a finite cyclic group which acts diagonally on $(X, \{0\})$. Then we can prove the following proposition by the same way as in Theorem 3.1.4 in [11].

Proposition 2.1. Suppose $h(x_0, x_1, x_2)$ is a quasi-homogeneous polynomial such that $(V(h), \{0\}) = \{h = 0\}$ has an isolated singularity, and $h = f + g$, where f belongs to one of the above eight classes and no monomial appears in both f and g. Suppose a finite cyclic group G acts diagonally on $V(h)$ and $V(f)$. Then $(V(h)/G, \pi(0))$ is $\boldsymbol{C}^*$-equivariantly diffeomorphic to $(V(f)/G, \pi(0))$.

The numerical data ($g, b, (\alpha_1, \beta_1), \cdots, (\alpha_s, \beta_s)$) (i.e., Seifert invariants) associated to the normal surface singularity with $\boldsymbol{C}^*$-action is uniquely determined by the diffeomorphism class of the singularity (see [14]). Hence, when we compute the weighted dual graph associated to $(X/G, \pi(0))$, we may only consider the quasi-homogeneous polynomials which belong to the eight classes in (2.1).

Notation. For an integer j, let $[j]$ be the integer satisfying $j \equiv [j] \mod 3$ and $0 \leq [j] \leq 2$ in this section.

2.1. Types of cyclic quotient singularities associated to $(X/G, \pi(0))$. From now on we compute the types of cyclic quotient singularities associated to $(X/G, \pi(0))$. Let f be a quasi-homogeneous polynomial of type $(d; q_0, q_1, q_2)$ which belongs to one of the eight classes in (2.1) and $X = \{f = 0\}$. Let $\eta : \boldsymbol{C}^3(y) \longrightarrow \boldsymbol{C}^3(x)$ be the covering map defined by $x_j = y_j^{q_j}$ ($j = 0, 1, 2$), so its covering transformation group is the abelian group $H =< e_{q_0} > \oplus < e_{q_1} > \oplus < e_{q_2} >$. Let $\tilde{f}(y_0, y_1, y_2)$ be the pull-back of f by π, and let $\tilde{X} = \{\tilde{f} = 0\}$ and $\tilde{C} = \tilde{X} - \{0\}/\boldsymbol{C}^* \subset \boldsymbol{P}^2$. Let $\sigma : W \longrightarrow \boldsymbol{C}^3(y)$ be the blowing-up centered at $\{0\} \in \boldsymbol{C}^3$ and let S the proper transformation of $\tilde{X}$ by σ. Then W is the total space of the dual hyperplane bundle over $\boldsymbol{P}^2$. Let $\varphi : W \longrightarrow \boldsymbol{P}^2$ be a bundle map. The exceptional set of the restriction map $\bar{\sigma} : S \longrightarrow \tilde{X}$ is $\tilde{C}$. Let $[\zeta_0{:}\zeta_1{:}\zeta_2]$ be the homogeneous coordinate of $\boldsymbol{P}^2$, and let $U_j =\{[\zeta_0{:}\zeta_1{:}\zeta_2] \in \boldsymbol{P}^2 | \, \zeta_j \neq 0 \}forj = 0, 1, 2$. If we put $W_j = \varphi^{-1}(U_j) = U_j \times \boldsymbol{C}$, $u_{j1} = \zeta_{[j+1]}/\zeta_j, u_{j2} = \zeta_{[j+2]}/\zeta_j$ and $t_j = \zeta_j$, then (u_{j1}, u_{j2}, t_j) gives a coordinate of W_j. These W_0, W_1 and W_2 form a coordinate system of $W$and σ is expressed by

$$\begin{tikzcd}
W \arrow[rrr, "\sigma"] & & & \boldsymbol{C}^3 \arrow[ddddd] \\
\cup & & & \\
S \arrow[rr] & & \tilde{X} \arrow[ur, phantom, sloped, "\subset"] \arrow[ddd] & \\
\cup & & & \\
\tilde{C} & \{0\} \arrow[ur, phantom, sloped, "\in"] & & \\
& & X \arrow[r, phantom, "\subset"] & \boldsymbol{C}^3
\end{tikzcd}$$

these coordinates as follows : $y_0 = t_0 = u_{12}t_1 = u_{21}t_2, y_1 = u_{01}t_0 = t_1 = u_{22}t_2$, $y_2 = u_{02}t_0 = u_{11}t_1 = t_2$. Put $h_0 = (e_{q_0}, 1, 1)$, $h_1 = (1, e_{q_1}, 1)$ and $h_2 = (1, 1, e_{q_2})$, so $H =< h_0, h_1, h_2 >$. The action of h_k ($k =$ 0,1,2) can be lifted to W_j by the relation as above. Then the lifting $\tilde{h}_k$ can be written as follows :

$$\tilde{h}_j = (e_{q_j}^{-1}, e_{q_j}^{-1}, e_{q_j}), \;\; \tilde{h}_{[j+1]} = (e_{q_{[j+1]}}, 1, 1), \;\; \tilde{h}_{[j+2]} = (1, e_{q_{[j+2]}}, 1).$$

Here we put $\tilde{H}_j =< \tilde{h}_j, \tilde{h}_{[j+1]}, \tilde{h}_{[j+2]} >$ (j=0,1,2).

Now let $G =< g >=< (e_n^{i_0}, e_n^{i_1}, e_n^{i_2}) >$ be a cyclic group acting diagonally on $X \subset \boldsymbol{C}^3$. If $p \in X - \{0\}$ is a fixed point of G, then the $\boldsymbol{C}^*$-orbit of p is contained in the fixed point set of G. Then $\pi(p)$ is a ramification point of $\pi : X \longrightarrow X/G$, but not a singular point. Therefore X/G has an isolated singular point only at $\pi(0)$. We denote the following integers :

$$L = l.c.m.(q_0, q_1, q_2), \;\; N = Ln, \;\; I_j = Li_j/q_j \;\; (j = 0, 1, 2) \quad \text{and} \quad V = |V(G)|,$$

where $V(G)$ is the Veronese subgroup of G. Then the action of G can be lifted to $\boldsymbol{C}^3(y)$ with the form $\tilde{G} =< (e_N^{I_0}, e_N^{I_1}, e_N^{I_2}) >$. The action of $\tilde{G}$ can be lifted onto W_j with the form $\tilde{G}_j =< \tilde{g}_j >=< (e_N^{I_{[j+1]}-I_j}, e_N^{I_{[j+2]}-I_j}, e_N^{I_j}) >$. Since $V(G) =< (e_V^{q_0}, e_V^{q_1}, e_V^{q_2}) >$, the lifting onto W_j of the generator $g_v = (e_V^{q_0}, e_V^{q_1}, e_V^{q_2})$ is given by $\tilde{g}_v = (1, 1, e_V)$. Let $\bar{W}_j$ ($\simeq \boldsymbol{C}^3$) be the quotient space of W_j by the group generated by the reflections $\tilde{g}_v, \tilde{h}_{[j+1]}$ and $\tilde{h}_{[j+2]}$, and let $\lambda_j : W_j \longrightarrow \bar{W}_j$ be the quotient map. Further let $\bar{G}_j =< \tilde{G}_j, \tilde{H}_j > / < \tilde{g}_v, \tilde{h}_{[j+1]}, \tilde{h}_{[j+2]} >$, where $< \tilde{G}_j, \tilde{H}_j >$ is the group generated by $\tilde{G}_j$ and $\tilde{H}_j$. If we put $v_{j1} = u_{j1}^{q_{[j+1]}}, v_{j2} = u_{j2}^{q_{[j+2]}}$ and $\tau_j = t_j^V$, then (v_{j1}, v_{j2}, τ_j) gives a coordinate on $\bar{W}_j$. Moreover $\tilde{g}_j$ and $\tilde{h}_j$ are expressed on $\bar{W}_j$ as follows :

$$\bar{g}_j = (e_N^{(I_{[j+1]}-I_j)q_{[j+1]}}, e_N^{(I_{[j+2]}-I_j)q_{[j+2]}}, e_N^{I_jV}),$$
$$\bar{h}_j = (e_{q_j}^{-q_{[j+1]}}, e_{q_j}^{-q_{[j+2]}}, e_{q_j}^{V}), \qquad (2.2).$$

Now we consider the action of $\bar{G}_j =< \bar{g}_j, \bar{h}_j >$ on the algebraic surface $\bar{S}_j := \lambda_j(S \cap W_j) = (S \cap W_j)/ < \tilde{g}_v, \tilde{h}_{[j+1]}, \tilde{h}_{[j+2]} > \subset \bar{W}_j$, and we compute stabilizers of $\bar{G}_j$ at fixed points in $\bar{S}_j$. The algebraic curve $\bar{C}_j := \lambda_j(\tilde{C} \cap W_j)$ is defined by the equation $\tau_j = f(1, v_{j1}, v_{j2}) = 0$ in $\bar{W}_j$. Then, by using the Euler's formula $\sum_{i=0}^{k} q_i x_i \partial f/\partial x_i(x) = d \cdot f(x)$, we can easily see that $\bar{C}_j$ is non-singular. The algebraic surface $\bar{S}_j$ is defined by $f(1, v_{j1}, v_{j2}) = 0$ in W_j (so $\bar{S}_j \simeq \bar{C}_j \times \boldsymbol{C}$) and is non-singular.

$$\begin{array}{ccccl}
C \cap W_j & \subset & S \cap W_j & \subset & W_j \quad (\simeq \boldsymbol{C}^3) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{C}_j & \subset & \bar{S}_j & \subset & \bar{W}_j = W_j / < \tilde{g}_V, \tilde{h}_{[j+1]}, \tilde{h}_{[j+2]} > \quad (\simeq \boldsymbol{C}^3) \\
\downarrow & & \downarrow & & \downarrow / \bar{G}_j \\
\bar{C}_j/\bar{G}_j & \subset & \bar{S}_j/\bar{G}_j & \subset & \bar{W}_j/\bar{G}_j = W_j / < \tilde{G}_j, \tilde{H}_j > .
\end{array}$$

Since $\deg(\bar{C}_j \longrightarrow \bar{C}_j/\bar{G}_j) = \deg(\bar{S}_j \longrightarrow \bar{S}_j/\bar{G}_j) = q_j|G|/V$, $\bar{G}_j$ doesn't contain any reflection with the form $(1, 1, e_s^i)$ except for the identity. Thus, if $p = (v_{j1}, v_{j2}, 0) \in \bar{S}_j$ is a fixed point of the action of $\bar{G}_j$, then $v_{j1} = 0$ or $v_{j2} = 0$. Hence we may only compute the stabilizer of $\bar{G}_j$ at the points in $\bar{S}_j$ with coordinate $(v_{j1}, 0, 0)$ or $(0, v_{j2}, 0)$ or (0,0,0). But a point with coordinate $(0, v_{j2}, 0)$ ($v_{j2} \neq 0$) in $\bar{W}_j$ is mapped to a point $(v_{[j+1]1}, 0, 0)$ in $\bar{W}_{[j+2]}$ by the coordinate transformation, so we may only consider the point with the coordinate $(v_{j1}, 0, 0)$ or (0,0,0) as the fixed points of $\bar{G}_j$ in $\bar{S}_j$.

First we consider the fixed point $p \in \bar{S}_j$ with the coordinate $(a, 0, 0), a \neq 0$. The tangent plane $T_p(\bar{S}_j)$ at p is equal to the affine plane $\{v_{j1} = a\}$ in $\bar{W}_j = \boldsymbol{C}^3(v_{j1}, v_{j2}, \tau_j)$. Hence, in order to compute the cyclic quotient singularity $C_{\alpha_j:1,\beta_j}$ we must compute the action of the stability group $\bar{G}_{j,p}$ at p on $T_p(\bar{S}_j) = \{v_{j1} = a\} \simeq \{a\} \times \boldsymbol{C}^2$ (see [3]). By Lemma 1.3 we can find at most two generators of $\bar{G}_{j,p}$, so we can compute the type (α_j, β_j) for $C_{\alpha_j:1,\beta_j}$ ($\simeq T_p(\bar{S}_j)/\bar{G}_{j,p}$) by Remark 1.4.

Second we consider the fixed point $p = (0, 0, 0) \in \bar{S}_j$ for $\bar{G}_j$. The tangent plane $T_p(\bar{S}_j)$ at p is given by $\{v_{j1} = 0\}$ or $\{v_{j2} = 0\}$ in $\bar{W}_j$. If $T_p(\bar{S}_j) = \{v_{j1} = 0\}$ (resp. $T_p(\bar{S}_j) = \{v_{j2} = 0\}$), then the cyclic quotient is given by the action :

$$< (e_N^{(I_{[j+2]}-I_j)q_{[j+2]}}, e_N^{I_jV}), (e_{q_j}^{-q_{[j+2]}}, e_{q_j}^{V}) > \qquad (2.3).$$
$$(\text{ resp. } < (e_N^{(I_{[j+1]}-I_j)q_{[j+1]}}, e_N^{I_jV}), (e_{q_j}^{-q_{[j+1]}}, e_{q_j}^{V}) >)$$

From Remark 1.4, we can compute the type (α_j, β_j) of the cyclic quotient singularity $C_{\alpha_j:1,\beta_j}$ ($\simeq T_p(\bar{S}_j)/\bar{G}_j$).

2.2. Number of cyclic quotient singularities. On each $\bar{W}_j$, if $p = (0,0,0) \in \bar{S}_j$ is a fixed point of $\bar{G}$ and if $T_p(\bar{S}_j) = \{v_{jk} = 0\}$, then we put $n_{jk} = 1$ ($j = 0,1,2$ and $k = 1,2$). It shows the number of the cyclic quotient singularities associated with the fixed point of such type. It is determined according to the class in (2.1), and it will be listed in table 1.

Next we consider a fixed point $p = (a,0,0) \in \bar{S}_j$ ($a \neq 0$). Let $\mu_j : \bar{W}_j \longrightarrow \bar{W}_j/\bar{G}_j$ be the quotient map by $\bar{G}_j$. Let n_j be the number of cyclic quotient singularities corresponding to the orbits of fixed points $\{(v_{j1},0,0) \in S_j | v_{j1} \neq 0\}$. Then $n_j = \#\mu_j(\{(v_{j1},0,0) \in \bar{S}_j | v_{j1} \neq 0\})$ for $j = 0,1$ and 2. Let $C = X - \{0\}/\boldsymbol{C}^*$. Then C is a smooth curve in $\boldsymbol{P}(q_0,q_1,q_2)$, and it is the central curve associated to (X,x). Furthermore let

$$C_j = \{[x_0:x_1:x_2] \in C \subset \boldsymbol{P}(q_0,q_1,q_2) | x_j \neq 0, x_{[j+1]} \neq 0 \text{ and } x_{[j+2]} = 0\},$$

and let $\ell_j = \#C_j$. Let L_j be the number of points in one orbit on C_j by G. Then, by the same way as in Lemma 1.2, L_j is given as follows :

$$L_j = \min\left\{ s \in N; \begin{array}{c} q_k x/(q_j, q_{[j+1]}) \equiv i_k \bmod n/s \ (k = j, [j+1]) \\ \text{for some } x \in \boldsymbol{Z} \end{array} \right\} \qquad (2.4).$$

We can easily see that n_j is equal to $\#\pi(C_j)$ which is equal to the number of orbits on C_j by the action of G. Hence

$$n_j = \ell_j / L_j \qquad (2.5),$$

and for the eight classes in (2.1), ℓ_j is given by the following table 1.

class	ℓ_0	n_{01}	n_{02}	ℓ_1	n_{11}	n_{12}	ℓ_2	n_{21}	n_{22}
I	(a_0, a_1)			(a_1, a_2)			(a_2, a_0)		
II	(a_0, a_1)			$(a_1 - 1, a_2)$					1
III				$(a_1 - 1, a_2 - 2)$	1				1
IV	$(a_0 - 1, a_1)$					1			1
V		1			1			1	
VI					1				1
VII	$(a_0 - 1, a_1)$					1	$(a_2, a_0 - 1)$	1	
VIII	$(a_0 - 1, a_1 - 1)$	1				1		1	

(We abbreviate "0" in the above.) Table 1.

2.3. Self-intersection number and genus of the central curve. Now let f, $(X,\{0\})$ and G be as in **2.1**. Further let $V(G)$ be the Veronese subgroup of G (i.e., any element of $V(G)$ has trivial action on $\boldsymbol{P}(q_0,q_1,q_2)$). Let $C_{\alpha_j:1,\beta_j}$ ($j = 1,\cdots,s$) be the cyclic quotient singularities associated to $(X/G,\pi(0))$. Let $N_0 = \sum_{i=0}^{2} n_i + \sum_{i=0}^{2}\sum_{j=1}^{2} n_{ij}$ and $N_1 = \sum_{i=0}^{2} \ell_i + \sum_{i=0}^{2}\sum_{j=1}^{2} n_{ij}$. Then the self-intersection number $-b$ of the central curve C associated to $(X/G, \pi(0))$ and the genus of C are given by the following.

Proposition 2.4. (i) $\quad b = d|V(G)|^2/q_0q_1q_2|G| + \sum_{j=1}^{s} \alpha_j/\beta_j$.
(ii) $\quad 2g(C/G) - 2 - N_1 = |V(G)|/|G|\{2g(C) - 2 - N_0\}$.

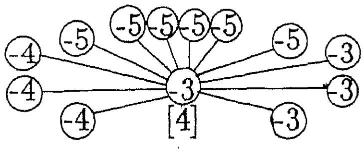
Proof. (i) Let $\tilde{X}, S, W, C$ and $\tilde{C}$ be as in **2.1**. Consider the following diagram :

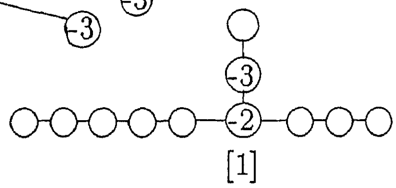
$$\begin{tikzcd}
 & & W \arrow[rr, "\sigma"] & & \boldsymbol{C}^3 \\
 & \hat{S} \arrow[r] \arrow[dr] \arrow[dd] & S \arrow[u, hook] \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[u, hook] \arrow[d] \\
 & & \hat{C} \arrow[r, "\phi"] & \tilde{C} & X \arrow[d] \\
\tilde{S} \arrow[r, "\psi"] & \hat{S}/< \tilde{G}, \tilde{H} > \arrow[rrr] & & & X/G,
\end{tikzcd}$$

where ϕ is the normalization, $\hat{S} = S \times_{\tilde{C}} \hat{C}$ and ψ is the minimal resolution of $\boldsymbol{C}_{\alpha_j;1,\beta_j}(j = 1, \cdots, s)$. For each case of the eight classes in (2.1), by using the local uniformization at the singular point of $\tilde{C}$ we can easily check that the action of $< \tilde{G}, \tilde{H} >$ can be lifted to the action on $\hat{S}$. For the Veronese subgroup $V(\tilde{G}, \tilde{H})$, we have $|V(\tilde{G}, \tilde{H})| = |V(\tilde{G})|$. The self-intersection number of $\hat{C}$ in the quotient space $\hat{S}/V(\tilde{G}, \tilde{H})$ is $d|V(G)|$, and the degree of $\hat{S}/V(\tilde{G}, \tilde{H}) \to \hat{S}/ < \tilde{G}, \tilde{H} >$ is $q_0q_1q_2|G|/|V(G)|$ because $\deg(\hat{S} \to \hat{S}/ < \hat{G}, \hat{H} >) = \deg(\tilde{X} \to X \to X/G) = q_0q_1q_2|G|$. By Orlik and Wagreich's result (Theorem 4.3 in [13]), we obtain our formula.

(ii) If we know the ramification indices at fixed points on C by $\bar{G}(:= G/V(G)$), we can obtain the value of $g(C/\bar{G})$ from the Riemann-Hurwitz formula $2g(C) - 2 = |\bar{G}|(2g(C/\bar{G}) - 2) - \sum_{p \in F}(e(p) - 1)$, where F is the fixed points set of $\bar{G}$ and $e(p)$ is the ramification index ($= |\bar{G}_p|$) at $p \in F$. Since G acts diagonally on $(X, \{0\})$, any fixed point on C by $\bar{G}$ is contained in the subset $\{x_j = 0\} \cap C \subset \boldsymbol{P}(q_0, q_1, q_2)$ for $j \in \{0, 1, 2\}$. The order of the stability group (i.e., the ramification index) at [1:0:0] or [0:1:0] or [0:0:1] is equal to $|\bar{G}|$. The order of stability group at the point $[\omega{:}1{:}0] \in C \subset \boldsymbol{P}(q_0, q_1, q_2)$ $(\omega \neq 0$) (resp. $[0{:}\omega{:}1]$, $[1{:}0{:}\omega]$) is given by $|\bar{G}|/L_0$ (resp. $|\bar{G}|/L_1, |\bar{G}|/L_2$), where L_j is the integer given by (2.4). The number of the points in C with the coodinate $[\omega{:}1{:}0]$ (resp. $[0{:}\omega{:}1]$, $[1{:}0{:}\omega]$) is equal to ℓ_0 (resp. ℓ_1, ℓ_2), whose value was shown in Table 1. From these considerations and (2.5) we can obtain our formula. q.e.d

By Proposition 3.5.1 in [12] we can obtain the value of g(C). Then we can obtain the value of $g(C/G)$. All preparations for the computation of the w.d.graph of $(X/G, \pi(0))$ are now completed.

Example 2.5. Let $(X, x) = \{x_0^{15} + x_1^{18} + x_2^{24} = 0\}$, then the w.d.graph of (X, x) is given by 

. Let $G =< (e_{12}^4, e_{12}^{10}, e_{12}^{11}) >$, then the w.d.graph of $(X/G, \pi(x))$ is given by 

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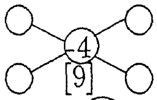
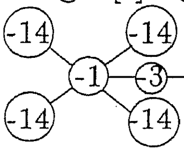
Example 2.6. Let $(X, x) = \{x_0^4 + x_1^8 + x_1 x_2^{14} = 0\}$, then the w.d.graph of (X, x) is given by 

. Let $G =< (e_7, e_7^4, e_7^4) >$, then the w.d.graph of $(X/G, \pi(x))$ is given by 

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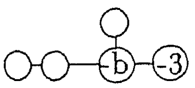
Example 2.7 (quotient singularities, cf. [2], [18]). Let G be a finite subgroup of $GL(2, \boldsymbol{C})$. We consider the quotient singularity $(\boldsymbol{C}^2/G, \pi(0))$, where $\pi : \boldsymbol{C}^2 \longrightarrow \boldsymbol{C}^2/G$ is the quotient map. Let $G_0 = G \cap SL(2, \boldsymbol{C})$, then it is well known that $(\boldsymbol{C}^2/G_0, \varphi(0))$ is a hypersurface singularity (i.e., rational double point). Since $GL(2, \boldsymbol{C})/SL(2, \boldsymbol{C}) = \boldsymbol{C}^*, G/G_0$ is a finite cyclic group $< g >$.

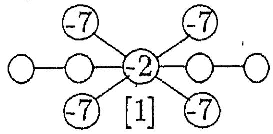
$$
\begin{array}{ccc}
\boldsymbol{C}^2 & \xrightarrow{\varphi} & \boldsymbol{C}^2/G_0 \\
\pi \downarrow & \swarrow \phi & \\
\boldsymbol{C}^2/G & &
\end{array}
$$

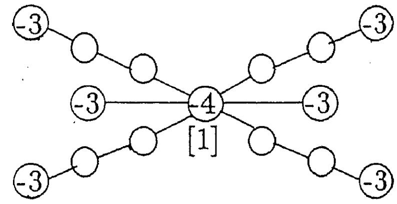
Let F_0, F_1 and F_2 be generators of the invariant subring $\boldsymbol{C}[x_0, x_1, x_2]^{G_0}$ by G_0. Then the action by g to F_j is written by $g^* F_j = e_n^{i_j} \cdot F_j$ $(j = 0, 1, 2)$ for some $(i_0, i_1, i_2; n)$. Therefore any quotient singularity is represented as a diagonal cyclic quotient of a rational double point. In [23], J.Wahl showed that the quotient map $\phi : \boldsymbol{C}^2/G_0 \longrightarrow \boldsymbol{C}^2/G$ by G/G_0 gives the canonical cover of $(\boldsymbol{C}^2/G, \pi(0))$. We can see that the quotient singularity with the w.d.graph 

(b$\geq$ 2) is given as a cyclic quotient $(X/G, \pi(0))$, where $(X, \{0\}) = \{x_0^2 + x_1^3 + x_2^3 = 0\}$ (i.e., rational double point of type D_4) and $G =< (e_{18b-27}^3, e_{18b-27}^2, e_{18b-27}^{6b-7}) >$. Moreover, by using the Pinkham's construction, we can easily check that any quotient singularity except for the above case is a Veronese quotient of a rational double point. For example, a cyclic quotient singularity with the w.d.graph 

is given as the cyclic quotient of $(X, \{0\}) = \{x_0^{2n} + x_1 x_2 = 0\}$ by $G =< (e_2, e_2, e_2) >$. It is well known that we can choose various weights for the polynomial. If n is odd (resp. even), we choose $(2n; 1, n, n)$ (resp. $(2n; 1, 3, 2n-3)$) as the weight of $x_0^{2n} + x_1 x_2$. Then G is the Veronese subgroup of the order 2.

Remark 2.8. Let $R = R(C, D)$ be the Pinkham's construction of a normal surface singularity with $\boldsymbol{C}^*$-action (X, x). If a finite group G is Veronese (i.e., $G = V(G)$), the graded ring associated to $(X/G, \pi(x))$ is equal to $R(C, dD)$, where $d = |G|$. Therefore, when G is a Veronese group and the w.d.graph associated to (X, x) is already known, the w.d.graph associated to $(X/G, \pi(x))$ is easily computed. For example, let $f = x_0^4 + x_1^{12} + x_2^{14}$ and $G =< (e_5, e_5^2, e_5) >$. Then G is the Veronese group of order 5 on $(X, \{0\}) = \{f = 0\}$.

The w.d.graph associated to $(X, \{0\})$ is given by . Then the $\boldsymbol{Q}$-divisor D of the Pinkham's construction is equal to $D_0 - \sum_{i=1}^{4} 1/7 \cdot p_i - \sum_{i=5}^{6} 2/3 \cdot p_i$, where $\deg D_0 = 2$. Since $5D = (5D_0 - 3p_5 - 3p_6) - \sum_{i=1}^{4} 5/7 \cdot p_i - \sum_{i=5}^{6} 1/3 \cdot p_i$, the w.d.graph associated to $(X/G, \pi(0))$ is given by



2.4. Brieskorn type. Let's consider the case of type (I) in (2.1). Let G be an abelian group generated by the elements $g_1 = (e_m^{a_1}, e_n^{b_1}), \cdots, g_r = (e_m^{a_r}, e_n^{b_r})$ acting naturally on $\boldsymbol{C}^2$.

Lemma 2.9. The quotient singularity $(\boldsymbol{C}^2/G, \pi(0))$ is a cyclic quotient singularity whose cyclic order is a divisor of $g.c.d.(m, n)$.

Proof. Let H be the subgroup of G generated by all reflections (i.e, the element whose fixed point set is a line in $\boldsymbol{C}^2$). Then we have $\boldsymbol{C}^2/H \simeq \boldsymbol{C}^2$ and $\boldsymbol{C}^2/G \simeq \boldsymbol{C}^2/(G/H)$. Hence we may assume that G doesn't contain any reflection. Let φ be a homomorphism from G to $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ defined by $(e_m^i, e_n^j) \in G \longrightarrow e_m^i \in \boldsymbol{C}^*$. Then the kernel of φ is trivial and φ is an isomorphism from G onto $\varphi(G)$. Since any finite subgroup of $\boldsymbol{C}^*$ is cyclic, G is a cyclic group whose order is a divisor of m. By considering the map to the second factor $(e_m^i, e_n^j) \longrightarrow e_n^j$, the order $|G|$ is also a divisor of n. q.e.d.

Theorem 2.10. Let $f = x_0^{a_0} + x_1^{a_1} + x_2^{a_2}, (X, \{0\}) = \{f = 0\}$ and let $G =< (e_n^{i_0}, e_n^{i_1}, e_n^{i_2}) >$ be a cyclic group acting diagonally on $(X, \{0\})$. Let $C_{\alpha_j:1,\beta_j}$ $(j = 0, 1, 2)$ be the three types of cyclic quotient singularities (or smooth point) associated to $(X/G, \pi(0))$. If we take a suitable change of j, then the cyclic order α_j is a divisor of a_j for $j = 0, 1, 2$ and the number n_j of $C_{\alpha_j:1,\beta_j}$ is equal to $g.c.d.(a_j, a_{[j+1]}, |s_{[j+2]}|)$, where $s_j = (a_{[j+2]}i_{[j+2]} - a_{[j+1]}i_{[j+1]})/n$.

Proof. Let us consider the situation of **2.1**. By (2.2) we have $\bar{g} = (e_N^{(I_1 - I_0)q_1}, e_N^{(I_2 - I_0)q_2}, e_N^{I_0 V})$, $\bar{h} = (e_{q_0}^{-q_1}, e_{q_0}^{-q_2}, e_{q_0}^{V})$ on $\bar{\tilde{W}}_0$, and $\bar{C}_0 = \lambda_0(C \cap \tilde{W}_0) = \{1 + v_{01}^{a_1} + v_{02}^{a_2} = 0\} \subset \bar{W}_0 = \boldsymbol{C}^3(v_{01}, v_{02}, \tau_0)$. Then

$$\bar{g} = (e_{a_1}^{-s_2}, e_{a_2}^{-s_1}, e_{nq_0}^{i_0 V}) \text{ and } \bar{h} = (e_{a_1}^{-a_0}, e_{a_2}^{-a_0}, e_{nq_0}^{nV}).$$

Consider the fixed point $p = (\omega, 0, 0) \in C_0$ with $\omega^{a_1} = -1$ and consider the associated cyclic quotient singularity $C_{\alpha:1,\beta}$ attached to p, then the cyclic order α is a divisor of a_2 by Lemma 1.3 and Lemma 2.9. Moreover we can see that the number of orbits by the action of $\bar{G} =< \bar{g}, \bar{h} >$ on the set $\{(\omega, 0, 0) | \omega^{a_1} = -1\}$ is equal to $g.c.d.(a_0, a_1, |s_2|)$. q.e.d.

Remark 2.11. Let $(X, \{0\}) = \{x_0^{p_0} + x_1^{p_1} + x_2^{p_2} = 0\}$, where p_0, p_1 and p_2 are pairwise coprime positive integers (≥ 2). Let $G \subset GL(3, \boldsymbol{C})$ be a finite group which acts $\boldsymbol{C}^*$-equivariantly on $(X, \{0\})$. Then G is a Veronese group $< (e_n^{p_1 p_2}, e_n^{p_2 p_0}, e_n^{p_0 p_1}) >$, where n is the order of G. In fact, any element of G is diagonal from Preliminaries. Let $g =< (e_m^{i_0}, e_m^{i_1}, e_m^{i_2}) >$ be any element of G. Since $g.c.d.(p_1p_2, p_2p_0, p_0p_1)$=1, there exist integers α_0, α_1 and α_2 satisfying $\alpha_0 p_1 p_2 + p_0 \alpha_1 p_2 + p_0 p_1 \alpha_2$=1. We put $\gamma = \alpha_0 i_0 + \alpha_1 i_1 + \alpha_2 i_2$. Then it satisfies $\gamma \cdot (p_0p_1, p_1p_2, p_2p_0) \equiv (i_0, i_1, i_2)$ mod m. Then $< g >=< (e_{\bar{m}}^{p_1 p_2}, e_{\bar{m}}^{p_0 p_2}, e_{\bar{m}}^{p_2 p_0}) >$, where $\bar{m} = m/g.c.d.(m, \gamma)$. From this we can easily see the above.

3. The condition for $(X/G, \pi(0))$ to be Gorenstein and a formula for pluri-general $\delta_m(X/G, \pi(0))$, $m \geq 1$.

Definition 3.1. Let Y $\subset \boldsymbol{C}^{k+1}$ be an analytic subvariety and $G \subset GL(k+1, \boldsymbol{C})$ a finite group acting naturally on Y. For an element $g \in G$, if the fixed point set of g on Y contains a subvariety of codimension 1, we say g a *reflection* on Y. Moreover if G doesn't contain any reflection, we say that G is *small*.

From now on let $f \in \boldsymbol{C}[x_0, \cdots, x_k]$ be a quasi-homogeneous polynomial of type $(d; q_0, \cdots, q_k)$ such that $(X, \{0\}) = \{f = 0\} \subset \boldsymbol{C}^{k+1}$ has an isolated singularity at $\{0\}$. Assume $G =< (e_n^{i_0}, \cdots, e_n^{i_k}) >$ acts diagonally on $(X, \{0\})$. We may consider only cases such that the group G doesn't contain any reflection of the type : $(1, \cdot\cdot 1, \overset{i-th.}{e_s}, 1, \cdot\cdot, 1)$ $(i = 0, 1, \cdots, k)$. If it is not so, we can reduce to the case where G doesn't contain such reflections. However there exist reflections of another type. For example, let $f = x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0$ and $G =< g >=< (e_7, 1, e_7^3) >$, then g is a reflection whose fixed point set is the x_1-axis. If we consider the polynomial of Brieskorn type, we may always assume that G is small.

A normal isolated singularity (X, x) of dimension k is called a Gorenstein singularity if the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay and there exists a holomorphic k-form on $X - \{x\}$ which doesn't vanish anywhere. The pluri-genera δ_m is an analytic invariant of the normal isolated singularity which was defined by Ki. Watanabe [25]. Although we don't recall the definition, we note that δ_1 is equal to the geometric genus $p_g = \dim_C R^{k-1}\pi_* \mathcal{O}_{\tilde{X},x}$, where $\pi : \tilde{X} \longrightarrow (X, x)$ is a resolution.

Let $f, (X, \{0\})$ and G be as above. Let's consider a non-vanishing holomorphic k-form ω on $X - \{0\}$ which is defined by

$$\omega|_{U_i} = (-1)^i \frac{dx_0 \wedge \cdot\cdot \overset{i}{\vee} \cdot\cdot \wedge dx_k}{\partial f / \partial x_i}$$

on $U_i = \{\partial f/\partial x_i \neq 0\} \cap X$ $(i = 0, 1, \cdots, k)$. For $a = (a_0, \cdots, a_k)$ and $b = (b_0, \cdots, b_k) \in \boldsymbol{Z}^{k+1}$, we denote $[a, b] = \sum_{i=0}^{k} a_i b_i$. Then we have the following.

Proposition 3.2. Under the situation as above, assume G is small.

(i) $(X/G, \pi(0))$ is a Gorenstein singularity if and only if ω is a G-invariant form.

(ii) For any positive integer m,

$$\delta_m(X/G, \pi(0)) = \#\{ \lambda \in \boldsymbol{N}^{k+1} | m \cdot a(X) \geq [q, \lambda] \text{ and } m \cdot a(G) \equiv [i, \lambda] \bmod n \} - \#\{ \lambda \in \boldsymbol{N}^{k+1} | m \cdot a(X) - d \geq [q, \lambda] \text{ and } m \cdot a(G) - s \equiv [i, \lambda] \bmod n \},$$

where $\lambda = (\lambda_0, \cdots, \lambda_k)$, $i = (i_0, \cdots, i_k)$, $q = (q_0, \cdots, q_k)$, and where we denote $a(X) = d - \sum_{j=0}^{k} q_j$ and $a(G) = s - \sum_{j=0}^{k} i_j$.

Proof. By the condition for G to be small, the covering map $\pi : X - \{0\} \longrightarrow X - \{0\}/G$ is unramified. From the ramification formula (p. 41 in [1]) we have

$$K_{X-\{0\}} = \pi^*(K_{X/G-\pi(0)}) \qquad (3.1).$$

(i) If ω is G-invariant, we have a non-zero holomorphic k-form ω^G on $X/G - \pi(0)$. By the result of [9], the invariant local ring $\mathcal{O}_{X,x}^G$ is Cohen-Macaulay. Then $(X/G, \pi(0))$ is Gorenstein.

Conversely if $(X/G, \pi(0))$ is a Gorenstein singularity, then there exists a non-zero holomorphic k-form η on $X/G - \pi(0)$. Then $\pi^*(\eta)$ is a non-zero holomorphic k-form on $X - \{0\}$ from (3.1). We put $h = \pi^*(\eta)/\omega$ on $X - \{0\}$. Then h is a non-zero holomorphic function on $X - \{0\}$. Since $(X, \{0\})$ is a normal singularity, h can be extended to X. Let $\bar{h}$ be a holomorphic function in a neighborhood of $\{0\} \in \boldsymbol{C}^{k+1}$ whose restriction to X is h. Let $\bar{h} = a_0 + a_1 x^{I_1} + a_2 x^{I_2} + \cdots$ ($a_0 \neq 0$) be the expansion of power series of $\bar{h}$ at $\{0\}$. We put $g^*(\bar{h}) = a_0 + e_n^{b_1} a_1 x^{I_1} + e_n^{b_2} a_2 x^{I_2} + \cdots$ and $g^*(\omega) = e_n^{\lambda}\omega$, where $\lambda, b_1, b_2, \cdots$ are integers. Since $h\omega$ is G-invariant, we can write as follows :

$$h\omega = g^*(h) \cdot g^*(\omega) = (a_0 + e_n^{b_1} a_1 x^{I_1} + e_n^{b_2} a_1 x^{I_2} + \cdots)|_X \cdot e_n^{\lambda}\omega,$$

Therefore $a_0 e_n^{\lambda} = a_0$, so $\lambda \equiv 0 \bmod n$, and then $b_j \equiv 0 \bmod n$ for $j = 1, 2, \cdots$. Hence h is G-invariant, so is ω.

(ii) Let U be a Stein neighborhood of $\{0\}$ in X. Then, from (3.1) we have

$$\Gamma((X - \{0\})/G, \mathcal{O}(mK_{(X-\{0\})/G})) \simeq \Gamma(X - \{0\}, \mathcal{O}(m \cdot K_{X-\{0\}})^G).$$

Moreover, since G is a finite group,

$$L^{2/m}(U/G - \pi(0)) \simeq L^{2/m}(U - \{0\})^G,$$

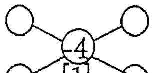
where $L^{2/m}(U - \{0\})$ is the set of all $L^{2/m}$-integrable m-ple holomorphic k-forms on $U - \{0\}$. Then we can obtain our formula by the same way as in the proof of Theorem 1.13 of [25]. q.e.d

Remark 3.3. In [8], V.A. Hinic implicitly proved the "only if part" of (i) of the above proposition in a more general content.

Corollary 3.4. Suppose $G=< g >=< (e_n^{i_0},\cdots,e_n^{i_k}) >$ is small and $g^*f = e_n^s \cdot f$ for some integer $s \geq 0$.

(i) Then $(X/G,\pi(0))$ is a Gorenstein singularity if and only if $i_0+\cdots+i_k \equiv s$ mod n.

(ii) $p_g(X/G,\pi(0)) = \#\{\lambda \in \boldsymbol{N}^{k+1} | a(X) \geq [q,\lambda]$ and $a(G) \equiv [i,\lambda]$ mod n $\}$.

Example 3.5. Let $(X,0) = \{x_0^4+x_1^4+x_2^4 = 0\}$ and $G =< (1,e_2,e_2) >$, then the w.d.graph of $(X/G,\pi(0))$ is given by . From Proposition 3.2, $(X,\pi(0))$ is Gorenstein and $\delta_1 = p_g = 2, \delta_2 = 6, \delta_3 = 10, \delta_4 = 18, \delta_5 = 26$. Moreover, the embedding dimension is 4, so by Serre's result $(X/G,\pi(0))$ is a complete intersection. Then $(X/G,\pi(0)) = \{z_2^2 - z_1z_3 = 0, z_0^4+z_1^2+z_3^2 = 0\} \subset \boldsymbol{C}^4$. We can easily check that $(X/G,\pi(0))$ is a maximally elliptic singularity (see [26]).

4. **Cyclic quotients of simple elliptic singularity $\tilde{\mathbf{E}}_8$.** Suppose (X,x) is a normal surface singularity and $(\tilde{X},E) \longrightarrow (X,x)$ the minimal resolution. If the exceptional set E is a smooth elliptic curve, then (X,x) is called a simple elliptic singularity (K. Saito [19]). The self-intersection number E^2 is equal to -1 or -2 or -3 if and only if (X,x) is a hypersurface singularity. It is called $\tilde{E}_8, \tilde{E}_7$ and $\tilde{E}_6$ respectively. In this section we classify all cyclic groups $< (e_n^{i_0},e_n^{i_1},e_n^{i_2}) >$ which act naturally on $\tilde{E}_8$ and are small (i.e., do not contain any reflection), and classify all rational singularities obtained by the quotients of the above actions (Theorem 4.4). If G is not small, then $\delta_m(X/G,\pi(0)) = 0$ for any m. Hence $(X/G,\pi(0))$ is a quotient singularity by the result in [25]. Since $p_g(X/G,\pi(0)) \leq p_g(X,\{0\})$, if $(X/G,\pi(0)$ is not a rational singularity, then $(X/G,\pi(0))$ is a simple elliptic singularity.

Now let $(X,\{0\}) = \{f=0\}$ be a simple elliptic singularity of the type $\tilde{E}_8$. Since $\tilde{E}_8$ is defined by a quasi-homogeneous polynomial of the type (6;3,2,1) (see [19]), if a finite subgroup of $GL(3,\boldsymbol{C})$ acts $\boldsymbol{C}^*$-equivariantly on $(X,\{0\})$, it is a cyclic group generated by a diagonal element $(e_n^{i_0},e_n^{i_1},e_n^{i_2})$ as in section 1.

Suppose $G =< g >=< (e_n^{i_0},e_n^{i_1},e_n^{i_2}) >$ is a cyclic group which acts diagonally on $(X,\{0\})$ and satisfies the following three conditions :

$$\begin{array}{ll} \text{(i)} & G \text{ is small,} \\ \text{(ii)} & (X/G,\pi(0)) \text{ is a rational singularity,} \qquad\qquad (4.1). \\ \text{(iii)} & g.c.d.(i_0,i_1,i_2,n) = 1, \end{array}$$

Hereafter we usually abbreviate the generator $g = (e_n^{i_0},e_n^{i_1},e_n^{i_2})$ as $g = (i_0,i_1,i_2;n)$. The defining polynomial of $\tilde{E}_8$ is given by

$$x_0^2+x_1^3+a_1x_2^6+a_2x_0x_2^3+a_3x_1x_2^4+a_4x_1^2x_2^2+a_5x_0x_1x_2,$$

where $a_i \in \boldsymbol{C}$ $(i = 1, \cdots, 5)$ are constants. Then g satisfies the following conditions :

$$2i_0 \equiv 3i_1 \bmod \text{n and } i_0 \not\equiv i_1 + i_2 \bmod \text{n} \qquad (4.2).$$

Lemma 4.1. By a suitable G-equivariant coodinate transformation, the defining polynomial f can be transformed to the following polynomial :

$$\text{(i)} \quad x_0^2 + x_1^3 + x_2^6 + cx_1^2x_2^2 \ (\ c \in \boldsymbol{C}\ ) \quad \text{or} \quad \text{(ii)} \quad x_0^2 + x_1^3 + x_1x_2^4 .$$

Proof. If $a_5 \neq 0$, then $2i_0 \equiv i_0 + i_1 + i_2 \bmod n$, so $a_5 = 0$. From (4.1), if $a_2 \neq 0$ and $a_3 \neq 0$ (or $a_2 \neq 0$ and $a_4 \neq 0$), then $(X/G, \pi(0))$ is a simple elliptic singularity. Since $(X/G, \pi(0))$ is a rational singularity, the defining polynomial of $(X, \{0\})$ has one of the following forms :

$$\text{(I) } f = x_0^2 + x_1^3 + a_1x_2^6 + a_2x_0x_2^3,$$
$$\text{(II) } f = x_0^2 + x_1^3 + a_1x_2^6 + a_3x_1x_2^4 + a_4x_1^2x_2^2 .$$

In case (I), we may assume $a_2 \neq 0$. Then $(i_0, i_1, i_2; n)$ satisfies $i_0 \equiv 3i_2 \bmod \text{n}$. Since $f = (x_0 + a_2/2 \cdot x_2^3)^2 + x_1^3 + (a_1 - a_2/4) \cdot x_2^6$ and a coordinate transformation $z_0 = x_0 + a_2/2 \cdot x_2^3, z_1 = x_1$ and $z_2 = cx_2$ is G-equivariant, f is transformed to $z_0^2 + z_1^3 + z_2^6$. In case (II), we may assume $a_3 \neq 0$. Hence we may only consider the following two cases :

$$\text{(II-i) } a_1 \neq 0 \text{ , and (II-ii) } a_1 = 0 \text{ and } a_4 \neq 0.$$

For these cases we have $i_1 \equiv 2i_2 \bmod \text{n}$. Then we get the G-equivariant coordinate transformation : $z_0 = x_0, z_1 = x_1 - tx_2^2, z_2 = x_2$ ($t \in \boldsymbol{C}$). We have $f = z_0^2 + z_1^3 + (3t + a_4)z_1^2z_2^2 + (3t^2 + 2a_4t + a_3)z_1z_2^4 + (t^3 + a_4t^2 + a_3t + a_1)z_2^6$. We choose t satisfying $3t^2 + 2a_4t + a_3 = 0$. q.e.d.

By choosing the suitable generator of G, we classify the actions $(i_0, i_1, i_2; n)$ acting on $\{x_0^2 + x_1^3 + x_2^6 = 0\}$.

Lemma 4.2. Suppose $G = < (i_0, i_1, i_2; n) >$ acts on $(Y, y) = \{x_0^p + x_1^q + x_2^{pq} = 0\}$, where $(p, q) = 1$. Then there exists a generator of G with the form

$$(q + \lambda n/(p, n), p + \mu n/(q, n), 1; n),$$

where λ (resp. μ) is an integer with $0 \leq \lambda \leq (p, n)$ (resp. $0 \leq \mu \leq (q, n)$).

Proof. We have $pi_0 \equiv qi_1 \equiv pqi_2 \bmod \text{n}$. Let $n_1 = n/(p, n)$ and $n_2 = n/(q, n)$, then $i_0 \equiv qi_2 \bmod n_1$ and $i_1 \equiv pi_2 \bmod n_2$ from $(p, q) = 1$. From this we obtain a generator with the form $g = (qi_2 + an_1, pi_2 + bn_2, i_2; n)$ for some $a, b \in \boldsymbol{Z}$. From (iii) of (4.1), $g.c.d.(i_2, n_1, n_2) = 1$. We can easily see that $(n, i_2) = 1$ since G is small. Thus there exists an integer α satisfying $\alpha i_2 \equiv 1 \bmod n$ so that α is prime to n and g^α is a generator of G. Let $\lambda \equiv \alpha a \bmod n$ ($0 \leq \lambda \leq (p, n)$) and $\mu \equiv \alpha b \bmod n$ ($0 \leq \mu \leq (q, n)$), then it completes our proof. q.e.d.

Let's consider the case $p = 2$ and $q = 3$ in the above lemma, then $g = (3 + \lambda n/(2, n), 2 + \mu n/(3, n), 1; n)$, where $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 2$. If $\lambda = \mu = 0$, then G is a Veronese group for $(X, \{0\})$, so $(X/G, \pi(0))$ is a simple elliptic singularity. Hence we may only consider the following five cases :

$$(1)\ \lambda=0 \text{ and } \mu=1,\ (2)\ \lambda=0 \text{ and } \mu=2,\ (3)\ \lambda=1 \text{ and } \mu=0,$$
$$(4)\ \lambda=1 \text{ and } \mu=1,\ (5)\ \lambda=1 \text{ and } \mu=2, \qquad (4.3).$$

Consider the case (1). We have $n \equiv 0 \bmod 3$. If we put $n=3m$, then $g=(3,2+m,1;3m)$. From the condition for $(X/G,\pi(0))$ to be small, $m \equiv 0$ or $2 \bmod 3$. If $m \equiv 0 \bmod 3$ and $m=3\ell$, then $(3\ell+1,n)=1$. From this $g^{3\ell+1}=(3,2,3\ell+1;9\ell)$ is a generator of G. If $m \equiv 2 \bmod 3$ and $m=3\ell+2$, then $(6\ell+5,n)=1$. Hence $g^{6\ell+5}=(3,2,6\ell+5;9\ell+6)$ is a generator of G. By the same way we can obtain the following generators for the five cases in (4.3).

$$\begin{array}{lll}
(1) & (3,2,3\ell+1;9\ell) & \text{or } (3,2,6\ell+5;9\ell+6),\\
(2) & (3,2,6\ell+1;9\ell) & \text{or } (3,2,3\ell+2;9\ell+3),\\
(3) & (3,2,2\ell+1;4\ell), & \qquad\qquad\qquad (4.4).\\
(4) & (3,2,30\ell+1;36\ell) & \text{or } (3,2,6\ell+5;36\ell+24),\\
(5) & (3,2,6\ell+1;36\ell) & \text{or } (3,2,30\ell+11;36\ell+12),
\end{array}$$

We can see that (3) is the only case which acts on $(X,\{0\})=\{x_0^2+x_1^3+x_2^6+cx_1^2x_2^2=0\}$, where $c \in \boldsymbol{C}^*$.

Next let's classify the group $G=<(i_0,i_1,i_2;n)>$ which acts on $(X,\{0\})=\{x_0^2+x_1^3+x_1x_2^4=0\}$ so that G satisfies the condition $2i_0 \equiv 3i_1 \equiv i_1+4i_2 \bmod n$. If $2i_0 \equiv 6i_2 \bmod n$, then G has already appeared in the above classification (4.4). Therefore we may only classify the actions which satisfy the condition :

$$2i_0 \equiv 3i_1 \equiv i_1+4i_2 \not\equiv 6i_2 \bmod n \qquad (4.5).$$

Lemma 4.3. There exists a generator of G with the form $(3,2,1+2\lambda t;8t)$, where $\lambda=1$ or 3.

Proof. Let $g=(i_0,i_1,i_2;n)$ be a generator of G. Suppose n is odd. Then $i_1 \equiv 2i_2 \bmod n$, so $3i_1 \equiv 6i_2$ mod n. This contradicts to (4.5), so n is even. We put $n=2m$. Suppose m is odd, then $i_1 \equiv 2i_2$ mod m. If i_1 is even ($=2J$), then $J \equiv i_2$ mod m. Thus $i_1 \equiv 2i_2 \bmod n$, which contradicts (4.5). If i_1 is odd ($=2J+1$), then $2i_0 \equiv 6J+3$ mod 2m. Hence $3 \equiv 0$ mod 2,which is a contradiction, so m is even. We put $m=2s$, so $<g>=<(i_0,i_1,i_2;4s)>$. Then $i_1 \equiv 2i_2$ mod 2s. We put $i_1=2i_2+2bs$ for some $b \in \boldsymbol{Z}$. Then $i_0 \equiv 3i_2+3bs$ mod 2s. If we write $i_0=3i_2+as$, then $G=<g>=<(3i_2+as,2i_2+2bs,i_2;4s)>$. Since $g.c.d.(i_0,i_1,i_2,4s)=1,(i_2,s)=1$. Let p,q be integers satisfying $pi_2-qs=1$. We may assume that p is odd. Because if p is even, we may exchange p,q for $p(i_2+1)-1, q(i_2+1)$ respectively. Since p is prime to the order $4s$, g^p is a generator. Then we have

$$G=<g^p>=<(3+cs,2+2ds,1+es;4s)> \qquad (4.6),$$

for suitable constants c,d and e. From (4.5) we have $c \equiv d \not\equiv e \bmod 2$ and $0 \le d \le 1$. Hence we may only consider the following three cases :

$$(1)\ c=2 \text{ and } d=0,\ (2)\ c=d=1,\ (3)\ c=3 \text{ and } d=1.$$

For (1), the $2s+1-th$ power of the form in (4.6) is $g_1=(3,2,1+\lambda s;4s)$. If λ is even

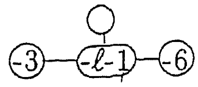
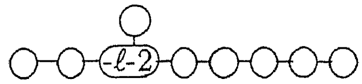
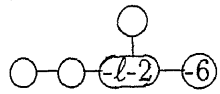
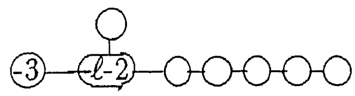
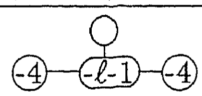
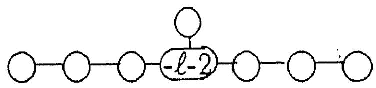
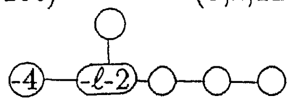
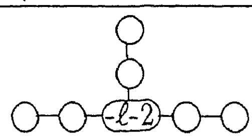
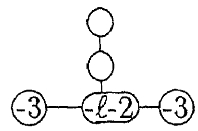
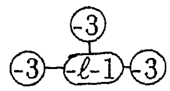
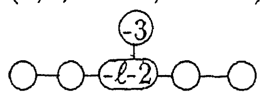
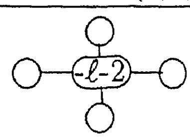
($= 2\mu$), then $2i_0 \equiv 6 \equiv 6(1+2\mu s) \equiv 6i_2 \bmod 4s$, which contradicts the condition (4.5). Then λ is 1 or 3. If s is odd ($= 2t+1$), G contains a reflection $g_1^{4t+2} = (e_2, 1, 1)$. It contradicts the assumption that G is small, so s is even ($s = 2t$). For cases (2) and (3), s is even. In fact, if s is odd, then G contains a reflection $g^{2s} = (1, 1, e_2)$. For case (2), let $p = s+1$ (resp. $s-1$) if $s \equiv 0$ (resp. 2) mod 4. Then $(p, n) = 1$ and the $p-th$ power of the form in (4.6) has a desired form. By the same way we can show the case (3). q.e.d.

Therefore we may only consider the action of the following four types.

$$\begin{array}{llll} (1) & (3,2,4\ell+1;16\ell), & (2) & (3,2,12\ell+1;16\ell), \\ (3) & (3,2,4\ell+3;16\ell+8), & (4) & (3,2,12\ell+7;16\ell+8) \end{array} \qquad (4.7).$$

From the considerations until now, we obtain the following.

Theorem 4.4. For the finite cyclic group acting diagonally on $\tilde{E}_8$, assume the action is small and the quotient is a rational singularity. Then the group and the w.d.graph of the rational singularity are classified as follows :

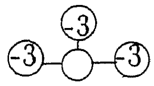
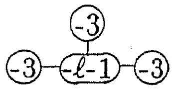
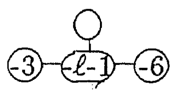
Defining polynomial of $\tilde{E}_8$	$\lvert G/V(G)\rvert$	Weighted dual graph and group< $(i_0, i_1, i_2 : n)$ >
$x_0^2 + x_1^3 + x_2^6$	6	 (3,2,$6\ell+1;36\ell$)  (3,2,$30\ell+1;36\ell$)  (3,2,$6\ell+5;36\ell+24$)  (3,2,$30\ell+11;36\ell+12$)
$x_0^2 + x_1^3 + x_1x_2^4$	4	 (3,2,$4\ell+1;16\ell$)  (3,2,$12\ell+1;16\ell$)  (3,2,$4\ell+3;16\ell+8$) or $(3,2,12\ell+7;16\ell+8)$
$x_0^2 + x_1^3 + x_2^6$	3	 (3,2,$6\ell+1;9\ell$)  (3,2,$6\ell+5;9\ell+6$)  (3,2,$3\ell+1;9\ell$)  (3,2,$3\ell+2;9\ell+3$)
$x_0^2 + x_1^3 + x_2^6 + cx_1^2x_2^2$ ($c \in \boldsymbol{C}$)	2	 (3,2,$2\ell+1;4\ell$)

Remark 4.5. The above rational singularities were already found and discussed from several points of view ([10], [11], [21], [22] and [25]). Theorem 4.4 gives a concrete representation of these singularities.

Under the situation as above, by proposition 3.2, we can see that

$$\delta_m(X/G, \pi\{0\}) = \begin{cases} 0 & \text{if } m \not\equiv 0 \bmod |G|/|V(G)| \\ 1 & \text{if } m \equiv 0 \bmod |G|/|V(G)| \end{cases}$$

(see [25]). In Theorem 4.4 we can see that the elliptic curve associated to the $\tilde{E}_8$ has a special analytic type when the order $|G/V(G)|$ is 3 or 4 or 6, and the only case which has moduli is the case of $|G/V(G)| = 2$. This fact corresponds to the fact that the induced group $G/V(G)$ on the elliptic curve is the complex multiplicative group when $|G/V(G)|$ is 3 or 4 or 6, and $G/V(G)$ is the group generated by the involution when $|G/V(G)| = 2$. From this we can clearly understand the reason why the period of 0 and 1 of δ_m is 2 or 3 or 4 or 6 for these rational singularities.

Remark 4.6. In [23], J. Wahl defined the canonical cover of the normal surface singularity. As an example he showed that the canonical cover of the rational singularity with the w.d.graph  is $\underset{[1]}{-3}$. Recently, M. Tomari and K-i. Watanabe [20] proved the following fact. If (X,x) (resp. (Y,y)) is a normal (resp. normal Gorenstein) isolated singularity with $\boldsymbol{C}^*$-action and if $\varphi : (Y,y) \longrightarrow (X,x)$ is the finite cyclic covering which is unramified outside $x = \varphi(y)$, then φ is decomposed by the canonical covering of (X,x). From this we can easily check that the canonical cover of any rational singularity $(X/G, \pi(0))$ in Theorem 4.4 is given by $\underset{[1]}{-v}$, where $v = |V(G)|$ the Veronese order of G. For example, the canonical cover of  (resp. ) is given by $\underset{[1]}{-3\ell}$, (resp. $\underset{[1]}{-6\ell}$)

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Chapter II

ON A CLASS OF NORMAL SURFACE SINGULARITIES DETERMINED BY WEIERSTRASS POINTS ON ALGEBRAIC CURVES.

Abstract Let x_o be a point on a smooth algebraic curve C with positive genus. We investigate the structure of the surface singularities with the affine ring $\bigoplus_{k=0}^{\infty} H^o(C,\mathcal{O}([\frac{ek}{d}]x_o)\cdot t^k$. We study the embedding dimensions of these singularities, and determine the condition for these singularities to be complete intersecion when the associated semi-group $H(x_o)$ is generated by two elements.

§0. Introduction.

Let C be a smooth algebraic curve over C (or compact Riemann surface) and $x_o \in C$ be a point. Let (X,x) be the normal C^*-surface singularity associated to the 2-dimensional normal graded ring : $\bigoplus_{k=0}^{\infty} H^o(C,\mathcal{O}([\frac{ek}{d}]x_o)\ t^k$, where d and e are positive integers satisfying $d>e\geq 1$ and $(d,e)=1$, and [ ] is the Gaussian symbol. If the genus of C is 0 (i.e., $C= P^1$), then these singularities are cyclic quotient singularities. For cyclic quotient singularities we know their embedding dimension (see [2] or [12]) and the condition to be complete intersecton (i.e., A_k-type). In this paper we assume that the genus of C is positive. If $(\tilde{X},A) \longrightarrow (X,x)$

is the minimal resolution of this singularity, then the weighted dual graph of the exeptional set $A = \bigcup_{i=0}^{r} A_i$ (A_i is the irreducible component of A) is given by the following graph (cf [11]):

$$\overset{A_0}{\underset{[g]}{(-1)}} \;\text{———}\; \overset{A_1}{(-b_1)} \;\text{—}\; \cdots \;\text{———}\; \overset{A_r}{(-b_r)} \qquad \cdots (0.1),$$

where $A_o = C$, $A_o \cap A_1 = \{x_o\}$, the normal bundle of A_o in $\tilde{X}$ is given by $-[x_o]$, further $A_i = P^1$($i = 1, \cdots, r$), g=genus of C and $\frac{d}{d-e} = b_1 - 1 \rfloor b_2 - \cdots - 1 \rfloor b_r$ ($= b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$: continued fractional expantion).

In this paper we use the following notations :

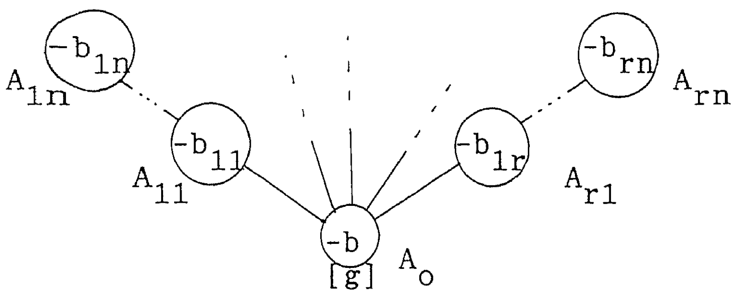
$$R(d,e) = \bigoplus_{k=0}^{\infty} H^o(C, \mathcal{O}([\tfrac{ek}{d}]x_o) \cdot t^k \subseteqq k(A_o)[t],$$

$$R(1,1) = \bigoplus_{k=0}^{\infty} H^o(C, \mathcal{O}(kx_o)) \cdot t^k \subseteqq k(A_o)[t]$$

and for a real number x, let $\{x\}$ be the least integer n which satisfies $n \geq x$. Moreover let $H\langle x_o \rangle$ be a semi-group whose elements are meromorphic functions with a pole only at x_o, and $H(x_o)$ a semi-group whose elements are integers $\mathrm{ord}_{x_o}(f)$ (= order of f at x_o) for $f \in H\langle x_o \rangle$.

In §1 we study the generators of R(d,e) and obtain the embedding dimension of R(d,e) (= emb(R(d,e))) from d, e, $p_1, \cdots, p_s$ when the semi-group $H(x_o)$ is generated by integers $p_1, \cdots, p_s$. In [14] F. VanDyke obtained an interesting formula for the embedding dimensions of some normal C^*- surface singularities. Let us review it briefly. Let (X,x) be a normal C^*- surface singularity and

$\pi:(\tilde{X}, A) \longrightarrow (X,x)$ the minimal good resolution ($\pi^{-1}(x) = A$). Then the weighted dual graph of the exceptional set A is given by the following star-shaped graph ([10],[11]):



..... (0.2),

where $A_{ij} \simeq P^1$ for $i= 1,\cdots,r$; $j= 1,\cdots,n$. Let (Y_i,y_i) be the cyclic quotient singularity which is obtained by the blowing-down of the i-th branch $(-b_{i1})-\cdots-(-b_{in})$ $(i=1,\cdots,r)$, and ℓ_i the embedding dimension of (Y_i,y_i) (i.e., $\ell_i = 3+\sum_{j=1}^{n_i}(b_{ij}-2)$ from [2],[12]).

Theorem (F.VanDyke [13]). If $b \geq 2g+r+1$, then

$$\mathrm{emb}((X,x)) = b-g+1+\sum_{i=1}^{r}(\ell_i-3) = b-g+1+\sum_{i=1}^{r}\sum_{j=1}^{n_i}(b_{ij}-2).$$

In paticular if we insert the component (-2) between A_o and A_{i1}, or A_{ij} and A_{ij+1} $(i= 1,\cdots,r$; $j= 1,\cdots,n_i-1)$ of the above graph, the singularity associated to the graph has the same embedding dimension as that of (X,x). We can see that this property does not hold for the singularity associated to R(d,e). However if we insert the component (-2) between the central curve A_o and the curve A_1 in (0.1), then the embedding dimension does not change (Remark 1.7).

In §2 we first study the defining ideal of R(d,e), and prove the following (Cor.2.3):

If $R(d,e)$ is a complete intersection for some d,e, then $R(1,1)$ is a comlete intersection.

Next we want to consider the converse of the above statement. That is to say, what condition on (d,e) do we need for $R(d,e)$ to be a complete intersection when $R(1,1)$ is a complete intersection ? In §3 we investigate this problem when $R(1,1)$ is a hypersurface (i.e., the semi-group $H(x_o)$ is generated by two elements), and obtain the numerical conditions (Th.3.7) for $R(d,e)$ to be a complete intersection.

In §4 we give a formula of Poincare series of $R(d,e)$, which is written by the non-gap values of x_o and d,e.

The author would like to express his thanks to Prof. Kei-ichi Watanabe for his helpful advice and encouragements during the preparation of this paper.

§1. Generators and the embedding dimension for R(d,e).

First we give an example to show how we can find the generators and the embedding dimension of $R(d,e)$ ($=\mathrm{emb}(R(d,e))$) .

Example 1.1 Let C be a hyperelliptic curve of genus 3 and x_o a Weierstrass point on C. Further let f and g be meromorphic function on C which generate the semi-group $H\langle x_o\rangle$, so their order of f, g at x_o are 2, 7 respectively. Suppose that d=4, e=3, then we have the followings:

k	$[\frac{3k}{4}]$	basis of $H^0(C,\mathcal{O}([\frac{3k}{4}]x_0))\cdot t^k$
1	0	t
2	1	t^2
3	2	$t^3,\ ft^3$
4	3	$t^4,\ ft^4$
5	3	$t^5,\ ft^5$
6	4	$t^6,\ ft^6,\ f^2t^6$
7	5	$t^7,\ ft^7,\ f^2t^7$
8	6	$t^8,\ ft^8,\ f^2t^8,\ f^3t^8$
9	6	$t^9,\ ft^9,\ f^2t^9,\ f^3t^9$
10	7	$t^{10},\ ft^{10},\ f^2t^{10},\ f^3t^{10},\ gt^{10}$
11	8	$t^{11},\ ft^{11},\ f^2t^{11},\ f^3t^{11},\ gt^{11},\ f^4t^{11}$
$\vdots$	$\vdots$	....

We can choose the generators of $R(d,e)=\bigoplus_{k=0}^{\infty} H^0(C,\mathcal{O}([\frac{3k}{4}]x_0))\cdot t^k$ as follows: $z_0=t$, $z_1=ft^3$, $z_2=f^3t^8$ and $z_3=gt^{10}$, and so emb(R(d,e))= 4. We can see that the base z_k $(1\le k\le 3)$ has the form $f^ig^jt^{\{\frac{4(2i+7j)}{3}\}}$ for $(i,j)=(1,0),(3,0)$ and $(0,1)$.

In the following let C be a smooth curve of genus g and $x_0\in C$ a point on C. Let $\langle f_1,\cdots,f_s\rangle$ be a generator system of $H\langle x_0\rangle$, let $p_i=\mathrm{ord}_{x_0}(f_i)$ $(i=1,\cdots,s)$, and let $p_1<\cdots<p_s$. Furthermore we say that a monomial $f_1^{i_1}\cdots f_s^{i_s}\cdot t^m$ is a "generator" if it is a member of minimal generating system of $R(d,e)$.

Proposition 1.2. $\mathrm{emb}(R(d,1))=\mathrm{emb}(R(1,1))$.

Proof. The ring is generated by $t, f_1t^{p_1}, f_2t^{p_2}, \cdots, f_st^{p_s}$. If $e=1$, then t, $f_1t^{dp_1}$, $f_2t^{dp_2}$, $\cdots$, $f_st^{dp_s}$ generate the ring $R(d,1)$. Q.E.D.

Moreover, let $\varphi : C[z_o, z_1, \cdots, z_s] \longrightarrow C[w_o, w_1, \cdots, w_s]$ be the inclusion map given by $z_o = w_o^d$, $z_1 = w_1, \cdots, z_s = w_s$ and $R(1,1) \simeq C[z_o, z_1, \cdots, z_s]/I$, then we have $R(d,1) \simeq C[w_o, w_1, \cdots, w_s]/\langle\varphi(I)\rangle$, where $\langle\varphi(I)\rangle$ is the ideal generated by $\varphi(I)$.

Proposition 1.3. If $M = f_1^{i_1}\cdots f_s^{i_s}\cdot t^m$ is a generator of $R(d,e)$, then $m=\{\frac{d\sum i_jp_j}{e}\}$.

Proof. Set $n=\{\frac{d\sum i_jp_j}{e}\}$. Since $M \in R(d,e)$, we have $m \geq n$. If $m > n$, then $M = (f_1^{i_1}\cdots f_s^{i_s}\cdot t^n)\cdot t^{m-n}$, which shows that M is not a generator. Q.E.D.

Proposition 1.4. The set of elements

$$\{ t,\ f_1^{i_1}\cdots f_s^{i_s}t^{\{\frac{d\Sigma p_ji_j}{e}\}} \mid 0\leq i_j\leq e,\ j=1,\cdots,s\ ,\ (i_1,\cdots,i_s)\neq(0,\cdots,0)\}$$

contains a generator system of the graded ring $R(d,e)$.

Proof. Let $f_1^{i_1}\cdots f_s^{i_s}\cdot t^k$ be an element of $H^o(C,\mathcal{O}([\frac{ek}{d}]x_o))\cdot t^k$, then $[\frac{ek}{d}]\geq \sum_{j=1}^{s} i_jp_j$, and so $k\geq\{\frac{d\Sigma i_jp_j}{e}\}$. If we put $i_j=\beta_je+\gamma_j$ ($0\leq\gamma_j<e$), then $k \geq d\sum_{j=1}^{s}\beta_jp_j+\{\frac{d\cdot\Sigma\gamma_jp_j}{e}\}$. Hence if we put $c=k-d\sum_{j=1}^{s}\beta_jp_j-\{\frac{d\cdot\Sigma\gamma_jp_j}{e}\}$, then $f_1^{i_1}\cdots f_s^{i_s}\cdot t^k=$ $(f_1^{\gamma_1}\cdots f_s^{\gamma_s}\cdot t^{\{\frac{d\cdot\Sigma\gamma_jp_j}{e}\}}).(f_1^e\cdot t^{p_1})^{\beta_1}\cdots(f_s^e\cdot t^{p_s})^{\beta_s}\cdot t^c$. Q.E.D.

Corollary 1.5. $\mathrm{emb}(R(d,e)) \leq \min\{(e+1)^d,\ d(\sum_{j=1}^{s} p_j)\}$.

From Proposition 1.4, the ring $R(d,e)$ has a generator system consisting of the following homogeneous elements:

$$z_0 = t,\ z_1 = f_1 t^{\{\frac{dp_1}{e}\}}, \cdots, z_s = f_s t^{\{\frac{dp_s}{e}\}},\ \text{and}$$

$$z_{s+1} = f_1^{i_{s+1,1}} \cdots f_s^{i_{s+1,s}} \cdot t^{\{\frac{d\Sigma i_{s+1,j}p_j}{e}\}}, \cdots,\ z_\ell = f_1^{i_{\ell,1}} \cdots f_s^{i_{\ell,s}} \cdot t^{\{\frac{d\Sigma i_{\ell,j}p_j}{e}\}} \qquad \cdots\cdots (1.1),$$

where $0 \leq i_{j,k} \leq e$ for $j=s+1,\cdots,\ell; k=1,\cdots,s$, and further $\ell+1 = \mathrm{emb}(R(d,e))$ and $\deg z_{s+1} < \cdots < \deg z_\ell$. Hence we have $\mathrm{emb}(R(d,e)) \geq \mathrm{emb}(R(1,1))$.

Next we consider what conditions are required for degrees of generators of $R(d,e)$. From Corollary 1.5, we may assume that the degree of any generator is less than or equal to $d(\sum_{j=1}^{s} p_j)$. Moreover we have $\dim H^0(C, \mathcal{O}([\frac{ek}{d}]x_0)) - \dim H^0(C, \mathcal{O}([\frac{e(k-1)}{d}]x_0)) \leq 1$ for any k. Therefore if M_1 and M_2 are elements of same homogeneous generator system of $R(d,e)$, then $\deg M_1 \neq \deg M_2$.

Definition 1.6. When the semi-group $H(x_0)$ is generated by $p_1,\cdots,p_s$, we define

$$I(x_0) = \{\ i \in H(x_0, d(\sum_{j=1}^{s} p_j))\ |\ \{\frac{di}{e}\} < \{\frac{d\ell}{e}\} + \{\frac{d(i-\ell)}{e}\} \text{ for any } \ell \in H(x_0, i-1)\}$$

$$= \{\ i \in H(x_0, d(\sum_{j=1}^{s} p_j))\ |\ \{\frac{ri}{e}\} < \{\frac{r\ell}{e}\} + \{\frac{r(i-\ell)}{e}\} \text{ for any } \ell \in H(x_0, i-1)\},$$

where $H(x_0,k) = H(x_0) \cap \{1,2,\cdots,k\}$ and $r \equiv d \pmod e$ with $0 \leq r < e$.

We can easily see that any element of $I(x_o)$ is the degree of the generator of $R(d,e)$, so we obtain the following results.

Proposition 1.7. (i) $\mathrm{emb}(R(d,e)) = \#I(x_o)$, where "#" is the number of elements in the set. (ii) If $d_1 \equiv d_2 \bmod e$, then $\mathrm{emb}(R(d_1,e)) = \mathrm{emb}(R(d_2,e))$, where x_o is the same point in both cases.

Remark 1.8. Under the situations of Proposition 1.7 (ii), if $d_1 \equiv d_2 \bmod e$ ($d_1 > d_2$), then $d_1 - d_2 = ke$ for some $k>0$. Therefore if

$$\frac{d_2}{d_2 - e} = b_1 - \cfrac{1}{b_2 - \cdots - \cfrac{1}{b_n}},$$

then

$$\frac{d_1}{d_1 - e} = \underbrace{2 - \cfrac{1}{2 - \cdots - \cfrac{1}{2}}}_{k\text{-times}} - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cdots \cfrac{1}{b_n}}}.$$

Hence if we change the branch of the exceptional set

from the form : $\underset{[g]}{(-1)} - \overset{A_1}{(-b_1)} - \cdots - \overset{A_n}{(-b_n)}$

to the form : $\underset{[g]}{(-1)} - (-2) - \cdots - (-2) - (-b_1) - \cdots - (-b_n)$

and if we do not change other conditions (the central curve A_o, the intersection point x_o between A_o and the branch, and the normal bundle $[-x_o]$ of A_o in $\tilde{X}$), then the embedding dimension does not change. However if we insert the component (-2) between A_i and A_{i+1} $(i>0)$, the above property does not hold generally.

In the remainder of this section we consider the case when x_o is a point whose semi-group is generated by two elements. Hence if the genus of C>1, then x_o is a Weierstrass point on C. In the following let f, g be generators of $H<x_o>$, and p, q their orders at x_o.

Example 1.9 (I. Ono - Ki. Watanabe). This example was given in [9] and we recall thier results briefly. Let C_o be a affine curve defined by the equation : $x^p + \prod_{i=1}^{q}(y - \alpha_i) = 0$ (q>p>1, p and q are relatively prime, and $\alpha_i \neq \alpha_j$ if $i \neq j$) and $\bar{C}_o = \{ [x:y:z] \in \mathbb{P}^2 \mid x^p z^{q-p} + \prod_{i=1}^{q}(y - \alpha_i z)=0 \}$ the closure of C_o in $\mathbb{P}^2$. If q=p+1, then $\bar{C}_o$ is non-singuler. If q > p+1, then $\bar{C}_o$ has one singular point [1:0:0]. Let $\phi : C \longrightarrow \bar{C}_o$ be the normalization and $x_o = \phi^{-1}([1:0:0])$, then C is a compact smooth curve of genus $\frac{(p-1)(q-1)}{2}$. If we denote $\phi^*(\frac{y}{z})$ (resp. $\phi^*(\frac{x}{z})$) by f (resp. g), then they are meromorphic function on C which have the poles of order p and q respectively at x_o and satisfy the relation : $g^p + \prod_{i=1}^{q}(f - \alpha_i) = 0$. If $\frac{(p-1)(q-1)}{2} > 1$, that is, the genus of C > 1, then x_o is a Weierstrass point on C whose semi-group is generated by f and g. If the genus of C = 1, then (p,q) = (2,3). Ono - Watanabe showed that the ring R(1,1) = $\bigoplus_{k=0}^{\infty} H^o(C, \mathcal{O}(kx_o)) \cdot t^k$ is a hypersurface defined by the equation : $x^p + \prod_{i=1}^{q}(y + \alpha_i z^p) = 0$. The Brieskorn type singularity with the form : $x^p + y^q + z^{pq} = 0$ is an example of this procedure.

Notations. Set $e_1 = \frac{e}{(e,p)}$, $e_2 = \frac{e}{(e,q)}$, $p_1 = \frac{p}{(e,p)}$ and $q_1 = \frac{q}{(e,q)}$, where (a,b) is the greatest common factor of a and b.

Proposition 1.10. (i) The set of homogeneous elements

$$\left\{ t,\ f^{e_1}t^{dp_1},\ \text{and}\ f^i g^j t^{\{\frac{d(ip+jq)}{e}\}} \ \middle|\ \begin{array}{l} i=0,1,\cdots,e_1-1 \\ j=0,1,\cdots,p-1 \end{array},\ (i,j)\neq(0,0) \right\}$$

generates the graded ring $R(d,e)$.

(ii) The set of homogeneous elements

$$\left\{ t,\ f^{e_1}t^{dp_1},\ g^{e_2}t^{dq_1}\ \text{and}\ f^i g^j t^{\{\frac{d(ip+jq)}{e}\}} \ \middle|\ \begin{array}{l} i=0,1,\cdots,e_1-1 \\ j=0,1,\cdots,e_2-1 \end{array},\ (i,j)\neq(0,0) \right\}$$

generates the graded ring $R(d,e)$.

(iii) $\mathrm{emb}(R(d,e)) \leq \min\{ \frac{pe}{(e,p)}+1\ ,\ \frac{e^2}{(e,p)\cdot(e,q)}+1 \}$.

The proof of this proposition is similar to that of Proposition 1.4, so we omit it.

Definition 1.11. We define the set of integers as follows :

$$\begin{aligned} S_o(e,r,p) &= \{0,1\} \cup \{i \mid 2\leq i\leq e_1,\ ip \in I(x_o)\} \\ &= \{0,1\} \cup \{i \mid 2\leq i\leq e_1,\ \{\tfrac{rp\varphi}{e}\}+\{\tfrac{rp(i-\varphi)}{e}\}>\{\tfrac{rpi}{e}\}\ \text{for}\ \varphi=1,\cdots,i-1\}, \end{aligned}$$

$$S_j(e,r,p,q) = \{i \mid 0\leq i<e_1,\ ip+jq \in I(x_o)\}$$

$$= \left\{ i \ \middle|\ 0\leq i<e_1,\ \begin{array}{l} \{\frac{r(p\varphi+q\psi)}{e}\}+\{\frac{r(p(i-\varphi)+q(j-\psi))}{e}\}>\{\frac{r(pi+qj)}{e}\} \\ \text{for any}\ \varphi=0,1,\cdots,i\ ;\psi=0,1,\cdots,j,\ (\varphi,\psi)\neq(0,0),(i,j) \end{array} \right\}$$

for $j=1,\cdots,p-1$, and we define the numbers :

$$s_o = s_o(e,r,p) = \#S_o(e,r,p),$$

$$s_j = s_j(e,r,p,q) = \#S_j(e,r,p,q) \qquad (j=1,\cdots,p-1).$$

We can easily obtain the next proposition from Proposition 1.7.

Proposition 1.12. $\mathrm{emb}(R(d,e)) = \sum_{j=0}^{p-1} s_j$.

Example 1.13. (p = 2 :hyperelliptic point case)

Let p=2, q=2m+1 and q>e>1 (m is the genus of the curve C), then

$$S_0(e,r,2)=\{\ 0,1\ \} \cup \{\ i \mid 2\le i\le e_1, \{\tfrac{2r\varphi}{e}\}+\{\tfrac{2r(i-\varphi)}{e}\}>\{\tfrac{2ri}{e}\} \text{ for } \varphi=1,\cdots,i-1$$

$$S_1(e,r,2,q)=\left\{\ i \;\middle|\; \begin{array}{l} 0\le i<e_1, \{\frac{r(2\varphi+q\psi)}{e}\}+\{\frac{r(2(i-\varphi)+q(1-\psi))}{e}\}>\{\frac{r(2i+q)}{e}\} \\ \quad \text{for any } \varphi=0,\cdots,i\ ;\psi=0,1\ ,(\varphi,\psi)\neq(0,0),(i,1) \end{array}\right\}$$

$$=\{\ i \mid 0\le i<e_1,\ \{\tfrac{2r\varphi}{e}\}+\{\tfrac{r(2(i-\varphi)+q)}{e}\}>\{\tfrac{r(2i+q)}{e}\} \text{ for } \varphi=1,\cdots,i\ \}$$

$$\cap\ \{\ i \mid 0\le i<e_1, \{\tfrac{r(2\varphi+q)}{e}\}+\{\tfrac{2r(i-\varphi)}{e}\}>\{\tfrac{r(2i+q)}{e}\} \text{ for } \varphi=0,\cdots,i-1\ \}$$

$$=\{\ i \mid 0\le i<e_1,\ \{\tfrac{2r\varphi}{e}\}+\{\tfrac{r(2(i-\varphi)+q)}{e}\}>\{\tfrac{r(2i+q)}{e}\} \text{ for } \varphi=1,\cdots,i\ \}\ .$$

Here we suppose that e =7, $d \equiv 1 \bmod 7$ (d > 7) and q > 7, then

$$S_0(7,1,2) = \{\ 0,\ 1,\ 2,\ 3,\ 7\ \},$$

so $s_0 = 5$, and further

$$S_1(7,1,2,q) = \{\ i \mid 0\le i<7,\ \{\tfrac{2\varphi}{7}\}+\{\tfrac{2(i-\varphi)+q}{7}\}>\{\tfrac{2i+q}{7}\} \text{ for } \varphi=1,\cdots,i\}.$$

Therefore if we put v to be $v\equiv q \bmod 7$ $(0\le v<7)$, then we have the following TABLE 1.

TABLE 1

v	$S_1(7,1,2,q)$	$\mathrm{emb}(R(d,7))= s_0+s_1$
0	0	6
1	0, 1, 2, 3	9
2	0, 1, 2, 6	9
3	0, 1, 2	8
4	0, 1, 5	8
5	0, 1	7
6	0, 6	7

From TABLE 1 we can write generator systems of these rings. For example, when $v = 1$, the generators of the rings are listed as follows :

$$t\ ,\quad ft^{\frac{2d+5}{7}},\quad f^2t^{\frac{4d+3}{7}},\quad f^3t^{\frac{6d+1}{7}},\quad f^7t^{2d},$$

$$gt^{\frac{dq+6}{7}},\quad gft^{\frac{d(q+2)}{7}},\quad gf^2t^{\frac{d(q+4)+2}{7}},\quad gf^3t^{\frac{d(q+6)}{7}}.$$

Moreover we obtain TABLE 2 which shows emb(R(d,e)) for low values of d and e in the case of $p = 2$ (hyperelliptic point cases).

We put $m = \frac{q-1}{2}$, which is the genus of C (see Lemma 3.3).

TABLE 2

d \ e	1	2	3	4	5	6
2	3					
3	3	3				
4	3	*	3(m=1) 4(m≡0mod3) 5(m≢0mod3)			
5	3	3	5(m≡1mod3) 6(m≡0mod3) 7(m≡2mod3)	4(m≡1mod2) 5(m≡0mod2)		
6	3	*	*	*	5(m≡2mod5) 6(m≡1,4mod5) 7(m≡0,3mod5)	
7	3	3	3(m=1) 4(m≡1mod3) (m>1) 5(m≢1mod3)	4(m≡0mod2) 5(m≡1mod2)	3(m=2) 4(m≡2mod5) (m>1) 5(m≢2mod5)	5(m≡2mod3) 6(m≡1mod3) 7(m≡0mod3)
8	3	*	5(m≡1mod3) 6(m≡0mod3) 7(m≡2mod3)	*	7(m≡2mod5) 8(m≡1mod5) 9(m≡0mod5) 10(m≡4mod5) 11(m≡3mod5)	*

§ 2 The defining ideal of $R(d,e)$.

In this section we investigate the property of the defining ideal of $R(d,e)$. Let $< f_1,\cdots, f_s >$ be a generator system of semi-group $H<x_o>$, and p_i the order of the pole of f_i at x_o ($i= 1,2,\cdots,s$), and assume $p_1<p_2<\cdots<p_s$. Let I be the ideal generated by relations between $f_1,\cdots,f_s$ and $F_1,\cdots,F_m$ a generator system of I. Let S be a graded polynomial ring $C[z_0,\cdots,z_s]$ with weights $1,p_1,\cdots,p_s$. The set of elements t, $f_1t^{p_1},\cdots, f_st^{p_s}$ generate the ring $R(1,1)$, so we have $R(1,1) \simeq S / \tilde{I}$, where $\tilde{I}$ is a homogeneous ideal in S which is generated by homogeneous elements $\tilde{F}_1,\cdots, \tilde{F}_m$, and where $\tilde{F}_i$ is an element induced from F_i ($i=1,\cdots,m$).

Now we define a generator system of homogeneous elements of $R(d,e)$ as follows (see §1):

$$W_0= t \ , \ W_1= f_1t^{\{\frac{dp_1}{e}\}} ,\cdots, W_s= f_st^{\{\frac{dp_s}{e}\}}, \text{ and}$$

$$W_{s+1}= f_1^{m_{s+1,1}}\cdot\cdot f_s^{m_{s+1,s}}t^{\{\frac{d\cdot\sum m_{s+1,i}p_i}{e}\}},\cdots,W_\ell= f_1^{m_{\ell,1}}\cdot\cdot f_s^{m_{\ell,s}}t^{\{\frac{d\cdot\sum m_{\ell,i}p_i}{e}\}} \qquad \cdots\cdots\cdot (2.1).$$

Let $R(d,e) \simeq C[W_o,\cdots,W_\ell]/J$, where J is the defining ideal of $R(d,e)$. In the following we consider what generator system of J there is. For the element $G(W_o,\cdots,W_\ell) \in C[W_o,\cdots,W_\ell]$, let $\bar{G}(f_1,\cdots,f_s)$ be the element of $C[f_1,\cdots,f_s]$ which is obtained by substituting as follows :

$$W_o=1,\ W_1=f_1,\cdots,W_s=f_s, \text{ and}$$

$$W_{s+1}=f_1^{m_{s+1,1}}\cdots f_s^{m_{s+1,s}} \ ,\ \cdots \ ,\ W_\ell=f_1^{m_{\ell,1}}\cdots f_s^{m_{\ell,s}} \qquad \cdots (2.2).$$

For an element $F \in I$ there is an element $G \in J$ which satisfies $\overline{G} = F$. Because, if $F = \sum_{\alpha} a_{K^{\alpha}} f^{\alpha}$ ($\alpha=(\alpha_1,\cdots,\alpha_s)$), then

$G := \sum_{\alpha} a_{K^{\alpha}} (f_1 t^{\{\frac{dp_1}{e}\}})^{\alpha_1} \cdots (f_s t^{\{\frac{dp_s}{e}\}})^{\alpha_s} \cdot t^{(b-\sum_{i=1}^{s}\alpha_i\{\frac{dp_i}{e}\})}$ satisfies this property, where $b = \max\{ \sum_{i=1}^{s} \alpha_i \{\frac{dp_i}{e}\} \mid \alpha \in K \}$. With respect to the degree, let F^* be the least among elements satisfying $\overline{F^*} = F$ in J. The element F^* is not uniquely determined by F. However if $\overline{F_1^*} = \overline{F_2^*} = F$, then $F_1^* - F_2^*$ can be written by the elements of type II and III in (2.3). Next we consider the relations associated to the basis $W_j = f_1^{m_{j,1}} \cdots f_s^{m_{j,s}} \cdot t^{\{\frac{d\sum m_{j,i} p_i}{e}\}}$ ($j=s+1,\cdots,\ell$). Let $m_{j,k} > 0$ for some $k \le s$, then the element $f_1^{m_{j,1}} \cdots f_k^{m_{j,k}-1} \cdots f_s^{m_{j,s}} \cdot t^{\{\frac{d\sum m_{j,i}p_i - dp_k}{e}\}}$ can be written as $W_1^{\varphi_1} \cdots W_{j-1}^{\varphi_{j-1}}$ for some integers $\varphi_1,\cdots,\varphi_{j-1}$, and then

$$\sum_{i=1}^{s} \varphi_i \{\frac{dp_i}{e}\} + \sum_{i=s+1}^{j-1} \varphi_i \{\frac{d\cdot\sum_{\alpha} m_{i,\alpha}p_\alpha}{e}\} = \{\frac{d\cdot\sum_{\alpha} m_{j,\alpha}p_\alpha - dp_k}{e}\}$$

$= \{\frac{d\cdot\sum_{\alpha} m_{j,\alpha}p_\alpha}{e}\} - \{\frac{dp_k}{e}\} + \varepsilon$, where $\varepsilon = 0$ or 1. Hence we have $W_o^{\varepsilon} W_j = W_1^{\varphi_1} \cdots W_k^{\varphi_k+1} \cdots W_{j-1}^{\varphi_{j-1}}$. Then we have $\varepsilon=1$, because W_j can not be written by $W_1, \cdots, W_{j-1}$. Therefore we have the following relations ;

$$R_j : W_o W_j = W_1^{k_1} \cdots W_{j-1}^{k_{j-1}} \quad (j = s+1, \cdots, \ell)$$

for non-negative integers $k_1, \cdots, k_{j-1}$. Then we have the relations of three types as follows :

(I) $\quad F_i^*$ $\quad (i=1,\cdots,m),$

(II) $\quad R_i$ $\quad (i=s+1,\cdots,\ell),$

(III) $\quad W_o^{a_o} W_{i_1}^{a_1} \cdots W_{i_u}^{a_u} - W_{j_1}^{b_1} \cdots W_{j_v}^{b_v} \quad (1 < \sum_{k=1}^{u} a_k \ , 1 < \sum_{k=1}^{v} b_k)$

$\cdots\cdots (2.3),$

where $\overline{W_o^{a_o} \cdots W_{i_u}^{a_u}} = \overline{W_{j_1}^{b_1} \cdots W_{j_v}^{b_v}}$ in (III).

Lemma 2.1. (i) Let F and G be homogeneous elements of J with same degree. If $\bar{F} = \bar{G}$, then $F - G$ is contained in the ideal generated by the elements of type II and III in (2.3).

(ii) The ideal J is generated by elements of type I, II and III.

Proof. The proof of (i) is obvious, so we show (ii). If $P \in J$, then $\bar{P} \in I$. We can write it as $\bar{P} = \sum_{i=1}^{m} B_i \cdot F_i$ for $B_1,\cdots,B_m \in C[f_1,\cdots,f_s]$. Now if we put $P_1 = \sum_{i=1}^{m} B_i^* \cdot F_i^*$, then $P - P_1$ can be written by elements of type II and III. Then P can be written by elements of type I, II and III. Q.E.D.

Theorem 2.2 There is a generator system which contains the set $\{ F_1^*, \cdots, F_m^* , R_{s+1}, \cdots ,R_\ell \}$ as a part of it.

Proof. First we show that we need the elements R_i $(i=s+1,\cdots,\ell)$ as generators. Here we must note the facts : $\deg R_i = \deg W_i + 1$ $(i= s+1, \cdots,\ell)$ and $\deg R_{s+1} < \cdots < \deg R_\ell$. Let us assume that the element R_k can be written by elements of type I, III and type II except for R_k , then we can write as

$$R_k = \sum_{i=1}^{m} A_i \cdot F_i^* + \sum_{j=1}^{k-1} B_j \cdot R_j + G \quad \cdots\cdots\cdots (2.4),$$

where A_i and B_j are the homogeneous elements and G is a polynomial which is written by the elements of type III. If we put $W_o = 1$ in (2.4), then we have the following :

$$W_k = W_1^{m_{k,1}} \cdots W_{k-1}^{m_{k,k-1}} + \sum_{i=1}^{m} \tilde{A}_i \cdot \tilde{F}_i^* + \sum_{j=1}^{k-1} \tilde{B}_j \cdot \tilde{R}_j + \tilde{G} \quad \cdots\cdots (2.5),$$

where $\tilde{P}$ is the polynomial obtained from P by substituting 1 for W_o. Suppose that $\tilde{A}_{i_o} \cdot \tilde{F}_{i_o}^*$ contains the variable W_k for some i_o. Then $\tilde{A}_{i_o}$ does not contain W_k, because F_i has a monomial whose degree is greater than one and deg $A_{i_o} \cdot F_{i_o}^*$ = deg $W_k + 1$. If $\tilde{F}_{i_o}^*$ contains W_k, then $F_{i_o}^*$ has a monomial which contains the monomial $W_o W_k$. Hence $A_{i_o} \in C^*$. By the relation $\sum_{i=1}^{m} \bar{A}_i \cdot F_i = 0$ in $C[f_1, \cdots, f_s]$ obtained from (2.5), we have $F_{i_o} = -\frac{1}{A_{i_o}} \sum_{i \neq i_o} \bar{A}_i \cdot F_i$, where $\bar{A}_i$ is obtained by substituting (2.2) for A_i. This contradicts to the fact that $F_1, \cdots, F_m$ are mutually independent, so $\sum_{i=1}^{m} \tilde{A}_i \cdot \tilde{F}_i^*$ does not contain the variable W_k. Moreover it is obvious that $\sum_{j=1}^{k-1} \tilde{B}_j \cdot \tilde{R}_j + \tilde{G}$ does not contain W_k. Then $W_k \in C[W_1, \cdots, \overset{k}{\vee}, \cdots, W_\ell]$, which is a contradiction.

Next we consider the element F_1^* of type I. Suppose that F_1^* can be written by $F_2^*, \cdots, F_m^*$ and elements of type II and III: $F_1^* = \sum_{i=2}^{m} C_i \cdot F_i^* + H$, where H is written by elements of type II and III. By substituting (2.2) for F_1^*, we have $F_1 = \sum_{i=2}^{m} C_i \cdot F_i$, this is a contradiction. Therefore when we consider the generator system of J consisting of the elements of type I, II and III, we can not delete any element of type I and II. Hence there is a generator system which contains any element of type I and II as the member of it. Q.E.D

Corollary 2.3. (i) If the graded ring $R(d,e)$ is a complete intersection for some integers d and e ($d>e\geq 1$), then the graded ring $R(1,1)$ is a complete intersection .

(ii) When $R(1,1)$ is a complete intersection, $R(d,e)$ is a complete intersection if and only if J is generated by the elements of type I and II.

§ 3. Complete intersections (when the semi-group is generated by two elements).

In this section we determine the condition for $R(d,e)$ to be a complete intersection, when the semi-group $H(x_0)$ is generated by two integers p and q. Let C be a curve of genus g and x_0 a point on C as in §1. Let $a(R)$ be the invariant which was defined for finitely generated graded rings in [5]. It is very important and useful for us. We refer to [5] for its definition and properties. In our situation we have the following formula of $a(R)$:

$$a(R(d,e)) = [\frac{d(2g-1)-1}{e}] \qquad \cdots\cdots (3.1),$$

from the result of Ke. Watanabe [17, Th.2.8]. Moreover we can easily obtain the following equivalent condition for $R(d,e)$ to be Gorenstein from [17,Cor.2.9] (also see [4]).

Proposition 3.1. Let C be a compact smooth curve of genus g and x_0 a point on C.

(i) If g=1, then R(d,e) is a Gorenstein ring if and only if d-1 is divisible by e.

(ii) If $g \geq 2$, then R(d,e) is a Gorenstein ring if and only if x_0 is a Weierstrass point and R(1,1) is a Gorenstein ring and d(2g-1)-1 is divisible by e.

The numerical condition for R(1,1) to be Gorenstein is given in [7] as the symmetricity of the semi-group $H\langle x_0\rangle$.

Proposition 3.2 ([5],Remark 3.1.6). If $R(d,e) \simeq$ $C[z_0,\cdots,z_n]/(F_1,\cdots,F_{n-1})$ is a complete intersection, then we have

$$a(R(d,e)) = \sum_{i=1}^{n-1} \deg F_i - \sum_{j=0}^{n} \deg z_i .$$

From now on we assume that $H\langle x_0\rangle$ is generated by two meromorphic functions f and g, which have the pole of the order p and q at x_0 respectively. If the genus g of C is greater than one, then x_0 is the Weierstrass point. The following lemma is already well-known ([1]).

Lemma 3.3. If there is a point of the above type on a curve C, then we have the equality;

$$\text{the genus of } C = \frac{(p-1)(q-1)}{2} .$$

From these results we have

$$a(R(d,e)) = [\frac{d(pq-p-q)-1}{e}] \quad \cdots (3.2),$$

and we put $a(R)=a(R(d,e))$ in the following. From Proposition (3.1) we have the following.

Lemma 3.4. When $H(x_0)$ is generated by two elements p and q, $R(d,e)$ is a Gorenstein ring if and only if $e|d(pq-p-q)-1$.

Lemma 3.5. If $R(d,e)$ is a complete intersection and $\mathrm{emb}(R(d,e)) = 4$, then the element of the form $f^i g^j t^{\{\frac{d(ip+jq)}{e}\}}$ $(i>0, j>0)$ can not be a generator of $R(d,e)$.

Proof. If $e|p$ or $e|q$, then it is obvious that $f^i g^j t^{\{\frac{d(ip+jq)}{e}\}}$ is not a generator of $R(d,e)$. So we assume that $e|p$ and $e|q$. If $f^i g^j t^{\{\frac{d(ip+jq)}{e}\}}$ $(i>0, j>0)$ is a generator, then

$$f^{q-1} t^{\{\frac{d(q-1)p}{e}\}} = (f t^{\{\frac{dp}{e}\}})^{q-1} \quad \text{and} \quad g^{p-1} t^{\{\frac{d(p-1)q}{e}\}} = (g t^{\{\frac{dq}{e}\}})^{p-1}.$$

Hence

$$(q-1)\{\frac{dp}{e}\} = \{\frac{d(q-1)p}{e}\} = \{\frac{dq+1}{e}\} + a(R) = \{\frac{dq}{e}\} + a(R) \quad \cdots (3.3),$$

also $(p-1)\{\frac{dp}{e}\} = \{\frac{dq}{e}\} + a(R)$, and then $p\{\frac{dq}{e}\} = q\{\frac{dp}{e}\}$. If we put $\{\frac{dp}{e}\} = \frac{dp+s}{e}$ and $\{\frac{dq}{e}\} = \frac{dq+t}{e}$, then we easily get $qs=pt$, so $p|s$ and $q|t$. Then we have $s = pc$ and $t = qc$ for the same positive integer c. From (3.3), $(q-1)(\frac{dp+pc}{e}) = \frac{dq+qc}{e} + a(R)$, and so $c(p-1)(q-1)-c = -1$. This is a contradiction, because the left hand side is non- negative. Q.E.D.

Lemma 3.6. If $R(d,e)$ is a complete intersection, then $\mathrm{emb}(R(d,e)) \leq 5$.

Proof. By (ii) of Corollary 2.3, we can write as follows:

$$R(d,e) \simeq C[W_0,\cdots,W_\ell]/(F,R_3,\cdots,R_\ell)$$

where $\ell+1 = \mathrm{emb}(R(d,e))$ and R_i is the relation associated with the generator W_i for $i=3,\cdots,\ell$. From Proposition 3.2 and (3.2),

$$\frac{d(pq-p-q)-1}{e} = \deg F + \sum_{i=3}^{\ell} \deg R_i - \sum_{i=0}^{\ell} \deg W_i$$

$$= \deg F + 1 + \{\frac{dp}{e}\} + \{\frac{dq}{e}\} + (\ell-2) \quad \cdots \quad (3.4).$$

The inequality $\deg F \geq \{\frac{dpq}{e}\}$ holds, then

$$-(\{\frac{dpq}{e}\} - \frac{dpq}{e}) + (\{\frac{dp}{e}\} - \frac{dp}{e}) + (\{\frac{dq}{e}\} - \frac{dq}{e}) + \frac{e-1}{e} \geq \ell-2 \quad \cdots \quad (3.5),$$

so that $\ell+1 \leq 5$. Q.E.D.

Theorem 3.7. (i) If the ring $R(d,e)$ is a complete intersection, then $\mathrm{emb}(R(d,e)) \leq 4$.

(ii) $\mathrm{emb}(R(d,e)) = 3$ (i.e., a hypersurface) if and only if one of following three conditions holds.

(ii-1) $e = 1$.

(ii-2) $e = p$ and $e \mid dq+1$.

(ii-3) $e = q$ and $e \mid dp+1$.

(iii) The ring $R(d,e)$ is a complete intersection ring of $\mathrm{emb}(R(d,e)) = 4$ if and only only if one of following four conditions holds.

(iii-1) $e \mid p$, $e \mid dq+1$ and $p>e$.

(iii-2) $e|q$, $e|dp+1$ and $q>e$.

(iii-3) $q|e$, $e|dp+1$, $e|(d+1)q$ and $q<e$.

(iii-4) $p|e$, $e|dq+1$, $e|(d+1)p$ and $p<e$.

Proof. (i) If $e|p$ or $e|q$, then we can easily see that $\mathrm{emb}(R(d,e)) \leq 4$ by (3.4), so we may assume that $e|p$ and $e|q$. In the following we assume that $R(d,e)$ is a complete intersection and $\mathrm{emb}(R(d,e))=5$. Then we have $R(d,e) \simeq C[W_0,\cdots,W_4]/(F,R_3,R_4)$ from (ii) of Corollary 2.3. If $\deg F \geq \{\frac{dpq}{e}\}+1$, then $\mathrm{emb}(R(d,e)) \leq 4$ from (3.4). Hence we may also assume that $\deg F = \{\frac{dpq}{e}\}$, so that $f^q t^{\{\frac{dpq}{e}\}}$ and $g^p t^{\{\frac{dpq}{e}\}}$ can not be generators. Moreover from (3.4), we obtain the equality

$$-(\{\frac{dpq}{e}\}-\frac{dpq}{e})+(\{\frac{dp}{e}\}-\frac{dp}{e})+(\{\frac{dq}{e}\}-\frac{dq}{e})+\frac{e-1}{e}=2 \quad \cdots (3.6),$$

then $q\{\frac{dp}{e}\} > \{\frac{dpq}{e}\}$ and $p\{\frac{dq}{e}\} > \{\frac{dpq}{e}\}$. Therefore we need two generators $f^a t^{\{\frac{dap}{e}\}}$ and $g^b t^{\{\frac{dbq}{e}\}}$ ($1<a,b$; $a<\min(e,q-1)$; $b<p$) in order to express $f^q t^{\{\frac{dpq}{e}\}}$ and $g^p t^{\{\frac{dpq}{e}\}}$ respectively by the generators. Let $e=\lambda a+\mu$ ($0 \leq \mu < a$), then $f^e t^{dp} = (f^a t^{\{\frac{dap}{e}\}})^\lambda \cdot (f t^{\{\frac{dp}{e}\}})^\mu$, so $\lambda(\{\frac{dap}{e}\}-\frac{dap}{e}) + \mu(\{\frac{dp}{e}\}-\frac{dp}{e}) = 0$. Hence

$$\mu = 0 \text{ (i.e, } a|e \text{) and } e|ap \quad \cdots\cdots (3.7),$$

because $e|p$ and $(d,e)=1$. Suppose $b > e$, then $p > b > e$ and $g^e t^{dq} = (g t^{\{\frac{dq}{e}\}})^e$, so $e|q$. This is a contradiction, which implies $b \leq e$. If we put $e = \xi b + \eta$ ($0 \leq \eta < b$), then as in (3.7)

$$\eta = 0 \ (\text{ i.e, } b|e \ ) \text{ and } e|bq \quad \cdots\cdots (3.8).$$

On the other hand, $(ft^{\{\frac{dp}{e}\}})^{a-1} = f^{a-1}t^{\{\frac{d(a-1)p}{e}\}}$, so $(a-1)\{\frac{dp}{e}\} = \{\frac{d(a-1)p}{e}\} = \frac{dap}{e} - \{\frac{dp}{e}\} + 1$. From (3.7), we have

$$\{\frac{dp}{e}\} - \frac{dp}{e} = \frac{1}{a} \quad \cdots\cdots (3.9).$$

For the element $g^{b-1}t^{\{\frac{d(b-1)q}{e}\}}$, we obtain

$$\{\frac{dq}{e}\} - \frac{dq}{e} = \frac{1}{b} \quad \cdots\cdots (3.10),$$

from (3.8). Further from (3.6), we have

$$\frac{e-1}{e} - (\{\frac{dpq}{e}\} - \frac{dpq}{e}) = 2 - (\frac{1}{a} + \frac{1}{b}) \quad \cdots\cdots (3.11).$$

Here the left hand side of the above is less than one. On the contrary, the right hand side of the above is larger or equal to one. This yields a contradiction.

(ii) Let emb(R(d,e)) = 3, then the defining ideal J of R(d,e) is generated by an element F. Suppose $e < q$, then $f^e t^{dp} = (ft^{\{\frac{dp}{e}\}})^e$. So $e|p$ and $e|dq+1$ by Lemma 3.4. Then $\deg F = \max(p\{\frac{dq}{e}\}, q\{\frac{dp}{e}\}) = p\{\frac{dq}{e}\}$, so from (3.4) we have

$$(p-1)(\{\frac{dq}{e}\} - \frac{dq}{e}) - \frac{e-1}{e} = 0 \quad \cdots\cdots (3.12).$$

Therefore if $e \neq 1$, then $\{\frac{dq}{e}\} = \frac{dq+1}{e}$ and $e = p$ by (3.12). So if $e < q$, then $e = 1$ or p. Next suppose $e > q$, then

$$f^{q-1}g^{p-1}t^{\{\frac{d((q-1)p+(p-1)q)}{e}\}} = (ft^{\{\frac{dp}{e}\}})^{q-1}\cdot(gt^{\{\frac{dq}{e}\}})^{p-1}, \text{ so}$$

$$(p-1)(\{\frac{dq}{e}\} - \frac{dq}{e}) + (q-1)(\{\frac{dp}{e}\} - \frac{dp}{e}) < 1 \quad \cdots\cdots (3.13).$$

Furthermore from (3.4) and (3.13),

$$\max\left(p\{\tfrac{dq}{e}\},\ q\{\tfrac{dp}{e}\} \right) = \frac{d(pq-p-q)-1}{e} + 1 + \{\tfrac{dp}{e}\} + \{\tfrac{dq}{e}\}$$
$$> p\{\tfrac{dq}{e}\} + q\{\tfrac{dp}{e}\} - \frac{dpq+1}{e} \quad \cdots\cdots (3.14).$$

So if $p\{\frac{dq}{e}\} \geq q\{\frac{dp}{e}\}$, then $q\{\frac{dp}{e}\} < \frac{dpq+1}{e}$. On the other hand, $e|p$ by the assumption, then $\{\frac{dp}{e}\} \geq \frac{dp+1}{e}$. This is a contradiction. Therefore if we assume that emb(R(d,e))= 3, then $e \leq q$. If we assume that $e = q$, then $e|dp+1$ by Lemma 3.4.

Conversely if we assume (ii-1) $e = 1$, then it is obvious that R(d,1) is a hypersurface. Next we assume that (ii-2) $e = p$ and $e|dq+1$. If we put $dq+1 = \ell e$ ($\ell \in \mathbb{Z}$), then

$$f^i g^j t^{\{\frac{d(ip+jq)}{e}\}} = (ft^d)^i \cdot g^j t^{\{\frac{\ell ej-j}{e}\}} = (ft^d)^i \cdot (gt^{\ell})^j$$

for $j = 1,\cdots,p-1=e-1$, and for any i. So, by Proposition 1.10 (i), the ring R(d,e) is generated by three elements t, ft^d and $gt^{\frac{dq+1}{e}}$. For the case (ii-3) $e=q$ and $e|dp+1$, we can prove as in (ii-2), and so omit it.

(iii) We assume that R(d,e) is a complete intersection ring of emb(R(d,e))=4. Then R(d,e) has three bases $W_0=t$, $W_1=ft^{\{\frac{dp}{e}\}}$ and $W_2=gt^{\{\frac{dq}{e}\}}$, and another generator $W_3=f^a t^{\{\frac{dap}{e}\}}$ or $g^b t^{\{\frac{dbq}{e}\}}$ ($a,b>1$) from Lemma 3.5. From (ii) of Corollary 2.3 we have

$$R(d,e) \simeq \mathbb{C}[W_0,W_1,W_2,W_3]/(F,R) \quad \cdots\cdots (3.15),$$

where F is an element of type (i) in (2.3) and R is an element of type (ii) which is associated with W_3. From (3.4) we may assume that

$$\deg F = \{\tfrac{dpq}{e}\} \quad \text{or} \quad \{\tfrac{dpq}{e}\} + 1 \quad \cdots\cdots (3.16).$$

If $e|p$, then $e|dq+1$ from Lemma 3.4, and $e < p$ from (ii) of this Theorem. Similarly if $e|q$, then we have $e|dp+1$ and $e < q$. So we may assume that $e \nmid p$ and $e \nmid q$ in the following. First we consider the case $W_3 = f^a t^{\{\frac{dap}{e}\}}$ $(a>1)$. If we write the element $f^e t^{dp}$ by generators as $f^e t^d = (f^a t^{\{\frac{dap}{e}\}})^i \cdot (ft^{\{\frac{dp}{e}\}})^j \cdot (gt^{\{\frac{dq}{e}\}})^k$, then $pe = (ai+j)p + kq$, and $i(\{\frac{dap}{e}\} - \frac{dap}{e}) + j(\{\frac{dp}{e}\} - \frac{dp}{e}) + k(\{\frac{dq}{e}\} - \frac{dq}{e}) = 0$.

Therefore

$$j = k = 0 \quad \text{and} \quad e|ap \quad \cdots\cdots \quad (3.17),$$

because $e|p$, $e|q$ and $(d,e)=1$. Further if we write the element $g^e t^{dq}$ as follows : $g^e t^{dq} = (f^a t^{\{\frac{dap}{e}\}})^{\ell} \cdot (ft^{\{\frac{dp}{e}\}})^m \cdot (gt^{\{\frac{dq}{e}\}})^n$, then

$$m = n = o \quad \text{and} \quad p|e \quad \cdots\cdots \quad (3.18).$$

Here we put $ap = eL$ for some positive integer L. If $L > 1$, then $\frac{e}{p} < a$ and so $f^{e/p} t^d = (ft^{\{\frac{dp}{e}\}})^{e/p}$. This induces $\{\frac{dp}{e}\} = \frac{dp}{e}$ and $e|p$. This yields a contradiction. Therefore

$$L=1 \text{ and } a = \frac{e}{p} \quad \cdots\cdots \quad (3.19),$$

so that $f^{e/p-1} t^{\{\frac{d(e/p\ -1)p}{e}\}} = (ft^{\{\frac{dp}{e}\}})^{(e/p\ -1)}$. Then $\{\frac{dp}{e}\} = \frac{(d+1)p}{e}$,

so $$e|(d+1)p \quad \cdots\cdots \quad (3.20).$$

Here suppose that $\deg F = \{\frac{dpq}{e}\}$ in (3.16), then $a(R) = \{\frac{dpq}{e}\} - \{\frac{dp}{e}\} - \{\frac{dq}{e}\}$ from (3.4). Further $(gt^{\{\frac{dq}{e}\}})^{p-1} = g^{p-1} t^{\{\frac{d(p-1)q}{e}\}}$, so $p\{\frac{dq}{e}\} - \{\frac{dq}{e}\} = \{\frac{dpq-dq}{e}\} = \{\frac{dp}{e}\} + a(R) = \{\frac{dpq}{e}\} - \{\frac{dq}{e}\}$, then

$$\{\frac{dpq}{e}\} = p\{\frac{dq}{e}\} \quad \cdots\cdots \quad (3.21).$$

From (3.4), $\{\frac{dpq}{e}\} - \{\frac{dq}{e}\} = a(R) + \{\frac{dp}{e}\}$, and so from (3.2),(3.20),(3.21)

$(p-1)\{\frac{dq}{e}\} = \frac{(dpq-dp-dq)-1}{e} + \frac{(d+1)p}{e} = \frac{(p-1)(dq+1)}{e}$, then

$$e \mid dq+1 \quad \cdots\cdots \quad (3.22).$$

If $\deg F = \{\frac{dpq}{e}\}+1$ in (3.16), then $\{\frac{dpq-dp}{e}\}=\{\frac{dpq}{e}\}-\{\frac{dq}{e}\}+1$ by (3.4). As in (3.22), we have

$$\{\frac{dpq}{e}\} = p\{\frac{dq}{e}\} - 1 \quad \cdots\cdots \quad (3.23).$$

From (3.4) and $\{\frac{dp}{e}\} = \frac{(d+1)p}{e}$, we have $e \mid dq+1$. Therefore we obtain the condition of (iii-4). If we consider the case that $g^b t^{\{\frac{dbq}{e}\}}$ ($b>1$) is a generator, then we obtain the condition of (iii-3). The proof is similar to the above case, so we omit it.

Now we consider the converse. For the cases (iii-1) and (iii-2), we omit the proof, because it can be done as in the proof of (ii).

We consider the condition (iii-3) $q \mid e$, $e \mid dp+1$, $e \mid (d+1)q$ and $q<e$. Then $e \mid (q-1)(dp+1)-(d+1)q=d(pq-p-q)-1$, so by Lemma 3.4, $R(d,e)$ is Gorenstein. Then from Serre's result (i.e., if R is Gorenstein and has the embedding codimension =2, then it is a complete intersection.) we have to show that the $\mathrm{emb}(R(d,e))=4$. In the following we prove that $R(d,e)$ is generated by four elements t, $ft^{\{\frac{dp}{e}\}}$, $gt^{\{\frac{dq}{e}\}}$ and $g^{e/q}t^d$. We show that $f^a g^b t^{\{\frac{d(ap+bq)}{e}\}}$ can be written by these four elements. First we consider the case $a<q$. If we put $b = L\cdot(\frac{e}{q}) + R$ ($0 \le R < \frac{e}{q}$), then

$$\{\frac{d(ap+bq)}{e}\} = dL + \{\frac{dqR+dap}{e}\} = dL + \{\frac{(d+1)qR-qR+(dp+1)a-a}{e}\}$$

$$= dL + \frac{(d+1)qR}{e} + \frac{(dp+1)a}{e} - [\frac{qR+a}{e}] = dL + \frac{(d+1)qR}{e} + \frac{(dp+1)a}{e}$$

because $qR+a < (R+1)q \le \frac{e}{q}\cdot q = e$. Hence

$$f^a g^b t^{\{\frac{d(ap+bq)}{e}\}} = (g^{e/q}t^d)^L \cdot (gt^{\{\frac{dq}{e}\}})^R \cdot (ft^{\{\frac{dp}{e}\}})^a,$$

because $\{\frac{dq}{e}\} = \frac{(d+1)q}{e}$ and $\{\frac{dp}{e}\} = \frac{dp+1}{e}$.

Next we consider the case $a \geq q$. For the element $\bar{F}$ of $C[f,g]$ associated with F in (3.15), we can write it as follows:

$$\bar{F} = f^q + g^p + \sum_{pi+qj<pq} a_{ij} f^i g^j .$$

Therefore if we put $a = \alpha q + \gamma$ $(0 \leq \gamma < q)$, then $f^a g^b t^{\{\frac{d(ap+bq)}{e}\}}$ $= (-g^p - \sum_{pi+qj<pq} a_{ij} f^i g^j)^{\alpha} \cdot g^b t^{\{\frac{d(ap+bq)}{e}\}}$. So this case $a \geq q$ is reduced to the above case $a<q$. The proof of the case (iii-4) is similar to (iii-3), so we omit it. Q.E.D.

Remark 3.8. (i) When $R(d,e)$ is a complete intersection whose embedding dimension is four, the degree of the defining equation F of type I in (3.15) is given by $\{\frac{dpq}{e}\}$. Because if $R(d,e)$ belongs to (iii-1) of the above theorem, then from (3.4)

$$\begin{aligned} \deg F &= \frac{d(pq-p-q)-1}{e} + \{\frac{dp}{e}\} + \{\frac{dq}{e}\} \\ &= \frac{d(pq-p-q)-1}{e} + \frac{dp}{e} + \frac{dq+1}{e} = \frac{dpq}{e} = \{\frac{dpq}{e}\}. \end{aligned}$$

Also if $R(d,e)$ belongs to (iii-3), then from (3.4)

$$\deg F = \frac{d(pq-p-q)-1}{e} + \frac{dp+1}{e} + \frac{dq+q}{e} = \frac{dpq+q}{e} = \{\frac{dpq}{e}\}.$$

In the same way, we can check the other two cases (iii-2) and (iii-4).

(ii). If we assume the condition (iii-3) in the above theorem, then we easily see that $e|(p-1)q$. Furthermore if we assume (iii-4), then we have $e|(q-1)p$.

Remark 3.10. Let C be the curve of Example 1.9, which is defined by the equation $y^p + x^q + 1 = 0$, and x_o the Weierstrass point of Example 1.9. When $R(d,e)$ is a complete section, we can write the defining equation as follows:

emb(R(d,e))=3 (i.e., hypersurface cases) :

$$Z_2^p + Z_1^q + Z_0^{dpq} = 0 \qquad \cdots\cdot \text{(ii-1)},$$

$$Z_2^p + Z_1^q Z_0 + Z_0^{dq+1} = 0 \qquad \cdots\cdot \text{(ii-2)},$$

$$Z_1^q + Z_2^p Z_0 + Z_0^{dp+1} = 0 \qquad \cdots\cdot \text{(ii-3)},$$

where $Z_0 = t$, $Z_1 = ft^{\{\frac{dp}{e}\}}$ and $Z_2 = gt^{\{\frac{dq}{e}\}}$,

emb(R(d,e))=4 :

$$Z_2^p + Z_3^{q/e} + Z_0^{dpq/e} = 0\ ,\ Z_0Z_3 - Z_1^e = 0 \qquad \cdots\cdot \text{(iii-1)},$$

where $Z_0 = t$, $Z_1 = ft^{(dp+1)/e}$, $Z_2 = gt^{dq/e}$ and $Z_3 = f^e t^{dp}$,

$$Z_1^q + Z_3^{p/e} + Z_0^{dpq/e} = 0\ ,\ Z_0Z_3 - Z_2^e = 0 \qquad \cdots\cdot \text{(iii-2)},$$

where $Z_0 = t$, $Z_1 = ft^{dp/e}$, $Z_2 = gt^{(dq+1)/e}$ and $Z_3 = g^e t^{dq}$,

$$Z_1^q + Z_2Z_3^{(p-1)q/e} + Z_0^{(dpq+q)/e} = 0\ ,\ Z_0Z_3 - Z_2^{e/q} = 0 \qquad \cdots\cdot \text{(iii-3)},$$

where $Z_0 = t$, $Z_1 = ft^{(dp+1)/e}$, $Z_2 = gt^{(dq+q)/e}$ and $Z_3 = g^{e/q}t^d$,

$$Z_1^p + Z_1Z_3^{(q-1)p/e} + Z_0^{(dpq+p)/e} = 0\ ,\ Z_0Z_3 - Z_1^{e/p} = 0 \qquad \cdots\cdot \text{(iii-4)}$$

where $Z_0 = t$, $Z_1 = ft^{(dp+p)/e}$, $Z_2 = gt^{(dq+1)/e}$ and $Z_3 = f^{e/p}t^d$.

Example 3.11. We give an example whose embedding dimension is 4 , but not Gorenstein. Let us consider the case : p=5, q=9, d=7 and e=5, then $e \mid d(pq-p-q)-1$, so this ring is not Gorenstein from Lemma 3.4. We can easily see that the embedding dimension is equal to 4 and the generator system is given by $Z_0 = t$, $Z_1 = ft^7$, $Z_2 = gt^{13}$ and $Z_3 = g^3t^{38}$. If we consider the curve and the Weierstrass point of Example 1.8, then we have the following independent relations :

$$Z_0Z_3 - Z_2^3 = 0,\quad Z_2^2Z_3 - Z_0Z_1^9 + Z^{64} = 0 \quad \text{and} \quad Z_3^2 + Z_1^9Z_2 + Z_0^{63}Z_2 = 0.$$

§ 4. Poincare series of R(d,e).

Let C be a curve and x_o a point on C and let $H(x_o) \cap \{1,2,\cdots,2g\} = \{\varphi_1,\varphi_2,\cdots,\varphi_{2g}\}$, where $\varphi_1 < \varphi_2 < \cdots < \varphi_{2g}$ (i.e., φ_i is the non-gap value of x_o for any i). The series $P_R(t) = \sum_{k\geq 0} a_k t^k$ is called the Poincare series of R, where $a_k = \dim_{\mathbb{C}} R_k = \dim_{\mathbb{C}} H^0(C, \mathcal{O}([\frac{ke}{d}]x_o))$. This series is important in the theory of graded rings. In this section we express the Poincare series $P_R(t)$ by g, d, e, $\varphi_1,\cdots,\varphi_{2g}$.

If $[\frac{ek}{d}] \geq 2g-1$, then $a_k = [\frac{ek}{d}]-g+1$ from the Riemann-Roch theorem of curves ([7]). Let us decompose the ring R into the sum of two modules : $R = \overset{\alpha}{\underset{k=0}{+}} R_k t^k + (\overset{\infty}{\underset{k=\alpha+1}{+}} R_k t^k)$, where $\alpha = \{\frac{d(2g-1)-1}{e}\}$. First we compute the latter :

$$\sum_{k=\alpha+1}^{\infty} a_k t^k = \sum_{k=\alpha+1}^{\infty} ([\frac{ke}{d}]-g+1)t^k$$

$$= t^{\alpha+1}\{(1-g)\sum_{k=0}^{\infty} t^k + \sum_{i=0}^{\infty}\sum_{j=0}^{d-1}(\ ie + [\frac{e(\alpha+j+1)}{d}])t^{id+j}\}$$

$$= \frac{(1-g)t^{\alpha+1}}{1-t} + t^{\alpha+1}\{e(\sum_{i=0}^{\infty} it^{id})(\sum_{j=0}^{d-1} t^j) + (\sum_{i=0}^{\infty} t^{id})(\sum_{j=0}^{d-1}[\frac{e(\alpha+j+1)}{d}]t^j)\}$$

$$= \frac{(1-g)t^{\alpha+1}}{1-t} + t^{\alpha+1}\{\frac{et^d}{(1-t)(1-t^d)} + \frac{1}{1-t^d}\sum_{j=0}^{d-1}[\frac{e(\alpha+j+1)}{d}]t^j\}$$

$$= \frac{(1-g)t^{\alpha+1}}{1-t} + \frac{t^{\alpha+1}}{1-t^d}\{\frac{et^d}{1-t} + \sum_{j=0}^{d-1}[\frac{e(\alpha+j+1)}{d}]t^j\} \qquad \cdots\cdots \quad (4.1).$$

Next we compute the former. If we denote integers $\{\frac{d\varphi_i}{e}\}$ by s_i and $s_i - s_{i-1}$ by m_i for $i = 1,\cdots,g$ $(s_0=0)$, then

$$\sum_{k=0}^{\alpha} a_k t^k = \sum_{i=1}^{g} i\Big(\sum_{j=0}^{m_i-1} t^j\Big) t^{s_{i-1}} = \sum_{i=1}^{g} i t^{s_{i-1}} \cdot \frac{(t^{m_i}-1)}{t-1}$$

$$= \frac{1}{t-1}\sum_{i=1}^{g} i(t^{s_i} - t^{s_{i-1}}) = \frac{gt^{s_g}}{t-1} - \frac{1}{t-1}\sum_{i=0}^{g-1} t^{s_i} \quad \cdots \ (4.2).$$

From (4.1) and (4.2) we have the following formula.

Theorem 4.1. Under the above notations,

$$P_R(t) = \frac{1}{1-t}\Big\{\sum_{i=0}^{g-1} t^{s_i} - gt^{s_g} + (1-g)t^{\alpha+1}\Big\}$$

$$+ \frac{t^{\alpha+1}}{1-t^d} \cdot \sum_{j=0}^{d-1}\Big[\frac{e(\alpha+j+1)}{e}\Big]t^j + \frac{et^{\alpha+d+1}}{(1-t)(1-t^d)} \quad \cdots\cdots \ (4.3).$$

Example 4.2. Let $p=2$, $q=7$, $d=5$ and $e=4$, then $g=3$, $\alpha=7$, $s_1=3$, $s_2=5$ and $s_3=8$. From (4.3),

$$P_R(t) = \frac{1}{1-t}\{1+t^3+t^5-5t^8\} + \frac{t^8}{1-t^5}\{6+7t+8t^2+8t^3+9t^4\} + \frac{4t^{13}}{(1-t)(1-t^5)}$$

$$= \frac{(1+t^3)(1+t^9)}{(1-t)(1-t^5)} = \frac{(1-t^6)(1-t^{18})}{(1-t)(1-t^3)(1-t^5)(1-t^9)} .$$

In this case, the singularity is a quasi-homogeneous complete intersection (see Theorem 3.7 (iii-4)), so we can also obtain the above rational function by using the other formula (see [3] p.57).

Example 4.3. Let us consider the case : $p=4$, $q=5$, $d=7$ and $e=4$. By the definition we have that $g=6$, $\alpha=20$, $s_1=7$, $s_2=9$, $s_3=14$, $s_4=16$, $s_5=18$ and $s_6=21$. Then, by using the above formula, we have

$$P_R(t) = \frac{1}{(1-t)(1-t^7)}\{ 1+t^9+t^{18}+t^{27}\} = \frac{(1-t^{36})}{(1-t)(1-t^7)(1-t^9)} .$$

It is easy to check that the above rational function $P_R(t)$ satisfies the Stanley's condition for R to be Gorenstein ([12]). Then this ring is Gorenstein. Of course we can also check it by Proposition 3.4.

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