# THE PRINCIPAL 2-BLOCKS OF FINITE GROUPS

# WITH ABELIAN SYLOW 2-SUBGROUPS

BY

SHIGEO KOSHITANI

A thesis submitted in partial fulfilment of

the requirement for the degree of

DOCTOR OF SCIENCE

at the

UNIVERSITY OF TSUKUBA

1980

# THE PRINCIPAL 2-BLOCKS OF FINITE GROUPS WITH ABELIAN SYLOW 2-SUBGROUPS

By

Shigeo KOSHITANI

# THE PRINCIPAL 2-BLOCKS OF FINITE GROUPS WITH ABELIAN SYLOW 2-SUBGROUPS

By

Shigeo KOSHITANI

## Introduction

Let $G$ be a finite group, $p$ a prime number and $B$ a $p$-block of $G$ with defect group $D$. There is an important problem in representation theory of finite groups that is to give a description of $B$ when the structure of $D$ is given. Concerning with this problem there are some successful results. E.C. Dade [9] proved his results when $D$ is cyclic. R. Brauer [6] proved his results for the case where $p = 2$ and $D$ is dihedral by making use of his powerful methods ([3], [4], [5]). Using Brauer's methods J.B. Olsson [18] obtained his results when $p = 2$ and $D$ is generalized quaternion or quasidihedral. In [3, IV] R. Brauer investigated $B$ when $p = 2$ and $D$ is elementary abelian of order 4.

In the present paper we study $B$ when $p = 2$ and $B$ is the principal 2-block of $G$ with an abelian Sylow 2-subgroup $P$. Let $e(G) = |N_G(P) : C_G(P)|$. Let $B_o(G)$ be the principal 2-block of $G$, and let $O(G)$ and $O'(G)$ be the maximal normal subgroup of $G$ of odd order and the minimal normal subgroup of $G$ of odd index, respectively. By the results on finite groups with abelian Sylow 2-subgroups ([2], [16], [17], [20], [21]), the structure of $O'(G/O(G))$ is almost determined. In general, however, $B_0(G)$ is different from $B_0(S)$ where $S = O'(G/O(G))$. The main purpose of this paper is to investigate the relation between $B_0(G)$ and $B_0(S)$. In particular we shall prove that $B_0(G)$ is isomorphic to $B_0(S)$ for the cases where $e(G) = e(S) =$ prime, 9 and 21.

In section 1 we shall state several lemmas and propositions which will be useful for our aim. One of them is Alperin's theorem on isomorphic principal blocks [1]. Let $S = O'(G/O(G))$. In section 2 we shall consider $B_0(G)$ for the case where $e(G) = 2^m-1$. In particular, we shall prove that if $G$ is nonsolvable and if $e(G)$ is prime then $e(G) = 2^m-1$ for some $m \geq 2$ and $B_0(G)$ is isomorphic to $B_0(S)$. In sections 3 and 4 we shall investigate $B_0(G)$ for the cases when $e(G) = 9$ and 21, respectively. Indeed, we shall prove that if $e(G) = e(S) = 9$ or 21 then $B_0(G)$ is isomorphic to $B_0(S)$. It is noted that when $e(G) \neq e(S)$, $B_0(G)$ is not necessarily isomorphic to $B_0(S)$. In sections 5 and 6 we shall determine $B_0(G)$ when $P$ is elementary abelian of order 8 and 16, respectively.

Throughout this paper we shall use the following notation. When $S$ is a subset of $G$, $N_G(S)$ and $C_G(S)$ denote the normalizer and the centralizer of $S$ in $G$, respectively. Specially, for each $x \in G$ we write $C_G(x)$ for $C_G(\{x\})$. If $x, y \in G$, we write $x^y$ for $y^{-1}xy$. When $S$ is a subset of $G$, $\langle S \rangle$ denotes the subgroup of $G$ generated by $S$. When $x_1, \ldots, x_n$ are elements of G and S is a subset of G, we also write $\langle x_1, \ldots, x_n, S\rangle$ for the subgroup of G generated by $\{x_1, \ldots, x_n\} \cup S$. The cyclic group of order n is denoted $Z_n$ for a positive integer n. We write G' and Z(G) for the commutator subgroup of G and the center of G, respectively. We denote by Aut(G) the group of all automorphisms of G. Let us denote by $O_{p'}(G)$ the maximal normal subgroup of G of order prime to p, and by $O^{p'}(G)$ the minimal normal subgroup of G of index prime to p. In particular, for p = 2 we write O(G) and O'(G) for $O_{2'}(G)$ and $O^{2'}(G)$, respectively. When P is an abelian Sylow 2-subgroup of G, we write e(G) (or shortly e ) for $|N_G(P) : C_G(P)|$. When B is a p-block of G, let us denote by Irr(B) the set of all irreducible complex characters in B, by IBr(B) the set of all irreducible Brauer characters in B, by k(B) the number of elements of Irr(B), by k'(B) the number of elements of Irr(B) with degree one, and by $\ell(B)$ the number of elements of IBr(B). We write $B_0(G)$ (or shortly $B_0$ ) for the principal p-block of G, and for each $x \in G$ we write $b_x$ for $B_0(C_G(x))$. When $\psi_1$ and $\psi_2$ are complex characters of G, let $(\psi_1, \psi_2) = (1/|G|)\Sigma_{g\in G}\, \psi_1(g)\psi_2(g^{-1})$, that is to say, $(\psi_1, \psi_2)$ is the inner product of $\psi_1$ and $\psi_2$. We write $1_G$ for the trivial complex (or Brauer) character of G. When H is a normal subgroup of G, $\psi|_H$ denotes the restriction of $\psi$ to H for a character $\psi$ of G, $W|_H$ denotes the restriction of W to H for a representation W of G, and $I_G(\tilde{\psi})$ denotes the inertial group of $\tilde{\psi}$ in G for a character $\tilde{\psi}$ of H, that is to say, $I_G(\tilde{\psi}) = \{g \in G \mid \tilde{\psi}^g = \tilde{\psi}\}$, where $\tilde{\psi}^g$ is the conjugate of $\tilde{\psi}$.

## 1. Preliminaries

In this section we state some lemmas and propositions which will be needed for our aim. We fix a prime number $p$ and we consider p-modular representations of a finite group $G$.

Lemma 1.1. Let $G$ be a finite group with a Sylow p-subgroup $P$, and let $K = O_{p'}(G)$, $\overline{G} = G/K$ and $\overline{P} = (PK)/K$. Then we have the following.

(i) $B_0(G) = B_0(\overline{G})$.

(ii) $N_G(P)/C_G(P) \cong N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$.

Proof. We get (i) by [10, Theorem 65.2] and [11, V (4.3)]. Since $N_{\overline{G}}(\overline{P}) = (N_G(P)\cdot K)/K$ from [15, I 7.7 Hilfssatz (c)] and since $C_{\overline{G}}(\overline{P}) = (C_G(P)\cdot K)/K$ from [19, Lemma 2.2], we easily get (ii).

We shall frequently use the next four propositions in order to prove our main theorems.

Proposition 1.2 (Brauer). Let $G = QC_G(Q)$ where $Q$ is a p-group, and let $\overline{G} = G/Q$. Then $\ell(B_0(G)) = \ell(B_0(\overline{G}))$.

Proof. See [10, Lemma 64.5 and Theorem 65.2(2)].

Proposition 1.3 (Brauer). Let $H$ be a normal subgroup of $G$. If $W$ is an ordinary or modular irreducible representation in $B_0(G)$, then any irreducible constituent of $W|_H$ lies in $B_0(H)$.

Proof. This is the special case of [3, I Lemma 1].

Proposition 1.4 (Brauer). Let $H$ be a normal subgroup of $G$. Then for any $\tilde{\chi} \in \mathrm{Irr}(B_0(H))$, there is some $\chi \in \mathrm{Irr}(B_0(G))$ such that $(\chi|_H, \tilde{\chi}) \neq 0$.

Proof. This is the special case of [3, II Lemma 1].

Proposition 1.5 (Brauer). Let $P$ be a Sylow p-subgroup of $G$, and let $PC_G(P) = P \times V$. Then $k'(B_0(G)) = |G:VG'|$.

Proof. See [3, IV Proposition (4G)].

Next, we state Alperin's theorems on isomorphic principal p-blocks which are very important for our aim.

Let $F$ be an algebraically closed field of characteristic $p$ and $FG$ the group algebra of $G$ over $F$. Let $H$ be a normal subgroup of $G$ with $p \nmid |G:H|$. We write $B_0(G) \cong B_0(H)$, if the category of all finitely generated FG-modules in $B_0(G)$ is isomorphic to the category of all finitely generated FH-modules in $B_0(H)$ and if the isomorphism is given by the restriction from $G$ to $H$ (cf. [1]).

Proposition 1.6 (Alperin). Let $F$ be as above, and let $P$ be a Sylow p-subgroup of $G$. If $H$ is a normal subgroup of $G$ which satisfies the conditions that $p \nmid |G:H|$, $G/H$ is solvable and $G = HC_G(P)$, then we get the following.

(i) $B_0(G) \cong B_0(H)$.

(ii) $A_0(G) \cong A_0(H)$ as F-algebras, where $A_0(G)$ and $A_0(H)$ are the block ideals of $FG$ and $FH$ corresponding to $B_0(G)$ and $B_0(H)$, respectively.

Proof. See [1, Theorems 1 and 2].

Corollary 1.7 (Alperin). Let $H$ be a normal subgroup of $G$ of prime index $q$ with $q \neq p$. Let $B_0 = B_0(G)$ and $b_0 = B_0(H)$. Assume that $k(B_0) = k(b_0)$ and $\ell(B_0) = \ell(b_0)$, and that

$I_G(\tilde{\chi}) = G$ for every $\tilde{\chi} \in \mathrm{Irr}(b_o)$. Then we have the following.

(i) The correspondence $\mathrm{Irr}(B_o) \longrightarrow \mathrm{Irr}(b_o)$ given by $\chi \longmapsto \chi|_H$ is a bijection.

(ii) The correspondence $\mathrm{IBr}(B_o) \longrightarrow \mathrm{IBr}(b_o)$ given by $\phi \longmapsto \phi|_H$ is a bijection.

(iii) $B_o \cong b_o$.

Proof. (i) Since $I_G(\tilde{\chi}) = G$ for every $\tilde{\chi} \in \mathrm{Irr}(b_o)$, the correspondence is surjective by Clifford's theorem, [8, (53.17) Theorem] and Propositions 1.3 and 1.4. Since $k(B_o) = k(b_o)$, we obtain (i).

(ii) By (i), [1, Lemma 1] holds. Thus, by the proof of [1, Lemma 3], the correspondence is surjective. Hence (ii) holds since $\ell(B_o) = \ell(b_o)$.

(iii) Since [1, Lemmas 1 and 3] hold, we get (iii) by the proofs of Alperin's theorems [1, Theorems 1 and 2].

In the remainder of this paper we assume $p = 2$ and let $G$ and $P$ be a finite group and its abelian Sylow 2-subgroup of order $2^n$, respectively. We use the notation $B_o$ and $e$ for $B_o(G)$ and $e(G)$, respectively.

Corollary 1.8 (Alperin). Let $H$ be a normal subgroup of $G$ of odd prime index. Let $B_o = B_o(G)$ and $b_o = B_o(H)$. Assume that $k(B_o) = k(b_o)$ and $\ell(B_o) = \ell(b_o)$, and that $H$ has an involution $x$ such that $\chi(x) = \pm 1$ for every $\chi \in \mathrm{Irr}(B_o)$ and $\tilde{\chi}(x) = \tilde{\chi}'(x) = \pm 1$ for all $\tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_o)$ with $\tilde{\chi}(1) = \tilde{\chi}'(1)$. Then $B_o \cong b_o$.

Proof. By Clifford's theorem and Proposition 1.3, we have

$\chi|_H \in \mathrm{Irr}(b_0)$ for all $\chi \in \mathrm{Irr}(B_0)$. Thus, by Proposition 1.4, $I_G(\tilde{\chi}) = G$ for all $\tilde{\chi} \in \mathrm{Irr}(b_0)$. Thus the corollary is proved by Corollary 1.7 (iii).

Lemma 1.9. Let $P$ be an abelian Sylow 2-subgroup of $G$. Suppose that $k(B_0) = |P|$ and that $G$ has an involution $x$ with $\ell(b_x) = 1$. Then $\chi(x) = \pm 1$ for all $\chi \in \mathrm{Irr}(B_0)$.

Proof. Since $\ell(b_x) = 1$, $b_x$ has the unique Cartan invariant $|P|$. Hence, by [10, Theorems 63.3(2), 63.2 and 65.4], we get $\Sigma\, \chi(x)^2 = |P|$ where the sum runs through all $\chi \in \mathrm{Irr}(B_0)$. By [4, II (7A) and (4C)], $\chi(x)$ is a nonzero integer for every $\chi \in \mathrm{Irr}(B_0)$ since $|x| = 2$. Therefore, the assumption $k(B_0) = |P|$ implies the lemma.

Proposition 1.10 (Bender, Janko, Janko-Thompson, Walter, Ward). If $G$ has abelian Sylow 2-subgroups, then $O'(G/O(G))$ is a direct product of an abelian 2-group and simple groups of one of the following types;

(1) the special linear group $SL(2,2^n)$ for $n \geqslant 2$,

(2) the projective special linear group $L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod 8$,

(3) the Janko's first simple group $J_1$,

(4) the simple group $R(q)$ of Ree type.

Proof. For groups of types (1) and (2), see [14, p.40]. For $J_1$ see [16], and for $R(q)$ see [21]. The proposition is obtained from [2], [16], [17], [20] and [21].

In the rest of this paper we use the notation $SL(2,2^n)$, $L_2(q)$, $J_1$ and $R(q)$ as in Proposition 1.10 (cf. [13, p.415]).

We also use the notation $GL(m,2)$ for the general linear group (cf. [14, p.40]).

The next lemma shows that Brauer's conjecture on heights of irreducible complex characters in p-blocks with abelian defect groups is affirmative for the principal 2-blocks of finite groups with abelian Sylow 2-subgroups.

Lemma 1.11. If $G$ has abelian Sylow 2-subgroups, then all irreducible complex characters in $B_0(G)$ have height zero.

Proof. We may assume $O(G) = 1$ by Lemma 1.1. Let $H$ be a normal subgroup of $G$ of odd index. If $\chi \in Irr(B_0(G))$, then there is some $\tilde{\chi} \in Irr(B_0(H))$ with $\chi(1) = m\tilde{\chi}(1)$ for a positive integer $m$ from Clifford's theorem and Proposition 1.3. By [8, (53.17) Theorem], $m$ divides $|G:H|$. This shows that if $\tilde{\chi}(1)$ is odd then $\chi(1)$ is also odd. Thus, we may assume $O'(G) = G$. Then, by Proposition 1.10, we can write $G = Q \times (\prod S_i)$ where $Q$ is an abelian 2-group and each $S_i$ is a simple group of one of the following types;

(i) $SL(2,2^n)$ for $n \geqslant 2$,

(ii) $L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod 8$,

(iii) $J_1$,

(iv) $R(q)$.

When $S_i$ is of type (i) or (ii), every $\chi \in Irr(B_0(S_i))$ has odd degree from [10, Theorems 38.2 and 38.1]. When $S_i$ is of type (iii) or (iv), every $\chi \in Irr(B_0(S_i))$ has odd degree from [16, Lemma 5.1] and [21, Chap.I], respectively. These show that every $\chi \in Irr(B_0(G))$ has odd degree. This completes the proof.

The next three lemmas are useful in order to obtain $e = e(G)$.

Lemma 1.12. Let $P$ be a Sylow 2-subgroup of $G$.

(i) If $G = SL(2,2^n)$ for $n \geqslant 2$, then $P$ is elementary abelian of order $2^n$ and $N_G(P)/C_G(P)$ is cyclic of order $2^n-1$.

(ii) If $G = L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod 8$, then $P$ is noncyclic of order $4$ and $N_G(P)/C_G(P)$ is cyclic of order $3$.

(iii) If $G = J_1$ or $R(q)$, then $P$ is elementary abelian of order $8$ and $N_G(P)/C_G(P)$ is noncyclic of order $21$.

Proof. (i) By [14, Theorems 2.8.1 and 2.8.3], $P$ is elementary abelian of order $2^n$. Let $q = 2^n$, and let $F_q$ be the finite field of $q$ elements. We may assume that $P = \{\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in F_q\}$ (cf. the proof of [14, Theorem 2.8.3]). Clearly, $C_G(P) = P$. Let $u$ be a generator of the multiplicative group $F_q-\{0\}$, and let $s = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ in $G$. Then, $N_G(P) = \langle p,s \rangle$ and $s$ has order $q-1$. Hence we get that $N_G(P)/P$ is cyclic of order $q-1$.

(ii) $P$ is noncyclic of order $4$ from [14, Lemma 15.1.1]. Hence $Aut(P)$ is isomorphic to the symmetric group of degree $3$. Since $G$ is not 2-nilpotent, we get (ii).

(iii) If $G = J_1$, we obtain (iii) from [16, VI p.160]. Assume $G = R(q)$. By [21, p.63], $P$ is elementary abelian of order $8$ and $|N_G(P) : C_G(P)| = 21$. Then we know that $N_G(P)/C_G(P)$ is noncyclic since $Aut(P) \cong GL(3,2) \hookrightarrow GL(4,2) \cong A_8$ from [15, II 2.5 Satz] where $A_8$ is the alternating group of degree $8$.

Lemma 1.13. (i) $GL(4,2) \cong A_8$, the alternating group of degree $8$.

(ii) If $H$ is a subgroup of $A_8$ of odd order, then $|H| = 1, 3, 5, 7, 9, 15$ or $21$.

(iii) $A_8$ has subgroups of orders $1, 3, 5, 7, 9, 15$ and $21$, and the subgroups of order $9$ and the subgroups of order $21$ are noncyclic.

Proof. (i) We have already showed (i) in the proof of Lemma 1.12(iii).

(ii) Since $|A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, $|H| = 1, 3, 5, 7, 9, 15, 21, 35, 35, 63, 105$ or $315$. Since the groups of order $35$ are cyclic, $|H| \neq 35$. By elementary calculations, $A_8$ has no subgroups of order $45$, so that $|H| \neq 45$. Similarly, $|H| \neq 63$. If $|H| = 105$, then $H$ has an element of order $35$. Evidently, this is a contradiction. Hence $|H| \neq 105$. If $|H| = 315$, then $H$ has an element of order $35$, and this is a contradiction. So that $|H| \neq 315$.

(iii) By Sylow's theorem, $A_8$ has subgroups of orders $3, 5, 7$ and $9$. Since $A_8$ has no elements of order $9$, Sylow $3$-subgroups of $A_8$ are noncyclic of order $9$. If $G = SL(2,2^4)$, then $P$ is elementary abelian of order $16$ and $N_G(P)/C_G(P)$ is cyclic of order $15$ from Lemma 1.12(i). Thus, by (i), $A_8$ has subgroups of order $15$. Let $H = \langle (124)(536), (1234567) \rangle$. Then $H$ is a noncyclic subgroup of $A_8$ of order $21$. Since $A_8$ has no elements of order $21$, all subgroups of $A_8$ of order $21$ are noncyclic.

Lemma 1.14. (i) If $H$ is a subgroup of $GL(3,2)$ of odd order, then $|H| = 1, 3, 7$ or $21$.

(ii) $GL(3,2)$ has subgroups of orders $1, 3, 7$ and $21$, and the subgroups of order 21 are noncyclic.

Proof. (i) By [10, Lemma 35.2(1)], $|GL(3,2)| = 2^3 \cdot 3 \cdot 7$. So that we easily get (i).

(ii) By the proof of (i) and Sylow's theorem, $GL(3,2)$ has subgroups of orders 3 and 7. By Lemma 1.12(iii), $GL(3,2)$ has noncyclic subgroups of order 21. Since $GL(3,2) \hookrightarrow GL(4,2)$, all subgroups of $GL(3,2)$ of order 21 are noncyclic from Lemma 1.13(i) and (iii).

The next two lemmas are useful in order to determine $B_o$ when Sylow 2-subgroups of $G$ are elementary abelian of order 8 or 16.

Lemma 1.15. Let $P$ be an abelian Sylow 2-subgroup of $G$, and let $B_o = B_o(G)$. Assume that $G$ has an involution $x$ with $\ell(b_x) = 1$.

(1) If $|P| = 8$, then $k(B_o) = 8$.

(2) If $|P| = 16$, then $k(B_o) = 8$ or 16.

Proof. Let $\{\chi_1, \ldots, \chi_{k(B_o)}\} = \mathrm{Irr}(B_o)$. Since $\ell(b_x) = 1$, by [10, Theorems 63.2 and 65.4], for each $\chi_i$ let $d^x_{i1}$ be the generalized decomposition number of $B_o$ relative to $x$. By Lemma 1.11 and [4, II (7A) and (4C)], every $d^x_{i1}$ is an odd integer. Since $b_x$ has the unique Cartan invariant $|P|$, by [10, Theorem 63.3], $\Sigma_{i=1}^{k(B_o)} (d^x_{i1})^2 = |P|$. These imply (1) and (2).

Lemma 1.16. Let $G = L_2(q)$ for $q > 3$ with $q \equiv 3$ or 5 (mod 8), and let $B_o = B_o(G)$. Then we have the following.

(i) $\ell(B_o) = 3$ and the degrees of all irreducible Brauer characters in $B_0$ are 1, (q-1)/2 and (q-1)/2.

(ii) The decomposition matrix of $B_0$ is as follows:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad\qquad \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} .$$

$$3 < q \equiv 3 \pmod 8 \qquad\qquad 3 < q \equiv 5 \pmod 8$$

Proof. Since G is not 2-nilpotent, $\ell(B_0) > 1$ from [10, Corollary 65.3]. Thus $k(B_0) = 4$ and $\ell(B_0) = 3$ by [3, IV Proposition (7D)].

Case 1. $3 < q \equiv 3 \pmod 8$: Let $\mathrm{Irr}(B_0) = \{\chi_1, \ldots, \chi_4\}$. By [10, Theorem 38.1], we may assume $\chi_1 = 1_G$, $\chi_2(1) = \chi_3(1) = (q-1)/2$ and $\chi_4(1) = q$. By [14, Theorem 2.8.2], G has a Frobenius subgroup E of order $q(q-1)/2$. We know the character tables of E and $L_2(q)$ from [10, Theorems 13.8 and 38.1]. Thus, by [8, §84 Exercise 2], $\chi_2|_{G_0}$ and $\chi_3|_{G_0}$ are both irreducible Brauer characters of G, where $\chi_i|_{G_0}$ is the restriction of $\chi_i$ to the set $G_0$ of all 2'-elements of G. Since $\chi_2 \neq \chi_3$ on $G_0$, and since $\chi_4 = \chi_1 + \chi_2 + \chi_3$ on $G_0$, we know (i) and the decomposition matrix of $B_0$.

Case 2. $3 < q \equiv 5 \pmod 8$: As in Case 1 we get (i) and the decomposition matrix of $B_0$.

Remark 1. If G has an abelian Sylow 2-subgroup P and if $e(G) = 1$, then $B_0(G) \cong B_0(P)$ since G is 2-nilpotent by [10, Theorem 18.7].

## 2. The case $e = 2^m - 1$

In this section we consider the case when $e = 2^m-1$ for $m \geqslant 2$. We use the notation $G$, $P$, $n$, $e$ and $B_0$ as before, that is to say, $P$ is an abelian Sylow 2-subgroup of $G$ with order $2^n$ $(n \geqslant 2)$, $e = e(G)$ and $B_0 = B_0(G)$. To begin with we state the next three lemmas which will be needed for the main result of this section.

Lemma 2.1. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong SL(2,2^n)$ for some $n \geqslant 3$. Assume $e = 2^n-1$. Then $B_0 \cong B_0(S)$.

Proof. We may assume $S = SL(2,2^n)$. There are an element $t \in N_S(P)$ and an involution $x \in P$ such that $N_S(P) = \langle t, C_S(P) \rangle$ and $P = \{1, x, x^t, \ldots, x^{t^{2^n-2}}\}$ (cf. the proof of Lemma 1.12(i)). Since $e = 2^n-1$, $N_G(P) = \langle t, C_G(P) \rangle$. Clearly, $y^t \neq y$ for all $y \in P-\{1\}$, so that $N_M(P) = C_M(P)$ where $M = C_G(x)$. Hence $M$ is 2-nilpotent from [10, Theorem 18.7]. Thus, by [10, Corollary 65.3], $\ell(b_x) = \ell(B_0(M)) = 1$. Now, we prove the lemma by induction on $|G|$. Suppose $G \neq S$. Since $|G/S|$ is odd, by [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index $\ell$ with $S \subseteq H$. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. Hence, by the character table of $SL(2,2^n)$ [10, Theorem 38.2], we get

| | $1$ | $x$ | |
|---|---|---|---|
| $1_H$ | $1$ | $1$ | |
| $\tilde{\theta}_i$ | $2^n-1$ | $-1$ | for $i = 1,\ldots,2^{n-1}$ |
| $\tilde{\chi}_j$ | $2^n+1$ | $1$ | for $j = 1,\ldots,2^{n-1}-1$ |

where $\{1_H, \tilde{\theta}_i, \tilde{\chi}_j \mid i = 1,\ldots,2^{n-1};\ j = 1,\ldots,2^{n-1}-1\} = \mathrm{Irr}(b_0)$.

Let $C_G(P) = P \times V$. If $G = VH$, then $G = C_G(P) \cdot H$, so that $B_o \cong b_o$ from Proposition 1.6. Hence we may assume $G \neq VH$. Then $H = VH$, so that $C_H(P) = P \times V$. Thus, by Proposition 1.5, $k'(b_o) = |H:VH'|$. Since $b_o \cong B_o(S)$, $k'(b_o) = 1$. Thus, $H = VH'$. This implies $H = VG'$ since $G/H$ is cyclic. Hence $k'(B_o) = \ell$ from Proposition 1.5. By Clifford's theorem and Proposition 1.3, for each $\chi \in \mathrm{Irr}(B_o)$ one of the following five cases occurs:

(a) $\chi|_H = 1_H$,

(b) $\chi|_H = \tilde{\theta}_i$ for some $i$,

(c) $\chi|_H = \tilde{\theta}_{i_1} + \dots + \tilde{\theta}_{i_\ell}$ for $i_1 < \dots < i_\ell$, and all $\tilde{\theta}_{i_k}$ are G-conjugate,

(d) $\chi|_H = \tilde{\chi}_j$ for some $j$,

(e) $\chi|_H = \tilde{\chi}_{j_1} + \dots + \tilde{\chi}_{j_\ell}$ for $j_1 < \dots < j_\ell$, and all $\tilde{\chi}_{j_k}$ are G-conjugate.

Since $k'(b_o) = 1$, for each $\chi \in \mathrm{Irr}(B_o)$ $\chi(1) = 1$ if and only if $\chi|_H = 1_H$. Let $r$, $s$, $u$ and $v$ be the numbers of $\chi \in \mathrm{Irr}(B_o)$ of types (b), (c), (d) and (e), respectively. Since $\ell(b_x) = 1$, as in the proof of Lemma 1.9, $\Sigma\, \chi(x)^2 = 2^n$ where the sum runs through all $\chi \in \mathrm{Irr}(B_o)$. This shows $\ell + r + s\ell^2 + u + v\ell^2 = 2^n$. On the other hand, by Proposition 1.4, for every $\tilde{\chi} \in \mathrm{Irr}(b_o)$ there is some $\chi \in \mathrm{Irr}(B_o)$ with $(\chi|_H, \tilde{\chi}) \neq 0$. So that $k(b_o) \leqslant 1 + r + s\ell + u + v\ell$. Since $k(b_o) = 2^n$, we have a contradiction. This completes the proof.

Remark 1. We can not remove the assumption $e = 2^n - 1$ in Lemma 2.1. Indeed, let $S = SL(2,8)$ and $P = \{\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in F_8\}$ where $F_8$ is the finite field of 8 elements. Let $u$ be a generator of the multiplicative group $F_8 - \{0\}$. There is an automprphism $h$ of $F_8$ with $h(u) = u^2$. For each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$ let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^h = \begin{pmatrix} h(a) & h(b) \\ h(c) & h(d) \end{pmatrix}$. Then we can consider $h \in Aut(S)$ and $h|_P \in Aut(P)$ where $h|_P$ is the restriction of $h$ to $P$. Hence there is a semi-direct product $G$ of its normal subgroup $S$ by $\langle h \rangle$. Then $O'(G) = S = SL(2,8)$ and $e(G) = 21 \neq 2^3-1$. By [10, Theorem 38.2], $\ell(B_0(S)) = 7$. But we shall afterwards show that $\ell(B_0(G)) = 5$, and this shows $B_0(G) \not\cong B_0(S)$.

Lemma 2.2. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong L_2(q) \times (P/(Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$, or $S \cong SL(2,2^m) \times (P/(\underbrace{Z_2 \times \ldots \times Z_2}_{m}))$ for some $m \geqslant 3$. Assume $e = e(S)$. Then $k(B_0) = 2^n$ and $\ell(B_0) = e$.

Proof. Let $L = L_2(q)$ for $m = 2$, and let $L = SL(2,2^m)$ for $m \geqslant 3$. Let $R$ be a Sylow 2-subgroup of $L$. We can write $S = L \times Q$ and $P = R \times Q$. We use induction on $n$. If $n = m = 2$, then the lemma is proved by [3, IV Proposition (7D)]. If $n = m \geqslant 3$, by Lemma 2.1, $B_0 \cong B_0(S)$, so that $k(B_0) = 2^n$ and $\ell(B_0) = 2^n-1 = 2^m-1$ (cf. [10, Theorem 38.2]). Next, suppose $n > m$. There are an element $t \in N_L(R)$ and an involution $x \in R$ such that $N_L(R) = \langle t, C_L(R) \rangle$ and $R = \{1, x, x^t, \ldots, x^{t^{2^m-2}}\}$. Since $e = e(S)$, $N_G(P) = \langle t, C_G(P) \rangle$. Let $Q = \{1=y_1, y_2, \ldots, y_{2^{n-m}}\}$. Then, by [10, Lemma 18.5], the G-conjugate classes of $P$ are as follows:

$$\{1\}$$

$$\{y_i\} \quad \text{for} \quad i = 2,\ldots,2^{n-m}$$

$$\{xy_i, x^t y_i, \ldots, x^{t^{2^m-2}} y_i\} \quad \text{for} \quad i = 1,\ldots,2^{n-m}.$$

Then, by [10, Theorems 68.4 and 65.4],

$$k(B_o) = \ell(B_o) + \Sigma_{i=2}^{2^{n-m}} \ell(b_{y_i}) + \Sigma_{i=1}^{2^{n-m}} \ell(b_{xy_i}).$$

Fix any $i$ with $2 \leqslant i \leqslant 2^{n-m}$, and let $M = C_G(y_i)$. Since $y_i \in Z(S)$, let $\overline{S} = S/\langle y_i \rangle$. Similarly, let $\overline{M} = M/\langle y_i \rangle$, $\overline{P} = P/\langle y_i \rangle$ and $\overline{Q} = Q/\langle y_i \rangle$. Since $\overline{S} \cong L \times \overline{Q}$, we get $e(\overline{S}) = e(L) = 2^m-1$. Since $\overline{S} \subseteq \overline{M}$, the canonical homomorphism $N_{\overline{S}}(\overline{P})/C_{\overline{S}}(\overline{P}) \longrightarrow N_{\overline{M}}(\overline{P})/C_{\overline{M}}(\overline{P})$ is monomorphic. This shows $(2^m-1) | e(\overline{M})$. On the other hand, by [15, I 7.7 Hilfssazt (c)], we get $N_{\overline{M}}(\overline{P}) = (N_M(P)\cdot\langle y_i \rangle)/\langle y_i \rangle$. This implies that the canonical homomorphism $N_M(P)/C_M(P) \longrightarrow N_{\overline{M}}(\overline{P})/C_{\overline{M}}(\overline{P})$ is epimorphic. Hence $e(\overline{M}) | e(M)$. Since $S \subseteq M \subseteq G$ and $e = e(S) = 2^m-1$, we have $e(M) = e(S) = 2^m-1$ by considering the canonical monomorphisms as above. Thus $e(\overline{M}) = 2^m-1$. Hence we get $\ell(B_o(\overline{M})) = 2^m-1$ by induction. Thus, $\ell(b_{y_i}) = \ell(B_o(M)) = 2^m-1$ from Proposition 1.2. We may assume $O(G) = 1$ by Lemma 1.1. Since $Q \neq 1$, there is an involution $y_j \in Q$. By Z*-theorem [10, Theorem 67.1], $y_j \in Z(G)$. Hence $\ell(B_o) = \ell(b_{y_j}) = 2^m-1$. Next, we consider $\ell(b_{xy_i})$ for each $i = 1,\dots,2^{n-m}$. For an integer $k$ it is seen that $(xy_i)^{t^k} = xy_i$ if and only if $(2^m-1)|k$. Hence $N_U(P) = C_U(P)$ where $U = C_G(xy_i)$. Then $U$ is 2-nilpotent from [10, Theorem 18.7], so that $\ell(b_{xy_i}) = \ell(B_o(U)) = 1$ by [10, Corollary 65.3]. These imply $k(B_o) = 2^n$.

Lemma 2.3. Assume as in Lemma 2.2. Then $B_o \cong B_o(S)$.

Proof. We use the same notation as in the proof of Lemma 2.2. We prove the lemma by induction on $|G|$. Suppose $G \neq S$. By [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index with $S \subseteq H$. Let $b_o = B_o(H)$. By induction, $b_o \cong B_o(S)$. It follows from Lemma 2.2 that $k(B_o) = k(b_o) = 2^n$ and that $\ell(B_o) = \ell(b_o) = 2^m-1$. By the proof of Lemma 2.2, there is an involution $x \in G$ with $\ell(b_x) = 1$. Hence $\chi(x) = \pm 1$ for all $\chi \in \mathrm{Irr}(B_o)$ from Lemma 1.9. Thus, by Corollary 1.8, it is sufficient to show that

(*) if $\tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_o)$ with $\tilde{\chi}(1) = \tilde{\chi}'(1)$, then $\tilde{\chi}(x) = \tilde{\chi}'(x) = \pm 1$.

Let $\{\theta_1, \ldots, \theta_{2^{n-m}}\}$ be the set of all irreducible complex characters of $Q$.

Case 1. $m = 2$: By the character table of $L_2(q)$ (cf. [10, Theorem 38.1]), we can write

| | $1$ | $x$ |
|---|---|---|
| $\zeta_1$ | $1$ | $1$ |
| $\zeta_2$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\zeta_3$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\zeta_4$ | $q$ | $\varepsilon$ , |

$$\varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod 8 \\ 1 & \text{if } q \equiv 5 \pmod 8 \end{cases}$$

where $\{\zeta_1, \ldots, \zeta_4\} = \mathrm{Irr}(B_o(L_2(q)))$. Since $b_o \cong B_o(S)$ and since $S = L_2(q) \times Q$, we may write $\mathrm{Irr}(b_o) = \{\tilde{\chi}_{ij} \mid i = 1,\ldots,4;\ j = 1,\ldots,2^{n-2}\}$ such that $\tilde{\chi}_{ij}|_S = \zeta_i\theta_j$ for all $i, j$. Then

$$\tilde{\chi}_{ij}(1) = \begin{cases} 1 & \text{for } i = 1 \\ (q+\varepsilon)/2 & \text{for } i = 2, 3 \\ q & \text{for } i = 4 \end{cases}$$

and

$$\tilde{\chi}_{ij}(x) = \begin{cases} 1 & \text{for } i = 1 \\ -\varepsilon & \text{for } i = 2, 3 \\ \varepsilon & \text{for } i = 4 \end{cases}$$

These imply (*).

Case 2. $m \geqslant 3$: By the character table of $SL(2,2^m)$ (cf. [10, Theorem 38.2]), we know

| | $1$ | $x$ | |
|---|---|---|---|
| $1$ | $1$ | $1$ | |
| $\tilde{\theta}_i$ | $2^m-1$ | $-1$ | for $i = 1,\ldots,2^{m-1}$ |
| $\tilde{\chi}_j$ | $2^m+1$ | $1$ | for $j = 1,\ldots,2^{m-1}-1$ |

where $\{1, \tilde{\theta}_i, \tilde{\chi}_j \mid i = 1,\ldots,2^{m-1};\ j = 1,\ldots,2^{m-1}-1\} =$ $\mathrm{Irr}(B_0(SL(2,2^m)))$. Using this we can show (*) as in Case 1. This completes the proof.

Now, the above lemmas imply the next main result of this section.

Theorem 2.4. Let $P$ be an abelian Sylow 2-subgroup of $G$. Assume that $e$ is prime. Then we have the following.

(1) $\ell(B_0) = e$. And if $G$ is nonsolvable then $k(B_0) = |P|$.

(2) When $G$ is nonsolvable, one of the following holds:

(i) $e = 3$, and $B_0 \cong B_0(L_2(q) \times (P/(Z_2 \times Z_2)))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$,

(ii) $e = 2^m-1$ for some $m \geqslant 3$, and $B_0 \cong B_0(SL(2,2^m) \times (P/(\underbrace{Z_2 \times \cdots \times Z_2}_{m})))$.

Proof. We can assume $O(G) = 1$ by Lemma 1.1. Let $S = O'(G)$. Firstly assume that $S$ is solvable. Then $S = P$, so that $C_G(P) = P$. Hence $G$ is a semi-direct product of its normal subgroup $P$ by $Z_e$. This shows $\ell(B_0) = e$. So it is enough to consider the case where $G$ is nonsolvable. Since $e$ is prime, $e = e(S)$. By Proposition 1.10 and Lemma 1.12, one of the following two cases occurs:

(i) $e(S) = 3$, and $S \cong L_2(q) \times (P/(Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$,

(ii) $e(S) = 2^m - 1$ for some $m \geqslant 3$, and $S \cong SL(2,2^m) \times (P/(\underbrace{Z_2 \times \ldots \times Z_2}_{m}))$.

Hence we obtain (1) and (2) from Lemmas 2.2 and 2.3, respectively.

Remark 2. For the case where $G$ is solvable, the latter half of Theorem 2.4(1) does not hold in general. Indeed, let $P$ be an elementary abelian group of order 16 with $P = \langle x, y, z, w \rangle$. Let $t \in Aut(P)$ such that $x^t = y$, $y^t = xy$, $z^t = w$ and $w^t = zw$. There is a semi-direct product $G$ of its normal subgroup $P$ by $\langle t \rangle$. Then $G$ is solvable and $e = |G:P| = 3$. Since $u^t \neq u$ for all $u \in P - \{1\}$, we shall show that $k(B_o) = 8 \neq 16$ (cf. Proposition 6.1). As another example, let $P$ be the same as above, and let $t \in Aut(P)$ with $|t| = 5$. If $G$ is a semi-direct product of $P$ by $\langle t \rangle$ and $G$ is not the direct product $P \times Z_5$, then we shall show that $k(B_o) = 8 \neq 16$ (cf. Proposition 6.3).

## 3. The case $e = 9$

In this section we consider the case when $e = e(S) = 9$, where $S = O'(G/O(G))$. We use the notation $G$, $P$, $n$, $e$ and $B_o$ as in §2.

Lemma 3.1. Let $P$ be an elementary abelian Sylow 2-subgroup of $G$ of order 16. If $e = 9$, then $k(B_o) = 16$ and $\ell(B_o) = 9$.

Proof. By Lemma 1.13, $Aut(P)$ has noncyclic Sylow 3-subgroups of order 9. Hence we may assume that $N_G(P) = \langle s, t, C_G(P)\rangle$ for some $s, t \in N_G(P)$, $P = \langle x, y, z, w\rangle$, $x^s = x$, $y^s = y$, $z^s = w$, $w^s = zw$, $x^t = y$, $y^t = xy$, $z^t = z$ and $w^t = w$. By [10, Lemma 18.5 and Theorems 68.4 and 65.4],

$$k(B_o) = \ell(B_o) + \ell(b_x) + \ell(b_z) + \ell(b_{xz}).$$

Since $e(C_G(xz)) = 1$, $\ell(b_{xz}) = 1$ from [10, Theorem 18.7 and Corollary 65.3]. Since $e(C_G(x)) = e(C_G(z)) = 3$, it follows from Theorem 2.4 that $\ell(b_x) = \ell(b_z) = 3$. By [10, Corollary 65.3], $\ell(B_o) \geqslant 2$ since $e = 9$. Hence, by Lemma 1.15(2), $k(B_o) = 16$, so that $\ell(B_o) = 9$.

Lemma 3.2. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod 8$ and $q' \equiv 3$ or $5 \pmod 8$. If $e = 9$, then $k(B_o) = 2^n$ and $\ell(B_o) = 9$.

Proof. We may assume $S = L_2(q) \times L_2(q') \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2 \times Z_2)$. We use induction on $n$. If $n = 4$, Sylow 2-subgroups of $G$ are elementary abelian of order 16, so that the lemma is proved by Lemma 3.1. Suppose $n > 4$. Let $R_1$ and $R_2$ be Sylow 2-subgroups of $L_2(q)$ and $L_2(q')$, respectively. We may assume $P = R_1 \times R_2 \times Q$. We can write $R_1 = \{1, x, x^s, x^{s^2}\}$ for some $s \in L_2(q)$ and for an involution $x \in R_1$. Similarly, $R_2 = \{1, y, y^t, y^{t^2}\}$ for some $t \in L_2(q')$ and for an involution $y \in R_2$. Since $e = e(S) = 9$, we know that $N_G(P) = \langle s, t, C_G(P)\rangle$ and that $N_G(P)/C_G(P)$ is elementary abelian of order 9. Let $Q = \{1{=}z_1, z_2, \ldots, z_{2^{n-4}}\}$. By [10, Lemma 18.5], $\{z_i, xz_i, yz_i, xyz_i \mid i = 1, \ldots, 2^{n-4}\}$ is the set of all representatives of G-conjugate classes of P. Thus, by [10, Theorems 68.4 and 65.4],

$$k(B_0) = \ell(B_0) + \Sigma_{i=2}^{2^{n-4}} \ell(b_{z_i}) + \Sigma_{i=1}^{2^{n-4}} \{\ell(b_{xz_i}) + \ell(b_{yz_i}) + \ell(b_{xyz_i})\}.$$

As in the proof of Lemma 2.2, by induction, we get $\ell(b_{z_i}) = 9$ for all $i = 2,\ldots,2^{n-4}$. By Lemma 1.1, we may assume $O(G) = 1$. Since $Q \neq 1$, as in the proof of Lemma 2.2, by making use of $Z^*$-theorem [10, Theorem 67.1], we have $\ell(B_0) = 9$. Since $s \notin C_G(xz_i)$ and since $t \in C_G(xz_i)$, we obtain $e(C_G(xz_i)) = 3$. Hence $\ell(b_{xz_i}) = 3$ for all $i = 1,\ldots,2^{n-4}$ from Theorem 2.4(1). Similarly, by Theorem 2.4(1), $\ell(b_{yz_i}) = 3$ for all $i = 1,\ldots,2^{n-4}$. Fix any $i$ with $1 \leqslant i \leqslant 2^{n-4}$. For integers $j$ and $k$, it is seen that $(xyz_i)^{s^j t^k} = xyz_i$ if and only if $3|j$ and $3|k$. Hence as in the proof of Lemma 2.2, $\ell(b_{xyz_i}) = 1$ for all $i = 1,\ldots,2^{n-4}$. Thus $k(B_0) = 2^n$. This finishes the proof.

Lemma 3.3. Assume as in Lemma 3.2. Then $B_0 \cong B_0(S)$.

Proof. We use the same notation as in the proof of Lemma 3.2. We prove the lemma by induction on $|G|$. Assume $G \neq S$. By [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index with $S \subseteq H$. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. By the proof of Lemma 3.2, there is an involution $xy \in G$ with $\ell(b_{xy}) = 1$. It follows from Lemmas 3.2 and 1.9 that $\chi(xy) = \pm 1$ for all $\chi \in Irr(B_0)$. By Lemma 3.2, $k(B_0) = k(b_0)$ and $\ell(B_0) = \ell(b_0)$. Thus, by Corollary 1.8, it is enough to prove that

$$(*)\qquad \begin{array}{l} \text{if } \tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_0) \text{ with } \tilde{\chi}(1) = \tilde{\chi}'(1), \\ \text{then } \tilde{\chi}(xy) = \tilde{\chi}'(xy) = \pm 1. \end{array}$$

As in the proof of Lemma 2.3 we know the character tables of $L_2(q)$ and $L_2(q')$. Thus we can write

| | 1 | x |
|---|---|---|
| $\eta_1$ | 1 | 1 |
| $\eta_2$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\eta_3$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\eta_4$ | q | $\varepsilon$ |

$$\varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod 8 \\ 1 & \text{if } q \equiv 5 \pmod 8 \end{cases}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\} = \mathrm{Irr}(B_0(L_2(q)))$, and

| | 1 | y |
|---|---|---|
| $\zeta_1$ | 1 | 1 |
| $\zeta_2$ | $(q'+\varepsilon')/2$ | $-\varepsilon'$ |
| $\zeta_3$ | $(q'+\varepsilon')/2$ | $-\varepsilon'$ |
| $\zeta_4$ | $q'$ | $\varepsilon'$ |

$$\varepsilon' = \begin{cases} -1 & \text{if } q' \equiv 3 \pmod 8 \\ 1 & \text{if } q' \equiv 5 \pmod 8 \end{cases}$$

where $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} = \mathrm{Irr}(B_0(L_2(q')))$. Let $\{\theta_1, \ldots, \theta_{2^{n-4}}\}$ be the set of all irreducible complex characters of Q. Since $b_0 \cong B_0(S)$, we may write $\mathrm{Irr}(b_0) = \{\tilde{\chi}_{ijk} \mid i = 1,\ldots,4;\ j = 1,\ldots,4;\ k = 1,\ldots,2^{n-4}\}$ such that $\tilde{\chi}_{ijk}|_S = \eta_i\zeta_j\theta_k$ for all i, j, k.

Case 1. $\varepsilon = -1$ and $\varepsilon' = 1$: In order to show (*) it is enough to prove that $\{1, (q-1)/2, q', (q-1)q'/2, q(q'+1)/2\} \cap \{(q'+1)/2. q, (q-1)(q'+1)/4, qq'\} = \emptyset$ since $\tilde{\chi}_{ijk}(1) = \eta_i(1)\zeta_j(1)$ and $\tilde{\chi}_{ijk}(xy) = \eta_i(x)\zeta_j(y)$ for all i, j, k. We can prove it.

Case 2. $\varepsilon = \varepsilon' = -1$: We know that $\{1, (q-1)/2, (q'-1)/2, (q-1)(q'-1)/4, qq'\} \cap \{q, q', (q-1)q'/2, q(q'-1)/2\} = \emptyset$. This implies (*) as in Case 1.

Case 3. $\varepsilon = \varepsilon' = 1$: Since $\{1, q, q', (q+1)(q'+1)/4, qq'\} \cap$

$\{(q+1)/2,\ (q'+1)/2,\ (q+1)q'/2,\ q(q'+1)/2\} = \emptyset$, we can show (*). This completes the proof of the lemma.

The above lemmas imply the next main result of this section.

Theorem 3.4. Let $P$ be an abelian Sylow 2-subgroup of $G$. Assume $e = e(S) = 9$, where $S = O'(G/O(G))$. Then we have the following.

(1) $k(B_0) = |P|$ and $\ell(B_0) = 9$.

(2) $B_0 \simeq B_0(L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2)))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod 8$ and $q' \equiv 3$ or $5 \pmod 8$.

Proof. We may assume $O(G) = 1$ by Lemma 1.1. Since $e(S) = 9$, by Proposition 1.10 and Lemma 1.12, we get that $S \simeq L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod 8$ and $q' \equiv 3$ or $5 \pmod 8$. Hence we obtain (1) and (2) from Lemmas 3.2 and 3.3, respectively.

## 4. The case $e = 21$

In this section we deal with the case when $e = e(S) = 21$, where $S = O'(G/O(G))$. As in §1, let $J_1$ and $R(q)$ be the Janko's first simple group and the simple groups of Ree type, respectively (cf. [16], [21] and [13]). We use the notation $G$, $P$, $n$, $e$ and $B_0$ as before.

Lemma 4.1. Let $P$ be an abelian Sylow 2-subgroup of $G$ of order 8. If $e = 21$, then $k(B_0) = 8$ and $\ell(B_0) = 5$.

Proof. By Lemma 1.14, $N_G(P)/C_G(P)$ is noncyclic of order 21. Hence we can write that $N_G(P) = \langle s, t, C_G(P)\rangle$,

$P = \{1, x, x^s, x^{s^2}, z, xz, x^sz, x^{s^2}z\} = \{1, z, z^t, \ldots, z^{t^6}\}$ for some $s, t \in N_G(P)$ and involutions $x, z \in P$ with $z^s = z$. Then, by [10, Theorems 68.4 and 65.4], $k(B_o) = \ell(B_o) + \ell(b_z)$. Since $e(C_G(z)) = 3$, $\ell(b_z) = 3$ from Theorem 2.4(1). The calculation of the generalized decomposition matrix of $B_o$ relative to $z$ is due to J.B. Olsson [18, Theorems 3.15, 3.16 and 3.17]. Let $M = C_G(z)$, $\overline{M} = M/\langle z\rangle$ and $\overline{b}_z = B_o(\overline{M})$. By [10, Theorem 66.3], there is a basic set $\overline{W}$ of $\overline{b}_z$ such that $\overline{W}$ contains the trivial Brauer character and the Cartan matrix of $\overline{b}_z$ with respect to $\overline{W}$ has the form

$$\begin{matrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{matrix} .$$

Then, by [10, Lemma 66.1], there is a basic set $W$ of $b_z$ such that $W$ contains the trivial Brauer character and the Cartan matrix $C_z$ of $b_z$ with respect to $W$ has the form

$$(*) \qquad \begin{matrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{matrix} .$$

We use the following notation here. For an integer $r \geq 0$ and a p-block $B$, let $E_B(p^r)$ denote the multiplicity of $p^r$ as an elementary divisor of the Cartan matrix of $B$. If $Q$ is a p-subgroup of a finite group $A$ and if $B$ is a p-block of $A$, let $n_B(Q)$ denote the multiplicity of $Q$ as a lower defect group of $B$ (cf. [5]. In [5], $n_B(Q)$ is denoted by $m_B^{(1)}(Q)$ ). By [8, (89.8) Theorem], $E_{B_o}(8) = 1$. Since all involutions in $G$ are conjugate, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $E_{B_o}(2) = n_{b_z}(\langle z\rangle)$. Since every lower defect group of a 2-block of $G$ contains all 2-subgroups $U$ of

$G$ with $U \subseteq Z(G)$, by [5, (7G)], $E_{b_z}(2) = n_{b_z}(\langle z\rangle)$. By (*), $E_{b_z}(2) = 2$. Thus $E_{B_0}(2) = 2$, so that $\ell(B_0) \geqslant 3$. This shows $k(B_0) \geqslant 6$. Let $\{\chi_i \mid i = 1,\ldots,k(B_0)\} = \mathrm{Irr}(B_0)$. Since $\ell(b_z) = 3$, let $N = (n_{i\alpha})_{\substack{1\leqslant i\leqslant k(B_0)\\ 1\leqslant\alpha\leqslant 3}}$ be the matrix of the generalized decomposition numbers of $B_0$ relative to $z$ with respect to $W$. Since $|z| = 2$, every $n_{i\alpha}$ is an integer. By [4, II (7A) and (4C)], $(n_{i1}, n_{i2}, n_{i3}) \neq (0, 0, 0)$ for every $\chi_i$. For $\chi_i$, $\chi_j$ let $a_{ij} = \Sigma_{1\leqslant\alpha,\beta\leqslant 3}\, 8n_{i\alpha}u_{\alpha\beta}n_{j\beta}$, where $C_z^{-1} = (u_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 3}$. By Lemma 1.11 and [4, II (7A) and (5G)], all $a_{ii}$ are odd integers. Hence $n_{i1} + n_{i2} + n_{i3}$ is odd for every $\chi_i$. Let $N_\alpha$ be the $\alpha$-th column of $N$ for each $\alpha$, and let $N_\alpha N_\beta = \Sigma_{i=1}^{k(B_0)} n_{i\alpha}n_{i\beta}$ for all $\alpha$, $\beta$. By [10, Theorem 63.3(2)], ${}^tNN = C_z$ where ${}^tN$ is the transposed matrix of $N$. So $N_\alpha N_\beta = 4$ if $\alpha = \beta$, and $N_\alpha N_\beta = 2$ if $\alpha \neq \beta$. Clearly, $12 = \mathrm{tr}(C_z) = \Sigma_{i,\alpha}\, n_{i\alpha}^2$ where $\mathrm{tr}(C_z)$ is the trace of $C_z$. Then the next three possibilities arise for the nonzero entries of $N$:

(i) 2 entries are $\pm 2$, and 4 entries are $\pm 1$.

(ii) 1 entry is $\pm 2$, and 8 entries are $\pm 1$.

(iii) 12 entries are $\pm 1$.

By elementary calculations as in [18, Theorems 3.15, 3.16 and 3.17] we can write

$$N = \begin{matrix} \delta_1 & 0 & 0 \\ \delta_2 & 0 & 0 \\ 0 & \delta_3 & 0 \\ 0 & \delta_4 & 0 \\ 0 & 0 & \delta_5 \\ 0 & 0 & \delta_6 \\ \delta_7 & \delta_7 & \delta_7 \\ \delta_8 & \delta_8 & \delta_8 \end{matrix}$$

where $\delta_i = \pm 1$. This shows $k(B_o) = 8$, so that $\ell(B_o) = 5$. This completes the proof.

Lemma 4.2. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$ or $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e = 21$, then $k(B_o) = 2^n$ and $\ell(B_o) = 5$.

Proof. We may assume $S = R \times Q$ where $R = J_1$ or $R(q)$ and $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. Let $T$ be a Sylow 2-subgroup of $R$ with $T \times Q = P$. By Lemma 1.12(iii), $N_R(T)/C_R(T)$ is noncyclic of order 21. Hence we can write $N_R(T) = \langle s, t, C_R(T) \rangle$ and $T = \{1, x, x^s, x^{s^2}, z, xz, x^s z, x^{s^2} z\} = \{1, x, x^t, \ldots, x^{t^6}\}$ for some $s, t \in N_R(T)$ and for involutions $x, z \in T$ with $z^s = z$. Since $e = 21$, $N_G(P) = \langle s, t, C_G(P) \rangle$. We prove the lemma by induction on $n$. If $n = 3$, the lemma is proved from Lemma 4.1 because $P = T$ and $P$ is elementary abelian of order 8 from Lemma 1.12(iii). Suppose $n > 3$. Let $Q = \{1 = y_1, y_2, \ldots, y_{2^{n-3}}\}$. By [10, Lemma 18.5], $\{y_i, zy_i \mid i = 1, \ldots, 2^{n-3}\}$ is the set of all representatives of G-conjugate classes of $P$. Then, by [10, Theorems 68.4 and 65.4],

$$k(B_o) = \ell(B_o) + \Sigma_{i=2}^{2^{n-3}} \ell(b_{y_i}) + \Sigma_{i=1}^{2^{n-3}} \ell(b_{zy_i}).$$

As in the proof of Lemma 2.2, by induction we get $\ell(b_{y_i}) = 5$ for all $i = 2, \ldots, 2^{n-3}$. We can assume $O(G) = 1$ by Lemma 1.1. Since $Q \neq 1$, it follows from $Z^*$-theorem that $\ell(B_o) = 5$. Since $s \in C_G(zy_i)$ and $t \notin C_G(zy_i)$, we have $e(C_G(zy_i)) = 3$. Hence $\ell(b_{zy_i}) = 3$ for all $i = 1, \ldots, 2^{n-3}$ from Theorem 2.4(1). Thus $k(B_o) = 2^n$.

Lemma 4.3. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e = 21$, then $B_0 \cong B_0(S)$.

Proof. We can assume $S = J_1 \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. We use induction on $|G|$. Assume $G \neq S$. By [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index $\ell$ with $S \subseteq H$. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. Let $s$, $t$, $x$, $z$ and $y_i$ be the same as in the proof of Lemma 4.2. Since $z$ is an involution in $J_1$, by [16, Theorem], $C_{J_1}(z) = A_5 \times \langle z \rangle$ where $A_5$ is the alternating group of degree 5. Hence $C_S(z) = A_5 \times \langle z \rangle \times Q$. Let $M = C_G(z)$. Clearly $C_S(z) \cong A_5 \times (P/(Z_2 \times Z_2))$ and $C_S(z)$ is a normal subgroup of $M$ of odd index. By the proof of Lemma 4.2, $e(M) = 3$. Hence, by Lemma 2.3, we get that $b_z = B_0(M) \cong B_0(A_5 \times (P/(Z_2 \times Z_2)))$ since $A_5 \cong L_2(5)$. By Lemma 1.16(ii), the Cartan matrix of $B_0(A_5)$ has the form

$$\begin{array}{cccc} & 1 & & \\ 1 & 4 & 2 & 2 \\ & 2 & 2 & 1 \\ & 2 & 1 & 2 \end{array} .$$

Thus, by [10, Lemma 66.1], the Cartan matrix $C_z$ of $b_z$ has the form

$$\begin{array}{cccc} & 1 & & \\ 1 & 2^n & 2^{n-1} & 2^{n-1} \\ & 2^{n-1} & 2^{n-1} & 2^{n-2} \\ & 2^{n-1} & 2^{n-2} & 2^{n-1} \end{array} .$$

By Lemma 4.2, $k(B_0) = 2^n$. Let $\{\chi_1, \ldots, \chi_{2^n}\} = \mathrm{Irr}(B_0)$. We can write $\mathrm{IBr}(b_z) = \{\phi_1^z = 1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1) = \phi_3^z(1) = 2$ from Lemma 1.16(i). For each $\chi_i$ and $\phi_\alpha^z$, let $n_{i\alpha} = d_{i\alpha}^z$ be the generalized decomposition number of $B_0$ relative to $z$.

Since $|z| = 2$, every $n_{i\alpha}$ is an integer. Let $N = (n_{i\alpha})_{1\leqslant i\leqslant 2^n, 1\leqslant\alpha\leqslant 3}$, $N_\alpha = (n_{i\alpha})_{1\leqslant i\leqslant 2^n}$ for each $\alpha$, and $N_\alpha N_\beta = \Sigma_{i=1}^{2^n} n_{i\alpha}n_{i\beta}$ for each $\alpha, \beta$. It follows from [10, Theorems 63.3(2), 63.2 and 65.4] that $N_1N_1 = 2^n$, $N_2N_2 = N_3N_3 = 2^{n-1}$, $N_1N_2 = N_1N_3 = 2^{n-1}$ and $N_2N_3 = 2^{n-2}$. For each $\chi_i$, $\chi_j$, let $a_{ij} = \Sigma_{1\leqslant\alpha,\beta\leqslant 3}\ 2^n n_{i\alpha}u_{\alpha\beta}n_{j\beta}$, where $C_z^{-1} = (u_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 3}$. Then

$$\begin{aligned} a_{ii} &= 3n_{i1}^2 + 4(n_{i2}^2 + n_{i3}^2) - 4(n_{i1}n_{i2} + n_{i1}n_{i3}) \\ &\equiv n_{i1}^2 \equiv n_{i1} \pmod 2 \end{aligned}$$

for all $\chi_i$. By Lemma 1.11, every $\chi_i$ has height zero. Hence, by [4, II (7A) and (5G)], every $a_{ii}$ is odd, so that $n_{i1}$ is odd for all $i = 1,\ldots,2^n$. Since $N_1N_1 = 2^n$, $n_{i1} = \pm 1$ for all $i = 1,\ldots,2^n$. Let $\delta_i = n_{i1}$ and $u_i = n_{i2}\delta_i$ for each $i$.

Since $N_1N_2 = N_2N_2 = 2^{n-1}$, $\Sigma_{i=1}^{2^n} u_i = \Sigma_{i=1}^{2^n} u_i^2$. Thus, $u_i = 1$ or $0$ for all $i = 1,\ldots,2^n$. Hence exactly $2^{n-1}$ $u_i$'s are 1 and the other $u_i$'s are 0 since $N_1N_2 = 2^{n-1}$. Then we may assume

$$n_{i2} = \begin{cases} \delta_i & \text{for } i = 1,\ldots,2^{n-1} \\ 0 & \text{for } i = 2^{n-1}+1,\ldots,2^n. \end{cases}$$

Similarly, exactly $2^{n-1}$ $(n_{i3}\delta_i)$'s are 1 and the other $(n_{i3}\delta_i)$'s are 0. Since $N_2N_3 = 2^{n-2}$, we may assume

$$n_{i3} = \begin{cases} \delta_i & \text{for } i = 1,\ldots,2^{n-2} \text{ and for } i = 2^{n-1}+1,\ldots,3\cdot 2^{n-2} \\ 0 & \text{for } i = 2^{n-2}+1,\ldots,2^{n-1} \text{ and for } i = 3\cdot 2^{n-2}+1,\ldots,2^n. \end{cases}$$

Since $\chi_i(z) = n_{i1} + 2(n_{i2}+n_{i3})$ for each $i$, we get

$$\chi_i(z) = \begin{cases} \pm 5 & \text{for } i = 1,\ldots,2^{n-2} \\ \pm 3 & \text{for } i = 2^{n-2}+1,\ldots,3\cdot 2^{n-2} \\ \pm 1 & \text{for } i = 3\cdot 2^{n-2}+1,\ldots,2^n. \end{cases}$$

Let $C_G(P) = P \times V$. When $G = VH$, $G = C_G(P)\cdot H$, so that $B_0 \cong b_0$

from Proposition 1.6. Thus, we may assume $G \neq VH$. Hence $C_H(P) = P \times V$. Since $b_o \cong B_o(S)$, it follows from Proposition 1.5 that $|H:VH'| = k'(b_o) = 2^{n-3}$. By [10, Theorem 18.4], $P \cap G' = \{1, x, x^t, \ldots, x^{t^6}\}$. Then the order of Sylow 2-subgroups of $G'$ is 8. This implies $2^{n-3} \mid |G:VG'|$ and $2^{n-2} \nmid |G:VG'|$. Thus, by Proposition 1.5, $k'(B_o) = |G:VG'| = \ell 2^{n-3}$ where $\ell = |G:H|$. Since $b_o \cong B_o(S)$, by Clifford's theorem, Proposition 1.3 and the character table of $J_1$ [16, p.148], we get that $\chi_i(z) = 1$ for every $\chi_i \in \mathrm{Irr}(B_o)$ with degree one. These show that the number of $\chi_i \in \mathrm{Irr}(B_o)$ with $\chi_i(z) = 1$ is at least $\ell 2^{n-3}$. However, $\chi_i(z) = \pm 1$ only for $i = 3 \cdot 2^{n-2}+1, \ldots, 2^n$. This is a contradiction since $\ell 2^{n-3} > 2^{n-2}$. This completes the proof.

Lemma 4.4. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e = 21$, then $B_o \cong B_o(S)$.

Proof. Let $R = R(q)$. We may assume $S = R \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. We prove the lemma by induction on $|G|$. Assume $G \neq S$. By [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index $\ell$ with $S \subseteq H$. Let $b_o = B_o(H)$. By induction, $b_o \cong B_o(H)$. Let $s$, $t$, $x$, $z$ and $y_i$ be the same as in the proof of Lemma 4.2. Since $z$ is an involution in $R$, $C_R(z) = L_2(q) \times \langle z \rangle$ from [21, p.62 III]. (It is noted that we use the notation $R(q)$ as in the sense of [13]). Hence $C_S(z) = L_2(q) \times \langle z \rangle \times Q$. Let $M = C_G(z)$. Then $C_S(z)$ is a normal subgroup of $M$ of odd index and $C_S(z) \cong L_2(q) \times (P/(Z_2 \times Z_2))$. By the proof of Lemma 4.2, $e(M) = 3$. Then, by Lemma 2.3, $b_z = B_o(M) \cong B_o(L_2(q) \times (P/(Z_2 \times Z_2)))$. By [21, Theorem (1)],

$3 < q \equiv 3 \pmod 8$, so that as in the proof of Lemma 4.3 the Cartan matrix $C_z$ of $b_z$ has the form

$$\begin{matrix} 2^{n-1} & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-2} & 2^{n-1} \end{matrix} .$$

By Lemma 4.2, $k(B_o) = 2^n$. Let $\{\chi_1, \ldots, \chi_{2^n}\} = \mathrm{Irr}(B_o)$. We can write $\mathrm{IBr}(b_z) = \{\phi_1^z = 1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1) = \phi_3^z(1) = (q-1)/2$ from Lemma 1.16(i). Let $n_{i\alpha}$, $N$, $N_\alpha$ and $N_\alpha N_\beta$ be the same as in the proof of Lemma 4.3. Every $n_{i\alpha}$ is an integer. As in the proof of Lemma 4.3 we get $N_\alpha N_\alpha = 2^{n-1}$ for all $\alpha = 1, 2, 3$, and $N_\alpha N_\beta = 2^{n-2}$ if $\alpha \neq \beta$. Let $C_G(P) = P \times V$. As in the proof of Lemma 4.3 we may assume $G \neq VH$. Since $b_o \cong B_o(S)$, $k'(b_o) = 2^{n-3}$. So that $k'(B_o) = |G:VG'| = \ell \cdot 2^{n-3}$ as in the proof of Lemma 4.3, where $\ell = |G:H|$. Since $b_o \cong B_o(S)$, by [21, p.74 and pp.87-88], we can write $\{\tilde{\chi}_{ij} \mid i = 1,\ldots,8;\ j = 1,\ldots,2^{n-3}\} = \mathrm{Irr}(b_o)$ and

| | $1$ | $z$ |
|---|---|---|
| $\tilde{\chi}_{ij}$ | $1$ | $1$ |
| $\tilde{\chi}_{2j}$ | $q^2-q+1$ | $-1$ |
| $\tilde{\chi}_{3j}$ | $q^3$ | $q$ |
| $\tilde{\chi}_{4j}$ | $q(q^2-q+1)$ | $-q$ |
| $\tilde{\chi}_{5j}$ | $(q-1)m(q+1+3m)/2$ | $-(q-1)/2$ |
| $\tilde{\chi}_{6j}$ | $(q-1)m(q+1+3m)/2$ | $-(q-1)/2$ |
| $\tilde{\chi}_{7j}$ | $(q-1)m(q+1-3m)/2$ | $(q-1)/2$ |
| $\tilde{\chi}_{8j}$ | $(q-1)m(q+1-3m)/2$ | $(q-1)/2$ |

for $j = 1,\ldots,2^{n-3}$, where $q = 3^{2k+1}$ and $m = 3^k$ for some $k \geqslant 1$ (cf. [21, Theorem]). By Clifford's theorem, Proposition

1.3 and the above table, we know that if $\chi_i(1) = 1$ then $\chi_i(z) = 1$. When $n_{i1} = 0$, $\chi_i(z) = (n_{i2} + n_{i3})(q-1)/2$. Thus $n_{i1} \neq 0$ if $\chi_i(z) = \pm 1$. Hence the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $n_{i1} \neq 0$ is at least $\ell 2^{n-3}$. Since $N_1 N_1 = 2^{n-1}$, we get $\ell = 3$. Fix any $\chi_i$. If $\chi_i|_H = \tilde{\chi}_{2j}$ for some $j$ with $1 \leqslant j \leqslant 2^{n-3}$, then $n_{i1}^2 \geqslant 1$ since $\chi_i(z) = -1$. If $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j'} + \tilde{\chi}_{2j''}$ for some $j, j', j''$ with $1 \leqslant j < j' < j'' \leqslant 2^{n-3}$, then $n_{i1}^2 \geqslant 9$ since $\chi_i(z) = -3$. Let $u$ be the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $\chi_i|_H = \tilde{\chi}_{2j}$, and let $v$ be the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j'} + \tilde{\chi}_{2j''}$ for $j < j' < j''$. Since $N_1 N_1 = 2^{n-1}$, and since $1 < q^2-q+1 < 3(q^2-q+1)$, we have

$$2^{n-1} = \Sigma_{i=1}^{2^n} n_{i1}^2 \geqslant k'(B_0) + u + 9v = 3 \cdot 2^{n-3} + u + 9v.$$

Then $2^{n-3} \geqslant u + 9v$. By Proposition 1.4, for every $\tilde{\chi}_{2j}$ there is some $\chi_i$ with $(\chi_i|_H, \tilde{\chi}_{2j}) \neq 0$, so that, by Clifford's theorem and Proposition 1.3, $\chi_i|_H = \tilde{\chi}_{2j}$ or $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j}{}^{g} + \tilde{\chi}_{2j}{}^{g^2}$ where $g$ is an element of $G$ with $G = \langle g, H\rangle$. By considering the degrees of $\tilde{\chi}_{ij}$, we get that $\tilde{\chi}_{2j}{}^{g}$ and $\tilde{\chi}_{2j}{}^{g^2}$ are both in $\{\tilde{\chi}_{2j'} \mid j' = 1,\ldots,2^{n-3}\}$. Thus $2^{n-3} \leqslant u + 3v$, so that $v = 0$ and $u = 2^{n-3}$. This implies that the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $\chi_i(z) = -1$ is at least $2^{n-3}$, so that the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $\chi_i(z) = \pm 1$ is at least $2^{n-1}$. Then the number of $\chi_i \in \mathrm{Irr}(B_0)$ with $n_{i1} \neq 0$ is at least $2^{n-1}$. Since $N_1 N_1 = 2^{n-1}$, we may assume

$$n_{i1} = \begin{cases} \delta_i & \text{for } i = 1,\ldots,2^{n-1} \\ 0 & \text{for } i = 2^{n-1}+1,\ldots,2^n \end{cases}$$

where $\delta_i = \pm 1$. Thus $\chi_i(z) = \pm 1$ for all $i = 1,\ldots,2^{n-1}$. For all $i = 1,\ldots,2^{n-1}$, $\chi_i(z) = \delta_i + (n_{i2} + n_{i3})(q-1)/2$, so that $n_{i2} + n_{i3} = 0$ since $(q-1)/2 \geqslant 13$. Consequently, $N_1N_2 + N_1N_3 = \Sigma_{i=1}^{2^n} n_{i1}(n_{i2} + n_{i3}) = \Sigma_{i=1}^{2^{n-1}} \delta_i(n_{i2} + n_{i3}) = 0$. This is a contradiction since $N_1N_2 = N_1N_3 = 2^{n-2}$. This completes the proof.

Lemma 4.5. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong L_2(q) \times SL(2,8)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$. If $e = 21$, then $B_o \cong B_o(S)$.

Proof. Let $R_1$ and $R_2$ be Sylow 2-subgroups of $L_2(q)$ and $SL(2,8)$, respectively. We may assume $S = L_2(q) \times SL(2,8)$ and $P = R_1 \times R_2$. There are an element $s \in L_2(q)$ and an involution $x \in R_1$ with $R_1 = \{1, x, x^s, x^{s^2}\}$. Similarly, we can write $R_2 = \{1, y, y^t, \ldots, y^{t^6}\}$ for some $t \in SL(2,8)$ and for an involution $y \in R_2$. Since $e = 21$, $N_G(P) = \langle s, t, C_G(P)\rangle$ and $N_G(P)/C_G(P)$ is cyclic of order 21. By [10, Lemma 18.5], $\{1, x, y, xy\}$ is the set of all representatives of G-conjugate classes of $P$. Hence, by [10, Theorems 68.4 and 65.4], $k(B_o) = \ell(B_o) + \ell(b_x) + \ell(b_y) + \ell(b_{xy})$. Since $s \notin C_G(x)$ and $t \in C_G(x)$, we have $e(C_G(x)) = 7$. Thus $\ell(b_x) = 7$ from Theorem 2.4(1). Similarly, $\ell(b_y) = 3$ from Theorem 2.4(1). For integers $i$ and $j$, $(xy)^{s^i t^j} = xy$ if and only if $3|i$ and $7|j$. This implies $N_M(P) = C_M(P)$ where $M = C_G(xy)$. Thus, by [10, Theorem 18.7 and Corollary 65.3], $\ell(b_{xy}) = \ell(B_o(M)) = 1$. Since $G$ is nonsolvable, $\ell(B_o) \geqslant 2$ from [10, Corollary 65.3], so that $k(B_o) \geqslant 13$.

Now, we prove the lemma by induction on $|G|$. Assume $G \neq S$.

By [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index $\ell$ with $S \subseteq H$. Let $b_o = B_o(H)$. We know $b_o \cong B_o(S)$ by induction. From the character tables of $L_2(q)$ and $SL(2,8)$ (cf. [10, Theorems 38.1 and 38.2]), we can write

| | 1 | x |
|---|---|---|
| $\theta_1$ | 1 | 1 |
| $\theta_2$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\theta_3$ | $(q+\varepsilon)/2$ | $-\varepsilon$ |
| $\theta_4$ | $q$ | $\varepsilon$ |

$$\varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod 8 \\ 1 & \text{if } q \equiv 5 \pmod 8 \end{cases}$$

where $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \mathrm{Irr}(B_o(L_2(q)))$ and

| | 1 | y | |
|---|---|---|---|
| $\zeta_1$ | 1 | 1 | |
| $\zeta_j$ | 7 | -1 | for $j = 2, 3, 4, 5$ |
| $\zeta_j$ | 9 | 1 | for $j = 6, 7, 8$ |

where $\{\zeta_1, \ldots, \zeta_8\} = \mathrm{Irr}(B_o(SL(2,8)))$. Since $b_o \cong B_o(S)$, we may write $\mathrm{Irr}(b_o) = \{\tilde{\chi}_{ij} \mid i = 1,\ldots,4;\ j = 1,\ldots,8\}$ with $\tilde{\chi}_{ij}|_S = \theta_i\zeta_j$ for all $i, j$. Hence the degrees of all $\tilde{\chi}_{ij}$ are 1, 7, 9, $(q+\varepsilon)/2$, $7(q+\varepsilon)/2$, $9(q+\varepsilon)/2$, $q$, $7q$ and $9q$. Next, we want to show that

$$(*) \qquad \begin{array}{l} \text{if } \tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_o) \text{ with } \tilde{\chi}(1) = \tilde{\chi}'(1) \\ \text{then } \tilde{\chi}(xy) = \tilde{\chi}'(xy) = \pm 1. \end{array}$$

Case 1. $\varepsilon = 1$: Clearly $\{1, 9, 7(q+1)/2, 9, 9q\} \cap \{7, (q+1)/2, 9(q+1)/2, 7q\} = \emptyset$. Hence, by considering the values $\tilde{\chi}_{ij}(1)$ and $\tilde{\chi}_{ij}(xy)$, we get (*).

Case 2. $\varepsilon = -1$: Since $\{1, 9, (q-1)/2, 9(q-1)/2, 7q\} \cap \{7, 7(q-1)/2, q, 9q\} = \emptyset$, we obtain (*) as in Case 1.

We get from Clifford's theorem, Proposition 1.3, (*) and the above character tables of $L_2(q)$ and $SL(2,8)$ that $\chi(xy) = \pm 1$ or $\pm \ell$ for every $\chi \in \mathrm{Irr}(B_o)$. Let $k = k(B_o)$, and let $m$ be the number of $\chi \in \mathrm{Irr}(B_o)$ with $\chi(xy) = \pm 1$. Hence we can write $\mathrm{Irr}(B_o) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_m, \chi_{m+1}, \ldots, \chi_k\}$ such that

$$\chi_i(xy) = \begin{cases} \pm 1 & \text{for } i = 1,\ldots,m \\ \pm \ell & \text{for } i = m+1,\ldots,k. \end{cases}$$

Since $\ell(b_{xy}) = 1$, as in the proof of Lemma 1.9,

$$(**) \qquad 32 = \Sigma_{i=1}^{k} \chi_i(xy)^2 = m + (k-m)\ell^2.$$

Firstly, suppose $k = m$. Then $\chi_i(xy) = \pm 1$ for all $\chi_i \in \mathrm{Irr}(B_o)$. Since $k = m = 32$ and since $b_o \cong B_o(S)$, we have $k(B_o) = k(b_o) = 32$. Hence $\ell(B_o) = 21$, so that $\ell(B_o) = \ell(b_o)$ since $b_o \cong B_o(S)$. Thus, by (*) and Corollary 1.8, $B_o \cong b_o$. Thus, we may assume $k > m$. Since $k \geqslant 13$, by (**), $\ell = 3$. So that $k-m = 1$ or $2$. Let $C_G(P) = P \times V$. Since $k > m$ and $b_o \cong B_o(S)$, we know $B_o \not\cong b_o$. Hence $G \neq VH$ from Proposition 1.6. This shows $C_H(P) = P \times V$. Thus, by Proposition 1.5, $|H:VH'| = k'(b_o) = 1$ since $b_o \cong B_o(S)$. Then $H = VH'$. Since $G/H$ is cyclic, $VG' = VH' = H$. Hence $k'(B_o) = |G:VG'| = \ell = 3$ by Proposition 1.5. So that we may assume that $\chi_1(1) = \chi_2(1) = \chi_3(1) = 1$ and $\chi_i(1) > 1$ for all $i = 4,\ldots,k$.

Case A. $k-m = 1$: By (**), we get $m = 23$ and $k = 24$. Then $\chi_i(xy) = \pm 1$ for $i = 1,\ldots,23$ and $\chi_{24}(xy) = \pm 3$. Since $b_o \cong B_o(S)$, by Clifford's theorem and Proposition 1.3,

$\chi_i|_H = 1_H$ for $i = 1, 2, 3$

$\chi_i|_H \neq 1_H$ and $\chi_i|_H \in \mathrm{Irr}(b_o)$ for $i = 4,\ldots,23$

$\chi_{24}|_H = \tilde{\chi}_{i_1} + \tilde{\chi}_{i_2} + \tilde{\chi}_{i_3}$ where $\tilde{\chi}_{i_1}, \tilde{\chi}_{i_2}$ and $\tilde{\chi}_{i_3}$ are distinct G-conjugate elements in $\mathrm{Irr}(b_o)$. On the other hand, it follows from Proposition 1.4 that for every $\tilde{\chi} \in \mathrm{Irr}(b_o)$ there is some $\chi_i \in \mathrm{Irr}(B_o)$ with $(\chi_i|_H, \tilde{\chi}) \neq 0$. These show $k(b_o) \leqslant 1+20+3 = 24$. But $k(b_o) = 32$ since $b_o \cong B_o(S)$. Then we have a contradiction.

Case B. $k-m = 2$: We have from (**) that $\chi_i(xy) = \pm 1$ for $i = 1,\ldots,14$, $\chi_{15}(xy) = \pm 3$ and $\chi_{16}(xy) = \pm 3$. Hence as in Case A we get $k(b_o) \leqslant 1+11+6 = 18$. This is a contradiction as in Case A. This completes the proof.

Lemma 4.6. Let $S$ be a normal subgroup of $G$ of odd index such that $S \cong L_2(q) \times SL(2,8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$. If $e = 21$, then $k(B_o) = 2^n$ and $\ell(B_o) = 21$.

Proof. If $n = 5$, we can prove the lemma by Lemma 4.5 (cf. Lemma 1.12 and Theorem 2.4). If $n > 5$, we can prove the lemma by induction on $n$ as in the proof of Lemma 4.2.

Lemma 4.7. Assume as in Lemma 4.6. Then $B_o \cong B_o(S)$.

Proof. We may assume $S = L_2(q) \times SL(2,8) \times Q$ with $Q \cong P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2)$. We use induction on $|G|$ as before. Assume $G \neq S$. Hence $G$ has a normal subgroup $H$ of odd prime index with $S \subseteq H$ from [12, Theorem]. Let $b_o = B_o(H)$. By induction, $b_o \cong B_o(S)$. Let $x$ and $y$ be involutions in $L_2(q)$ and $SL(2,8)$, respectively. As in the proof of Lemma 4.2, $\ell(b_{xy}) = 1$. By Lemma 4.6, $k(B_o) = 2^n$. Thus, $\chi(xy) = \pm 1$ for all $\chi \in \mathrm{Irr}(B_o)$ from Lemma 1.9. By Lemma 4.6, $k(B_o) = k(b_o)$ and $\ell(B_o) = \ell(b_o)$. Since $b_o \cong B_o(S)$, as in the proof of Lemma 4.5, we get that if $\tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_o)$ with $\tilde{\chi}(1) = \tilde{\chi}'(1)$ then $\tilde{\chi}(xy) = \tilde{\chi}'(xy) = \pm 1$. These imply $B_o \cong b_o$ from Corollary 1.8.

This completes the proof.

Next, we state the following main result of this section. That is proved by making use of Lemmas 4.2-4.7.

Theorem 4.8. Let $P$ be an abelian Sylow 2-subgroup of $G$, and let $S = O'(G/O(G))$. If $e = e(S) = 21$, then we have the following.

(1) $k(B_0) = |P|$ and

$$\ell(B_0) = \begin{cases} 5 & \text{if } N_G(P)/C_G(P) \text{ is noncyclic} \\ 21 & \text{if } N_G(P)/C_G(P) \text{ is cyclic.} \end{cases}$$

(2) One of the following holds:

(i) $B_0 \cong B_0(J_1 \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(ii) $B_0 \cong B_0(R(q) \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(iii) $B_0 \cong B_0(L_2(q) \times SL(2,8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2)))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$.

Proof. By Lemma 1.1, we may assume $O(G) = 1$. By Proposition 1.10 and Lemma 1.12, one of the following holds:

(i) $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$,

(ii) $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$,

(iii) $S \cong L_2(q) \times SL(2,8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$. Then we can prove the theorem by Lemmas 4.2-4.7.

## 5. The case when $P$ is elementary abelian of order 8

In this section we consider the case when $G$ has elementary abelian Sylow 2-subgroups of order 8. In particular, we shall determine $B_0$ in the case when $G$ is nonsolvable, $e = 21$ and

$e(S) \neq 21$ where $S = O'(G/O(G))$. Throughout this section we assume that $G$ has an elementary abelian Sylow 2-subgroup $P$ of order 8 and we use the notation $e$ and $B_0$ as before.

By Lemma 1.14 and Remark 1 of §1, it is sufficient to consider the cases when $e = 3$, 7 and 21.

Proposition 5.1. (i) If $e = 3$, then $k(B_0) = 8$ and $\ell(B_0) = 3$.

(ii) If $e = 7$, then $k(B_0) = 8$ and $\ell(B_0) = 7$.

(iii) If $e = 21$, then $k(B_0) = 8$ and $\ell(B_0) = 5$.

Proof. (i) We can write $N_G(P) = \langle s, C_G(P)\rangle$ for some $s \in N_G(P)$. There is an involution $x \in P$ with $x^s \neq x$. Hence $\ell(b_x) = 1$ as in the proof of Lemma 2.1. Then $k(B_0) = 8$ from Lemma 1.15(1). On the other hand, $\ell(B_0) = 3$ by Theorem 2.4(1).

(ii) We can verify (ii) as in (i).

(iii) We have already proved (iii) in Lemma 4.1.

Proposition 5.2. There is a basic set $W$ of $B_0$ such that $W$ contains the trivial Brauer character and the decomposition matrix of $B_0$ with respect to $W$ has the form

$$
\begin{array}{cccc}
1_G & 1 & 0 & 0 \\
 & \delta_2 & 0 & 0 \\
 & 0 & \delta_3 & 0 \\
 & 0 & \delta_4 & 0 \\
 & 0 & 0 & \delta_5 \\
 & 0 & 0 & \delta_6 \\
 & \delta_7 & \delta_7 & \delta_7 \\
 & \delta_8 & \delta_8 & \delta_8
\end{array},
\qquad
\begin{array}{ccccc}
1 & & & & \\
 & \delta_2 & & \text{\huge 0} & \\
 & & \delta_3 & & \\
 & \text{\huge 0} & & \ddots & \\
 & & & & \delta_7 \\
\delta_8 & \delta_8 & \delta_8 & \cdots & \delta_8
\end{array},
\qquad
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 & 0 \\
0 & 0 & \delta_3 & 0 & 0 \\
0 & 0 & 0 & \delta_4 & 0 \\
0 & 0 & 0 & 0 & \delta_5 \\
\delta_6 & 0 & 0 & \delta_6 & \delta_6 \\
0 & \delta_7 & 0 & \delta_7 & \delta_7 \\
0 & 0 & \delta_8 & \delta_8 & \delta_8
\end{array}
$$

$e = 3 \qquad\qquad e = 7 \qquad\qquad e = 21$

where $\delta_i = \pm 1$.

Proof. Case 1. $e = 3$: Clear from Proposition 5.1(i) and the proof of Lemma 4.1.

Case 2. $e = 7$: By Proposition 5.1(ii), $k(B_0) = 8$. Let $\{\chi_1, \ldots, \chi_8\} = \mathrm{Irr}(B_0)$. By the proof of Proposition 5.1(ii), $G$ has an involution $x$ with $\ell(b_x) = 1$. By Lemma 1.9, we get $\chi_i(x) = \pm 1$ for all $i$. On the other hand, $\Sigma_{i=1}^{8} \chi_i(x)\chi_i = 0$ on 2'-elements of $G$ from [10, Theorem 63.3(1)]. Thus, the assertion is proved.

Case 3. $e = 21$: Let $z$ ba an involution in $G$. By the proof of Lemma 4.1, the generalized decomposition matrix of $B_0$ relative to $z$ with respect to some basic set of $b_z$ has the same form as in Case 1. Hence, by [10, Theorem 63.3(1)], we can verify the proposition.

Lemma 5.3. Assume $e = 21$, $O(G) = 1$, $O'(G) = SL(2,8)$ and $G$ has a normal subgroup $H$ of odd prime index with $e(H) = 7$. Then for any involution $z$ in $G$ we get

$$\chi_i(1) = \begin{cases} 1 & \text{for } i = 1,2,3 \\ 7 & \text{for } i = 4,5,6 \\ 21 & \text{for } i = 7 \\ 27 & \text{for } i = 8 \end{cases} \qquad \chi_i(z) = \begin{cases} 1 & \text{for } i = 1,2,3 \\ -1 & \text{for } i = 4,5,6 \\ -3 & \text{for } i = 7 \\ 3 & \text{for } i = 8 \end{cases}$$

where $\{\chi_1 = 1_G, \chi_2, \ldots, \chi_8\} = \mathrm{Irr}(B_0)$.

Proof. Let $S = O'(G) = SL(2,8)$. By Lemmas 1.12 and 1.14, we can write $N_S(P) = \langle s, C_S(P)\rangle$, $N_H(P) = \langle s, C_H(P)\rangle$ and $N_G(P) = \langle s, t, C_G(P)\rangle$ for some $s \in N_S(P)$ and $t \in N_G(P)$ such that $s$ and $t$ have orders 7 and 3 modulo $C_G(P)$, respectively. Clearly, $G/H = \langle tH\rangle$. Let $b_0 = B_0(H)$, and let $C_G(P) = P \times V$. By Proposition 5.2, $B_0 \not\approx b_0$. Hence $VH = H$ from Proposition 1.6.

Then $C_G(P) = C_H(P)$ and $|G:H| = 3$. We may assume $z \in P$. Let $M = C_G(z)$. By the proof of Lemma 2.1, $C_S(z)$ is a 2-nilpotent normal subgroup of $M$, so that $M$ is solvable. By the proof of Lemma 4.1, $e(M) = 3$. Thus, by Lemma 1.1, $B_o(M) \cong B_o(PZ_3)$ where $PZ_3$ is the semi-direct product of its normal subgroup $P$ by $Z_3$ and it is not the direct product $P \times Z_3$. Thus, as in the proof of Lemma 4.1 we know the generalized decomposition numbers of $B_o$ relative to $z$. So we can write

$$(*) \qquad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i = 1,\dots,6 \\ \pm 3 & \text{for } i = 7,8 \end{cases}$$

for suitable indexing of $\chi_2, \dots, \chi_8$. By Lemma 2.1, $b_o \cong B_o(S)$. Hence, by [10, Theorem 38.2],

$$(**) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i = 1 \\ 7 & \text{for } i = 2,\dots,5 \\ 9 & \text{for } i = 6,7,8 \end{cases} \qquad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i = 1 \\ -1 & \text{for } i = 2,\dots,5 \\ 1 & \text{for } i = 6,7,8 \end{cases}$$

where $\{\tilde{\chi}_1, \dots, \tilde{\chi}_8\} = \mathrm{Irr}(b_o)$. Since $|G:VH| = |G:H| = 3$, we get $3 \mid |G:VG'|$. By Proposition 1.5, $k'(B_o) = |G:VG'|$. By (**), $k'(b_o) = 1$, so that $|G:VG'| = 3$ from Frobenius reciprocity. So we may assume that $\chi_1|_H = \chi_2|_H = \chi_3|_H = \tilde{\chi}_1$ from (*), (**) and Proposition 1.3. Similarly, we may also assume that $\chi_7|_H = \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5$ and $\chi_8|_H = \tilde{\chi}_6 + \tilde{\chi}_7 + \tilde{\chi}_8$. Then we get $\chi_4|_H = \chi_5|_H = \chi_6|_H = \tilde{\chi}_2$. This completes the proof.

The next theorem is the main result of this section.

Theorem 5.4. Let $\overline{G} = G/O(G)$ and $S = O'(\overline{G})$. If $G$ is nonsolvable, $e = 21$ and $e(S) \neq 21$, then we have the following.

(i) $S \cong SL(2,8)$.

(ii) For any subnormal subgroup $\overline{L}$ of $\overline{G}$ of odd index with

$e(\bar{L}) = 21$, $B_0 \cong B_0(\bar{L})$.

Proof. We may assume $O(G) = 1$ by Lemma 1.1, so that $S = O'(G)$.

(i) Noncyclic groups of order 21 have no normal subgroups of order 3. Thus, by Lemma 1.14, $e(S) = 7$. Then $S \cong SL(2,8)$ from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to show that

$$(*) \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an} \\ \text{odd prime and } e(L) = 21 \text{ and if } H \text{ is a normal subgroup} \\ \text{of } L \text{ such that } |L:H| \text{ is an odd prime and } e(H) = 7, \\ \text{then } B_0 \cong B_0(L). \end{cases}$$

Let $b_0 = B_0(L)$, and let $z$ ba an involution in $G$. By Lemma 5.3, we get

$$(**) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i = 1,2,3 \\ 7 & \text{for } i = 4,5,6 \\ 21 & \text{for } i = 7 \\ 27 & \text{for } i = 8 \end{cases} \qquad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i = 1,2,3 \\ -1 & \text{for } i = 4,5,6 \\ -3 & \text{for } i = 7 \\ 3 & \text{for } i = 8 \end{cases}$$

where $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_8\} = \mathrm{Irr}(b_0)$. As in the proof of Lemma 5.3, using the generalized decomposition numbers of $B_0$ relative to $z$,

$$(***) \qquad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i = 1,\ldots,6 \\ \pm 3 & \text{for } i = 7, 8 \end{cases}$$

where $\{\chi_1, \ldots, \chi_8\} = \mathrm{Irr}(B_0)$. Since $|G:L|$ is an odd prime, $I_G(\tilde{\chi}_7) = I_G(\tilde{\chi}_8) = G$ from (**). Thus, by Proposition 1.4, Clifford's theorem, (**) and (***), we may assume that $\chi_7|_L = \tilde{\chi}_7$ and $\chi_8|_L = \tilde{\chi}_8$. By Clifford's theorem, Proposition 1.3, (***) and (**), we have $\chi_i|_L \in \mathrm{Irr}(b_0)$ for $i = 1,\ldots,6$. Thus, by Proposition 1.4, we may assume that $\chi_i|_L = \tilde{\chi}_i$ for $i = 1,\ldots,6$.

These show $I_G(\tilde{\chi}_j) = G$ for all $\tilde{\chi}_j \in \mathrm{Irr}(b_o)$. By Proposition 5.1(3), $k(B_o) = k(b_o)$ and $\ell(B_o) = \ell(b_o)$. Thus, $B_o \simeq b_o$ from Corollary 1.7. Then, (*) is proved. Since $G/S$ is solvable by [12, Theorem], by repeating the above way, we can prove (ii).

Remark 1. If $G$ is solvable, we easily know $B_o$ since we may assume $O(G) = 1$ from Lemma 1.1. Assume $G$ is nonsolvable. If $e = 3$ or $7$, we know $B_o$ from Theorem 2.4. If $e = 21$, we know $B_o$ from Theorems 4.8 and 5.4.

Remark 2. By Remark 1 of §2, there is a finite group $G$ with elementary abelian Sylow 2-subgroups of order 8 such that $e(G) = 21$ and $e(S) = 7$ where $S = O'(G/O(G))$.

## 6. The case when $P$ is elementary abelian of order 16

In this section we deal with the case when $G$ has elementary abelian Sylow 2-subgroups of order 16. Specially, we are interested in the case where $e$ is not prime. When $e$ is $9$ or $21$, the similar phenomenon to Theorem 5.4 occurs. Throughout this section we assume that $G$ has an elementary abelian Sylow 2-subgroup $P$ of order $16$ and we use the notation $e$ and $B_o$ as usual.

By Lemma 1.13 and Remark 1 of §1, it is enough to consider the cases when $e = 3, 5, 7, 9, 15$ and $21$.

Proposition 6.1. If $G$ is solvable and $e = 3$, then one of the following holds.

(i) $B_o \simeq B_o(M)$ where $M$ is a semi-direct product of its normal subgroup $P$ by $\langle t\rangle$ such that $P = \langle x, y, z, w\rangle$ is elementary abelian of order 16, $\langle t \rangle$ is cyclic of order 3, $x^t = y$, $y^t = xy$, $z^t = w$ and $w^t = zw$. In this case $k(B_o) = 8$.

(ii) $B_o \cong B_o(L)$ where $L$ is a semi-direct product of its normal subgroup $P$ by $\langle t \rangle$ such that $P = \langle x, y, z, w \rangle$ is elementary abelian of order 16, $\langle t \rangle$ is cyclic of order 3, $x^t = x$, $y^t = y$, $z^t = w$ and $w^t = zw$. In this case $k(B_o) = 16$.

Proof. By Lemma 1.1, we may assume $O(G) = 1$. Hence $G$ is a semi-direct product of its normal subgroup $P$ by $Z_3$ and $G$ is not the direct product $P \times Z_3$. Let $G = P\langle t \rangle$ where $\langle t \rangle$ is cyclic of order 3, and let $P = \langle x, y, z, w \rangle$. We may assume that

(i) $x^t = y$, $y^t = xy$, $z^t = w$, $w^t = zw$

or

(ii) $x^t = x$, $y^t = y$, $z^t = w$, $w^t = zw$.

Then we can easily prove the assertion.

Proposition 6.2. Let $D$ be the decomposition matrix of $B_o$. If $e = 3$, then we have the following.

(i) When $G$ is solvable, $D$ has the form

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} , \text{ or } \qquad \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} .$$

$k(B_o) = 8$ $\qquad$ $k(B_o) = 16$

(ii) When $G$ is nonsolvable, we obtain $D$ from Theorem 2.4(2) and Lemma 1.16(ii).

Proof. The assertion is proved by Proposition 6.1.

Proposition 6.3. If $e = 5$, then $G$ is solvable, $B_0 \cong B_0(PZ_5)$ where $PZ_5$ is the semi-direct product of its normal subgroup $P$ by $Z_5$ and it is not the direct product $P \times Z_5$, and the decomposition matrix of $B_0$ has the form

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} .$$

Proof. By Proposition 1.10 and Lemma 1.12, $G$ is solvable since we may assume $O(G) = 1$ by Lemma 1.1. Hence $G$ is the semi-direct product of $P$ by $Z_5$, and it is not the direct product $P \times Z_5$. The decomposition matrix of $B_0$ is easily obtained.

Proposition 6.4. If $e = 7$, then there is a basic set $W$ of $B_0$ such that $W$ contains the trivial Brauer character and the decomposition matrix of $B_0$ with respect to $W$ has the form

$$\begin{array}{cccccc} & 1_G & & & & \\ 1_G & 1 & & & & \\ & \delta_2 & & O & & \\ & & \delta_3 & & & \\ & & \delta_4 & \ddots & & \\ & & O & & \ddots & \delta_{13} \\ & & & & & \delta_{14} \\ & \delta_{15} & \delta_{15} & \cdots & & \delta_{15} \\ & \delta_{16} & \delta_{16} & \cdots & & \delta_{16} \end{array}$$

where $\delta_i = \pm 1$.

Proof. As in the proof of Lemma 4.1 we can prove the assertion by Proposition 5.2.

Proposition 6.5. Suppose $k(B_0) = 16$.

(1) If $G$ has an involution $x$ with $b_x \cong B_0(PZ_3)$ where $PZ_3$ is a semi-direct product of $P$ by $Z_3$ and it is not the direct product $P \times Z_3$, then the generalized decomposition matrix $D^x$ of $B_0$ relative to $x$ has the form (*).

(2) If $G$ has an involution $x$ with $b_x \cong B_0(Z_2 \times Z_2 \times L_2(q))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$, then the generalized decomposition matrix $D^x$ of $B_0$ relative to $x$ is as follows:

(i) When $3 < q \equiv 3 \pmod 8$, $D^x$ has the form (*).

(ii) When $3 < q \equiv 5 \pmod 8$, $D^x$ has the form (**).

$$
\begin{array}{c|ccc}
 & 1_M & & \\
1_G & 1 & 0 & 0 \\
 & \delta_2 & 0 & 0 \\
 & \delta_3 & 0 & 0 \\
 & \delta_4 & 0 & 0 \\
 & 0 & \delta_5 & 0 \\
 & 0 & \delta_6 & 0 \\
 & 0 & \delta_7 & 0 \\
 & 0 & \delta_8 & 0 \\
 & 0 & 0 & \delta_9 \\
 & 0 & 0 & \delta_{10} \\
 & 0 & 0 & \delta_{11} \\
 & 0 & 0 & \delta_{12} \\
 & \delta_{13} & \delta_{13} & \delta_{13} \\
 & \delta_{14} & \delta_{14} & \delta_{14} \\
 & \delta_{15} & \delta_{15} & \delta_{15} \\
 & \delta_{16} & \delta_{16} & \delta_{16}
\end{array}
\qquad
\begin{array}{c|ccc}
 & 1_M & & \\
1_G & 1 & 0 & 0 \\
 & \delta_2 & 0 & 0 \\
 & \delta_3 & 0 & 0 \\
 & \delta_4 & 0 & 0 \\
 & \delta_5 & \delta_5 & 0 \\
 & \delta_6 & \delta_6 & 0 \\
 & \delta_7 & \delta_7 & 0 \\
 & \delta_8 & \delta_8 & 0 \\
 & \delta_9 & 0 & \delta_9 \\
 & \delta_{10} & 0 & \delta_{10} \\
 & \delta_{11} & 0 & \delta_{11} \\
 & \delta_{12} & 0 & \delta_{12} \\
 & \delta_{13} & \delta_{13} & \delta_{13} \\
 & \delta_{14} & \delta_{14} & \delta_{14} \\
 & \delta_{15} & \delta_{15} & \delta_{15} \\
 & \delta_{16} & \delta_{16} & \delta_{16}
\end{array}
$$

(*) (**)

where $\delta_i = \pm 1$ and $M = C_G(x)$.

Proof. (1) By Proposition 6.2(i), we know the Cartan matrix of $b_x$. Hence the assertion is proved as in the proof of Lemma 4.1.

(2) We obtain the Cartan matrix of $b_x$ from Lemma 1.16(ii). Thus we can verify (2) as in the proof of (1).

Lemma 6.6. Assume $e = 9$, $O(G) = 1$, $O'(G) = Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$, and $G$ has a normal subgroup $H$ of odd prime index with $e(H) = 3$. Let $b_o = B_o(H)$, and let $x$ and $z$ be involutions in $Z(O'(G))$ and $L_2(q)$, respectively. Then we have the following.

(i) $\chi_i|_H = \chi_{i+1}|_H = \chi_{i+2}|_H = \tilde{\chi}_i$ for $i = 1, 5, 9, 13$

$\chi_j|_H = \tilde{\chi}_{j-2} + \tilde{\chi}_{j-1} + \tilde{\chi}_j$ for $j = 4, 8, 12, 16$

and the values $\tilde{\chi}_i(1)$, $\tilde{\chi}_i(x)$, $\tilde{\chi}_i(z)$ and $\tilde{\chi}_i(xz)$ are as follows:

| | 1 | x | z | xz |
|---|---|---|---|---|
| $\tilde{\chi}_1$, $\tilde{\chi}_2$ | 1 | 1 | 1 | 1 |
| $\tilde{\chi}_3$, $\tilde{\chi}_4$ | 1 | -1 | 1 | -1 |
| $\tilde{\chi}_5$, $\tilde{\chi}_6$ | $(q+\varepsilon)/2$ | $(q+\varepsilon)/2$ | $-\varepsilon$ | $-\varepsilon$ |
| $\tilde{\chi}_7$, $\tilde{\chi}_8$ | $(q+\varepsilon)/2$ | $-(q+\varepsilon)/2$ | $-\varepsilon$ | $\varepsilon$ |
| $\tilde{\chi}_9$, $\tilde{\chi}_{10}$ | $(q+\varepsilon)/2$ | $(q+\varepsilon)/2$ | $-\varepsilon$ | $-\varepsilon$ |
| $\tilde{\chi}_{11}$, $\tilde{\chi}_{12}$ | $(q+\varepsilon)/2$ | $-(q+\varepsilon)/2$ | $-\varepsilon$ | $\varepsilon$ |
| $\tilde{\chi}_{13}$, $\tilde{\chi}_{14}$ | $q$ | $q$ | $\varepsilon$ | $\varepsilon$ |
| $\tilde{\chi}_{15}$, $\tilde{\chi}_{16}$ | $q$ | $-q$ | $\varepsilon$ | $-\varepsilon$ |

where $\{\chi_1 = 1_G, \chi_2, \ldots, \chi_{16}\} = \mathrm{Irr}(B_o)$, $\{\tilde{\chi}_1 = 1_H, \tilde{\chi}_2, \ldots, \tilde{\chi}_{16}\} = \mathrm{Irr}(b_o)$, and $\varepsilon = -1$ if $q \equiv 3 \pmod 8$; $\varepsilon = 1$ if $q \equiv 5 \pmod 8$.

(ii) $\phi_i|_H = \phi_{i+1}|_H = \phi_{i+2}|_H = \tilde{\phi}_{(i+2)/3}$ for $i = 1, 4, 7$

$\tilde{\phi}^G_{j/3} = \phi_{j-2} + \phi_{j-1} + \phi_j$ for $j = 3, 6, 9$

where $\{\phi_1 = 1_G, \phi_2, \ldots, \phi_9\} = IBr(B_o)$ and $\{\tilde{\phi}_1 = 1_H, \tilde{\phi}_2, \tilde{\phi}_3\} = IBr(b_o)$.

Proof. Let $S = O'(G) = \langle x, y\rangle \times L_2(q)$ and $P = \langle x, y, z, w\rangle$ where $\langle z, w\rangle$ is a Sylow 2-subgroup of $L_2(q)$. We can write $N_S(P) = \langle s, C_S(P)\rangle$ for some $s \in N_S(P)$. We may assume $z^s = w$ and $w^s = zw$. We can also write $N_G(P) = \langle s, t, C_G(P)\rangle$ for some $t \in N_G(P)$ where $s$ and $t$ have order 3 modulo $C_G(P)$ since $e = 9$ (cf. Lemma 1.13). We may assume $x^t = y$, $y^t = xy$, $z^t = z$ and $w^t = w$. As in the proof of Lemma 5.3, we get $G/H = \langle tH\rangle$, $C_G(P) = C_H(P)$ and $|G:H| = 3$. By [10, Lemma 18.5], $\{1, x, z, xz\}$ is the set of all representatives of G-conjugate classes of P. As before, $\ell(b_x) = \ell(b_z) = 3$ and $\ell(b_{xz}) = 1$. By Lemma 3.1, $k(B_o) = 16$. Since S is normal in $C_G(x)$ and $e(C_G(x)) = 3$, it follows from Lemma 2.3 that $b_x \simeq B_o(Z_2 \times Z_2 \times L_2(q))$. Thus, by Lemmas 1.16(i) and 6.5(2), we may assume

$$(*) \qquad \chi_i(x) = \begin{cases} \pm 1 & \text{for } i = 1,\ldots,4 \\ \pm(q+\varepsilon)/2 & \text{for } i = 5,\ldots,12 \\ \pm q & \text{for } i = 13,\ldots,16. \end{cases}$$

Since $e(H) = 3$, by Lemma 2.3, $b_o \simeq B_o(S)$. Let $C_G(P) = P \times V$. By [10, Theorem 18.4], $P \cap G' = P$, so that $|G:VG'|$ is odd. Since $b_o \simeq B_o(S)$ and since $C_H(P) = P \times V$, by Proposition 1.5, $|H:VH'| = 4$. Thus, $|G:VG'| = 3$, so that $k'(B_o) = 3$ from Proposition 1.5. Since $b_o \simeq B_o(S)$, by [10, Theorem 38.1], we know the values of $\tilde{\chi}_i|_S$ for all $i$. Then we get the table in (i). Using this we may assume that

$$I_G(\tilde{\chi}_i) = G \quad \text{for} \quad i = 1,5,9,13$$

$$(**) \qquad \left.\begin{array}{l} I_G(\tilde{\chi}_{j-2}) = I_G(\tilde{\chi}_{j-1}) = I_G(\tilde{\chi}_j) = H \\ \tilde{\chi}^t_{j-2} = \tilde{\chi}_{j-1}, \quad \tilde{\chi}^t_{j-1} = \tilde{\chi}_j \end{array}\right\} \quad \text{for} \quad j = 4,8,12,16.$$

By (*) and (**), we may assume that $\chi_1|_H = \chi_2|_H = \chi_3|_H = \tilde{\chi}_1$. Since $\tilde{\chi}_2(x) + \tilde{\chi}_3(x) + \tilde{\chi}_4(x) = -1$, by Proposition 1.4, (*) and (**), we get $\chi_4|_H = \tilde{\chi}_2 + \tilde{\chi}_3 + \tilde{\chi}_4$. Similarly, we may assume that $\chi_j|_H = \tilde{\chi}_{j-2} + \tilde{\chi}_{j-1} + \tilde{\chi}_j$ for $j = 8, 12, 16$. We may also assume that $\chi_i|_H = \chi_{i+1}|_H = \chi_{i+2}|_H = \tilde{\chi}_i$ for $i = 5, 9, 13$ using Frobenius reciprocity, (*) and (**). This completes the proof of (i). Since $b_0 \cong B_0(S)$, by Lemma 1.16(i), $\tilde{\varnothing}_2(1) = \tilde{\varnothing}_3(1) = (q-1)/2$. Thus, $I_G(\tilde{\varnothing}_j) = G$ for $j = 1, 2, 3$ since $|G:H| = 3$. For all $\varnothing_i \in IBr(B_0)$ we have $\varnothing_i|_H \in IBr(b_0)$ by Clifford's theorem since $|G:H| = 3$. Thus, by [15, V 16.6 Satz], we get (ii) for suitable indexing of $\varnothing_2, \ldots, \varnothing_9$. This completes the proof of the lemma.

Proposition 6.7. Assume as in Lemma 6.6. Then the decomposition matrix $D$ of $B_0$ is as follows.

(i) $3 < q \equiv 3 \pmod 8$:

$$D = \begin{array}{ccc} \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} & O & O \\ O & \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} & O \\ O & O & \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} \\ \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} & \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} & \begin{matrix} 1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{matrix} \end{array} .$$

(ii) $3 < q \equiv 5 \pmod 8$:

$$D = \begin{array}{ccccccccc}
1&0&0&&&&&&\\
0&1&0&&\mathbf{0}&&&\mathbf{0}&\\
0&0&1&&&&&&\\
1&1&1&&&&&&\\
1&0&0&1&0&0&&&\\
0&1&0&0&1&0&&\mathbf{0}&\\
0&0&1&0&0&1&&&\\
1&1&1&1&1&1&&&\\
1&0&0&&&&1&0&0\\
0&1&0&&\mathbf{0}&&0&1&0\\
0&0&1&&&&0&0&1\\
1&1&1&&&&1&1&1\\
1&0&0&&&&1&0&0\\
0&1&0&&(*)&&0&1&0\\
0&0&1&&&&0&0&1\\
1&1&1&1&1&1&1&1&1
\end{array},$$

where (*) is one of the following types

$$\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array},\quad \begin{array}{ccc}1&0&0\\0&0&1\\0&1&0\end{array},\quad \begin{array}{ccc}0&1&0\\1&0&0\\0&0&1\end{array},\quad \begin{array}{ccc}0&1&0\\0&0&1\\1&0&0\end{array},\quad \begin{array}{ccc}0&0&1\\1&0&0\\0&1&0\end{array},\quad \begin{array}{ccc}0&0&1\\0&1&0\\1&0&0\end{array}.$$

Proof. We use the same notation as in Lemma 6.6. Let $\tilde{D}$ be the decomposition matrix of $b_o$, and let $D = (d_{j\lambda})_{j,\lambda}$. Let $\chi_j|_H = \Sigma_i \ell_{ij}\tilde{\chi}_i$ for each $j$, and $L = (\ell_{ij})_{i,j}$. Similarly, let $\tilde{\varnothing}_\kappa^G = \Sigma_\lambda \beta_{\kappa\lambda}\varnothing_\lambda$ for each $\kappa$, and $B = (\beta_{\kappa\lambda})_{\kappa,\lambda}$. By [7, §26],

(1) $$\tilde{D}B = LD.$$

(i) Since $b_o \cong B_o(S)$, by (1) and Lemmas 1.16(ii) and 6.6,

(2) $$D = \begin{array}{ccccccccc}
1&0&0&&&&&&\\
0&1&0&&\mathbf{0}&&&\mathbf{0}&\\
0&0&1&&&&&&\\
1&1&1&&&&&&\\
&&&1&0&0&&&\\
&\mathbf{0}&&0&1&0&&\mathbf{0}&\\
&&&0&0&1&&&\\
&&&1&1&1&&&\\
&&&&&&1&0&0\\
&\mathbf{0}&&&\mathbf{0}&&0&1&0\\
&&&&&&0&0&1\\
&&&&&&1&1&1\\
&?&&&?&&&?&\\
1&1&1&1&1&1&1&1&1
\end{array}$$

and

(3) $$d_{13,\lambda} + d_{14,\lambda} + d_{15,\lambda} = 1 \quad \text{for all } \lambda = 1,\ldots,9.$$

By Lemma 6.6,

$$(4) \qquad D^{xz} = (d^{xz}_{i1})_i = \begin{matrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{matrix}$$

where $D^{xz}$ is the generalized decomposition matrix of $B_o$ relative to xz. Clearly, S is normal in $C_G(x)$. By the proof of Lemma 3.1, $e(C_G(x)) = 3$. Since $q \equiv 3 \pmod 8$, by Lemmas 2.3 and 6.5(2),

$$(5) \qquad (d^x_{\gamma_i,\alpha})_{i,\alpha} = \begin{matrix} & \phi^x_1 & \phi^x_2 & \phi^x_3 \\ 1_G = {}^x\gamma_1 & 1 & 0 & 0 \\ {}^x\gamma_2 & \delta_2 & 0 & 0 \\ {}^x\gamma_3 & \delta_3 & 0 & 0 \\ {}^x\gamma_4 & \delta_4 & 0 & 0 \\ {}^x\gamma_5 & 0 & \delta_5 & 0 \\ {}^x\gamma_6 & 0 & \delta_6 & 0 \\ {}^x\gamma_7 & 0 & \delta_7 & 0 \\ {}^x\gamma_8 & 0 & \delta_8 & 0 \\ {}^x\gamma_9 & 0 & 0 & \delta_9 \\ {}^x\gamma_{10} & 0 & 0 & \delta_{10} \\ {}^x\gamma_{11} & 0 & 0 & \delta_{11} \\ {}^x\gamma_{12} & 0 & 0 & \delta_{12} \\ {}^x\gamma_{13} & \delta_{13} & \delta_{13} & \delta_{13} \\ {}^x\gamma_{14} & \delta_{14} & \delta_{14} & \delta_{14} \\ {}^x\gamma_{15} & \delta_{15} & \delta_{15} & \delta_{15} \\ {}^x\gamma_{16} & \delta_{16} & \delta_{16} & \delta_{16} \end{matrix}$$

where $d^x_{i\alpha}$ are the generalized decomposition numbers of $B_o$ relative to $x$, $\delta_i = \pm 1$, $\chi_{\gamma_1} = \chi_1 = 1_G$, $\{\chi_{\gamma_2}, \ldots, \chi_{\gamma_{16}}\} = \{\chi_2, \ldots, \chi_{16}\}$, $\{\phi^x_1 = 1_M, \phi^x_2, \phi^x_3\} = \mathrm{IBr}(b_x)$ and $M = C_G(x)$. Let $c = \overline{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$ in $L_2(q)$. Then, by [10, Theorem 38.1] and Lemma 1.16, we may assume that

(6) $$\phi^x_2(c) = (-1+\sqrt{-q})/2 \quad \text{and} \quad \phi^x_3(c) = (-1-\sqrt{-q})/2.$$

By Lemma 6.6 and (5), $\{\chi_{\gamma_1},\ldots,\chi_{\gamma_4}\} = \{\chi_1,\ldots,\chi_4\}$ and $\{\chi_{\gamma_{13}},\ldots,\chi_{\gamma_{16}}\} = \{\chi_{13},\ldots,\chi_{16}\}$. We may assume that $\chi_5 \in \{\chi_{\gamma_5},\ldots,\chi_{\gamma_8}\}$. By Lemma 6.6, $\chi_5|_H = \chi_6|_H = \chi_7|_H$. Thus, by (5) and (6), we get that $\chi_6$ and $\chi_7$ are both in $\{\chi_{\gamma_5},\ldots,\chi_{\gamma_8}\}$. Similarly, none of $\{\chi_9, \chi_{10}, \chi_{11}\}$ are in $\{\chi_{\gamma_5},\ldots,\chi_{\gamma_8}\}$. Hence, by (2), (5) and (6), we know $\{\chi_{\gamma_5},\ldots,\chi_{\gamma_8}\} = \{\chi_5,\ldots,\chi_8\}$. Thus, $\{\chi_{\gamma_9},\ldots,\chi_{\gamma_{12}}\} = \{\chi_9,\ldots,\chi_{12}\}$. Hence we may assume that $\chi_{\gamma_i} = \chi_i$ for all $i = 1,\ldots,16$. Therefore, by Lemma 6.6,

(7) $$(d^x_{i\alpha})_{i,\alpha} = \begin{array}{ccc} \phi^x_1 & \phi^x_2 & \phi^x_3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{array} .$$

Next, we want to know the generalized decomposition numbers $d^z_{i\alpha}$ of $B_o$ relative to $z$. Let $L = C_G(z)$. As for $x$, $e(L) = 3$ and

$\ell(B_o(L)) = 3$. Since $N_L(P) = \langle t, C_L(P)\rangle$ and $z^t = z$, we get from Proposition 6.1 and Theorem 2.4(1) that $k(B_o(L)) = k(b_z) = 16$. By Theorem 2.4(2) and Lemmas 6.6 and 6.5, L is solvable, so that $b_z \cong B_o(PZ_3)$ from Proposition 6.1 where $PZ_3$ is a semi-direct product of its normal subgroup P by $Z_3$ and it is not the direct product $P \times Z_3$. Thus, by Lemma 6.5,

$$(8) \qquad (d^z_{\gamma_i,\alpha})_{i,\alpha} = \begin{array}{c|ccc} & \phi^z_1 & \phi^z_2 & \phi^z_3 \\ 1_G = \chi_{\gamma_1} & 1 & 0 & 0 \\ \chi_{\gamma_2} & 0 & \delta_2 & 0 \\ \chi_{\gamma_3} & 0 & 0 & \delta_3 \\ \chi_{\gamma_4} & \delta_4 & \delta_4 & \delta_4 \\ \chi_{\gamma_5} & \delta_5 & 0 & 0 \\ \chi_{\gamma_6} & 0 & \delta_6 & 0 \\ \chi_{\gamma_7} & 0 & 0 & \delta_7 \\ \chi_{\gamma_8} & \delta_8 & \delta_8 & \delta_8 \\ \chi_{\gamma_9} & \delta_9 & 0 & 0 \\ \chi_{\gamma_{10}} & 0 & \delta_{10} & 0 \\ \chi_{\gamma_{11}} & 0 & 0 & \delta_{11} \\ \chi_{\gamma_{12}} & \delta_{12} & \delta_{12} & \delta_{12} \\ \chi_{\gamma_{13}} & \delta_{13} & 0 & 0 \\ \chi_{\gamma_{14}} & 0 & \delta_{14} & 0 \\ \chi_{\gamma_{15}} & 0 & 0 & \delta_{15} \\ \chi_{\gamma_{16}} & \delta_{16} & \delta_{16} & \delta_{16} \end{array}$$

where $\delta_i = \pm 1$, $\chi_{\gamma_1} = \chi_1 = 1_G$, $\{\chi_{\gamma_2},\ldots,\chi_{\gamma_{16}}\} = \{\chi_2,\ldots,\chi_{16}\}$ and $\{\phi^z_1, \phi^z_2, \phi^z_3\} = IBr(b_z)$. Clearly, $\phi^z_1(1) = \phi^z_2(1) = \phi^z_3(1) = 1$. Hence, by Lemma 6.6 and (8),

$$(9) \qquad \begin{aligned} \{\chi_{\gamma_4}, \chi_{\gamma_8}, \chi_{\gamma_{12}}, \chi_{\gamma_{16}}\} &= \{\chi_4, \chi_8, \chi_{12}, \chi_{16}\} \\ \{\delta_4, \delta_8, \delta_{12}, \delta_{16}\} &= \{1, 1, 1, -1\}. \end{aligned}$$

By Lemma 6.6, $\chi_i(z) = \chi_{i+1}(z) = \chi_{i+2}(z) = 1$ for $i = 1, 5, 9$. So it follows from (4), (7), (8) and [10, Theorem 63.3] that $\delta_4 = \delta_8 = \delta_{12} = 1$ and $\delta_{16} = -1$. Thus, again by (4), (7), (8) and [10, Theorem 63.3],

$$
(10) \qquad (d^z_{i\alpha})_{i,\alpha} = \begin{array}{ccc}
\phi^z_1 & \phi^z_2 & \phi^z_3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
-1 & -1 & -1
\end{array}
$$

for suitable indexing. By (2), (3), (10) and [10, Theorem 63.3],

$$
D = \begin{array}{c|ccccccccc}
\chi_{13} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\chi_{14} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\chi_{15} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\chi_{16} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array} .
$$

This completes the proof of (i).

(ii) Since $b_o \cong B_o(S)$, as in the proof of (i) we get

$$
(11) \qquad D = \begin{array}{ccccccccc}
1 & 0 & 0 & & & & & & \\
0 & 1 & 0 & & 0 & & & 0 & \\
0 & 0 & 1 & & & & & & \\
1 & 1 & 1 & & & & & & \\
 & ? & & & ? & & & 0 & \\
1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & ? & & & 0 & & & ? & \\
1 & 1 & 1 & & & & 1 & 1 & 1 \\
 & ? & & & ? & & & ? & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

and

$$(12)\qquad \begin{aligned} &d_{5\lambda} + d_{6\lambda} + d_{7\lambda} = 1 \quad \text{for} \quad \lambda = 1,\ldots,6 \\ &d_{9\lambda} + d_{10,\lambda} + d_{11,\lambda} = 1 \quad \text{for} \quad \lambda = 1,2,3,7,8,9 \\ &d_{13,\lambda} + d_{14,\lambda} + d_{15,\lambda} = 1 \quad \text{for} \quad \lambda = 1,\ldots,9. \end{aligned}$$

As in the proof of (i), we have

$$(13)\quad (d^{xz}_{i1})_i = \begin{array}{r} 1\\1\\1\\-1\\-1\\-1\\-1\\1\\-1\\-1\\-1\\1\\1\\1\\1\\-1, \end{array} \quad (d^{x}_{i\alpha})_{i,\alpha} = \begin{array}{rrr} \varnothing^x_1 & \varnothing^x_2 & \varnothing^x_3 \\ 1&0&0\\1&0&0\\1&0&0\\-1&0&0\\1&1&0\\1&1&0\\1&1&0\\-1&-1&0\\1&0&1\\1&0&1\\1&0&1\\-1&0&-1\\1&1&1\\1&1&1\\1&1&1\\-1&-1&-1, \end{array} \quad (d^{z}_{i\alpha})_{i,\alpha} = \begin{array}{rrr} \varnothing^z_1 & \varnothing^z_2 & \varnothing^z_3 \\ 1&0&0\\0&1&0\\0&0&1\\1&1&1\\-1&0&0\\0&-1&0\\0&0&-1\\-1&-1&-1\\-1&0&0\\0&-1&0\\0&0&-1\\-1&-1&-1\\1&0&0\\0&1&0\\0&0&1\\1&1&1 \end{array}$$

where $d^{xz}_{i1}$, $d^{x}_{i\alpha}$, $d^{z}_{i\alpha}$, $\varnothing^x_\alpha$ and $\varnothing^z_\alpha$ are the same as in (i). By Lemmas 6.6 and 1.16, we know the degrees of all $\chi_i$ and $\varnothing_\alpha$. Thus, by (11), (12), (13) and [10, Theorem 63.3], we may assume

$$(14)\qquad D = \begin{array}{c|cccccc|c} \chi_5 & 1&0&0&1&0&0 & \\ \chi_6 & 0&1&0&0&1&0 & 0 \\ \chi_7 & 0&0&1&0&0&1 & \\ \chi_8 & 1&1&1&1&1&1 & \end{array} .$$

Similarly, we may assume

$$(15)\qquad D = \begin{array}{c|ccc|c|ccc} \chi_9 & & & & & 1&0&0 \\ \chi_{10} & & ? & & 0 & 0&1&0 \\ \chi_{11} & & & & & 0&0&1 \\ \chi_{12} & 1&1&1 & & 1&1&1 \end{array} .$$

So, by (11), (12), (13), (14), (15) and [10, Theorem 63.3],

$$(16)\qquad D = \begin{array}{c|ccc|cccccc} \chi_{13} & & & & 1&0&0&1&0&0 \\ \chi_{14} & & ? & & 0&1&0&0&1&0 \\ \chi_{15} & & & & 0&0&1&0&0&1 \\ \chi_{16} & 1&1&1 & 1&1&1&1&1&1 \end{array} .$$

Thus, considering the degrees of $\chi_i$ and $\phi_\alpha$, by (11)-(16) we get the following six cases:

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_9$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\chi_{10}$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| $\chi_{11}$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| $\chi_{12}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_{13}$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\chi_{14}$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| $\chi_{15}$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| $\chi_{16}$ | 1 | 1 | 1, | 1 | 1 | 1, | 1 | 1 | 1, | 1 | 1 | 1, | 1 | 1 | 1, | 1 | 1 | 1 . |
| | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_1$ | $\phi_2$ | $\phi_3$ |

Thus, for suitable indexing of $\chi_i$ and $\phi_\alpha$, we obtain (ii).

The following theorem is one of the main results of this section.

Theorem 6.8. Let $\overline{G} = G/O(G)$ and $S = O'(\overline{G})$. If $G$ is nonsolvable, $e = 9$ and $e(S) \neq 9$, then we have the following.

(i) $S \cong Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod 8$.

(ii) For any subnormal subgroup $\overline{L}$ of $\overline{G}$ of odd index with $e(\overline{L}) = 9$, $B_0 \cong B_0(\overline{L})$.

Proof. We may assume $O(G) = 1$ by Lemma 1.1, so that $S = O'(G)$.

(i) Since $G$ is nonsolvable, $e(S) = 3$. Thus, we get (i) from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to prove that

(*) if $L$ is a normal subgroup of $G$ such that $|G:L|$ is an odd prime and $e(L) = 9$ and if $H$ is a normal subgroup of $L$ such that $|L:H|$ is an odd prime and $e(H) = 3$, then $B_0 \cong B_0(L)$.

Let $b_o = B_o(L)$. By Lemma 3.1, $k(B_o) = k(b_o) = 16$ and $\ell(B_o) = \ell(b_o) = 9$. We may write $O'(G) = S = \langle x, y\rangle \times L_2(q)$ and $P = \langle x, y, z, w\rangle$ where $\langle z, w\rangle$ is a Sylow 2-subgroup of $L_2(q)$. By the proof of Lemma 6.6, we may assume $x^s = x$, $y^s = y$, $z^s = w$, $w^s = zw$, $x^t = y$, $y^t = xy$, $z^t = z$ and $w^t = w$ where $s, t \in N_L(P)$ and $N_L(P) = \langle s, t, C_L(P)\rangle$. So $N_G(P) = \langle s, t, C_G(P)\rangle$. By the proof of Lemma 3.1, $\ell(b_{xz}) = 1$. Thus, by Lemma 1.9, $\chi_i(xz) = \pm 1$ for all $\chi_i \in \mathrm{Irr}(B_o)$. By Lemma 6.6, we know the values $\tilde{\chi}_j(1)$ and $\tilde{\chi}_j(xz)$ for all $\tilde{\chi}_j \in \mathrm{Irr}(b_o)$. Using this, if $\tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_o)$ and $\tilde{\chi}(1) = \tilde{\chi}'(1)$, then $\tilde{\chi}(xz) = \tilde{\chi}'(xz) = \pm 1$. Hence it follows from Corollary 1.8 that $B_o \cong b_o$. Thus we get (*). On the other hand, G/S is solvable from [12, Theorem]. Hence we can verify (ii) by repeating the above way. This completes the proof.

Proposition 6.9. Let $D$ be the decomposition matrix of $B_o$, and let $S = O'(G/O(G))$. If $e = 9$, then we have the following.

(i) When $G$ is solvable,

$$D = \begin{pmatrix}
1&0&0&&&&&&\\
0&1&0&&\bigcirc&&&\bigcirc&\\
0&0&1&&&&&&\\
&&&1&0&0&&&\\
&\bigcirc&&0&1&0&&\bigcirc&\\
&&&0&0&1&&&\\
&&&&&&1&0&0\\
&\bigcirc&&&\bigcirc&&0&1&0\\
&&&&&&0&0&1\\
1&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&1&1&1&1&1&1&1&1
\end{pmatrix}.$$

(ii) When $G$ is nonsolvable and $e(S) = 9$, we know $D$ from

Theorem 3.4(2) and Lemma 1.16(ii).

(iii) When $G$ is nonsolvable and $e(S) = 3$, we know $D$ from Theorem 6.8 and Proposition 6.7.

Remark 1. There is a finite group $G$ with an elementary abelian Sylow 2-subgroup $P$ of order 16 such that $e(G) = 9$ and $O'(G/O(G)) \cong Z_2 \times Z_2 \times L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5$ (mod 8). Let $\langle z, w\rangle$ be a Sylow 2-subgroup of $L_2(q)$. and let $S = \langle x, y\rangle \times L_2(q)$ and $P = \langle x, y, z, w\rangle$ where $\langle x, y\rangle$ is elementary abelian of order 4. There is an automorphism $r$ of $\langle x, y\rangle$ with $x^r = y$ and $y^r = xy$. We can consider that $r \in \mathrm{Aut}(S)$ if we consider that $r$ is trivial on $L_2(q)$. So there is a semi-direct product $G$ of its normal subgroup $S$ by $\langle r\rangle$. Then, $e(G) = 9$ and $O'(G) = S = Z_2 \times Z_2 \times L_2(q)$.

The next theorem is one of the main results of this section.

Theorem 6.10. If $G$ is nonsolvable and $e = 15$, then $B_0 \cong B_0(SL(2,16))$.

Proof. By Lemma 1.1, we may assume $O(G) = 1$. Let $S = O'(G)$. Since $G$ is nonsolvable, it follows from Proposition 1.10 and Lemma 1.12 that $e(S) \neq 1$ and $e(S) \neq 5$. So that $e(S) = 3$ or 15. Firstly, suppose $e(S) = 3$. By Proposition 1.10 and Lemma 1.12, $S \cong Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or 5 (mod 8). Thus, there is an involution $x \in P \cap Z(S)$. We can write $N_S(P) = \langle s, C_S(P)\rangle$ for some $s \in N_S(P)$. Thus, $x^s = x$. Since $e = 15$, we can write $N_G(P) = \langle t, C_G(P)\rangle$ for some $t \in N_G(P)$. Since $N_S(P)/C_S(P)$ can be considered as a subgroup of $N_G(P)/C_G(P)$ through the canonical mononorphism, we get that

$s \equiv t^{5i} \pmod{C_G(P)}$ for some integer $i$ with $i \not\equiv 0 \pmod 3$. Thus, $x = x^{t^{5i}}$. This is a contradiction. Hence $e(S) = 15$, so that $S \cong SL(2,16)$ from Proposition 1.10 and Lemma 1.12.

We prove $B_0 \cong B_0(S)$ by induction on $|G|$. Let $G \neq S$. Since $G/S$ is solvable by [12, Theorem], $G$ has a normal subgroup $H$ of odd prime index. Let $b_0 = B_0(H)$, and let $z$ be an involution in $P$. Since $b_0 \cong B_0(S)$ by induction, we get $k(b_0) = 16$ and

$$(*) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i = 1 \\ 15 & \text{for } i = 2,\dots,9 \\ 17 & \text{for } i = 10,\dots,16, \end{cases} \quad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i = 1 \\ -1 & \text{for } i = 2,\dots,9 \\ 1 & \text{for } i = 10,\dots,16 \end{cases}$$

using [10, Theorem 38.2], where $\{\tilde{\chi}_1,\dots,\tilde{\chi}_{16}\} = \mathrm{Irr}(b_0)$. Since all involutions in $P$ are G-conjugate, $P \cap G' = P$ by [10, Theorem 18.4]. Thus, $k'(B_0)$ is odd from Proposition 1.5. Now, we want to claim that $k(B_0) = 16$. If $k'(B_0) = 1$, we get from Propositions 1.5 and 1.6 that $k(B_0) = 16$. Suppose $k(B_0) \neq 16$. Since $e = 15$, $\ell(b_z) = 1$. Thus, by Lemma 1.15(2), $k(B_0) = 8$. So that $k'(B_0) = 3$, 5 or 7. Let $\{\chi_1,\dots,\chi_8\} = \mathrm{Irr}(B_0)$.

Case 1. $k'(B_0) = 7$: We may assume $\chi_1(1) = \dots = \chi_7(1) = 1$ and $\chi_8(1) > 1$. By Clifford's theorem, Proposition 1.3 and (*), we have $\chi_1|_H = \dots = \chi_7|_H = \tilde{\chi}_1$. Thus, by Proposition 1.4, $(\chi_8|_H, \tilde{\chi}_j) \neq 0$ for $j = 2,\dots,16$. Then we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$.

Case 2. $k'(B_0) = 5$: We may assume $\chi_i(1) = 1$ for $i = 1,\dots,5$ and $\chi_j(1) > 1$ for $j = 6, 7, 8$. As in case 1 we know $\chi_1|_H = \dots = \chi_5|_H = \tilde{\chi}_1$. Since $k(B_0) \neq k(b_0)$, $B_0 \not\cong b_0$. So that we get from Proposition 1.6 that $G \neq VH$ where $V$ is a subgroup of $G$ with $C_G(P) = P \times V$. Since $k'(B_o) = 5$, $|G:H| = 5$ by Proposition 1.5. So, by Clifford's theorem and Proposition 1.4,

$$\chi_6|_H = \tilde{\chi}_2 + \cdots + \tilde{\chi}_6, \quad \chi_7|_H = \tilde{\chi}_7 + \cdots + \tilde{\chi}_{11},$$
$$\chi_8|_H = \tilde{\chi}_{12} + \cdots + \tilde{\chi}_{16}$$

for suitable indexing of $\tilde{\chi}_2,\ldots,\tilde{\chi}_{16}$. Hence we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$.

Case 3. $k'(B_o) = 3$: Let $\chi_i(1) = 1$ for $i = 1, 2, 3$ and $\chi_j(1) > 1$ for $j = 4,\ldots,8$. As in Case 2, $|G:H| = 3$. Then, by Proposition 1.4, for suitable indexing of $\tilde{\chi}_2,\ldots,\tilde{\chi}_{16}$, we get

$$\chi_4|_H = \tilde{\chi}_2 + \tilde{\chi}_3 + \tilde{\chi}_4,\ \chi_5|_H = \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_7,\ \chi_6|_H = \tilde{\chi}_8 + \tilde{\chi}_9 + \tilde{\chi}_{10}$$
$$\chi_7|_H = \tilde{\chi}_{11} + \tilde{\chi}_{12} + \tilde{\chi}_{13},\ \chi_8|_H = \tilde{\chi}_{14} + \tilde{\chi}_{15} + \tilde{\chi}_{16}.$$

Then we have a contradiction as in Case 2.

Thus, $k(B_o) = 16$. Let $\{\chi_1,\ldots,\chi_{16}\} = \mathrm{Irr}(B_o)$. Since $\ell(b_z) = 1$, $\chi_i(z) = \pm 1$ for $i = 1,\ldots,16$ from Lemma 1.9. Thus, we know from Clifford's theorem, Proposition 1.3 and (*) that $\chi_i|_H \in \mathrm{Irr}(b_o)$ for all $i = 1,\ldots,16$. Hence, by Proposition 1.4, we may assume that $\chi_i|_H = \tilde{\chi}_i$ for all $i = 1,\ldots,16$. This shows $k'(B_o) = 1$. So that $B_o \cong b_o$ from Propositions 1.5 and 1.6. This completes the proof of the theorem.

Proposition 6.11. If $e = 15$, then there is a basic set $W$ of $B_o$ such that $W$ contains the trivial Brauer character and the decomposition matrix of $B_o$ with respect to $W$ has the form

$$\begin{array}{c} 1_G \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{ccccc} 1 & & & & \\ & \delta_2 & & O & \\ & & \ddots & & \\ & O & & \ddots & \\ & & & & \delta_{15} \\ \delta_{16} & \delta_{16} & \cdots & & \delta_{16} \end{array}$$

where $\delta_i = \pm 1$.

Proof. The proof is similar to that of Proposition 5.2 (cf. the case when $e = 7$ in Proposition 5.2).

Lemma 6.12. If $e = 21$, then there is an involution $z \in P$ and there are two elements $s, t \in N_G(P)$ such that $N_G(P) = \langle s, t, C_G(P) \rangle$, $z^s = z$ and $z^t = z$.

Proof. Firstly, we want to prove that

(*) there is an involution $u \in P$ and there are two elements $s, t \in N_G(P)$ such that $N_G(P) = \langle s, t, C_G(P) \rangle$, $s$ and $t$ have orders 3 and 7 modulo $C_G(P)$ respectively, and $u^s = u$.

We may assume $O(G) = 1$ by the proof of Lemma 1.1. Since $S = O'(G)$ is normal in $G$, $e(S) = 1$, 7 or 21. When $e(S) = 7$ or 21, we get (*) from Proposition 1.10 and Lemma 1.12. Assume $e(S) = 1$. Then $P$ is normal in $G$ and $|G:P| = 21$. We can write $G = \langle s, t, P \rangle$ for $s, t \in G$ such that $s$ and $t$ have orders 3 and 7 modulo $P$, respectively. Clearly, there is an involution $y \in P$ with $y^t = y$. Suppose $x^s \neq x$ for all involutions $x \in P$. Then, $e(C_G(y)) = 7$. By Proposition 5.2 and [10, Lemma 66.1], the Cartan matrix of $b_y$ has 2 as an elementary divisor of multiplicity 6. Thus, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $\ell(B_0) \geq 7$ since all involutions in $G$ are conjugate. On the other hand, $\ell(B_0) = 5$ since $G/P$ is noncyclic of order 21 (cf. Lemma 1.13). This is a contradiction. Hence we obtain (*).

Next, we prove the lemma. There is an involution $z \in P$ with $z^t = z$. By (*), there are other two involutions $v, w \in P$

such that $v^s = v$, $w^s = w$ and $u, v, w$ are all distinct. It suffices to show $z \in \{u, v, w\}$. Suppose $z \notin \{u, v, w\}$. Since $z^s \neq z$, we know that $\{1, u, v, w, z, uz, vz, wz\}$ is the set of all representatives of $\langle s\rangle$-conjugate classes of $P$. Since $u \neq z$, we get $u^t \neq u$. Thus, $\{1, u, uz, z\}$ is the set of all representatives of $\langle t\rangle$-conjugate classes of $P$. Hence, by elementary calculation, we get that $v \in \{uz, u^t z, \ldots, u^{t^6} z\}$. Hence no two elements in $\{u, v, w\}$ are conjugate in $G$. These show that all $G$-conjugate classes of $P$ are $\{1\}$, $\{z\}$, $\{u, u^t, \ldots, u^{t^6}\}$ and $\{uz, u^t z, \ldots, u^{t^6} z\}$. Thus, $z^s = z$. This is a contradiction. This completes the proof.

Lemma 6.13. If $e = 21$, then $k(B_0) = 16$ and $\ell(B_0) = 5$.

Proof. Let $s$, $t$ and $z$ be the same as in Lemma 6.12. Hence $s$ and $t$ have orders 3 and 7 modulo $C_G(P)$, respectively. There is an involution $x \in P$ with $x^s = x$ and $x^t \neq x$. Thus, $\{1, x, xz, z\}$ is the set of all representatives of conjugate classes of $G$ of 2-elements by [10, Lemma 18.5]. We may assume $O(G) = 1$ by Lemma 1.1. By Z*-theorem [10, Theorem 67.1], $z \in Z(G)$. These imply from [10, Theorems 68.4 and 65.4] that $k(B_0) = 2\ell(B_0) + \ell(b_x) + \ell(b_{xz})$. Since $e(C_G(x)) = 3$, $\ell(b_x) = 3$ by Theorem 2.4(1). Similarly, $\ell(b_{xz}) = 3$. Since $z \in Z(G)$, as in the proof of Lemma 2.2, we get from Lemma 4.1 and Proposition 1.2 that $\ell(B_0) = 5$, so that $k(B_0) = 16$.

Lemma 6.14. Assume $e = 21$, $O(G) = 1$, $O'(G) = Z_2 \times SL(2,8)$ and $G$ has a normal subgroup $H$ of odd prime index with $e(H) = 7$. Then for any involution $z$ in $SL(2,8)$, we have

$$\chi_1(1) = \dots = \chi_6(1) = 1, \quad \chi_7(1) = \dots = \chi_{12}(1) = 7,$$
$$\chi_{13}(1) = \chi_{14}(1) = 21, \quad \chi_{15}(1) = \chi_{16}(1) = 27,$$
$$\chi_1(z) = \dots = \chi_6(z) = 1, \quad \chi_7(z) = \dots = \chi_{12}(z) = -1,$$
$$\chi_{13}(z) = \chi_{14}(z) = -3, \quad \chi_{15}(z) = \chi_{16}(z) = 3$$

where $\{\chi_1 = 1_G, \chi_2, \dots, \chi_{16}\} = \mathrm{Irr}(B_o)$.

Proof. Let $b_o = B_o(H)$, $S = O'(G) = \langle w\rangle \times SL(2,8)$ and $P = \langle w, x, y, z\rangle$ where $\langle x, y, z\rangle$ is a Sylow 2-subgroup of $SL(2,8)$. As in the proof of Lemma 5.3, $G/H = \langle rH\rangle$ for some $r \in N_G(P)$, $C_G(P) = C_H(P)$ and $|G:H| = 3$. We can write $N_S(P) = \langle t, C_S(P)\rangle$ for some $t \in N_S(P)$. Since $P \cap Z(S) = \langle w\rangle$, it follows from Lemma 6.12 and Z*-theorem [10, Theorem 67.1] that $w \in Z(G)$. Then, by the proof of Lemma 6.13, we may assume that $z^t \neq z$, $z^r = z$ and $\ell(b_z) = 3$. By the proof of Lemma 2.2, $\ell(B_o(C_S(z))) = 1$. So that $C_S(z)$ is 2-nilpotent from [10, Corollary 65.3]. Hence $C_G(z)$ is solvable. Since $e(C_G(z)) = 3$, by Proposition 6.1, $b_z \cong B_o(PZ_3)$ where $PZ_3$ is a semi-direct product of its normal subgroup $P$ by $Z_3$ and it is not the direct product $P \times Z_3$. Then, by Lemma 6.5, we know the generalized decomposition numbers of $B_o$ relative to $z$. This implies

$$(*) \qquad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i = 1,\dots,12 \\ \pm 3 & \text{for } i = 13,\dots,16 \end{cases}$$

for suitable indexing of $\chi_2,\dots,\chi_{16}$. By Lemma 2.3, $b_o \cong B_o(S)$. So, by [10, Theorem 38.2],

$$\tilde{\chi}_1(1) = \tilde{\chi}_2(1) = 1, \quad \tilde{\chi}_3(1) = \dots = \tilde{\chi}_{10}(1) = 7,$$
$$\tilde{\chi}_{11}(1) = \dots = \tilde{\chi}_{16}(1) = 9,$$
$$(**) \qquad \tilde{\chi}_1(z) = \tilde{\chi}_2(z) = 1, \quad \tilde{\chi}_3(z) = \dots = \tilde{\chi}_{10}(z) = -1$$
$$\tilde{\chi}_{11}(z) = \dots = \tilde{\chi}_{16}(z) = 1$$

where $\{\tilde{\chi}_1 = 1_H, \tilde{\chi}_2, \ldots, \tilde{\chi}_{16}\} = \mathrm{Irr}(b_o)$. We can write $C_G(P) = P \times V$. By Theorem 2.4 and Lemma 6.13, we get $\ell(B_o) \neq \ell(b_o) = \ell(B_o(S))$. Hence $B_o \not\cong b_o$, so that $VH \neq G$ by Proposition 1.6. Thus, $|G:VH| = |G:H| = 3$. Hence, by Proposition 1.5, $k'(B_o)$ is divisible by 3. Since $|G:H| = 3$, it follows from Frobenius reciprocity, Proposition 1.3 and (**) that $k'(B_o) \leqslant 6$. By observing the conjugate classes of $G$ of 2-elements, we know $P \cap G' \neq P$ from [10, Theorem 18.4]. Hence $|G:VG'|$ is divisible by 2. Thus, $k'(B_o) = |G:VG'| = 6$ from Proposition 1.5. Then, by (*) and (**), we may assume that

$$\chi_1|_H = \chi_2|_H = \chi_3|_H = \tilde{\chi}_1, \quad \chi_4|_H = \chi_5|_H = \chi_6|_H = \tilde{\chi}_2.$$

Similarly, we may assume that

$$\chi_{13}|_H = \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_7, \quad \chi_{14}|_H = \tilde{\chi}_8 + \tilde{\chi}_9 + \tilde{\chi}_{10},$$

$$\chi_{15}|_H = \tilde{\chi}_{11} + \tilde{\chi}_{12} + \tilde{\chi}_{13}, \quad \chi_{16}|_H = \tilde{\chi}_{14} + \tilde{\chi}_{15} + \tilde{\chi}_{16}.$$

Hence we may assume that

$$\chi_7|_H = \chi_8|_H = \chi_9|_H = \tilde{\chi}_3, \quad \chi_{10}|_H = \chi_{11}|_H = \chi_{12}|_H = \tilde{\chi}_4.$$

Therefore the lemma is proved by (**).

Now, we state the next theorem which is one of the main results of this section.

Theorem 6.15. Let $\overline{G} = G/O(G)$ and $S = O'(\overline{G})$. If $G$ is nonsolvable, $e = 21$ and $e(S) \neq 21$, then we have the following.

(i) $S \cong Z_2 \times SL(2,8)$.

(ii) For any subnormal subgroup $\overline{L}$ of $\overline{G}$ of odd index with $e(\overline{L}) = 21$, $B_o \cong B_o(\overline{L})$.

Proof. We can assume $O(G) = 1$ by Lemma 1.1. Hence

$S = O'(G)$.

(i) By Lemma 1.13, $e(S) = 7$. Hence, by Proposition 1.10 and Lemma 1.12, $S \cong Z_2 \times SL(2,8)$.

(ii) Firstly, we want to show that

$$(*) \quad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \\ \text{is an odd prime and } e(L) = 21, \text{ and if } H \text{ is a} \\ \text{normal subgroup of } L \text{ such that } |L:H| \text{ is an odd} \\ \text{prime and } e(H) = 7, \text{ then } B_o \cong B_o(L). \end{cases}$$

Let $b_o = B_o(L)$. By Lemma 6.13, $k(B_o) = k(b_o) = 16$ and $\ell(B_o) = \ell(b_o) = 5$. Let $S = O'(G) = \langle w\rangle \times SL(2,8)$ and $P = \langle w, x, y, z\rangle$ where $\langle x, y, z\rangle$ is a Sylow 2-subgroup of $SL(2,8)$. As in the proof of Lemma 6.14,

$$(**) \qquad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i = 1,\ldots,12 \\ \pm 3 & \text{for } i = 13,\ldots,16 \end{cases}$$

where $\{\chi_1, \ldots, \chi_{16}\} = \mathrm{Irr}(B_o)$. Let $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{16}\} = \mathrm{Irr}(b_o)$. By Lemma 6.14, we may assume that

$$(***) \quad \begin{aligned} &\tilde{\chi}_1(1) = \ldots = \tilde{\chi}_6(1) = 1, \quad \tilde{\chi}_7(1) = \ldots = \tilde{\chi}_{12}(1) = 7, \\ &\qquad \tilde{\chi}_{13}(1) = \tilde{\chi}_{14}(1) = 21, \quad \tilde{\chi}_{15}(1) = \tilde{\chi}_{16}(1) = 27, \\ &\tilde{\chi}_1(z) = \ldots = \tilde{\chi}_6(z) = 1, \quad \tilde{\chi}_7(z) = \ldots = \tilde{\chi}_{12}(z) = -1, \\ &\qquad \tilde{\chi}_{13}(z) = \tilde{\chi}_{14}(z) = -3, \quad \tilde{\chi}_{15}(z) = \tilde{\chi}_{16}(z) = 3. \end{aligned}$$

Thus, as in the proof of Theorem 5.4, by (**), (***), Clifford's theorem and Proposition 1.4, we may assume that $\chi_i|_L = \tilde{\chi}_i$ for all $i = 1,\ldots,16$. Hence we get $B_o \cong b_o$ by Corollary 1.7. This proves (*). Since $G/S$ is solvable by [12, Theorem], we can verify (ii).

Remark 2. There is a finite group $G$ with elementary abelian Sylow 2-subgroups of order 16 such that $e(G) = 21$ and $O'(G/O(G)) \cong Z_2 \times SL(2,8)$. We know it as in Remark 1 of §2.

Proposition 6.16. If $e = 21$, then there is a basic set $W$ of $B_0$ such that $W$ contains the trivial Brauer character and the decomposition matrix of $B_0$ with respect to $W$ has the form

$$
\begin{array}{c|ccccc}
1_G & 1 & 0 & 0 & 0 & 0 \\
 & \delta_2 & 0 & 0 & 0 & 0 \\
 & 0 & \delta_3 & 0 & 0 & 0 \\
 & 0 & \delta_4 & 0 & 0 & 0 \\
 & 0 & 0 & \delta_5 & 0 & 0 \\
 & 0 & 0 & \delta_6 & 0 & 0 \\
 & 0 & 0 & 0 & \delta_7 & 0 \\
 & 0 & 0 & 0 & \delta_8 & 0 \\
 & 0 & 0 & 0 & 0 & \delta_9 \\
 & 0 & 0 & 0 & 0 & \delta_{10} \\
 & \delta_{11} & 0 & 0 & \delta_{11} & \delta_{11} \\
 & \delta_{12} & 0 & 0 & \delta_{12} & \delta_{12} \\
 & 0 & \delta_{13} & 0 & \delta_{13} & \delta_{13} \\
 & 0 & \delta_{14} & 0 & \delta_{14} & \delta_{14} \\
 & 0 & 0 & \delta_{15} & \delta_{15} & \delta_{15} \\
 & 0 & 0 & \delta_{16} & \delta_{16} & \delta_{16}
\end{array}
$$

where $\delta_i = \pm 1$.

Proof. We can verify the proposition as in Proposition 5.2.

## References


[1] Alperin, J.L.: Isomorphic blocks. J. Algebra 43 (1976), 694-698.

[2] Bender, H.: On groups with abelian Sylow 2-subgroups. Math. Z. 117 (1970), 164-176.

[3] Brauer, R.: Some applications of the theory of blocks of characters of finite groups I-IV. J. Algebra 1 (1964), 152-167; ibid. 1 (1964), 307-334; ibid. 3 (1966), 225-255; ibid. 17 (1971), 489-521.

[4] Brauer, R.: On blocks and sections in finite groups I-II. Amer. J. Math. 89 (1967), 1115-1136; ibid. 90 (1968), 895-925.

[5] Brauer, R.: Defect groups in the theory of representations of finite groups. Illinois J. Math. 13 (1969), 53-73.

[6] Brauer, R.: On 2-blocks with dihedral defect groups, Symposia Mathematica 13 (1974), 367-393.

[7] Brauer, R. and Nesbitt, C.: On the modular characters of groups. Ann. of Math. (2) 42 (1941), 556-590.

[8] Curtis, C.W. and Reiner, I.: Representation theory of finite groups and associative algebras. Interscience, New York, 1962.

[9] Dade, E.C.: Blocks with cyclic defect groups. Ann. of Math. (2) 84 (1966), 20-48.

[10] Dornhoff, L.: Group representation theory (parts A and B). Dekker, New York, 1971-72.

[11] Feit, W.: Representations of finite groups (mimeographed notes). Yale University, Connecticut, 1969.

[12] Feit, W. and Thompson, J.G.: Solvability of groups of odd order. Pacific J. Math. 13 (1963), 775-1029.

[13] Fong, P.: On decomposition numbers of $J_1$ and $R(q)$. Symposia Mathematica 13 (1974), 415-422.

[14] Gorenstein, D.: Finite groups. Harper and Row, New York, 1968.

[15] Huppert, B.: Endliche Gruppen I. Springer, Berlin, 1967.

[16] Janko, Z.: A new finite simple group with abelian Sylow 2-subgroups and its characterization. J. Algebra 3 (1966), 147-186.

[17] Janko, Z. and Thompson, J.G.: On a class of finite simple groups of Ree. J. Algebra 4 (1966), 274-292.

[18] Olsson, J.B.: On 2-blocks with quaternion and quasidihedral defect groups. J. Algebra 36 (1975), 212-241.

[19] Reynolds, W.F.: A block correspondence and isometries of group characters. Math. Z. 113 (1970), 1-16.

[20] Walter, J.H.: The characterization of finite groups with abelian Sylow 2-subgroups. Ann. of Math. (2) 89 (1969), 405-514.

[21] Ward, H.N.: On Ree's series of simple groups. Trans. Amer. Math. Soc. 121 (1966), 62-89.