Product formulas for integral transformations associated with classical mechanics

by

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THESIS

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CONTENTS

Introduction

How should differential operators be constructed ?

Although this problem seems to be rather abrupt, it may be seen in the following stream: Given a differential (or psuedo-differential) operator, we usually study its analytic properties (spectrum, signature, index etc.), and then we take out the inherent geometrical characters and invariants.

L:differential operators

---→ analytic properties

[spectrum, index, signature, etc]

---→ geometry

This may be the standard method of studying the differential equations. However, many geometers may have a sort of mysterious for this procedure. For explaining this, we shall try to place the converse to the above way. After obtaining the construction of the differential operator L through a specific way using the geometrical properties, as a result, we may naturally hold the analytic properties for L.

Now, to put our plan into the concrete shape, we will borrow an idea in the quantum mechanics. The first and simplest problem of quantization concerns the kinematic relationships between the classical and quantum domains. When we learn quantum mechanics, we are told to forget our naive, classical idea of particles traveling on trajectories. That is, to consider that a particle might be here at one time invite contradiction and confusion. So, at the quantum level, the states of a physical system should be represented by the rays in a Hilbert space H and the observables by the symmetric operators Q on H, which in the limiting classical description, the quantum states and observables corresponding to the classical ones.

The idea of Feynman's path integral has been known as the one of the most powerful tool to treat the quantization problem. Though it seems hard to be justified rigorously, the expression of the path integral have a interesting form at the geometrical point of view. Also, it is felt to fit our beginning problem, even if we leave the physical problem.

Therefore, throughout of this paper, we shall generalize the idea of Feynman's path integral more generally and gives a rigorous meaning for it. For future interests, there may be many applications, for examples, to

(i) Geometrical constructions of the fundamental solutions for the evolutional equations (parabolic, hyperbolic, or Schrödinger equations).

(ii) The invariant theory for the non-compact Riemannian manifolds.

(iii) Construction problems of operators from variational problems (including the field equations).

The best explanation of the path integrals may be found in Feynman's paper [10]. In contrasting with the classical mechanics which can be described as the variational formula using a Lagrangian function $L(x,\dot{x})$ considered as a function on the tangent bundle TM over the configuration space M, the probability amplitude at a time t corresponding to the quantized one, for two points in M, is given by

$$(1) \qquad G(t;x,y)= \sum_{\substack{x(\cdot) \\ x(0)=y,x(t)=x}} e^{\lambda S[\dot{x}()]} ,$$

$$S(x(\cdot)) = \int_0^t L(x(\tau),\dot{x}(\tau))d\tau .$$

This is a sum over paths, or histories, of $e^{\lambda S[\dot{x}()]}$ with a all paths satisfying $x(0)=y, x(t)=x$, entering the sum. Here the symbol Σ is used to avoid giving the impression that we have a bona fide measure. Also, we remark that the parameter λ in (1) will be considered mainly for the case $\lambda<0$ or $\lambda=i/h$, $h\in\mathbf{R}$.

Now, we shall explain Feynman's original idea of the path integrals for the free particles. Consider the Euclidean n-space $\mathbf{R}^n$ and the Lagrangian

(2) $L(\dot{x},x) = \frac{1}{2}|\dot{x}|^2 : TR^n \dashrightarrow \mathbf{R}$.

Then, the action integral (the least action) corresponding to (2) is given by

(3) $$S(t,x,y) = \inf_{\substack{\dot{x}() \\ x(0)=y, x(t)=x}} \int_0^t L(x(\tau),\dot{x}(\tau))d\tau = \frac{|x-y|^2}{2t} .$$

Set the following integral transformation, for t>0,

(4) $$H(t)f(x) = \left(\frac{1}{2\pi\lambda t}\right)^{\frac{m}{2}} \int_{\mathbf{R}^n} e^{\lambda S(t,x,y)} f(y)dy \ , \ f\in C_0^\infty(\mathbf{R}^n),$$

where $\mathrm{Re}\lambda \leq 0$. Then, (4) has the following properties :

(i) H(t) is a bounded linear operator on $L^2(\mathbf{R}^n)$ for each t>0.

(ii) $\lim_{t\to 0+} \|H(t)f-f\| = 0$ for $f\in L^2(\mathbf{R}^n)$,

(iii) $H(t+s)f(x) = H(t)H(s)f(x)$ for $f\in L^2(\mathbf{R}^n)$.

(iv) $\frac{\partial}{\partial t}H(t)f(x)_{|t=0} = \frac{1}{2}\Delta f(x)$, $\Delta = \Sigma^n{}_{i=1} \frac{\partial^2}{\partial x_i^2}$

As we obtain the above properties , Feynman's original idea is to consider (1) as a sort of 'Riemannian integration' in L^2-scheme which may be described as follows :

(P.1) Construct the (approximate) operator H(t) on a Hilbert space for sufficiently small times t>0 using the given classical mechanics.

(P.2) Iterate H(t) with respect to the division of t and make an evolutional operator.

(P.3) Compute the infinitesimal generator to get the \`observable\`.

Distincting the above example (4), we cannot expect the evolutional property in general. So, the main investigation is to study the convergence for (P.2). Namely, denoting $H(t;x,y)$ the kernel function of $H(t)$ in (P.2), we consider the following iterated integral : For a division Δ of the interval $[0,t]$,

$$(5) \quad H(\Delta;t;x,y) = \int\cdots\int H(\tfrac{t}{N};x,y_1)\cdots H(\tfrac{t}{N},y_{N-1},y)dy_1\cdots dy_{N-1} .$$

If (5) convergences as $N\to\infty$, then we may define its limit by (1).

Putting the above idea in our mind, we will investigate the convergence in the sense of (5) for the various integral transformation which arises from the classical mechanics.

In chapter I, we generalize the above example and reformulate the path integral in a curved space, though ih^{-1} is replaced by $-\lambda(\lambda>0)$. Here, we consider a certain integral transformation on a given Riemannian manifold , which is given by the action integral determined by the geodesics. This method is based on the variational problem and gives a rigorous meaning to the Feynman's original idea. We can extend our situation to the one which acts on the sections of the vector bundles, and it gives the construction of the differential operator acting on systems. In the frame work of this investigation, we give the construction of the fundamental solutions for a heat type equation on the non-compact Riemannian manifold and the asymptotic estimate for this. Also, we remark that this procedure can be extended to the general Lagrangian function.

The other description of the classical mechanics is given by the Hamiltonian formulation, where the ' action integral ' can be also defined as an analogue to the Lagrangian mechanics. In chapter II, we discuss the integral transformation which uses the action defined by the given Hamiltonian. Here, we restrict our concerns to the case that the configuration space is compact and the degree of Hamiltonian is less than one. Our integral transformation considered here can be described as a Fourier integral operator. Since the group of invertible Fourier integral operator of order zero is a infinite dimensional Lie group having a nice property which fits to our scheme (Cf 2.1 Theorem B), we can prove the product formula for the above integral transformation as a kernel function.

Chapter I Path integral formulation from the Lagrangian mechanics.

1.0 Preliminaries and the statement of results

In this chapter, we give a rigorous meaning to the convergence of path integral in a non-compact curved space. Though comparing with Feynman's original idea, we consider the case where ih^{-1} is replaced by $-\lambda(\lambda>0)$. Namely, we considered a certain integral transformations (to those which acts on sections of a general vector bundle), associated with a given Lagrangian function of the form; $L(x,\dot{x}) = g_{ij}(x)\dot{x}_i\dot{x}_j$, where $G = (g_{ij}(x))$ defines a Riemannian metric, and show the convergence of its product integral in a refined topology (pointwise convergence of the kernel function).

Let (M,g) be a smooth, Riemannian m-manifold and let E be a vector bundle over M with a linear connection D. Suppose that E is furnished with an inner product preserved by D, which is denoted by $\langle\ ,\ \rangle_x$ at each fiber E_x, $x \in M$. We extend the action of D to tensor fields on M with values in E. Using the connection D, we can consider the parallel translation along the minimal geodesic γ from y to x, which maps an element of E_y to that of E_x. We denote it by $P(x,y)$.

Denote by $C_o(M)$ the set of all continuous sections of E with the compact support and by $C^\infty(E)$ that of all smooth sections of E. Put $C^{\infty}_{0}(E) = C_0(E) \;\; C^{\infty}(E)$. For $\xi \in C_0(E)$, we define the L^2-norm as

$$\|\xi\|_{L^2(E)} = [\int_M \langle \xi(x), \xi(x) \rangle_x d\mu_g(x)]^{\frac{1}{2}} ,$$

where $d\mu_g(x)$ denotes the canonical measure defined by the Riemannian metric g. We denote by $L^2(E)$ the Hilbert space of sections ξ of E such that $\|\xi\|_{L^2(E)} < +\infty$.

Now, consider the following integral transformation in $L^2(E)$ with 1-parameters s,t, $0 \leqq s < t$, and $\lambda > 0$ (Cf. [18]),

(0.1) $$H(\lambda;t,s)\xi(x)$$

$$= (2\pi\lambda^{-1})^{-m/2} \int_M \rho(t,s;x,y)[\exp - \lambda S(t,s;x,y)]$$

$$\times P(x,y)\xi(y)d\mu_g(y).$$

for $\xi(x) \in C^{\infty}_{0}(E)$: Here $S(t,s;x,y) = d^2(x,y)/(2(t-s))$, where $d(x,y)$ is the distance function and $\rho(t,s;x,y)$ is defined by

(0.2) $$\rho(t,s;x,y)$$

$$= |\det[-\partial_x \partial_y S(t,s;x,y)]/\mu_g(x)\mu_g(y)|^{1/2} ,$$

where $\mu_g(x) = [\det (g_{ij}(x))]^{1/2}$ ((0.2) is assumed to be well-defined here. In fact, it is guaranteed under the assumption (A.0) which is stated in § 2. Cf. [5] and [18]).

The kernel function of $H(\lambda;t,s)$ will be denoted by $H(\lambda;t,s;x,y)$ which may be considered as a section on E E; the vector bundle over $M\times M$ whose fiber at $(x,y)\in M\times M$ is given by the tensor product $E_x \boxtimes E_y^*$.

We consider the product integral for the above operator (0.1). Namely, let σ_N be the N-equal subdivision of the interval $[0,t]$ for given $t>0$ and any positive integer N,

$$\sigma_N \quad :0=t_0<t_1<\dots<t_{N-1}<t_N = t \ ,t_j = (j/N)t \quad ,$$

We set

$$(0.3) \qquad H(\lambda;\sigma_N|t)$$

$$=H(\lambda;t,t_{N-1})H(\lambda;t_{N-1},t_{N-2}) \ \dots \ H(\lambda;t_1,0),$$

and denote by $H(\lambda;\sigma_N|t;x,y)$ the kernel function of (0.5).

In order to state our results, we introduce the following assumptions :

(A.0) (M,g) is a connected, simply connected, complete Riemannian manifold and has non-positive sectional curvature.

(A.1) There exists a positive constant k_1 such that for any multi-index $\alpha = (\alpha_1,\dots,\alpha_m)$, $0\leq|\alpha|\leq 3$ and $x\in M$,

$$(0.4) \qquad |\nabla^\alpha R^h_{i\,jk}(x)|_x \leq k_1 \quad ,$$

where ∇ and $R^{h}_{ijk}(x)$ is the Riemannian connection and the curvature tensor defined by g respectively.

(A.2) There exists a positive constant k_2 such that the curvature 2-form Ω of D satisfies, for $0\leqq|\alpha|\leqq 3$, and $x\in M$

$$(0.5)\qquad |D^{\alpha}\Omega(x)|_x \leqq k_2 \quad .$$

We can state the main theorem in chapter 1.

Theorem A. Let (M,g) be a m-dimensional Riemannian manifold and E be a vector bundle over M which satisfy (A.0)-(A.2). Fix T>0 arbitrarily. Then, the limit

$$(0.6)\qquad H(\lambda;t;x,y) = \lim_{N\to\infty} H(\lambda;\sigma_N|t;x,y)$$

exists as an section of E E for any t, 0<t<T. Moreover, $H(\lambda;t;x,y)$ gives a fundamental solution of the following parabolic equation :

$$(0.7)\qquad [(\frac{\partial}{\partial t} - \lambda^{-1}A_x]H(\lambda;t;x,y) = 0,$$

$$\lim_{t\to 0+} H(\lambda;t;x,y) = \delta_k(x)\times \mathrm{Id}_y ,$$

where

$$(0.8)\qquad A_x = (1/2)\Delta^D_x - (1/12)\mathrm{Scal}_g(x) \ ,\Delta^D = -D^*D ,$$

D^* is the adjoint operator of D with respect to the inner product on $L^2(E)$ and $Scal_g(x)$ is the scalar curvature.

On the other hand, in the course of the proof of the main theorem, we can get the asymptotic behavior of $H(\lambda;t,s;x,y)$ as $t\to 0$, which is a partial extension of results in Molchanov [25] who treated the case where $E=M\times C$.

<u>Corollary 0.1.</u> Under the same assumption as in the main theorem, the fundamental solutions $H(\lambda;t,s;x,y)$ of (0.7) satisfies, for any ε, $0<\varepsilon< 1/2$,

$$(0.9)\qquad |H(\lambda;t;x,y) - (2\pi\lambda^{-1}t)^{-m/2}\rho(x,y)[\exp-\lambda(d^2(x,y)/2t)]P(x,y)|_{(x,y)}$$

$$\leq \gamma' t^{\frac{-(m-3)}{2}}[\exp-\lambda\varepsilon^{**}d^2(x,y)/2t] \quad ,$$

for any $x,y\in M$, with some positive constant γ', where $\varepsilon^{**}= 1-2\varepsilon$, $\rho(x,y) = |\det_g(dExp_x^{-1})_y|^{1/2}$, Exp_x is the exponential mapping defined by g and $|\quad|_{(x,y)}$ is the norm in $E_x\ E_y^*$ (Cf. § 2).

1.1 Outline of the proof of Theorem A and related remarks.

In this section, we state our plan to prove the main theorem in §1.0. First, in §1.3, we show the following basic properties of $H(\lambda;t,s)$ defined by (0.1).

<u>Proposition 1.1</u> Assume (A.0)-(A.2). On fixing $T>0$ arbitrarily, the following properties hold for $0=<s<t<T$:

(a) The integral transformation $H(\lambda;t,s)$ defines a bounded linear operator in $L^2(E)$.

(b) $$\lim_{t \downarrow s}\|H(\lambda;t,s)\xi - \xi\|_{L^2(E)} = 0,$$

$$\lim_{s \uparrow t}\|H(\lambda;t,s)\xi - \xi\|_{L^2(E)} = 0.$$

Let $\mathcal{B}(L^2(E))$ be the set of all bounded linear operators on $L^2(E)$ and we introduce the topology by the operator norm in it. By Proposition 1.1, $H(\lambda;t,0)$ can be considered as a curve in $\mathcal{B}(L^2(E))$ starting from the identity operator. Now, we may consider the convergence of the product integral of $H(\lambda;t,0)$ in $(L^2(E))$. So, we prove the following in § § 1.4-1.5, which is one of the key results :

<u>Theorem 1.2.</u> Under the same assumptions as in Proposition 1.1, the following properties hold :

(a) There exists a positive constant $C_0=C_0(\lambda;T)$ such that

(1.1) $$\|H(\lambda;t+t',s) - H(\lambda;t+t',t')H(\lambda;t',s)\xi\|_{L^2(E)}$$

$$\leq C_0[(t+t'-s)^{3/2} - t^{3/2} + (t'-s)^{3/2}]\|\xi\|_{L^2(E)}$$

for any $\xi\in L^2(E)$ and $0\leq s<t'<t+t'<T$.

(b) There exists a limit $H(\lambda;t) = \lim_{N\to\infty} H(\lambda;t,t_{N-1})\ldots H(\lambda;t_1,0)$, $t_j = (j/N)t$, $j=1, \ldots ,N-1$, in $(L^2(E))$ for any $t>0$. Therefore, $\{H(\lambda;t)\}_{t\geq 0}$ with $H(\lambda;0)$ = the identity operator, is a C^0 semi-group in $L^2(E)$.

(c) The infinitesimal generator $\lambda^{-1}A$ of $H(\lambda;t)$ is given by

(1.2) $$\lambda^{-1}(A\xi)(x)$$

$$= [(\frac{\partial}{\partial t}H(\lambda;t)\xi(x)_{t=0}]$$

$$= \lambda^{-1}[(1/2)\Delta_x^D - (1/12)\mathrm{Scal}_g(x)]\xi(x) \quad .$$

Theorem 1.2 shows that the product integral of (0.6) determines a fundamental solution of the heat type equation (0.8) in the distribution sense. To show the regularity, we construct a kernel function by another method which is rather standard in the theory of partial differential equation (Cf. Friedman [12]). Using this estimate, we prove the main theorem. Namely, we show in §1.7 the following :

<u>Theorem 1.3.</u> Under the same assumptions as in Proposition 1.1, we can construct a fundamental solution $H(\lambda;t)$

with the following estimate : For any ε, $0<\varepsilon<\frac{1}{4}$, there exists a positive constant $\gamma=\gamma(\lambda;T,\varepsilon)$ which dose not depend on σ_N such that

$$(1.3) \qquad |H(\lambda;x,y) - H(\lambda;\sigma_N|t;x,y)|_{(x,y)}$$

$$\leq \gamma t^{\frac{-(m-3)}{2}} N^{-1/2}[\exp(-\lambda\varepsilon^{(4)}(d^2(x,y)/2t))] ,$$

where $\varepsilon^{(4)}= 1-4\varepsilon$ and $H(\lambda;t;x,y)$ is the kernel function of $H(\lambda;t)$.

Remark 1. We cannot prove the convergence of $H(\lambda;\sigma_N|t;x,y)$ directly, which may be still an interesting problem.

For the sake of our computations, we shall introduce the normal coordinate. Given $\bar{x}\in M$, let U be a local coordinate neighborhood of $\bar{x}$ with the coordinate $(x^1,\ldots,x^m)$ such that $E_{|U}$ is trivialized as $E_{|U}\cong U\times F$, where F is the standard fiber of E. Taking a frame field $\{e_a(x)\}$ of $E_{|U}$ (i.e. $e_a(x)$ depends smoothly on $x\in U$ and $\{e_a(x)\}$ forms a basis on F for any $x\in U$.

Denote by $\Gamma_{jb}^{a}(x)$ the component of $D_j = D_{(\frac{\partial}{\partial x^j})}$. Therefore, for each $\xi\in C^\infty(E)$, its covariant derivative $D_j\xi$ can be expressed by

$$(1.4) \qquad D_j\xi^a(x) = \partial_j\xi^a(x) + \Gamma_{jb}^{a}\xi^b(x) \quad .$$

Also, for any $\psi\in\Omega^1(E)$, a E-valued 1-form, writting by $\psi(x)= \psi_i(x)dx^i$, $\psi_i= \psi_i^a(x)e_a(x)$, we have

$$(1.5)\qquad D_j\psi_i^a(x) = \partial_j\psi_i^a(x) - \Gamma_{ji}^k\psi_k^a(x) - \Gamma_{jb}^a(x)\psi_i^b(x) ,$$

where Γ_{ji}^k is the Christoffel symbol of g. Moreover, the local coordinate expression of the covariant derivatives for any tensor field with values in E is obtained similarly. Using these notations, Δ^D can be expressed as

$$\Delta^D\xi^a(x) = g^{ij}(x)[\delta_b^a\partial_i + \Gamma_{ib}^a(x)][\delta_c^b\partial_j + \Gamma_{jc}^b(x)]\xi^c(x) ,$$

for any $\xi\in C^\infty(E)$.

Finally, we give some remarks about the main theorem.

<u>Remark 2.</u> (i) <u>Trivial bundle</u>, $E = M\times\mathbf{R}$ (or $M\times\mathbf{C}$). A section of the trivial bundle can be identified with a function on M and $C(E)\cong C(M)$. Taking the trivial connection, i.e. $P(x,y) = id.$, we get a integral transformation acting for functions on M, which is considered in [18]. So, in this case, the limit

$$\lim_{N\to\infty}\int_M\int_M\cdots H(\lambda;t,t_{N-1};x;z_{N-1})\cdots H(\lambda;t_1,0;z_1,y)$$

$$d\mu_g(z_{N-1})\cdots d\mu_g(z_1)$$

exists as a function on $M\times M$ for fixed t, $0<t<T$, under the assumption (A.1). We may denote its limit by

$$\sum_{\substack{x(\cdot)\\ x(0)=y,x(t)=x}} [\exp-\lambda S(x(\cdot))] \quad \text{(Cf. Feynman [11])} ,$$

(ii) The bundle of p-forms, $E = \Lambda^p T^* M$. In this bundle, we can induce the inner product $\langle\ ,\ \rangle_x$ and the connection D canonically by g. Namely, for

$$\xi(x)=\xi_{i_1\ldots i_p}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_p},$$

$$\eta(x)=\eta_{i_1\ldots i_p}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_p}\in C(E),$$

we define

$$\langle\xi(x),\eta(x)\rangle_x = g^{i_1 j_1}(x)\ldots g^{i_p j_p}(x)\xi_{i_1\ldots i_p}(x)\eta_{j_1\ldots j_p}(x), \tag{1.6}$$

and

$$D_j\xi_{i_1\ldots i_j p}(x) \tag{1.7}$$

$$= \partial_j\xi_{i_1\ldots i_p}(x) - \Gamma_{ji_1}^{k}(x)\xi_{ki_2\ldots i_p}(x) - \ldots$$

$$- \Gamma_{ji_p}^{k}(x)\xi_{i_1\ldots i_{p-1}k}(x) \quad .$$

Then, we get the operator $A_x = -(1/2)\Delta_L - (1/12)\mathrm{Scal}_g(x)$ along the path-integral approach : Here Δ_L is the rough Laplacian defined by g (Cf. [21]), and it is given by

$$\Delta_L\xi_{i_1\ldots i_p}(x) \tag{1.8}$$

$$= \Delta_H\xi_{i_1\ldots i_p}(x) + \Sigma_{r=1}^{p} R_{i_r}{}^{m}(x)\xi_{i_1\ldots m\ldots i_p}(x)$$

$$= \Sigma_{m>k} R_{i_m i_k}{}^{uv}(x)\xi_{i_1\ldots u\ldots v\ldots i_p}(x) ,$$

where $R_{ij}(x)$ and $R_{ijk}{}^{h}(x)$ are the component of the Ricci tensor and the curvature tensor of g respectively, and Δ_H denotes the Hodge-de Rham operator.

(iii) As a generalization of (ii), our method constructs the fundamental solution of the parabolic equation whose infinitesimal generator is the following :

(a) The Lichnerowicz Laplacian acting on tensor fields.

(b) The spinorial Laplacian of Lichnerowicz when M admits a spinorial structure. (Cf. [21])

1.2 Preliminaries for classical action and the parallel translation.

Throughout the rest of chapter, notations and definitions concerning the differential geometry will follow the references [5] and [18]. We shall use well-known facts in [5],[6] and §2 of [18] without proof.

Let (M,g) be a Riemannian manifold. First, we recall a geodesic, i.e. a curve $\gamma(u)$ which satisfies the following differential equation,

$$\frac{\delta^2 x^i(u)}{\delta^2 u} = \frac{d^2 x^i(u)}{du^2} + \Gamma^i_{jk}(x(u)) \frac{dx^j}{du}\frac{dx^k}{du} = 0, \tag{2.1}$$

where $\frac{\delta}{\delta u}$ denotes the covariant derivative along a curve. Given $x \varepsilon M$, we define a mapping Exp_x from the tangent space $T_x M$ into M by $Exp_x uX = x(u)$, where x(u) satisfies (2.1) with the initial condition $x(0) = x$ and $\frac{dx}{du}(0) = X \varepsilon T_x M$. Under the assumptions (A.1)-(A.2), Exp_x gives a diffeomorphism from $T_x M$ into M for each point $x \varepsilon M$. Denote by d(x,y) the Riemannian distance between x and y. Then, the function S(t,s;x,y) defined by (1.3) is given by $S(t,s;x,y) = \frac{d^2(x,y)}{2t}$.

For each $X \varepsilon T_x M$, identifying $T_X(T_x M)$ with T_M, we may induce naturally the scalar product in $T_X(T_x M)$. Now, for fixed $x \varepsilon M$, we denote by $(dExp_x^{-1})_y$, the differential mapping of Exp_x^{-1} at y. Define also the function $\theta(x,y)$ on M×M by $\theta(x,y) = |\det_g(dExp_x)_X|$, $Exp_x X = y$ (Cf. See [4]). Then, the function $\tilde{\rho}(t,s;x,y)$ defined by (1.5) can be written by

$$\tilde{\rho}(t,s;x,y)=(t-s)^{-m/2}\rho(x,y), \tag{2.2}$$

where $\rho(x,y) = \theta(x,y)^{\frac{-1}{2}}$.

Recall the alternative representation of the function $\rho(x,y)$, or $\theta(x,y)$. Let $\{J_2(u),\dots,J_m(u)\}$ be a (m-l)-Jacobi field with the initial conditions

(i) $J_i(0)=0$ $(i=2,\dots,m)$

(ii) $\{\dot{J}_i(0)\}_{i=2,\dots,m}$ forms an orthonormal basis of the orthogonal complement of $\dot{x}(0)$ in T_M.

Then, we have

$$\theta(x,y) = r^{1-m}\|\det((J_i(r),J_j(r))_y\|, \quad r=d(x,y). \tag{2.3}$$

Remark that the assumption (A.3) implies that the any sectional curvature is bounded below by some constant $-k^2$ $(k>0)$. Therefore, by the well-known Rauch comparison theorem, we get (Cf. [5] and [6])

<u>Lemma 2.1.</u> Assume that (A.0)-(A.2). Let $J(u)$ be a Jacobi field along geodesic $x(u)$ with arclength parameter and satisfying the initial conditions $J(0)=0$, $\dot{J}(0)=0$. Then,

(i) There exists a positive constant k such that

$$r\|\dot{J}(0)\|_x \le \|J(r)\|_y \le \left(\frac{\sinh kr}{kr}\right)\|\dot{J}(0)\|_x. \tag{2.4}$$

(ii) Particularly, we have

(2.5) $$1 \leqq \theta(x,y) \leqq \left(\frac{\sinh kr}{kr}\right)^{(m-1)/2},$$

(2.5') $$\left(\frac{\sinh kr}{kr}\right)^{-(m-1)/2} \leqq \rho(x,y) \leqq 1.$$

Denote by SM and S_xM the unit sphere bundle over M and the fibre of SM at x respectively. By the same proof as in Lemma 4.1 of [18], we have

<u>Lemma 2.2</u> Notations and assumptions being as in Lemma 2.1, there exists a positive constant k_1 such that

(2.6) $$\|\dot{J}(r)\|_y \leqq (k_1 \exp k_1 r)\|\dot{J}(0)\|_x, \quad r=d(x,y).$$

Moreover, we have

(2.7) $$\|\theta_r(x,y)\| \leqq k_1 \exp k_1 r, \qquad r=d(x,y),$$

where $\theta_r = \frac{d}{dr}\theta(x, \mathrm{Exp}_x\, r\omega)$, $\mathrm{Exp}\, r\omega = y$, $\omega \in S_xM$.

Now, we give the estimate of higher order derivatives of the functions $\rho(x,y)$:

<u>Proposition 2.3.</u> Assume that (M,g) satisfies (A.0)-(A.2). Then, there exists a positive constant k_0

(2.8) $$\|\nabla_y^j \rho(x,y)\| \leqq k_0 \exp k_0 r, \quad r=d(x,y),$$

for $0 \leqq j \leqq s$ and for any $x,y \in M$.

To give the estimate for $\nabla_y^j \rho(x,y)$, we need several steps as below. From now on, we assume that x is fixed in M.

By the assumptions (A.0)-(A.2), we deduce that the exponential mapping is a diffeomorphism from T_xM onto M. Thus, we can introduce the normal coordinate around x. (Cf. See [18] for the precise notation). By the identification $T_xM = S_xM\times\mathbf{R}$, we shall use the normal polar coordinate (r,ω), where $\omega = (\omega^2,\ldots,\omega^m)$ in a local coordinate of $S_xM = \{Y\varepsilon T_xM;\ \|Y\|_x = 1\}$ and $r\varepsilon\ \mathbf{R}_+$. Choosing an orthonormal vectors $\{e_2(\omega),\ldots,e_m(\omega)\}$, at a point (r,ω), which is perpendicular to radial axis, we may assume that $\{e_2(\omega),\ldots,e_m(\omega)\}$ depends smoothly on ω locally. (For example, we consider a neighborhood of ω in S_xM and orthonormal basis with ω as a first vector, and we will deform by using the Gram-Schmit orthogonalization). We put , for $a = 2,\ldots,m$,

(2.9) $$K_a(u,\varepsilon_1) = \mathrm{Exp}_x u(\omega + (\frac{\varepsilon_1}{r})e_a(\omega)) \quad ,$$

for sufficiently small ε_1, Since (2) is an geodesic variation, $\frac{\partial K_a}{\partial\varepsilon_1}$ is a Jacobi field along the curve $K_a(u,\varepsilon_1)$ for each fixed ε_1 and has the initial conditions

(2.10) $$\frac{\partial K_a}{\partial\varepsilon_1}(0,0) = 0 \ ,\frac{\delta}{\delta u}\frac{\partial}{\partial\varepsilon_1}K_a(0,\varepsilon_1) = \omega+(\frac{\varepsilon_1}{r})e_a(\omega) \ .$$

Therefore, for sufficiently small ε_1,

$$\|(\frac{\delta}{\delta u})(\frac{\partial}{\partial\varepsilon_1})K_a(0,\varepsilon_1)\|_x = 0 \quad .$$

Also, we can apply Lemma 2.1 and Lemma 2.2, we have

$$\|\frac{\partial}{\partial\varepsilon_1}K_a(u,0)\|_{x(u)} \leq K_1'' \exp kr \ ,$$

(2.11)

$$\left\| \frac{\delta}{\delta u} \frac{\partial}{\partial \varepsilon_1} K_a(u,0) \right\|_{X(u)} \leq K_1 \exp kr \ ,$$

with some constants K_1 and k.

Let us use the indices $A,B,C,\ldots = 1,2, \ \ldots \ ,m$ and $a,b,c,\ldots = 2,3, \ \ldots \ ,m$. Denote by g_{AB} the component of Riemannian metric g with respect to the coordinate (r,ω) i.e.

$$g_{AB} = \begin{bmatrix} g_{11}(r,\omega) & , & g_{1a}(r,\omega) \\ g_{a1}(r,\omega) & , & g_{ab}(r,\omega) \end{bmatrix} \tag{2.12}$$

where

$$= g_{11}(r,\omega) = g_{(r,\omega)}((dExp_X)_{r\omega}\omega,(dExp_X)_{r\omega}\omega) = 1 \ ,$$

$$\begin{aligned} g_{1a}(r,\omega) &= g_{a1}(r,\omega) \\ &= g_{(r,\omega)}((dExp_X)_{r\omega}\omega,(dExp_X)_{r\omega}e_a(\omega)) \\ &= 0, \end{aligned}$$

$$\begin{aligned} g_{ab}(r,\omega) \\ = \ g_{(r,\omega)}((dExp_X)_{r\omega}e_a(\omega),(dExp_X)_{r\omega}e_b(\omega)). \end{aligned} \tag{2.13}$$

<u>Lemma 2.5.</u> Assume that (M,g) satisfies (A.0)-(A.1). Then, there exists a positive constant K_2 such that, for $2 \leq a,b \leq m$,

$$|g_{ab}(r,\omega)| \leqq K_2 \exp K_2 r,$$

(2.14) $$|\partial_r g_{ab}(r,\omega)| \leqq K_2 \exp K_2 r,$$

$$|\partial_\omega g_{ab}(r,\omega)| \leqq K_2 \exp K_2 r,$$

Proof. By estimate of Jacobi field in Lemma 2.3, we obtain the first inequality of (2.14). For the second one of (2.14), we obtain it by the following computation

$$\frac{d}{dr} g_{ab}(r,\omega) = \frac{d}{du}\left(\frac{\partial}{\partial \varepsilon_1} K_a|_{\varepsilon_1=0}, \frac{\partial}{\partial \varepsilon_1} K_b|_{\varepsilon_1=0}\right)_{x(u)}|_{u=r}$$

$$= \left(\frac{\delta}{\delta u} \frac{\partial}{\partial \varepsilon_1} K_a|_{\varepsilon_1=0}, \frac{\partial}{\partial \varepsilon_1} K_b|_{\varepsilon_1=0}\right)_{x(r)}$$

$$+ \left(\frac{\partial}{\partial \varepsilon_1} K_a|_{\varepsilon_1=0}, \frac{\delta}{\delta u} \frac{\partial}{\partial \varepsilon_1} K_b|_{\varepsilon_1=0}\right)_{x(r)}$$

Thus, we have the second one of (2.14) by (2.11). To obtain the last one of (2.14), we take a curve $\omega(\varepsilon_2)$ in S_xM for sufficiently small ε_2, and $\omega(\varepsilon_2) = \omega + \varepsilon_2 e_c(\omega)$, $c = 2,\ldots,m$. Consider

(2.15) $$K_a(u,\varepsilon_1,\varepsilon_2) = \mathrm{Exp}_x\, u(\omega(\varepsilon_2) + (\frac{\varepsilon_1}{r}) e_a(\omega(\varepsilon_2))).$$

Then, $K_a(u,\varepsilon_1,\varepsilon_2)$ is also geodesic variation in two parameters $\varepsilon_1,\varepsilon_2$ and has the following initial conditions :

(2.16) $$\frac{\partial}{\partial \varepsilon_1} K_a(0,\varepsilon_1,\varepsilon_2) = 0, \quad \frac{\partial}{\partial \varepsilon_2} K_a(0,\varepsilon_1,\varepsilon_2) = 0,$$

$$\frac{\delta}{\delta u} \frac{\partial}{\partial \varepsilon_1} K_a(0,\varepsilon_1,\varepsilon_2) = \frac{1}{r}\, e_a(\omega(\varepsilon_1)) ,$$

(2.17)

$$\frac{\delta}{\delta u}\frac{\partial}{\partial \varepsilon_2}K_a(0,\varepsilon_1,\varepsilon_2) = \omega'(\varepsilon_2) + (\frac{\varepsilon_1}{r})\frac{d}{d\varepsilon_2}e_a(\omega(\varepsilon_2)).$$

By differentiating Jacobi equation with respect to ε_1 or ε_2, and putting $\varepsilon_1=\varepsilon_2=0$, we get

(2.18) $\quad F_{a,i,j}(u)$

$$= \frac{\delta^2}{\delta\varepsilon_i\delta\varepsilon_j}\frac{\delta^2}{\delta u^2}K_a(u,0,0) + R(\dot{x}(u),\frac{\delta^2}{\delta\varepsilon_i\delta\varepsilon_j}K_a(u,0,0))\dot{x}(u)$$

where $F_{a,i,j}(u)$ is the function of R, ∇R, $\frac{\partial}{\partial\varepsilon_i}K_a$, $\frac{\partial}{\partial\varepsilon_j}K_a$, $\frac{\delta^2}{\delta u\delta\varepsilon_i}K_a$, $\frac{\delta^2}{\delta u\delta\varepsilon_j}K_a$, and we have the following estimate, by Lemma 2.1 and Lemma 2.2,

(2.19) $\quad \|F_{a,i,j}(u)\|_{x(u)} \leqq k_2' \exp k_2' u$

Therefore, we get by variation of constant

(2.20) $\quad \|\frac{\delta^2}{\delta\varepsilon_i\delta\varepsilon_j}K_a(u,0,0)\|_{x(u)} \leqq k_2''\exp ku \quad ,$

with some positive constants k_2'' and k. Then, in accordance of $\omega = \omega^a e_a$, $\omega = (\omega^2,\ldots,\omega^m)$ as the coordinate of ω. using (2.20) and

(2.21) $\quad \partial_c g_{ab}(r,\omega) = \frac{d}{d\varepsilon_2}g_{ab}(r,\omega(\varepsilon_2))\,|_{\varepsilon_2=0}$

$$= (\frac{\delta^2}{\delta\varepsilon_2\delta\varepsilon_1}K_a(r,0,0),\frac{\delta}{\delta\varepsilon_1}K_b(r,0,0))_{x(r)}$$

$$+ (\frac{\delta}{\delta\varepsilon_1}K_a(r,0,0),\frac{\delta^2}{\delta\varepsilon_2\delta\varepsilon_1}K_b(r,0,0))_{x(r)},$$

we get (7) by Lemma 2.2.

Remark. In the above proof, we shall use the inverse matrix of fundamental solution of Jacobi equation, because we use the method of variation of constant. To get this, we use the estimate of Jacobi field $J(s)$ with the initial condition $J(0)=0$, $\dot{J}(0)=0$, which is also exponetial grouwth at infinity (Cf. See also Cheeger-Ebin [6]). Also, we use that the initial value of $\frac{\partial}{\partial u}\frac{\partial^2}{\partial\varepsilon_i\partial\varepsilon_j}K_a$ is bounded, which may be assumed by the appropriately choosing of $e_a(\omega)$, $a=2, \ldots, m$.

Lemma 2.6. Given any $x\varepsilon M$, take a normal polor coordninate (r,ω) around x. Then, there exists a positive constant k_3''' such that the following estimates holds :

$$|g^{AB}(r,\omega)| \leq k_3''' \exp k_3''' r, \quad r=d(x,y) \quad , \tag{2.22}$$

$$|\Gamma^A_{BC}(r,\omega)| \leq k_3''' \exp k_3''' r \quad ,r=d(x,y) \tag{2.23}$$

where $g^{AB}(r,\omega)$ and $\Gamma^A_{BC}(r,\omega)$ are the inverse matrix of $g=(g_{AB}(r,\omega)$ and the Christofel symbol with respect to the coordinate $(r,\omega^2,\ldots,\omega^m)$ respectively.

Moreover, for the function $\rho(x,y)$ defined by (2.5), there exists a positive constant k_3 such that for any $x,y\varepsilon M$,

$$|\nabla_y \rho(x,y)|_y \leq k_3 \exp k_3 r \quad , \quad r=d(x,y) \tag{2.24}$$

Proof. We shall only show (2.24). Recalll $\rho(x,y) = \theta(x,y)^{-\frac{1}{2}}$ and $\theta(x,y)\geq 1$ by (A.0)-(A.1). Take normal polar coordinate (r,ω) around x and use the same notation as in (2.9), we have

$$\theta(x,y)$$

$$= |\det[g_{(r,\omega)}(\frac{\partial}{\partial\varepsilon_1}K_a(r,0)\ ,\ \frac{\partial}{\partial\varepsilon_2}K_b(r,0))_y]_{a,b=2,\dots,m}|$$

$$= |\det(g_{ab}(r,\omega))|,$$

Differentiating directly and applying Lemma 2.5 and (2.23), we get the disired results.

Now, let $\omega^2,\dots,\omega^m$ be the coordinates on part of S_xM. We denote by $D^{p,v}$ the differnetial operator , $v=(v_2,\dots,v_m)$

$$(\frac{\partial}{\partial r})^t(\frac{\partial}{\partial\omega^2})^{v_2}(\frac{\partial}{\partial\omega^m})^{v_m}$$

To obtain Proposition 2.3, we only get the following :

<u>Proposition 2.7.</u> Under the same assumptions and notations as in Proposition 2.3, there exists a positive constant k_0 such that

(2.25) $\quad \|D^{p,v}J_a(r,\omega)\|_{x(r)} \leqq k_0 \exp k_0 r$

for $p+|v| \leqq L$.

Differentiating the Jacobi equation and putting $J_a(u,\omega) = \frac{\partial}{\partial\varepsilon_1}K_a(u,0)$, we get

(2.26) $\quad D^{t,v}J_a''(u,\omega) + R(u,\omega)D^{t,v}J_a(u,\omega) = F_{a,t,v}(u,\omega)$,

where

$$F_{a,t,v}(r,\omega)$$

$$= \sum_{\substack{p+q+|v'|+|v''|=t+|v|+1 \\ p+|v'|\leq t+|v| \\ q+|v''|\leq t+|v|}} D^{p,v'}R\, D^{q,v''}J_a(u,\omega).$$

J_a'' is the 2nd derivative with respect to u, and $R(u,\omega)X(u) = R(\dot{x}(u),X(u))\dot{x}(u)$. By Lemma 2.6, we have

$$(2.27) \quad |D^{p,v'}R(u,\omega)|$$

$$\leq [k_3' \exp k_3' r]\Sigma_{p+|v'|\leq L}\|\nabla^j R\|$$

for some constant k_3'.

Now, we show (2.25) by induction. Recall that (2.25) holds for $p+|v|^{\sim}=1$. Assume that (2.25) holds for $p+|v|\leq L-1$. by (2.27), we have

$$\|F_{a,t,v}(u,\omega)\| \leq k_0' \exp k_0' u$$

for some positive constant k_0'. Then, by the variation of constant for (2.26), we get the estimate (2.27) for $p+|v| = L$ (In this case, we choose $J_a(0,\omega)$ and $\dot{J}_a(0,\omega)$ to be bounded).

Remark. Berȧrd [3] has similar estimate for $\rho(x,y)$ for the case that M is a compact manifold without conjugate points.

Next, we shall recall the parallel transformation of a section of the vector bundle E over M. Given a curve $x(\tau)$ on M such that $x(s) = y$, $x(t) = x$ $(s<t)$, and $\xi \epsilon\ \Lambda_y^p(M)$, define $\tilde{\xi}(\tau) \epsilon\ \Lambda^p_{x(\tau)}(M)$ by

$$(2.28) \qquad \frac{\delta}{\delta\tau}\tilde{\xi}(\tau) = 0, \qquad \tilde{\xi}(0)=\xi(y).$$

Now, we write the above one by $\tilde{\xi}(\tau) = P_s^t(\nabla;x(\tau))\xi(y)$. Since (2.28) the first order ordinary differential equation, the solution of (2.28) exists uniquely for given curve $x(\tau)$. Particularly, if $\gamma_c(\tau)$ be a classical path which attains the infimum of (1.3), we can denote by $P_s^t(\nabla;\gamma_c)\xi(y)$ $\tilde{\xi}(t)$. On the other hand, consider also a geodesic $x(u)= \mathrm{Exp}_x\ uX$, where $\mathrm{Exp}_x = y$, and the parallel transport $P_0^1(\nabla;x(u))\xi(y)$ for $x(u)$. Now, easily we get $P_s^t(\nabla;\gamma_c) = P_0^1(\nabla;x(u))$ if we assume that (M,g) satisfies (A.0)-(A.1). Moreover, we write $P_0^1(\Delta;x(u))$ by $P(x,y)$ for simplicity .

Let $E \times E^*$ be a vector bundle over $M{\times}M$, with the fibre $E_x{\times}E_y^*$. Since each vector spaces E_x and E_y^* equips with inner product $(\ ,\ )_x$ (or $(\ ,\ )_y$), we give the inner product on each fibre $E_x \times E_y^*$. This is, given $\tilde{\xi}(x,y) = \xi(x) \times \xi(y)^*$, $\tilde{\eta}(x,y) = \eta(x) \times \eta(y)^*$, we define

$$(2.29) \qquad (\tilde{\xi}(x,y),\tilde{\eta}(x,y))_{(x,y)}=(\xi(x),\eta(x))_x(\xi(y)^*,\eta(y)^*)_y.$$

and

$$(2.30) \qquad \|\tilde{\xi}(x,y)\|_{(x,y)}=(\xi(x,y),\xi(x,y))^{\frac{1}{2}}_{(x,y)}$$

The following lemma is easily obtained (Cf. See Berger [4]).

Lemma 2.8. Assume (A.0)-(A.2). Then, we get the following :

(i) $P(x,y)\epsilon C^{\infty}(E \boxtimes E^*)$,

(ii) For $\xi\epsilon C_0^{\infty}(E)$, we have $\|P(x,y)\xi(y)\|_x=\|\xi(y)\|_y$ and $\|P(x,y)\|_{(x,y)}=m=\dim M$.

(iii) $P(x,x)=$ Id., the identity operator on E_x.

(iv) $(\nabla_x d^2(x,y), \nabla_x P(x,y))_x = 0$,

where ∇_x is canonically extended Riemannian connection by g with respect to x-variables on $E \times E^*$.

Lemma 2.9. Assume (A.0)-(A.1). Then, we have

(2.31) $\nabla_x P(x,y)|_{x=y} = 0, \quad \Delta_x P(x,y)|_{x=y} = 0.$

(2.32) $\nabla_y P(x,y)|_{y=x} = 0, \quad \Delta_x P(x,y)|_{x=y} = 0.$

Proof. Let $\{e_a\}_{a=1}^{p}$ be an orthonormal basis at E_y, where $\dim E_y = p$. Extend $\{e_a\}$ to a local frmame field so that they are parallel. Take a normal coordinate $(Y^1,\dots,Y^m)$ at y and denote by Γ_{ja}^{b} the coefficients of D. By putting $\tilde{\xi}(\tau) = \tilde{\xi}^a(\tau)e_a(\gamma(\tau))$, (2.8) can be written as

(2.33) $$\frac{d\tilde{\xi}^a}{d\tau} + \Gamma_{jb}^{a}(\gamma(\tau))\frac{d\gamma^j}{d\tau}\tilde{\xi}^b(\tau) = 0 \quad .$$

So, using the Taylor expansion, we can write $\tilde{\xi}(z)$ by

(2.34) $$\tilde{\xi}^a(z)$$

$$= \tilde{\xi}^a(0) + (\tilde{\xi}^a)'(0)z$$

$$+ (1/2!)(\tilde{\xi}^a)''(0)z^2 + O(z^3) \quad ,$$

with respect to z, we get

(2.35) $$(\tilde{\xi}^a)'(0) = - \Gamma^a_{jb}(y)Y^j\xi^b(Y) \quad ,$$

(2.36) $$(\tilde{\xi}^a)''(0)$$

$$= - [\partial_u\Gamma^a_{jb}(y)Y^uY^j + \Gamma^a_{jc}(y)\Gamma^c_{ub}Y^{j^u}] \quad ,$$

where $Exp_y Y = x$, because $\Gamma^i_{jk}(y) = 0$, where Γ^i_{jk} is the Christoffel symbol of g. Substituting (2.35) and (2.36) into (2.34) and putting $z=1$, we have

(2.37) $$\tilde{\xi}^a(1) = P^a_b(x,y)\xi^b(y)$$

$$= [\delta^a_b - \Gamma^a_{jb}(y)Y^j$$

$$- (1/2)[\partial_u\Gamma^a_{jb}(y)Y^uY^j + \Gamma^a_{jc}\Gamma^c_{ub}(y)Y^uY^j$$

$$+ O(Y^3]\xi(y) \quad ,$$

where $P^a_b(x,y)$ is the component of $P(x,y)$ with respect to

$\{e_a\}$. Recall

(2.38) $D_{x,j}P^a_b(x,y)$

$$= \partial_{Y^j}P^a_b(x,y) + \Gamma^{\,a}_{jc}(Y)P^c_b(x,y) \quad , \mathrm{Exp}_y Y = x \; .$$

Combining (2.37) and (2.38), we get the first inequality of (2.31). Similarly, we have

(2.39) $D_{x,k}D_{x.j}P^a_b(x,y)$

$$= \partial_k\partial_j P^a_b(x,y) + \partial_k\Gamma^{\,a}_{jc}(Y)P^c_b(x,y)$$

$$+ \Gamma^{\,a}_{jc}(Y)\partial_k P^c_b(x,y) + \Gamma^{\,a}_{kc}(Y)\partial_j P^c_b(x,y)$$

$$= -\,(1/2)[\partial_k\Gamma^{\,a}_{jb}(y) + \partial_j\Gamma^{\,a}_{kb}(y)$$

$$+ 2\Gamma^{\,a}_{kc}(y)\Gamma^{\,c}_{jb}(y) + \partial_k\Gamma^{\,a}_{jb}(y) + \Gamma^{\,a}_{jb}(y)$$

$$+ \; O(Y)$$

which proves the second equality of (2.32). For (2.14), remark that for any $x\in M$ and $y,z\in\widetilde{V}$, we have

(2.40) $P(x,z)P(z,y) = P(x,x)=\mathrm{id}.$

Differentiating (2.40) covariantly and using Lemma 2.3 (ii) and (2.31), we obtain (2.32).

Lastly, we give the estimate of higher order derivatives of $P(x,y)$.

Proposition 2.5. Let (M,g) satisfies (A.1)-(A.3). Then, there exists a positive constant K_0 such that

$$\text{(2.41)} \qquad \|\nabla_x^j P(x,y)\|_{(x,y)} \leq K_0 \exp K_0 r \ , \quad r=d(x,y),$$

for $0 \leq j \leq 3$.

Remark. The norm $\| \ \|_{(x,y)}$ is the extended one canonically for tensor field. Hereafter, we use the same notation without explaining it.

To obtain the above proposition, we first get the following

Lemma 2.6. There exists a positive constant K_1 such that the following estimates hold :

$$\text{(2.42)} \qquad \begin{aligned} \|\nabla_x P(x,y)\|_{(x,y)} &\leq K_1 \exp K_1 r \ , \\ \|\nabla_y P(x,y)\|_{(x,y)} &\leq K_1 \exp K_1 r, \ r=d(x,y) \end{aligned}$$

Proof. Let $\{ e_1(y), \ldots, e_m(y)\}$ be an orthonormal basis of T_yM and put $\tilde{\xi}_i(x,y) = P(x,y)e_i(y)$. Take $\{f_1(x), \ldots, f_m(x)\}$ as an orthonormal basis of T_xM also. Let $x_j(\varepsilon_1)$ be a C^1-curve such that $\xi_j(0)=x$, $(\frac{d}{d\varepsilon_1})x_j(0) = f_j(x)$, $j=1, \ldots, m$. Then, we get

$$\nabla_{f_j(x)}\tilde{\xi}(x,y) = \frac{\partial}{\partial \varepsilon_1} \tilde{\xi}_i(x_j(\varepsilon_1),y)_{|\varepsilon_1=0}.$$

Consider the variation $K_j(u,\varepsilon_1) = \mathrm{Exp}_y u(\omega+(\frac{\varepsilon_1}{r})\tilde{f}_j(y))$, where

$\mathrm{Exp}_y\, r\omega = x$ and $\tilde{f}_j(y)$ is the parallel transport along the geodesic from x to y, i.e. $\tilde{f}_j(y) = P(x,y)f_j(x)$. Also, we define $\xi_j(u,\varepsilon_1)\varepsilon E_{K_j(u,\varepsilon_1)}$, $j=1, \ldots ,m$ by

$$\frac{d}{du}\tilde{\xi}_j(u,\varepsilon_1) = 0 \quad \tilde{\xi}_j(u,\varepsilon_1) = e_j(y)$$

for each fixed ε_1. In terms of local coordinates, $\tilde{\xi}_j(u,\varepsilon_1) = \xi_{j,i_1,\ldots,i_p}(u,\varepsilon_1)dx^{i_1}\wedge\cdots\wedge dx^{i_p}$, we get

$$(2.43) \qquad \frac{\delta}{\delta u}\frac{\delta}{\delta\varepsilon_1}\tilde{\xi}_j(u,\varepsilon_1)$$

$$= \frac{\delta}{\delta\varepsilon_1}\frac{\delta}{\delta u}\tilde{\xi}_j(u,\varepsilon_1) - \bar{R}(\frac{\delta}{\delta u}K_j,\frac{\delta}{\delta\varepsilon_1}K_j)\tilde{\xi}_j(u,\varepsilon_1)$$

$$= -\bar{R}(\frac{\delta}{\delta u}K_j,\frac{\delta}{\delta\varepsilon_1}K_j)\tilde{\xi}_j(u,\varepsilon_1)$$

where

$$(\bar{R}(X,Y)\tilde{\xi}_j)_{i_1,\ldots,i_p}$$

$$= R_{jki_1}{}^{h}X^{j}\tilde{\xi}_{ji_2\ldots i_p} + \cdots + R_{jki_p}{}^{h}X^{j}Y^{k}\tilde{\xi}_{ji_1\ldots i_{p-1}h},$$

$X=X^j(\frac{\partial}{\partial X_j})$, $Y=Y^k(\frac{\partial}{\partial X_k})$. Remark that

$$\|(\frac{\delta}{\delta u}K_j(0,\varepsilon_1)\| = \|\omega+(\frac{\varepsilon_1}{r})\tilde{f}_j(y)\| \le \|1+(\frac{\varepsilon_1}{r})\|$$

and $(\frac{\delta}{\delta\varepsilon_1})K_j{}_{|\varepsilon_1=0}$ is a Jacobi field along $x(u)= \mathrm{Exp}_y u\omega$ with the initial conditions $(\frac{\delta}{\delta\varepsilon_1})K_j(0,0) = 0$, $\frac{\delta^2}{\delta u\varepsilon_1}K_j(0,0) = \frac{1}{r}$.

Then, we get by Lemma 2.4,

(2.44) $$\frac{d}{du}\|\frac{\delta}{\delta\varepsilon_1}\tilde{\xi}_j|_{\varepsilon_1=0}\|$$

$$\leq 2\|R\|_{x(u)}\|\frac{\delta}{\delta u}K_j(u,0)\|_{x(u)}\|\frac{\delta}{\delta\varepsilon_1}K_j(u,0)\|_{x(u)}\|\tilde{\xi}_j\|_{x(u)}$$

$$\leq 2K_1' \exp K_1' u$$

with some constant $K_1' > 0$. Thus, we have

(2.45) $$\|\nabla_x P(x,y)\|^2 = \Sigma_{i,j=1}^m \|\nabla_{f_j(x)}\tilde{\xi}_i(x,y)\|_x^2$$

$$\leq 2K_1'' \exp K_1'' r, \quad r=d(x,y)$$

For the second inequality of (2.42) is easily obtained by (2.46).

<u>Proof of Proposition 2.5.</u> Now, use the same notations as in the proof of Lemma A.6. To prove (2.412.41), we shall succed to differentiate (2.10) covariantly. Let $X(\varepsilon_1,\ldots,\varepsilon_t)$ be a C^1-curve such that $X(0,\ldots,0)=0$, and $\frac{\partial X}{\partial\varepsilon_1}(0,\ldots,0) = f_{j_1}(x)$, $\frac{\partial X}{\partial\varepsilon_t}(0,\ldots,0) = f_{j_t}(x)$. Define $\tilde{\xi}_j(0,\varepsilon_1,\ldots,\varepsilon_t)$ samely as in (23), by

(2.47) $$\frac{\delta}{\delta u}\tilde{\xi}_j(u,\varepsilon_1,\ldots,\varepsilon_t) = 0,\ \tilde{\xi}_j(0,\varepsilon_1,\ldots,\varepsilon_t) = e_j(y)$$

along

$$K_j(u,\varepsilon_1,\ldots,\varepsilon_t) = \mathrm{Exp}_y\ u[\omega+(\frac{1}{r})(\varepsilon_1\tilde{f}_{i_1}(y)+\ldots+\varepsilon_t\tilde{f}_{i_t}(y))].$$

Differentiating (26) covariantly, we have by Ricci identity,

(2.49) $$\frac{\delta}{\delta u}\frac{\delta^t}{\delta\varepsilon_1\cdots\delta\varepsilon_t}\tilde{\xi}_j(u,0,\ldots,0)$$

$$=\Sigma\nabla^t R[(\frac{\delta}{\delta u})(\frac{\delta}{\delta\varepsilon})^{\alpha}K_j\ |_{\varepsilon_j=0}](\frac{\delta}{\delta\varepsilon})^{\beta}\tilde{\xi}_j(u,0,\ldots,0)$$

where $|\alpha|\leqq t$, $|\mu|\leqq t$, and $(\frac{\delta}{\delta\varepsilon})^{\alpha}=\frac{\delta^{|\alpha|}}{(\delta\varepsilon_1)^{\alpha_1}\cdots(\delta\varepsilon_t)^{\alpha_t}}$, where $\alpha=(\alpha_1,\ldots,\alpha_t)$. Then, we get inductively, if the estimate $\|(\frac{\delta}{\delta\varepsilon})^{\alpha}\tilde{\xi}_j(r,0,\ldots,0)\|\leqq K'_{\alpha}\exp K'_{\alpha}r$, $r=d(x,y)$ for $|\alpha|\leqq t-1$. So, we get by (2.49),

(2.50) $$\|(\frac{\delta}{\delta\varepsilon})^{\alpha}\tilde{\xi}_j(r,0,\ldots,0)\|\leqq K'_{\alpha}\exp K'_{\alpha}r,\ |\alpha|\leqq t,\ r=d(x,y).$$

Thus, we have easily the Proposition 2.5.

1.3 Basic properties of $H(\lambda;t,s)$

Recall the operator $H(\lambda;t,s)$ in (1.2). Using the notation as in §2, it can be written by the following :

(3.1) $H(\lambda;t,s)\xi(x)$

$$= (2\pi\lambda^{-1})^{-m/2}\int\tilde{\rho}(t,s;x,y)\ \exp -\lambda\frac{d^2(x,y)}{2(t-s)}\ P(x,y)\xi(y)d\mu_g(y),$$

for $\xi\varepsilon C_0^\infty(E)$.

In this section, we shall give some basic properties of (3.1), using the result in §2. First, we prove the part (a) in Theorem A.

<u>Proposition 3.1.</u> Let us assume that (M,g) satisfies (A.0)-(A.2) and we fix $T>0$ arbitrary. Then, the operator $H(\lambda;t,s)$ is stable, that is, there exists a positive constant $C_0 = C_0(\lambda;T)$ such that

$$\|H(\lambda;t,s)\|_{L^2(E)} \leq e^{C_0(t-s)}\ \|\xi\|_{L^2(E)}, \tag{3.2}$$

for $0\leq s<t<T$ and $\xi\varepsilon\ C_0^\infty(E)$

Before proving above, we recall the useful lemma in [18]. Set the function $\tilde{h}(\lambda;t,s;x,y)$ by

$$\tilde{h}(\lambda;t,s;x,y) = (2\pi\lambda^{-1})^{-m/2}\tilde{\rho}(t,s;x,y)\ e^{-\lambda S(t,s;x,y)}.$$

By using Proposition 2.3 and the same computation as in Proposition 2.1 in [18], we get

Lemma 3.2. Under the same assumptions as in Proposition 3.1, there exists a positive constant $C_0' = C_0'(\lambda;T)$ such that

$$(3.3)\qquad \begin{aligned}\int_M \|\tilde{h}(\lambda;t,s;x,y)\|d\mu_g(y) &\leq (1+C_0'(t-s)),\\ \int_M \|h(\lambda;t,s;x,y)\|d\mu_g(x) &\leq (1+C_0'(t-s)).\end{aligned}$$

Proof of Proposition 3.1. The kernel function $H(\lambda;t,s;x,y)$ of $H(\lambda;t,s)$ is given by $H(\lambda;t,s;x,y) = \tilde{h}(\lambda;t,s;x,y)P(x,y)$. Thus, for $\xi \in C_0^\infty(E)$, we have

$$(3.4)\qquad \|H(\lambda;t,s)\xi(x)\|_x \leq \int_M \tilde{h}(\lambda;t,s;x,y)\|P(x,y)\xi(y)\|_x d\mu_g(y)$$

$$= \int_M \tilde{h}(\lambda;t,s;x,y)\|\xi(y)\|_y d\mu_g(y),$$

because of Lemma 2.1. Thus, by Schwartz' inequality and Lemma 3.2, we have

$$\|H(\lambda;t,s)\xi(x)\|_x^2$$

$$\leq [\int_M \tilde{h}(\lambda;t,s;x,y)^{\frac{1}{2}}\tilde{h}(\lambda;t,ss;x,y)^{\frac{1}{2}}\|\xi(y)\|_y d\mu_g(y)]^2$$

$$\leq (1+C_0'(t-s))\int_M \tilde{h}(\lambda;t,s;x,y)\|\xi(y)\|_y^2 d\mu_g(y).$$

Then, we get by Fubini's theorem,

$$\|H(\lambda' t,s)\xi\|_{L^2(E)}^2 = \int_M \|H(\lambda;t,s;)\xi(x)\|_x^2 d\mu_g(x).$$

$$\leq (1+C_0' (t-s))\int_M [\int_M \tilde{h}(\lambda;t,s;x,y)d\mu_g(x)\|\xi(y)\|_y^2]d\mu_g(y)$$

$$\leq (1+C_0' (t-s))^2\|\xi\|^2_{L^2(E)} ,$$

which implies (4.2).~

Next, we shall study the behavior of $H(\lambda;t,s)$ as $t\to s$ and $s\to t$. Namely, we have

<u>Proposition 3.3.</u> Under the same assumptions as in Proposition 4.1, we have for $0\leq s<t<T$,

$$\text{(3.5)}\qquad \begin{aligned} &\lim_{t\downarrow s} \|H(\lambda;t,s)\xi-\xi\|_{L^2(E)} = 0 , \\ &\lim_{s\uparrow t} \|H(\lambda;t,s)\xi-\xi\|_{L^2(E)} = 0 , \end{aligned}$$

for any $\xi\in L^2(E)$. Therefore, for fixed $s\geq 0$, putting $H(\lambda;s,s)$ = the identity transformation, we have the mapping from $t\in [s,T]$ to $H(\lambda;t,s)\xi\in L^2(E)$, strongly continuous in t for each $\xi\in L^2(E)$. Also, similiar statement in s as above holds.

<u>Proof</u>. By proposition 3.1, it is sufficient to prove (3.5) for each $\xi\in C_0^\infty(E)$. We define a cut off function $\chi\in C_0^\infty(E)$. as $\chi(x) = 1$ if $d(x,\mathrm{supp}\xi)\leq 2$ and $= 0$ if $d(x,\mathrm{supp}\xi)\geq 3$. We shall show the following :

$$\text{(3.6)}\qquad \lim_{t\downarrow s} \|H_1(\lambda;t,s)\xi-\xi\|_{L^2(E)} = 0,\quad \lim_{s\uparrow t} \|H_1(\lambda;t,s)\xi-\xi\|_{L^2(E)} = 0,$$

$$\text{(3.7)}\qquad \lim_{t\downarrow s} \|H_2(\lambda;t,s)\xi\|_{L^2(E)} = 0,\quad \lim_{s\uparrow t} \|H_2(\lambda;t,s)\xi\|_{L^2(E)} = 0,$$

where $H_1(\lambda;t,s)\xi(x)=\chi(x)H(\lambda;t,s)\xi(x)$ and $H_2(\lambda;t,s)\xi(x)$ $= (1-\chi(x))H(\lambda;t,s)\xi(x)$.

Proof of (3.6). Putting $y=\mathrm{Exp}_x\, r\omega$, $\omega\varepsilon\, S_xM$, we get

$$\tilde{\xi}(x,y) = \xi(x) + \xi_1(x;r\omega) ,$$
$$\theta^{\frac{1}{2}} = 1 + \theta_1(x;r\omega) ,$$

where $\tilde{\xi}(x,y) = P(x,y)\xi(y)\ \varepsilon E_x$. Then, by mean value theorem, we have

$$\tilde{\xi}_1(x;r\omega) = \int_0^r \frac{\partial}{\partial u}\tilde{\xi}(x,x(u))du = \int_0^r (\nabla_y\xi(x,x(u),\dot{x}(u))_{x(u)}\ du ,$$

$$\theta_1(x;r\omega) = \int_0^r \frac{d}{du}\theta^{\frac{1}{2}}(x,x(u))du$$

$$= \int_0^r (\tfrac{1}{2})\theta^{-\frac{1}{2}}(x,x(u))\ (\frac{d}{du})\theta(x,x(u))du,$$

where $x(u)= \mathrm{Exp}_x u\omega$, So, we get

$$H_1(\lambda;t,s)u(x)$$

$$= \chi(x)(2\pi\lambda^{-1})^{-m/2}\int_0^\infty\int_{S_xM}[\xi(x) + \tilde{H}_{(x;r^{\omega})}]r^{m-1}\exp -\frac{r^2}{2t}drd\omega ,$$

where $H_1(x;r\omega)=\tilde{\xi}_1(x;r\omega)+\theta_1(x;r\omega)\tilde{\xi}(x;r\omega)$. Using Lemma 2.2, (ii) and Proposition 2.4, we get

$$\|\tilde{H}_1(x;r\omega)\| \leq \|\tilde{\xi}_1(x;r\omega)\|_x + \|\theta_1(x;r\omega)\|\times\|\tilde{\xi}(x;r\omega)\|_x$$

$$\leq C_1' \; r \exp Kr[\sup_{x\in M} \|\nabla\xi\|_x + \sup_{x\in M} \|\xi\|_x] ,$$

$r = d(x,y)$, for some constant $K > 0$. Remarking $(2\pi\lambda^{-1})^{-m/2}$ $\int_0^\infty\int_{S^{m-1}} r^{m-1} e^{-\lambda r^2/2t} dr d\omega = 1$, we get

$$\|H_1(\lambda;t,s)\xi(x) - \xi(x)\|_x$$

$$\leq C_1' \; \chi(x) \; \mathrm{vol}(S^{m-1})(t-s)^{1/2} \sup_{x\in M} [|\nabla\xi|_x + |\xi|_x]$$

$$\times\int_0^\infty r^m \exp[-\lambda\frac{r^2}{2t} + K(t-s)^{1/2} r] dr .$$

Therefore, for $0\leq s<t<T$, there exists a constant $C_1'' = C_1''(\lambda;T,\xi)$ depending on the support of ξ such that

$$(3.8) \qquad \|H_1(\lambda;t,s)\xi - \xi\|_{L^2(E)} \leq C_1''(t-s)^{1/2} \sup_{x\in M} [\|\nabla\xi\|_x + \|\xi\|_x] ,$$

which implies (3.6).

<u>Proof of (3.7).</u> Define other cut off function $\varphi(y)$ as $\varphi(y) = 1$ on $d(y,\mathrm{supp}\xi)\leq 1$ and $= 0$ on $d(x,\mathrm{supp}\xi)\geq 3/2$. Remark that $\varphi(y)\xi(y) = \xi(y)$ and we have

$$H_2(\lambda;t,s)\xi(x) = \int_M F(\lambda;t,s;x,y)P(x,y)\xi(y)d\mu_g(y),$$

where

$$F(\lambda;t,s;x,y)$$

$$= (2\pi\lambda^{-1})^{-m/2}(1 - \chi(x))\varphi(y)\tilde{\rho}(t,s;x,y)\exp -[\frac{\lambda d^2(x,y)}{2(t-s)}] .$$

Since $F(t,s;x,y)=0$ for $d(x,y)\leqq 1/2$, we have $F(t,s;x,y) \leqq (2\pi\lambda^{-1})^{-m/2}(1 - \chi(x))\ \varphi(y)\rho(x,y)\ \exp -\lambda d^2(x,y)$, where $\rho(x,y)$ is defined by (2.5) for small $0\leqq s<t$. Moreover, $|F(t,s;x,y)| \to 0$, as $t\downarrow s$ $s\uparrow t$ for each $(x,y)\ \varepsilon\ M\times M$. So, we have , by Lesbegue's dominated convergence and the similar argument in the proof of Proposition 3.1, we get (3.7). Then, we get Proposition 3.3.

For later use, we shall give the some properties about $H(\lambda;t,s;x,y)$. Denote by $H(\lambda;t,s;x,y)$ the kernel function of $H(\lambda;t,s)$, which can be consider as the section of $E \boxtimes E^*$. Let $\xi(\tau,y)$ be a mapping from $[s,t] \times M$ to E such that for each fixed $\xi(\tau,y)$ is the section of E, which will be called a <u>parametrized section</u> of E. Given, any $y\ \varepsilon\ M$, denote by $\#_y$ the interior product between E_y and E^*_y

<u>Lemma 3.4</u> Let $\xi(\tau,y)$ be a continuous, bounded parametrized section defined for $\tau\varepsilon\ [s,t]$, $0\leqq s<t<T$ and put,

$$\xi(t,\tau;x) = \int_M H(\lambda;t,s;x,y)\#_y\xi(\tau,y)d\mu_g(y) \quad , \quad s<\tau<t \ .$$

Then, we have

$$(3.9) \qquad \frac{\partial\xi}{\partial t} = \int_M \frac{\partial}{\partial t}H(\lambda;t,s;x,y)\#_y\xi(\tau,y)\ d\mu_g(y) \ ,$$

$$(3.10) \qquad \nabla^j_x\xi = \int_M \nabla^j_x H(\lambda;t,s;x,y)\#_y\xi(\tau,y)\ d\mu_g(y) \quad ,$$

$$(3.11) \qquad \lim_{t\geqq\tau\downarrow s} \xi(t,\tau,x) = \xi(s;x) \ , \quad \lim_{s\geqq\tau\uparrow t} \xi(t,\tau,x)=\xi(t,x) \ .$$

Proof is obvious by the same computation as in Proposition 3.3. So, we omit here.

Similarly, let $\xi^*(\tau,x)$ be a mapping from $[s,t] \times M$ to E^* such that for each fixed $\xi^*(\tau,y)$ is a section of E^*, which is called also a <u>parametrized section of E*</u>. Now, we have

<u>Lemma 3.5.</u> For the parametrized section ξ^* of E^*, we have,

$$(3.12) \qquad \lim_{s<\tau<t,\, s\uparrow t} \int_M \xi^*(\tau,x) \underset{x}{\#} H(\lambda,t,s;x,y)\,d\mu_g(x) = \xi^*(t,y) \quad ,$$

$$(3.13) \qquad \lim_{s<\tau<t,\, t\downarrow s} \int_M \xi^*(\tau,x) \underset{x}{\#} H(\lambda,t,s;x,y)\,d\mu_g(x) = \xi^*(s,y) \quad .$$

1.4 Convergence of the product integral in the operator norm.

In this section, we shall prove the parts (c) and (d) in Theorem A. Take T>0 arbitrary and fixed it. We divide a closed interval [s,t], $0 \leq s<t<T$ into subintervals, i.e.

$$(4.1) \qquad \sigma_N \; ; \quad s=t_0<t_1<\dots<t_{N-1}<t_N=t \quad ,$$
$$t_j=s+\left(\frac{j(t-s)}{N}\right), \quad j=0,\dots,N \; .$$

And we define the operator

$$(4.2) \qquad H(\lambda;\sigma_N|t,s) = H(\lambda;t,t_{N-1})\dots\cdot H(\lambda;t_1,s)$$

Now, we prove the following , which is (d) in Theorem A:

<u>Proposition 4.1.</u> Assume that (M,g) satisfies (A.1)-(A.3). Then, there exists a C^0 semi-group $H(\lambda;t)$ $t>0$, on $L^2(E)$ such that, for any $t>0$, we have

$$(4.3) \qquad \lim_{N\to\infty}\|H(\lambda;t) - H(\lambda;\sigma_N)\|_{(L^2(E))} = 0 \; ,$$

where $\| \quad \|_{(L^2(E))}$ is the operator norm in the space of all bounded operators on $L^2(E)$. Moreover, we have

$$(4.4) \qquad \|H(\lambda;t) - H(\lambda;\Delta|t,0)\|_{(L^2(E))} \leq C_2 t N^{-1/2} \exp C_2 t^{1/2} \quad ,$$

for some constant $C_2 = C_2(\lambda;T)$.

To prove the above proposition, we need several steps as below. First, recall the kernel function $H(\lambda;t,s;x,y)$ of

$H(\lambda;t,s)$,i.e.

(4.5) $\quad H(\lambda;t,s,x,y)$

$$= (2\pi\lambda^{-1})^{-m/2}\tilde{\rho}(t,s;x,y)\ e^{-\lambda S(t,s;x,y)}P(x,y).$$

By the direct computation using the Hamilton-Jacobi equation for $S(t,s;x,y)$ and the continuity equation for $\rho(t,s;x,y)$, (Cf. Lemma 1.1 and Lemma 1.5 in [18]), we have the following :

<u>Lemma 4.2</u> The above $H(\lambda;t,s;x,y)$ satisfies the following :

(4.6) $\quad \frac{\partial}{\partial t} H(\lambda;t,s;x,y)$

$$= (\frac{\lambda^{-1}}{2})\Delta_y H(\lambda;t,s;x,y)$$

$$- [-(\frac{\lambda^{-1}}{2})\Delta_y\tilde{\rho}(t,x,;x,y)\ e^{-\lambda S(t,s;x,y)}P(x,y).$$

$$+ \ e^{-\lambda S(t,s;x,y)}(\nabla_y\rho(t,s;x,y),\nabla_y P(x,y))_y$$

$$- (\frac{\lambda^{-1}}{2})\tilde{\rho}(t,x,;x,y)\ e^{-\lambda S(t,s;x,y)}\Delta_y P(x,y)].$$

and

(4.7) $\quad \frac{\partial}{\partial s} H(\lambda;t,s;x,y)$

$$= -(\frac{\lambda^{-1}}{2})\ \Delta_x H(\lambda;t,s;x,y)$$

$$+ (2\pi\lambda^{-1})^{-m/2}[(\frac{\lambda^{-1}}{2})\Delta_x\tilde{\rho}(t,x,;x,y)\ e^{-\lambda S(t,s;x,y)}P(x,y).$$

$$- e^{-\lambda S(t,s;x,y)}(\nabla_y\tilde{\rho}(t,s;x,y),\nabla_x P(x,y))_x$$

$$+ (\frac{\lambda^{-1}}{2})\tilde{\rho}(t,x,;x,y)\ e^{-\lambda S(t,s;x,y)}\Delta_x P(x,y)],$$

because of Lemma 2.3 (iv).

For $\xi\varepsilon\ C_0^\infty(E)$, and $0\leq s<t'<t$, we may write

$$(4.8)\qquad H(\lambda;t+t',s)\xi(x) - H(\lambda,t+t',t')H(\lambda;t's)\xi(x)$$

$$= \int_M H(\lambda;t,t',s;x,y)\xi(y)d\mu_g(y),$$

where

$$H(\lambda;t,t',s;x,y)$$

$$= H(\lambda;t+t',s;x,y)$$

$$- \int_M H(\lambda;t+t',t';x,z)\underset{z}{\#}H(\lambda;t',s;z,x)d\mu_g(z),$$

where $\underset{z}{\#}$ means that the interior product between E_z^* and E_z. Since $H(\lambda;t,s;x,y)$ has a singularity at $t=s$, we define, for positive ε,

$$(4.9)\qquad H^\varepsilon(\lambda;t,t',s;x,y)$$

$$= - \int_{s+\varepsilon}^{t} \frac{d}{d\sigma} \int_M H(\lambda;t+t',\sigma;x,z)\underset{z}{\#}H(\lambda;\sigma,s,x;z,y)d\mu_g(y),$$

which satisfies $\lim_{\varepsilon\to 0} H^{\varepsilon}(\lambda;t,t',s;x,y)=H(\lambda;t,t',s;x,y)$ for any (t,t',s,x,y) and $x=y$. Exchanging $\frac{d}{d\sigma}$ and the integral in (4.9), we have, by Lemma 4.1,

$$(4.10)\qquad H^{\varepsilon}(\lambda;t,t',s;x,y)$$

$$= -\int_{s+\varepsilon}^{t'} (2\pi\lambda^{-1}(t+t'-\sigma))^{-m/2}(2\pi\lambda^{-1}(\sigma-s))^{-m/2}$$

$$\times\Sigma_{i=1}^{3} \int_M h_i^{\varepsilon}(\lambda;t,t',s;x,y,z)d\mu_g(z)d\sigma$$

$$(4.11)\qquad h_1^{\varepsilon}(\lambda;t,t',s;x,y,z)$$

$$= (\frac{\lambda^{-1}}{2})[\Delta_z\rho(x,z) - \Delta_z(y,z)]$$

$$\times\exp -\lambda[S(t+t',\sigma,x,z)+S(\sigma,s,z,y)]P(x,z)\underset{z}{\#}P(z,y)$$

$$(4.12)\qquad h_2^{\varepsilon}(\lambda;t,t',s,\sigma,x,y,z)$$

$$= (\frac{\lambda^{-1}}{2})\exp -\lambda[S(t+t',\sigma;x,z)+S(\sigma,s;z,y)]$$

$$\times(\nabla_z\rho(x,z),\nabla_z P(x,z))_z P(z,y)\underset{z}{\#}P(x,z)$$

$$- P(x,z)\underset{z}{\#}(\nabla_z\rho(y,z),\nabla_z(y,z))_z$$

$$(4.13)\qquad h_3^{\varepsilon}(\lambda;t,t',s,\sigma;x,y,z)$$

$$= \exp -\lambda[S(t+t',\sigma;x,z)+S(\sigma,s;z,y)]\rho(x,z)\rho(z,y)$$

$$\times[\Delta_z P(x,z)\underset{z}{\#}P(z,y) - P(x,z)\underset{z}{\#}\Delta_z P(z,y)],$$

Because of Lemma 2.3,(iii), where $\rho(x,y)$ is defined in (2.5).

<u>Lemma 4.4.</u> For arbitrary $T>0$, there exists a positive constant $C'_2=C'_2(\lambda;T)$ such that

$$\lim_{\varepsilon\downarrow 0}\int_M \|H^{\varepsilon}(\lambda;t,t',s;x,y)\|_{(x,y)}d\mu_g(y)$$

$$\leq C_2' \; [(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}] \quad ,$$

(4.14)

$$\lim_{\varepsilon\downarrow 0}\int_M \|H^{\varepsilon}(\lambda;t,t',s;x,y)\|_{(x,y)}d\mu g(x)$$

$$\leq C_2' \; [(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}].$$

Consider each term $h_i^{\varepsilon}(\lambda;t,t',s,\sigma;x,y,z)$, $i=1,2,3$, in (4.11)-(4.13). First, remark

$$|\Delta_z\rho(x,z)-\Delta_z\rho(y,z)|$$

$$\leq |\Delta_z\rho(x,z)-\Delta_z\rho(x,z)_{|z=x}| + |\Delta_z\rho(x,z)_{|z=x}-\Delta_z\rho(y,z)_{|z=y}|$$

$$+ |\Delta_z\rho(y,z)-\Delta_z\rho(y,z)_{|z=y}|$$

Recall that $\Delta_\rho(x,z)_{|z=x}=(\frac{1}{6})\mathrm{Scal}_g(x)$ (Cf. See [5] and [18]).

Taking a geodesic $x(u)=\mathrm{Exp}_x u\omega'$, $z=\mathrm{Exp}_x r\omega'$, $\omega'\varepsilon S_xM$, we have

$$|\Delta_z\rho(x,z)-\Delta_\rho(x,z)_{|z=x}| \leqq \int_0^r(\nabla_z\Delta_z\rho(x,x(u)),\dot{x}(u))_{x(u)}du$$

$$\leqq kr' \exp kr' \quad , \quad r'=d(x,z) \quad .$$

and

$$|\Delta_z\rho(x,z)_{|z=x}-\Delta_\rho(y,z)_{|z=y}| = (\tfrac{1}{6})|\mathrm{Scal}_g(x)-\mathrm{Scal}_g(y)|$$

$$\leqq kr \exp kr \quad , \quad r=d(x,y).$$

Thus, we have

$$\|\Delta_z\rho(x,z)-\Delta_z\rho(y,z)\|$$

$$\leqq k_3[d(x,z) \exp kd(x,z) + d(y,z) \exp kd(y,z)]$$

for some constants k_3 and k. Therefore, we get

(4.15) $$\|h_1^\varepsilon(\lambda;t,t',s,\sigma:x,y,z)\|_{(x,y)}$$

$$\leqq(\frac{\lambda^{-1}}{2})m\ k_3[d(x,z)\exp kd(x,z) + d(y,z)\exp kd(y,z)]$$

$$\times e^{-\lambda[S(t+t',\sigma;x,z)+S(\sigma,s;z,y)]},$$

for some constant $k_3^{'} > 0$. Then, we have

$$\Sigma_{i=1}^3 \int_M\int_M \|h_i^\varepsilon(\lambda;t,t',s,\sigma;x,y,z)\|_{(x,y)}d\mu_g(z)d\mu_{(y)}$$

$$\leq (2\pi\lambda^{-1}(t+t'-\sigma))^{-m/2}(2\pi\lambda^{-1}(s-\sigma))^{-m/2}k_4$$

$$\times[\int_M\int_M d(x,z)\ e^{-\lambda[S(t+t'-\sigma;x,z)+S(\sigma,s;z,y)+kd(y,z)]}$$

$$d\mu_g(z)d\mu_g(y)$$

and

$$(4.16)\qquad \int_M\int_M d(y,z)\ \exp{-\lambda[S(t+t'-\sigma;x,z)+S(\sigma,s;x,z)+kd(x,z)]}$$
$$d\mu_g(z)d\mu_g(y)$$

$$\leq C_2''(\lambda,T)[(t+t'-\sigma)^{1/2} + (\sigma-s)^{1/2}]$$

because of $S(t,s;x,z)= d^2(x,z)/2(t-s)$. Therefore, we get

$$(4.17)\qquad \int_M |H\varepsilon(\lambda;t,t',s;x,y)|_{(x,y)} d\mu_g(y)$$

$$\leq C_2''(\lambda,T)\int_\varepsilon^s [(t+t'-\sigma)^{1/2} + (\sigma-s)^{1/2}]d\sigma \quad .$$

Taking a limit as $\varepsilon\downarrow 0$, we get the first inequality of (4.14). By a similar computation, we have the Lemma 4.3. ~

For fixed t,t',s and $x\varepsilon M$, we see that $\|H^\varepsilon(t,t',s;x,y)\|_{(x,y)}$ is bounded away by integral function in y-variables, $\Sigma_{i=1}^3 \int_0^s h_i(\lambda;t,t',s,\sigma;x,y,z)d\mu_g(z)d\sigma$. Thus, using Lebesgue's dominated convergence theorem and a.e.,

$$\lim_{\varepsilon\to 0} H^\varepsilon(\lambda;t,t',s,\sigma;x,y) = H(\lambda;t,t's,\sigma;x,y) \quad .$$

we have

$$(4.18)\qquad \int_M \|H^{\varepsilon}(\lambda;t,t',s,\sigma;x,y)\|_{(x,y)}d\mu_g(y)$$

$$\leq C_4[(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}]$$

because of Lemma 4.3. By the same computation, we have

$$(4.19)\qquad \int_M \|H^{\varepsilon}(\lambda;t,t',s,\sigma;x,y)\|_{(x,y)}d\mu_g(x)$$

$$\leq C_4[(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}]$$

So , we have shown (c) of Theorem A, by using (4.18)-(4.19), and the same computation as in the proof of Proposition 3.1, i.e.

<u>Lemma 4.4.</u> For any t,t',s, $0\leq s<t'<t<T$, $t+t'<T$ and $\xi\varepsilon L^2(E)$, we have

$$(4.20)\qquad \|H(\lambda;t+t',s)\xi - H(\lambda;t',s)\xi\|_{L^2(E)}$$

$$\leq C_2[(t+t'-s)^{3/2} - t^{3/2} + (t'-s)^{3/2}]\|\xi\|_{L^2(E)} .$$

<u>Lemma 4.5.</u> Under the same assumptions as in Proposition 5.1, $H(\lambda;\sigma_N|t,0)$ forms a Caushy sequence in $\mathcal{B}(L^2(E))$ in operator norm, uniformly in $t\varepsilon$ $[0,T)$, where $\mathcal{B}(L^2(E))$ denotes the space of bounded linear operator in $L^2(E)$ with the operator norm. Moreover, its limit $H(\lambda;t)$ satisfies estimate (4.4)

<u>Proof</u>. We follow the similar computations as in Lemma 3.6 in [18], which now we recall it. As

(4.21) $$H(\lambda;t,s) - H(\lambda;\sigma_N|t,s)$$

$$= \Sigma_{j=0}^{N-1} H(\lambda;t,t_j)$$

$$- H(\lambda;t_j,t_{j-1})H(\lambda;t_{j-1},t_{j-2})$$

$$\times H(\lambda;t_{j-2},t_{j-3}) \ldots\ldots H(\lambda;t_1,0)$$

we get

$$\|H(\lambda;t,s)\xi - H(\lambda;\sigma_N|t,s)\xi\|_{L^2(E)}$$

$$\leq [C_1 \exp C_0(t-s)][(t-s)^{\frac{3}{2}} + (N-1)(\frac{(t-s)}{N})^{\frac{3}{2}}]\|\xi\|_{L^2(E)} ,$$

by Lemma 4.4 and Proposition 3.1. Also, we have for large integers N,M,

$$\|H(\lambda;\sigma_N|t,0)\xi - H(\lambda;\sigma_{NM}|t,0)\xi\|_{L^2(E)}$$

$$\leq \Sigma_{j=0}^{N-1} \|[H(\lambda;t,\frac{(N-1)t}{N}))\ldots H(\lambda;\frac{(N-j)t}{N},\frac{(N-j-1)t}{N})]$$

$$\times[H(\lambda;\frac{(N-j-1)t}{N},\frac{(N-j-2)t}{N}) - H(\lambda;\sigma_M|\frac{(N-j-1)t}{N},\frac{(N-j-2)t}{N})]$$

$$\times H(\lambda;\Delta_{jM}|\frac{(N-j-2)t}{N},0)\xi\|_{L^2(E)}$$

$$\leq [C_1 \exp C^0T]t^{\frac{3}{2}}(N^{-\frac{1}{2}} + (NM)^{-\frac{1}{2}}\|\xi\|_{L^2(E)}$$

by (4.21). Therefore, we get

(4.22) $$\|H(\lambda;\sigma_N|t,0)\xi - H(\lambda;\sigma_M|t,0)\xi\|_{L^2(E)}$$

$$\leq \|H(\lambda;\sigma_M|t,0)\xi - H(\lambda;\sigma_{NM}|t,0)\xi\|_{L^2(E)}$$

$$+ \|H(\lambda;\sigma_M|t,0)\xi - H(\lambda;\sigma_{MN}|t,o)\|_{L^2(E)}$$

$$+ \|H(\lambda;\sigma_{NM}|t,0)\xi - H(\lambda;\sigma_N|t,0)\xi\|_{L^2(E)}$$

Thus, $\{H(\lambda;\sigma_N|t,0)\}_N$ is a Caushy sequence uniformly in $t \in [t,T)$, in the operator norm. Therefore it converges to a limit $H(\lambda;t)$. Letting M tend to ∞ in (4.22), we get (4.4).

Remark. By a slight modification of the above proof, we can generalize Proposition 4.1 for arbitrary subdivision of $[0,t]$ (Cf. See [13] and [18]).

1.5 Computation of the infinitesimal generator.

It is easily seen that $H(\lambda;t)$ given in §5 is C^0 -semi group. Therefore, to finish the proof of Theorem 1.2, we only compute the infinitesimal generator of $H(\lambda;t)$. Namely, we get

<u>Proposition 5.1.</u> Assume that (M,g) satisfies (A.0)-(A.2). then, for any $\xi \varepsilon C_0^\infty(E)$, we have

(5.1) $$\frac{\partial}{\partial t} H(\lambda;t)\xi(x)$$

$$= [(\frac{\lambda}{2})(\Delta_L)_x - (\frac{\lambda}{12})\mathrm{Scal}_g(x)]H(\lambda;t)\xi(x),$$

where $(\Delta_L)_x$ is the rough Laplacian, defined by g (Cf. §1).

We shall prepare some lemmas to prove the above.

<u>Lemma 5.2</u> For any $\xi \varepsilon C_0^\infty(E)$, we have

$$\frac{\partial}{\partial t} H(\lambda;t)\xi(x)_{|t=0} = \frac{\partial}{\partial t}H(\lambda;t,0)\xi(x)_{|t=0}$$

<u>Proof.</u> For each N, we have

$$H(\lambda;t)\xi(x) - \xi(x)$$

$$= [H(\lambda;t) - H(\lambda:\sigma_N|t,0)]\xi(x) - [H(\lambda;\Delta|t,0) - I]\xi(x) \quad ,$$

for subdivision $\sigma_N:0=t_0< \ldots <t_N=t,\ t_j=jt/N$. By proposition 4.1, we remark

(5.2) $$\|H(\lambda;t) - H(\lambda;\sigma_N|t,0)\|_{L^2(E)} \leqq C_a t\ N^{\frac{1}{2}} \exp C_5 t^{1/2}.$$

On the other hand, we get

$$H(\lambda;\sigma_N|t,0)\xi(x) - \xi(x)$$

$$= \Sigma_{j=0}^{N-1} H(\lambda;t,\frac{(N-1)t}{N})\ldots H(\lambda;\frac{(N-j+1)t}{N},\frac{(N-j)t}{N})$$

$$\times[H(\lambda;\frac{(N-1)t}{N}, \frac{(N-j-1)t}{N}) - I]\xi(x)$$

Therefore, we have

$$(5.3) \qquad (\frac{1}{t})[H(\lambda;\sigma_N|t,0) - I]\xi(x)$$

$$= \Sigma_{j=0}^{N-1} H(\lambda;t,\frac{(N-1)t}{N})\ldots H(\lambda;\frac{(N-j+1)t}{N},\frac{(N-1)t}{N})$$

$$\times(\frac{1}{N})[\frac{1}{\frac{t}{N}}(H(\lambda;\frac{(N-j)t}{N},\frac{(N-j-1)t}{N}) - I)\xi(x)$$

Since $H(\lambda;t,(N-j)t/N)\xi(x)\to\xi(x)$ as $N\to\infty$ by Lemma 4.4, we get

$$\frac{\partial}{\partial t} H(\lambda;\sigma_N|t,0)\xi(x)_{|t=0} = (\frac{1}{N})\Sigma_{j=0}^{N-1}\frac{\partial}{\partial t}H(\lambda;t,0)\xi(x)_{|t=0}$$

$$= \frac{\partial}{\partial t}H(\lambda;t,0)\xi(x)_{|t=0} \quad .$$

Combining with (5.2) and (5.3), we get Lemma 6.2 . ~

To prove Proposition 5.1, we only prove the following:

<u>Proposition 5.3.</u> Under the same assumptions as in Proposition 5.1, we have for any $\xi\in C_o^\infty(E)$ and $x\in M$,

$$(5.4) \qquad H(\lambda;t,0)\xi(x) - \xi(x) = t\lambda^{-1}A\xi(x) + tG(t;\xi) \quad ,$$

where $A\xi(x) = [(1/2)(\Delta_L)_x - (i/12)Scal_g(x)]\xi(x)$ and $G(t;\xi)$

satisfies

$$(5.5)\qquad \lim_{t\to 0}\|G(t;\xi)\|_{L^2(E)} = 0.$$

Proof. Recall (4.6), we get by using integration by parts,

$$(5.6)\qquad \frac{\partial}{\partial t}H(\lambda;t,0)\xi(x) - \left(\frac{\lambda^{-1}}{2}\right)H(\lambda;t,0)\Delta_L\xi(x)$$

$$= \lambda^{-1}(2\pi\lambda^{-1}t)^{-m/2}\int_M \exp -\lambda S(t,0;x,y)\ Q(x,y)\xi(y)d\mu_g(y)\quad ,$$

where

$$(5.7)\qquad Q(x,y)$$

$$= -\frac{1}{2}\Delta_y\rho(x,y)P(x,y) + (\Delta_y\rho(x,y),\Delta_y P(x,y))_y$$

$$+ \frac{1}{2}\rho(x,y)\Delta_y P(x,y)\ .$$

Therefore, we have , by Lemma 4.4,

$$(5.8)\qquad \lambda\widetilde{G}(t;\xi)$$

$$= \frac{\partial}{\partial t}H(\lambda;t,0)\xi(x) - \frac{1}{2}H(\lambda;t,0)\Delta_L\xi(x)$$

$$+ \frac{1}{12}Scal_g(x)\xi(x),$$

where

$$(5.9)\qquad \widetilde{G}(t;\xi)$$

$$= \frac{1}{2}[H(\lambda;t,0) - I]\xi(x)$$

$$-\frac{1}{2}(2\pi\lambda^{-1}t)^{-m/2}\int_M \exp -\lambda S(t,0;x,y)\Sigma_{i=1}^{3}\ g_i(x,y)\xi(y)d\mu_g(y) \quad ,$$

(5.10) $g_1(x,y) = [\Delta_y\rho(x,y) - \Delta_y\rho(x,y)_{|y=x}$,

(5.11) $g_2(x,y) = (\Delta_x\rho(x,y),\Delta_x P(x,y))_x$,

(5.12) $g_3(x,y) = \rho(x,y)\Delta_x P(x,y)$.

Using Proposition 3.3, we have

(5.13) $\|H(\lambda;t,0)\Delta_L\xi(x) - \Delta_L\xi(x)\|_{L^2(E)}$

$$\leq Ct^{\frac{1}{2}}\ \underset{x\in M}{p}\ [\|\Delta\xi\|_x+\|\xi\|_x] + o(t;\xi),$$

$$\lim_{t\to 0} o(t;\xi) = 0 \quad ,$$

Also, by Proposition 2.4 , we have

(5.14) $\|g_i(x,y)\|_{(x,y)} \leq K \exp Kd(x,y) \quad ,i=1,2,3$,

for any $x,y\in M$ with some constant $K>0$. Then, we get with some constant $C_6 = C_6(\lambda;T)$

(5.15) $\|G(t;\xi)\|_{L^2(E)} \leq C_6 t^{\frac{1}{2}} \exp Kt^{\frac{1}{2}}\|\xi\|_{L^2(E)}$.

Remarking $H(\lambda;t,o)\xi(x) - \xi(x) = \int_0^t (d/d\sigma)H(\lambda;t,0)\xi(x)d\sigma$, we have the desired results. ~

By Lemma 5.2 and Proposition 5.3, we get Proposition 5.1. So, throughout §4- §6, we get Theorem A completely.

Now, for later use in §6, we prepare the following : Let $\xi(\tau,y)$ be a parametrized section of E (Cf. Lemma 3.4), which satisfies the following conditions:

(i) $\xi(\tau,y)$ is Horder continuous in $[s,T)\times M$.

(ii) Given any closed interval $[s_1,t_1]$ $[s,T)$, $\xi(\tau,y)$ is bounded on $[s_1,t_1]\times M$.

(iii) For any $t\in[s,T)$,

$$\int_s^t d\tau\int_M \|\xi(\tau,y)\|_y d\mu_g(y)<+\infty, \quad \int_s^t d\tau\int_M \|\xi(\tau,y)\|_y d\mu_g(y)<+\infty.$$

Proposition 5.4 Assume that (M,g) satisfies (A.1)-(A.3). Let $\xi(\tau,y)$ be as above with the conditions (i)-(iii). Now, define xi$(t,\tau;x)$ and $\xi(t,x)$ by

$$\xi(t,\tau;x) = \int_M H(\lambda;t;x,y)\xi(\tau,y)d\mu_g(y),$$

$$\Xi(t,x) = \int_s^t \xi(t,\tau;x)d\tau \quad .$$

Then, there exists a positive constant C_8 depending only the closed interval $[s_1,t_1]$ such that

$$\text{(5.17)} \qquad \frac{\partial}{\partial t}\|\xi(t;\tau;x)\|_x \leq C_8(t-\tau)^{-(1-\frac{\delta}{2})} \quad ,s_1\leq\tau\leq t\leq t_1 \ .$$

where δ is the Horder exponent of ξ at (t,x). Also, in (5.17), the same inequality holds replacing $\frac{\partial}{\partial t}$ by ∇_x and Δ_x. Moreover, we have

$$(5.18) \qquad A\,\Xi(t,x) = \int_s^t A\xi(t,\tau;x)d\tau \ ,$$

$$(5.19) \qquad \frac{\partial}{\partial t}\,\Xi(t,x) = \xi(t,x) + \int_s^t \frac{\partial}{\partial t}\xi(t,\tau;x)d\tau \quad .$$

Proof. Given any $(t,x)\ \varepsilon[s,T)\times M$, let δ be Horder exponent of ξ at this point. Take a closed interval $[s_1,t_1]$ such that $s<s_1<t_1<T$. Then, there exists C_8' and $0<\delta<1$ such that $s_1<t-\delta$ and if $|t-\tau|<\delta$ and $d(x,y)<\delta$, then,

$$(5.20) \qquad \|P(x,y)\xi(\tau,y) - \xi(t,x)\|_x$$

$$\leq C_8' \ (|t-\tau|^{\delta} + d^{\delta}(x,y))$$

and for $t-\delta<\tau<t\leq t'$, we have

$$\int_M \frac{\partial}{\partial t}H(\lambda;t',\tau;x,y)\xi(\tau,y)d\mu_g(y) = I_1 + I_2 + I_3 \ ,$$

where

$$I_1 = \int_{d(x,y)<\delta} \frac{\partial}{\partial t'}\ H(\lambda;t',\tau,x,y)[\xi(\tau,y)-P(y,x)\xi(\tau,x)]d\mu_g(y) \ ,$$

$$I_2 = \int_{d(x,y)\leq\delta} \frac{\partial}{\partial t'}H(\lambda;t',\tau;x,y)[\xi(\tau,y) - P(y,x)\xi(\tau,x)]d\mu_g(y) \ ,$$

$$I_3 = H(\lambda;t',\tau;x,y)P(y,x)\xi(\tau,x)d\mu_g(y) \ .$$

Combining with the above , we have that there exists a constant C_8'', such that

$$\|I_1\|_x \leq C_8''(t'-\tau)^{-(1-\frac{\delta}{2})} ,$$

(5.21) $$\|I_2\|_x \leq C_8'' ,$$

$$\|I_3\|_x \leq C_8''(t'-\tau) ,$$

which implies if $t-\delta\leq t'$,

$$\|\frac{\partial}{\partial t'}\xi(\lambda;t,\tau;x)\|_x \leq C_8''' \quad (t'-\tau)^{-(1-\frac{\delta}{2})} ,$$

with some constant C_8'''. On the other hand, if $s_1\leq\tau\leq t-s$, then $t'-\tau\geq\delta>0$. Therefore, we see that $(\partial/\partial t')\xi(\lambda;t',\tau;x)$ is uniformly in (t',τ,x), because of the form $(\partial/\partial t)H(\lambda;t',\tau,x,y)$ and (ii). So, we have the estimate (5.17). Other estimates are obviously obtained. Now, by assumptions (ii) and (iii) of $\xi(t,x)$, we have, for some constant C_9,

$$\|\nabla_x\xi(\lambda;t,\tau;x)\|_x \leq C_9(t-\tau)^{-(1-\frac{\delta}{2})} ,$$

$$\|\Delta_x\xi(\lambda;t,\tau;x)\|_x \leq C_9(t-\tau)^{-(1-\frac{\delta}{2})} , \quad s_1\leq\tau<t<t_1 .$$

and

$$\|\nabla_x\xi(\lambda;t,\tau;x)\|_x \leq C_9\int_M \|\xi(\tau,y)\|_y d\mu_g(y) ,$$

$$\|\Delta_x\xi(\lambda;t,\tau;x)\|_x \leq C_9\int_M \|\xi(\tau,y)\|_y d\mu_g(y) , \quad s_1\leq\tau<t<t_1 .$$

Remarking $\int_s^t (t-\tau)^{-(1-\delta/2)} d\tau < +\infty$, we can interchange the operation **A** and the integration, we have (5.18). Similarly, we have (5.19).

As a direct consequence of Proposition 5.4 and Proposition 3.4, we have

<u>Corollary 5.5.</u> Let $\xi(\tau,x)$ and $\Xi(\tau,x)$ be as in Proposition 5.4. Then, we have

(5.22) $\quad \mathbf{A}\,\Xi(t,x)$

$$k = \int_s^t d\tau \int_M AH(\lambda;t,\tau;x,y)\xi(\tau,y)d\mu_g(y).$$

(5.23) $\quad \frac{\partial}{\partial t}\Xi(t,x)$

$$= \xi(t,x) + \int_s^t d\tau \int_M \frac{\partial}{\partial t} H(\lambda;t,s;x,y)\xi(\tau,y)d\mu_g(y) \quad .$$

1.6 Construction of the fundamental solution.

To prove Theorem 1.3, we shall construct the fundamental solution for the following parabolic equation:

(6.1) $\frac{\partial}{\partial t}\xi = \lambda^{-1}A\xi \quad , \xi(0,x) = \xi_0(x) \in C^\infty(E)$,

where $A = \frac{1}{2}\Delta_L - \frac{1}{12}\,Scal_g(x)$.

With a slight modification of a standard construction of a fundamental solution of parabolic equation for functions, we estimate it (Cf. See for example, Friedman [12]).

Now, let (M,g) satisfies the assumptions (A.0)-(A.2). Also, for simplicity, we denote by L the differential operator $-\frac{\partial}{\partial t} + \lambda^{-1}A$. Recall the approximate kernel function of $H(\lambda;t,s)$, $0 \leq s < t < T$, i.e.

$$H(\lambda;t,s,;x,y)$$

$$= (2\pi\lambda^{-1}(t-s))^{-m/2}\rho(x,y)\ e^{-\lambda S(t,s;x,y)}P(x,y) \ ,$$

which is considered as the section of $E \times E^*$. Put

(6.2) $$J_0(\lambda;t,s;x,y)$$

$$= LH(\lambda t,s;x,y) = -(\frac{\partial}{\partial t} - \lambda^{-1}A)H(\lambda;t,s;x,y) \ .$$

<u>Lemma 6.1.</u> For any $0 \leq s < t < T$ and $x,y \in M$, there exists a positive constant $M_0 = M_0(\lambda;T)$ such that for any $0 < \varepsilon < 1$, we have

(6.3) $$\|H(\lambda;t,s;x,y)\|_{(x,y)} \leq M_0(t-s)^{-m/2}\exp\frac{-\lambda d^2(x,y)}{2(t-s)}\ .$$

and

(6.4) $$\|J_0(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0M_1(\lambda;T,\varepsilon)(t-s)^{\frac{-m+1}{2}}\exp -[\frac{\lambda\varepsilon^* d^2(x,y)}{2(t-s)}]\ ,$$

$\varepsilon^* = 1-\varepsilon$, where $M_1(\lambda;T,\varepsilon) = C_7(\lambda;T)\varepsilon^{-1}\exp k_3\varepsilon^{-1}T$, for some positive constants $C_7=C_7(\lambda;T)$ and $k_3>0$.

Proof. By Lemma 2.3, and taking $M_0=(2\pi\lambda^{-1})^{-m/2}m^{1/2}$, we get (6.1). Computing (6.1) exactly, we have

$$J_0(\lambda;t,s;x,y)$$

$$= [2\pi\lambda^{-1}(t-s)]^{-m/2}\exp\frac{\lambda d^2(x,y)}{2(t-s)}$$

$$[(\nabla_x\rho(x,y),\nabla_x P(x,y))_x + \rho(x,y)\Delta_x P(x,y)$$

$$-\frac{1}{2}\Delta_x\rho(x,y)P(x,y) - \frac{1}{12}\mathrm{Scal}_g(x)]$$

Then, by Proposition 1.4 and 2.6, there exists a positive constant k_4 such that

(6.5) $$\|J_0(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0k_4(t-s)^{-m/2}d(x,y)\exp -[\frac{\lambda d^2(x,y)}{2(t-s)}-k_4 d(x,y)]\ .$$

Now, put the function F(r) by

$$F(r) = r \exp [k_4 r - \frac{\lambda\varepsilon d^2(x,y)}{2(t-s)}] \quad ,r>0.$$

Then, we get

$$F(r) \leq (2\pi\varepsilon)^{-1}(t-s)^{\frac{1}{2}}[k_4(t-s)^{\frac{1}{2}} + (k_4^2(t-s)+4\lambda\varepsilon)]^{\frac{1}{2}}$$

$$\times\exp (4\lambda\varepsilon)^{-1}[k_4^2(t-s)+k_4(t-s)^{\frac{1}{2}}k_4^2(t-s)+4\lambda\varepsilon]^{\frac{1}{2}}.$$

$$\leq C_7^{'} (\lambda;T)\varepsilon^{-1}(t-s)^{\frac{1}{2}}\exp k_3^{'} \varepsilon^{-1}T ,$$

for some constant $C_7^{'} = C_7^{'}(\lambda;T)$. Substituting (6.6) into (6.5), we get , for any $x,y \in M$,

(6.7) $\quad \|J_0(\lambda;t,s;x,y)\|_{(x,y)}$

$$\leq M_0 k_4 C_7^{'} \varepsilon^{-1}(t-s)^{\frac{-m+1}{2}} \exp k_3^{'} \varepsilon^{-1}T \exp \frac{-\lambda\varepsilon^* d^2(x,y)}{2(t-s)} .$$

So, by putting $M_1(\lambda;T,\varepsilon) = k_4 C_7^{'} \varepsilon^{-1}\exp k_3^{'} T$, we get Lemma 6.1. ~

Next, we put

(6.8) $$J_1(\lambda;t,s;x,y) = \int_s^t dz\int_M J_0(\lambda;t,z;x,z)\underset{z}{\#}J_0(\lambda;t,s;z,y)d\mu_g(z) ,$$

where $\underset{z}{\#}$ denotes the interior product between E_z and E_z^*.

<u>Lemma 6.2.</u> For any $0\leq s<t<T$, and $x,y\in M$, there exists a

positive constant $M_2=M_2(\lambda;T,\varepsilon)$ such that, for any $0<\varepsilon<1/2$, $0<\varepsilon<1/4$,

(6.9) $$\|J_1(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leqq M_0^2 M_2(\lambda;T,\varepsilon)^2(t-s)^{\frac{-m+2}{2}} B(\tfrac{3}{2}:\tfrac{3}{2})$$

$$\times\exp\left[-\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}\right].$$

where $\varepsilon^{**}=1-2\varepsilon$, and $M_2(\lambda;T,\varepsilon)=C_7(\lambda;T)\varepsilon^{-1}\exp k_3(\lambda)\varepsilon^{-1}T$.

<u>Proof.</u> First, we put

$$J_1(\lambda;t,s,\sigma;x,y) = \int_M J_0(\lambda;t,\sigma;x,z)\underset{z}{\#}J_0(\lambda;\sigma,s;z,y)d\mu_g(z) .$$

By the comparison theorem, we have $d^2(z,y)\geqq\|Z-Y\|^2$, where $z=\mathrm{Exp}_x Y$, and $z=\mathrm{Exp}_x Z$. Thus, we get

$$\|J_1(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leqq \int_M \|J_0(\lambda;t,\sigma;x,z)\|_{(x,z)}\|J_0(\lambda;\sigma,s;z,y)\|_{(z,y)}d\mu(z)$$

$$\leqq M_0^2 M_1(\lambda;T)^2(t-\sigma)^{\frac{-m+1}{2}}(\sigma-s)^{\frac{-m+1}{2}}$$

$$\times\int_{T_xM}\exp -\left[\frac{\lambda\varepsilon^*\|Z\|^2}{2(t-\sigma)}+\frac{\lambda\varepsilon^*\|Z-Y\|^2}{2(\sigma-s)} -k\ \|Z\|\right]dZ$$

with some constant k. Since

$$\frac{\|Z\|^2}{2(t-\sigma)} + \frac{\|Z-Y\|^2}{2(\sigma-s)}$$

$$= \frac{(t-s)}{2(t-\sigma)(\sigma-s)}\|Z - \frac{t-\sigma}{(t-s)}Y\|^2 + \frac{1}{2(t-s)}\|Y\|^2 \ ,$$

we have

$$(6.10) \qquad \|J_1(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leqq M_0^2 M_1(\lambda;T,\varepsilon)^2 (t-\sigma)^{\frac{-m+1}{2}} (\sigma-s)^{\frac{-m+1}{2}} \exp-[\frac{\lambda\varepsilon^* d^2(x,y)}{2(t-s)}]$$

$$\times \int_{T_xM} \exp-[\frac{(t-s)\lambda\varepsilon^*}{2(t-\sigma)(\sigma-s)}\|Z-\frac{t-\sigma}{t-s}Y\|^2 - k\|Z\|]dZ$$

$$\leqq M_0^2 M_1(\lambda;T,\varepsilon)^2 (2\lambda^{-1}\varepsilon^{*^{-1}})^{m/2}(t-s)^{-m/2}(t-\sigma)^{\frac{1}{2}}(\sigma-s)^{\frac{1}{2}}$$

$$\exp-[\frac{\lambda\varepsilon^* d^2(x,y)}{2(t-s)} - kd(x,y)]$$

$$\times \quad \int_{T_xM} \exp-[\|Z'\|^2 - k[\frac{2(t-\sigma)(\sigma-s)}{\lambda\varepsilon^*(t-s)}]^{\frac{1}{2}}\|Z'\|dZ',$$

because of $0<\frac{t-\sigma}{t-s}\leqq 1$.

Let $F_2(r)$ be a function on $[0,\infty)$ defined by

$$F_2(r) = -\frac{1}{2}r^2 + k[\frac{2(t-\sigma)(\sigma-s)}{\lambda\varepsilon^*(t-s)}]^{\frac{1}{2}} r.$$

Then, $F_2^{\cdot}(r)=0$ means that $r_0=k[2(t-\sigma)(\sigma-s)/\lambda\varepsilon^*(t-s)]^{1/2}$.
So, we get

$$(6.11) \qquad \|F_2(r)\| \leqq k^2\left[\frac{(t-\sigma)(\sigma-s)}{\lambda\varepsilon^*(t-s)}\right] \leqq \frac{k^2}{4\lambda\varepsilon^*}(t-s) \leqq \frac{Tk^2}{4\lambda\varepsilon^*}.$$

Substituting (6.11) into (6.10), we have

$$(6.12) \qquad \|J_1(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leqq M_0^2M_1^2(t-\sigma)^{\frac{1}{2}}(\sigma-s)^{\frac{1}{2}}(t-s)^{-\frac{m}{2}}(\lambda\varepsilon^*)^{-\frac{m}{2}}$$

$$\times \mathrm{vol}(S^{m-1})\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\exp\ \frac{(t-s)}{4\lambda\varepsilon^*}k^2$$

$$\times\int_M \exp -\left[\frac{\lambda\varepsilon^* d^2(x,y)}{2(t-s)}-kd(x,y)\right]$$

Choosing $M_2 = M_2(\lambda;T,\varepsilon) \geqq M_1(\lambda\varepsilon^*)^{-\frac{m}{2}}\ \mathrm{vol}(S^{m-1})\ (2\pi)^{\frac{1}{2}}\ \exp\ \frac{Tk^2}{2\lambda\varepsilon}$,
we have

$$\|J_1(\lambda;t,s,z;x,y)\|_{(x,y)}$$

$$\leqq M_0^2M_2(\lambda;T,\varepsilon)^2(t-s)^{-m/2}\exp -\left[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}\right]$$

$$\times\int_s^t (t-\sigma)^{1/2}(\sigma-s)^{1/2}d\sigma$$

$$\leqq M_2(\lambda;T,\varepsilon)^3(t-s)^{\frac{-m+2}{2}}\ B\left(\frac{3}{2}:\frac{3}{2}\right)\exp -\left[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}\right],$$

where $\varepsilon^{**} = 1-\varepsilon$, which gives Lemma 6.2.

Successively, we define

(6.13) $\quad J_n(\lambda;t,s;x,y)$

$$= \int_s^t d\sigma \int_M J_0(\lambda;t,\sigma;x,z) \underset{z}{\#} J_{n-1}(\lambda;\sigma,s;z,y)\, d\mu_g(z), \qquad (n\geq 1).$$

Now, we have the following :

Proposition 6.3. We have the following estimate, for any $0\leqq s<t,T$, and $x,y\varepsilon M$, for $0<\varepsilon<\frac{1}{3}$,

(6.14) $\quad \|J_n(\lambda;t,s;x,y)\|_{(x,y)}$

$$\leqq M_0^n M_2(\lambda;T,\varepsilon)^n (t-s)^{\frac{-m+n+1}{2}}$$

$$\times \Pi_{a=1}^n B\left(\frac{3}{2}:\frac{2+a}{2}\right) \exp-\left[\frac{\lambda\varepsilon^{**} d^2(x,y)}{2(t-s)}\right], \quad (n\geq 2)$$

where $\varepsilon^{**} = 1-2\varepsilon$.

Proof. It has been shown (6.14) for the case $n=1$. We assume (6.20) for the case $n-1\geqq 1$. Put for $0\leqq s<\sigma<t<T$,

(6.15) $\quad J_n(\lambda;t,s,\sigma;x,y)$

$$= \int_M J_0(\lambda;t,\sigma;x,z) \underset{z}{\#} J_{n-1}(\lambda;\sigma,s;z,y)\, d\mu_g(z).$$

By (6.5) and the assumption, we have

$\|J_n(\lambda;t,s,\sigma;x,y)\|_{(x,y)}$

$$\leqq M_0^n M_2^{n-1} (t-\sigma)^{\frac{-m+1}{2}} (\sigma-s)^{\frac{-m+n-1}{2}} \Pi_{a=1}^{n-1} B\left(\frac{3}{2}:\frac{a+2}{2}\right)$$

$$\times \exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}] \int_{T_xM} \exp -[\frac{\lambda\varepsilon^{**}(t-s)}{2(t-\sigma)(\sigma-s)}\|Z'\|^2]dZ'$$

$$\leq M_0^n M_2^n (t-s)^{-m/2}(t-\sigma)^{\frac{1}{2}}(\sigma-s)^{\frac{n+1}{2}}$$

$$\times \Pi_{a=1}^{n-1} B(\tfrac{1}{2}:\tfrac{a+2}{2}) \exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}].$$

Therefore, we have

$$\|J_n(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0^n M_2^n (t-s)^{-\frac{m}{2}} \Pi_{a=1}^{n-1} B(\tfrac{1}{2}:\tfrac{a+2}{2})$$

$$\times \exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}] \int_s^t (t-\sigma)^{\frac{1}{2}}(\sigma-s)^{\frac{n+1}{2}} d\sigma$$

$$\leq M_0^n M_2^n (t-s)^{\frac{-m+n+1}{2}} \Pi_{a=1}^{n} B(\tfrac{1}{2}:\tfrac{a+2}{2}) \exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}],$$

which gives Proposition 6.3. ~

Remark that

$$\Pi_{a=1}^{n} B(\tfrac{1}{2}:\tfrac{a+2}{2}) = \Pi_{a=1}^{n} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+2}{2})}{\Gamma(\frac{a+3}{2})} = \frac{2\Gamma(\frac{1}{2})^n\Gamma(\frac{3}{2})}{(n+1)\Gamma(\frac{n+1}{2})} .$$

Then, there exists a positive constant $M_3 = M_3(\lambda,T,\varepsilon)$ such that

(6.16) $\quad \Sigma_{n=0}^{\infty} \|J_n(\lambda;t,s;x,y)\|_{(x,y)}$

$$\leq M^0 M_3 (t-s)^{\frac{-m+1}{2}} \exp M_2 (t-s)^{\frac{1}{2}} \exp -[\frac{\lambda\varepsilon^{**} d^2(x,y)}{2(t-s)}].$$

Thus, on $\{(t,s)|0\leq s<t<T\}\times M\times M$, we can define a function

(6.17) $$K(\lambda;t,s;x,y) = \Sigma_{n=0}^{\infty} J_n(\lambda;t,s;x,y)$$

and for any $C>1$, on $\{(t,s)|0\leq s<t<T, C^{-1}\leq t-s\leq C\}\times M\times M$, the infinite sum of (6.17) is convergence uniformly on each compact set, and we have

(6.18) $$\|K(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0 M_3 (t-s)^{\frac{-m+1}{2}} \exp M_3 T^{\frac{1}{2}} \exp -[\frac{\lambda\varepsilon^{**} d^2(x,y)}{2(t-s)}].$$

Moreover, by direct computation, we get

<u>Lemma 6.4.</u> Let $J_n(\lambda;t,s;x,y)$ be the function defined by (6.15). For any $0\leq s<t<T$, there exists a constant $M_3^{'} = M_3^{'}(\lambda;T,\varepsilon)$,

(6.19) $$\Sigma_{n=0}^{\infty}\int_M \|J_n(\lambda;t,s;x,y)\|_{(x,y)} d\mu_g(y)$$

$$\leq M_0 M_3^{'} (t-s)^{\frac{1}{2}} \exp M_3^{'} (t-s)^{\frac{1}{2}}.$$

(6.20) $$\Sigma_{n=0}^{\infty}\int_M \|J_n(\lambda;t,s;x,y)\|_{(x,y)} d\mu_g(x)$$

$$\leq M_0 M_3^{'} (t-s)^{\frac{1}{2}} \exp M_3^{'} (t-s)^{\frac{1}{2}}.$$

Moreover, we have

$$(6.21)\qquad \int_M \|K(\lambda;t,s;x,y)\|_{(x,y)}d\mu_g(y)$$

$$\leq M_0M_3^{'}\ (t-s)^{\frac{1}{2}}\exp M_3^{'}\ (t-s)^{\frac{1}{2}},$$

$$(6.22)\qquad \int_M \|K(\lambda;t,s;x,y)\|_{(x,y)}d\mu_g(x)$$

$$\leq M_0M_3^{'}\ (t-s)^{\frac{1}{2}}\exp M_3^{'}\ (t-s)^{\frac{1}{2}}.$$

Now, fix (s,y) and consider $\xi(t,z) = K(\lambda;t,s;x,y)$. Applying Corollary 5.5, we have

$$(6.23)\qquad A_x\int_s^t dz\int_M H(\lambda;t,z;x,z)\underset{z}{\#}K(\lambda;z,s;z,y)d\mu_g(z)$$

$$= \int_s^t dz\int_M A_xH(\lambda;t,z;x,z)\underset{z}{\#}K(\lambda;z,s;z,y)d\mu_g(z),$$

where $A_x=\frac{1}{2}\Delta_x - \frac{1}{12}\ \mathrm{Scal}_g(x)$. Thus, we get

$$(6.24)\qquad \frac{\partial}{\partial t}\int_s^t dz\int_M H(\lambda;t,z;x,z)\underset{z}{\#}K(\lambda;z,s;z,y)d\mu_g(z)$$

$$= K(\lambda;t,s;x,y) + \int_s^t dz\int_M \frac{\partial}{\partial t}H(\lambda;t,z;x,z)\underset{z}{\#}K(\lambda;z,s;z,y)d\mu_g(z).$$

Therefore, we have

$$-\ [\tfrac{\partial}{\partial t}-\lambda^{-1}A]\int_s^t dz\int_M H(\lambda;t,z,x,z)\#K(\lambda;z,s;z,y)d\mu_g(z)$$

$$= -K(\lambda;t,s;x,y)+\int_s^t\int_M J_0(\lambda;t,\tau;x,z)\underset{z}{\#}K(\lambda;\tau,s;z,y)d\mu_g(z)$$

$$= -K(\lambda;t,s;x,y)+\Sigma_{n=1}^{\infty}J_n(\lambda;t,s;x,y)$$

$$= -J_0(\lambda;t,s;x,y).$$

Then, we obtain the following

<u>Proposition 6.5.</u> Under the same assumptions and notations as above, put

(6.25) $$H(\lambda;t,s;x,y)$$

$$= H(\lambda;t,s;x,y)+\int_s^t d\tau\int_M H(\lambda;t,\tau;x,z)\underset{z}{\#}K(\lambda;\tau,s;z,y)d\mu_g(z).$$

Then, we have

(i) $H(\lambda;t,s;x,y)$ is continuous in $\{(t,s)|0\leq s<t<T\}\times M\times M$.

(ii) $H(\lambda;t,s;x,y)$ satisfies

(6.26) $$\frac{\partial}{\partial t} H(\lambda;t,s;x,y) = \lambda^{-1}AH(\lambda;t,s;x,y),$$

where $A=\frac{1}{2}\Delta_L - \frac{1}{2}\,\mathrm{Scal}_g(x)$.

(iii)There exists a positive constant $M_4=M_4(\lambda;T,\varepsilon)$ such that

(6.27) $$\|H(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0M_4(t-s)^{-\frac{m}{2}}\exp M_4(t-s)^{\frac{1}{2}}\exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}],$$

$$(6.28) \qquad \int_M \|H(\lambda;t,s;x,y)\|_{(x,y)} d\mu_g(y) \leq M_0 M_4 \exp M_4 (t-s)^{\frac{1}{2}},$$
$$\int_M \|H(\lambda;t,s;x,y)\|_{(x,y)} d\mu_g(x) \leq M_0 M_4 \exp M_4 (t-s)^{\frac{1}{2}},$$

Remark. Therefore, defining $H(\lambda;t,s)\xi(x) = \int_M H(\lambda;t,s;x,y)\xi(y)d\mu_g(y)$, we have a bounded linear operator $H(\lambda;t,s)$ on $L^2(E)$ and is C^0 semi-group with infinitesimal generator $\lambda^{-1}A$ in (6.26).

By similar argument of Lemma 4.4, we have

Lemma 6.6. Let $\xi(z,z)$ be a bounded continuous parametrized section on $[s,t]\times M$. Then, we have

$$(6.29) \qquad \lim_{z\downarrow s}\int_M \xi(z,z)\#_z H(\lambda;z,s;z,y)d\mu_g(z) = \xi(s,y),$$
$$\lim_{z\downarrow s}\int_M d\mu_g(z) = \xi(t,x).$$

uniformly on an compact set of M.

Finally, we can state the following :

Theorem 6.7. Assume that (M,g) satisfies (A.1)-(A.3). Then, $H(\lambda;t,s;x,y)$ defined by (6.28) is the fundamental solution for the parabolic equation (6.1).

As a direct results, we get the other proof of a partial result of Molchanov [25] (Cf. See also Cheng et al [7]).

Corollary 6.8. Under the same assumptions as in

Theorem 6.7, there exists a positive constant $C_{10} = C_{10}(\lambda;T,\varepsilon)$ such that the kernel $H(\lambda;t,s;x,y)$ of the fundamental solution of (6.1) has the following asymptotic expansion: for any $T>0$, $0\leqq s<t<T$ and any $x,y \in M$,

(6.30) $$\|H(\lambda;t,s;x,y) - (2\pi\lambda^{-1}(t-s)^{-m/2}\rho(x,y)\exp -[\frac{\lambda d^{(x,y)}}{2(t-s)}]P(x,y)\|_{(x,y)} \leqq C_{10}(t-s)^{-\frac{m-3}{2}}$$

1.7 Convergence of path integral as the kernel function.

In this section, we shall prove Theorem B, using the fundamental solution considered in §6. Let $[0,t]$ be any closed interval such that $0<t<T$, and T be any fixed positive number. Let σ_N be a N-equal subdivision of $[0,t]$,

$$(7.1)\qquad \sigma_N : 0=t_0<t_1<\dots<t_{N-1}<t_N=t,\quad t_j=\frac{j}{N}t,\quad j=0,\dots,N.$$

We define a operator $H(\lambda|\sigma_N|t,s)$ associated with the subdivision σ_N:

$$(7.2)\qquad H(\lambda;\sigma_N|t) = H(\lambda;t,t_{N-1})\dots H(\lambda;t_1,0),$$

and we denote by $H(\lambda;\sigma_N|t;x,y)$ the kernel function of the operator (7.2), i.e.

$$(7.3)\qquad H(\lambda;\sigma_N|t;x,y)$$

$$= \int_M \dots \int_M H(\lambda;t,t_{N-1};x,z_{N-1}) \underset{z_{N-1}}{\#} H(\lambda;t_{N-1},t_{N-2};z_{N-1},z_{N-2})\dots$$

$$\underset{z_1}{\#} H(\lambda;t_1,0;z_1,y)d\mu_g(z_{N-1})\dots d\mu_g(z_1),$$

where $H(\lambda;t,s;x,y)$ is defined in (1.7).

To prove Theorem B, we shall show the following :

<u>Proposition 7.1.</u> Let (M,g) satisfies (A.1)-(A.3). For any fixed $T>0$, there exists a positive constant $\gamma=\gamma(\lambda;T,\varepsilon)$ such that for $0<\varepsilon<\frac{1}{8}$,

(7.4) $$\|H(\lambda;t;x,y) - H(\lambda;\sigma_N|t;x,y)\|_{(x,y)}$$

$$\leq \gamma t^{-m/2}N^{-\frac{1}{2}}T\exp-[\frac{\lambda\varepsilon^{***}d^2(x,y)}{2t}].$$

$0\leqq s<t<T$ and $\varepsilon^{***}=1-3\varepsilon$. Here γ dose not independent of t and the sub division σ_N.

We needs several steps to prove the above proposition. First, put

(7.5) $$R(\lambda;t,s) = H(\lambda;t,s) - H(\lambda;t,s)$$

and denote by $R(\lambda;t,s;x,y)$ the kernel function of (7.5). Then, we get

<u>Lemma 7.2.</u> For any ε, $0<\varepsilon<1/4$, there exists a positive constant $\gamma_1=\gamma_1(\lambda;T,\varepsilon)$ such that

(7.6) $$\|R(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq M_0\gamma_1(t-s)^{\frac{-m+3}{2}}\exp-[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}],$$

where $\varepsilon^{**}=1-2\varepsilon$.

<u>Proof.</u> Since

$$R(\lambda;t,s;x,y) = \int_s^t d\sigma\int_M H(\lambda;t,\sigma;x,z)\underset{z}{\#}K(\lambda;\sigma,s;z,y)d\mu_g(z),$$

we have

$$\|R(\lambda;t,s;x,y)\|_{(x,y)}$$

$$\leq \int_s^t d\sigma \int_M \|H(\lambda;t,\sigma;x,z)\|_{(x,z)} \|K(\lambda;\sigma,s;z,y)\|_{(z,y)} d\mu_g(z)$$

$$\leq M_0^2 M_3 \gamma_1 \exp M_2(t-s) \int_s^t (t-\sigma)^{-\frac{m}{2}} (\sigma-s)^{\frac{-m+1}{2}} d\sigma$$

$$\times \int_M \exp -[\frac{\lambda d^2(x,z)}{2(t-\sigma)} - \frac{\lambda \varepsilon^{**} d^2(z,y)}{2(\sigma-s)}] d\mu_g(z)$$

$$\leq M_0 \gamma_1(\lambda;T,\varepsilon)$$

$$\times \int_s^t (t-\sigma)^{-\frac{m}{2}} (\sigma-s)^{\frac{-m+1}{2}} d\sigma \int_{T_x M} \exp -[\frac{\lambda\varepsilon^{**}\|Z\|^2}{2(t-\sigma)} - \frac{\lambda\varepsilon^{**}\|Z-Y\|^2}{2(\sigma-s)}] dZ$$

$$\leq M_0 \gamma_1(\lambda;T,\varepsilon)(t-s)^{\frac{-m+3}{2}} \exp -[\frac{\lambda\varepsilon^{**}d^2(x,y)}{2(t-s)}],$$

by the same computation as in §7. Thus, we get (7.6)

Now, we obtain

(7.7) $\quad H(\lambda;\sigma_N|t) - H(\lambda;t,0)$

$$= H(\lambda;t,t_{N-1})\ldots H(\lambda;t_1,0) - H(\lambda;t,0)$$

$$= [H(\lambda;t,t_{N-1})+R(\lambda;t,t_{N-1})]\ldots.[H(\lambda;t_1,0)+R(\lambda;t_1,0)] - H(\lambda;t,o).$$

Using the evolution property of $H(\lambda;t,s)$, we shall write down the right hand side of (7.6). Let

(7.8)

$$\mathfrak{Z} = \{(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_{k+1}); k=1,\ldots,N,\ \Sigma_{i=1}^{k}[\alpha_i+\beta_i]+\beta_{k+1}=N\}.$$

Also, we denote by

(7.9) $A_j = \alpha_1+\ldots+\alpha_j,\quad B_j = \beta_1+\ldots+\beta_j, j\geq 1, A_0 = B_0 = 0.$

Thus, $A_k + B_{k+1}=N$. The right hand side of (7.6) is written by

(7.10) $H(\lambda;\sigma_N) - H(\lambda;t,0)$

$$= \Sigma_{(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_{k+1})\varepsilon\mathfrak{Z}}\ \mathfrak{G}(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_{k+1}),$$

where

(7.11) $\mathfrak{G}(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_{k+1})$

$$= H(\lambda;t,\frac{(A_k+B_k)t}{N})$$

$$R(\lambda;\frac{(A_k+B_k)t}{N},\frac{(A_k-1+B_k)t}{N}).\ldots.R(\lambda;\frac{(A_{k-1}+1+B_k)t}{N},\frac{(A_{k-1}+B_k)t}{N})$$

$$H(\lambda;\frac{(A_{k-1}+B_k)t}{N},\frac{(A_{k-1}+B_{k-1})t}{N})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R(\lambda;\frac{(A_1+B_1)t}{N},\frac{(A_1-1+B_1)t}{N})\ldots R(\lambda;\frac{(1+B_1)t}{N},\frac{B_1t}{N})$$

$$H(\lambda;\frac{B_1t}{N},0).$$

Now, we put

(7.12) $\quad R^{(j)}$

$$= R(\lambda;\frac{(A_j+B_j)t}{N},\frac{(A_j-1+B_j)t}{N})\cdots\cdots$$

$$\cdots R(\lambda;\frac{(A_{j-1}+1+B_j)t}{N},\frac{(A_{j-1}+B_j)t}{N})$$

and denote by $R^{(j)}(\lambda;x,y)$ the kernel function of (7.12).

Lemma 7.3. Given any ε, $0<\varepsilon<1/4$, there exists a positive constant $\gamma_2 = \gamma_2(\lambda;T,\varepsilon)$ such that

(7.13) $\quad \|R_{(1)}(\lambda;x,y)\|_{(x,y)}$

$$\leq M_0^{\alpha_j}\gamma_2^{\alpha_j}(\frac{t}{N})^{\frac{3}{2}\alpha_j}(\frac{\alpha_j}{N}t)^{-\frac{m}{2}}\exp -[\frac{\lambda\varepsilon^{***}d^2(x,y)}{(2t\alpha_j)/N}$$

where $\varepsilon^{***}=1-3\varepsilon$.

Proof. Generally, take $t_1,\ldots,t_a \in [0,t)$, $0<t_1<,\ldots,<t_a<T$. Put,

(7.14) $\quad R(\lambda;t_1,\ldots t_a) = R(\lambda;t_a,t_\nu)\ldots R(\lambda;t_2,t_1)$.

We denote by $R(\lambda;t_1,\ldots,t_a;x,y)$ the kernel function of (7.14). To prove (7.13), it is sufficient to get the following estimate for (7.14).

(7.15) $\quad \|R(\lambda;t_1,\ldots,t_a;x,y)\|_{(x,y)}$

$$\leq M_0^a \gamma_1^a \, \Pi_{j=1}^a (t_{j+1}-t_j)^{\frac{3}{2}} \, (t_{a+1}-t_1)^{-\frac{m}{2}} \exp -[\frac{\lambda\varepsilon^{***} d^2(x,y)}{2(t_{a+1}-t_1)}].$$

We shall show (7.15) by induction. Remark that (7.15) holds for $a=l$ by Lemma 7.2. Assume that (7.15) holds for $a-1\geq 1$. Then, by the similar computation as in Lemma 7.2, we have

$$(7.16) \qquad \|R(\lambda;t_1,\ldots,t_a;x,y)\|_{(x,y)}$$

$$\leq M_0^a \gamma_2^{a-1} \gamma_1 \, \Pi_{j=1}^{a-1} (t_{j+1}-t_j 0^{\frac{3}{2}})(t_a-t_{a-1})^{\frac{-m+3}{2}} (t_{a-1}-t_1)^{-\frac{m}{2}}$$

$$\text{tmes} \int_M \exp -[\frac{\lambda\varepsilon^{**} d^2(x,z)}{2(t_{a+1}-t_a)} + \frac{\lambda\varepsilon^{***} d^2(z,y)}{2(t_a-t_1)}] d\mu_g(z)$$

$$\leq M_0^a \gamma_2^{a-1} \gamma_1 \exp \frac{kT}{2} \, \Pi_{j=1}^{a-1} (t_{j+1}-t_j)^{\frac{3}{2}} (t_a-t_{a-1})^{\frac{3}{2}} (t_a-t_1)^{-\frac{m}{2}}$$

$$\times \int_{T_xM} \exp -\lambda\varepsilon^{***} [\frac{\|Z\|^2}{2(t_a-t_{a-1})} - \frac{\|Z-Y\|^2}{2(t_a-t_1)}],$$

because $\exp -[\frac{\|Z\|^2}{2(t_a-t_{a-1})} -k\|Z\|] \leq \exp \frac{kT}{2}$. Thus, we have

$$(7.17) \qquad \|R(\lambda;t_1,\ldots,t_a;x,y)\|_{(x,y)}$$

$$\leq M_0^a \gamma_1^{a-1} \exp \frac{kT}{2} \, \Pi_{j=1}^a (t_j-t_{j-1})^{\frac{3}{2}} (t_a-t_1)^{-\frac{m}{2}}$$

$$\times \exp -[\frac{\lambda\varepsilon^{***} d^2(x,y)}{2(t_{a+1}-t_1)}] \int_{T_xM} \exp -[\frac{\lambda\varepsilon^{***} \|Z'\|^2}{2}] dZ'.$$

Choosing $\gamma_2 \geq \gamma_1 \exp \frac{kT}{2} \int_{T_xM} \exp -[\lambda\varepsilon^{***} \frac{\|Z'\|^2}{2}] dZ'$, we get

(7.15).

Define a operator $S^{(j)}(\lambda)$ by

(7.18) $$S^{(j)}(\lambda)$$

$$= H(\lambda;\frac{(A_j+B_{j+1})}{N}t,\frac{(A_j+B_j)}{N}t)R^{(j)}(\lambda)\ldots$$

$$\ldots H(\lambda;\frac{(A_1+B_1)}{N}t,\frac{(A_1+B_1)}{N}t)R^{(1)}(\lambda)$$

and we denote by $S^{(j)}(\lambda;x,y)$ the kernel function of $S^{(j)}(\lambda)$.

Using (7.13), we have

(7.19) $$\|S^{(1)}(\lambda;x,y)\|_{(x,y)}$$

$$\leq M_0^{\alpha_1+1}\gamma_2^{\alpha_1}(\frac{N}{t})^{\frac{3}{2}\alpha_j}(\frac{\alpha_j}{N}t)^{-\frac{m}{2}}(\frac{\beta_j}{N}t)^{-\frac{m}{2}}$$

$$\int_M \exp -[\frac{\lambda d^2(x,z)}{2\frac{\beta_j}{N}t} - \frac{\lambda\varepsilon^{***}d^2(z,y)}{2\frac{\alpha_j}{N}t}]d\mu_g(z)$$

$$\leq M_0^{\alpha_1+1}\gamma_2^{\alpha_1+1}(\frac{N}{t})^{\frac{3}{2}\alpha_1}(\frac{\alpha_1+\beta_1}{N}t)^{-\frac{m}{2}}\exp[\frac{\lambda\varepsilon^{***}d^2(x,y)}{2(\frac{\alpha_j+\beta_j}{N})t}].$$

By the similar computations as above, we obtain the following :

<u>Lemma 7.4.</u> Given ε, $0<\varepsilon<1/4$, there exists a positive constants $\gamma_3=\gamma_3(\lambda;T,\varepsilon)$ such that

(7.20) $$\|S^{(j)}(\lambda;x,y)\|_{(x,y)}$$

$$\leq M_0^{A_j+j}\gamma_3^{A_j+j}\left(\frac{N}{t}\right)^{\frac{3}{2}A_j}\left(\frac{A_j+B_j}{N}t\right)^{-\frac{m}{2}} \exp -\left[\frac{\lambda\varepsilon^{***}d^2(x,y)}{2(A_j+B_j)t/N}\right].$$

Proof of Proposition 7.1.

Combining with (7.11) and Lemma 7.4, we get

(7.21) $$\Sigma\ \|\mathfrak{g}(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_{k+1}:x,y)\|_{(x,y)}$$

$$\leq \Sigma\ M_0^{A_k+k+1}\ \gamma_3^{A_k+k+1}\ \left(\frac{N}{t}\right)^{\frac{3}{2}A_k} t^{-\frac{m}{2}} \exp -\left[\frac{\lambda\varepsilon^{(4)}d^2(x,y)}{2t}\right]$$

$$\leq M_0\ \Pi_j^N\ [(1+M_0\gamma_3\ (\tfrac{t}{N})^{\frac{3}{2}})^N - 1] \exp -\left[\frac{\lambda\varepsilon^{(4)}d^2(x,y)}{2t}\right]$$

$$\leq M_0\ \gamma_3 t^{\frac{3}{2}}\ N^{-\frac{1}{2}} \exp M_0\gamma_3 tN^{\frac{1}{2}} \exp -\left[\frac{\lambda\varepsilon^{(4)}d^2(x,y)}{2t}\right],$$

where $\varepsilon^{(4)} = 1-4\varepsilon$. This proves the Proposition 7.1. ~

Remark. The above computation can be moved slightly for general subdivision $\sigma_N:0=t_0<t_1<\dots<t_{N-1}<t_N=t$, $\delta(\sigma_N) = \max |t_{j+1}-t_j|$, replacing N to be $\delta(\sigma_N)$ in (7.4). Also, it is easily seen that for fixed t>0, $H(\lambda;\sigma_N|t;x,y)$ converges uniformly on any compact set on M×M to $H(\lambda;t;x,y)$ and $H(\lambda;\sigma_N|t)$ defines a bounded linear operator on $L^2(E)$ Also, there exists a positive constant $\gamma_5 = \gamma_5(\lambda;T,\varepsilon)$ such that

(7.22) $$\|H(\lambda;\sigma_N;t)\|_{\mathcal{B}(L^2(E))} \leq \gamma_5 \exp \gamma_5 t^{\frac{3}{2}}.$$

1.8 Generalizations and Concluding remarks

Finally, we shall give some generalizations and remarks for Theorem 1.2.

(i) First, we can generalize Theorem 1.2 if we insert the cut off function. Namely, let $\chi(x,y)$ be a cut off function defined by $\chi(x,y) = 1$ on $d(x,y)<\delta$, and $= 0$ on $d(x,y)\geqq 2\delta$, where δ is the injectivity radius of (M,g) (Cf. By the assumption (A.3), we see that δ is a positive constant). And consider the following integral transformation

(8.1) $\quad \tilde{H}(t)\xi(x)$

$$= (2\pi\lambda^{-1})^{-\frac{m}{2}}\int_M \chi(x,y)\rho(x,y)\exp -\lambda S(t,s;x,y)P(x,y)\xi(y)d\mu_g(y)$$

Then, we get the same results as in Theorem 1.2 by following the same computation through § §1.1-1.5.

(ii) Also, our scheme can be examined for the more general Lagrangian function. Moreover, we can construct the infinitesimal generator on the intrinsic Hilbert space as follows : Consider the following situation.

(M) M is a smooth, simply-connected and connected d-dimensional manifold.

(L.I) $L(\gamma,\dot{\gamma})$ is represented by

(8.2) $\quad L(\gamma,\dot{\gamma}) = L^0(\gamma,\dot{\gamma}) - V(\gamma), \quad L^0(\gamma,\dot{\gamma}) = (1/2)g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j$

for $(\gamma,\dot{\gamma})\in TM$. (Hereafter, we use Einstein's convention to contract indices.) Moreover,

(L.II) $ds^2 = g_{ij}(x)dx^i dx^j$ defines a complete Riemannian metric on M.

(L.III) There exists a constant $k>=0$ such that for any 2-plane π, the sectional curvature K_π satisfies $-k^2 <= K_\pi <= 0$.

(L.IV) Denote by $R(\cdot)$ the curvature tensor of g. Then, there exists a constant C_0 such that

$$|\nabla^\alpha R(\cdot)| \leq C_0 \qquad \text{for} \, 0<|\alpha|<=3,$$

where α is a multi-index, $\nabla^\alpha = \nabla_1^{\alpha_1}\cdots\nabla_d^{\alpha_d}$ and ∇_j represents the covariant derivation in the direction of x^j for any local chart at $x=(x^1,\ldots,x^d)$.

(L.V) $V\in C_0^\infty(M)$ are real valued.

For any natural measure μ on M, we consider the following transformation in $L^2(M,d\mu)$ with parameters $t>0$ and $\lambda>0$. For any $f\in C_0^\infty(M)$ and sufficiently small $t>0$, we put

(8.3) $(H_t^\lambda(L;\mu)f)(x)$

$$= (2\pi\lambda)^{-d/2}\int_M \rho(L;\mu)(t,x,y)\exp\{-\lambda^{-1}S(L)(t,x,y)\}\cdot f(y)d\mu(y).$$

Here we denote

$$(8.4) \qquad S(L)(t,x,y) = \inf\{\int_0^t L(\gamma(z),\dot{\gamma}(z))dz : \gamma(z)\in\Omega_{t,x,y}\},$$

$$\dot{\gamma}(z)=d\gamma(z)/dz,$$

$$(8.5) \qquad \Omega_{t,x,y}$$

$$= \{\gamma(\cdot)\in C([0,t]\to M) : \text{absolutely continuous in } z$$

$$\text{with } \gamma(0)=y,\ \gamma(t)=x, \text{ and } \int_0^t \langle\dot{\gamma}(z),\dot{\gamma}(z)\rangle_{\gamma(z)}dz < +\infty\}$$

and

$$(8.6) \qquad \rho(L;\mu)(t,x,y)$$

$$= [\det\{-\partial_{x^i}\partial_{y^a} S(L)(t,x,y)\}/\mu(x)\mu(y)]^{1/2}$$

where $\mu(x)$ is the density of μ at x, i.e. $d\mu(x)=\mu(x)dx^1\wedge\ldots\wedge dx^d$, ∂_{x^i} denotes the partial derivation in the direction of x^i at $x=(x^1,\ldots,x^d)$, and $\langle X,Y\rangle_x$ is the Riemannian scalar product at x for $X,Y\in T_xM$.

Then, we get the following :

Theorem 8.1. Let M and L be given satisfying Assumptions (M) and (L.I)-(L.V). Then, we have the following:

(a) There exists a positive number $T>0$ such that, for any natural measure μ, the operator $H_t^\lambda(L;\mu)$ defines a bounded linear operator in $L^2(M,d\mu)$ for $0<t<T$.

(b) $$\lim_{t\to 0} \|H_t^\lambda(L;\mu)f - f\|_{L^2(M,d\mu)} = 0$$

for all $f \in L^2(M,\mu)$.

(c) There exist positive constants C and C' depending on T independent of μ such that

(8.7) $$\|H_{t+s}^\lambda(L;\mu)f - H_t^\lambda(L;\mu)H_s^\lambda(L;\mu)f\|$$

$$\leqq [C\{(t+s)^{3/2} - t^{3/2} + s^{3/2}\} + C'(t+s)s]\|f\|$$

for $0<t+s<T$. Moreover, we take $C'=0$ for $V=0$.

(d) There exists a limit $H_t^\lambda(L;\mu) = \lim_{n\to\infty} [H_{t/n}^\lambda(L;\mu)]^n$ in the operator norm in $L^2(M,d\mu)$ for any $t>0$. Moreover, $\{H_t^\lambda(L;\mu)\}_{t\geqq 0}$ with $H_0^\lambda(L;\mu)$ = the identity operator, forms a C^0 semi-group in $L^2(M,d\mu)$.

(e) For any two natural measures μ and ν on M, we have

(8.8) $$H_t^\lambda(L;\mu) = U_{\nu\mu}^{-1} H_t^\lambda(L;\nu) U_{\nu\mu}$$

where $U_{\nu\mu}$ is an isomorphism from $L^2(M,d\mu)$ onto $L^2(M,d\nu)$, defined by

(8.9) $$(U_{\nu\mu}f)(x) = f(x)(\mu(x)/\nu(x))^{1/2} \quad \text{for } f \in L^2(M,d\mu).$$

(f) The infinitesimal generator $A^\lambda(L;\mu)$ of $H_t^\lambda(L;\mu)$ is given by

(8.10) $$\partial_t(H_t^\lambda(L;\mu)f)_{|t=0}$$

$$= A^{\lambda}(L;\mu)f = U^{-1}_{\mu_g\mu}A^{\lambda}(L;\mu_g)U_{\mu_g\mu}f \quad \text{for} \quad f\in C_0^{\infty}(M),$$

$$(A^{\lambda}(L;\mu_g)f)(x) = \lambda^2(\Delta_g/2 - R(x)/12)f(x) + V(x)f(x).$$

Here Δ_g is the negative Laplace-Beltrami operator associated with g.

In other word, the above procedure defines a C^0 semi-group $H_t^{\lambda}(L)$ and its infinitesimal generator $A^{\lambda}(L)$ on the intrinsic Hilbert space H (M) such that if H (M) is trivialized by a natural measure μ as $L^2(M,d\mu)$, then $H_t^{\lambda}(L)$ and $A^{\lambda}(L)$ are represented by $H_t^{\lambda}(L;\mu)$ and $A^{\lambda}(L;\mu)$ on $L^2(M,d\mu)$.

The above theorem gives that the old and debated question whether the Schrödinger equation in the curved space contains the term with $h^2R(\cdot)$ will be solved completely if we could proceed as same as above for $\lambda=ih$.

For the proof of this Theorem, see Inoue- Maeda [18].

(iii) In stead of the argument in our discussions, we may produce any multiple of $R(\cdot)$, if we change the order of our procedure and we content with the convergence of $H_n^{\lambda}(t)$ only in the strong sense.

To make our point clear, we consider the case where V=0.

For any $\beta\in\mathbf{R}$, we define an operator $H_t^{\lambda}(\beta)$ as

(8.11) $\quad (H_t^{\lambda}(\beta)f)(x)$

$$=(2\pi\lambda t)^{-d/2}\int_M \rho^0(x,y)^\beta$$

$$\times\exp(-\lambda^{-1}S^0(t,x,y))f(y)d\mu_g(y) \quad ,$$

for $f\in C_0^\infty(M)$, where $t^{d/2}\rho^0(t,x,y)$ is independent of t and simply denoted by $\rho^0(x,y)$. In this case, as we may put $\lambda=1$ without loss of generality, we denote $H_t^1(\beta)$ simply by $H_t(\beta)$. And we drop the supper index 0 above for notational simplicity.

Theorem 8.2. Under Assumptions (M),(L.I)-(L.IV), we have the following : Fix $T>0$ arbitrarily. For any $\beta\in\mathbf{R}$,

(a) $H_t(\beta)$ defines a bounded linear operator in $L^2(M,d\mu_g)$ for $0<t<T$.

Moreover, there exists a constant C_{10} such that

(8.12) $$\|H_t(\beta)f\|\leq\exp C_{10}t\cdot\|f\|$$

for $0<t<T$ and $f\in C_0^\infty(M)$.

(b) $\lim_{t\to 0}\|H_t(\beta)f-f\|=0$ for $f\in L^2(M,d\mu_g)$

(c) $$\partial_t(H_t(\beta)f)(x)|_{t=0} = [\Delta/2 - (1-(\beta/2))R(x)/6]f(x)$$

$$= (\mathbf{A}_\beta f)(x) \quad \text{for} \quad f\in C_0^\infty(M).$$

(d) There exists a limit $\operatorname{s-lim}_{n\to\infty}(H_{t/n}(\beta))^n f$, denoted by

$H_t(\beta)f$ for each $f\in C_0^\infty(M)$. $\{H_t(\beta)\}_{t\geq 0}$ with $H_0(\beta)$ = the identity operator, forms a C^0-semi group in $L^2(M,d\mu_g)$ with the infinitesimal generator given in (c).

Remark. Comparing above theorem with Theorem 8.1, we remark that the order of statements is changed. And in proving (d), we use the fact that the Laplace-Beltrami operator Δ is self-adjoint in $L^2(M,d\mu_g)$ under our assumptions (This fact is proved in the previous sections but we need that fact in order to prove (d).)

Proof of (a),(b). In our case, $t^d\Theta^0(t,x,y)$ is independent of t and denoted simply by $\Theta(x,y)$. We may rewrite the operator $H_t(\beta)$ by using normal polar coordinate at x and $\mathrm{Exp}_x X=\Phi_{t,x}(x)$ as

(8.11)' $(H_t(\beta)f)(x)$

$$= (2\pi t)^{d/2}\int_0^\infty\int_{S^{d-1}}\Theta(x,\mathrm{Exp}_x r\omega)^{1-(\beta/2)}$$

$$\times\exp(-d^2(x,\mathrm{Exp}_x r\omega)/2t)r^{d-1}drd\omega \quad .$$

To prove the statements (a) and (b), we proceed analogously as proving Proposition 2.1 and 2.2. But for $\beta\geq 2$, we use the fact $\Theta(x,y)>=1$ for estimating $\Theta(x,\mathrm{Exp}_x r\omega)^{1-(\beta/2)}$. (As V=0, we may take $\Theta(x,y)\geq 1$ in Proposition 1.10.)

Proof of (c). Take a function $\nu(x,y)\in C^\infty(M\times M)$, $0\leq\nu(x,y)\leq 1$, satisfying

$$\nu(x,y) = \begin{cases} 1 & \text{if } d(x,y)\leq 1 \\ 0 & \text{if } d(x,y)\geq 3 \end{cases}$$

Define operators $H_1(t,\beta)$ and $H_2(t,\beta)$ as follows :

(8.13) $\quad (H_1(t,\beta)f)(x)$

$$= (2\pi t)^{-d/2}\int_M \nu(x,y)\rho(x,y)^{\beta}\exp(-d^2(x,y)/2t)f(y)d\mu_g(y) \quad ,$$

(8.14) $\quad (H_2(t,\beta)f)(x)$

$$= (2\pi t)^{-d/2}\int_M (1-\nu(x,y))\rho(x,y)^{\beta}\exp(-d^2(x,y)/2t)f(y)d\mu_g(y) \quad .$$

Now, we claim the following :

(8.15) $\quad (H_1(t,\beta)f)(x)$

$$= f(x) + t(\mathbf{A}_{\beta}f)(x) + tG_1(t,f)(x) \quad \text{for} \quad f\in C_0^{\infty}(M) \quad .$$

(8.16) $\quad \lim_{t\to 0}\|G_1(t,f)(\cdot)\| = 0 \;,$

and

(8.17) $\quad \lim_{t\to 0}\|t^{-1}(H_2(t,\beta))f(\cdot)\| = 0 \quad .$

By Taylor's expansion, we get

(8.18) $\quad f(y)$

$$= f(x) + (\partial_{X^i}f)(x)X^i + 1/2(\partial X^i\partial X^j f)(x)X^iX^j + F(x,X) \quad ,$$

where $y=\mathrm{Exp}_xX$. $(\partial_{X^i}f)(x) = \partial_{X^i}f(\mathrm{Exp}_xX)|_{X=0}$, and

$$F(x,X) = (1/6)\int_0^1 [\partial_{X^i}\partial X^j \partial X^k f(\mathrm{Exp}_x sX)]ds X^i X^j X^k \quad .$$

Then, it is clear that $\tilde{F}(x,X)=\nu(x,\mathrm{Exp}_x X)F(x,X)$ is a smooth function in x and X with compact support.

Analogously, we have

$$(8.19)\quad \Theta(x,y)^{1-(\beta/2)} = 1-(1/6)(1-(\beta/2))R_{ij}(x)Y^iY^j+\Theta_\beta(x,y)$$

$$= 1 + \tilde{\Theta}_\beta(x,y) \quad ,$$

where

$$\Theta_\beta(x,y) = (1/6)\int_0^1 \partial_{Y^i}\partial_{Y^j}\partial_{Y^k}\Theta(x,\mathrm{Exp}_x sY)^{1-(\beta/2)}ds Y^iY^jY^k.$$

By Assumption (L.IV), there exist constants C_{11} and κ such that

$$(8.20)\qquad |\Theta_\beta(x,y)| \le C_{11}\exp\kappa|Y|$$

for any $x\in M$ and any $Y\in T_xM$. Inserting (8.18) and (8.19) into (8.13), we get (8.15) by defining $G_1(t,f)$ as

$$(8.21)\qquad tG_1(t,f)$$

$$= -f(x)(2\pi t)^{-d/2}\int_{T_xM}(1-\nu(x,y))$$

$$\times[1+(1/6)(1-(\beta/2))R_{ij}(x)Y^iY^j]e^{-|Y|^2/2t}\,dY$$

$$+ (\partial_{Y^i} f)(x)(2\pi t)^{-d/2}\int_{T_xM}\nu(x,y)Y^i\Theta(x,y)^{1-(\beta/2)}e^{-|Y|^2/2t}\,dY$$

$$- (1/2)(\partial Y^i \partial^j f)(x)(2\pi t)^{-d/2}$$

$$\int_{T_xM}[(1-\nu(x,y)] + \nu(x,y)\tilde{\Theta}_\beta(x,y)]Y^iY^je^{-|Y|^2/2t}dY$$

$$+ (2\pi t)^{-d/2}\int_{T_xM}\tilde{F}(x,Y)\nu(x,Y)^{1-(\beta/2)}e^{-|Y|^2/2t}\,dY,$$

where $\nu(x,y)=\nu(x,\mathrm{Exp}_xY)$ etc.

By (8.20) and the property of $\tilde{F}(x,y)$, we have the estimate in (8.16) readily.

The estimate (8.17) is a easy consequence of the introduction of $\nu(x,y)$.

Proof of (d). Under Assumptions (M),(L.I)-(L.IV), it is well-known that Δ is self-adjoint in $L^2(M,d\mu_g)$. So $\mathbf{A}_\beta$ is also self-adjoint. Moreover as $\mathbf{A}_\beta$ is bounded from below, $\mathbf{A}_\beta$ generates a C^0-semi group . This and the facts (a)-(c) guarantee us to apply the generalized Lax theorem to our case (cf. p. 214, Chorin et al [8]). So we proved our Theorem 8.1.

CHAPTER II Regular Frechet Lie groups and Product integrals.

1.0 Path integral formulation from Hamiltonian mechanics.

In this chapter, we will show that the construction procedure in Chapter I works well even when we replace there by using the generating function for the symplectic transformation corresponding to the given Hamiltonian. Moreover, we can consider it for the case $\lambda=i/h$. Here, we can use the remarkable properties for the infinite dimensional group $G\mathfrak{F}_0^0$ of invertible Fourier integral operators of order zero on the compact manifold. As is shown in the series of the works in Omori, Maeda, Yoshioka and Kobayashi ([23],[27]-[32] and [36]), we can introduce the topology in $G\mathfrak{F}_0^0$ by the kernel function (Cf. §2.2). Therefore, it is possible to investigate the convergence of the iterated integral as a kernel function.

Let N be a closed smooth n-manifold with an arbitrarily fixed Riemannian metric g. We denote by T*M and TN the cotangent bundle and the tangent bundle over M respectively. A point of TN (resp. T^*N) is denoted by $(x;X)$ (resp. $(x;\xi)$). Consider the time dependent Hamiltoninan function $H(t,x;\xi)$. Here we assume the following :

(H.0) $H(t,x;\xi)$ is a smooth function on $R\times T^*M$.

(H.1) $H(t,x;\xi)$ has an asymptotic expansion for $\xi\to\infty$;

$$H(t,x;\xi) \underset{|\xi|\to\infty}{\sim} H_0(t,x;\xi) + \dots + H_{-N}(t,x;\xi) + \dots ,$$

where $H_{-N}(t,x;\xi)$ is an homogeneous function with respect to ξ of degree $-N$.

Let $\varphi(t;x;\xi) = (\varphi_1(t;x,\xi),\varphi_2(t,x;\xi))$ be the symplectic transformation corresponding to the Hamiltonian $H(t,x;\xi)$, i.e. it satisfies

$$(0.1)\qquad \begin{aligned} \frac{d\varphi_1(t,x;\xi)}{dt} &= \partial_\xi H(t,\varphi(t,x;\xi) \\ \frac{d\varphi_2(t,x;\xi)}{dt} &= -\partial_x H(t,\varphi(t,x,\xi) \end{aligned}$$

with initial condition :

$$\varphi(0,x;\xi) = (x;\xi) \quad .$$

Therefore, we get the generating function $S(t,x;\xi)$ for $\varphi(t,x;\xi)$ which corresponds to the action integral in the Lagrangian mechanics. That is, put

$$(0.2)\qquad S(t,x;\xi) = \int_0^t (\theta(X_H) - H)\ (\varphi(-z,x;\xi))\ dz,$$

where X_H denotes the Hamiltonian vector field defined by (0.1) Associated to this, we consider the following integral transformation ;

$$(0.3)\qquad E(h;t)f(x) = \int_{T_x^*M} e^{\frac{i}{h}S(t,x;\xi)} (\nu\tilde{u})^h(x;\xi)d\xi \quad ,$$

Here $(\nu\tilde{u})^h(x;\xi)$ is a sort of the Fourier transformation which is defined by

$$(0.4)\quad (\nu\tilde{u})^h(x;\xi) = (2\pi h)^{-n}\int_N e^{-\frac{i}{h}\langle\xi|X\rangle}\nu(x,z)u(z)dz,\ Exp_x X=z.$$

where $\nu(x,z)$ is a smooth cut off function on $M\times M$ with the breadth ε (Cf §2.1), and Exp is denoted by the exponential mapping by g (Cf. chapter I. in this chapter we use the notation $\cdot_x X$ instead of $Exp_x X$ for simplicity.).

Then, the main problem we shall consider is the convergence of the successive integral ; for a division Δ,

$$\Delta : 0=t_0<t_1<\ldots<t_N=t,\quad t_j=(t/N)j \quad ,$$

$$(0.5)\qquad E_N(\Delta;h,t)u(x)$$

$$= E(h,t/N)\ldots.E(t,t/N)u(x) \qquad (N\text{-times})$$

Now, we can state our main Theorem in this chapter, which is the similar results as in Chapter I.

<u>Theorem B.</u> Let the assumptions (H.0)-(H.1) be satisfies and let $0<h\leq 1$. Then, for any $T>0$ and $t\in[-T,T]$, $E_N(\Delta;h,t)$ defines by (0.5) converges to a Fourier integral operator $U(h,t)$ in $G\mathcal{F}_0^0$ by the topology in $G\mathcal{F}_0^0$ where N tends to infinity. Moreover, we have the following properties:

(i) U(h,t) defines a bounded linear operator in $L^2(M)$, and we have

(0.6) $$\|U(h,t) - E_N(h,t)\| \le \gamma bh|t|N^{-1}[e^{bh|t|/2}] ,$$

where γ and b is some positive constant independent of N.

(ii) for any $t\in\mathbf{R}$ and $u\in C^\infty(M)$, we have

(0.7) $$\left(\frac{i}{h}\right)\frac{\partial}{\partial t}U(h,t) = H(h,t)U(h,t)u(x) ,$$

where H(h,t) is the psuedo-differential operator defined by

(0.8) $$H(h,t)u(x) = \int_{T^*N} H(t,x;\xi)(\nu\tilde{u})^h(x,\xi))d\xi .$$

As a direct consequence of this theorem, we have

Corollary. Let $H(t,x;\xi)$ be a Hamiltonian function on T^*M which satisfies (H.0)-(H.1) and for any $t\in\mathbf{R}$, for any $u,v\in C^\infty$,

(0.9) $$\langle H(h,t)u(x),v(x)\rangle = \langle u(x),H(h,t)v(x)\rangle .$$

where H(h,t) is defined one by (0.8). Then, H(h,t) is the essential self-adjoint operator in $L^2(M)$.

2.1 Infinite dimensional Lie group of invertible Fourier integral operators.

Throughout this chapter, we use mainly the same notations as in [27]. Let N be a closed C^∞ riemannian manifold and TN and T^*N be the tangent bundle and the cotangent bundle of N respectively. A point of TN (resp. T^*N) is denoted by (x,X) (resp. (x,ξ)). Denote by $\overset{o}{T}{}^*N$ the complement of the zero section in T^*N, i.e., $T^*N-\{0\}$ in the notation of [27]. A symplectic diffeomorphism φ of T^*N is called to be <u>positively homogeneous</u> of degree one, if it commutes with multiplication by positive scalars. That is, if we write φ as $\varphi(x;\xi) = (\varphi_1(x;\xi);\varphi_2(x;\xi))$, then it satisfies $\varphi_1(x;r\xi) = \varphi_1(x;\xi)$, $\varphi_2(x,r\xi) = r\varphi_2(x;\xi)$,. for any $r>0$.

Let $\mathcal{D}_\Omega^{(1)}$ be the totality of symplectic diffeomorphisms of $\overset{o}{T}{}^*N$ of positively homogeneous of degree one. Then, we have proved that $\mathcal{D}_\Omega^{(1)}$ <u>is naturally identified with</u> $\mathcal{D}_\omega(S^*N)$, <u>the group of all contact transformations on the unit sphere bundle</u> S^*N, <u>and</u> $\mathcal{D}_\Omega^{(1)}$ <u>is a regular</u> **Fréchet-Lie group** (the precise definition of regular Fréchet-Lie group will be stated in §2.2 cf. [26] and Theorem 6.4 in [30]).

Now, in this paper, all derivatives of functions, tensors, etc., on TN, T^*N and S^*N, etc. are taken by using a normal coordinate system at the considered point (cf. [27], §1, and [29], §1, (15)).

We have restricted our concern to Fourier-integral operators on N with the following expressions:

(1.1) $(F_\varphi u)(x)$

$$= \sum_\alpha \iint \lambda_\alpha a(x;\xi;X) \times e^{-i\langle\varphi_2(x;\xi)|X\rangle - i|\xi|A_\alpha(\varphi_1(x;\xi);X)} (\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi$$

$$+ \ (K\ u)(x),$$

where we use the following notations:

(F.1) ν is a cut off function (cf. [27], p.365) with the small breadth ε, $0 < \varepsilon < r_1/12$, where r_1 is a small constant which depends only on the riemannian metric of N (cf. §4.2). $(\nu u)^\cdot(x;X) = \nu(x,\cdot_x X)u(\cdot_x X)$ (cf. [27], p.359).

(F.2) $a(x;\xi;X)$ is an element of $\tilde{\Sigma}^0_\varphi$, a class of amplitude functions (cf. [27], p.366, (13)).

(F.3) $K\in C^\infty(N\times N)$ and K u is an integral operator with the kernel $K(x,y)$ (cf. [27], (12)).

$\{\lambda_\alpha(x;\xi)\}$ is an appropriate partition of unity of $\overset{\circ}{T}{}^*N$ (cf. [27], p.373) such that $\lambda_\alpha(x;r\xi) = \lambda_\alpha(x;\xi)$ for any $r>0$, and $A_\alpha(y;X)$'s are quadratic forms written in the form $A_\alpha(y;X) = \sum_{i,j}A^{(\alpha)}_{ij}(y)X^iX^j$ added to $\langle\varphi_2(x;\xi)|X\rangle$ in order to make the phase function nondegenerate (cf. [27], pp.366-368).

<u>Remark.</u> There are in general a lot of ambiguities in

the choice of $\{A_\alpha\}$ and hence $\{\lambda_\alpha\}$. The expression (1.1) is one of the way of describing operators whose wave front set is given by graph $\varphi\subset T^*(N\times N)$ (cf. [9],[15],[16],[33]).

However, if φ is sufficiently close to the identity, one can set $A_\alpha = 0$, hence (1.1) can be written in the form:

(1.2) $(F_\varphi u)(x)$

$$= \iint a(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi$$

$$+ \quad (K\ u)(x).$$

Moreover, we can always eliminate the variables X in the amplitude a (cf. [27], §4 and Corrections). Thus, (1.2) can be rewritten as follows:

(1.3) $$(F_\varphi u)(x) = \int_{T^*N} b(x;\xi)\nu\tilde{u}(\varphi(x;\xi))d\xi + (K\ u)(x),$$

where

(1.4) $$\nu\tilde{u}(y;\eta) = \int_N e^{-i\langle\eta|Y\rangle}\nu(y,z)u(z)dz, \quad \cdot_y Y = z.$$

Now, the above expression (1.3) can be written as a composition of more "elementary operators". Remark that $\overset{\circ}{T}{}^*N$ is naturally diffeomorphic to $(0,\infty)\times S^*N$. We denote by $\mathcal{S}_N$ the space of all C^∞ functions f on $[0,\infty)\times S^*N$ such that $f(r,\omega)$ is rapidly decreasing as $r\to\infty$. In other words, by identifying $[0,\infty)$ with $[0,1)$ (cf. [27], p.364, (10)),

$\mathcal{S}_N$ is the space of all C^∞ functions on $[0,1]\times S^*N$ which are flat at $\{1\}\times S^*N$. Also, $\mathcal{S}_N$ is a Frèchet space and $\mathcal{D}_\Omega^{(1)}$ acts effectively and smoothly on $\mathcal{S}_N$ by $\varphi^*f(x;\xi) = f(\varphi(x;\xi))$, $\varphi\in\mathcal{D}_\Omega^{(1)}$, $f\in\mathcal{S}_N$. Note that the amplitude function $b(x;\xi)$ in (1.3) is an element of Σ_C^0 (cf. [27], p.365). For each $b\in\Sigma_C^0$, we shall denote by $b\cdot$ the multiplication operator by b. Then, $b\cdot$ is a continuous linear operator of $\mathcal{S}_N$ into itself.

Define maps π: $\mathcal{S}_N \to C^\infty(N)$, and ι:$C^\infty(N) \to \mathcal{S}_N$ as follows:

(1.5) $$\pi f(x) = \int_{T_x^*N} f(x;\xi)d\xi,$$

(1.6) $$\iota u(x;\xi) = \nu\tilde{u}(x;\xi) \qquad \text{(cf. (1.4))}.$$

By the formula of Fourier transformation, we have

(1.7) $$\pi\iota = \text{id}.$$

Using these operators (1.5) and (1.6), one can write (1.3) by

(1.8) $$F_\varphi = \pi\, b\cdot \varphi^* \iota + K .$$

Remark. (i) The above expression (1.2) or (1.3) still have ambiguities. Using F_φ, one can only know φ and the asymptotic expansion of b. Namely, one can replace (b,K) by another (b',K') to obtain the same operator F_φ (cf. [27] and Corrections).

(ii) By (1.7), the operator $\iota\pi$: $\mathcal{S}_N \to \mathcal{S}_N$ is a projection operator, i.e., $(\iota\pi)^2 = \iota\pi$.

Now, we shall state the main theorem. Let $\mathfrak{U}$, V_1, U_0 be a connected neighborhood of the identity of $\mathcal{D}_{\Omega}^{(1)}$, a neighborhood of 1 in Σ_C^0,. a neighborhood of 0 in $C^{\infty}(N\times N)$ respectively. Denote by $(\mathfrak{U}, V_{1,}U_0)$ the set of all Fourier-integral operators of the form (1.8) such that $\varphi\in\mathfrak{U}$, $a\in V_1$, $K\in U_0$. Note that if $\mathfrak{U}$, V_1, U_0 are sufficiently small, then every element in $\mathcal{N}(\mathfrak{U}, V_1, U_0)$ is invertible and the inverse is again in $\mathcal{N}(\mathfrak{U}, V_1, U_0)$. Also, denote by $G\mathfrak{J}_0^0$ the group generated by $\mathcal{N}(\mathfrak{U}, V_1, U_0)$. Then, theorem B in [28] shows that every element of $G\mathfrak{J}_0^0$ can be written in the form (1.1).

Now, the goal of §§2.1-2.7 is as follows:

<u>Theorem C.</u> $G\mathfrak{J}_0^0$ <u>is a regular Fréchet-Lie group</u>.

<u>Remark.</u> Once a manifold structure is established on $G\mathfrak{J}_0^0$, Proposition A in [29] shows that $\sqrt{-1}\,\mathfrak{g}^1$ is its tangent space at the identity. Hence, by Lemma 2.2 in [30], $\sqrt{-1}\,\mathfrak{g}^1$ is the Lie algebra of $G\mathfrak{J}_0^0$.

2.2 Regular Frèchet-Lie groups and its properties.

First, we shall give the definition of Regular Frèchet-Lie group.

An group is called FL-group if it is a topological group and a C^∞ Frèchet manifold such that the group operations are C^∞. Now, we start with considering a division $\Delta: a=t_0<t_1<\ldots<t_m=b$, of a closed interval $J=[a,b]$, we denote by $|\Delta|$ the maximum of $|t_{j+1}-t_j|$.

Let G be an FL-group and be its Lie algebra. A step function defined on $[0,\varepsilon]\times[a,b]$ is a pair (h,Δ) and division of $[a,b]$ such that $|\Delta|<\varepsilon$ and a mapping $h: [0,\varepsilon]\times[a,b] \dashrightarrow G$ satisfying the following:

(i) $h(0,t)=e$ for all $t\in[a,b]$, and $h(s,t)$ in C^1 for each fixed t.

(ii) $h(s,t)=h(s,t_j)$ for $(s,t)\in[0,\varepsilon]\times[t_j,t_{j+1})$.

Denote by $J=[a,b]$. A mapping $h:[0,\varepsilon]\times J \dashrightarrow G$ will be called a C^1 hair at e if

(i) $h(0,t)=e$ for all $t\in J$, and $h(s,t)$ is C^1 in s for each fixed t.

(ii) $h(s,t)$ and $(\frac{\partial h}{\partial s})(s,t)$ is C^0 with respect to $(s,t)\in[0,\varepsilon]\times J$.

Let ρ be a right-invariant metric on G mentioned in the previous section, and d a metric on by which is a

Frèchet space. Define a metric $\tilde{\rho}$ on the space of the union of C^1 hairs at e defined on $[0,\varepsilon]\times J$ as follows:

$$\tilde{\rho}(h,h')$$

$$= \max_{[0,\varepsilon]\times J} \rho(h(s,t),h'(s,t))$$

$$+ \max_{[0,\varepsilon]\times J} \left(\frac{\partial h(s,t)}{\partial s}h(s,t)^{-1}, \frac{\partial h'(s,t)}{\partial s}h'(s,t)^{-1}\right).$$

Given a C^1 hair h and a division Δ: of J, we define a step function $(\sigma_\Delta(h),\Delta)$ by

(*) $\quad \sigma_\Delta(h)(s,t) = h(s,t_j) \quad$ for $\quad t\in[t_j,t_{j+1})$.

An FL-group G will be called a <u>regular</u> <u>Frèchet</u>-<u>Lie</u> <u>group</u>, if the following condition is satisfied: Let $\{(h_n,\Delta_n)\}$ be any sequence in the set of the all step functions satisfying (*) for some ε and $J=[a,b]$ such that $\lim_{n\to\infty}|\Delta| = 0$ and $\lim_{n\to\infty} h_n = h$ in the topology defined by $\tilde{\rho}$. Then, the product integral

$$\Pi_a^t(h_n,\Delta_n) = h(t-t_k)h(\Delta t_k,t_{k-1})\ldots(t_1-s)$$

convergence uniformly in $t\in[a,b]$.

Regular Frèchet-Lie groups have many useful properties (cf. [30]). Here, we shall explain about the extension of regular Frèchet-Lie groups as a useful tool.

Define a mapping $\Phi: G\mathfrak{J}_0^0 \to \mathfrak{D}_\Omega^{(1)}$ by

(2.1) $$\Phi(F_\varphi) = \varphi^{-1}, \qquad \varphi\in \mathfrak{D}_\Omega^{(1)}$$

Then, in view of theorem 5.5 in [30], Φ is a well-defined homomorphism, and the image of Φ is the identity component of $\mathcal{D}_{\Omega}^{(1)}$. The kernel of Φ is $G\mathfrak{J}^0$, the group of invertible pseudo-differential operators of order 0 (cf. [30], (38)). Since $\mathcal{D}_{\Omega}^{(1)}$ is naturally isomorphic to $\mathcal{D}_{\omega}(S^*N)$, we have an exact sequences as follows:

$$(2.2)\quad 1 \dashrightarrow G\mathfrak{J}^0 \dashrightarrow G\mathfrak{J}_0^0 \dashrightarrow \mathcal{D}_{\omega}(S^*N) - \to 1,$$

where the dotted arrow indicates that the image of Φ is an open subgroup.

We note here that $\mathcal{D}_{\omega}(S^*N)$ <u>is a regular Fréchet-Lie group</u> (cf. [26],[30]) <u>and also that</u> $G\mathfrak{J}^0$ <u>is a regular Fréchet-Lie group</u>. Indeed, in [36], we have seen that $G\mathfrak{J}_{(m)}^0$ is a regular Fréchet-Lie group for $m \leqq -\dim N - 1$, and that $G\mathfrak{J}^0$ is a regular Fréchet-Lie group obtained by the inverse limit of $\{G\mathfrak{J}_{(m)}^0 ; m \leqq -\dim N - 1\}$.

<u>Remark.</u> In view of the arguments in [36], we can easily check the following. For every $m \leqq 0$, $G\mathfrak{J}_{(m)}^0$ is an open subset of $\mathfrak{J}(m)^0$ and is an FL-group (cf. [30]). The condition $m \leqq -\dim N - 1$ is used only to ensure the convergence of product integrals.

Now, we define a mapping $\gamma: \mathfrak{U} \to G\mathfrak{J}_0^0$ by

$$(2.3)\qquad \gamma(\varphi) = \pi\, \varphi^{-1*}\, \iota.$$

Obviously, $\Phi\gamma = \mathrm{id.}$, and γ gives a local cross section of (2.2). Define a mapping r_{γ} by

$$(2.4) \qquad r_\gamma(\varphi,\psi) = \gamma(\varphi\psi)^{-1}\gamma(\varphi)\gamma(\psi).$$

As $\Phi: G\mathfrak{J}_0^0 \to \mathcal{D}_\Omega^{(1)}$ is a homomorphism, r_γ is a mapping of $\mathfrak{U} \times \mathfrak{U}$ into $G\mathfrak{J}^0$.

On the other hand, define $\alpha_\gamma(\varphi,A)$, for every $\varphi \in \mathfrak{U}$, $A \in G\mathfrak{J}$ by

$$(2.5) \qquad \alpha_\gamma(\varphi,A) = \gamma(\varphi)^{-1}A\gamma(\varphi) \in G\mathfrak{J}^0.$$

Recall that the topology of $G\mathfrak{J}^0$ is obtained by the inverse limit of $\{G\mathfrak{J}^0_{(m)}; m \leq 0\}$. Hence, recalling Proposition 5.2 and Theorem 5.4 in [30], to obtain Theorem A, we have only to show the following:

Proposition 2.1. The mappings r_γ and α_γ, defined by (2.4) and (2.5) respectively, have the following properties:

(Ext. 1) $r_\gamma: \mathfrak{U} \times \mathfrak{U} \to G\mathfrak{J}^0$ is a C^∞ mapping of $\mathfrak{U} \times \mathfrak{U}$ into $G\mathfrak{J}^0_{(m)}$ for every $m \leq 0$.

(Ext. 2) $\alpha_\gamma: \mathfrak{U} \times G\mathfrak{J}^0 \to G\mathfrak{J}^0$ can be extended to a C^∞ mapping of $\mathfrak{U} \times G\mathfrak{J}^0_{(m)}$ into $G\mathfrak{J}^0_{(m)}$ for every $m \leq 0$.

Remark. By the above proposition, we see also that there is $G\mathfrak{J}^0_{(m)}$-extension of the identity component of $\mathcal{D}_\Omega^{(1)}$, which is an FL-group for each $m \leq 0$, and a regular Frèchet-Lie group for $m \leq -\dim N - 1$. We shall denote this extension by $G\mathfrak{J}^0_{0(m)}$. $G\mathfrak{J}$ is indeed the inverse limit of $\{G\mathfrak{J}^0_{0(m)}; m \leq 0\}$.

To prove (Ext. 1-2) in Proposition 2.1, we have to know first the inverse of $\gamma(\varphi)$. To do that, set

$$(2.6) \qquad \Xi(\varphi) = \pi(\varphi^{*} \iota \pi \varphi^{*-1})\iota,$$

then we shall show the following in §2.6:

<u>Proposition 2.2.</u> Notations being as above, we have

(a) if φ is sufficiently close to the identity, then $\Xi(\varphi)$ is a pseudo-differential operator of order zero, i.e., $\Xi(\varphi)\in \mathfrak{Z}^{0}(N)$.

(b) $\Xi: \mathcal{U} \to \mathfrak{Z}^{0}_{(m)}$, defined by $\Xi(\varphi)$ in (2.6), is smooth for every $m\leq 0$. Therefore, since $\Xi(\mathrm{id.}) = 1$,

(c) $\Xi(\varphi)$ is invertible if φ is sufficiently close to the identity.

By using Proposition 2.2, (c), we obtain for sufficiently small $\varphi\in \mathcal{D}^{(1)}_{\Omega}$,

$$(2.7) \qquad \gamma(\varphi)^{-1} = \Xi(\varphi)^{-1}\gamma(\varphi^{-1}).$$

Hence, if φ,ψ are sufficiently close to the identity, then $\Xi(\varphi\psi)$ is invertible. Thus, we obtain

$$(2.8) \qquad r_{\gamma}(\varphi,\psi) = \Xi(\varphi\psi)^{-1}\pi\{(\varphi\psi)^{*}\tau\pi(\varphi\psi)^{*-1}\}(\varphi^{*}\tau\pi\psi^{*-1})\tau.$$

On the other hand, any $A\in G\mathfrak{Z}^{0}$ can be expressed as follows:

$$(2.9) \qquad A = \pi a\cdot \tau + K ,$$

where $a\in\Sigma^{0}_{C}$ and $K\in C^{\infty}(N\times N)$. Hence, we have

$$(2.10)\qquad \alpha_\gamma(\varphi,A) = \Xi(\varphi)^{-1}\pi(\varphi^* z\pi\varphi^{*-1})(\varphi^* a\cdot\varphi^{*-1})(\varphi^* \iota\pi\varphi^{*-1})z$$

$$+\ \Xi(\varphi)^{-1}\pi(\varphi^* \iota K\ \pi\varphi^{*-1})z.$$

Note that $\varphi^* a\cdot\varphi^{*-1} = (\varphi^* a)\cdot$, and one may write

$$(2.11)\qquad \alpha\gamma(\varphi,A) = \Xi(\varphi)^{-1}\pi(\varphi^* z\pi\varphi^{*-1})(\varphi^* a)\cdot(\varphi^* z\pi\varphi^{*-1})z$$

$$+\ \Xi(\varphi)^{-1}\pi(\varphi^* zK\ \pi\varphi^{*-1})z.$$

The above computations show that operators of the form

$$\varphi^* \iota\pi\varphi^{*-1},\qquad (\varphi^* a)\cdot,\qquad \varphi^* \iota K\ \pi\varphi^{*-1}$$

and their composition rules play an important role in studying r_γ and α_γ. Thus, we shall set up a certain class of operators $\mathcal{M}$, containing $\varphi^* z\pi\varphi^{*-1}$ for every φ which is sufficiently close to the identity. $\mathcal{M}$ is indeed a C^∞ Frèchet manifold and a local semi-group with <u>smooth</u> semi-group operations (cf. §§2.4-2.6). Moreover, we shall see is closed under the multiplication by $\varphi^* a$. This is indeed smooth with respect to φ, a and $p\in\mathcal{M}$(cf. §2.6-2.7). Next, we shall prove that the "projection" $\mathcal{M}\to G\mathcal{J}^0_{(m)}$, $m\leq 0$, $P\to\pi P_z$ is smooth (cf. §2.6, Proposition 6.2).

Denote by $E(\varphi) = \varphi^* z\pi\varphi^{*-1}$. Then, E can be regarded as a smooth mapping of $\mathcal{U}$ into $\mathcal{M}$ (cf. §6.6). Thus, by using these smoothness properties of $\mathcal{M}$, we see that (2.8) and the first term of (2.11) are smooth. To treat the second term of (2.11), we shall need the following proposition which will be proved in §2.6 as well as some other smoothness properties stated above:

Proposition 2.3. For every $\varphi \in \mathcal{D}_{\Omega}^{(1)}$, and $K \in C^{\infty}(N \times N)$, put $\Lambda(\varphi, K) = \pi(\varphi^{*} zK\, \pi\varphi^{*-1})z$. Then, we get

(a) $\Lambda(\varphi, K)$ is a linear operator on $C^{\infty}(N)$ with C^{∞} kernel $L(\varphi, K)$ (cf. Lemma 6.4).

(b) The mapping $L: \mathcal{D}_{\Omega}^{(1)} \times C^{\infty}(N \times N) \to C^{\infty}(N \times N)$ is smooth.

2.3 Several properties of primordial operators.

First, we shall compute the kernel of $\varphi^* \tau\pi\varphi^{*-1}$ and $\varphi^* \tau K\, \pi\varphi^{*-1}$. Recall the definition of π and τ (cf. (1.5) and (1.6)). The, we have

$$(3.1)\quad (\tau\pi f)(x:\xi) = \iint \nu(x,y)e^{-i\langle\xi|\cdot^{x}y\rangle} f(y;\eta)dyd\eta,\ f\in \quad N'$$

where $\cdot^{x}y = Y$ implies $\cdot_{x}Y = y$. Let $\tau_0(x;\xi,y)$ be a smooth extension of $\langle\xi|\cdot^{x}Y\rangle$ onto $\overset{o}{T}{}^*N\times\overset{o}{T}{}^*N$ such that

$$(P.0)\quad \tau_0(x;r\xi,y) = r\tau_0(x;\xi,y), \qquad r>0.$$

Then, τ_0 has the following properties:

<u>Lemma 3.1.</u> For given $r_1>0$ in (F.1), if $d(x,y) < 2r_1/3$, then $\tau_0(x;\xi,y)$ has no critical point in $(x;\xi)$ for every y and $\tau_0(x;\xi,y)$ has no critical point in y for every $(x;\xi)\in\overset{o}{T}{}^*N$.

Now, $\tau\pi$ can be regarded as an integral operator with smooth kernel $\nu(x,y)e^{-i\tau_0(x;\xi,y)}$, hence the kernel of $\varphi^*\tau\pi\varphi^{*-1}$ is given by $(\varphi^*\nu)e^{-i\varphi^*\tau_0}$ because of $\varphi^*dyd\eta =$ $dyd\eta$, where $(\varphi^*\nu)(x;\xi,y;\eta) = \nu(\varphi_1(x;\xi),\varphi_1(y,\eta))$.

Similarly, the kernel of $\tau K\,\pi$ is given by

$$(3.2)\qquad a_K(x;\xi,y) = \int \nu(x,z)K(z,y)e^{-i\tau_0(x;\xi,z)}dz.$$

Since $\nu(x,z)K(z,y)$ has a compact support in z, $a_K(x;\xi,y)$ is rapidly decreasing in $|\xi|$. Hence, the kernel of $\varphi^* zK\, \pi\varphi^{*-1}$ is given by

$$(3.3) \qquad (\varphi^* a_K)(x;\xi,y;\eta) = a_K(\varphi(x;\xi),\varphi(y;\eta)).$$

To unify z_0 and $\varphi^* z_0$, we introduce a class of functions, which correspond to "phase functions " defined later.

Let $z(x;\xi,y;\eta)$ be a smooth function on $\overset{\circ}{T}{}^*N\times\overset{\circ}{T}{}^*N$ which satisfies

(P.1) $z(x;r\xi,y:s\eta) = rz(x;\xi,y;\eta)$ for any $r>0$, $s>0$.

The above z is considered as a smooth function on $[0,\infty)^2\times(S^*N)^2$ by putting

$$(3.4) \qquad \tilde{z}(r,s,x;\hat{\xi},y;\hat{\eta}) = z(x;r\hat{\xi},y;s\hat{\eta}).$$

For above z, define a subset $C(z)$ of $\overset{\circ}{T}{}^*N\times\overset{\circ}{T}{}^*N$ by

$$(3.5) \qquad C(z) = \{(x;\xi,y;\eta)\in\overset{\circ}{T}{}^*N\times\overset{\circ}{T}{}^*N \,;\, \nabla_{(x;\xi)}z = 0 \text{ or } \nabla_{(y;\eta)}z = 0\}.$$

Then, $C(z)$ is <u>conic</u>, that is, $(x;\xi,y;\eta)\in C(z)$ if and only if $(x;r\xi,y;s\eta)\in C(z)$ for every $r>0$, $s>0$.

Consider the following property for z;

(P.2) $C(z)$ is bounded away from the diagonal set.

It is obvious that z_0 satisfies (P.1) and (p.2), and that such properties are invariant under the action of $\mathcal{D}_\Omega^{(1)}$, hence $\varphi^* z_0$ satisfies (P.1-2) for every $\varphi\in\mathcal{D}_\Omega^{(1)}$.

However, what we shall need in the computation is not a general z with (P.1-2) but $\varphi^* z_0$, $\varphi \in \mathcal{D}_\Omega^{(1)}$, or "0". Thus, we have to consider $\varphi^* z_0$ more precisely.

First of all recall that each $\varphi \in \mathcal{D}_\Omega^{(1)}$ leaves the canonical 1-form θ invariant, where θ is given locally by $\theta = \sum \xi_i dx^i$. This fact gives the following:

<u>Lemma 3.2.</u> For each $\varphi \in \mathcal{D}_\Omega^{(1)}$, $\varphi^* z_0$ can be written by

$$\text{(P.3)} \qquad \varphi^* z_0 = z_0 + Q(z)(x;\xi,y;\eta),$$

and Q vanishes at $(x;\xi) = (y;\eta)$ up to the first derivatives.

<u>Proof</u>. Use a normal coordinate system $(y^1,\ldots,y^n)$ at x and its dual coordinate system $(\xi_1,\ldots,\xi_n)$. Then, we get

$$\langle \xi \,|\, \cdot^x y \rangle = \xi_i y^i, \quad \text{and} \quad \theta = \xi_i dy^i.$$

For $\varphi \in \mathcal{D}_\Omega^{(1)}$, we use a normal coordinate system and dual coordinate system at $\varphi_1(x;\xi)$. Denote $\varphi(y;\eta)$ by $(\bar{y}^1,\ldots,\bar{y}^n,\bar{\xi}_1,\ldots,\bar{\xi}_n)$. Then, letting $\bar{\xi} = \varphi_2(x;\xi)$, we have

$$\langle \varphi_2(x;\xi) \,|\, \cdot^{\varphi_1(x;\xi)} \varphi_1(y;\eta) \rangle = \bar{\xi}_i \bar{y}^i.$$

Remark that $\varphi^*\theta = \theta$ means that

$$\bar{\xi}_i \frac{\partial \bar{y}^i}{\partial y^j} = \xi_j, \qquad \bar{\xi}_i \frac{\partial \bar{y}^i}{\partial \eta_j} = 0 \qquad \text{(cf. [27], (25))}.$$

Put

$$\bar{y}^i = \frac{\partial \bar{y}^i}{\partial y^j}(o;\xi) y^j + \frac{\partial \bar{y}^i}{\partial \eta_j}(0;\xi)(\eta_j - \xi_j) + H(\varphi)(y,\eta-\xi),$$

where $H(\varphi)(y,\eta-\xi)$ is the quadratic term with respect to y and $\eta-\xi$. Then, we have

$$\varphi^* z_0 = \langle \varphi_2(x;\xi) | \cdot^{\varphi_1(x;\xi)} \varphi_1(y;\eta) \rangle$$

$$= \xi_i y^i + \langle \xi | H(\varphi)(y,\eta-\xi) \rangle .$$

So, $Q(\varphi)$ is given by the last term of the above equality.

We define amplitude functions associated with z in 3.2. Let z be a C^∞ functions on $\overset{o}{T}{}^*N \times \overset{o}{T}{}^*N$ which satisfies (P.1-2) in 3.2. (Remark that the property (P.3) is not used in this section.) For above z, we denote by $\mathcal{B}(z)$ the linear space of smooth functions h on $\overset{o}{T}{}^*N \times \overset{o}{T}{}^*N$ such that

(B.1) h is a C^∞ function on $[0,\infty)^2 \times (S^*N)^2$ and all derivatives of h are bounded.

(B.2) There exists a conic neighborhood V_h of $C(z)$ on which $h(r,x;\hat{\xi},y;\eta) = h(x;r\hat{\xi},y;\eta)$, $r\hat{\xi} = \xi$, is rapidly decreasing as $r\to\infty$.

Recall the kernels obtained in 3.1 and we know the significance of the following:

<u>Lemma 3.3.</u> (a) $\nu(x,y) \in \mathcal{B}(z_0)$. (b) $\varphi^*\nu \in \mathcal{B}(\varphi^* z_0)$ for any $\varphi \in \mathcal{D}^{(1)}_\Omega$. (c) $\varphi^* a_K(x;\xi,y) \in \mathcal{B}(0)$.

<u>Proof</u>. Since $\nu=0$ on a neighborhood $C(z_0)$, we get (a). (b) and (c) are easily obtained by a direct computation of derivatives.

Let z satisfy (P.1-2) in (3.2) and let $a\in \quad (z)$. Consider the following operator

$$(3.6) \qquad P(a,z)f(x;\xi) = \iint a(x;\xi,y;\eta)e^{-iz(x;\xi,y;\eta)}f(y;\eta)dyd\eta, \qquad f\in \mathcal{S}_N.$$

By Lemma 3.2, $\varphi^{*}\iota\pi\varphi^{*-1}$, and $\varphi^{*}\iota K\,\pi\varphi^{*-1}$ are written in the above form (3.6), which will be called <u>primordial</u> <u>operators</u> in this paper.

Now, we can give a rigid meaning of (3.6) as an operator as follows:

<u>Proposition 3.4.</u> Let z satisfy (P.1-2) and let $a\in \mathcal{B}(z)$. Then, $P(a,z)$ in (3.6) defines a linear operator on $_N$ into itself.

<u>Proof</u>. Let $\varphi(x;\xi,y;\eta)$ be a smooth function such that $\varphi(x;r\xi,y;s\eta) = \varphi(x;\xi,y,\eta)$, $r>0$, $s>0$, and $\varphi\equiv 1$ on a neighborhood of $C(z)$ and $\operatorname{supp}\varphi\subset V_a$ (cf. (B.2) for the notation V_a).

Divide (3.6) into two parts:

$$(3.7) \qquad P(a,z)f(x;\xi) = \iint \varphi a e^{-iz} f dyd\eta + \iint (1-\varphi)ae^{-iz}fdyd\eta$$
$$= P_1 + P_2.$$

Since φae^{-iz} is rapidly decreasing in $|\xi|$, we see that $P_1\in \mathcal{S}_N$ for every $f\in \quad _N$. Now, consider P_2. Remark that on the support of $(1-\varphi)a$, z has no critical point in (y,η). So, let

$$L_z = \frac{1 + ir(\nabla_{\tilde{\eta}} z \, \nabla_{\tilde{\eta}} + \nabla_y z \, \nabla_y)}{1 + r^2 |\nabla_{(y;\hat{\eta})} z|^2}, \qquad r = |\xi|, \qquad z = z(x;\hat{\xi},y;\hat{\eta}).$$

Then, $L_z e^{-iz} = e^{-iz}$ and the coefficients of the operator L_z can be bounded by r^{-1} for sufficiently large $r>0$. So, P_2 can be written as

$$(3.8) \qquad P_2(x;\xi) = \int_{S^*N}\int_0^\infty (1-p)a(x;r\hat{\xi},y;s\hat{\eta})(L_z)^l$$

$$\times e^{-irz(x;\hat{\xi},y;\hat{\eta})} f(y;s\hat{\eta}) s^{n-1} dy ds d\hat{\eta}.$$

Repeating the integration by parts, we see that $P_2(x;\xi)$ is rapidly decreasing in $|\xi|$. Smoothness at $r=0$ of $P_2(x;\xi)$ follows from those of $\varphi(x;r\hat{\xi},y;\eta)$ and $a(x;r\hat{\xi},y;\eta)$ at $r=0$.

Finally, we remark that in what follows we shall restrict our concern to much narrower class of amplitudes. The main reason to do so is that $\mathcal{B}(z)$ is not invariant under $\mathcal{D}_\Omega^{(1)}$. The restricted class is invariant under $\mathcal{D}_\Omega^{(1)}$ and contains Σ_C^0, though $\varphi^* a \in \Sigma_C^0$ even if $a \notin \Sigma_C^0$.

2.4 Phase functions of primordial operators.

Now, to fix the restricted class of primordial operators, we shall introduce a class of phase functions and study the properties of compositions of phase functions induced by the composition of primordial operators.

Let Θ be the space of all C^∞ functions z on $\overset{\circ}{T}{}^*N\times\overset{\circ}{T}{}^*N$ satisfying (p.1) in 3.2. Since such z is uniquely determined by the values on $\overset{\circ}{S}{}^*N\times\overset{\circ}{S}{}^*N$, we shall give a topology fo Θ by using the C^∞ topology on $S^*N\times S^*N$. Denote by Θ_0 the closed affine subspace of Θ defined by

$$(4.1)\qquad \Theta_0 = \{z\in\Theta : z-z_0 \text{ vanishes on the diagonal set up to the first derivatives}\}.$$

Remark that every $z\in\Theta_0$ satisfies (P.2) and (P.3) in Lemma 3.2. $\mathcal{D}_\Omega^{(1)}$ acts on Θ by the following: Given $\varphi\in\mathcal{D}_\Omega^{(1)}$, $z\in\Theta$, we define mapping ev: $\mathcal{D}_\Omega^{(1)}\times\Theta\to\Theta$ by

$$(4.2)\qquad ev(\varphi,z) = \varphi*z(x;xi,y;\eta) = z(\varphi(x;\xi),\varphi(y;\eta)).$$

Then, we have

<u>The mapping</u> ev: $\mathcal{D}_\Omega^{(1)}\times\Theta\to\Theta$ is a C^∞ mapping and which leaves Θ_0 invariant.

<u>Proof</u>. The smoothness of ev is obvious by that of composition of mappings (cf. [2], [24],[31]). The invariance of Θ_0 follows from Lemma 3.1.

Given $z_1, z_2 \in \Theta$, we define a composition $z_1 + z_2$ as a function on $\overset{\circ}{T}{}^*N \times \overset{\circ}{T}{}^*N \times \overset{\circ}{T}{}^*N$ by

$$(4.3) \qquad z_1 + z_2(x;\xi,y;\eta,z;\zeta) = z_1(x;\xi,y;\eta) + z_2(y;\eta,z;\zeta).$$

For a later use, we have to know at first the critical point and the critical value of (4.3) with respect to $(y;\eta)$. However, this is not so easy in general. Thus, we shall do this under the assumption that $z_1, z_2 \in \Theta_0$ and they are sufficiently close to z_0. Moreover, we shall restrict the domain of $z_1 + z_2$ Onto $d(x,y) \leq r_1/2$, $d(x,z) \leq r_1/4$, where r_1 is constant depending only on the riemannian structure of N, which will be given below.

On this restricted domain, one may set $y = \cdot_x X$, $z = \cdot_y Y = \cdot_x Z$ and $(y;Y,\eta) = \cdot_x(X,Y',\eta')$ by using the normal coordinate system at x. Y' is given by $Y' = \tilde{S}(x;Z,X) = \tilde{S}_1(x;Z,X)(Z-X)$ (cf. [27], p.360, (3)). The constant r_1 is defined by the supremum of r such that $\partial_X \tilde{S}|_{X=0}$ and $\tilde{S}_1(x;Z,X)$ are invertible matrices whenever $d(x,z) \leq r$. For the standard sphere, $r_1 = \pi/2$ and for many riemannian manifolds, r_1 is given as a half of the injectivity radius.

Set $z_i = z_0 + Q_i$ $(i=1.2)$. Then, $z_1 + z_2$ can be written in the form

$$(4.4) \qquad \langle \xi | X \rangle + \langle \eta' | \tilde{S}(x;Z,X) \rangle + Q_1(x;\xi,\cdot_x(X,\eta')) + Q_2(\cdot_x(X,\eta')z;\zeta).$$

Thus, consider the equations

(4.5) $\quad \partial_X(z_1+z_2) \quad = \quad \xi + \langle\eta' | \partial_X\tilde{S}\rangle + \partial_X Q_1 + \partial_X Q_2 = 0,$

(4.6) $\quad \partial_{\eta'}(z_1+z_2) \quad = \quad \tilde{S}(x;Z,X) + \partial_{\eta'} Q_1 + \partial_{\eta'} Q_2 = 0.$

Lemma 4.2. Suppose z_1 and z_2 are sufficiently close to z_0 in Θ_0 and suppose $d(x,y) \leq r_1/2$, $d(x,z) \leq r_1/4$. Then, we obtain the following:

(i) The equation $\partial_y(z_1+z_2) = 0$ can be solved uniquely with respect to η. Let $\bar{\eta}$ be its solution. Then, $\bar{\eta} = \bar{\eta}(x;\xi,y,z;\zeta)$ is C^∞ and $\bar{\eta}(x;r\xi,y,z;s\zeta) = r\bar{\eta}(x;\xi,y,z;\zeta)$ for any r>0, s>0.

(ii) There are constants C>0, M>0 such that

(4.7) $$|\partial_y(z_1+z_2)| \leq M(|\xi|+|\eta|)$$
$$\text{if} \quad |\eta| \geq C|\xi| \quad \text{or} \quad |\eta| \leq C^{-1}|\xi|.$$

Proof. One may assume that there are small $\delta>0$ and a constant $K>0$ such that $|\partial_X(Q_1+Q_2)| \leq \delta(|\xi|+|\eta'|)$, $K^{-1} \leq |\partial_X\tilde{S}| \leq K$, $K^{-1} \leq |(\partial_X\tilde{S})^{-1}| \leq K$. By (4.5), we see easily that if $\bar{\eta}$ exists then $\bar{\eta}$ must satisfy

$$\tfrac{1}{2}C^{-1}|\xi| \leq |\bar{\eta}'| \leq 2C|\xi|, \qquad \cdot_X(X,\bar{\eta}') = (y;\bar{\eta})$$

for some constant C≧2. Moreover, on this domain on may set

$$|\partial_{\eta'}\partial_X(z_1+z_2) - \partial_X\tilde{S}| \leq \delta(\tfrac{|\eta'|}{|\xi|} + 1) \leq \delta(2C + 1).$$

It follows that $\partial_{\eta'}\partial_X(z_1+z_2)$ is non-singular matrix on the conical domain: $d(x,y) \leq r_1/2$, $d(x,z) \leq r_1/4$, $\frac{1}{2}C^{-1}|\xi| \leq |\eta| \leq 2C|\xi|$.

Suppose $Q_1 = Q_2 = 0$ in (4.5). Then, it has the unique solution $\bar{\eta} = -\xi(\partial_x \tilde{S})^{-1}$. By means of the implicit function theorem (cf. [31], Lemma 4.9) on the above conical domain, we obtain the unique existence of $\bar{\eta}$. Smoothness of $\bar{\eta}$ follows from the regularity of $\partial_y \partial_{y'}(z_1+z_2)$, and the homogeneity of $\bar{\eta}$ follows from those of z_1, z_2.

Now, suppose $|\eta| \geq C|\xi|$ or $\eta \leq C^{-1}|\xi|$. Then, $\partial_y(z_1+z_2)$ cannot attain 0. Hence, there must be a constant M such that $|\partial_y(z_1+z_2)| \geq M(|\xi|+|\eta|)$.

<u>Proposition 4.3.</u> Suppose z_1, z_2 are sufficiently close to z_0 in θ_0. If $d(x,y) \leq r_1/2$, $d(x,z) \leq r_1/4$, then

(i) the function z_1+z_2 has only one critical point $(y_c;\eta_c)$, which is non-degenerate;

(ii) the critical point $(y_c;\eta_c)$ depends smoothly on $(x;\xi,z:\zeta)$ and satisfies

$$(4.8)\qquad \begin{cases} y_c(x;r\xi,z;s\zeta) = y_c(x;\xi,z;\zeta) \\ \eta_c(x;r\xi,z;s\zeta) = r\eta_c(x;\xi,z;\zeta) \end{cases} \qquad r>0, \quad s>0;$$

(iii) the critical value $z_{12} = (z_1+z_2)(z;\xi,y_c;\eta_c,z;\zeta)$ has the properties (P.1-3) in the variables $(x,\xi,z;\zeta)$.

<u>Proof</u>. We substitute $\eta' = \bar{\eta}'(x,\xi,y,z;\zeta)$ into (4.6). Note that $\partial_{\eta'}(z_1+z_2)(x;\xi,y;\bar{\eta},z;\zeta)$ is homogeneous of degree zero with respect to ξ. Suppose $Q_1 = Q_2 = 0$. Then, (4.6) has the unique solution $X_c = Z$, i.e., $y_c = z$. Recall that

$\partial_X \partial_{\eta'}(z_1+z_2)$ is invertible. Hence, the implicit function theorem (cf. [31], Lemma 4.9) implies (i). The uniqueness of $(y_c;\eta_c)$ and the homogeneity of z_1, z_2 yields (ii), which indicates that z_{12} satisfies (P.1). As for (P.2), (P.3) in (iii), we may consider near the diagonal set. Put $(z;\zeta) = (z;\xi)$, i.e., $\cdot_X(Z,\zeta') = \cdot_X(0,\xi)$ in (4.5) and (4.6). Then, the first derivatives of Q_1, Q_2 vanish at $(y;\eta) = (x;\xi)$, so we get $\cdot_X(X_c,\eta_c') = \cdot_X(o,\xi) = (x;\xi)$. Hence, the Taylor expansion of (X_c,η_c') with respect to (Z,ζ') at (o,ξ) is

$$\begin{cases} X_c = aZ + b(\xi-\eta') + \dots, \\ \eta_c' = \xi + cZ + d(\xi-\zeta') + \dots \end{cases}$$

Substituting this into z_{12}, we see that z_{12} has the properties (P.2) and (P.3).

Next, we shall observe the critical value z_{12} more carefully. Choose a C^∞ function ψ on **R** such that $\psi \equiv 1$ on $|t| \leq r_1/5$ and $\psi \equiv 0$ on $|t| \geq r_1/4$, and define a function $c(z_1,z_2)$ by

$$(4.9) \quad c(z_1,z_2) = \psi(d(x,z))z_{12}(x;\xi,z;\zeta) + (1-\psi(d(x,z))z_0(x;\xi,z).$$

c can be regarded as a function of z_1, z_2. By Proposition 4.3, we see also

<u>Lemma 4.4.</u> $c(z_1,z_2) \in \Theta_0$ for z_1, z_2 sufficiently close to

z_0. c is a C^∞ mapping of $U_{z_0} \times U_{z_0}$ into Θ_0, such that $c(z_0, z_0) = z_0$, where U_{z_0} is a small neighborhood of z_0 in Θ_0.

Proof. The desired smoothness follows from the implicit function theorem (cf. [31], Lemma 4.9). The property $c(z_0, z_0) = z_0$ is obtained by the computations in the case $Q_1 = Q_2 = 0$.

The following is s special case of Proposition 4.3.

Corollary 4.5. Let $z \in \Theta_0$ be sufficiently close to z_0. Then, $c(z_1, z_0)$ does not involve the ζ-variable, i.e., $c(z, z_0) = c(z, z_0)(x;\xi, z)$. Moreover, it is written in the form $z_0(z;\xi, z) + Q(z;\xi, z)$, where Q satisfies $Q(x;\xi, x) = 0$, $(\partial Q/\partial Z)_{z=0}(z;\xi, \cdot_x Z) = 0$.

Now, set $T = z_1 + z_2 - c(z_1, z_2)$. Using Proposition 4.3 and Lemma 4.2, we have the following properties of T.

Corollary 4.6. With the same notations as in Proposition 4.3, T has the following properties:

(T.1) $T(x;\xi, y;\eta, z;\zeta)$ is positively homogeneous of degree 1 in $\theta = (\xi, \eta)$ and degree zero in ζ.

(T.2) There are constants $C>0$, $M>0$ such that $|\partial_y T| \geq M(|\xi|+|\eta|)$ if $|\eta| \leq C^{-1}|\xi|$ or $|\eta| \leq 2C|\xi|$.

(T.3) If $\frac{1}{2}C^{-1}|\xi| \leq |\eta| \leq 2C|\xi|$, then on any conical subset in $\overset{\circ}{T}{}^*N \times \overset{\circ}{T}{}^*N \times \overset{\circ}{T}{}^*N$ bounded away from the critical set

$\{x;\xi,y_c;\eta_c,z;\zeta)\}$, there is $\delta>0$ such that $|\nabla_{(y;\eta)}\hat{T}| \geqq \delta$ on $\frac{1}{2}C^{-1} \leqq |\eta| \leqq 2C$, where $\hat{T} = T(a;\hat{\xi},y;\eta,z;\hat{\zeta})$.

<u>Proof</u>. We have only to show (T.3). Since T has no critical point on the considered domain and $(x;\hat{\xi},y,\eta,z;\hat{\zeta})$ moves in a compact set, we see the existence of $\delta>0$.

We continue to assume that z_1,z_2 are sufficiently close to z_0 in Θ_0, and let r_1 be as in 4.2. Let $(y_c;\eta_c)$ be the critical point in the domain $d(x,y) \leqq r_1$, $d(x,z) \leqq r_1/4$. Recall that if $Q_1 = Q_2 = 0$, then $(y_c;\eta_c) = \cdot_X(Z,-\xi(\partial_X\tilde{S})^{-1}|X+Z)$. Therefore, one may assume that there is $\delta>0$ such that $|X_c-Z| \leqq \delta$, $|\eta_c' + \xi(\partial_X\tilde{S})^{-1}| \leqq \delta|\xi|$ in general, whenever z_1,z_2 are sufficiently close to z_0.

Denote by D_δ the domain given by

$$(4.10) \qquad D_\delta = \{(x;\xi,\cdot_X(X,\eta'),\cdot_X(Z,\zeta')) \; ; \; |X| \leqq r_1/2,\ |Z| \leqq r_1/4,\ |\eta' + \xi(\partial_X\tilde{S})^{-1}| \leqq \delta|\xi|\}.$$

Obviously, $(x;\xi,\cdot_X(X_c,\eta_c'),z;\zeta) \in D_\delta$. Moreover, the index of the critical point $(y_c;\eta_c)$ is the same as that of z_0+z_0 and hence 0. Thus, by a suitable change of coordinate on a neighborhood of $(y_c;\eta_c)$, T can be expressed in the form $-\langle\eta'-\eta_c'|X-X_c\rangle$. This is known as the Morse lemma. However, the proof of the Morse lemma shows more precisely the following:

<u>Proposition 4.7.</u> Suppose that $\delta>0$ is sufficiently small. There are an open neighborhood D' of D_δ and a C^∞

diffeomorphism $\tilde{\Phi}$ of D' into $(\overset{\circ}{T}{}^*N)^3$ such that $\tilde{\Phi}(x;\xi,y;\eta,z;\zeta)$ $= (x;\xi,\tilde{\Phi}_1(*);\tilde{\Phi}_2(*),z;\zeta)$ and satisfy the following

(i) $\tilde{\Phi}(D') \supset D_\delta$.

(ii) $\tilde{\Phi}_1(z;r\xi,y,r\eta,z;s\zeta) = \tilde{\Phi}_1(x;\xi,y;\eta,z;\zeta)$,

(iii) $\tilde{\Phi}_2(x;r\xi,y;r\eta,z;s\zeta) = r\tilde{\Phi}_2(x;\xi,y;\eta,z;\zeta)$

for any $r>0$, $s>0$.

(iv) $\tilde{\Phi}$ <u>depends smoothly on</u> z_1, z_2.

(v) $\tilde{\Phi}^*T = -\langle\eta'-\eta_c'|X-X_c\rangle$.

The above proposition will be proved in several lemmas below. At first, denote $T_0 = z_0+z_0 - c(z_0,z_0)$. Since the critical point $(y_c;\eta_c) = \cdot_X(X_c,\eta_c')$ in this case, is given by $(X_c,\eta_c') = (Z,-\xi(\partial_X\tilde{S})^{-1}|_{X=Z})$, we see that

$$T_0(x;\xi,\cdot_X(X,\eta'),z;\zeta) = \langle\xi-\eta'\tilde{S}_1(x;Z,X)|X-Z\rangle$$

$$= \langle-\eta_c'(\partial_X\tilde{S})_{|X=Z} - \eta'\tilde{S}_1(x;Z,X)|X-X_c\rangle,$$

where $\tilde{S}(x;Z,X) = \tilde{S}_1(x;Z,X)(Z-X)$. Using $\tilde{S}_1(x;Z,Z) = -\partial_X\tilde{S}|_{X=Z}$, we see that T_0 can be written in the form

$$T_0 = -\langle\eta'-\eta_c'|X-X_c\rangle + S_c(X-X_c)^2,$$

where $S_c = S_c(x;\xi,\cdot_X(X,\eta'),z;\zeta)$ and $S_c = O(|\eta'|)$.

<u>Lemma 4.8.</u> On a neighborhood D' of D_δ, $T(x;\xi,\cdot_X(X,\eta'),z;\zeta)$ can be written as

$$T = A_1(\eta'-\eta_c')^2 + (-1+A_2)(\eta'-\eta_c')(X-X_c) + (S_c+A_3)(X-X_c)^2,$$

where $A_i = A_i(x;\xi,\cdot_x(X,\eta'),z:\zeta)$ and A_1, A_2, A_3 are positively homogeneous of degree -1, 0, 1, respectively with respect to the combined variable $\theta=(\xi,\eta')$ and of degree 0 with respect to ζ. Moreover, if $z_1 \to z_0$, $z_2 \to z_0$, then $|A_1||\theta| \to 0$, $|A_2| \to 0$, $|A_3|/|\theta| \to 0$ uniformly on D'.

Proof is easy by using Taylor's theorem at (X_c,η_c').

Now, consider a quadratic form $h(\xi,X)$ on $\mathbf{R}^n\times\mathbf{R}^n$ such that

$$h = P^{ij}\xi_i\xi_j + (\delta^i_j + l^i_j)\xi_i X^j + R_{ij}X^iX^j,$$

where (δ^i_j) is the identity matrix.

<u>Lemma 4.9.</u> Suppose $|l^i_j|$ and $|P^{ij}R_{kl}|$ are sufficiently small for all i, j, k, l. Then, there are matrices (a^i_j), (f^i_j) depending smoothly on (P^{ij}), (l^i_j), (R_{ij}) such that

$$h = (\xi_i + a_{ij}X^j)(f^i_k X_k + P^{il}\xi_l)$$

and $|f^i_k - \delta^i_k|$ are sufficiently small.

<u>Proof</u> has been done by using the implicit function theorem (cf. [29], pp. 243-244). We have only to solve

$$(4.11) \qquad f^l_j + P^{li}(f^{-1})^k_i R_{kj} = \delta^l_j + l^l_j,$$

and set $a_{ij} = (f^{-1})^k_i R_{kj}$.

Set $(a_{ij}) = \Phi(P,l,R)$, $(f^i_j) = \Psi(P,l,R)$ and apply the above lemma to our $-T$, then, we have the following:

<u>Lemma 4.10.</u> On the domain D', $-T$ can be expressed in the form

$$-T = \langle \eta' - \eta_c' + \Phi(A_1,A_2,A_3)(X-X_c) \,|\, \Psi(A_1,A_2,A_3)(X-X_c) - A_1(\eta'-\eta_c') \rangle .$$

Moreover, $\Phi(A_1,A_2,A_3)$ (resp. $\Psi(A_1,A_2,A_3)$) is positively homogeneous of degree 1 (resp. 0) in the variable θ and Φ, Ψ are positively homogeneous of degree 0 in the variable ξ'.

<u>Proof</u>. We have only to show the second statement. Recall the homogeneity property of A_i. Since $(P^{ij}) = -A_1$, $(l^i_j) = -A_2$, $(R_{ij}) = -(S_c+A_3)$, the equation (4.11) shows that (f^i_j) is positively homogeneous of degree 0 with respect to θ. Hence, by the equality $a = f^{-1}R$, we get the desired property.

<u>Proof of Proposition 4.7.</u>

Now, set

$$(4.12) \qquad \begin{cases} \bar{\eta}' - \eta_c' = \eta' - \eta_c' + \Phi(A_1,A_2,A_3)(X - X_c) \\ \\ \bar{X} - X_c = \Psi(A_1,A_2,A_3)(X - X_c) - A_1(\eta' - \eta_c') \end{cases}$$

The estimates for A_i's in Lemma 4.8 yield that the Jacobian $\frac{D(\bar{\eta},\bar{X})}{D(\eta',X)}$ never vanishes. So the above equation can be solved reversely with respect to (X,η') by using the implicit function theorem. Moreover, by the implicit function theorem. Moreover, by the implicit function theorem given in [31], Lemma 4.9, we see that

$$\eta' = \eta'(x;\xi,\cdot_X(\bar{X},\bar{\eta}'),z;\zeta;A_1,A_2,A_3)$$

$$X = X(x;\xi,\cdot_X(\bar{X},\eta'),z;\zeta;A_1,A_2,A_3)$$

are smooth. Thus, remarking that A_i's depend smoothly on z_1, z_2, we see η', X depend smoothly on $(x;\xi,\cdot_X(\bar{X},\bar{\eta}'),z;\zeta,z_1,z_2)$. Since z_1, z_2 are sufficiently close to z_0, one may assume that the domain of η', X contains D_δ.

2.5 Amplitude functions of primordial operators.

In this section, we shall fix a class of amplitude functions of promordial operators. Roughly speaking, functions in such a class are obtained by the blowing up of usual amplitude functions The main reason for using such functions is to make the class invariant under the natural action of $\underset{\Omega}{(1)}$ and to make it closed under the multiplication.

Recall that $\overset{o}{T}{}^*N$ is naturally diffeomorphic to $R_+ \times S^{*N}$, where $R_+ = (0,\infty)$. Hence for a positive integer k, $(\overset{o}{T}{}^*N)^k = \underbrace{\overset{o}{T}{}^*N \times \ldots \times \overset{o}{T}{}^*N}_{k}$ can be viewed as $R_+^k \times (S^*N)^k$. Here, we shall give a compactification of R_+^k.

Take a positive constant K, K>1. For each integer l, $0 \leq l \leq k$, and each l-tuple of ordered integers $I = (i_1, \ldots, i_l)$, $1 \leq i_1, \ldots, i_l \leq k$, which are mutually distinct. (If l=0, we write simply by $I = \phi$.) We define a subset $\Delta_{k,I}$ by

$$\Delta_{k,I}$$

$$= \{(s_1, \ldots, s_k) \in R_+^k ; s_{i_1} \geq K^{-1}, d_{i_j} >= D^{-1} s_{i_{j-1}},$$

$$j = 2, \ldots, l, \text{ and } 0 < s_j \leq K \text{ for } j \in I\}.$$

Then, it is easily seen that $\bigcup_{I:\text{all ordering}, 0 \leq l \leq k} \Delta_{k,I} = R_+^k$. Define maps Define maps $i_{k,I} : \Delta_{k,I} \dashrightarrow [0,K]^k$ for

$I = (i_1, \ldots, i_l) \neq \Phi$, by

(5.1) $$i_{k,I}(s_1, \ldots, s_k) = (r^{-1}, t_1, \ldots, \check{t}_{i_1}, \ldots, t_k),$$

where

(5.2) $$r = s_{i_1}, \quad t_{i_2} = s_{i_1}/s_{i_2}, \ldots, t_{i_l} = s_{i_{l-1}}/s_{i_l},$$
$$\text{and} \quad s_j = t_j \quad \text{for} \quad j \neq i_1, \ldots, i_l.$$

Moreover, for $I=\Phi$, we define $i_{k,\Phi}$ by

(5.3) $$i_{k,\Phi}(s_1, \ldots, s_k) = (t_1, \ldots, t_k), \qquad s_j = t_j$$
$$\text{for} \quad j=1, \ldots, k.$$

Remark. (i) We put a coordinate on $\Delta_{k,I}$ by using variables $r^{-1}, t_1, \ldots, t_k$. But one of these is not used for each I (see List 5.1). (ii) To give a compactification of $\mathbf{R}^k_+$, we use the variable r^{-1} instead of r.

To simplify the notation, we often write $(t_1, \ldots, t_k)$ by $\mathbf{t}$, a point $(x_1;\xi_1, \ldots, x_k;\xi_k)$ of $(T^*N)^k$ by $(\mathbf{x};\xi)$ and a point $(x_1;\hat{\xi}_1, \ldots, x_k;\hat{\xi}_k)$ of $(S^*N)^k$ by $(\mathbf{x};\hat{\xi})$, respectively.

By attaching $r^{-1}=0$, $t_1=\ldots=t_k=0$, we obtain a compactification of $\mathbf{R}^k_+$. Remark that the above compactification of $\mathbf{R}_+$ is natural two points compactification $[0,\infty]$.

Since our compactification is complicated, we shall list up the exact domains and used variables of $\Delta_{k,I}$ for the case k=2,3 for our later use:

<u>List 5.1.</u> (A) $k=2$;

$$\Delta_{2,\Phi} = \{(t_1,t_2);0<t_1,t_2\leq K\},$$

$$\Delta_{2,(1)}= \{(r_1,t_2);0<r^{-1},t_2\leq K\},$$

$$\Delta_{2,(2)}= \{(t_1,r);0<r^{-1},t_1\leq K\},$$

$$\Delta_{2,(1,2)}= \{(r,r/t_2);0<r^{-1},t_2\leq K\},$$

$$\Delta_{2,(2,1)}= \{(r/t_1,r);0<r^{-1},t_1\leq K\},$$

(B) $k=3$;

$$\Delta_{3,\Phi} = \{(t_1,t_2,t_3);0<t_i\leq K,\ i=1,2,3\},$$
$$\Delta_{3,(1)}= \{(r,t_2,t_3);0<r^{-1},t_2,t_3\leq K\},$$
$$\Delta_{3,(2)}= \{(t_1,r,t_3);0<r^{-1},t_1,t_3\leq K\},$$
$$\Delta_{3,(3)}= \{(t_1,t_2,r);0<r^{-1},t_1,t_2\leq K\},$$
$$\Delta_{3,(1,2)}= \{(r,r/t_2,t_3);0<r^{-1},t_2,t_3\leq K\},$$
$$\Delta_{3,(2,1)}= \{(r/t_1,r,t_3);0<r^{-1},t_1,t_3\leq K\},$$
$$\Delta_{3,(2,3)}= \{(t_1,r,r/t_3);0<r^{-1},t_1,t_3\leq K\},$$
$$\Delta_{3,(3,1)}= \{(r/t_1,t_2,r);0<r^{-1},t_1,t_2\leq K\},$$
$$\Delta_{3,(1,3)}= \{(r,t_2,r/t_3);0<r^{-1},t_2,t_3\leq K\},$$
$$\Delta_{3,(1,2,3)}= \{(r,r/t_2,r/t_2,t_3);0<r^{-1},t_2,t_3\leq K\},$$
$$\Delta_{3,(1,3,2)}= \{(r,r/t_2,t_3,r/t_3);0<r^{-1},t_2,t_3\leq K\},$$
$$\Delta_{3,(2,1,3)}= \{(r/t_1,r,t_1,t_3);0<r^{-1},t_1,t_3\leq K\},$$
$$\Delta_{3,(2,3,1)}= \{(r/t_1,t_3,r,r/t_3);0<r^{-1},t_1,t_3\leq K\},$$
$$\Delta_{3,(3,1,2)}= \{(r/t_1,r/t_1,t_2,r);0<r^{-1},t_1,t_2\leq K\},$$
$$\Delta_{3,(3,2,1)}= \{(r/t_1,t_2,r/t_2,r);0<r^{-1},t_1,t_2\leq K\},$$

Now, by the identification $(\overset{o}{T}{}^*N)^k = R_+^k \times (S^*N)^k$ and the above compactification of R_+^k, we get a compactification of $(\overset{o}{T}{}^*N)^k$. Namely, for $I = (i_1,\ldots,i_l)$, $l \leq k$, we use a set $\Delta_{k,I} \times (S*N)^k$, and a map $i_{k,I} \times id.$: $\Delta_{I,I} \times (S*N)^k \longrightarrow [0,K]^k \times (S*N)^{kK}$ and compactify $(\ddot{T}*N)^k$. Hereafter, we shall use the same notations $\Delta_{k,I}$ and $i_{k,I}$ instead of $\Delta_{k,I} \times (S*N)^k$ and $i_{k,I} \times id.$.

Now, each C^∞ function f on $(\overset{o}{T}{}^*N)^k$ can be regarded as a function on $R_+^k \times (S*N)^k$ and therefore, we write it by the same letter f if it is not confused, i.e.,

(5.4) $\quad f(\mathbf{s},\mathbf{x};\hat{\xi}) = f(x_1;s_1\hat{\xi}_1,\ldots,x_k;s_k\hat{\xi}_k), \quad \mathbf{s} = (s_1,\ldots,s_k),$

$$(\mathbf{x};\hat{\xi}) = (x_1;\hat{\xi}_1,\ldots,x_k;\hat{\xi}_k).$$

For $I = (i_1,\ldots,i_l)$, $l \leq k$, consider $i_{k,I}^{-1} * (f|\Delta_{k,I})$, where $i_{k,I}$ is defined by (5.2) and (5.3). We often write by $\tilde{f}_{k,I}$ instead of $i_{k,I}^{-1} * (f|\Delta_{k,I})$ for the sake of simplicity.

<u>Definition 5.2.</u> $f \in C^\infty((\ddot{T}*N)^k)$ is called an <u>amplitude function</u>. if the following conditions are satisfied:

(A.1) For each $I = (i_1,\ldots,i_l)$, $0 \leq l \leq k$, $\tilde{f}_{k,I}$ can be extended smoothly at $t_j = 0$ $(j=1,\ldots,k)$.

(A.2) For each $I = (i_1,\ldots,i_l)$, $0 < l \leq k$, $\tilde{f}_{k,I}$ has an asymptotic expansion as follows:

(5.5) $\tilde{f}_{k,I}(r^{-1},t,x;\hat{\xi}) \approx \Sigma_{j\leq 0}A_j(t,x;\hat{\xi})r^j$,

where $A_j(t,x,\hat{\xi})$ are C^∞ functions on $[0,K]^{k-1}\times(S*N)^k$.

Remark The condition (5.5) means that $\tilde{f}_{k,I}$ is smooth at $r=\infty$.

<u>Definition 5.3.</u> (i) We denote by $\mathfrak{A}^k$ the totality of amplitude functions which satisfy (A.1) and (A.2) in the above definition.

(ii) For each $I = (i_1,\ldots,i_l)$, $0<l\leq k$, we denote by $\mathfrak{A}^k_{I,m}$ the totality of C^∞ functions $\tilde{f}_{k,I}$ on $[0,K]^k\times(S^*N)^k$ such that for non-positive integer m, $\tilde{f}_{k,I}$ has the following asymptotic expansion:

(5.6) $\tilde{f}_{k,I} \approx \Sigma_{j\leq m}A_j(t,x;\hat{\xi})r^j$, $A_j \in C^\infty([0,K]^{k-1}\times(S^*N)^k)$.

(iii) For small $\varepsilon_1,\ldots,\varepsilon_{k-1} > 0$, denote by $\mathfrak{A}^k(\varepsilon_1,\ldots,\varepsilon_{k-1})$ the space of all functions $f\in\mathfrak{A}^k$ such that

(5.7) $f(x;\xi) \equiv 0$

if $d(x_i,x_{i+1}) > \varepsilon_i$ for some $i=1,\ldots,k-1$.

<u>Remark.</u> By Definition 5.3 and the remark in 5.1, $f(x;\xi)\in\mathfrak{A}^1$ if and only if it satisfies, for $t>0$

(a) $\tilde{f}(t,x;\hat{\xi}) = f(x;t\hat{\xi})$ can be extended smoothly on $[0,\infty)\times S^*N$;

(b) f has an asymptotic expansion, for large $r>0$,

$\mathcal{F} \supset \Sigma_{j} \leqq_{0} A_{j}(x;\hat{\xi})r^{j}$.

Next, we shall put a system of norms on $\mathcal{Q}^{k}$. Let $f \in \mathcal{Q}^{k}$. Then, for every $I = (i_1, \ldots, i_l)$, $I \neq 0\Phi$, and any non-positive integer m, $\mathcal{F}_{k,I} = i_{k,I}^{-1*}(f|\Delta_{k,I})$ in (5.5) can be written in the following form: For fixed C^{∞} function $\varphi(r)$ such that $\varphi(r) \equiv 0$ on $0 \leqq r \leqq 2K^{-1}$, and $\equiv 1$ on $r \geqq 3K^{-1}$, we have

$$(5.8) \quad \mathcal{F}_{k,I} = \varphi(r)(\operatorname{sum}_{m \leqq j \leqq 0} A_{j}(t,x,;\hat{\xi})r^{j}) + \mathcal{F}_{I,m-1}(r^{-1},t,x;\xi),$$

where $\mathcal{F}_{I,m-1} \in \tilde{\mathcal{Q}}^{k}_{I,m-1}$.

Let $|A_j|_s$ be the C^s-norm on $[0,K]^{k-1} \times (S^*N)^k$.

<u>Definition 5.4.</u> For each function $\mathcal{F}_{k,I}$ on $[0,K]^k \times (S^*N)^k$, we define a norm $\|\mathcal{F}_{k,I}\|_{m,s}$, $s \geqq 0$, $m \leqq 0$, $I \neq$, as follows:

$$(5.9) \quad \|\mathcal{F}_{k,I}\|_{m,s} = \sum_{m \leqq j \leqq 0} |A_j|_s + \mathcal{F}_{I,m-1\ s};$$

$$(5.10) \quad \mathcal{F}_{I,m-1\ s} = \sup_{\substack{r>0,\, p+|\alpha|=s \\ (x;\xi)\in(S^*N)^k}} |(1+r)^{-m+p+1}(\partial/\partial r)^{p} D^{\alpha}_{(t,x,\hat{\xi})} \mathcal{F}_{I,m-1}|,$$

where $D_{(t,x;\hat{\xi})}$ is the derivative on $[0,K]^{k-1} \times (S^*N)^k$ by using a normal coordinate system.

<u>Definition 5.5.</u> For each $f \in \mathcal{Q}^{k}$, we define a norm $\|f\|_{m,s}$, $s \geqq 0$, $m \leqq 0$ by

$$(5.11) \quad \|f\|_{m,s} = \sum_{\substack{I=\{i_1,\ldots,i_l\} \\ 1 \leqq l \leqq k}} \|\mathcal{F}_{k,I}\|_{m,s} + \|\mathcal{F}_{\varphi}\|_{s},$$

where $\|\tilde{f}_\varphi\|_s$ is the C^s-norm of $\tilde{f}_\varphi = i_{k,\varphi}^{-1*}(f|\Delta_{k,\varphi})$ on $[0,K]^k \times (S^*N)^k$, and the summation of the first term of (5.11) is taken by all l-tuple of mutually distinct indices in$\{1,\ldots,k\}$.

For every m≦0, the system of norms $\{\|\cdot\|_{m,s}; s=0,1,2,\ldots\}$ give a topology T_m on $\mathfrak{A}^k$. We denote by $\mathfrak{A}^k_{(m)}$ the completion of $(\mathfrak{A}^k, T_m)$. An element of $\mathfrak{A}^k_{(m)}$ will be called an <u>extended amplitude function</u> on $(T^*N)^k$. It is not hard to see that $\bigcup_m \mathfrak{A}^k_{(m)} = \mathfrak{A}^k$. Thus, we define the inverse limit topology for $\mathfrak{A}^k$. As a result, $\mathfrak{A}^k$ has a <u>Frèchet structure by above system of norms</u>. Also, we denote by $\mathfrak{A}^k_{(m)}(\varepsilon_1,\ldots,\varepsilon_{k-1})$, for m≦0, the closure of $\mathfrak{A}^k(\varepsilon_1,\ldots,\varepsilon_{k-1})$ in $\mathfrak{A}^k_{(m)}$.

<u>Remark.</u> By the definition of amplitude functions, it is easily seen that $\mathfrak{A}^k$ is invariant under any permutation of variables.

In the following, we shall investigate the differentiability of some operations on $\mathfrak{A}^k$.

Given f, g∈ $\mathfrak{A}^k$, denote by $f\cdot g$ the natural pointwise multiplication of f and g. The, it is easily seen that $f\cdot g \in \mathfrak{A}^k$. Moreover, we have the following:

<u>Lemma 5.6.</u> The multiplication map $M: \mathfrak{A}^k \times \mathfrak{A} \longrightarrow \mathfrak{A}^k$, defined by $M(g,g) = f\cdot g$, can be extenced to a continuous bilinear mapping of $\mathfrak{A}^k_{(m)} \times \mathfrak{A}^p_{(m)}$, for every m≦0.

For each $\mathcal{Q}^k$, $k\geq 1$, $\mathcal{Q}^{k-1}$ can be embedded smoothly in $\mathcal{Q}^k$ as follows: Let $p_j:(\overset{o}{T}{}^*N)^k \dashrightarrow (\overset{o}{T}{}^*N)^{k-1}$ be the projection efined by, for $j=1,\ldots,k$,

$$\text{5.12)} \qquad p_j(x;\xi) = (x_1;\xi_1,\ldots,\check{x}^j;\check{\xi}^j,\ldots,x_k;\xi_k),$$

here $\check{x}^j$, $\check{\xi}^j$ mean that x_j, ξ_j are omitted.

For p_j, we have the following:

<u>Lemma 5.7.</u> Given $f\in \mathcal{Q}^{k-1}$, $p_j^*f\in \mathcal{Q}^k$. Moreover, the apping $p_j^*: \mathcal{Q}^{k-1}\to \mathcal{Q}^k$ can be extended to a continuous inear mapping of $\mathcal{Q}^{k-1}_{(m)}$ into $\mathcal{Q}^k_{(m)}$ for every $m\leq 0$.

<u>Proof</u>. Let $I = (i_1,\ldots,i_l)$ be l-tuple of indices. By he remark in 5.2, we may assume that $i_1<i_2<\ldots<i_l$. Then, or any $f\in \mathcal{Q}^{k-1}$, we have

$$\text{5.13)} \qquad (P_j^*f)_{k,I}$$

$$\begin{cases} f^{-1}_{k-1,I}(r,t_1,\ldots,\check{t}^j,\ldots,t_k,x_1;\hat{\xi}_1,\ldots,\check{x}^j;\check{\xi}^j,\ldots,x_k;\hat{\xi}_k), \\ \qquad\qquad\qquad\qquad j\in I; \\ f_{k-1,I}((r/t_{i_2})^{-1},t_1,\ldots,\check{t}^j,\ldots,t_k,x_1;\hat{\xi}_1,\ldots,\check{x}^j;\check{\xi}^j,\ldots,x_k;\hat{\xi}_k), \\ \qquad\qquad\qquad\qquad j=i_1; \\ f_{k-1,I}(r^{-1},t_1,\ldots,t_{i_{m-1}},\check{t}^j,t_{i_m},t_{i_{m+1}},\ldots,t_k,x_1;\hat{\xi}_1,\ldots,\check{x}^j;\check{\xi}^j,\ldots,x_k;\hat{\xi}_k), \\ \qquad\qquad\qquad\qquad j=i_m \ (m\geq 2). \end{cases}$$

rom this, we get the lemma.

Next, we give a diagonalized operation. Given positive integer i, $1 \leq i \leq k-1$, define a map $d_i : (\overset{o}{T}{}^*N)^k \dashrightarrow (\overset{o}{T}{}^*N)^{k+1}$ by

$$(5.14) \qquad d_i(x;\xi) = (x_1;\xi_1,\ldots,x_i;\xi_i,x_{i+1};\xi_{i+1},\ldots,x_k;\xi_k).$$

Denote by $d_i^*: \mathcal{O}^{k+1} \dashrightarrow \mathcal{O}^k$ the pull-back mapping induced from d_i. By a similar computation as above, we get the following:

<u>Lemma 5.8.</u> For every $f \in \mathcal{O}^{k+1}$, $d_i^* f \in \mathcal{O}^k$ ($i=1,\ldots,k-10$). The mapping $d_i^*: \mathcal{O}^{k+1} \dashrightarrow \mathcal{O}^k$ can be extended to a continuous linear mapping of $\mathcal{O}^{k+1}_{(m)}$ into $\mathcal{O}^k_{(m)}$ for every $m \leq 0$.

Now, for $f \in \mathcal{O}^k$, $g \in \mathcal{O}^{k'}$, define a map $\times: \mathcal{O}^k \times \mathcal{O}^{k'} \longrightarrow \mathcal{O}^{k+k'-1}$ by

$$(5.15) \qquad f \times g(x_1;\xi_1,\ldots,x_{k+k'-1};\xi_{k+k'-1})$$

$$= f(x_1;\xi_1,\ldots,x_k;\xi_k)g(x_k;\xi_k,\ldots,x_{k+k'-1};\xi_{k+k'-1}).$$

Namely,

$$(5.16) \qquad f \times g = d_k^* M(p_{k,k'+k}^* f, p_{k',k'+k}^* g).$$

Hence, from Lemma 5.7-8, we have the following:

<u>Corollary 5.9.</u> The mapping $\times: \mathcal{O}^k \times \mathcal{O}^{k'} \longrightarrow \mathcal{O}^{k+k'-1}$ can be extended to a continuous bilinear mapping of $\mathcal{O}^k_{(m)} \times \mathcal{O}^{k'}_{(m)}$ into $\mathcal{O}^{k+k'-1}_{(m)}$ for every $m \leq 0$.

Finally, we shall state about the differentiability of the action of $\mathcal{D}^{(l)}_{\Omega}$ on $\mathcal{O}^k$. Namely, we get the following:

Lemma 5.10. For each $\varphi \in \mathcal{D}_{\Omega}^{(1)}$ and $f \in \mathcal{O}\!\mathcal{L}^{k}$, $\varphi^{*}f$ is an element of $\mathcal{O}\!\mathcal{L}^{k}$. Moreover, the mapping ev: $\mathcal{D}_{\Omega}^{(1)} \times \mathcal{O}\!\mathcal{L}^{k} \dashrightarrow \mathcal{O}\!\mathcal{L}^{k}$, defined by $ev(\varphi, f) = \varphi^{*}f$ can be extended to a C^{∞} mapping of $\mathcal{D}_{\Omega}^{(1)} \times \mathcal{O}\!\mathcal{L}^{k}_{(m)}$ for every $m \leq 0$.

Proof. Let $\varphi \in \mathcal{D}_{\Omega}^{(1)}$. Write $\varphi(x;\xi)$ by $(\varphi_1(x;\xi);\varphi_2(x;\xi))$. Putting $\mu(x;\hat{\xi}) = |\varphi_2(x;\hat{\xi})|$, we see $\mu(x;\hat{\xi})>0$ and φ maps $r,x;\hat{\xi})$ to $(\mu(x;\hat{\xi})r, \hat{\varphi}(x;\hat{\xi}))$ where $\hat{\varphi}\ (x;\hat{\xi}) = (\varphi_1(x;\hat{\xi});\mu(x;\hat{\xi})^{-1}\varphi_2(x;\hat{\xi}))$. Hence, we have

$$(5.17)\qquad \varphi^{*}f(x_1;r_1\hat{\xi}_1,\ldots,x_k;r_k\hat{\xi}_k)$$

$$= f(\varphi_1(x_1;\hat{\xi}_1);\mu(x_1;\hat{\xi}_1)r_1\cdot\hat{\varphi}_2(x_1;\hat{\xi}_1),\ldots$$

$$\ldots,\varphi_1(x_k;\hat{\xi}_k);\mu(x_k;\hat{\xi}_k)r_k\cdot\hat{\varphi}_2(x_k;\hat{\xi}_k)).$$

Thus, for any $I = (i_1,\ldots,i_l)$, $i_1<\ldots<i_l$, we have

$$(5.18)\qquad (\widetilde{\text{phi}}^{*}f)_{k,I}(r^{-1},t,x,\hat{\xi}) =$$

$$\tilde{f}_{k,I}((\mu(x_{i_1};\hat{\xi}_{i_1})r)^{-1};t',\hat{\varphi}(x_1;\hat{\xi}_1),\ldots,\hat{\varphi}(x_k;\hat{\xi}_k))$$

where $t_i'=t_i$ for $i\in I$, and $t_{i_2}' = \mu(x_{i_1};\hat{\xi}_{i_1})t_{i_2}/\mu(x_{i_2};\hat{\xi}_{i_2})$ $,\ldots,$ $t_{i_l}' = \mu(x_{i_{l-1}};\hat{\xi}_{i_{l-1}})t_{i_l}/\mu(x_{i_l};\hat{\xi}_{i_l})$. For the other case of I, the computation is similar. By the differentiability of (5.18) for each I, we obtain the desired results.

For our later use in §2.7, we shall modify Lemma 5.10 to a certain local form. First of all, we remark the following:

<u>Lemma 5.11.</u> Suppose $f \in C^{\infty}((\overset{o}{T}{}^{*}N)^2)$ satisfies the following conditions:

(LA.1) $f \equiv 0$ if $|\xi|^2 + |\eta|^2 \leqq R^2$, or $|\eta|/|\xi| \geqq C$, $C>=1$;

(LA.2) Put $F(r,\theta,x;\hat{\xi},y;\hat{\eta}) = f(x;r(\cos\theta)\hat{\xi},y;r(\sin\theta)\hat{\eta})$. Then F has an asymptotic expansion

$$F \approx \Sigma_{j\leqq 0} A_j(\theta,x;\hat{\xi},y;\hat{\eta})r^j \qquad (r>>0).$$

<u>Then</u>, $f\in \;^2$.

<u>Proof</u>. Set $\tilde{f}(r_1,r_2,x;\hat{\xi},y;\hat{\eta}) = f(x;r_1\hat{\xi},y;r_2\hat{\eta})$, and recall List 5.1. On $\Delta_{2,\Phi}$ $\Delta_{2,(1)}$ $\Delta_{2,(2)}$, there is no problem because $\tilde{f}_{2,I}$ on each domain is identically zero. Hence, we have only to check that $\tilde{f}(r?t,r,x;\hat{\xi},y;\hat{\eta})$ and $\tilde{f}(r,r/t,x;\hat{\xi},y;\hat{\eta})$ above asymptotic expansions requested in Definition 5.2. However, these functions are zero whenever $t\leqq C^{-1}$ or $t \geqq C$. Thus, we have the desired expansion by using (LA.2).

Now, denote by $D_{\varepsilon,\delta}$ the domain $\{(x;\xi,\cdot_X(X,\eta')); |X| \leqq epsilon, |\xi-\eta'| \leqq \delta(|\xi|+|\eta'|)\}$, where $\varepsilon>0$, $\delta>0$. We consider C^{∞} functions f such that supp fsubset$D_{\varepsilon,\delta}$ and f satisfies (LA.1-2) in the above lemma. For such a class of functions, we give the restricted topology of $\;^2_{(m)}$.

Let Φ be a C^{∞} diffeomorphisms of $D_{\varepsilon,\delta}$ into an open neighborhood of $D_{\varepsilon,\delta}$ such that $\Phi(x;r\xi,y;r\eta) = (PHI_1;r\Phi_2,\Phi_3;r\Phi_4)$ where $\Phi_i = \Phi_i(x;\xi,y;\eta)$. Such a class of diffeomorphisms can be topologized by the standard C^{∞} topology by which it turns out ot be an open set of a Frèchet space. For such Φ, and for such f defined above, $\Phi^* f$ is again a C^∞ function on $(\overset{\circ}{T}{}^*N)^2$ satisfying (LA.1-2). Moreover, by the smoothness of compositions, we have

<u>Lemma 5.12.</u> Notations and assumptions being as above, $\Phi^* f$ is smooth with respect to Φ and f for every $m \leqq 0$.

2.6 Proof of Theorem C.

In this section, we shall prove Proposition 2.2-3 and finally give the proof of Theorem A by assuming the smoothness property of some oscillatory integral (Cf. Proposition 6.1). This smoothness property will be proved in the next section.

(a) Choose ε_1, $\varepsilon_2>0$ so that $\varepsilon_1<r_{1/4}$, where r_1 is given in §2.4. Recall the definition of $\mathcal{O}\!\mathcal{L}^3(\varepsilon_1,\varepsilon_2)$ and $\mathcal{O}\!\mathcal{L}^3_{(m)}(\varepsilon_1,\varepsilon_2)$ (cf. Definition 5.3 and 5.5). Let z_1, z_2 be elements of Θ_0 which are sufficiently close to z_0. Given $a\in\mathcal{O}\!\mathcal{L}^3(\varepsilon_1,\varepsilon_2)$, consider the following integral

$$(6.1)\qquad \langle ae^{-iz_1\boxplus z_2}\rangle(x;\xi,z;\zeta)$$
$$= \mathrm{Os}\text{-}\iint a(x;xi,y;\eta,z;\zeta)e^{-iz_1\boxplus z_2(x;\xi,y;\eta,z;\zeta)}dyd\eta.$$

The above integral can be defined as the oscillatory integral for any fixed $(x;\xi)$, (z,ζ) and it will be called the <u>contraction integral of</u> a <u>by</u> z_1+z_2.

First of all, we state the following, which will be proved in §2.7:

<u>Proposition 6.1.</u> (i) For z_1, $z_2\in\Theta_0$, sufficiently close to z_0, and $a\in\mathcal{O}\!\mathcal{L}^3(\varepsilon_1,\varepsilon_2)$, $\langle ae^{-iz_1\boxplus z_2}\rangle$ can be written by

$$(6.2)\qquad \langle ae^{-iz_1\boxplus z_2}\rangle(x;\xi,z;\zeta)$$
$$= b(x;\xi,z,\zeta)e^{-ic(z_1,z_2)(x;\xi,z;\zeta)},$$

where $b \in {}^2(\varepsilon_1+\varepsilon_2)$ and $c(z_1,z_2)$ is defined in (4.9.

(ii) For a sufficiently small neighborhood U_{z_0} of z_0 in Θ_0, the mapping $A(a,z_1,z_2) = b$ can be extended to a C^∞ mapping of $\mathcal{O}\!\mathcal{L}^3_{(m)}(\varepsilon_1,\varepsilon_2)\times U_{z_0}\times U_{z_0}$ into $\mathcal{O}\!\mathcal{L}^2_{(m)}(\varepsilon_1+\varepsilon_2)$ for every $m\leq 0$.

(b) Next integral is much simpler that the above case (a). Now, we denote by $\mathcal{O}\!\mathcal{L}^3(\infty,\varepsilon_2)$ the totality of $a\in\mathcal{O}\!\mathcal{L}^3$ such that

(6.3) $\quad a(x;\xi,y;\eta,z;\zeta) \equiv 0 \quad$ for $\quad d(y,z)>\varepsilon_2$.

Denote by $\mathcal{O}\!\mathcal{L}^3_{(m)}(\infty,\varepsilon_2)$ the closure of $\mathcal{O}\!\mathcal{L}^3(\infty,\varepsilon_2)$ in $\mathcal{O}\!\mathcal{L}^3_{(m)}$ for each $m\leq 0$. For $a\in\mathcal{O}\!\mathcal{L}^3(\infty,\varepsilon_2)$, we consider the following integral

(6.4) $\quad \langle ae^{-iz_0}\rangle(x;\xi,z;\zeta)$

$$= \mathrm{Os}-\iint a(x;\xi,y;\eta,z;\zeta)e^{-iz_0(y,\eta,z)}\,dyd\eta.$$

As in the case of $\langle ae^{-iz_1\boxplus z_2}\rangle$, (6.4) is well-defined as an oscillatory; integral, which will be called also the <u>contraction integral of a by</u> z_0. This integral has the following property:

<u>Proposition 6.2.</u> (i) For every $a\in\mathcal{O}\!\mathcal{L}^3(\infty,\varepsilon_2)$, $\langle ae^{-iz_0}\rangle$ is contained in $\mathcal{O}\!\mathcal{L}^2$.

(ii) The mapping $\langle *e^{-iz_0}\rangle:\mathcal{O}\!\mathcal{L}^3(\infty,\varepsilon_2) \longrightarrow \mathcal{O}\!\mathcal{L}^2$ can be extended to a continuous linear mapping of $\mathcal{Q}^3_{(m)}(\infty,\varepsilon_2)$ into $\mathcal{Q}^2_{(m)}$ for every $m\leq 0$.

(iii) If $a\in\mathcal{Q}^3(\infty,\varepsilon_2)$ is rapidly decreasing in $|\xi|$, then so is $\langle ae^{-iz_0}\rangle$ in $|\xi|$.

<u>Proof</u>. (iii) is trivial, since $\langle ae^{-iz_0}\rangle$ is defined as an oscillatory integral. To prove (i), (ii), we have only to repeat the standard technique on each local coordinate system $\Delta_{3,l}$ (cf. List 5.1), by finding operators L such that $Le^{-iz_0} = e^{-iz_0}$ and repeating the integration by parts. We omit here the precise procedure of these, for these will be discussed again more precisely in the next section §2.7.

In the previous papers, pseudo-differential operators of order 0 have been defined as operator with symbols contained in Σ^0_C (cf. [36]). Here, we shall remark the same operators can be defined by using $a\in\mathcal{Q}^1$ instead of $a\in\Sigma^0_C$.

Recall the definition of $\mathcal{Q}^1$ and the remark in 5.2. Given $a\in\mathcal{Q}^1$, we define a linear operator Q(a) on C^∞ as follows:

$$(6.5)\qquad (Q(a)u)(x) = \mathrm{Os}\text{-}\iint a(x;\xi)\nu(x,y)e^{-iz_0(x;\xi,y)}u(y)dyd\xi .$$

Now, fix a C^∞ function $\varphi(x,\xi)$ on $\overset{\circ}{T}{}^*N$ such that $\varphi\equiv 1$ on $|\xi|\leq K$ and $\varphi\equiv 0$ on $|\xi|\geq 2K$ where K is a positive constant. Divide (6.5) into two parts:

$$(6.6) \qquad (Q(a)u)(x) = \iint \varphi a\nu e^{-iz_0} u dy d\xi + \iint (1-\varphi)a\nu e^{-iz_0} u dy d\xi$$
$$= Q_1 + Q_2.$$

Since $(1-\varphi)a\nu \in \tilde{\Sigma}^0_{\mathbf{C}}$ (cf. [27], p. 365), Q_2 is a pseudo-differential operator of order 0, because $z_0(x;\xi,y) = \langle \xi | \overset{x}{\cdot} y \rangle$ on $\mathrm{supp}(1-\varphi)a\nu$. By Kuranishi's technique (cf. [29], p.269), we can eliminate the y-variable in the amplitude $(1-\varphi)a\nu$ and obtain a pseudo-differential operator with the amplitude contained in $\Sigma^0_{\mathbf{C}}$.

On the other hand, Q_1 is smoothing operator with the kernel

$$(6.7) \qquad K_{Q_1}(x,y) = \int \varphi(x;\xi)a(x;\xi)nu(x,y)e^{-iz_0(x;\xi,y)} d\xi,$$

which is obviously smooth. Hence recalling hw we defined the norm $\| \ \|_{m,s}$ on the space ^{0} and using Lemma 1 in [36], we obtain easily the following:

<u>Lemma 6.3.</u> Let $a \in \mathcal{O\!L}^1$. Then $Q(a)$ is a pseudo-differential operator of order zero on N and the mapping $Q\colon \mathcal{O\!L}^1 \dashrightarrow \mathfrak{z}^0$ can be extended to a continuous linear mapping from $\mathcal{O\!L}^1_{(m)}$ into $\mathcal{O\!L}^0_{(m)}$ for every $m \leq 0$.

For $K \in C^\infty(N\times N)$ and $\varphi \in \mathcal{D}^{(1)}_\Omega$, we shall consider the following operator

$$(6.8) \qquad \lambda(\varphi,K) = \varphi^* \iota K \, \pi\varphi^{*-1} \colon \mathcal{S}_N \dashrightarrow \mathcal{S}_N.$$

Then, recalling the statement of Proposition 2.3, we have

$\Lambda(\varphi,K) = \pi\ \lambda(\text{phi},K)\ \iota$. By (3.3), we have

(6.9) $$(\lambda(\varphi,K)f)(x;\xi) = \iint(\varphi^{*}a_K)(x;xi,y;\eta)f(y;\xi)dyd\eta,$$

where

$$a_K(x;\xi,y) = \iint\nu(x,z)K(z,y)e^{-i\langle\xi|\overset{x}{\cdot}z\rangle}dz.$$

First, we compute $\lambda(\text{phi},K)\iota$. The, we have for $u\in C^{\infty}(N)$ that

(6.10) $(\lambda(\varphi,K)\iota u)(x;\xi)$

$$= \iiint A(\varphi,K)(x;\xi,y;\eta,z)e^{-iz_0(y;\eta,z)}u(z)dyd\eta dz,$$

where

(6.11) $$A(\varphi,K)(x;\xi,y;\eta,z) = (\varphi^{*}a_K)(x;\xi,y;\eta)\nu(y,z).$$

By Corollary 5.9 and Lemma 5.10, we see $A(\varphi,K)\in\mathcal{O}^3(\infty,\varepsilon)$ and $A(\varphi,K)$ is rapidly decreasing in $|\xi|$, for so is a_K. Hence by Proposition 6.2, we have $\langle A(\varphi,K)e^{-iz_0}\rangle\in\mathcal{O}^2$ and rapidly decreasing in $|\xi|$. Moreover this is smooth with respect to φ and K.

Since $\Lambda(\varphi,K) = \pi\ \lambda(\varphi,K)\ \eta$, the kernel of $\Lambda(\varphi,K)$ is give by

(6.12) $$L(\varphi,K)(x,z) = \int\langle A(\varphi,K)e^{-iz_0}\rangle(x;\xi,z)d\xi,$$

which is obviously smooth on $N\times N$. Thus, we get the following, which proves Proposition 2.3:

<u>Lemma 6.4.</u> Let $K \in C^{\infty}(N \times N)$ and $\varphi \in \mathcal{D}_{\Omega}^{(1)}$. Then, $\Lambda(\varphi, K) = \pi\ \lambda(\varphi, K)\ \iota$ is an linear operator with a smooth kerned $L(\varphi, K)$ defined by (6.12). Moreover the mapping $L: \mathcal{D}_{\Omega}^{(1)} \times C^{\infty}(N \times N) \dashrightarrow C^{\infty}(N \times N)$ is a smooth mapping.

Before proving Proposition 2.2, we shall remark some properties of a certain oscillatory integral. Namely, consider the following linear operator on $C^{\infty}(N)$:

$$(6.13) \qquad (\mu(a,z)u)(x) \quad = \quad \iint a(x;\xi,y)e^{-iz(x;\xi,y)}u(y)dyd\xi,$$

where $a(x;\xi,y) \in \mathcal{O}\mathcal{L}^{2}$(epsilon) and $z \in \Theta_0$ do not involve η-variable and z sufficiently close to z_0.

Remark that on the support of a, $z(x;\xi,\cdot_x Y)$ can be expressed as

$$(6.14) \qquad z(x;\xi,\cdot_x Y)$$

$$= \langle \xi | Y \rangle + \langle \xi | Q(x;xi,Y)Y^2 \rangle = \langle \xi | Y+QY^2 \rangle = \langle \xi | (I+QY)Y \rangle.$$

Since $z-z_0$ is small and $|Y|<\varepsilon$, one may assume that $I + Q(z;xi,Y)Y$ is an invertible matrix. Set $\xi' = \xi(I+QY)$. Then by the implicit function theorem (cf. [31]), ξ can be expressed as a C^{∞} function $\Psi_z(x;\xi',y)$ depending smoothly on z. Let $D(\varepsilon)$ be the domain $\{(x;\xi,y) \in (\overset{o}{T^*}N) \times N; d(x,y) <= \varepsilon\}$. Then Ψ_z is actually a C^{∞} diffeomorphism of $D(\varepsilon)$ onto itself and positively homogeneous of degree 1. Hence, we have the following:

<u>Lemma 6.5.</u> For $z\in\Theta_0$, sufficiently close to z_0, there exists a C^∞ diffeomorphism Φsuz of $D(\varepsilon)$ onto itself such that $\Psi_z^* z = z_0$ and Ψsugz is positively homogeneous of degree one. Moreover, Ψ_z is smooth with respect to z under the C^∞ topology for Ψ_z.

Now, using the above lemma, we rewrite (6.13) as follows:

(6.15) $\quad (\mu(a,z)u)(x)$

$$= \iint (\Psi_z^* a)(z;\xi,y)|\det D\Psi_z| e^{-iz_0(x;\xi,y)} u(y)dyd\xi,$$

where we see easily that $(\Psi_z^* a)|\det D\Psi_z| \in \mathcal{O}\!\mathcal{L}^2(\varepsilon)$ and does not involve η-variable, and $(\Psi_z^* a)|\det D\Psi_z|$ depends smoothly on z (cf. Lemma 5.10).

Thus, using Kuranishi's technique, one can eliminate the y-variable in the amplitude $(\Psi_z$suop$*a)|\det D\Psi_z|$. Thus, by th same computation as in 6.2, we obtain the following:

<u>Lemma 6.6.</u> (i) For $z\in\Theta_0$, sufficiently close to z_0, and $a\in\mathcal{O}\!\mathcal{L}^2(\varepsilon)$ which do not contain η-variable, $\boldsymbol{\mu}(a,z)$ is a pseudo-differential operator of order 0. (ii) The mapping $\mu:\mathcal{O}\!\mathcal{L}_\eta^2(\varepsilon)\times\Theta_0 \dashrightarrow \mathfrak{F}^0$ can be extended to a C^∞ mapping of $\mathcal{O}\!\mathcal{L}_{\eta,(m)}^2(\varepsilon)\times\Theta_0$ into $\mathfrak{F}_{(m)}^0$, where $\mathcal{O}\!\mathcal{L}_\eta^2(\varepsilon)$ is the totality of $a\in\mathcal{O}\!\mathcal{L}^2(\varepsilon)$ which does not involve η-variable and $\mathcal{O}\!\mathcal{L}_{\eta,(m)}^2(\varepsilon)$ is its closure in $\mathcal{O}\!\mathcal{L}_{(m)}^2(\varepsilon)$.

Denote by $E(\varphi)$, for $\varphi\in\mathcal{D}_{\Omega(1)}$, th linear operator on $\mathcal{S}_N$

(6.16) $\qquad E(\varphi) \;=\; \varphi^{*}\iota\pi\varphi^{*-1} \qquad$ (cf. 2.3).

Recall the argument in 3.1. $E(\varphi)$ is an integral operator with a smooth kernel $\varphi^{*}\nu e^{-i\varphi^{*}z_0}$. By Lemma 4.1, $\varphi^{*}z_0\in\Theta_0$, and by Lemma 5.10, $\varphi^{*}\nu\in\ ^{\varepsilon})$ if φ is sufficiently close to the identity. Moreover, $\Xi(\varphi)$ of (2.6) is written as $\pi E(\varphi)\iota$, hence we have

(6.17) $\qquad (\Xi(\varphi)u)(x) \;=\; \iint B(\varphi)(x;\xi,z)u(z)dzd\xi,$

where

(6.18) $\qquad B(\varphi)(x;\xi,z) \;=\; \langle\varphi^{*}\nu\times\nu e^{-i\varphi^{*}z_0\boxplus z_0}\rangle.$

Note that $\varphi^{*}\nu\times\nu\in\mathcal{O}^{3}(\varepsilon,\varepsilon)$ and does not involve ξ-variable. The, using Proposition 6.1, we have

(6.19) $\qquad B(\varphi)(x;\xi,z) \;=\; b(\varphi)(x;\xi,yz)e^{-ic(\varphi^{*}z_0,z_0)},$

$$b(\varphi)\in\mathcal{O}^{2}(2\varepsilon).$$

It is easy to see that $bv(\varphi)$ does not involve ξ-variable, because so do $\varphi^{*}\nu\times\nu$ and $\varphi^{*}z_0+z_0$.

Also by Lemma 4.4, we have $c(\varphi^{*}z_0,z_0)\ \in\Theta_0$ is sufficiently close to $z_0(x;\xi,z)$, if φ is sufficiently close to the identity. Thus, by Lemma 6.6, we get Proposition 2.2, (a), (b). Proposition 2.2, 9c) is obvious, because

$\Xi(\mathrm{id.}) = \mathrm{id.}$ and $G\mathfrak{F}^0_{(m)}$ is an open subset of $\mathfrak{F}^0_{(m)}$ for every $m<=0$.

Now, we shall give the proof of the main theorem. As in 2.3, recall the operators r_γ, α_γ in (2.8), (2.11). We denote by $\mathfrak{M}$ the pairs (a,z) where $a\in \mathcal{O}^2(\varepsilon)$, $z\in U_{z_0}$, a sufficiently small open neighborhood of z_0 in Θ_0. By using Frèchet structures on $\mathcal{O}^2(\varepsilon)$ and Θ_0, $\mathfrak{M}$ captures a structure as an open set of a Frèchet space. Associating with $a\in \mathcal{O}^2(\varepsilon)$ and $z\in U_0$, we consider a primordial operator on $\mathcal{S}_N$ on the form

(6.20) $\quad (P(a,z)f)(x;\xi)$

$$= \iint a(x,\xi,y;\eta)e^{-iz(x;\xi,y;\eta)}f(y,\eta)dyd\eta,$$

and this plays an important role in the observation of r_γ and α_γ. Namely remark that

(6.21) $\quad \mathrm{rsug}\gamma(\varphi,\psi) = \Xi(\varphi\psi)^{-1}\pi E(\varphi\psi)E(\varphi)\iota,$

(6.22) $\quad \alpha_\gamma(\varphi,A) = \Xi(\varphi)^{-1}\pi E(\varphi)\varphi^{*}a\cdot E(\varphi)\iota + \Xi(\varphi)^{-1}\Lambda(\varphi,K),$

where $A = \pi a\iota + K$, $a\in\Sigma^0_{\mathbf{C}}$, $K\in C^\infty(N\times N)$. REmark also that $E(\varphi\psi)$, $E(\varphi)$ are primordial operators written in the form (6.20).

First of all, we shall observe (6.21). Remark that

(6.23) $\quad (E(\varphi\psi)E(\varphi)f)(x;\xi)$

$$= \iint ((\varphi\psi)^*\nu\times\varphi^*\nu)(x;\xi,y;\eta,z;\zeta)$$

$$\times e^{-i(\varphi\psi)^* z_0 + z_0} f(z;\zeta)dyd\eta dzd\zeta.$$

By the result in §2.4-5, we have $(\varphi\psi)^*\nu\times\varphi^*\nu \in \mathcal{Q}^3(\varepsilon,\varepsilon)$ and $(\varphi\psi)^* z_0$, $\varphi^* z_0 \in \Theta_0$ if φ, ψ are sufficiently close to the identity. Therefore, Proposition 6.1 can be applied in this case, and the kernel of (6.23) is given by the contraction integral

$$(6.24) \quad \langle(\varphi\psi)^*\nu\times\varphi^*\nu e^{-i(\varphi\psi)^* z_0 \times z_0}\rangle$$

$$= b(\varphi,\psi)e^{-ic((\varphi\psi)^* z_0, \varphi^* z_0)},$$

for some $b(\varphi,\psi)\in \mathcal{Q}^2(2\varepsilon)$. Thus, by Lemma 4.4, Proposition 6.1 and Lemma 5.10, we obtain the following:

Lemma 6.7. There exists a neighborhood V of the identity in $\mathcal{D}_\Omega^{(1)}$ such that the mapping of V×V into $\mathcal{Q}^2(2\varepsilon)\times\Theta_0$ defined by (φ,ψ) --> $(b(\text{phi},\psi), c((\varphi\psi)^* z_0, \varphi^* z_0))$ in (6.24) is a smooth mapping of V×V into $\mathcal{Q}^2_{(m)}(2\varepsilon)\times\Theta_0$ for every m<=0.

Now, we shall compute $\pi E(\varphi\psi)E(\varphi)\iota$. Set $z' = c((\varphi\psi)^* z_0, \varphi^* z_0)$ in the above notation. Then, we obtain

$$(6.25) \quad (E(\varphi\psi)E(\varphi)\iota u)(x)$$

$$= \iint b(\varphi,\psi)(x;\xi,y;\eta)\nu(y,z)e^{-i(z'+z_0)(x;\xi,y;\eta,z)} u(z)dzdyd\eta.$$

Thus

(6.26) $(\pi E(\varphi\psi)E(\varphi)\iota u)(x)$

$$= \iint \langle b(\varphi,\psi)\times\nu e^{-iz'+z_0}\rangle(x;\xi,z)u(z)dzd\xi .$$

Since z' is sufficiently close to z_0, one can apply Proposition 6.1 again and obtain

(6.27) $$\langle b(\varphi,\psi)\times\nu e^{-iz'+z^0}\rangle = \tilde{b}(\varphi,\psi)e^{-ic(z',z_0)},$$
$$\tilde{b}(\varphi,\psi)\in \mathfrak{M}^2(3\varepsilon).$$

Remark that $\tilde{b}(\varphi,\psi)$ does not involve η-variable. Hence, by Lemma 6.6 we see that $\pi E(\varphi\psi)E(\varphi)\iota$ is a pseudo-differential operator of order 0 and the amplitude depends. smoothly on (φ,ψ). This prove the smoothness of $r_\gamma(\varphi,\psi)$, because the smoothness of $\Xi(\varphi\psi)^{-1}$ has been already obtained in 6.5.

Next, we shall consider (6.22). The smoothness of the second term has been give in 6.3 combined with Proposition 2.2. Thus, we have only to consider the first term. However, the smooth dependence of $E(\varphi)\varphi^*a\cdot E(\varphi)$ can be easily seen by the similar way as the above argument. Hence, we complete the proof of Theorem A.

Now, what remains to be prove is only Proposition 6.1. Though the proof of Proposition 6.2 is not precisely given, the detail of the computations on each coordinate neighborhood can be naturally understood from the computations in the next section.

2.7 Regularity of primordial operators.

Our goal in this section is to prove Proposition 6.1 in § 2.6.

Let z_1, z_2 be elements of Θ_0 and are sufficiently close to z_0. Given $a \in \mathcal{Q}^3(\varepsilon_1, \varepsilon_2)$, recall the following integral:

(7.1) $\langle ae^{-iz_1 \boxplus z_2} \rangle$

$$= \mathrm{Os}\text{-}\iint a(x;xi,y;\eta,z;\zeta)e^{-iz_1 \boxplus z_2(x;\xi,y;\eta,z;\zeta)}dyd\eta \quad .$$

The above integral is defined as the oscillatory integral. Now, we shall show Proposition 6.1 by several steps as below.

Put as in 4.3

(7.2) $T(x;\xi,y;\eta,z;\zeta)$

$$= z_1 \boxplus z_2(x;\xi,y;\eta,z;\zeta) - c(z_1,z_2)(x;\xi,z;\zeta) \ .$$

where $z_i \in \Theta_0$ (i= 1,2,3). One can define the factor topology on $\Theta^{(2)}$ by using that of Θ.

Also, rewrite (7.1) by the following:

(7.3) $\langle ae^{-iz_1 \boxplus z_2} \rangle$

$$= A(a,z_1,z_2)(x;\xi,z;\zeta)e^{-ic(z_1,z_2)(x;\xi,z;\zeta)} \quad ,$$

where

$$(7.4)\quad A(a,z_1,z_2)$$

$$= \text{Os-}\iint a(x;\xi,y;\eta,z;\zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta \ .$$

Therefore, to prove Proposition 6.1, we may prove the following:

<u>Proposition 7.1</u> Notations being as above, we have

(i) For $z_1, z_2 \in \mathcal{O}_0$, sufficiently close to z_0, and $a \in \mathcal{Q}^3(\varepsilon_1,\varepsilon_2)$, the integral $A(a,z_1,z_2) \in \mathcal{Q}^2(\varepsilon_1+\varepsilon_2)$.

(ii) The mapping

$$A\colon \mathcal{Q}^3(\varepsilon_1,\varepsilon_2)\times U_{z_0}\times U_{z_0} \dashrightarrow \mathcal{Q}^2(\varepsilon_1+\varepsilon_2)\ ,$$

defined by (7.4) can be extended to a C^∞ mapping from $\mathcal{Q}^3_{(m)}(\varepsilon_1,\varepsilon_2)\times U_{z_0}\times U_{z_0}$ into $\mathcal{Q}^2_{(m)}(\varepsilon_1+\varepsilon_2)$ for every $m \leq 0$.

As an easy remark, if the integral (7.4) can be defined, then it is easily obtained that $A=0$ on $d(x,z)>\varepsilon_1+\varepsilon_2$.

The above proposition will be proved by dividing the integral into several domains $D_{(j)}$ and by expressing A by $A_{(j)}$. So, in what follows, we shall denote by Lem.$A_{(j)}$. If Lem.$A_{(j)}$ holds for every j, then so dose Proposition 7.1.

First, we take a positive constant R and fix it. Let

ω_R be a C^∞ function on $(T^*M)^2$ such that $\omega_R \geqq 0$,

(7.5) $\omega_R(x;\xi,y;\eta) = 1$ on $d(x,y) =< \varepsilon_1$ and $|\xi|^2 + |\eta'|^2 =< R^2/2$,

where $\cdot_x(X,\eta') = (y,\eta)$ and

(7.6) $\text{supp}\,\omega_R \quad \{(x;\xi,y;\eta) \in (T^*N)^2; d(x,y) =< e\varepsilon^1, |\xi|^2 + |\eta'|^2 =< R^2\} >$

Using ω_R, we divide (7.4) into two parts;

(7.7) $A(a,z_1,z_2) = \text{Os-}\iint (1 - \omega_R) a e^{-iT} + \iint \omega_R a e^{-iT}$

$$= A_{(0)} + A_{(-1)} \quad .$$

Remark that the second term $A_{(-1)}$ in (7.7) is integrable in the usual sense. Hence, a direct computation shows that Lem.$A_{(-1)}$ holds.

Remark. In fact, $A_{(-1)}(x;\xi,z;\zeta)$ is bounded in $|\zeta|$ and rapidly decreasing in $|\xi|$.

Next, we divide $A_{(0)}$ in (7.7) into several parts. First, let $\varphi(x;\xi,y;\eta;z:\zeta)$ be a C^∞ function on $(\overset{o}{T^*N})^3$ satisfying

(i) $\text{supp}\,\varphi \quad \{(x;\xi,y;\eta,z;\text{zeta}) \; ; \; d(x,y) =< r_1\}$

(ii) $\varphi \geqq 0$ and $\varphi \equiv 1$ on $|\eta - \eta_c| \leqq \delta_1 |\xi|/2$ and $\equiv 0$ on $|\eta - \eta_c| \geqq \delta_{|}\xi|$, where $(y_c;\eta_c)$ is the critical point given by Proposition 4.3 and δ_1 is chosen to be a sufficiently small constant.

(iii) $\varphi(x;r\xi,y;r\eta,z;s\zeta) = \varphi(x;\xi,y;\eta,z;\zeta)$ for any $r,s>0$.

Then, it is easily obtained that critical point of T which is the same as that of $z_1 + z_2$, obtained in Lemma 4.3, is contained in suppφ. Therefore, we get

(7.8) $A_{(0)}(a,z_1,z_2)$

$$= \iint \varphi a' e^{-iT} + \mathrm{Os}-\iint (1-\varphi) a' e^{-iT}$$

$$= A^1 + A^2 \quad ,$$

where $a' = (1-\omega_R)a$. Easily $\varphi a'$, $(1 - \text{phi})a' \in \Omega^3(\varepsilon_1,\varepsilon_2)$ by Lemma 5.6-7. Moreover, we divide A^2 in(7.8) by using a partition of unity: Namely, we choose functions Ψ_i $(i=1,2,3)$ with the following properties:

(7.9) $\Sigma_{i=1}^3 \Psi_i \equiv 1 \quad , \quad \Psi_i \in \Omega^3$;

and

(7.10)

$$\mathrm{supp}\Psi_1 = \{(x;\xi,y;\eta,z;\zeta) \in (\overset{o}{T}{}^*N)^3; |\eta| \leq C^{-1}|\xi|\}$$

$$\mathrm{supp}\Psi_2 = \{(x;\xi,y;\eta,z;\zeta) \in (T^*N)^3; (1/2)C^{-1}|\xi| \leq |\eta| \leq 2C|\xi|\}$$

$$\mathrm{supp}\Psi_3 = \{(x;\xi,y:\eta,z;\zeta) \in (T^*M)^3; |\eta| \geq C|\xi|\}$$

where C is chosen in Corollary 4.6. Now, we put

(7.11) A^2

$$= \mathrm{Os}\text{-}\iint \Psi_1(1 - \varphi)a'e^{-iT} + \mathrm{Os}\text{-}\iint \Psi_2(1 - \varphi)a'e^{-iT}$$

$$+ \mathrm{Os}\text{-}\iint \Psi_3(1 - \varphi)a' \quad e^{-iT}$$

$$= A^{2,1} + A^{2,2} + S^{2,3} \quad .$$

Using Lemmas 5.7-8, we summarize the following:

<u>Lemma 7.2.</u> Suppose that $\varepsilon_1, \varepsilon_2 < r_1/4$ and fix functions ω_R, φ, Ψ_i (i=1,2,3) defined as above. Then, we have

(i) The mapping $a \dashrightarrow a' = (1 - \omega_R)a$ can be extended to a C^∞ mapping on $\mathcal{R}^3_{(m)}(\varepsilon_1, \varepsilon_2)$ for every $m \leq 0$.

(ii) $A_{(0)}(a, z_1, z_2)$ in (7.8) can be written by

(7.12) $$A_{(0)(a,}z_1, z_2) = A^1 + A^{2,1} + A^{2,2} + A^{2,3} \quad ,$$

where

(7.13) A^1

$$= \iint c_1(x;\xi,y;\eta,z;\zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta \quad ,(i=1,2,3) \quad ,$$

where $c_1 = \varphi(1 - \omega_R)a$, $c_{2,i} = (1 - \omega_R)(1 - \varphi)\Psi_i a$ are elements in $\mathcal{R}^3(\varepsilon_1, \varepsilon_2)$ respectively and

(7.15) $$\mathrm{supp}C_1 \subset D_1 = \{(x;\xi,y;\eta,z;\zeta) \in (T^*N)^3; d(x,y) \leq \varepsilon_1, d(y,z) \leq \varepsilon_2,$$

$$|\xi|^2 + |\eta'|^2 \geq R^2/2, |\eta - \eta_c| \leq \delta_1|\xi|\} \quad ,$$

(7.16) $\operatorname{supp} c_{2,1} \subset D_{2,1} = \{A \in (T^*N)^3; d(x,y) \leqq \varepsilon_1, d(y,z) \leqq \varepsilon_2,$

$$|\xi|^2 + |\eta'|^2 \geqq R^2/2, |\eta - \eta_c| \geqq (1/2)\delta_1|\xi|, |\eta| \leqq C^{-1}|\xi|\},$$

(7.17) $\operatorname{supp} c_{2,2} \subset D_{2,2} = \{(x;\xi,y;\eta,z;\zeta) \in (T^*N)^3; d(x,y) \leqq \varepsilon_1, d(y,z) \leqq \varepsilon_2,$

$$|\xi|^2 + |\eta'|^2 \geqq R^2/2, |\eta - \eta_c| \geqq (1/2)\delta_1|\xi| ,$$

$$(1/2)C^{-1}|\xi| \leqq |\eta| \leqq 2C|\xi|\} ,$$

(7.18) $\operatorname{supp} c_{2,3} \subset D_{2,3} = \{(x,\xi,y,\eta,z,\zeta) \in (T^*N)^3; d(x,y) \leqq \varepsilon_1, d(y,z) \leqq \varepsilon_2,$

$$|\xi|^2 + |\eta'|^2 \geqq R^2/2, |\eta - \eta_c| \geqq (1/2)\delta_1|\xi|, |\eta| \geqq C|\xi|\}.$$

(iii) Moreover, c_1, $c_{2,i}$ $(i=1,2,3) \in \mathcal{O}\!\ell^3(\varepsilon_1,\varepsilon_2)$ depend continuous- linearly on a in $\mathcal{O}\!\ell^3_{(m)}$ topology.

Lem.A^1. First, we remark that A^1 is integrable in the usual sense. We shall check the conditions (A.1-2) in Definition 5.2 by using coordinate system $\{\Delta_{2,I}\}$ (Cf. List 5.1 for k=2). Now, we may take R by R>4k. Then, $\Delta_{2,\varphi} \cap \operatorname{supp} c_1 = \varphi$ and we have only to investigate for four cases; $\Delta_{2,(1)}$, $\Delta_{2,(2)}$, $\Delta_{2,(1,2)}$, $\Delta_{2,(2,1)}$.

<u>On $\Delta_{2,(1)}$</u>: By using the variables (r^{-1},t) in $\Delta_{2,(1)}$, we have

$$(\tilde{A}^1)_{2,(1)}(r^{-1},t,x;\xi,z;\zeta)$$

$$= \iint_{D_1} c_1(x;r\xi,y;\eta,z;t\xi)e^{-iT(x;r\xi y;\eta,z;t\xi)}dyd\eta \quad .$$

By Proposition 4.7, if we take δ_1 as a sufficiently small constant, we get

(7.19) $(\tilde{A}^1)_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$

$$= \iint_{D_1'} c_1' (x;r\hat{\xi},y;\eta,z;t\hat{\xi})e^{-i\langle\eta'-\eta_c'|X-X_c\rangle}dyd\eta \ ,$$

where $c_1' = (c\ \tilde{\Phi})|\det D\tilde{\Phi}|$, $D_1' = \tilde{\Phi}^{-1}D_1$. Setting $\tilde{\eta}=(1/r)\eta$, we get

$(\tilde{A}^1)_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$

$$= \iint r^n c_1'(x;r\hat{\xi},y;r\hat{\eta},z;t\hat{\xi})e^{ir\langle\tilde{\eta}'-\eta_c'(x;\hat{\xi},z;\hat{\xi})|X-X_c(x;\hat{\xi},z;\hat{\xi})\rangle}dyd\tilde{\eta} \ ,$$

where $\cdot_X(X,\tilde{\eta}')$. Consider the function $c_1(x;r\hat{\xi},y;\hat{\eta},z;t\hat{\xi})$. By using Proposition 4.7 (ii), we have

$c_1'(x;r\hat{\xi},y;r\hat{\eta},z;t\hat{\xi})$

$$= c_1(x;r\hat{\xi},\tilde{\Phi}_1(x;\hat{\xi},y;\hat{\eta},z;\hat{\xi});r\tilde{\Phi}_2(x;\hat{\xi},y;\tilde{\eta},z;t\hat{\xi})$$

$$\times|\det D\tilde{\Phi}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|$$

where $\tilde{\Phi}^{-1}(y;r\tilde{\eta})$ moves in D_1. Thus, using (3.12), we have

$$C^{-1}\|\partial_X\tilde{S}^{-1}\| - N\delta_1 \le |\tilde{\eta}'| \le C\|\partial_X\tilde{S}^{-1}\| + N\delta_1$$

for certain constants C,N>0depending on the Riemannian structure. Since X,Z are sufficiently small, we put

(7.20) $\quad \tilde{r} = r|\tilde{\eta}|\ ,\tilde{t}_1 = |\tilde{\eta}|,\tilde{t}_3 = t,$

and define a function ρ by

$$\rho(\tilde{r}^{-1},\tilde{t}_1,\tilde{t}_3,x;\hat{\xi},y;\hat{\tilde{\eta}},z;\hat{\xi}) = c_1'\ (x;\tilde{}(r/\tilde{t}_1)\hat{\xi},y;\tilde{r}\hat{\tilde{\eta}},z;\tilde{}t_3\hat{\xi}).$$

Then, ρ is C^∞ on $\Delta_{3,(2,1)}$. By putting $\tilde{\eta}'' = \tilde{\eta}'-\eta_c'$ and $X''=X-X_c$, we get

$$(\tilde{A}^1)_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$=\iint\rho((r|\tilde{\eta}|)^{-1},|\tilde{\eta}|,t,x;\hat{\xi},\cdot_X(X,\tilde{\eta}')z,\hat{\xi}))$$

$$\times e^{ir\langle\tilde{\eta}''|X''\rangle}r^n d\tilde{\eta}''dX'' \quad .$$

Using the Taylor expansion of $\rho(\ldots)$ with respect to X" and integration by partss, we get

(7.21) $\quad (\tilde{A}^1)_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$

$$= \sum\frac{1}{\alpha!}\partial^\alpha_{\tilde{\eta}''}\partial^\alpha_{X''}\rho((r|\tilde{\eta}|)^{-1},|\tilde{\eta}|,t,x;\hat{\xi},\cdot_X(X,\tilde{\eta}'),z;\hat{\xi}))_{\substack{X''=0\\ \tilde{\eta}''=0}}(-ir)^{-|\alpha|}$$

$$+ R_{m-1} \quad ,$$

where R_{m-1} is the remainder term obtained by Taylor expansion and is of order $O(r^{m-1})$. Moreover, use that ρ can have the asymptotic expansion with respect to r. Then (A.1-2) are obvious.

<u>On $\Delta_{2,(2)}$</u> : Also, use the variables in $\Delta_{2,(2)}$ and the notation as in above. Then, we have

(7.22) $$(\tilde{A}^1)_{2,(2)}(r^{-1},t,x,\hat{\xi},z,\hat{\xi})$$

$$= \iint_{D_1} c_1(x;tz\hat{\imath},y;\eta,z;r\hat{\xi})e^{-iT(x;t\hat{\xi},y;\eta z;r\xi)}dyd\eta \quad .$$

By Proposition 4.7, we have

$$(\tilde{A}^1)_{2,(2)}$$

$$= \iint_{D_1} c_1(x;t\hat{\xi}\ ,y;\eta,z;r\hat{\xi})$$

$$\times e^{i\langle\eta'-\eta_c'(x;t\hat{\xi},z,\hat{\xi})|X-X_c(x;t\hat{\xi},z;\hat{\xi})\rangle}dyd\eta \quad ,$$

where $c_1'=(c_x\cdot\tilde{\Phi})|(\mathrm{Det}D\tilde{\Phi})|$. Also, by using Proposition 4.7 (ii), we get

$$c^{1\cdot}(x;t\hat{\xi},y;\eta,z;r\hat{\xi})$$

$$= c_1(x;\xi,\Phi_{\Upsilon}(x;t\hat{\xi},y;eta,z,\hat{\xi});\tilde{\Phi}_2(x;t\hat{\xi},y;eta,z;\hat{\xi})$$

$$\times|\mathrm{det}D\Phi(x;\hat{\xi},y;\eta,z;\xi)|$$

Since $|\eta|$ may be estimated by $|\eta|\leq K$, put

(7.23) $$\tilde{r} = r,t_{\Upsilon} = t,t_2 ltilde = |eta| \quad .$$

we see that

$$\rho(\tilde{r}^{-1},\tilde{t}_1,\tilde{t}_2,x;\hat{\xi},y;\hat{e}ta,z;\hat{\xi}) = c_1'(x;\tilde{t}_1,\hat{\xi},y;\tilde{t}_2,\hat{\eta},z;\tilde{r}\hat{\xi})$$

is smooth on $\Delta_{3,(3)}$. So, by using Taylor expansion, we get (A.1-2).

<u>On $\Delta_{2,(2)}$</u> : Using varibles in $\Delta_{2,(1,2)}$ and using Proposition4.7, we have

$$(\tilde{A}^1)_{2,(1,2)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{D_1} c_1(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi})$$

$$\times e^{\langle\eta'-r\eta_c'(x;\hat{\xi},z;\xi)|X-X_c(x;\hat{\xi},z;\hat{\xi})\rangle} dyd\eta \quad .$$

Bh changing variables $\eta=(1/r)\eta$, we have

(7.24)
$$(\tilde{A}^1)_{2,(1,2)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{D_1} c_1(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi})$$

$$\times e^{ir\langle\eta'-\eta_c'(x;\hat{\xi},z;\xi)|X-X_c(x;\hat{\xi},z;\hat{\xi})\rangle} r^n dyd\eta \quad .$$

where $C^{-1}\|\partial_X\tilde{S}^{-1}\|-N\delta_1 \leq|\tilde{\eta}|\leq C\|\partial_X\tilde{S}^{-1}\|+N\delta$. put

(7.25) $\tilde{r} = r|\eta|, t_\Upsilon = |\eta|, t_\mathfrak{Z} = t\rangle$

Then, we get

$$\rho(r^{-1},t_\Upsilon,t_\mathfrak{Z},x;\hat{\xi},y;\hat{\tilde{\eta}},z,\hat{\xi}) = c_1'(x;\tilde{(}r/t_\Upsilon)\hat{\xi},y;\widehat{\tilde{r}\eta},x;(\tilde{r}/t_\Upsilon t_\mathfrak{Z})\hat{\xi}) \quad ,$$

is smooth on $\Delta_{3,(2,1,3)}$, where

$$c_1'(x;r\hat{\xi},y;r\hat{\eta},z;(r/t)\hat{\xi})$$

$$= c_1(x;r\hat{\xi},\Phi_{\gamma_1}(x;\hat{\xi},y;\hat{\eta},z;\hat{\xi});r\Phi_{\gamma_2}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}),z;(r/t)\hat{\xi})$$

$$\times|\det D\Phi_{\gamma_1}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})| \ .$$

Therefore, by using Taylor expansion of ρ, we get (A.1-2) in $\Delta_{2,(1,2)}$.

<u>On</u> $\Delta_{2,(2,1)}$: Using variables in $\Delta_{2,(2,1)}$ and using Proposition 4.7, we have

$$(\tilde{A}^1)_{2(2,1)}(r^{-1},t,x;\hat{\xi},z;\xi)$$

$$= \iint_{D_1'} c_1'(x;(r/t)\hat{\xi},y;\eta,z;r\xi)$$

$$\times e^{i\langle\eta'-(r/t)\eta_c'(x;\hat{\xi},z;\hat{\xi})|X-X_c(x;\hat{\xi},z;\hat{\xi})\rangle} dyd\eta \quad ,$$

where $c^{1\prime} = (c_{\cdot}\tilde{\Phi})|(\det D\tilde{\Phi})|$. By changing variables $(r/t)\tilde{\eta}=\eta$, we ahve

$$(7.26) \quad (\tilde{A}^1)_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint c_1'(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})$$

$$\times e^{i(r/t)\langle\tilde{\eta}'-\eta_c'(x;x\hat{\imath},z;\hat{\xi})|X-X_c(x;\hat{\xi},z;\hat{\xi})\rangle}(r/t0^n dyd\eta \ ,$$

where $C^{-1}\|\partial_X\tilde{S}^{-1}\|-N\delta_1 \leq|\tilde{\eta}|\leq C\|\partial_X\tilde{S}^{-1}\|+N\delta_1$. If. $|\tilde{\eta}|/t\leq K$, then we put

$$(7.27)\quad \tilde{r}=r|\tilde{\eta}|/r\ ,\tilde{t}_1=t,\tilde{t}_3=|\tilde{\eta}|/t.$$

Therefore,

$$(7.28)\quad \rho(r^{-1},\tilde{t}_1,\tilde{t}_3,x;\hat{\xi},y;\hat{\tilde{\eta}},z;\xi)$$

$$= c_1'(x;9\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi},yk\tilde{;}r\hat{\tilde{\eta}},z;9\tilde{r}/\tilde{t}_3)\hat{\xi})$$

is smooth on $\Delta_{3,(2,3,1)}$. Also, if $|\tilde{\eta}|/t\geq K^{-1}$, then we put

$$(7.29)\quad \tilde{r}=r,\tilde{t}_1=|\tilde{\eta}|,\tilde{t}_2=t/|\tilde{\eta}|.$$

Therefore,

$$(7.30)\quad \rho(\tilde{r}^{-1},\tilde{t}_1,\tilde{t}_2,x;\hat{\xi},y;\hat{\tilde{\eta}},z;\hat{\xi})$$

$$= c_1'(x;(\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\xi},y;(\tilde{r}/\tilde{t}_2)\hat{\tilde{\eta}},z;\tilde{r}\hat{\xi})$$

is smooth on $\Delta_{3,(3,2,1)}$. Here

$$c_1'(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})$$

$$=c_1(x;(r/t)\hat{\xi},\tilde{\Phi}_1(x;\hat{\xi},y;\eta\text{tilde},z;\hat{\xi});(r/t)\tilde{\Phi}_2(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})$$

$$\times|\det D\tilde{\Phi}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|\ .$$

Applying Taylor expansion for both case (7.28) and (7.30), we get (A.1-2) in $\Delta_{2,(2,1)}$.

Lastly, we have to check the differentiability of A^1 with respect to a, z_1 and z_2. It is easily seen by differentiating (7.19), (7.21),(7.24) and (7.26) directly with respct to a, z_1, z_2 and by the same computations as above.

Lem. $A^{2,1}$. We shall consider the integral $A^{2,11}$ in Lemma 7.4, i.e.,

$$A^{2,1}(x;\xi,z;\zeta)$$

$$= \mathrm{Os}\text{-}\iint c_{2,1}(x;\xi,y;\eta,z;\zeta)e^{iT(x;\xi,y;\eta,z;\zeta)}dydet ,$$

where $c_{2,1}$ is defined by (7.14). To check the differentiability of $A_{2,1}$ with respect to a, z_1, and z_2, we shall check formally differentiability of $A^{2,1}$ with respect to a, z_1 and z_2. Then, it is easily seen that these derivatives can be written by the sum of the following integrals, for |alha|≧0

(7.31) $\quad \nabla^{\alpha}A^{2,1}(x;\xi,z;\zeta)$

$$= \mathrm{Os}\text{-}\iint c_{2,1}^{\alpha,T'}(x;\xi,y;\eta,z;\zeta)e^{iT(x;xi,y;\eta,z;\zeta)}dyd\eta ,$$

where $c_{2,1}^{\alpha,T'}$ can be described as follows :

(a) $\quad c_{2,1}^{\alpha,T'} = \tilde{c}_{2,1}(T')^{\alpha}$,

(b) $\quad (T')^{\alpha} = (T'_1)^{\alpha_1}\ldots(T')^{\alpha_k} ,T'_i \in \quad (2)$,

(c) $\tilde{c}_{2,1} \in \mathcal{Q}^{(3)}(\varepsilon_1,\varepsilon_2)$ and satisfies the same conditions for $c_{2,1}$ in (7.16).

Now, we shall prove that Lem. $A^{2,1}$ holds.

We shall observe $A^{2,1}$ (or $\nabla^{\alpha} A^{2,1}$) for each chart $\{\Delta_{2,I}\}$. Remark that on support of $c_{2,1}$, we have $|\xi| \geq R^2/2(1+C^{-2})$. Therefore, if we take $R \geq (2K(1+C^{-2}))^{1/2}$, then supp $c_{2,1} \cap \Delta_{2,\varphi} = \varphi$ and supp $c_{2,1} \cap \Delta_{2,(2)} = \varphi$. So, we shall only investigate $\nabla^{\alpha} A^{2,1}$ for the cases $\Delta_{2,(1)}, \Delta_{2,(1,2)}$ and $\Delta_{2,(2,1)}$.

On $\Delta_{2,(1)}$: Use the coordinate on $\Delta_{2,(1)}$. Then, we have

(7.32) $$\nabla^{\alpha} A^{\tilde{2},1}{}_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \text{Os-}\iint_{D_{2,1}} c_{2,1}^{\alpha,T`}(x;r\hat{\xi},y;\eta z;t\hat{\xi})e^{iT(x;r\hat{\xi},y;\eta,z;t\hat{\xi})}dyd\eta \quad .$$

Setting $\eta = r\tilde{\eta}$, we get

$$\nabla^{\alpha} A^{\tilde{2},1}{}_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{|\tilde{\eta}| \leq C^{-1}} c_{2,1}^{\alpha,T`}(x;r\hat{\xi},y;\eta z;t\hat{\xi})$$

$$\times e^{irT(x;\hat{\xi},y;\eta,z;\hat{\xi})} r^n dyd\eta \quad ,$$

where

$$c_{2,1}^{\alpha,T`}(x;r\hat{\xi},y;\eta z;t\hat{\xi})$$

$$= \tilde{c}_{2,1}(x,r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})r^{|\alpha|}(T`(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})^{alhpa} \quad .$$

By Corollary 4.6,(2), we put

$$(7.33) \qquad L_T = \frac{i\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})\cdot\partial_y}{r|\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|^2}$$

Then, $L_T e^{-irT=e^{-trT}}$ and $|\partial_y T|\geqq M$. Using the integration by parts, we get

$$(\nabla^{\alpha} \quad A^{\tilde{2},1})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{|\tilde{\eta}|\leqq C^{-1}} r^{n+|\alpha|}(L_T^*)^m[\tilde{c}_{2,1}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})$$

$$\times(T`(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}))^{\alpha}]e^{-irT(x;\hat{\xi},y;\tilde{\eta}z;\hat{\xi})}dyd\tilde{\eta} \quad ,$$

where L_T^* is the adjoint operator of L_T. Since K can be chosen as sufficiently large number, we put

$$(7.34) \qquad \tilde{r}-r|\tilde{\eta}| \;,\tilde{t}_1 = |\tilde{\eta}| \;<\tilde{t}_3 = t \quad .$$

Then, the function ρ defined by

$$\rho(r^{-1},\tilde{t}_1,\tilde{t}_2,x;\hat{\xi}ylk;\hat{\tilde{\eta}},z;\hat{\xi}) = \tilde{c}_{2,1}(x;(\tilde{r}/\tilde{t}_1)\hat{\xi},y;\tilde{r}\hat{\tilde{\eta}},z;\tilde{t}_3\hat{\xi})$$

is smooth on $\Delta_{3,(2,1)}$. By choosing m sufficiently large, we have

$$(7.35) \qquad (\nabla_{\alpha}A^{\tilde{2},1})_{2,(1)} = O(r^{-N}) \quad \text{for any } N\geqq 0 \ .$$

Also, by a similar computation, we get

$$(7.36) \qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha}A^{\tilde{2},1})_{2,(1)} = O(r^{-N}) \quad \text{for any } N\geqq 0 \ .$$

On $\Delta_{2,(1,2)}$: Use the coordinate on $\Delta_{2,(1,2)}$. Then, we have

(7.37) $$(\nabla^{\alpha}A^{\tilde{2},1})_{2,(1,2)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \mathrm{Os}-\iint c_{2,1}(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi})e^{-iT(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi})}dyd\eta \quad .$$

Setting $\eta=r\tilde{\eta}$, we have

$$(\nabla^{\alpha}A^{\tilde{2},1})_{2,(1,2)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{|\tilde{\eta}|\leq C^{-1}} c_{2,1}^{\alpha,T`i}(x;r\hat{\xi},y;\tilde{\eta},z;(r/t)\hat{\xi})$$

$$\times e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})}r^{n}dyd\tilde{\eta} \quad ,$$

where

$$c_{2,1}^{\alpha,T`i}(x;r\hat{\xi},y;\tilde{\eta},z;(r/t)\hat{\xi})$$

$$= \iint_{|\tilde{\eta}|\leq C^{-1}} c_{2,1}^{\alpha,T`i}(x;r\hat{\xi},y;\tilde{\eta},z;(r/t)\hat{\xi})$$

$$\times r^{|\alpha|}(T`(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}))^{\alpha} \quad .$$

By Corollary 4.6 (2), we use the operator L_T in (7.33). Remark that by putting

(7.28) $$\tilde{r} = r|\tilde{\eta}| \ ,\tilde{t}_1 = |\tilde{\eta}| \ ,\tilde{t}_3 = t,$$

the function

$$\rho(\tilde{r}^{-1},\tilde{t}_1,\tilde{t}_3,x;\hat{\xi},y;\hat{\tilde{\eta}},z;\hat{\xi})$$

$$= \tilde{c}_{2,1}(x;\tilde{}(r/\tilde{t}_1)\hat{\xi},y;\tilde{r}\hat{\tilde{\eta}} \quad ,z;(\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi})$$

is smooth on $\Delta_{3,(2,1,3)}$. The same computation as in (7.35) gives

(7.39) $$(\nabla^{\alpha}A^{\tilde{2},1})_{2,(1,2)} = O(r^{-N})$$

(7.40) $$D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha}A^{2,1})_{2,(1,2)} = O(r^{-N}) \quad \text{<.EN}$$

for any $N \geq 0$.

On $\Delta_{2,(2,1)}$: Use the coordinate on $\Delta_{2,(2,1)}$. Then, we get

(7.41) $$(\nabla^{\alpha}A^{2,1})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \mathrm{Os}\text{-}\iint_{D_{2,1}} c^{\alpha,T`}_{2,1}(x;(r/t)\hat{\xi},y;\eta,z;\hat{\xi})$$

$$\times e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\xi})}dy\,d\eta \quad .$$

Set $\eta=(r/t)\tilde{\eta}$, and we have

$$(\nabla^{\alpha}A^{2,1})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{|\tilde{\eta}|\leq C^{-1}} c^{\alpha,T`}_{2,1}(x;(r/t)\hat{\xi},(r/t)y;\eta,z;r\hat{\xi})$$

$$\times e^{-i(r/t)T(x;\hat{\xi},y;\hat{\eta}\qquad,z;\hat{\xi})}(r/t)^n dy d\tilde{\eta} \quad ,$$

where

$$c_{2,1}^{\alpha,T'}(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})$$

$$= \tilde{c}_{2,1}(x;8r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})(r/t)^{|\alpha|}(T'(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}))^{\alpha} \ .$$

Put

$$(7.42) \qquad L_T^{`} = \frac{it\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})\cdot\partial_y}{r|\partial_T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|^2} \quad .$$

Then, $L_T^{`}e^{-i(r/t)T} = e^{-t(r/t)T}$ and $|\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|\geq M$ on supp $c_{2,1}$ by Corollary 4.6. Thus , we get for any $m\geq 0$,

$$(\nabla^{\alpha}A^{\tilde{2},1})_{2,(2,1)}$$

$$= \iint_{|\tilde{\eta}|\leq C^{-1}} (L_T^{`}*)^m[c_{2,1}^{\alpha,T'}(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;(r/t)\hat{\xi})]$$

$$\times (r/t)^n e^{-i(r/t)T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})} dy d\tilde{\eta} \quad .$$

Put as in (7.37)-(7.40), we get

$$(7.43) \qquad (\nabla^{\alpha}A^{\tilde{2},1})_{2,(2,1)} = O(r^{-N}t^N) \quad \text{for any } N\geq 0 \ .$$

and similarly

$$(7.44) \qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha} \quad \tilde{A}^{2,1})_{2,(2,1)} = O(r^{-N}t^N)$$

for any $N \geq 0$.

what we have shown in the above argument is any formal differentials of $A^{2,1}$ with respect to (a, z_1, z_2) are well-defined in $\mathcal{O}^2(\varepsilon_1, \varepsilon_2)$ and theses differentials are continuous. To prove the differentiability of $A^{2,1}(a, z_1, z_2)$, we have to take the formal Taylor expansion and compute the remainder term (cf. [31]). However, the estimation of the remainder term of Taylor expansion of the integrand $A^{2,1}$ by similar computations as above. Thus, we obtain Lem. $A^{2,1}$.

Lem. $A^{2,3}$: Now, we shall consider the integral Asu2,3 in Lemma 7.4. As in 7.3, to consider the differentiability with respect to a, z_1, z_2, for $|\alpha| \geq 0$,

(7.45) $\nabla^{\alpha} A^{2,3}(x;\xi, z;\zeta)$

$$= \text{Os-}\iint_{Ds;ub2,3} c_{2,3}^{\alpha,T`}(x;\xi,y;\eta,z;\zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta \quad ,$$

where $c_{2,3}^{alph,T`}$ can be described as follows:

(a) $c_{2,3}^{\alpha,T`} = \tilde{c}_{2,3}(T`)^{\alpha}$,

(b) $(T`)^{\alpha} = (T`_1)^{\alpha_1} \ldots (T`_k)^{\alpha_k}$ $, T`_i \in$ (2) ,

(c) $\tilde{c}_{2,3} \in$ $(\varepsilon_1, \varepsilon_2)$ and satisfies the same condition for $c_{2,3}$ in (7.18).

Now, we shall prove that Lemm. $A^{2,3}$ holds by observing $A^{2,3}$

On $\Delta_{2,\varphi}$: Use the coordinate on $\Delta_{2,\varphi}$. The, we have

$$(\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}(t_1,t_2,x;\hat{\xi},z;\hat{\xi})$$

$$= Os\text{-}\iint c_{2,3}^{\alpha,T`}(x;t_1\hat{\xi},y;\eta,zt_2\hat{\xi})e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\xi})}dyd\eta \quad .$$

Now, we divide the above integral into two parts. Namely, by using a cut off function $\psi(y;\eta)$ such that $\operatorname{supp}\psi(y;\eta)\in\{(y;\eta)\in T^*N \ ; \ |\eta|=<K\}$, we get

$$(7.46)\quad (\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}(t_1,t_2,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint \psi c_{2,3}^{\alpha,T`}(x;t_1\hat{\xi},yk;\eta,z;t_2\hat{\xi})$$

$$\times e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\xi})}dyd\eta$$

$$+ Os\text{-}\iint (1-\psi)c_{2,3}^{\alpha,T`}(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\xi})$$

$$\times e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\xi})}dyd\eta$$

$$= (\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}^{1} + (\nabla A^{\tilde{2},3})_{2,\varphi}^{2} \quad .$$

It is easily seen that the first integral $(\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}^{1}$ is differentiable function in $(t_1,t_2,x;\hat{\xi},z;\hat{\xi})$ by using the coordinate $\Delta_{3,\varphi}$. For $(\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}^{2}$, use the operator

(7.47) $$L_T'' = \frac{i\partial_y T(x;t_1\hat{\xi},y;\eta,z;\hat{\xi})\cdot\partial_y}{|\partial_y T(x;t_1\hat{\xi},y;\eta,z;\hat{\xi})|^2} .$$

Then, $L_T'' e^{-iT}=e^{-iT}$ and $|\partial_y T(x;t_1\hat{\xi},y;\eta,z;\hat{\xi})| \geq M|\eta`|>0$ on $\Delta_{3,(2)}$. By using L_T'' and the fact that $\tilde{c}_{2,3}(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\xi})$ is differentiable on $\Delta_{3,(2)}$, we get $(\nabla^{\alpha}A^{\tilde{2},3})_{2,\varphi}$ is differentiable on $(t_1,t_2,x;\hat{\xi},z;\hat{\xi})$.

<u>On</u> $\Delta_{2,(1)}$: Use the coordinate on $\Delta_{2,}(1)$. Then, we get

$$(\nabla^{\alpha}\tilde{2},3)^{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \mathrm{Os}\text{-}\iint c_{2,3}^{\alpha,T`}(x;r\hat{\xi},y;\eta,z;t\hat{\xi})\exp -iT(x;r\hat{\xi},y;\eta,z;t\hat{\xi})dyd\eta.$$

Putting $\eta=r\tilde{\eta}$, we have

(7.48) $$(\nabla^{\alpha}A^{\tilde{2},3})_{2,(1)}(r^{-1},t,x;\hat{\xi},y;\eta,z;t\hat{\xi})$$

$$= \mathrm{Os}\text{-}\iint c_{2,3}^{\alpha,T`}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})}r^n dyd\tilde{\eta} ,$$

where

$$c_{2,3}^{\alpha,T`}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})$$

$$= \tilde{c}_{2,3}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})(T`(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}))^{\alpha}r^{|\alpha|} ,$$

and $|\tilde{\eta}`|\geq C>0$. Now, we use L_T in (7.33). Integrating by parts, we have

$$(\nabla^{\alpha}A^{\tilde{2},3})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \mathrm{Os}\text{-}\iint (L_T^*)^m (c_{2,3} su\alpha, T`(x;r\hat{\xi}, y;r\tilde{\eta}, z;t\hat{\xi}))$$
$$\times e^{-iT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})} r^n dyd\tilde{\eta}.$$

Put

(7.49) $\tilde{r} = r \ , \tilde{t}_2 = \frac{1}{|\tilde{\eta}|} \ . \tilde{t}_3 = t \ .$

Then, $\tilde{c}_{2,3}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})$ is smooth on $\Delta_{3,(1,2)}$. Therefore, we get

(7.50) $(\nabla^\alpha A^{2,3}) \quad 2,(1) = O(r^{-N})$ for any N>=0 .

Similary, we have

(7.51) $D^\mu_{(r,t,x;\hat{\xi},z;\hat{\xi})} (\nabla^\alpha A^{\tilde{2},3})_{2,(1)} = O(r^{-N})$ for any N>=0 .

On $\Delta_{2,(2)}$:

$(\nabla^\alpha A^{\tilde{2},3})_{2,(2)}(r^{-1},t,x;\hat{\xi},y;\eta,z;t\hat{\xi})$.

$$= \mathrm{Os}\text{-}\iint c_{2,3}^{\alpha,T`}(x;t\hat{\xi},y;\eta,z;t\hat{\xi}) e^{-iT(x;t\hat{\xi},y;\eta,z;\hat{\xi})} r^n dyd\tilde{\eta} \ ,$$

where $|\eta| \geqq Ct$. By using a cut off function $\psi(y;\eta)$ defined for $\Delta_{2,\varphi}$ in (7.46), we divide the above integraion as

(7.52) $(\nabla^\alpha A^{\tilde{2},3})_{2,(2)}$

$$= \iint_{Ct=<|\eta|=<K} \varphi \cdot c_{2,3}^{\alpha,T`} e^{-iT} + \mathrm{Os}\text{-}\iint_{|\eta| \geqq K^{-1}} (1-\psi) c_{2,3}^{\alpha,T`} e^{-iT}$$

$$= (\nabla^{\alpha} A^{\tilde{2},3})^{1}_{2,(2)} + (\nabla^{\alpha} A^{\tilde{2},3})^{2}_{2,(2)} \quad .$$

Use the fact that $\tilde{c}_{2,3}(x;t\hat{\xi},y;\eta,z;r\hat{\xi})$ is smooth with respect to $(r^{-1},|\eta|,t,x;\hat{\xi},y;\hat{\eta},z;\hat{\xi})$ on $\Delta_{3,(3)}$ if $|\eta| =< K$. Then, $(\nabla^{\alpha} A^{\tilde{2},3})^{1}_{2,(2)}$ is smooth and is $O(1)$. Nect, consider $(\nabla^{\alpha} A^{\tilde{2},3})^{2}_{2,(2)}$. Put, if $\frac{|\eta|}{r} =< K$,

(7.53) $\tilde{r} = |\eta| \ ,\tilde{t}_1 = t \ ,\tilde{t}_3 = \frac{|\eta|}{r}$;

and if $\frac{r}{|\eta|} =< K$,

(7.54) $\tilde{r} = r,\ \tilde{t}_1 = \tilde{t},\ t_3 = \frac{r}{|\eta|}$.

Then, $\tilde{c}_{2,3}(x;t\hat{\xi},y;\eta,z;r\hat{\xi})$ is smooth on $\Delta_{3,(2,3)}$ and $\Delta_{3,(3,2)}$. Use also $\ddot{L}_T$ defined in (7.47), and we get

(7.55) $\quad (\nabla^{\alpha} \tilde{2,3})^{2}_{2,(2)}) = O(1)$.

Therefore, $(\nabla^{\alpha} A^{\tilde{2},3})_{2,(2)} = O(1)$. Similarly, we get

(7.56) $\quad D_{(t,x;\hat{\xi},z;\hat{\xi})}^{\mu} (\nabla^{\alpha} A^{\tilde{2},3})_{2,(2)} = O(1)$.

<u>On</u> $\Delta_{2,(1,2)}$: Use the coordinate in $\Delta_{22,(1,2)}$. We have

$(\nabla^{\alpha} A^{\tilde{2},3})_{2,(1,2)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$

$$= Os\text{-}\iint c^{\alpha,T`}_{2,3}(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi}) e^{-iT(x;r\hat{\xi},y;\tilde{\eta},z;(r/ts)\hat{\xi})} dy d\eta \quad .$$

Setting $\eta = r\tilde{\eta}$, we get

(7.57) $(\nabla^{\alpha} A^{\tilde{2},3})_{2,(1,2)}$

$$= \mathrm{Os}\text{-}\iint_{|\eta|\geq C} c^{\alpha}_{2,3}{}^{,T^{\cdot}}(x;r\hat{\xi},y;r\tilde{\eta},z;(r/t)\hat{\xi})$$

$$\times r^{n} e^{-irT(x;\hat{\xi},y;\eta\text{tsilde},z;\hat{\xi})} dy d\eta \quad ,$$

where

$$c^{\alpha,T^{\cdot}}_{2,3}(x;r\hat{\xi},y;r\tilde{\eta},z;(r/t)\hat{\xi})$$
$$= \tilde{c}_{2,3}(x;r\hat{\xi},y;r\tilde{\eta},z;(r/t)\xi\text{hat}) r^{|\alpha|} (T^{\cdot})^{\alpha}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}) \quad .$$

Use L_T in (7.33) and note that $|\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})| \geq M(1+|\eta|)$ because of Corollardy 4.6 and also $|\partial^{\mu}_{y} jT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})| =< C_{\mu}(1+|\tilde{\eta}|)$. Therefore, integrating by parts, we have

(7.58) $(\nabla^{\alpha} A^{\tilde{2},3})_{2,(1,2)}(x;r\hat{\xi},z;(r/t)\hat{\xi} \ )$

$$= \iint (L_T^{*})^{m} [\tilde{c}_{2,3}(x;r\hat{\xi},y;r\tilde{\eta},z;(r/t)\hat{\xi})$$

$$\times r^{N+|\alpha|} (T^{\cdot})^{\alpha}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})]$$

$$\times e^{-irT(x;\xi,y;\tilde{\eta},z;\xi)} dy d\tilde{\eta} \quad .$$

Put, if $|\tilde{\eta}|t=<K$,

(7.59) $\tilde{r} = r,\ \tilde{t}_2 = \dfrac{1}{|\tilde{\eta}|},\ \tilde{t}_3 = t|\tilde{\eta}| +$

and if $|\tilde{\eta}|t\geq K^{-1}$,

(7.60) $\tilde{r} = r,\ \tilde{t}_2 = \dfrac{1}{|\tilde{\eta}|t},\ \tilde{t}_3 = t$.

The amplitude function $\tilde{c}_{2,3}$ in (7.58) is smooth in (r,t) on $\Delta_{3,(1,2,3)}$ and $\Delta_{3,(1,3,2)}$ for the cases (7.59) and (7.60), respectively. We have

$$\partial_r = \partial_{\tilde{r}}\ ,\partial_t = |\tilde{\eta}|\partial_{\tilde{t}_3} \quad \text{for (7.59)}$$

(7.61)

$$\partial_r = \partial_{\tilde{r}},\ \partial_t = -(1/t)\tilde{t}_2\partial_{\tilde{t}_2} + \partial_{\tilde{t}_3} \quad \text{for (7.60)} .$$

Remark that for the case (7.60), we get $(1/t) = \langle K|\tilde{\eta}|$. Differentiating (7.58) in $(r,t,x;\hat{\xi},z;\hat{\zeta})$, we get, by taking m so large,

(7.62) $D^{\mu}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\nabla^{\alpha}A^{\tilde{2},3})_{2,(1,2)} = O(r^{-N})$ for any $N \geq 0$.

<u>On</u> $\Delta_{2,(2,1)}$: Use the coordinate on $\Delta_{2,(2,1)}$. Then, we get

$$(\nabla^{\alpha}A^{\tilde{2},3})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;t\hat{\zeta})$$

$$= \text{Os-}\iint c^{\alpha,T'}_{2,3}(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\zeta})e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\zeta})}dyd\eta \quad .$$

Set $\eta = (r/t)\tilde{\eta}$, and we have

(7.63) $(\nabla^{\alpha}A^{\tilde{2},3})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\zeta})$

$$= \mathrm{Os}\text{-}\iint_{|\tilde\eta|\geq C} c^{\alpha,T^{\backprime}}_{2,3}(x;(r/t)\hat\xi,y;(r/t)\tilde\eta z;r\hat\xi)$$

$$\times e^{-i(r/t)T(x;\hat\xi,y;\tilde\eta,z;\hat\xi)}(r/t)^n dyd\tilde\eta \quad .$$

where

$$c^{\alpha,T^{\backprime}}_{2,3}(x;(r/t)\hat\xi,y;(r/t)\tilde\eta,z;r\hat\xi)$$

$$= \tilde c_{2,3}(x;(r/t)\hat\xi,y;(r/t)\tilde\eta,z;r\hat\xi)(T^{\backprime}(x;\hat\xi,y;\tilde\eta,z;\hat\xi)^{\alpha}(r/t)^{|\alpha|} \quad .$$

Now, put

(7.64) $\tilde r = r,\ \tilde t_1 = t,\ \tilde t_2 = =\dfrac{1}{|\tilde\eta|}$.

Then, the function

$$\rho(\tilde r^{-1},\tilde t_1,\tilde t_2, x;\hat\xi,y;\hat{\tilde\eta},z;\hat\xi)$$

$$= \tilde c_{2,3}(x;(\tilde r/\tilde t_1)\hat\xi \quad ,y;(\tilde r/\tilde t_1\tilde t_2)\hat{\tilde\eta},z;\tilde r\hat\xi) .$$

is smooth on $\Delta_{3,(3,1,2)}$. Since $\partial_r = \partial_{\tilde r}$ and $\partial_t = \partial\tilde t_1$, we have

(7.65) $|\partial_r^a\partial_t^b D^{\mu}_{(x;\hat\xi,z;\zeta)}\tilde c_{2,3}| =< C_{a,b,\mu}$

for some constant $C_{a,b,\mu}$. Use $L^{\backprime}_T$ in (7.42), and the fact that if $|\tilde\eta|\geq C$,

$$|\partial_y T(x;\hat\xi,y;\tilde\eta,z;\hat\xi)|\geq M|\tilde\eta| \quad ,$$

$$|\partial_y^{\mu} T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})| \leq C_{\mu}(1+|\tilde{\eta}|) \quad ,|\mu|\geq 1 \ .$$

for some constants C_{μ}. Integrating by parts with $L_T^{\grave{}}$, we have

$$(7.66) \quad (\nabla^{\alpha} A^{\tilde{2},3})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi}) = O(r^{-N}t^{N})$$

and

$$(7.67) \quad D^{\mu}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha} A^{\tilde{2},3})_{2,(2,1)} = O(r^{-N}t^{N}) \ ,$$

for any $N\geq 0$.

By the same reasoning as in the last paragraph of Lem. $A^{2,1}$, we cansee the differentiability of $A^{2,3}$ with respect to a,z_1,z_2).

Lem.$A^{2,2}$. Lastly, we shall consider the integral Asubp2,2 in Lemma 7.4. As in 7.3-4, to consider the differentiability with respect to a,z_1,z_2, we have to consider the following integral; for $|\alpha|\geq 0$,

$$(7.68) \quad \nabla^{\alpha} A^{2,2}(x;\xi,z;\xi)$$

$$\text{Os-}\iint \quad D_{2,2}c_{2,2}^{\alpha,T^{\grave{}}}(x;\xi,y;\eta,z;\xi)$$

$$\times e^{-iT(x,\xi,y;\eta,z;\xi)}dyd\eta \quad ,$$

where $c_{2,2}^{\alpha,T^{\grave{}}}$ can be described as follows:

(a) $c_{2,2}^{\alpha,T^{\grave{}}} = \tilde{c}_{2,2}(T^{\grave{}})^{\alpha} \quad ,$

(b) $(T')^{\alpha} = (T'_1)^{\alpha_1} \ldots (T' \text{subk})^{\alpha_k}$, $T'_i \in {}^{(2)}$,

(c) $\tilde{c}_{2,2} \in {}^{3}(\varepsilon_1, \varepsilon_2)$ and satisfies the same condition for c sub 2,2 in (7.1).

Now, we shall prove that Lem. $A_{2,2}$ holds:

Remark that on support of $c_{2,2}$, we have $|\xi|^2 \geq R^2/2(1+4C^2)$ Therefore, if we take $R \geq (2(1+4C^2)K)^{1/2}$, supp $c_{2,2} \cap \Delta_{2,\Phi} = \Phi$ and supp $c_{2,2} \cap \Delta_{2,(2)} = \Phi$. So, we have only to consider $A^{2,2}$ on the domains $\Delta_{2,(1)}$, $\Delta_{2,(1,1)}$ and $\Delta_{2,(2,1)}$.

On $\Delta_{2,(1)}$: Use the coordinate on $\Delta_{2,(1)}$. Then, we have

$$(\nabla^{\alpha} \tilde{A}^{2,2})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\xi})$$

$$= \text{Os-}\iint c^{\alpha, T'}_{2,1}(x; r\hat{\xi}, y; \eta, z; t\hat{\xi}) \times e^{-iT(x; r\hat{\xi}, y; \eta, z; t\hat{\xi})} dy d\eta \quad .$$

Set $\eta = r\tilde{\eta}$, and we have

$$(7.69) \quad (\nabla^{\alpha} \tilde{A}^{2,2})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\xi})$$

$$= \iint_{(1/2)C^{-1} =< |\tilde{\eta}| =< 2C} i r^n c^{\alpha, T'}_{2,1}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\xi}) \times e^{-irT(x; r\hat{\xi}, y; \tilde{\eta}, z; t\hat{\xi})} dy d\tilde{\eta} \quad .$$

where

$$c^{\alpha,T^{\cdot}}_{2,2}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})$$

$$= \tilde{c}_{2,2}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})(T^{\cdot})^{\alpha}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})r^{|\alpha|} \quad .$$

Remark that the above integral is well-defined. Now, put

$$(7.70) \qquad (\frac{i}{r})M[\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})\partial_y + \text{partila}_{\tilde{\eta}} T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})\partial_{\tilde{\eta}}]$$

where

$$M = [|\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|^2 + |\partial_{\tilde{\eta}} T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})|^2] \quad .$$

By Corollary 4.6, $|\partial_{yT}|^2 + \quad |\partial_{\tilde{\eta}} \; T|^2 \geq \delta > 0$, and we get easily

$$(7.71) \qquad |\nabla^{\alpha}_{(y;\tilde{\eta})} T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})| =< C_{\alpha}(1 + |\tilde{\eta}|) \;, \quad |\alpha| \geq 1.$$

Then, we get

$$(\nabla^{\alpha} A^{\tilde{2},2})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{(1/2)C^{-1} \leqq |\tilde{\eta}| \leqq 2C} (L_T^{(3)*})^m [c^{\alpha,T^{\cdot}}_{2,1}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\xi})r^n]$$

$$e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})}dyd\tilde{\eta} \quad .$$

Put as follows:

$$(7.72) \qquad \begin{array}{ll} \tilde{r} = r|\tilde{\eta}|, \; t_1 = |\tilde{\eta}|, \; \tilde{t}_3 = t, & \text{if } |\tilde{\eta}| =< K \\ \tilde{r} = , \; \tilde{t}_2 = 1/|\tilde{\eta}|, \; \tilde{t}_3 = t, & \text{if } |\tilde{\eta}| \geq K^{-1} > \end{array}$$

Then, the amplitude function $\tilde{s}_{2,2}$ in (7.69) is smooth on $\Delta_{3,(2,1)}$ and $\Delta_{3,(1,2)}$ for each case in (7.72) respectively.

Therefore, we get

(7.73) $\quad (\nabla^{\alpha}A^{\tilde{2},2})_{2,(1)} = O(r^{-N})$ for any $N>=0$.

By the same computations as above, combining (7.71), we have

(7.74) $\quad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha}A^{\tilde{2},2})_{2,(1)} = O(r^{-N})$ for any $N\geqq 0$.

On $\Delta_{2,(1,1)}$: Use the coordinate on $\Delta_{2,(1,2)}$. Then, we have

$$(\nabla^{\alpha}A^{\tilde{2},2})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \text{Os-}\iint c^{\alpha,T'}_{2,1}(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi}) \times e^{-iT(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi})}dyd\eta \quad .$$

Set $\eta = r\tilde{\eta}$, and we get

(7.75) $$(\nabla^{\alpha}A^{\tilde{2},2})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \iint_{(1/2)C^{-1}=<|\tilde{\eta}|=<2C} r^{n}c^{\alpha,T'}_{2,1}(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\xi}) \times e^{-irT(x;r\hat{\xi},y;\eta,z;t\hat{\xi})}dyd\eta \quad .$$

where

$$c^{\alpha,T'}_{2,2}(x;r\hat{\xi},y;r\tilde{\eta},z;(r/t)\hat{\xi})$$

$$= \tilde{c}_{2,2}(x;r\hat{\xi},yk;r\tilde{\eta},z;(r/t)\hat{\xi})r^{|\alpha|}(T')^{\alpha}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi}) \quad .$$

Put as follows:

$$\tilde{r} = r|\tilde{\eta}|, \tilde{t}_1 = |\tilde{\eta}|, \tilde{t}_3 = t \quad \text{if } |\tilde{\eta}| =< K,$$

(7.76) $$\tilde{r} = r,\ \tilde{t}_2 = 1/|\tilde{\eta}|,\ \tilde{t}_3 = t|\tilde{\eta}|, \quad \text{if} |\tilde{\eta}| \geq K^{-1}, |\tilde{\eta}|t =< K,$$
$$\tilde{r} = r,\ \tilde{t}_2 = 1/|\tilde{\eta}|t,\ \tilde{t}_3 = t, \quad \text{if } |\tilde{\eta}| \geq K^{-1},\ |\eta\text{tidle}|t \geq K^{-1}.$$

Then, the following functions ρ, ρ', ρ'' are smooth on $\Delta_{3(2,1,3)}$, $\Delta_{3,(1,2,3)}$ and $\Delta_{3,(1,3,2)}$, respectively:

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_3, x;\hat{\xi}, y;\hat{\tilde{\eta}}, x;\hat{\xi})$$
$$= \tilde{c}_{2,2}(x;(\tilde{r}/\tilde{t}_1)\hat{\xi}, y;\tilde{r}\hat{\tilde{\eta}}, z;(\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi}) \quad \text{for } \Delta_{3,(2,1,3)},$$

(7.77) $$\rho'(\tilde{r}^{-1}, \tilde{t}_2, \tilde{t}_3, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\xi})$$
$$= \tilde{c}_{2,2}(x;\tilde{r}\hat{\xi}, y;(\tilde{r}/\tilde{t}_2)\hat{\tilde{\eta}}, z;(\tilde{r}/\tilde{t}_3\tilde{t}_2)\hat{\xi} \quad \text{for } \Delta_{3,(1,2,3)},$$
$$\rho''(\tilde{r}^{-1}, \tilde{t}_2, \tilde{t}_3, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\xi})$$
$$= \tilde{c}_{2,2}(x;\tilde{r}\hat{\xi}, y;(\tilde{r}/\tilde{t}_2\tilde{t}_3)\hat{\tilde{\eta}}, z;(\tilde{r}/\tilde{t}_3)\hat{\xi}) \quad \text{for } \Delta_{3,(1,3,2)}.$$

Also, we have

$$\partial_r = |\tilde{\eta}|\partial_{\tilde{r}},\ \partial_t = \partial_{\tilde{t}_3} \quad \text{on } \Delta_{3,(2,1,3)},$$

(7.78) $$\partial_r = \partial_{\tilde{r}}, \qquad \partial_t = |\tilde{\eta}|\partial_{\tilde{t}_3} \quad \text{on } \Delta_{3,(1,2,3)},$$
$$\partial_r = \partial_{\tilde{r}},\ \partial_t = -(\tilde{t}_2/t) \qquad \partial_{\tilde{t}_2} + \partial_{\tilde{t}_3} \quad \text{on } \Delta_{3,(1,3,2)} >$$

Remark that in the case $\Delta_{3,(1,3,2)}$, $(1/t) \leqq K|\tilde{\eta}| \leqq 2KC$. Use $L_T^{(3)}$ in (7.70) and Lax technique . So, we get

(7.79) $$(\nabla^{\alpha} A^{\tilde{2},2})_{2,(1,2)} = O(\text{rskulp}-N) \quad \text{for any } N \geq 0,$$

and

(7.80) $D^{\mu}_{(r,t,x;\hat{\xi},z;\hat{\xi})}(\nabla^{\alpha}A^{\tilde{2},2})^{2,(1,2)} = O(r^{-N})$ for any $N \geq 0$.

On $\Delta_{2,(2,1)}$: Use the coordinate in $\Delta_{2,(2,1)}$. Then, we have

$$(\nabla^{\alpha}A^{\tilde{2},2})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$$

$$= \mathrm{Os}\text{-}\iint c^{\alpha,T`}_{2,1}(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\xi})$$

$$\times e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\xi})}dyd\eta \quad .$$

Set by $(r/t)\tilde{\eta} = \eta$, and we get

(7.81) $(\nabla^{\alpha}A^{\tilde{2},2})_{2,(2,1)}(r^{-1},t,x;\hat{\xi},z;\hat{\xi})$

$$= \iint_{(1/2)C^{-1} \leqq |\tilde{\eta}| \leqq 2C} (r/t)^{n} c^{\alpha,T`}_{2,2}(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})$$

$$\times e^{-i(r/t)T(x;r\hat{\xi},y;\eta,z;t\hat{\xi})}dyd\tilde{\eta} \quad .$$

where

(7.82) $c^{\alpha,T`}_{2,2}(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})$

$$= \tilde{c}_{2,2}(x;(r/t)\hat{\xi},y;(r/t)\tilde{\eta},z;r\hat{\xi})(T`)^{\alpha}(x;\hat{\xi},y;\tilde{\eta},z;\hat{\xi})(r/t)^{|\alpha|} \quad .$$

Put as follows:

$$\begin{aligned}
&\tilde{r} = r,\ \tilde{t}_1 = t,\ \tilde{t}_2 = r|\tilde{\eta}|/t, \quad \text{if } r|\tilde{\eta}|/t \leq K,\\
(7.83)\quad &\tilde{r} = r, \tilde{t}_1 = t,\ \tilde{t}_2 = 1/|\tilde{\eta}|,\\
&\quad \text{if } r|\tilde{\eta}|/t \geq K^{-1} < |\tilde{\eta}| \geq K^{-1},\\
&\tilde{r} = (r/t)|\tilde{\eta}|,\ \tilde{t}_1 = t,\ \tilde{t}_3 = |\tilde{\eta}|/t,\\
&\quad \text{if } r|\tilde{\eta}|/t \geq K^{-1},\ |\tilde{\eta}| \leq K,\ |\tilde{\eta}|/t \leq K,\\
&\tilde{r} = r,\ \tilde{t}_1 = |\tilde{\eta}|,\ \tilde{t}_2 = t/|\tilde{\eta}|,\\
&\quad \text{if } r|\tilde{\eta}|/t \geq K^{-1}, |\tilde{\eta}| \leq K,\ |\tilde{\eta}|/t \geq K^{-1}.
\end{aligned}$$

Also, the following functions ρ, ρ', ρ'' are smooth on $\Delta_{3,(3,1)}$, $\Delta_{3,(3,1,2)}$, and $\Delta_{3,(3,2,1)}$ for each case of (7.83) respectively.

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\zeta})$$

$$= \tilde{c}_{2,2}(x;(\tilde{r}/\tilde{t}_1)\hat{\xi}, y;\tilde{t}_2\hat{\tilde{\eta}}, z;\tilde{r}\hat{\zeta}) \quad \text{on } \Delta_{3,(3,1)},$$

$$\rho'(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\zeta})$$

$$= \tilde{c}_{2,2}(x;(\tilde{r}/\tilde{t}_1)\hat{\xi}, y;\tilde{t}_2;(\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\tilde{\eta}}, z;\tilde{r}\hat{\zeta}) \quad \text{on } \Delta_{3,(3,1,2)},$$

$$(7.84)\quad \rho''(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\zeta})$$

$$= \tilde{c}_{2,2}(x;(\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi}, y;\tilde{r}_2\hat{\tilde{\eta}}, z;(\tilde{r}/\tilde{t}_3)\hat{\zeta}) \quad \text{on } \Delta_{3,(2,3,1)},$$

$$\rho'''(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x;\hat{\xi}, y;\hat{\tilde{\eta}}, z;\hat{\zeta})$$

$$= \tilde{c}_{2,2}(x;(\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\xi}, y;(\tilde{r}/\tilde{t}_2)\hat{\tilde{\eta}}, z;\tilde{r}\hat{\zeta}) \quad \text{on } \Delta_{3,(3,2,1)},$$

Remark that

$$\partial_r = \partial_{\tilde r} + (|\tilde\eta|/t)\partial_{\tilde t_2}, \quad \partial_t = \partial_{\tilde t^1} - (\mathrm{tidle}_2/t)\partial_{\tilde t_2}$$

$$\partial_r = \partial\tilde r, \partial_t = \partial_{\tilde t_1} - (\tilde t_3/t)\partial_{\tilde t_3} \quad \text{for } \Delta_{3,(3,1,2)},$$

(7.85) $$\partial_r = (|\tilde\eta|/t)\partial_{\tilde r}, \ \partial_t = \partial_{\tilde t_1} - (\tilde t_3/t)\partial_{\tilde t_3} \quad \text{for } \Delta_{3,(2,3,1)},$$

$$\partial_r = \partial^{\tilde r}, \partial_t = -(\tilde t_2/|\tilde\eta|)\partial_{\tilde t_2} \quad \text{for } \Delta_{3,(3,2,1)} .$$

Use $L_T^{(3)}$ in (7.70) and Lax technique. So, we get

(7.86) $$(\nabla^\alpha A^{\tilde 2,2})_{2,(2,1)} = O(r^{-N}t^N) \quad \text{for any } N\geq 0 ,$$

and

(7.87) $$D^\mu_{(r,t,x;\hat\xi,z;\hat\xi)}(\nabla^\alpha A^{\tilde 2,2})_{2,(2,1)} = O(r^{-N}t^N) \text{ for any } N\geq 0 .$$

By the same reasoning as in the last paragraph of Lem. $A^{2,1}$, we can see the differentiability of $A^{(2,2)}$ with respect to (a,z_1,z_2).

By 7.1-7.5, we obtain Proposition 6.1, completely.

2.8 Proof of Theorem B

In this section, we shall show that the integral transformation given by (0.3) can be written as a Fourier-integral operator of our class $G\mathcal{F}^0_0$ and as the results, we can give the convergence of the path integral as the kernel function, which proves Theorem B.

Let $(\tilde{X},\tilde{\Xi})$ be the local coordinate of T^*X around x defined by $\cdot_x(\tilde{X},\tilde{\Xi}) = (x,\xi)$. We shall first investigate some properties of the classical orbit to the Hamiltonian $H(x;\xi)$ satisfying (H.0)-(H.1). Hereafter, the time parameters t are contained in the interval $[-T,T]$ for any fixed $T>0$. Denote by $\varphi(t,\tilde{X},\tilde{\Xi})$ the solution of the Hamiltonian equation (0.1) with initial condition $\varphi(0,\tilde{X},\tilde{\Xi}) = (\tilde{X},\tilde{\Xi})$. Then, for sufficiently small t and sufficiently small $\tilde{X}$, $\varphi(t,\tilde{X},\tilde{\Xi})$ can be written by

$$\varphi(t,\tilde{X},\tilde{\Xi}) = \cdot_x(\tilde{\varphi}_1(\tilde{X},\tilde{\Xi}),\tilde{\varphi}_2(t,\tilde{X},\tilde{\Xi})) \quad .$$

By the construction of successive approximation of the solution (0.1) as in [13], we get easily the following:

<u>Proposition 8.1</u> Let assumptions (H.0)-(H.1) be satisfied. Then, there exists a positive constant $\delta>0$ such that $\varphi(t,\tilde{X},\tilde{\Xi})$ can be defined for sufficiently small $\tilde{X}$ and for $|t|<0$. Moreover, it has the following asymptotic expansions for $|\xi|\to\infty$:

$$(8.1)\qquad \begin{aligned} \tilde{\varphi}_1(t,\tilde{X},\tilde{\Xi}) &\sim \Sigma_{i=0}^{\infty}\tilde{X}_{-i}(t,\tilde{X};\tilde{\Xi}) \\ \tilde{\varphi}_2(t,\tilde{X},\tilde{\Xi}) &\sim \Sigma_{i=0}^{\infty}\tilde{\Xi}(t,\tilde{X},\tilde{\Xi}) \quad , \end{aligned}$$

where $\tilde{X}_j$, $\tilde{\Xi}_j$ are of homogeneous degree j with respect to Ξ.

Also, using the similar computation as above, we get

Proposition 8.2 Under the same assumptions and notations as in Proposition 8.1, we get the following properties for $|t|<\delta$:

(i) For any fixed $t,\tilde{\Xi}$, the mapping

$$\tilde{Y}\in T_xN \longrightarrow \tilde{\varphi}_1(t,\tilde{Y},\tilde{\Xi})\in T_yN$$

is a C^{∞}- diffeomorphism for sufficiently small $\tilde{Y}$. We write the inverse diffeomorphism as $\tilde{Y} = \tilde{Y}(t,\tilde{X},\tilde{\Xi})$. Moreover, $\tilde{Y}(t,\tilde{X},\tilde{\Xi})$ has an asymptotic expansion

$$\tilde{Y}(t,\tilde{X},\tilde{\Xi}) \sim \Sigma_{i=0}^{\infty}\tilde{Y}_{-i}(t,\tilde{X},\tilde{\Xi}),$$

where $\tilde{Y}_j$ is a homogeneous function of degree j with respect to $\tilde{\Xi}$.

Similarly,

(ii) for any fixed $t,\tilde{X}$, the mapping

$$\tilde{\eta}\in T_x^*N-\{0\} \longrightarrow \tilde{\varphi}_2(t,\tilde{X},\tilde{\eta})\in T_y^*N-\{0\} \quad ,$$

where $y=\varphi_1(t,\tilde{X},\tilde{\eta})$, is a C^{∞}- diffeomorphism. We rewrite the inverse mapping as $\tilde{\eta} = \tilde{\eta}(t,\tilde{X},\tilde{\Xi})$. Moreover, $\tilde{\eta}(t,\tilde{X},\tilde{\Xi})$ has an asymptotic expansion:

$$\tilde{\eta}(t,\tilde{X},\tilde{\Xi}) \sim \Sigma_{i=0}^{\infty}\tilde{\eta}_{1-i}(t,\tilde{X},\tilde{\Xi}) \quad ,$$

where $\tilde{\eta}_j$ is a homogeneous function of degree j.

Next, we shall construct the generating function $S(t,x;\xi)$ for the Hamiltonian flow (0.1) for sufficiently small t (Cf. [1], [32], [20], [21]).

<u>Definition 8.3.</u> For sufficiently small t and small $\tilde{X}$, put

$$(8.2) \qquad u(t,\tilde{Y},\tilde{\eta}) = \langle\tilde{Y}|\tilde{\eta}\rangle + \int_0^t (\theta(X_H) - H)(\varphi(-z,\tilde{Y},\tilde{\eta}))dz \quad ,$$

and

$$(8.3) \qquad S(t,x;\xi) = u(t,\tilde{Y}(t,0,\xi),\xi)$$

where X_H denotes the Hamiltonian vector field corresponding to $H(x;\xi)$ and θ is the canonical 1-form on T^*N.

The second term on the right-hand side of (8.2) is the classical action along the classical path starting from the position $\cdot_x\tilde{X}$ at time 0 with the momentum $(dExp_x)^*_{\tilde{Y}}\tilde{\eta}$. Thus the corresponding term of S is exactly the classical action along the path $\varphi(t,\tilde{X},\tilde{\Xi})$ passing through the position $\cdot_x\tilde{X}$ at time t and having the momentum $(dExp_x)_{\tilde{X}}\tilde{\Xi}$. For later argument, we give some properties for S(t,x;xi), which is easily proved by the Hamilton-Jacobi theory (Cf. [1]).

<u>Proposition 8.4</u> Let t and $\tilde{X}$ be sufficiently small. Then,

(i) $S(0,x;\xi) = 0$.

(ii) $\partial_t S(t,x;\xi) + H(x,\partial_{S(t,x;}\xi)) = 0$.

(8.4) $(\partial_\xi S(t,x;\xi),\partial_x S(t,x;\xi)) = \cdot_x(\tilde{Y}(t,0,\xi),\tilde{\eta}(t,0,\xi))$.

(iii) $S(t,x;\xi)$ has the follwing asymptotic expansion for $|\xi|\to\infty$:

(8.5) $S(t,x;\xi) \sim \Sigma_{i=0}^{\infty} S_{1-i}(t,x;\xi)$,

where S_j is a homogeneous function of degree j with respect to ξ.

Now, recall the integral transformation (0.3), i.e.

$$U(h,t)u(x) = (2\pi h)^{-n}\iint e^{\frac{i}{h}S(t,x;\xi)} e^{\frac{i}{h}\langle Z|\xi\rangle} \nu u(x;Z)dZd\xi \quad .$$

By using the above notations and Proposition 8.3, $U(h,t)$ can be rewritten by the following:

(8.6) $$U(h,t)u(x) = \iint a(t,x;\xi)e^{i[\frac{1}{h}S_l(t,x;\xi|Z)]} \nu u(x;Z)dZd\xi \quad .$$

where $S_1(t;x;\xi|Z) = S_1(t,x;\xi)+\langle Z|\xi\rangle$ and $a(t,x;\xi) = e^{\frac{i}{h}(S-S_1)}$
Take points (x_α,ξ_α) of $T^*N-\{0\}$ and these neighbourhood U_α of (x_α,ξ_α) which cover $T^*N-\{0\}$ and choose a cut off function $\{\psi_\alpha\}$. Then, (8.6) can be written by

(8.7) $U(h,t)u(x) = \Sigma_\alpha U_\alpha(h,t)u(x)$

where

$$(8.8)\qquad U_\alpha(h,t) = \iint \psi_\alpha \cdot a(t,x;\xi) e^{i[\frac{1}{h}S_1(t,x;\xi|Z)]} \nu u(x;Z)\,dZ\,d\xi \quad .$$

The following is easily obtained by the Hamilton-Jacobi theory (cf. [15], [26]):

Lemma 8.4. For sufficiently small fixed t, there exists a symplectic transformation $\psi(t,x;\xi)$ such that on $U_\alpha \times V$, where V is a neighborhood of $\psi_1(t,x;\xi)$

(i) $S(t,x;\xi|Z) = \langle \psi_2(t,x;\xi)|X\rangle \quad ,X \in T_{\psi_1(x;\xi)}{}^N$

for $(x;\xi)\in U_\alpha$ and for sufficiently small Z.

(ii) $X = S(y,Y_1,\bar{Y}_0(x,\xi))$

(iii) $\cdot_y \bar{Y}_0(x,\xi) = \psi_1(x;\xi)$

(iv) $y = \psi_1(x_\alpha,\xi_\alpha) \quad$, where $\cdot_y S(y,Y_1,\bar{Y}_0) = \cdot_y Y_0$.

,

Remark. Lemma 8.4 is trivially obtained for the case $S(t,x;\xi|Z) = \langle Z|\xi\rangle$ because $\psi(0,x;\xi) = id$. The other case is given by usi ng the implicit function theorem to the space of generating functions of symplectic transformations.

By using Lemma 8.4 and changing the variables, U(h,t) is reduced to the following :

(8.9) $\quad U_\alpha(h,t)u(x)$

$$= \iint \tilde{a}(t,x;\xi|Y_1)e^{i\langle\psi_2(x,\xi)|Y_1\rangle}\nu u(\cdot_{\psi_1(x;\xi)}X)$$

Therefore, we get the following, which proves Theorem B:

<u>Theorem 8.5.</u> Assume that the Hamiltonian function $H(x,\xi)$ satisfies (H.0)- (H.1). For an arbitrary fixed positive constant T, we have the following:

(i) $U(h,t)$ is contained in $G\mathfrak{F}_0^0$ for sufficiently small t, and $U(h,t)$ can be extended to the bounded operator on $L^2(N)$.

(ii) $\lim_{t\to 0} U(h,t)u(x) = u(x)$, for any $u\in L^2(N)$.

(iii) There exists a limit $\lim_{N\to\infty} U(h,\frac{t}{N})^N$ in a topology of $G\mathfrak{F}_0^0$.

(iv) $(\frac{i}{h})\frac{\partial}{\partial t}U(h,t)u(x) = H(h)u(x)$

for any $u\in C_0^\infty(N)$, where $H(h)$ is the psuedo-differential operator defined by

(8.10) $$H(h)u(x) = \int H(x;\xi)\nu\tilde{u}^h(x;\xi)d\xi \quad .$$

<u>Proof.</u> (i) and (iii) are easily seen by the definition of $G\mathfrak{F}_0^0$ and by the results of [16]. (ii) is also obvious because $\psi(0,x;\xi) = (x;\xi)$ and $a(0,x,\xi|Z) = 1$. To complete the proof of Theorem 8.8, we only show (iv). Using Lemma 8.3, we get

$$(\frac{i}{h})\frac{\partial}{\partial t}U(h,t)u(x)$$

$$= \iint \frac{\partial}{\partial t} S(t,x;\xi) e^{\frac{i}{h}S(t,x;\xi)} e^{\frac{i}{h}\langle Z|\xi\rangle} \nu u(x;Z)\,dZd\xi$$

$$= \iint H(x,\partial_x S(t,x;\xi)) e^{\frac{i}{h}S(t,x;\xi)} e^{\frac{i}{h}\langle Z|\xi\rangle} \nu u(x;Z)\,dZd\xi$$

Taking $t \to 0$ and using Lemma 8.4 (iii), we get (iv).

Remark. We must elliminate of Y_1 in the amplitude function in (8.10). This can be done by similar way as in [27],Correction.

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