Tiled orders over a discrete valuation ring
and
Frobenius full matrix algebras with structure systems

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Contents

Chapter 0

Introduction.

In the study of non-commutative ring theory, there are a number of important examples of rings consisting of certain matrices over commutative rings. In this dissertation, we study tiled orders over a discrete valuation ring and Frobenius full matrix algebras with structure systems over a field, which we consider as two kinds of such important examples of non-commutative ring theory.

It is well-known that the ring of all algebraic integers in an algebraic number field is a Dedekind domain, that is, a Noetherian integral domain such that all ideals of it are invertible (or projective). Let R be a Dedekind domain with a quotient field K. As a generalization of that fact, the *maximal R-order* in a finite dimensional separable K-algebra A is investigated, that is, the ring Λ of all R-integral elements in A, which is a Noetherian R-order in A having global dimension $\mathrm{gl.dim}\,\Lambda = 1$, that is, every one-sided ideal is projective (see [2]). While a commutative integral domain R is a Dedekind domain if and only if $\mathrm{gl.dim}\,R = 1$, in the non-commutative situation, however, there are non-maximal R-orders Γ in A having $\mathrm{gl.dim}\,\Gamma = 1$, which are called *hereditary orders* studied by Harada [10]. (See [15] for related facts.)

In [4], answering a question of Kaplansky, for each positive integer n, Fields found an example of orders of global dimension n in a matrix ring over the quotient field of a discrete valuation ring. The study of such examples of orders of finite global dimension was taken over by Tarsy [20], [21], and in [20], he called such matrix orders *tiled orders.* (See Jategaonkar [12], [13], Roggenkamp [16], [17], Wiedemann and Roggenkamp [22], de la Peña and A. Raggi-Cárdenas [3], Kirkman and Kuzmanovich [14], Fujita [5], Rump [19], Jansen and Odenthal [11], Fujita [6] and Fujita and Oshima [8] for subsequent study on tiled orders of finite global dimension.)

Let D be a discrete valuation ring with a unique maximal ideal πD and a quotient filed K. Let $n \geq 2$ be an integer, and let $\{\lambda_{ij} \mid 1 \leq i,j \leq n\}$ be a set of $n \times n$ integers satisfying

$$\lambda_{ii} = 0, \quad \text{and} \quad \lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}$$

for all i,j,k $(1 \leq i,j,k \leq n)$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a semiperfect, Noetherian D-subalgebra of the full $n \times n$ matrix algebra $\mathbb{M}_n(K)$. We call such Λ a *tiled D-order* in $\mathbb{M}_n(K)$. Since any semiperfect ring is Morita equivalent to a basic one, we always assume that a tiled D-order is basic, which is equivalent to the condition $\lambda_{ij} + \lambda_{ji} > 0$ for all i,j $(1 \leq i,j \leq n)$ with $i \neq j$.

Following Fujita [6], we recall some facts on tiled orders of finite global dimension. In his study of global dimension of orders [20], Tarsy posed four conjectures. Three of them were solved soon by Tarsy [21] and Jategaonkar [12], [13], and it was remained open for about twenty years that if Λ is a D-order of finite global dimension in $\mathbb{M}_n(K)$, then $\operatorname{gl.dim}\Lambda \leq n-1$, which was the Tarsy's conjecture. It was proved by Jategaonkar [12] that for a fixed integer $n \geq 2$, there are, up to conjugate, only finitely many tiled D-orders of finite global dimension in $\mathbb{M}_n(K)$, so that there is a upper bound of finite global dimension, but it is not known what is the maximum. As a strategy to prove Tarsy's conjecture for tiled D-orders, Jategaonkar conjectured that if Λ is a tiled D-order of finite global dimension, there exists a primitive idempotent $e \in \Lambda$ such that $e\Lambda(1-e)$ or $(1-e)\Lambda e$ is $(1-e)\Lambda(1-e)$-projective. In some special cases, both conjectures were settled by some authors. However, Kirkman and Kuzmanovich [14] found a counterexample to Jategaonkar's conjecture. A counterexample to Tarsy's conjecture was also found by Fujita [5], by providing a tiled D-order in $\mathbb{M}_n(K)$ of global dimension n for all $n \geq 6$. It had been expected to find tiled D-orders with finite global dimension larger than n. In [19], Rump found a tiled D-order in $\mathbb{M}_8(K)$ of global dimension 9. In [11], Jansen and Odenthal found a series of tiled D-orders having global dimension $2N-8$ in $\mathbb{M}_N(K)$, for every even integer $N \geq 8$. Both examples were modifications of Fujita's counterexample to Tarsy's conjecture, and their computations of the global dimension were too complicated. In [6], Fujita improved their examples and the computation by using neat primitive idempotents (see [1] for neat idempotents), and he posed a question "Does any tiled D-order of finite global dimension have a neat primitive idempotent?", which can be considered as an improved version of Jategaonkar's conjecture disproved by Kirkman and Kuzmanovich.

Recently, in [8], Fujita and Oshima found a counterexample to the above question. Namely, for an arbitrary prime p, they constructed a tiled D-order Λ in $\mathbb{M}_n(K)$ such that $\operatorname{gl.dim}\Lambda = 5$ if characteristic $\operatorname{char} F \neq p$ and $\operatorname{gl.dim}\Lambda = \infty$ if $\operatorname{char} F = p$, where $F = D/\pi D$ and $n = 4p+5$. Moreover, they proved that if $\operatorname{char} F \neq p$, then Λ has no neat primitive idempotent. In order to compute the global dimension of Λ, they used the theory of Rump [19], where the computation is reduced to the case of a formal power series ring $D = F[[t]]$ and executed in the category of finite dimensional Ω-representations over F where Ω is the σ-poset determined by the exponent matrix of Λ. Since their computation is executed via the category of Ω-representations, it was not known what is an actual minimal projective resolution of the Jacobson radical of Λ.

In Chapter 1 of this dissertation, we construct certain exact sequences of modules by using finite submodules of a given module, and as an application of such an exact sequence, we compute directly a minimal projective resolution of the Jacobson radical of the tiled D-order Λ given by Fujita and Oshima.

In Chapter 2, we study Frobenius full matrix algebras with structure systems. The class of Gorenstein D-orders is important and studied by some authors. (See [18] and its references.) In the study of D-orders, it is standard to reduce homological properties of D-orders Λ to those of the factor F-algebras $\Lambda/\pi\Lambda$. For example, Gorenstein D-orders can be reduced to quasi-Frobenius F-algebras. As another example, we recall a theorem of Jategaonkar. It is proved in [12] that there are, up to conjugate, only finitely many tiled D-orders in $\mathbb{M}_n(K)$ having finite global dimension for a fixed integer $n \geq 2$. The key idea of its proof comes from a fact concerning the structure of the factor F-algebras $\Lambda/\pi\Lambda$ of a tiled D-order Λ. However, the study of such factor algebras is very limited, while its importance is well-recognized by many authors. (See [9] for related informations.)

In [7], Fujita introduced $\mathbb{A}$-full matrix algebras over a field F to provide a framework for such factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ.

Let F be a field and $n \geq 2$ an integer. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(F)$ $(1 \leq k \leq n)$ satisfying the following three conditions.

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $1 \leq i, j, k, l \leq n$.

(A2) $a_{kj}^{(k)} = a_{ij}^{(j)} = 1$ for all $1 \leq i, j, k \leq n$.

(A3) $a_{ii}^{(k)} = 0$ whenever $i \neq k$, $1 \leq i, k \leq n$.

Let $A = \bigoplus_{1 \leq i,j \leq n} F u_{ij}$ be an F-vector space with a basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$, and define the multiplication in A by

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Then A is an associative, connected basic F-algebra, which is called a *full matrix F-algebra* with a *structure system* $\mathbb{A}$ or an $\mathbb{A}$-*full matrix F-algebra.*

Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and define

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\ 0 & \text{if } \lambda_{ik} + \lambda_{kj} > \lambda_{ij} \end{cases}$$

for all $1 \leq i, j, k \leq n$. Then $\mathbb{A} = (A_1, \ldots, A_n)$ where $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(F)$ $(1 \leq k \leq n)$ is a structure system and the $\mathbb{A}$-full matrix F-algebra $A = \bigoplus_{1 \leq i,j \leq n} F u_{ij}$ is isomorphic to the factor F-algebra $\Lambda/\pi\Lambda$.

If Λ is a Gorenstein tiled D-order, then it is well-known that $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix F-algebra. In Section 2.1, we study the converse problem of that fact and prove the following theorem for $\mathbb{A}$-full matrix algebras with $(0,1)$-structure systems $\mathbb{A}$, that is, every entry of $\mathbb{A}$ is 0 or 1.

THEOREM. (1) *For every integer integer $2 \leq n \leq 7$, every Frobenius full $n \times n$ matrix F-algebra has corresponding Gorenstein tiled D-orders.*

(2) *For every integer $n \geq 8$, there is a Frobenius full $n \times n$ matrix F-algebra having no corresponding Gorenstein tiled D-orders.*

The results of Section 2.1 are obtained by joint work with H. Fujita.

In Section 2.2, we study $\mathbb{A}$-full matrix algebras whose structure systems $\mathbb{A}$ are not necessarily $(0,1)$-structure systems. Since we are able to treat the class of full matrix algebras with structure systems by elementary algebraic geometry technique, we introduce minor degenerations of the full matrix algebras. It turns out that, for a suitable choice of structure matrix $q = [q^{(1)}, \ldots, q^{(n)}]$, the algebra $\mathbb{M}_n^q(F)$ is a degeneration of the full matrix algebra $\mathbb{M}_n(F)$. So, we can consider the class of full matrix algebras with structure systems as a subclass of minor degenerations of the full matrix algebra $\mathbb{M}_n(F)$, that is, basic minor degenerations of $\mathbb{M}_n(F)$ are full matrix algebras with structure systems. In this section, among other things, we characterize Frobenius, basic minor degenerations of $\mathbb{M}_n(F)$ and we give the following example.

EXAMPLE. Assume that F is an infinite field. For each $n \geq 4$, there is a one-parameter F-algebraic family $\{C_\mu\}_{\mu \in F^*}$ of basic Frobenius F-algebras of the form $C_\mu = \mathbb{M}_n^{q_\mu}(F)$ such that $\sigma = (1, 2, \ldots, n)$ is the Nakayama permutation of C_μ and $C_\mu \ncong C_\nu$, if $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.

The results of Section 2.2 are obtained by joint work with H. Fujita and D. Simson.

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Chapter 1

An elementary exact sequence of modules with an application to tiled orders.

Let R be a ring with an identity, and let M be a right R-module. For R-submodules X, Y of M, there is an elementary short exact sequence

$$0 \longrightarrow X \cap Y \xrightarrow{\eta} X \oplus Y \xrightarrow{\varphi} X + Y \longrightarrow 0$$

where $\eta(t) = (t, -t)$ for $t \in X \cap Y$ and $\varphi(x, y) = x + y$ for $(x, y) \in X \oplus Y$. In this chapter, we extend the elementary short exact sequence to the case of more than two R-submodules of a given right R-module, and as an application, we compute a minimal projective resolution of Jacobson radical of a tiled order given by Fujita and Oshima [5], which provides a tiled order of finite global dimension without neat primitive idempotent, see [1], [4] for neat primitive idempotents, [3], [4], [6], [7], [8], [10], [13] for global dimension of tiled orders, and e.g. [11], [12], [14] ,[15] for further facts on tiled orders.

1.1 An elementary exact sequence of modules.

In [6], Jansen and Odenthal found a series of tiled orders having large global dimension. In order to compute global dimension of their tiled orders, they used a short exact sequence constructed with three irreducible lattices. We begin by clarifying the short exact sequence used in [6].

PROPOSITION 1.1.1. *Let X, Y, Z be R-submodules of a right R-module M. Let*

$$(X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y) \xrightarrow{\psi} X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \to 0$$

be a sequence of R-modules and R-homomorphisms defined by

$$\varphi(x, y, z) = x + y + z, \qquad \psi(x_0, y_0, z_0) = (x_0 - y_0,\ y_0 - z_0,\ z_0 - x_0)$$

for all $(x, y, z) \in X \oplus Y \oplus Z$ and $(x_0, y_0, z_0) \in (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y)$. Then

(1) $\operatorname{Ker}\psi \cong X \cap Y \cap Z$, $\operatorname{Im}\psi \subset \operatorname{Ker}\varphi$ *and* φ *is surjective.*

(2) *The following are equivalent.*

(a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$

(b) $(X+Y)\cap Z \subset X + (Y \cap Z)$

(c) *For any* $(x,y,z) \in \operatorname{Ker}\varphi$, *there exists* $x_0 \in X \cap Z$ *such that* $x_0 - x \in Y$.

(3) *If two of* X, Y, Z *are comparable by the relation of inclusion, then* $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.

(4) *Suppose that* $X \cap Y \cap Z = Y \cap Z$. *Then* $\operatorname{Im}\psi \cong (X\cap Z) \oplus (Y \cap X)$. *If the equivalent conditions of* (2) *hold, then there is a short exact sequence*

$$0 \longrightarrow (X \cap Z) \oplus (Y \cap X) \longrightarrow X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \longrightarrow 0.$$

Proof. (1) Straightforward.

(2) (a)⇒(b) Take an arbitrary $x + y = z \in (X+Y)\cap Z$ where $x \in X, y \in Y, z \in Z$. Then $(x,y,-z) \in \operatorname{Ker}\varphi$. Hence $(x,y,-z) = (x_0 - y_0,\, y_0 - z_0,\, z_0 - x_0)$ for some $(x_0,\, y_0,\, z_0) \in (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y)$. Then $z = x_0 - z_0 \in X + (Y \cap Z)$.

(b)⇒(c) Let $(x,y,z) \in \operatorname{Ker}\varphi$. Then $z = -x - y \in (X+Y)\cap Z \subset X + (Y \cap Z)$. Hence $z = -x_0 + z_0$ for some $x_0 \in X$ and $z_0 \in Y \cap Z$. Hence $x_0 - x = x_0 + y + z = y + z_0 \in Y$.

(c)⇒(a) Take an arbitrary $(x,y,z) \in \operatorname{Ker}\varphi$. Then we have $x_0 \in X \cap Z$ such that $x_0 - x \in Y$. Put $y_0 = x_0 - x$ and $z_0 = y_0 - y$. Then $y_0 = x_0 - x \in Y \cap X$ and $z_0 = y_0 - y = y_0 + x + z = x_0 + z \in Z \cap Y$. Hence $(x,y,z) = \psi(x_0, y_0, z_0) \in \operatorname{Im}\psi$.

(3) If $X \subset Y$, then $(X+Y)\cap Z = Y \cap Z \subset X + (Y\cap Z)$. If $Y \subset X$, then $(X+Y)\cap Z = X \cap Z \subset X + (Y \cap Z)$. If $X \subset Z$, then $(X+Y)\cap Z = X + (Y\cap Z)$ by the modular law. If $Z \subset X$, then $(X+Y)\cap Z \subset Z \subset X + (Y\cap Z)$. If $Y \subset Z$, then $(X+Y)\cap Z \subset X + Y = X + (Y \cap Z)$. If $Z \subset Y$, then $(X+Y)\cap Z \subset Z \subset X + (Y \cap Z)$.

(4) Since $X \cap Y \cap Z = Z \cap Y$, we can define an R-isomorphism

$$\theta : (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y) \to (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y)$$

by $\theta(x,y,z) = (x - z, y - z, z)$ for all $(x,y,z) \in (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y)$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccc}
0 & \longrightarrow & X\cap Y\cap Z & \xrightarrow{\eta} & (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y) \\
 & & \| & & \downarrow \theta \\
0 & \longrightarrow & X\cap Y\cap Z & \xrightarrow{i} & (X\cap Z)\oplus(Y\cap X)\oplus(Z\cap Y)
\end{array}$$

where $\eta(t) = (t, t, t)$ and $i(t) = (0, 0, t)$ for all $t \in X \cap Y \cap Z$. Hence we have

$$\operatorname{Im}\psi \cong \operatorname{Coker}\eta \cong \operatorname{Coker} i \cong (X \cap Z) \oplus (Y \cap X).$$

□

In what follows, D is a discrete valuation ring with a unique maximal ideal πD and a quotient field K.

Let $n \geq 2$ be an integer, and let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be a set of n^2 integers satisfying $\lambda_{ii} = 0$, $\lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}$ for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a semiperfect Noetherian D-subalgebra of the full $n \times n$ matrix algebra $\mathbb{M}_n(K)$, and Λ is a D-order in $\mathbb{M}_n(K)$, see [9]. Following [7] and [13], we call such a D-order Λ a *tiled D-order* in $\mathbb{M}_n(K)$, see also Chapter 13 of [11]. We note that Λ is *basic* if and only if $\lambda_{ij} + \lambda_{ji} > 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.

Let $V = K^n = (K, \ldots, K)$ be a simple right $\mathbb{M}_n(K)$-module. Let $a_1, \ldots, a_n$ be integers satisfying $a_i + \lambda_{ij} \geq a_j$ for all $1 \leq i, j \leq n$. Then $L = (\pi^{a_1} D, \ldots, \pi^{a_n} D)$ is a right Λ-submodule of V. We call L an *irreducible right Λ-lattice* in V, see [10] and [15].

The following fact is well known, see Lemmas 1.9, 1.10 of [6].

COROLLARY 1.1.2. *Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and let X, Y, Z be irreducible right Λ-lattices in $V = K^n$. Then the following statements hold.*

(1) *There is an exact sequence*

$$0 \to X \cap Y \cap Z \xrightarrow{\eta} (X \cap Z) \oplus (Y \cap X) \oplus (Z \cap Y) \xrightarrow{\psi} X \oplus Y \oplus Z \xrightarrow{\varphi} X + Y + Z \to 0$$

of right Λ-lattices.

(2) *Suppose that $X \cap Y \cap Z = Y \cap Z$. Then there is a short exact sequence*

$$0 \to (X \cap Z) \oplus (Y \cap X) \to X \oplus Y \oplus Z \to X + Y + Z \to 0$$

of right Λ-lattices.

Proof. (1) By Proposition 1.1.1 (1), it is sufficient to show that $\operatorname{Im}\psi = \operatorname{Ker}\varphi$. Put $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$, $Z = (Z_1, \ldots, Z_n)$ where X_j, Y_j, Z_j $(1 \leq j \leq n)$ are nonzero ideals of D. Let $(x, y, z) \in \operatorname{Ker}\varphi$, and let $x = (x_j), y = (y_j), z = (z_j)$ where $x_j \in X_j, y_j \in Y_j, z_j \in Z_j$ for $j = 1, \ldots, n$. Then $x_j + y_j + z_j = 0$, for each $1 \leq j \leq n$. Since D is a discrete valuation ring, X_j, Y_j, Z_j can be linearly ordered by inclusion, for each $1 \leq j \leq n$. Hence by (3) and (2) in Proposition 1.1.1, we can find $x_0 = (x_{0j}) \in X \cap Z$ such that $x_0 - x \in Y$. Hence it follows from Proposition 1.1.1 (2) that $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.

(2) This follows from (1) and Proposition 1.1.1 (4). □

LEMMA 1.1.3. *Let X, Y, Z be nonzero ideals of a principal ideal domain. Then $(X+Y)\cap Z \subset X+(Y\cap Z)$.*

Proof. Since each nonzero ideal of a principal ideal domain is generated by a product of prime elements, it is sufficient to show that $\max\{\min\{\alpha,\beta\},\gamma\} \geq \min\{\alpha,\max\{\beta,\gamma\}\}$, for any integers $\alpha,\beta,\gamma \geq 0$. If $\alpha \leq \beta \leq \gamma$, then $\max\{\min\{\alpha,\beta\},\gamma\} = \gamma \geq \alpha = \min\{\alpha,\max\{\beta,\gamma\}\}$. Similarly, we can check the remaining cases. □

REMARK 1.1.4. (1) The converse of Proposition 1.1.1 (3) does not hold in general. By Lemma 1.1.3 , we can find such examples among ideals of principal ideal domains. In fact, for example, let $R=\mathbb{Z}$ be the ring of integers, and let $X=2\mathbb{Z},\ Y=3\mathbb{Z},\ Z=5\mathbb{Z}$. Then $(2\mathbb{Z}+3\mathbb{Z})\cap 5\mathbb{Z} = 5\mathbb{Z} \subset \mathbb{Z} = 2\mathbb{Z}+(3\mathbb{Z}\cap 5\mathbb{Z})$, but any two of $2\mathbb{Z},\ 3\mathbb{Z},\ 5\mathbb{Z}$ are not comparable by the relation of inclusion.

(2) The sequence of Proposition 1.1.1 is not exact in general. In fact, let $R=\mathbb{Z}[t]$ be the polynomial ring over $\mathbb{Z}$ in the indeterminate t, and let $X=2R,\ Y=tR,\ Z=(2+t)R$. Then $(X+Y)\cap Z \not\subset X+(Y\cap Z)$, because $2+t \notin X+(Y\cap Z)$.

Next, we explore analogous elementary exact sequences constructed by using more than three submodules of a given module.

PROPOSITION 1.1.5. *Let R be an arbitrary ring and let $X_1,\ldots,X_m = X_0$ be R-submodules of a right R-module M, where $m \geq 3$. Let*

$$\bigoplus_{i=1}^{m}(X_i\cap X_{i-1}) \xrightarrow{\ \psi\ } \bigoplus_{i=1}^{m}X_i \xrightarrow{\ \varphi\ } \sum_{i=1}^{m}X_i \to 0$$

be a sequence of R-modules and R-homomorphisms defined by

$$\psi(y_1,\ldots,y_m) = (y_1-y_2,\ \ldots,\ y_{m-1}-y_m,\ y_m-y_1) \text{ and } \varphi(x_1,\ldots,x_m) = \sum_{i=1}^{m}x_i$$

for $(y_1,\ldots,y_m)\in\bigoplus_{i=1}^{m}(X_i\cap X_{i-1})$ and $(x_1,\ldots,x_m)\in\bigoplus_{i=1}^{m}X_i$. Then

(1) $\operatorname{Ker}\psi \cong \bigcap_{i=1}^{m}X_i$, $\operatorname{Im}\psi \subset \operatorname{Ker}\varphi$ *and φ is surjective.*

(2) *For any fixed $1\leq a\leq m$, the following two statements are equivalent.*

  (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$

  (b) *For any $(x_i)\in\operatorname{Ker}\varphi$, there exists $y\in X_a\cap X_{a-1}$ such that $y-(x_a+\cdots+x_t)\in X_{t+1}$ for all t $(a\leq t\leq a+m-3)$ where the indices are counted in modulo m.*

(3) *Suppose that there exist $1\leq a,b\leq m$ such that $X_a\subset X_i\subset X_b$ for all $1\leq i\leq m$. Then the following two statements are equivalent.*

  (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$

(b) $X_a \subset X_{a+1} \subset \cdots \subset X_{b-1} \subset X_b$ *and* $X_a \subset X_{a-1} \subset \cdots \subset X_{b+1} \subset X_b$, *where the indices are counted in modulo* m.

Proof. (1) Straightforward.

(2) We can assume that $a = 1$ by shifting the indices.

(a)$\Rightarrow$(b) Let $(x_i) \in \operatorname{Ker}\varphi$. Since $\operatorname{Ker}\varphi = \operatorname{Im}\psi$, $x_i = y_i - y_{i+1}$ $(1 \leq i \leq m)$, for some $(y_i) \in \bigoplus_{i=1}^m (X_i \cap X_{i-1})$, where $y_{m+1} := y_1$. Put $y := y_1 \in X_1 \cap X_m$. Then, for $t = 1, \ldots, m-2$,

$$\begin{aligned} y - (x_1 + \cdots + x_t) &= y_1 - \sum_{i=1}^{t} (y_i - y_{i+1}) \\ &= y_{t+1} \in X_{t+1} \cap X_t \subset X_{t+1}. \end{aligned}$$

(b)$\Rightarrow$(a) Let $(x_i) \in \operatorname{Ker}\varphi$. Then, by (a), there exists $y \in X_1 \cap X_m$ such that $y - (x_1 + \cdots + x_t) \in X_{t+1}$, for $1 \leq t \leq m-2$. Put $y_1 := y$ and $y_i := y - (x_1 + \cdots + x_{i-1})$, for $2 \leq i \leq m$. Then $y_1 = y = y - (x_1 + \cdots + x_m) = y_m - x_m$, and for $2 \leq i \leq m$, $y_i = y - (x_1 + \cdots + x_{i-1}) = y_{i-1} - x_{i-1}$. Hence $x_i = y_i - y_{i+1}$, for $1 \leq i \leq m$, where $y_{m+1} = y_1$. Since $y_1 \in X_1 \cap X_m$ and $y_{t+1} = y_t - x_t \in X_{t+1}$ for $1 \leq t \leq m-2$, then $y_i \in X_i \cap X_{i-1}$ for $1 \leq i \leq m-1$, and $y_m = y_{m-1} - x_{m-1} \in X_{m-1} \cap X_m$, because $y_m = y_1 + x_m \in X_m$. Hence $(x_i) = \psi(y_i) \in \operatorname{Im}\psi$.

(3) Without loss of generality, we can assume that $a = 1$.

($\Rightarrow$) Let $2 \leq r < b$ and take an arbitrary $x \in X_r$. Then, for $1 \leq i \leq m$, we set

$$x_i := \begin{cases} -x & \text{if } i = r \\ x & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_i) \in \operatorname{Ker}\varphi$. It follows from (2) that there exists $y \in X_1 \cap X_m$ such that $y + x = y - (x_1 + \cdots + x_r) \in X_{r+1}$. Hence $x \in X_{r+1}$, because $y \in X_1 \subset X_{r+1}$ and we get $X_r \subset X_{r+1}$. Therefore $X_1 \subset X_2 \subset \cdots \subset X_{b-1} \subset X_b$.

Let $b < s \leq m$ and take an arbitrary $x \in X_s$. Then for $1 \leq i \leq m$, set

$$x_i := \begin{cases} x & \text{if } i = s \\ -x & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_i) \in \operatorname{Ker}\varphi$. It follows from (2) that there exists $y \in X_1 \cap X_m$ such that $y + x = y + (x_m + \cdots + x_s) = y - (x_1 + \cdots + x_{s-1}) \in X_{s-1}$. Hence $X_s \subset X_{s-1}$ for all $b < s \leq m$.

($\Leftarrow$) Let $(x_i) \in \operatorname{Ker}\varphi$. Then put $y := x_1 \in X_1 = X_1 \cap X_m$. If $1 \leq t < b$, then $y - (x_1 + \cdots + x_t) = -(x_2 + \cdots + x_t) \in X_t \subset X_{t+1}$. If $b \leq t \leq m-2$, then $y - (x_1 + \cdots + x_t) = y + (x_m + \cdots + x_{t+1}) \in X_{t+1}$. Hence it follows from (2) that $\operatorname{Ker}\varphi = \operatorname{Im}\psi$. □

REMARK 1.1.6. We notice that the sequence of Proposition 1.1.5 is not always exact, even if $X_1, X_2, \ldots, X_m$ can be linearly ordered by inclusion. In fact, for example, consider the submodules

$X_1 = 4\mathbb{Z} \subset X_3 = 2\mathbb{Z} \subset X_2 = X_4 = \mathbb{Z}$ of $M = \mathbb{Z}$. Then it follows from Proposition 1.1.5 (3) that the sequence

$$\bigoplus_{i=1}^{4}(X_i \cap X_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{4} X_i \xrightarrow{\varphi} \sum_{i=1}^{4} X_i \to 0$$

is not exact. However, if we change the indices of X_2 and X_3, then $X_1 = 4\mathbb{Z} \subset X_2 = 2\mathbb{Z} \subset X_3 = X_4 = \mathbb{Z}$ and the sequence is exact.

The following is a generalization of Corollary 1.1.2.

COROLLARY 1.1.7. *Let* $\Lambda = (\pi^{\lambda_{ij}}D)$ *be a basic tiled* D*-order in* $\mathbb{M}_n(K)$*, and let* $L_1 = (L_{11}, \ldots, L_{1n}), \ldots, L_m = (L_{m1}, \ldots, L_{mn}) = L_0$ *be irreducible right* Λ*-lattices in* $V = K^n$*, where* $m \geq 3$*. For each* $1 \leq j \leq n$*, let* a_j, b_j *be integers in* $\{1, \ldots, m\}$ *such that* $L_{a_j,j} \subset L_{ij} \subset L_{b_j,j}$ *for all* $1 \leq i \leq m$*. Let*

$$\bigoplus_{i=1}^{m}(L_i \cap L_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{m} L_i \xrightarrow{\varphi} \sum_{i=1}^{m} L_i \to 0$$

be a sequence of Λ*-lattices and* Λ*-homomorphisms defined by*

$$\psi(y_1, \ldots, y_m) = (y_1 - y_2, \ldots, y_{m-1} - y_m, y_m - y_1) \text{ and } \varphi(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i$$

for $(y_1, \ldots, y_m) \in \bigoplus_{i=1}^m (L_i \cap L_{i-1})$ *and* $(x_1, \ldots, x_m) \in \bigoplus_{i=1}^m L_i$*.*

(1) *The following statements are equivalent.*

  (a) $\operatorname{Im}\psi = \operatorname{Ker}\varphi$.

  (b) *For each* $1 \leq j \leq m$*,* $L_{i,j} \subset L_{i+1,j}$ *for all* $i \in \{1, \ldots, m\}$ *with* $a_j \leq i < b_j (\bmod m)$ *and* $L_{i,j} \subset L_{i-1,j}$ *for all* $i \in \{1, \ldots, m\}$ *with* $a_j \geq i > b_j \pmod m$*.*

(2) *Suppose that the equivalent conditions of* (1) *hold. Then there is an exact sequence*

$$0 \longrightarrow \bigcap_{i=1}^{m} L_i \longrightarrow \bigoplus_{i=1}^{m}(L_i \cap L_{i-1}) \xrightarrow{\psi} \bigoplus_{i=1}^{m} L_i \xrightarrow{\varphi} \sum_{i=1}^{m} L_i \longrightarrow 0$$

of right Λ*-lattices. In particular, if* $\bigcap_{i=1}^m L_i = L_{m-1} \cap L_m$*, then there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{m-1}(L_i \cap L_{i-1}) \longrightarrow \bigoplus_{i=1}^{m} L_i \longrightarrow \sum_{i=1}^{m} L_i \longrightarrow 0$$

of right Λ*-lattices.*

Proof. Apply Proposition 1.1.5 and the arguments used in the proof of Corollary 1.1.2. □

REMARK 1.1.8. The condition (b) always holds, if $m = 3$.

1.2 An application to tiled orders.

As an application of our elementary exact sequence, we compute a minimal projective resolution of the Jacobson radical of a tiled D-order given by Fujita and Oshima [5].

We use the following notations. Let

$$\Lambda = (\pi^{\lambda_{ij}} D)$$

be a basic tiled D-order in $\mathbb{M}_n(K)$, and let $J(\Lambda)$ be the Jacobson radical of Λ. For each $1 \leq i \leq n$, let $e_i \in \mathbb{M}_n(K)$ be the matrix whose (i,i)-entry is 1 and the other entries are 0. For each $1 \leq i \leq n$, let P_i be the irreducible right Λ-lattice

$$P_i = (\pi^{\lambda_{i1}} D, \ldots, \pi^{\lambda_{in}} D)$$

in $V = K^n$, and let

$$J_i = P_i J(\Lambda) \simeq e_i J(\Lambda)$$

be the radical of $P_i \simeq e_i \Lambda$ for $1 \leq i \leq n$. Moreover we put

$$S_i := P_i / J_i$$

for each $1 \leq i \leq n$. Then P_i $(1 \leq i \leq n)$ are the indecomposable projective right Λ-modules, and S_i $(1 \leq i \leq n)$ are the simple right Λ-modules.

Example 1.2.1. We compute minimal projective resolutions of J_i $(1 \leq i \leq 13)$ of the following basic (0,1)-tiled D-order Λ in $\mathbb{M}_{13}(K)$ where $\boldsymbol{\pi} = \pi D$, see [15] and Chapter 13 of [11].

$$\Lambda := \begin{pmatrix}
D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D & D & D \\
\boldsymbol{\pi} & D & \boldsymbol{\pi} & D & D & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & D & D & D & D & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & D & \boldsymbol{\pi} & \boldsymbol{\pi} & D \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & D
\end{pmatrix}$$

Let $F := D/\boldsymbol{\pi}$ be the residue field, and let A be the F-algebra $\Lambda/\mathbb{M}_{13}(\boldsymbol{\pi})$. It follows from [2] that the link graph of Λ is obtained from the Gabriel quiver $\mathcal{Q}(A)$ of A by adding the arrows from non-domains in $\mathcal{Q}(A)$ to non-ranges in $\mathcal{Q}(A)$ to the set of allows of $\mathcal{Q}(A)$. Note that $\mathcal{Q}(A)$ is the following quiver

1 2 3

$\mathcal{Q}(A)$: 4 5 8 9 6 7

10 11 12 13

We check that

(1) If char$F \neq 2$, then gl.dim $\Lambda = 5$ and Λ has no neat primitive idempotent.

(2) If char$F = 2$, then gl.dim $\Lambda = \infty$.

Step 1. Since $J_1/J_1J(\Lambda) \cong S_4 \oplus S_5 \oplus S_8 \oplus S_9$, then J_1 has the projective cover

$$\varphi : P_4 \oplus P_5 \oplus P_8 \oplus P_9 \twoheadrightarrow J_1,\ (x_4, x_5, x_8, x_9) \mapsto x_4 + x_5 + x_8 + x_9.$$

Note also that the modules P_4, P_8, P_5, P_9 satisfy the condition (b) of Corollary 1.1.7 in that order. Moreover, note that $P_4 \cap P_9 = P_{10},\ P_8 \cap P_4 = P_{12},\ P_5 \cap P_8 = P_{11}, P_9 \cap P_5 = P_{13}$ and $P_4 \cap P_8 \cap P_5 \cap P_9 = J_{10}$. Hence, by Corollary 1.1.7, we have the exact sequence

$$0 \to J_{10} \xrightarrow{\eta} P_{10} \oplus P_{12} \oplus P_{11} \oplus P_{13} \xrightarrow{\psi} P_4 \oplus P_8 \oplus P_5 \oplus P_9 \xrightarrow{\varphi} J_1 \to 0.$$

Note that $P_{10} \oplus P_{12} \oplus P_{11} \oplus P_{13} \overset{\psi}{\twoheadrightarrow} \operatorname{Im}\psi$ is a projective cover, because $\operatorname{Im}\eta \subset (P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13})J(\Lambda)$. Similarly, we get the following exact sequences

$$0 \to J_{10} \to P_{12} \oplus P_{10} \oplus P_{11} \oplus P_{13} \to P_4 \oplus P_6 \oplus P_5 \oplus P_7 \to J_2 \to 0,$$

$$0 \to J_{10} \to P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13} \to P_6 \oplus P_8 \oplus P_7 \oplus P_9 \to J_3 \to 0.$$

Step 2. Note that J_i $(4 \leq i \leq 9)$ have the following projective covers

$$0 \to J_{10} \to P_{10} \oplus P_{12} \to J_4 \to 0,$$

$$0 \to J_{10} \to P_{11} \oplus P_{13} \to J_5 \to 0,$$

$$0 \to J_{10} \to P_{10} \oplus P_{11} \to J_6 \to 0,$$

$$0 \to J_{10} \to P_{12} \oplus P_{13} \to J_7 \to 0,$$

$$0 \to J_{10} \to P_{11} \oplus P_{12} \to J_8 \to 0,$$

$$0 \to J_{10} \to P_{10} \oplus P_{13} \to J_9 \to 0.$$

Step 3. Note that $X := (D, D, \ldots, D) \cong J_{10} = J_{11} = J_{12} = J_{13}$ and $X/XJ(\Lambda) \cong S_1 \oplus S_2 \oplus S_3$. Hence by Corollary 1.1.2, we get the following exact sequence

$$0 \to P_1 \cap P_2 \cap P_3 \xrightarrow{h} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \xrightarrow{g} P_1 \oplus P_2 \oplus P_3 \xrightarrow{f} X \to 0.$$

If we put $Y := \operatorname{Ker} f$, then we get two short exact sequences

$$0 \to Y \to P_1 \oplus P_2 \oplus P_3 \xrightarrow{f} X \to 0,$$

$$0 \to P_1 \cap P_2 \cap P_3 \xrightarrow{h} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \xrightarrow{g} Y \to 0.$$

Note that projective covers of $P_1 \cap P_3$, $P_2 \cap P_1$, and $P_3 \cap P_2$ are given by

$$0 \to J_{10} \to P_8 \oplus P_9 \to P_1 \cap P_3 \to 0,$$

$$0 \to J_{10} \to P_4 \oplus P_5 \to P_2 \cap P_1 \to 0,$$

$$0 \to J_{10} \to P_6 \oplus P_7 \to P_3 \cap P_2 \to 0.$$

Hence $(P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2)$ has the projective cover

$$0 \to J_{10} \oplus J_{10} \oplus J_{10} \to P \xrightarrow{\theta} (P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2) \to 0,$$

where $P := P_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8 \oplus P_9$. Note that

$$\operatorname{Im} h \subset [(P_1 \cap P_3) \oplus (P_2 \cap P_1) \oplus (P_3 \cap P_2)]J(\Lambda).$$

Hence, the projective cover of Y has the form

$$0 \longrightarrow Z \longrightarrow P \xrightarrow{\theta \circ g} Y \longrightarrow 0,$$

where

$$\begin{aligned} Z &:= \operatorname{Ker} \theta \circ g \\ &= \{(x_4, x_5, x_6, x_7, x_8, x_9) \in P \mid x_4 + x_5 = x_6 + x_7 = x_8 + x_9\}. \end{aligned}$$

Step 4. Note that

$$\begin{aligned} P_{10} + P_{12} = J_4 \subset P_4, \quad P_{11} + P_{13} = J_5 \subset P_5, \\ P_{10} + P_{11} = J_6 \subset P_6, \quad P_{12} + P_{13} = J_7 \subset P_7, \\ P_{11} + P_{12} = J_8 \subset P_8, \quad P_{10} + P_{13} = J_9 \subset P_9. \end{aligned}$$

Hence, we obtain a Λ-homomorphism $\alpha : P_{10} \oplus P_{11} \oplus P_{12} \oplus P_{13} \to Z$ defined by

$$\alpha : \begin{pmatrix} x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}.$$

CLAIM 1. *If* $\operatorname{char} F \neq 2$ *then* α *is an isomorphism.*

Proof. First note that $2 \in D \setminus \pi D$ and 2 is invertible in D, because $\operatorname{char} F \neq 2$.

Let $(x_{10}, x_{11}, x_{12}, x_{13}) \in \operatorname{Ker} \alpha$. Then we have $2x_{10} = x_{10} + x_{12} + x_{10} + x_{11} = 0$, and similarly $2x_{11} = 0,\ 2x_{12} = 0,\ 2x_{13} = 0$. Hence $(x_{10}, x_{11}, x_{12}, x_{13}) = (0, 0, 0, 0)$, so that α is a monomorphism.

Let $(x_4, x_5, x_6, x_7, x_8, x_9) \in Z$. Since $x_4 + x_5 = x_6 + x_7 = x_8 + x_9$, we put

$$\begin{array}{rclclcl} 2x_{10} & := & x_4 + x_6 - x_8 & = & x_9 - x_5 + x_6 & = & x_4 + x_9 - x_7, \\ 2x_{11} & := & x_5 + x_6 - x_9 & = & x_8 - x_4 + x_6 & = & x_5 + x_8 - x_7, \\ 2x_{12} & := & x_4 + x_7 - x_9 & = & x_8 - x_5 + x_7 & = & x_4 + x_8 - x_6, \\ 2x_{13} & := & x_5 + x_7 - x_8 & = & x_9 - x_4 + x_7 & = & x_5 + x_9 - x_6. \end{array}$$

Further, we put $x_i = (x_{i1}, \ldots, x_{i13}) \in P_i$ $(4 \leq i \leq 9)$. Then, for each $1 \leq j \leq 13$ with $j \neq 10$, we have $x_{4j} + x_{6j} - x_{8j} = x_{9j} - x_{5j} + x_{6j} = x_{4j} + x_{9j} - x_{7j} \in \pi D$. Hence $x_{10} \in P_{10}$. Similarly, we can check that $x_{11} \in P_{11}, x_{12} \in P_{12}, x_{13} \in P_{13}$. Hence α is an epimorphism, because $\alpha(x_{10}, x_{11}, x_{12}, x_{13}) = (x_4, x_5, x_6, x_7, x_8, x_9)$. $\square$

CLAIM 2. *If* $\operatorname{char} F \neq 2$, *then* $\operatorname{gl.dim} \Lambda = 5$ *and* Λ *has no neat primitive idempotent.*

Proof. It follows from Steps 3, 4 and Claim 1 that J_k $(10 \leq k \leq 13)$ has the following minimal projective resolution

$$0 \to \bigoplus_{i=10}^{13} P_i \to \bigoplus_{i=4}^{9} P_i \to \bigoplus_{i=1}^{3} P_i \to J_k \to 0.$$

Note that every P_i $(1 \leq i \leq 13)$ appears in the above resolution, and that minimal projective resolutions of J_k $(1 \leq k \leq 9)$ are given by connecting the above resolution to the sequences of Steps 1 and 2. Hence $\operatorname{gl.dim} \Lambda = \sup\{\operatorname{pd} J_i \mid 1 \leq i \leq 13\} + 1 = 5$, and it follows from Proposition 1 of [4] that every e_i $(1 \leq i \leq 13)$ is not neat. $\square$

Step 5. Note that for any $x_{10} = (x_{10,1}, \ldots, x_{10,13}) \in J_{10} \subset P_{10}$, $x_{10j} \in \pi D$ for each $1 \leq j \leq 13$. Hence we get a Λ-homomorphism $\beta : J_{10} \to Z$ defined by

$$\beta(x_{10}) = (x_{10}, 0, x_{10}, 0, x_{10}, 0).$$

CLAIM 3. *If* $\operatorname{char} F = 2$ *then* β *is a split monomorphism.*

Proof. For any $(x_4, x_5, x_6, x_7, x_8, x_9) \in Z$, put $y := x_4 + x_5 = x_6 + x_7 = x_8 + x_9$ and $z := x_4 - x_6 + x_8 = -x_5 + x_7 + x_8 = x_4 + x_7 - x_9$. Put $y = (y_1, \ldots, y_{13})$ and $z = (z_1, \ldots, z_{13})$. Then for each $1 \leq j \leq 13$ with $j \neq 12$, $z_j = x_{4j} - x_{6j} + x_{8j} = -x_{5j} + x_{7j} + x_{8j} = x_{4j} + x_{7j} - x_{9j} \in \pi D$. If $j = 12$, then $z_{12} = x_{4,12} - x_{6,12} + x_{8,12} = 2y_{12} - x_{5,12} - x_{6,12} - x_{9,12} \in \pi D$ because $2 \in \pi D$. Hence we get a Λ-homomorphism $\beta' : Z \to J_{10}$ defined by

$$\beta'(x_4, x_5, x_6, x_7, x_8, x_9) = x_4 - x_6 + x_8.$$

Since we can check that $\beta' \circ \beta = id_{J_{10}}$, then β is a split monomorphism. $\square$

CLAIM 4. *If* $\operatorname{char} F = 2$, *then* $\operatorname{gl.dim} \Lambda = \infty$.

Proof. It follows from Step 3 that there exist a long exact sequence

$$0 \to Z \to \bigoplus_{i=4}^{9} P_i \to \bigoplus_{i=1}^{3} P_i \to J_{10} \to 0.$$

It follows from Claim 3 that $Z \simeq J_{10} \oplus W$, for some right Λ-lattice W. Therefore pd $J_{10} = \infty$ and $\operatorname{gl.dim} \Lambda = \infty$. □

REMARK 1.2.2. Extending the $(0,1)$-tiled D-order Λ in $\mathbb{M}_{13}(K)$ of Example 1.2.1, for each $n \geq 14$, one can construct a basic $(0,1)$-tiled D-order Λ_n in $\mathbb{M}_n(K)$ such that $\operatorname{gl.dim} \Lambda = 5$ if $\operatorname{char} F \neq 2$ and that $\operatorname{gl.dim} \Lambda = \infty$ if $\operatorname{char} F = 2$. In fact, for example, let A_n be the F-algebra whose quiver $\mathcal{Q}(A_n)$ is obtained by adding arrows $n \to n-1 \to \cdots \to 13$ to the quiver $\mathcal{Q}(A)$ of the F-algebra $A = \Lambda/\mathbb{M}_{13}(\boldsymbol{\pi})$, and let Λ_n be the $(0,1)$-tiled D-order in $\mathbb{M}_n(K)$ such that $A_n = \Lambda_n/\mathbb{M}_n(\boldsymbol{\pi})$. Then one can conclude that $\operatorname{gl.dim} \Lambda_n = 5$ if $\operatorname{char} F \neq 2$ and that $\operatorname{gl.dim} \Lambda_n = \infty$ if $\operatorname{char} F = 2$ as in Example 1.2.1.

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Chapter 2

Frobenius full matrix algebras with structure systems.

In this chapter, we study Frobenius full matrix algebras with structure systems. In [7], Fujita introduced $\mathbb{A}$-full matrix algebras over a field F to provide a framework for factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ.

Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be a basic tiled D-order in $\mathbb{M}_n(K)$, and define

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\ 0 & \text{if } \lambda_{ik} + \lambda_{kj} > \lambda_{ij} \end{cases}$$

for all $1 \le i, j, k \le n$. Then $\mathbb{A} = (A_1, \ldots, A_n)$ where $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(F)$ $(1 \le k \le n)$ is a structure system and the $\mathbb{A}$-full matrix F-algebra $A = \bigoplus_{1 \le i,j \le n} F u_{ij}$ is isomorphic to the factor F-algebra $\Lambda/\pi\Lambda$. Moreover, if Λ is a Gorenstein tiled D-order, then it is well-known that $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix F-algebra.

In Section 2.1, we study the converse problem of that fact and prove the following theorem for $\mathbb{A}$-full matrix algebras with $(0,1)$-structure systems $\mathbb{A}$, that is, every entry of $\mathbb{A}$ is 0 or 1.

THEOREM. (1) *For every integer integer $2 \le n \le 7$, every Frobenius full $n \times n$ matrix F-algebra has corresponding Gorenstein tiled D-orders.*

(2) *For every integer $n \ge 8$, there is a Frobenius full $n \times n$ matrix F-algebra having no corresponding Gorenstein tiled D-orders.*

The results of Section 2.1 are obtained by joint work with H. Fujita.

In Section 2.2, we study $\mathbb{A}$-full matrix algebras whose structure systems $\mathbb{A}$ are not necessarily $(0,1)$-structure systems. Since we are able to treat the class of full matrix algebras with structure systems by elementary algebraic geometry technique, we introduce minor degenerations of the full matrix algebras. It turns out that, for a suitable choice of structure matrix $q = [q^{(1)}, \ldots, q^{(n)}]$, the algebra $\mathbb{M}_n^q(F)$ is a degeneration of the full matrix algebra $\mathbb{M}_n(F)$. So, we can consider the class of full matrix algebras with structure systems as a subclass of minor degenerations of the full matrix algebra $\mathbb{M}_n(F)$, that is, basic minor degenerations of $\mathbb{M}_n(F)$ are full matrix algebras with structure systems. In this section, among other things, we characterize Frobenius, basic minor degenerations of $\mathbb{M}_n(F)$ and we give the following example.

EXAMPLE. Assume that F is an infinite field. For each $n \geq 4$, there is a one-parameter F-algebraic family $\{C_\mu\}_{\mu \in F^*}$ of basic Frobenius F-algebras of the form $C_\mu = \mathbb{M}_n^{q_\mu}(F)$ such that $\sigma = (1, 2, \ldots, n)$ is the Nakayama permutation of C_μ and $C_\mu \not\cong C_\nu$, if $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.
The results of Section 2.2 are obtained by joint work with H. Fujita and D. Simson.

2.1 Gorenstein tiled orders and Frobenius full matrix algebras.

Let D be a discrete valuation ring with a unique maximal ideal πD, and let Λ be a D-order in a semisimple algebra. It is standard to reduce homological properties of Λ to those of the factor algebra $\Lambda/\pi\Lambda$ and such factor algebras are deserving of further study. (See [14].)

Let n be an integer with $n \geq 2$. An $n \times n$ $\mathbb{A}$-full matrix algebra over a field K is an $n \times n$ dimensional K-vector space with multiplication defined by a structure system $\mathbb{A}$, that is, an n-tuple of $n \times n$ matrices with certain properties. A prototype of $\mathbb{A}$-full matrix algebras is the class of factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ. Studying representation matrices of certain modules over $\mathbb{A}$-full matrix algebras, Frobenius $\mathbb{A}$-full matrix algebras are characterized by the shape of their structure systems $\mathbb{A}$. For a Gorenstein tiled D-order Λ, the factor algebra $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix algebra. For $n \leq 5$, a list of Frobenius $\mathbb{A}$-full matrix algebras is obtained, and they have corresponding Gorenstein tiled D-orders, which can be found in Examples of Roggenkamp, Kirichenko, Khibina and Zhuravlev [26]. (See Fujita [7].)

In this section, we study minimal Frobenius structure systems and show that for every integer $n \geq 8$, there exist Frobenius $\mathbb{A}$-full matrix algebras which have no corresponding Gorenstein tiled D-orders. Moreover for $n \leq 7$, we give some Gorenstein tiled D-orders and verify that their factor algebras provide a list of all Frobenius $\mathbb{A}$-full matrix algebras, up to isomorphism.

In Subsection 2.1.1, we first recall the definition of full matrix algebras with structure systems and a characterization of Frobenius full matrix algebras (Proposition 2.1.1). Let σ be a permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$, and let T be the set of all triples (i, k, j) of integers $1 \leq i, k, j \leq n$. Then σ defines a bijection $\varphi : T \to T, (i, k, j) \mapsto (k, j, \sigma(i))$. Structure systems of Frobenius full matrix algebras are determined by some φ-invariant subsets of T. Considering 4-tuples of φ-orbits, we formulate a procedure to check what φ-invariant subsets define structure systems of Frobenius full matrix algebras (Proposition 2.1.6). In Subsection 2.1.2, we clarify φ-orbits of T precisely for a cyclic permutation $\sigma = (1\ 2\ \cdots\ n)$. In Subsection 2.1.3, we introduce minimal Frobenius structure systems and determine what φ-invariant subsets define minimal Frobenius structure systems for a cyclic permutation $\sigma = (1\ 2\ \cdots\ n)$, using the results obtained in Subsection 2.1.1 and 2.1.2 (Theorem 2.1.15). In Subsection 2.1.4, we show that for every integer $n \geq 8$, there exist Frobenius $n \times n$ full matrix algebras having no corresponding Gorenstein tiled orders (Theorem 2.1.19). Our examples of such Frobenius full matrix algebras are defined by some minimal Frobenius structure systems. In Subsection 2.1.5, we study some Frobenius full matrix algebras having non-cyclic Nakayama permutations. For $n = 6, 7$, in Subsection 2.1.6, we show that every Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra A has a Gorenstein tiled D-order Λ such that $\Lambda/\pi\Lambda \cong A$. (For $2 \leq n \leq 5$, see Fujita [7].)

2.1.1 Preliminaries.

Let K be a field and n an integer with $n \geq 2$. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions.

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$, and

(A3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

Let $A = \bigoplus_{1 \leq i,j \leq n} K u_{ij}$ be a K-vector space with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. Then we define multiplication of A by using $\mathbb{A}$, that is,

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Then A is an associative, basic K-algebra. We call A an $n \times n$ $\mathbb{A}$-*full matrix algebra* with *structure system* $\mathbb{A}$. We note that each indecomposable projective right A-module and each indecomposable injective right A-module can be represented by some matrices made of a structure system $\mathbb{A}$. (See Propositions 2.2 and 2.3 of [7].)

In what follows, we assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \leq i, k, j \leq n$.

PROPOSITION 2.1.1. *Let A be an $n \times n$ $\mathbb{A}$-full matrix algebra. Then the following are equivalent.*

(1) A is a Frobenius algebra.

(2) There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i \in \{1, \ldots, n\}$ and that $a_{i\sigma(i)}^{(k)} = 1$ for all $i, k \in \{1, \ldots, n\}$.

In this case, σ is the Nakayama permutation of A. Furthermore, for all $i, j, k \in \{1, \ldots, n\}$, $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ holds, and $a_{ij}^{(k)} = 0$ holds whenever $j = \sigma(k)$ or $k = \sigma(i)$.

Proof. This follows from Lemma 4.2 and the proof of Lemma 4.5 of [7]. □

We utilize Proposition 2.1.1 for finding structure systems $\mathbb{A}$ of Frobenius $\mathbb{A}$-full matrix algebras. In the rest of this section, we formulate some notations which will be used throughout the paper.

Let n be an integer with $n \geq 2$, and let T be the set of triples (i, k, j) of integers i, k, j such that $1 \leq i, k, j \leq n$. Let σ be a permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$. Then σ defines a bijection $\varphi_\sigma : T \to T, (i, k, j) \mapsto (k, j, \sigma(i))$. We decompose T into φ_σ*-orbits* $\{T_\alpha\}_\alpha$, that is, $T_\alpha = \{\varphi_\sigma^t(i, k, j) \in T \mid t \in \mathbb{Z}\}$ for some $(i, k, j) \in T$ and T is a disjoint union of $\{T_\alpha\}_\alpha$. When the permutation σ is fixed or clear in the context, we sometimes omit the subscript σ of φ_σ, e.g., we will use φ*-orbit* instead of φ_σ-orbit.

Lemma 2.1.2. *The set T is a disjoint union of the following three φ-invariant subsets*

$$\begin{aligned} I &:= \cup\{T_\alpha \mid (i,i,j) \in T_\alpha \ \ \textit{for some} \ \ 1 \le i,j \le n\} \\ Z &:= \cup\{T_\alpha \mid (i,k,i) \in T_\alpha \ \ \textit{for some} \ \ 1 \le i,k \le n \ \textit{with} \ i \ne k\} \\ X &:= \cup\{T_\alpha \mid T_\alpha \not\subset I, T_\alpha \not\subset Z\}. \end{aligned}$$

Moreover, the following equations hold.

$$\begin{aligned} I &= \{(i,k,j) \in T \mid i = k, \ \textit{or}\ j = k, \ \textit{or}\ j = \sigma(i)\} \\ Z &= \{(i,k,j) \in T \mid i = j, i \ne k, \ \textit{or for distinct}\ i,k,j, \ j = \sigma(k) \ \textit{or}\ k = \sigma(i)\} \\ X &= \{(i,k,j) \in T \mid i,k,j \ \textit{are distinct}, \ j \ne \sigma(i), \ j \ne \sigma(k), \ k \ne \sigma(i)\}. \end{aligned}$$

Proof. Let T_α be a φ-orbit containing (i,i,j) for some $1 \le i,j \le n$. Any element of T_α is one of the following

$$(\sigma^t(i), \sigma^t(i), \sigma^t(j)), \quad (\sigma^t(i), \sigma^t(j), \sigma^{t+1}(i)), \quad \text{or} \quad (\sigma^t(j), \sigma^{t+1}(i), \sigma^{t+1}(i))$$

for some $t \ge 0$. Since $\sigma(k) \ne k$ for all $k = 1,\dots,n$, T_α does not contain (i',k',i') for any $1 \le i',k' \le n$ with $i' \ne k'$. Therefore $I \cap Z = \emptyset$. Since X is the complement of $I \cup Z$, T is a disjoint union of φ-invariant subsets I, Z, X.

The remaining part is left to the reader as an exercise. □

Proposition 2.1.3. *Let $\mathbb{A} = (A_1,\dots,A_n) = (a_{ij}^{(k)})$ be a structure system of a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ. Then there exists a φ-invariant subset Y of X such that*

$$a_{ij}^{(k)} = \begin{cases} 1 & \textit{if} \ \ (i,k,j) \in I \cup Y \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It follows from (A2), (A3) and Proposition 2.1.1 that $a_{ij}^{(k)} = 1$ for all $(i,k,j) \in I$ and that $a_{ij}^{(k)} = 0$ for all $(i,k,j) \in Z$. Let Y be the union of φ-orbits T_α contained in X such that T_α has an element (i,k,j) with $a_{ij}^{(k)} = 1$. Since $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ for all $(i,k,j) \in T$ by Proposition 2.1.1, Y is the desired subset of X. □

Proposition 2.1.4. *Let σ be a permutation of the set $\{1,\dots,n\}$ such that $\sigma(i) \ne i$ for all $i = 1,\dots,n$. Let Y be a φ-invariant subset of X, and let $\mathbb{A} = (A_1,\dots,A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(k = 1,\dots,n)$ defined by*

$$a_{ij}^{(k)} := \begin{cases} 1 & \textit{if} \ \ (i,k,j) \in I \cup Y \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\mathbb{A}$ is a structure system of a Frobenius full matrix algebra with Nakayama permutation σ if and only if $\mathbb{A}$ satisfies (A1).

Proof. First note that $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ for any $(i,k,j) \in T$, because Y is φ-invariant. For all $i,j,k \in \{1,\dots,n\}$, since $(k,k,j),(i,k,k) \in I$, $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$, so that (A2) holds. For all $i,k \in \{1,\dots,n\}$ with $i \neq k$, since $(i,k,i) \in Z$, $a_{ii}^{(k)} = 0$, so that (A3) holds. For all $i,k \in \{1,\dots,n\}$, since $(i,k,\sigma(i)) \in I$, $a_{i\sigma(i)}^{(k)} = 1$. Therefore, Proposition 2.1.1 completes the proof. □

When we check (A1) for $\mathbb{A}$ of Proposition 2.1.4, we need to consider 4-tuples of integers. Let F be the set of 4-tuples (i,k,j,l) of integers i,k,j,l such that $1 \leq i,k,j,l \leq n$. Let σ be a given permutation of the set $\{1,\dots,n\}$ such that $\sigma(i) \neq i$ for all $i = 1,\dots,n$. Let $\mathcal{F}$ be the set of 4-tuples $(T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4})$ of φ-orbits $T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4}$ of T. Then we have the following two bijections

$$\psi : F \to F, \quad (i,k,j,l) \mapsto (k,j,l,\sigma(i))$$

$$\theta : \mathcal{F} \to \mathcal{F}, \quad (T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4}) \mapsto (T_{\alpha_4},T_{\alpha_3},T_{\alpha_1},T_{\alpha_2})$$

and a map

$$\Phi : F \to \mathcal{F}, \ (i,k,j,l) \mapsto (T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4})$$

where $T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4}$ are defined by $(i,k,j) \in T_{\alpha_1}$, $(i,j,l) \in T_{\alpha_2}$, $(i,k,l) \in T_{\alpha_3}$, $(k,j,l) \in T_{\alpha_4}$.

LEMMA 2.1.5. *It holds that* $\Phi \circ \psi = \theta \circ \Phi$ *and* $\theta^4 = \mathrm{id}$.

Proof. Take an arbitrary $(i,k,j,l) \in F$, and let $\Phi(i,k,j,l) = (T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4})$. Then we have $(k,l,\sigma(i)) = \varphi(i,k,l) \in T_{\alpha_3}$, $(k,j,\sigma(i)) = \varphi(i,k,j) \in T_{\alpha_1}$, and $(j,l,\sigma(i)) = \varphi(i,j,l) \in T_{\alpha_2}$. Hence

$$\Phi \circ \psi(i,k,j,l) = \Phi(k,j,l,\sigma(i)) = (T_{\alpha_4},T_{\alpha_3},T_{\alpha_1},T_{\alpha_2}).$$

Therefore $\Phi \circ \psi = \theta \circ \Phi$. It is immediate to check that $\theta^4 = \mathrm{id}$. □

PROPOSITION 2.1.6. *Let* Y *be a* φ*-invariant subset of* X*, and let* $\mathbb{A}$ *be an* n*-tuple of* $n \times n$ *matrices defined as in Proposition 2.1.4. Let* $(\ ) : \{T_\alpha\}_\alpha \to K$ *be a map defined by*

$$(T_\alpha) := \begin{cases} 1 & \text{if } T_\alpha \subset I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

Then the following are equivalent.

(1) $\mathbb{A}$ *is a structure system of a Frobenius full matrix algebra.*

(2) For any $(T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4}) \in \mathrm{Im}\Phi$, $(T_{\alpha_1})(T_{\alpha_2}) = (T_{\alpha_3})(T_{\alpha_4})$ *holds.*

Proof. Since $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ for all $(i,k,j) \in T$, this follows from Proposition 2.1.4. □

PROPOSITION 2.1.7. *For any* $(T_{\alpha_1},T_{\alpha_2},T_{\alpha_3},T_{\alpha_4}) \in \mathrm{Im}\Phi$*, the following statements hold.*

(1) $T_{\alpha_1},T_{\alpha_2} \subset I \ \Leftrightarrow \ T_{\alpha_3},T_{\alpha_4} \subset I$.

(2) If $T_{\alpha_1} \subset I$ *and* $T_{\alpha_2} \subset X$*, then* $T_{\alpha_3} = T_{\alpha_2}$ *and* $T_{\alpha_4} \subset I$*, or else* $T_{\alpha_4} = T_{\alpha_2}$ *and* $T_{\alpha_3} \subset I$.

Proof. Let $(T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, T_{\alpha_4}) = \Phi(i,k,j,l)$ for some $(i,k,j,l) \in F$.

(1) Suppose that $T_{\alpha_1}, T_{\alpha_2} \subset I$. Then $(i,k,j), (i,j,l) \in I$. Since $(i,k,j) \in I$, we have $i = k$, or $k = j$, or $j = \sigma(i)$ by Lemma 2.1.2. Similarly, we have $i = j$, or $j = l$, or $l = \sigma(i)$. By the assumption of σ, we can exclude the case that $j = \sigma(i)$ and $i = j$. In the remaining eight cases, i.e., $i = k$ and $i = j$, etc., one can check that $(i,k,l), (k,j,l) \in I$ by Lemma 2.1.2. Hence $T_{\alpha_3}, T_{\alpha_4} \subset I$. We can show the converse implication using Lemma 2.1.5.

(2) Since $T_{\alpha_1} \subset I$ and $T_{\alpha_2} \subset X$, $(i,k,j) \in I$ and $(i,j,l) \in X$. Since $(i,j,l) \in X$, we have $j \neq \sigma(i)$ by Lemma 2.1.2. Therefore, since $(i,k,j) \in I$, we have $i = k$ or else $k = j$. In the case of $i = k$, $(i,i,l) = (i,k,l) \in T_{\alpha_3}$ and $(i,j,l) = (k,j,l) \in T_{\alpha_4}$, so that $T_{\alpha_3} \subset I$ and $T_{\alpha_4} = T_{\alpha_2}$. Similarly, in the case of $k = j$, we have $T_{\alpha_3} = T_{\alpha_2}$ and $T_{\alpha_4} \subset I$. □

As an application of Proposition 2.1.7 (1), we have the following corollary, which is shown in Theorem 4.4 of [7].

Corollary 2.1.8. *Let σ be an arbitrary permutation of the set $\{1, 2, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$. Then there exists a Frobenius $n \times n$ full matrix algebra with Nakayama permutation σ.*

Proof. Let Y be empty, and let $\mathbb{A}$ be an n-tuple of $n \times n$ matrices as defined in Proposition 2.1.4. Then it follows from Proposition 2.1.7 (1) that $\mathbb{A}$ is a structure system of a Frobenius full matrix algebra. □

Remark 2.1.9. Let $A = \bigoplus_{1 \le i,j \le n} K u_{ij}$ be a full matrix algebra with a structure system $\mathbb{A} = (A_1, \ldots, A_n)$ where $A_k = (a_{ij}^{(k)})$ $(1 \le k \le n)$, and let ρ be a permutation of the set $\{1, \ldots, n\}$. Then, let $\mathbb{B} = (B_1, \ldots, B_n)$ be an n-tuple of $n \times n$ matrices $B_k = (b_{ij}^{(k)})$ $(1 \le k \le n)$ defined by $b_{ij}^{(k)} := a_{\rho(i)\rho(j)}^{(\rho(k))}$ for all $(i,k,j) \in T$. It is obvious that $\mathbb{B}$ satisfies (A1), (A2) and (A3). Let $B = \bigoplus_{1 \le i,j \le n} K v_{ij}$ be a $\mathbb{B}$-full matrix algebra. Then there is a K-algebra isomorphism $f : A \to B, u_{ij} \mapsto v_{\rho^{-1}(i)\rho^{-1}(j)}$. We denote B by A_ρ. Let σ, τ be permutations such that their cycle decompositions have the same orders. Then there exists a permutation ρ such that $\tau = \rho^{-1}\sigma\rho$. If A is a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ, then it follows from Proposition 2.1.1 that A_ρ is a Frobenius full matrix algebra with Nakayama permutation τ. Therefore we can classify Frobenius full matrix algebras according to cycle decompositions of Nakayama permutations with a fixed numbering. Thus, for $n = 4, 5$, Example 4.7 of [7] provides a list of all Frobenius $\mathbb{A}$-full matrix algebras up to isomorphism.

2.1.2 φ-orbits for a cyclic permutation.

In this section, we clarify φ-orbits of the set T for a cyclic permutation $\sigma = (1\ 2\ \cdots\ n)$.

Lemma 2.1.10. *Let T_α be a φ-orbit of T. Then the number $|T_\alpha|$ of elements in T_α is $3n$ or n. More precisely, the following statements hold.*

(1) If $|T_\alpha|$ is divisible by 3, then $|T_\alpha| = 3n$.

(2) If $|T_\alpha|$ is not divisible by 3, then $|T_\alpha| = n$.

Proof. Take an element $(i,k,j) \in T_\alpha$. Since $\varphi^{3n}(i,k,j) = (\sigma^n(i), \sigma^n(k), \sigma^n(j)) = (i,k,j)$, $|T_\alpha| \leq 3n$. Let s be the smallest positive integer satisfying $\varphi^s(i,k,j) = (i,k,j)$. Then $|T_\alpha| = s$ and $s|3n$.

(1) Suppose that $s = 3t$ for some integer $t \geq 1$. Then

$$\varphi^s(i,k,j) = (\sigma^t(i), \sigma^t(k), \sigma^t(j)) = (i,k,j).$$

Hence $\sigma^t(i) = i$. Since $\sigma = (1\ 2\ \cdots\ n)$, $\sigma^t = \varepsilon$ (the identity permutation), so that $n|t$ and hence $s = 3n$.

(2) Suppose that $s = 3t+1$ for some integer $t \geq 1$. Then

$$\varphi^s(i,k,j) = (\sigma^t(k), \sigma^t(j), \sigma^{t+1}(i)) = (i,k,j).$$

Hence $i = \sigma^t(k) = \sigma^{2t}(j) = \sigma^{3t+1}(i) = \sigma^s(i)$, so that $n|s$. Since $s|3n$ and $s \not\equiv 0 \pmod 3$, we conclude that $s = n$.

In the case of $s = 3t+2$ for some integer $t \geq 1$, we can show that $s = n$ in a similar way. $\square$

LEMMA 2.1.11. *Every φ-orbit has elements $(1,k_1,j_1), (i_2,1,j_2), (i_3,k_3,1)$ for some $k_1, j_1, i_2, j_2, i_3, k_3 \in \{1,2,\ldots,n\}$.*

Proof. Take an element (i,k,j) in a φ-orbit T_α. Since $\varphi^3(i,k,j) = (\sigma(i), \sigma(k), \sigma(j))$ and $\sigma = (1\ 2\ \cdots\ n)$, T_α contains $\varphi^{3(n-i+1)}(i,k,j) = (1,k_1,j_1)$ for some $1 \leq k_1, j_1 \leq n$. Similarly, $(i_2,1,j_2), (i_3,k_3,1) \in T_\alpha$ for some $1 \leq i_2, j_2, i_3, k_3 \leq n$. $\square$

PROPOSITION 2.1.12. *T has a φ-orbit T_α with $|T_\alpha| = n$ if and only if n is not divisible by 3. In this case, T has a unique φ-orbit having n elements, which is contained in X.*

Proof. 'Only if' part follows from Lemma 2.1.10 (1). Suppose that $n = 3t+1$ for some $t \geq 1$. Let T_α be the φ-orbit containing $(1, 2t+2, t+2)$. Then one can check that $|T_\alpha| = n$. Conversely, let T_β be a φ-orbit with $|T_\beta| = n$. It follows from Lemma 2.1.11 that T_β contains an element $(1,k,j)$ for some k, j. Since $(1,k,j) = \varphi^n(1,k,j) = (\sigma^t(k), \sigma^t(j), \sigma^{t+1}(1))$, we have $j = t+2$ and $k = 2t+2$. Therefore $T_\beta = T_\alpha$, which is a unique φ-orbit of T having n elements. Since $(1, 2t+2, t+2) \in X$, $T_\alpha \subset X$.

In the case of $n = 3t+2$ for some $t \geq 1$, one can similarly show that the φ-orbit containing $(1, t+2, 2t+3)$ is a unique φ-orbit having n elements, which is contained in X. $\square$

PROPOSITION 2.1.13. *The following statements hold.*

(1) I has $n-1$ φ-orbits.

(2) Z has $n-2$ φ-orbits.

(3) Let $n \equiv 0 \pmod 3$. Then T has $n^2/3$ φ-orbits and X has $(n-3)^2/3$ φ-orbits.

(4) Let $n \not\equiv 0 \pmod 3$. Then T has $(n^2-1)/3+1$ φ-orbits and X has $(n-2)(n-4)/3+1$ φ-orbits.

Proof. (1) Let T_α be a φ-orbit contained in I. Then, by Lemma 2.1.11, $(1,1,i) \in T_\alpha$ for some $1 \le i \le n$. Suppose that $(1,1,j) \in T_\alpha$ for some $j \ne i$. Then $(1,1,j) = \varphi^s(1,1,i)$ for some $s \ge 1$. Considering the three cases of $s \equiv 0,1,2 \pmod 3$, one can show that $\{i,j\} = \{1,2\}$. Therefore T_α corresponds to a unique integer i with $2 \le i \le n$. Hence I has $n-1$ φ-orbits.

(2) Let T_α be a φ-orbit contained in Z. Then, by Lemma 2.1.11, $(1,k,1) \in T_\alpha$ for some $2 \le k \le n$. Suppose that $(1,l,1) \in T_\alpha$ for some $l \ne k$. Then one can similarly show that $\{k,l\} = \{2,n\}$. Therefore T_α corresponds to a unique integer k with $3 \le k \le n$. Hence Z has $n-2$ φ-orbits.

(3) It follows from Lemma 2.1.10 and Proposition 2.1.12 that every φ-orbit of T has $3n$ elements. Since $|T| = n^3$, T has $n^2/3$ φ-orbits. Since X is the complement of $I \cup Z$ in T, X has $(n-3)^2/3$ φ-orbits by (1) and (2).

(4) This can be shown as in (3). □

PROPOSITION 2.1.14. *Let T_α be a φ-orbit of T. For each $r = 1, \dots, n$, put $T_\alpha^{(r)} := \{(i,k,j) \in T_\alpha \mid k = r\}$. Then the following statements hold.*

(1) Suppose that $|T_\alpha| = 3n$. Then $|T_\alpha^{(r)}| = 3$ for each $r = 1, \dots, n$. If $(i,1,j) \in T_\alpha$, then

$$T_\alpha^{(1)} = \{(i,1,j), (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i)), (\sigma^{-i}(j), 1, \sigma^{-i+1}(1))\}.$$

(2) If $|T_\alpha| = n$, then $|T_\alpha^{(r)}| = 1$ for each $r = 1, \dots, n$.

Proof. Note that there is a bijection $T_\alpha^{(r)} \to T_\alpha^{(1)}, (i,r,j) \mapsto \varphi^{3(n-r+1)}(i,r,j) = (\sigma^{-r+1}(i), 1, \sigma^{-r+1}(j))$. Hence we have $|T_\alpha^{(r)}| = |T_\alpha^{(1)}|$ for each $r = 1, \dots, n$. Therefore for each $r = 1, \dots, n$, $|T_\alpha| = n|T_\alpha^{(r)}|$, so that $|T_\alpha^{(r)}| = 3$ (or 1) if $|T_\alpha| = 3n$ (or n).

Suppose that $|T_\alpha| = 3n$ and $(i,1,j) \in T_\alpha$. Note that $T_\alpha^{(1)}$ contains $(i,1,j)$,

$$\begin{aligned} \varphi^{3(n-j+1)+1}(i,1,j) &= (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i)) \text{ and} \\ \varphi^{3(n-i+1)-1}(i,1,j) &= (\sigma^{-i}(j), 1, \sigma^{-i+1}(1)). \end{aligned}$$

It is sufficient to show that the above three elements are distinct. Assume that

$$(i,1,j) = (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i)).$$

Then one can show that $3(j-1) \equiv 1 \pmod n$. Hence n is not divisible by 3. Assume that $n = 3t+1$ for some integer $t \ge 1$. Then we have $j = 2t+2$ and $i = t+1$. Hence, as shown in the proof of Proposition 2.1.12, T_α is the φ-orbit having n elements, which contradicts to $|T_\alpha| = 3n$. In the case of $n = 3t+2$, we get to a contradiction, in a similar way. Hence $(i,1,j) \ne (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i))$. We can show the remaining cases in a similar way. This completes the proof. □

2.1.3 Minimal Frobenius structure systems.

Let A be a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ. Then it follows from Proposition 2.1.3 that the structure system $\mathbb{A}$ is determined by a φ_σ-invariant subset Y of X. We call $\mathbb{A}$ a *minimal Frobenius structure system* if Y is minimal among non-empty φ_σ-invariant subsets of X which define Frobenius full matrix algebras with Nakayama permutation σ. In this section, we prove the following theorem.

THEOREM 2.1.15. *Let n be an integer with $n \geq 4$, and let $\sigma = (1\ 2\ \cdots\ n)$ be a cyclic permutation. Then the following statements hold.*

(1) Let n be even. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits contained in X.

(2) Let n be odd and $n = 2s+1$ for some s. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits X_β contained in X such that X_β does not contain any element of the form $(s+1, 1, k)$ for any k with $k \not\equiv s^2 + 1 \pmod{n}$.

We note that in the case of $n = 2, 3$, structure systems of Frobenius full matrix algebras are unique. (See Corollary 4.3 and Remark 4.6 of [7].)

In the rest of this section, we assume that $\sigma = (1\ 2\ \cdots\ n)$.

LEMMA 2.1.16. *For any φ-orbit $X_\beta \subset X$, the following are equivalent.*

(1) $(X_\beta, X_\beta, T_{\alpha_3}, T_{\alpha_4}) \in \mathrm{Im}\Phi$ for some φ-orbits $T_{\alpha_3}, T_{\alpha_4}$.

(2) There exist integers s, k such that $n = 2s+1$ and $(s+1, 1, k) \in X_\beta$.

In this case, $\Phi(s+1, \sigma^s(k), 1, k) = (X_\beta, X_\beta, T_{\alpha_3}, T_{\alpha_4})$ and $T_{\alpha_3} = T_{\alpha_4} \subset X$.

Proof. (1) $\Rightarrow$ (2) Suppose that $(X_\beta, X_\beta, T_{\alpha_3}, T_{\alpha_4}) = \Phi(i, k, j, l)$ for some $(i, k, j, l) \in F$. Using Lemma 2.1.5 as Lemma 2.1.11, we can assume that $i = 1$. Then $(1, k, j), (1, j, l) \in X_\beta$. Hence using Proposition 2.1.14, we can show that $(1, j, l)$ is one of

$$(1, k, j),\ (1, \sigma^{-k+1}(j), \sigma^{-k+1}(2)),\ \text{ or }\ (1, \sigma^{-j+1}(2), \sigma^{-j+2}(k)).$$

Since $k \neq j$ and $k \neq 1$, we conclude that $(1, j, l) = (1, \sigma^{-j+1}(2), \sigma^{-j+2}(k))$. Hence we have $2j \equiv 3 \pmod{n}$. Hence n is odd. Put $n = 2s+1$ for some integer $s \geq 1$. Then we have $j = s+2$. Hence $(s+1, 1, k) = \varphi^{-1}(1, k, s+2) \in X_\beta$.

(2) $\Rightarrow$ (1) Note that

$$(s+1, \sigma^s(k), 1) = \varphi^{3s+1}(s+1, 1, k) \in X_\beta.$$

Hence

$$\Phi(s+1, \sigma^s(k), 1, k) = (X_\beta, X_\beta, T_{\alpha_3}, T_{\alpha_4})$$

for some φ-orbits $T_{\alpha_3}, T_{\alpha_4}$ of T.

In this case, note that $(s+1, \sigma^s(k), k) \in T_{\alpha_3}$ and $(\sigma^s(k), 1, k) \in T_{\alpha_4}$. Since

$$\varphi^{3s+1}(\sigma^s(k), 1, k) = (s+1, \sigma^s(k), k),$$

we have $T_{\alpha_3} = T_{\alpha_4}$. Using Lemma 2.1.2 and $n = 2s+1$, one can check that $(s+1, 1, k) \in X$ implies $(\sigma^s(k), 1, k) \in X$. Hence $T_{\alpha_3} = T_{\alpha_4} \subset X$. □

LEMMA 2.1.17. *For any φ-orbit $X_\beta \subset X$, the following are equivalent.*

(1) $(X_\beta, X_\beta, X_\beta, X_\beta) \in \mathrm{Im}\Phi$.

(2) There exist integers s, k such that $n = 2s+1$, $(s+1, 1, k) \in X_\beta$ and that $k \equiv s^2 + 1 \pmod{n}$.

Proof. (1) $\Rightarrow$ (2) It follows from Lemma 2.1.16 that there exist integers s, k such that $n = 2s+1$, $(s+1, 1, k) \in X_\beta$ and that $(\sigma^s(k), 1, k) \in X_\beta$. Since $k \neq 1$ and $k \neq \sigma(s+1)$, using Proposition 2.1.14, we conclude that

$$(\sigma^s(k), 1, k) = (\sigma^{-k+1}(1), 1, \sigma^{-k+1}(s+2)).$$

Hence we have $2k \equiv s+3 \pmod{n}$. Since $2(s+1) \equiv 1 \pmod{n}$, we obtain $k \equiv s^2 + 1 \pmod{n}$.

(2) $\Rightarrow$ (1) Since $k \equiv s^2 + 1 \pmod{n}$, $2k \equiv s+3 \pmod{n}$. Hence

$$(\sigma^s(k), 1, k) = (\sigma^{-k+1}(1), 1, \sigma^{-k+1}(s+2)) = \varphi^{3(n-k+1)+1}(s+1, 1, k) \in X_\beta.$$

This completes the proof. □

Proof of Theorem 2.1.15. (1) Let n be even, and let X_β be a φ-orbit contained in X. For $Y = X_\beta$, we put $\mathbb{A}$ an n-tuple of $n \times n$ matrices defined as in Proposition 2.1.4. Since each nonempty φ-invariant subset of X contains a φ-orbit, it is sufficient to show that $\mathbb{A}$ is a structure system of a Frobenius full matrix algebra. Let $(T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, T_{\alpha_4})$ be an element in $\mathrm{Im}\Phi$ such that $(T_{\alpha_1})(T_{\alpha_2}) = 1$. In the case of $T_{\alpha_1}, T_{\alpha_2} \subset I$, it follows from Proposition 2.1.7 (1) that $T_{\alpha_3}, T_{\alpha_4} \subset I$, so that $(T_{\alpha_3})(T_{\alpha_4}) = 1$. In the case of $T_{\alpha_1} \subset I$ and $T_{\alpha_2} = X_\beta$, it follows from Proposition 2.1.7 (2) that one of $T_{\alpha_3}, T_{\alpha_4}$ is contained in I and the other one is equal to X_β, so that $(T_{\alpha_3})(T_{\alpha_4}) = 1$. It follows from Lemma 2.1.16 that $\mathrm{Im}\Phi$ does not contain $(X_\beta, X_\beta, T_\alpha, T_{\alpha'})$ for any φ-orbits $T_\alpha, T_{\alpha'}$ of T. Therefore, using Lemma 2.1.5, we can show that for any $(T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, T_{\alpha_4}) \in \mathrm{Im}\Phi$, $(T_{\alpha_1})(T_{\alpha_2}) = (T_{\alpha_3})(T_{\alpha_4})$ holds. Hence by Proposition 2.1.6, $\mathbb{A}$ is a structure system of a Frobenius full matrix algebra.

(2) Let n be odd and $n = 2s+1$ for some s. Let X_β be a φ-orbit contained in X such that X_β does not contain any element of the form $(s+1, 1, k)$ for any k with $k \not\equiv s^2 + 1 \pmod{n}$. Then, using Lemma 2.1.17, as in the proof of (1), we can show that X_β corresponds to a minimal Frobenius structure system.

Conversely, let Y be a φ-invariant subset of X corresponding to a minimal Frobenius structure system. Assume that Y contains a φ-orbit X_β such that $(s+1, 1, k) \in X_\beta$ for some k with $k \not\equiv s^2+1 \pmod{n}$. Then it follows from Lemma 2.1.16 that

$$\Phi(s+1, \sigma^s(k), 1, k) = (X_\beta, X_\beta, X_{\beta'}, X_{\beta'})$$

for some φ-orbit $X_{\beta'} \subset X$. Since $(X_\beta)(X_\beta) = (X_{\beta'})(X_{\beta'})$, we have $X_{\beta'} \subset Y$. Note that

$$(s+1, \sigma^{-k+1}(1), 1) = \varphi^{3(s-k+1)+2}(s+1, \sigma^s(k), k) \in X_{\beta'}.$$

Hence

$$\Phi(s+1, \sigma^{-k+1}(1), 1, k) = (X_{\beta'}, X_\beta, T_{\alpha_3}, T_{\alpha_4})$$

for some φ-orbits $T_{\alpha_3}, T_{\alpha_4}$ of T. Since $(X_{\beta'})(X_\beta) = (T_{\alpha_3})(T_{\alpha_4})$, using Lemma 2.1.5 and Proposition 2.1.7 (2), we obtain $T_{\alpha_3}, T_{\alpha_4} \subset X$. Then $(s+1, \sigma^{-k+1}(1), 1) \in T_{\alpha_3}$. Since $k \neq 1$ and $k \not\equiv s^2+1 \pmod{n}$, using Proposition 2.1.14, we can show that T_{α_3} does not contain any element of the form $(s+1, 1, l)$ for any l. Hence T_{α_3} corresponds to a minimal Frobenius structure system as shown above. This contradicts to that Y is minimal. This completes the proof. □

The following example illustrates Theorem 2.1.15 and gives all φ-orbits which define minimal Frobenius structure systems for $n = 5, 6, 7$.

EXAMPLE 2.1.18. (1) Let $n = 5$. As given in Example 4.7 (3) of [7], X has two φ-orbits X_1 and X_2. Then

$$X_1^{(1)} = \{(2,1,4), (3,1,5), (2,1,5)\} \quad \text{and} \quad X_2^{(1)} = \{(4,1,3)\}.$$

(See Propositions 2.1.12, 2.1.13 and 2.1.14, too.) Since $5 = 2 \cdot 2 + 1 = 2^2 + 1$, both of X_1 and X_2 define minimal Frobenius structure systems.

(2) Let $n = 6$. Then X has three φ-orbits, all of which define minimal Frobenius structure systems. (See §2.1.6 (1).)

(3) Let $n = 7$. Then X has 6 φ-orbits X_i $(1 \leq i \leq 6)$ such that

$$\begin{aligned}
X_1^{(1)} &= \{(4,1,3), (6,1,3), (6,1,5)\} \\
X_2^{(1)} &= \{(2,1,5), (3,1,7), (4,1,6)\} \\
X_3^{(1)} &= \{(2,1,6), (4,1,7), (3,1,5)\} \\
X_4^{(1)} &= \{(5,1,3), (5,1,4), (6,1,4)\} \\
X_5^{(1)} &= \{(2,1,4), (2,1,7), (5,1,7)\} \\
X_6^{(1)} &= \{(3,1,6)\}
\end{aligned}$$

Since $7 = 2 \cdot 3 + 1$ and $3^2 + 1 \equiv 3 \pmod{7}$, there are minimal Frobenius structure systems corresponding to X_1, X_4, X_5, X_6, but not to X_2, X_3.

2.1.4 Gorenstein tiled orders.

Let D be a commutative discrete valuation domain with a unique maximal ideal πD. Let $n \geq 2$ be an integer. Let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be the set of non-negative integers satisfying

$$\lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \quad \lambda_{ii} = 0 \quad \text{and} \quad \lambda_{ij} + \lambda_{ji} > 0 \ \text{ if } i \neq j$$

for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a D-subalgebra of $\mathbb{M}_n(D)$. We call Λ an $n \times n$ *tiled D-order*. (See e.g. Jategaonkar [7], Reiner [25], Simson [28].)

For a tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$, let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ defined by

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Lambda/\pi\Lambda$ is isomorphic to an $\mathbb{A}$-full matrix algebra. (See Example 3.2 of [7].)

A tiled D-order Λ is *Gorenstein* if $\mathrm{Hom}_D(\Lambda, D)$ is projective as a right (or left) Λ-module. It is known that Λ is a Gorenstein tiled D-order if and only if there exists a permutation σ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$ and that

$$\lambda_{ik} + \lambda_{k\sigma(i)} = \lambda_{i\sigma(i)}$$

for all $1 \leq i, k \leq n$. (See Theorem 1.4 of [26].) It is also known that Λ is Gorenstein if and only if $\Lambda/\pi\Lambda$ is Frobenius. (Note that, in our context, this follows from Proposition 2.1.1.)

In this section, we prove the following theorem.

THEOREM 2.1.19. *For every integer $n \geq 8$, there exists a Frobenius $n \times n$ full matrix algebra which has no corresponding Gorenstein tiled orders.*

We begin by studying the following system of linear equations with unknowns $\{x_{ij} \mid 1 \leq i, j \leq n\}$.

$$(*) \qquad \begin{cases} x_{ii} = 0 \\ x_{1j} = 0 \\ x_{ik} + x_{k\sigma(i)} - x_{i\sigma(i)} = 0 \end{cases}$$

where $i, j, k \in \{1, \ldots, n\}$.

LEMMA 2.1.20. *Assume that $\sigma = (1\ 2\ \cdots\ n)$, and let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be a solution of the system of linear equations $(*)$. Put $a_i := \lambda_{i1}$ $(2 \leq i \leq n)$. Then the following equations hold.*

(1) $\lambda_{i2} = 0$ *for all* $i = 1, \ldots, n$.

(2) $a_i = \lambda_{i1} = \lambda_{i,i+1} = \lambda_{2,i+1}$ *for all* $i = 2, \ldots, n-1$.

(3) $\lambda_{ik} = \sum_{l=1}^{k-2} (a_{k-l} - a_{i-l})$ *for all* i, k *with* $3 \leq k < i \leq n$.

(4) $\lambda_{ki} = \sum_{l=1}^{k-2} (a_{i-l} - a_{k-l}) + a_{i-k+1}$ *for all* i, k *with* $3 \leq k < k+1 < i \leq n$.

(5) $a_{n-i} = a_{i+2}$ *for all* $i = 0, 1, \ldots, [\frac{n-1}{2}] - 1$.

Conversely, for given $a_2, \ldots, a_m$ where $m = [\frac{n}{2}]$, we obtain a solution $\{\lambda_{ij}\}$ of $(*)$ using the equations (1)–(5).

Proof. (1) Since $\lambda_{1j} = 0$ for all $j = 1, \ldots, n$ and $2 = \sigma(1)$, we have $\lambda_{i2} = \lambda_{1i} + \lambda_{i2} = \lambda_{12} = 0$ for all $i = 1, \ldots, n$.

(2) Since $\lambda_{i1} = \lambda_{i1} + \lambda_{1\sigma(i)} = \lambda_{i\sigma(i)}$, we have $\lambda_{i1} = \lambda_{i,i+1}$ for all $i = 2, \ldots, n-1$. Hence, moreover by (1), $\lambda_{i,i+1} = \lambda_{i2} + \lambda_{2,i+1} = \lambda_{2,i+1}$ for all $i = 2, \ldots, n-1$.

(3) Note that $a_{k-1} = \lambda_{k-1,k} = \lambda_{k-1,i} + \lambda_{ik}$ and $a_{i-1} = \lambda_{i-1,i} = \lambda_{i-1,k-1} + \lambda_{k-1,i}$. Hence $\lambda_{ik} = a_{k-1} - a_{i-1} + \lambda_{i-1,k-1}$. Since $\lambda_{i-k+3,3} = a_2 - a_{i-k+2}$, we obtain the desired equations by induction.

(4) Since $\lambda_{ki} = a_{i-1} - \lambda_{i-1,k}$, we obtain the desired equations using (3).

(5) Since $\sigma(n) = 1$, $\lambda_{n1} = \lambda_{nk} + \lambda_{k1}$ for all $k = 1, \ldots, n$. Hence by (1) we have $a_n = \lambda_{n1} = \lambda_{21} = a_2$, and hence $\lambda_{nk} = a_2 - a_k$ for all $k = 3, \ldots, n-1$. On the other hand, by (3), we have $\lambda_{nk} = \sum_{l=1}^{k-2}(a_{k-l} - a_{n-l})$ for all $k = 3, \ldots, n-1$. Therefore, when $k = 3$, we have $a_2 - a_3 = \lambda_{n3} = a_2 - a_{n-1}$, so that $a_3 = a_{n-1}$. We obtain the remaining equations by induction. □

LEMMA 2.1.21. *Assume that $\sigma = (1\ 2\ \cdots\ n)$, and let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be a solution of $(*)$. Then the following equations hold.*

(1) $\lambda_{i1} + \lambda_{1j} - \lambda_{ij} = \lambda_{j,\sigma(i)}$ for all $1 \leq i, j \leq n$.

(2) $\lambda_{ij} = \lambda_{i,\sigma^{-j+i}(2)}$ for all $1 \leq i, j \leq n$.

(3) $\lambda_{ik} + \lambda_{kj} - \lambda_{ij} = \lambda_{kj} + \lambda_{j\sigma(i)} - \lambda_{k\sigma(i)}$ for all $1 \leq i, j, k \leq n$.

(4) For any φ-orbit T_α, if $(i, 1, j), (i', 1, j') \in T_\alpha$, then $\lambda_{j\sigma(i)} = \lambda_{j'\sigma(i')}$.

Proof. (1) Since $\lambda_{ij} + \lambda_{j\sigma(i)} = \lambda_{i\sigma(i)}$ and $\lambda_{i1} + \lambda_{1\sigma(i)} = \lambda_{i\sigma(i)}$, we have $\lambda_{i1} + \lambda_{1j} - \lambda_{ij} = \lambda_{i1} - \lambda_{ij} = \lambda_{j,\sigma(i)}$.

(2) One can check the equation using Lemma 2.1.20.

(3) This follows from $\lambda_{ij} + \lambda_{j\sigma(i)} = \lambda_{i\sigma(i)} = \lambda_{ik} + \lambda_{k\sigma(i)}$.

(4) This follows from (3) and (1). □

The following example illustrates the equations of Lemmas 2.1.20 and 2.1.21.

EXAMPLE 2.1.22. When $n = 8$, the matrix (λ_{ij}) of the solution of $(*)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & b & c & d & c & b \\ b & 0 & 0 & b & r & s & s & r \\ c & 0 & x & 0 & c & s & t & s \\ d & 0 & y & y & 0 & d & s & s \\ c & 0 & z & w & z & 0 & c & r \\ b & 0 & y & w & w & y & 0 & b \\ a & 0 & x & y & z & y & x & 0 \end{pmatrix}$$

where $x := a - b$, $y := a - c$, $z := a - d$, $w := a + b - c - d$, $r := b + c - a$, $s := c + d - a$, and $t := 2c + d - b - a$.

Proof of Theorem 2.1.19. Assume that $\sigma = (1\ 2\ \cdots\ n)$, and let X_β be a φ-orbit containing $(2, 1, 5)$. Then it follows from Proposition 2.1.14 that

$$X_\beta^{(1)} = \{(2, 1, 5), (n-3, 1, n-1), (3, 1, n)\}.$$

Since $n \geq 8$, it follows from Theorem 2.1.15 that X_β defines a minimal Frobenius structure system $\mathbb{A}$ as in Proposition 2.1.4. Suppose that there exists a Gorenstein tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$ corresponding to $\mathbb{A}$. Then by Lemma 1.1 of [7], we may assume that $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ is a solution of $(*)$. Since $a_{25}^{(1)} = 1$, it follows from Lemma 2.1.21 (1) that $\lambda_{53} = \lambda_{5,\sigma(2)} = 0$. Hence it follows from Lemma 2.1.21 (2) that $\lambda_{54} = 0$. Therefore it follows from Lemma 2.1.21 (1) that $a_{35}^{(1)} = 1$, so that $(3, 1, 5) \in X_\beta$, a contradiction. This completes the proof. □

REMARK 2.1.23. (1) We can find Gorenstein tiled D-orders $\Lambda = (\pi^{\lambda_{ij}} D)$ by solving some linear inequalities in integers. For example, by Example 2.1.22, a set of integers a, b, c, d satisfying some inequalities $a \geq b > 0, a \geq c > 0, a \geq d > 0, b + c \geq c + d$, etc., defines a Gorenstein tiled D-order $\Lambda(a, b, c, d)$. Note that the factor algebras $\Lambda(a, b, c, d)/\pi\Lambda(a, b, c, d)$ essentially depend on a, b, c, d and provide some Frobenius full matrix algebras. While we have shown in Theorem 2.1.19 that some Frobenius full matrix algebras do not have corresponding Gorenstein tiled orders, we can classify Gorenstein tiled orders (or e.g. the above a, b, c, d), using the classification of Frobenius full matrix algebras.

(2) Note that if (λ_{ij}) defines a Gorenstein tiled D-order Λ then for each integer $t \geq 1$, $(t \cdot \lambda_{ij})$ defines a Gorenstein tiled D-order Λ_t such that $\Lambda/\pi\Lambda \cong \Lambda_t/\pi\Lambda_t$. Hence if a Frobenius full matrix algebra has a corresponding Gorenstein tiled order, it has infinitely many corresponding Gorenstein tiled orders. The following example shows that the infinite series $\{\Lambda_t \mid t \geq 1\}$ is not necessarily unique for a Frobenius full matrix algebra.

EXAMPLE 2.1.24. Let Λ, Γ be Gorenstein tiled D-orders defined by the following exponent matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 2 \\ 2 & 0 & 0 & 2 \\ 3 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 3 \\ 3 & 0 & 0 & 3 \\ 5 & 0 & 2 & 0 \end{pmatrix}.$$

Then note that $\Lambda/\pi\Lambda \cong \Gamma/\pi\Gamma$ and that $\{\Lambda_t \mid t \geq 1\}$ and $\{\Gamma_t \mid t \geq 1\}$ are disjoint infinite series.

We end this section with the following example. (A similar example can be found in Shiba [27].)

EXAMPLE 2.1.25. Let σ be a permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$, and let $\mathbb{A}$ be a Frobenius structure system defined by the empty φ-invariant subset of X. (See Corollary 2.1.8.) Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be an $n \times n$ tiled D-order defined by

$$\lambda_{ij} := \begin{cases} 0 & \text{if } i = j \\ 2 & \text{if } j = \sigma(i) \\ 1 & \text{otherwise} . \end{cases}$$

Then $\Lambda/\pi\Lambda$ is isomorphic to a Frobenius $\mathbb{A}$-full matrix algebra.

2.1.5 Non-cyclic Nakayama permutation.

In this section, we study Frobenius full matrix algebras which are constructed from couples of Frobenius full matrix algebras.

Let $n = p + q$ with integers $p, q \geq 2$, and let B and C be Frobenius full matrix algebras of size $p \times p$ and $q \times q$, and having Nakayama permutations τ and ρ, respectively. Let σ be a permutation of the set $\{1, \ldots, n\}$ defined by

$$\sigma(i) := \begin{cases} \tau(i) & \text{if } 1 \leq i \leq p \\ p + \rho(i - p) & \text{if } p + 1 \leq i \leq n. \end{cases}$$

Let $\mathbb{B} = (b_{ij}^{(k)})$ and $\mathbb{C} = (c_{ij}^{(k)})$ be structure systems of B and C, respectively. Then we put $\mathbb{A} = (A_1, \ldots, A_n)$ an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ defined by

$$a_{ij}^{(k)} := \begin{cases} b_{ij}^{(k)} & \text{if } (i,k,j) \in T(p) \\ c_{i-p,j-p}^{(k-p)} & \text{if } (i,k,j) \in T(q) \\ 1 & \text{if } (i,k,j) \in I - (T(p) \cup T(q)) \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} T(p) &:= \{(i,k,j) \in T \mid 1 \leq i,j,k \leq p\} \\ T(q) &:= \{(i,k,j) \in T \mid p+1 \leq i,j,k \leq n\}. \end{aligned}$$

PROPOSITION 2.1.26. *$\mathbb{A}$ is a structure system of a Frobenius $n \times n$ full matrix algebra with Nakayama permutation σ.*

Proof. Note that $T(p), T(q)$ are φ-invariant subsets of T, so that $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ for all $(i,k,j) \in T$. Hence there is a φ-invariant subset Y of X such that $Y \subset T(p) \cup T(q)$ and

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i,k,j) \in I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

Let $(T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, T_{\alpha_4})$ be an arbitrary element of $\mathrm{Im}\Phi$ such that $(T_{\alpha_1})(T_{\alpha_2}) = 1$. If one of T_{α_1} and T_{α_2} is contained in I, then by Proposition 2.1.7, we have $(T_{\alpha_3})(T_{\alpha_4}) = 1$. Suppose that $T_{\alpha_1}, T_{\alpha_2} \not\subset I$. Then one can check that $T_{\alpha_1}, T_{\alpha_2} \subset Y \cap T(p)$ (or else $T_{\alpha_1}, T_{\alpha_2} \subset Y \cap T(q)$). Hence by Proposition 2.1.7, $T_{\alpha_3}, T_{\alpha_4} \subset Y \cap T(p)$ (or else $T_{\alpha_3}, T_{\alpha_4} \subset Y \cap T(q)$). Since $\mathbb{B}$ and $\mathbb{C}$ are structure systems, we have $(T_{\alpha_3})(T_{\alpha_4}) = 1$. Hence Lemma 2.1.5 and Proposition 2.1.6 complete the proof. □

We denote the above Frobenius $\mathbb{A}$-full matrix algebra by (B, C). The following theorem is a particular case where we can prove a converse of Proposition 2.1.26.

THEOREM 2.1.27. *Let A be a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ. Suppose that σ is decomposed into two cycles $(1 \cdots p)(p+1 \cdots n)$, where $n = p + q$ and $(p,q) = 1$. Then $A = (B, C)$ for some Frobenius full matrix algebras B and C with cyclic Nakayama permutations $(1 \cdots p)$ and $(1 \cdots q)$, respectively.*

In order to prove Theorem 2.1.27, we need the following three lemmas.

LEMMA 2.1.28. *Let $\mathbb{A} = (A_1, \ldots, A_n) = (a_{ij}^{(k)})$ be a structure system of a full matrix algebra. Then the following statements holds.*

(1) $a_{ij}^{(k)} a_{ik}^{(j)} = 0$ whenever $j \neq k$.

(2) For any $m \geq 1$, $a_{ix_1}^{(x_0)} a_{ix_2}^{(x_1)} \cdots a_{ix_0}^{(x_m)} = 0$ whenever $x_m \neq x_0$.

Proof. (1) It follows from (A1) and (A3) that $a_{ij}^{(k)}a_{ik}^{(j)} = a_{ik}^{(k)}a_{kk}^{(j)} = 0$ whenever $j \neq k$.

(2) We prove by induction on m. When $m = 1$, this follows from (1). Assume that $m \geq 2$. By (A1) for (i, x_{m-1}, x_m, x_0), we have

$$a_{ix_1}^{(x_0)} \cdots a_{ix_m}^{(x_{m-1})} a_{ix_0}^{(x_m)} = a_{ix_1}^{(x_0)} \cdots a_{ix_0}^{(x_{m-1})} a_{x_{m-1}x_0}^{(x_m)}.$$

Hence if $x_{m-1} = x_0$, $a_{x_{m-1}x_0}^{(x_m)} = 0$ by (A3), while if $x_{m-1} \neq x_0$, $a_{ix_1}^{(x_0)} \cdots a_{ix_0}^{(x_{m-1})} = 0$ by induction hypothesis. This completes the proof. □

In the rest of this section, let $\mathbb{A} = (A_1, \ldots, A_n) = (a_{ij}^{(k)})$ be a structure system of a Frobenius full matrix algebra A with Nakayama permutation $\sigma = (1 \cdots p)(p+1 \cdots n)$, where $n = p + q$ and $(p, q) = 1$.

LEMMA 2.1.29. *The following statements hold.*

(1) If $1 \leq i \leq p$, $p+1 \leq k, j \leq n$, *then* $a_{ij}^{(k)} = a_{i\sigma(j)}^{(\sigma(k))}$.

(2) If $1 \leq k, j \leq p$, $p+1 \leq i \leq n$, *then* $a_{ij}^{(k)} = a_{i\sigma(j)}^{(\sigma(k))}$.

Proof. Since $(p, q) = 1$, $ps + qt = 1$ for some integers s, t. Assume that $1 \leq i \leq p$ and $p + 1 \leq k, j \leq n$. Then $\sigma^{ps}(i) = i, \sigma^{ps}(k) = k$ and $\sigma^{qt}(j) = j$. Hence by Proposition 2.1.1, we have

$$a_{ij}^{(k)} = a_{\sigma(i)\sigma(j)}^{(\sigma(k))} = a_{\sigma^{ps}(i)\sigma^{ps}(j)}^{(\sigma^{ps}(k))} = a_{i\sigma^{ps}(j)}^{(\sigma^{ps}(k))} = a_{i\sigma^{ps+qt}(j)}^{(\sigma^{ps+qt}(k))} = a_{i\sigma(j)}^{(\sigma(k))}.$$

This completes the proof of (1). (2) is similar to (1). □

LEMMA 2.1.30. *The following statements hold.*

(1) If $1 \leq i \leq p$, $p+1 \leq k, j \leq n$ *and* $(i, k, j) \notin I$, *then* $a_{ij}^{(k)} = 0$.

(2) If $1 \leq k, j \leq p$, $p+1 \leq i \leq n$ *and* $(i, k, j) \notin I$, *then* $a_{ij}^{(k)} = 0$.

Proof. Assume that $1 \leq i \leq p$ and $p + 1 \leq k, j \leq n$. Then it follows from Lemma 2.1.29 (1) that (i, k, j) and $(i, \sigma^r(k), \sigma^r(j))$ are in the same φ-orbit for all integers r. Put $x_0 := k$ and $x_1 := j$. Since $p + 1 \leq k, j \leq n$, $x_1 = \sigma^{r_1}(x_0)$ for some r_1. Put $x_2 = \sigma^{r_1}(x_1)$. Then (i, x_0, x_1) and (i, x_1, x_2) are in the same φ-orbit. Iterating the argument to (i, x_1, x_2), we have integers s, t such that $0 \leq s < t \leq q$ and $x_s = x_t$. Since $(i, x_{t-1}, x_t) \notin I$, $x_{t-1} \neq x_t$. Hence by Lemma 2.1.28, we have

$$a_{ix_{s+1}}^{(x_s)} a_{ix_{s+2}}^{(x_{s+1})} \cdots a_{ix_t}^{(x_{t-1})} = 0.$$

Since $(i, x_s, x_{s+1}), \ldots, (i, x_{t-1}, x_t)$ are in the same φ-orbit, we have $a_{ix_{s+1}}^{(x_s)} = \cdots = a_{ix_t}^{(x_{t-1})} = 0$, so that $a_{ij}^{(k)} = 0$. This completes the proof of (1). (2) is similar to (1). □

Proof of Theorem 2.1.27. Let $\mathbb{B} = (B_1, \ldots, B_p)$ be a p-tuple of $p \times p$ matrices $B_k = (b_{ij}^{(k)})$ $(1 \leq k \leq p)$ defined by $b_{ij}^{(k)} := a_{ij}^{(k)}$ for all $(i, k, j) \in T(p)$. Then $\mathbb{B}$ is a structure system of a Frobenius $p \times p$ full matrix algebra B with Nakayama permutation $(1\ 2\ \cdots\ p)$. Similarly, using $a_{ij}^{(k)}$ where $(i, k, j) \in T(q)$, we have a Frobenius $q \times q$ $\mathbb{C}$-full matrix algebra C with Nakayama permutation $(1\ 2\ \cdots\ q)$. It follows from Lemma 2.1.30 that $a_{ij}^{(k)} = 0$ for all $(i, k, j) \in T - (I \cup T(p) \cup T(q))$, because the remaining four cases are obtained by applying φ to the cases of (1) and (2) of Lemma 2.1.30. Hence we have $A = (B, C)$. This completes the proof. □

2.1.6 The case of $n = 6, 7$.

Let K be a field and D a discrete valuation ring with a unique maximal ideal πD such that $D/\pi D \cong K$. For $n = 6, 7$, we can verify that every Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra A has a Gorenstein tiled D-order Λ such that $\Lambda/\pi\Lambda \cong A$. In this section, we execute its verification. Our strategy is as follows.

Let σ be a given permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$. Find out all φ-orbits $X_1, \ldots, X_m$ contained in X. Suppose that $\mathbb{A} = (A_1, \ldots, A_n)$ is a structure system such that $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ and

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i,k,j) \in I \\ 0 & \text{if } (i,k,j) \in Z \\ x_r & \text{if } (i,k,j) \in X_r \ (1 \leq r \leq m). \end{cases}$$

Deduce some relations of $x_1, \ldots, x_m$ from (A1)–(A3). Using the relations, find all candidates of $(x_1, \ldots, x_m)$. Eliminate any candidate if its structure system is, up to permutation, equal to a structure system obtained by one of the remaining candidates. (Note that at this stage, we have not checked yet whether each candidate determines a structure system.) For each candidate, find a Gorenstein tiled D-order Λ such that $\Lambda/\pi\Lambda$ has a structure system $\mathbb{A}$ determined by the candidate.

We may assume that $x_r \neq 0$ for some r $(1 \leq r \leq m)$ by Corollary 2.1.8 and Example 2.1.25. It is sufficient to consider the following eight cases by Remark 2.1.9. In what follows, we abbreviate a tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$ as $\Lambda = (\lambda_{ij})$.

(1) Let $n = 6$ and $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$. Then X has the following three φ-orbits.

$$\begin{aligned} X_1 &= \{\varphi^t(4,1,3) \mid 0 \leq t \leq 17\} \\ X_2 &= \{\varphi^t(2,1,5) \mid 0 \leq t \leq 17\} \\ X_3 &= \{\varphi^t(2,1,4) \mid 0 \leq t \leq 17\} \end{aligned}$$

Since $(2,4,1) \in X_1$ and $(2,1,4) \in X_3$, for $(2,1,4,1)$, we have $x_3 x_1 = a_{24}^{(1)} a_{21}^{(4)} = a_{21}^{(1)} a_{11}^{(4)} = 0$. Similarly, for $(2,1,5,1)$, we have $x_2 x_1 = a_{25}^{(1)} a_{21}^{(5)} = a_{21}^{(1)} a_{11}^{(5)} = 0$. Hence $(1,0,0), (0,1,0), (0,0,1), (0,1,1)$ are the candidates of (x_1, x_2, x_3). We have no candidates to eliminate. The following four Gorenstein tiled D-orders have the factor algebras isomorphic to Frobenius $\mathbb{A}$-full matrix algebras determined by $(x_1, x_2, x_3) = (1,0,0), (0,1,0), (0,0,1), (0,1,1)$, respectively.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 1 & 2 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore there are five Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(2) Let $n = 6$ and $\sigma = (1\ 2\ 3\ 4)(5\ 6)$. Then X has the following five φ-orbits.

$$\begin{array}{rclrcl} X_1 & = & \{\varphi^t(2,1,4) \mid 0 \le t \le 3\} & X_2 & = & \{\varphi^t(2,1,5) \mid 0 \le t \le 11\} \\ X_3 & = & \{\varphi^t(2,1,6) \mid 0 \le t \le 11\} & X_4 & = & \{\varphi^t(3,1,5) \mid 0 \le t \le 11\} \\ X_5 & = & \{\varphi^t(3,1,6) \mid 0 \le t \le 11\} & & & \end{array}$$

Since $(3,1,5),(1,3,5) \in X_4$ and $(3,1,6),(1,3,6) \in X_5$, for $(1,3,1,5)$, we have $x_4^2 = a_{15}^{(3)}a_{35}^{(1)} = a_{11}^{(3)}a_{15}^{(1)} = 0$ and for $(1,3,1,6)$, we have $x_5^2 = a_{16}^{(3)}a_{36}^{(1)} = a_{11}^{(3)}a_{16}^{(1)} = 0$, so that $x_4 = x_5 = 0$. Since $(2,6,4) \in X_2$ and $(1,6,4) \in X_5$, for $(2,1,6,4)$, we have $x_3x_2 = a_{26}^{(1)}a_{24}^{(6)} = a_{24}^{(1)}a_{14}^{(6)} = x_1x_5 = 0$, because $x_5 = 0$. Hence we have the following five candidates.

$$(x_1,x_2,x_3) = (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1)$$

We can eliminate $(0,0,1)$ and $(1,0,1)$. In fact, one can check that $\mathbb{A}$-full matrix algebras determined by $(0,1,0)$ and $(0,0,1)$ are isomorphic by permutation $\rho = (5\ 6)$, and that $\mathbb{A}$-full matrix algebras determined by $(1,1,0)$ and $(1,0,1)$ are isomorphic by permutation $\rho = (5\ 6)$. The following three Gorenstein tiled D-orders correspond to Frobenius $\mathbb{A}$-full matrix algebras determined by $(1,0,0),(0,1,0),(1,1,0)$, respectively.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 3 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 & 3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 2 & 4 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 4 & 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 3 & 1 & 4 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 4 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 4 & 0 & 2 & 2 & 4 & 0 \end{pmatrix}$$

Therefore there are four Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(3) Let $n = 6$ and $\sigma = (1\ 2\ 3)(4\ 5\ 6)$. Then X has the following six φ-orbits.

$$\begin{array}{rclrcl} X_1 & = & \{\varphi^t(4,1,3) \mid 0 \le t \le 8\} & X_2 & = & \{\varphi^t(5,1,3) \mid 0 \le t \le 8\} \\ X_3 & = & \{\varphi^t(6,1,3) \mid 0 \le t \le 8\} & X_4 & = & \{\varphi^t(4,1,6) \mid 0 \le t \le 8\} \\ X_5 & = & \{\varphi^t(5,1,4) \mid 0 \le t \le 8\} & X_6 & = & \{\varphi^t(6,1,5) \mid 0 \le t \le 8\}. \end{array}$$

For $(4,1,6,3)$, $x_1^2 = a_{43}^{(1)}a_{13}^{(6)} = a_{46}^{(1)}a_{43}^{(6)} = x_4^2$, so that $x_1 = x_4$. For $(5,1,4,3)$, $x_2^2 = a_{53}^{(1)}a_{13}^{(4)} = a_{54}^{(1)}a_{53}^{(4)} = x_5^2$, so that $x_2 = x_5$. For $(6,1,5,3)$, $x_3^2 = a_{63}^{(1)}a_{13}^{(5)} = a_{65}^{(1)}a_{63}^{(5)} = x_6^2$, so that $x_3 = x_6$. For $(4,1,3,2)$, we have $x_1x_2 = a_{43}^{(1)}a_{42}^{(3)} = a_{42}^{(1)}a_{12}^{(3)} = 0$, because $(4,1,2) \in Z$. For $(5,1,3,2)$, we have $x_2x_3 = a_{53}^{(1)}a_{52}^{(3)} = a_{52}^{(1)}a_{12}^{(3)} = 0$, because $(5,1,2) \in Z$. For $(6,1,3,2)$, we have $x_3x_1 = a_{63}^{(1)}a_{62}^{(3)} = a_{62}^{(1)}a_{12}^{(3)} = 0$, because $(6,1,2) \in Z$. Hence we have the following three candidates.

$$(x_1, x_2, x_3) = (1,0,0),\ (0,1,0),\ (0,0,1)$$

We can eliminate $(0,1,0)$ and $(0,0,1)$. The following Gorenstein tiled D-order corresponds to a Frobenius $\mathbb{A}$-full matrix algebra determined by $(1,0,0)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Therefore there are two Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(4) Let $n = 6$ and $\sigma = (1\ 2)(3\ 4)(5\ 6)$. Then X has eight φ-orbits.

$$\begin{array}{lllllll} X_1 & = & \{\varphi^t(3,1,5) \mid 0 \le t \le 5\} & \quad & X_2 & = & \{\varphi^t(3,1,6) \mid 0 \le t \le 5\} \\ X_3 & = & \{\varphi^t(4,1,5) \mid 0 \le t \le 5\} & & X_4 & = & \{\varphi^t(4,1,6) \mid 0 \le t \le 5\} \\ X_5 & = & \{\varphi^t(5,1,3) \mid 0 \le t \le 5\} & & X_6 & = & \{\varphi^t(5,1,4) \mid 0 \le t \le 5\} \\ X_7 & = & \{\varphi^t(6,1,3) \mid 0 \le t \le 5\} & & X_8 & = & \{\varphi^t(6,1,4) \mid 0 \le t \le 5\} \end{array}$$

We have the following relations.

$$\begin{array}{ll} x_1x_6 = x_1x_7 = x_1x_8 = 0 & \quad x_2x_5 = x_2x_6 = x_2x_8 = 0 \\ x_3x_5 = x_3x_7 = x_3x_8 = 0 & \quad x_4x_5 = x_4x_6 = x_4x_7 = 0 \end{array}$$

There are 34 candidates of $(x_1, \ldots, x_8)$. After elimination, we have the following nine candidates.

$$\begin{array}{lll} (1,0,0,0,0,0,0,0) & (1,1,0,0,0,0,0,0) & (1,0,1,0,0,0,0,0) \\ (1,0,0,1,0,0,0,0) & (1,0,0,0,1,0,0,0) & (1,1,1,0,0,0,0,0) \\ (1,1,0,1,0,0,0,0) & (1,0,1,1,0,0,0,0) & (1,1,1,1,0,0,0,0) \end{array}$$

The following nine Gorenstein tiled D-orders correspond to Frobenius $\mathbb{A}$-full matrix algebras determined by the above candidates, respectively.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 3 & 3 & 4 & 2 \\ 3 & 0 & 0 & 3 & 3 & 1 \\ 3 & 0 & 3 & 0 & 2 & 2 \\ 2 & 0 & 1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 4 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 4 & 0 & 2 & 3 \\ 3 & 0 & 2 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 3 & 2 & 4 & 1 \\ 2 & 0 & 0 & 2 & 2 & 1 \\ 3 & 0 & 3 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 1 & 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 5&0&2&3&3&2\\ 3&0&0&3&3&1\\ 2&0&2&0&1&2\\ 2&0&1&0&0&2\\ 3&0&0&2&3&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0\\ 3&0&2&1&2&1\\ 1&0&0&1&1&0\\ 2&0&2&0&1&1\\ 1&0&1&0&0&1\\ 2&0&1&1&2&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0\\ 3&0&2&1&2&1\\ 1&0&0&1&1&1\\ 2&0&2&0&2&1\\ 1&0&0&0&0&1\\ 2&0&1&0&2&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 5&0&3&2&2&3\\ 2&0&0&2&2&2\\ 3&0&3&0&2&3\\ 3&0&1&0&0&3\\ 2&0&0&0&2&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0\\ 5&0&2&3&3&2\\ 3&0&0&3&3&2\\ 2&0&2&0&2&2\\ 2&0&0&0&0&2\\ 3&0&0&1&3&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0\\ 2&0&1&1&1&1\\ 1&0&0&1&1&1\\ 1&0&1&0&1&1\\ 1&0&0&0&0&1\\ 1&0&0&0&1&0 \end{pmatrix}$$

Therefore there are 10 Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(5) Let $n = 7$ and $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. Then X has the following six φ-orbits.

$$\begin{array}{rclrcl} X_1 &=& \{\varphi^t(4,1,3) \mid 0 \le t \le 20\} & X_2 &=& \{\varphi^t(2,1,5) \mid 0 \le t \le 20\} \\ X_3 &=& \{\varphi^t(2,1,6) \mid 0 \le t \le 20\} & X_4 &=& \{\varphi^t(5,1,3) \mid 0 \le t \le 20\} \\ X_5 &=& \{\varphi^t(2,1,4) \mid 0 \le t \le 20\} & X_6 &=& \{\varphi^t(3,1,6) \mid 0 \le t \le 6\} \end{array}$$

We have the following relations.

$$x_2 = x_3, \quad x_2x_3 = x_5x_6, \quad x_1x_2 = x_1x_5 = x_2x_4 = x_4x_5 = 0$$

Hence we have the following eight candidates of $(x_1, \dots, x_6)$.

$$\begin{array}{llll} (1,0,0,0,0,0) & (0,0,0,1,0,0) & (1,0,0,1,0,0) & (0,0,0,0,1,0) \\ (0,0,0,1,1,0) & (0,0,0,0,0,1) & (1,0,0,0,0,1) & (0,1,1,0,1,1) \end{array}$$

The following Gorenstein tiled D-orders correspond to Frobenius $\mathbb{A}$-full matrix algebras determined by the above candidates, respectively.

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 5&0&5&2&3&3&2\\ 2&0&0&2&0&1&0\\ 3&0&3&0&3&1&1\\ 3&0&2&2&0&3&0\\ 2&0&2&1&2&0&2\\ 5&0&3&2&2&3&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 4&0&4&3&2&2&3\\ 3&0&0&3&1&0&1\\ 2&0&1&0&2&0&0\\ 2&0&2&2&0&2&1\\ 3&0&2&3&2&0&3\\ 4&0&1&2&2&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 2&0&2&1&1&1&1\\ 1&0&0&1&0&0&0\\ 1&0&1&0&1&0&0\\ 1&0&1&1&0&1&0\\ 1&0&1&1&1&0&1\\ 2&0&1&1&1&1&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0&0\\ 3&0&3&3&2&2&3\\ 3&0&0&3&2&1&2\\ 2&0&0&0&2&1&1\\ 2&0&1&1&0&2&2\\ 3&0&1&2&1&0&3\\ 3&0&0&1&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 2&0&2&2&1&1&2\\ 2&0&0&2&1&0&1\\ 1&0&0&0&1&0&0\\ 1&0&1&1&0&1&1\\ 2&0&1&2&1&0&2\\ 2&0&0&1&1&0&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0&0\\ 4&0&4&2&3&3&2\\ 2&0&0&2&1&2&1\\ 3&0&2&0&3&2&2\\ 3&0&1&1&0&3&1\\ 2&0&1&0&1&0&2\\ 4&0&2&1&1&2&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 3&0&3&1&2&2&1\\ 1&0&0&1&0&1&0\\ 2&0&2&0&2&1&1\\ 2&0&1&1&0&2&0\\ 1&0&1&0&1&0&1\\ 3&0&2&1&1&2&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0&0\\ 1&0&1&1&1&1&1\\ 1&0&0&1&1&1&1\\ 1&0&0&0&1&1&1\\ 1&0&0&0&0&1&1\\ 1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}$$

Therefore there are nine Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(6) Let $n = 7$ and $\sigma = (1\ 2\ 3\ 4\ 5)(6\ 7)$. Then X has the following five φ-orbits.

$$\begin{aligned} X_1 &= \{\varphi^t(2,1,4) \mid 0 \le t \le 14\} & X_2 &= \{\varphi^t(4,1,3) \mid 0 \le t \le 4\} \\ X_3 &= \{\varphi^t(2,1,6) \mid 0 \le t \le 29\} & X_4 &= \{\varphi^t(3,1,6) \mid 0 \le t \le 29\} \\ X_5 &= \{\varphi^t(4,1,6) \mid 0 \le t \le 29\} \end{aligned}$$

We have relations $x_3 = x_4 = x_5 = 0$ by Theorem 2.1.27 and $x_1x_2 = 0$. Hence we have two candidates $(x_1, x_2, x_3, x_4, x_5) = (1,0,0,0,0), (0,1,0,0,0)$. The following Gorenstein tiled D-orders correspond to Frobenius $\mathbb{A}$-full matrix algebras determined by $(1,0,0,0,0), (0,1,0,0,0)$, respectively.

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 5&0&5&5&5&4&4\\ 5&0&0&5&5&3&3\\ 5&0&0&0&5&2&2\\ 5&0&0&0&0&1&1\\ 4&0&1&2&3&0&4\\ 4&0&1&2&3&4&0 \end{pmatrix} \qquad \begin{pmatrix} 0&0&0&0&0&0&0\\ 10&0&10&5&5&6&6\\ 5&0&0&5&0&2&2\\ 5&0&5&0&5&3&3\\ 10&0&5&5&0&4&4\\ 6&0&4&3&2&0&6\\ 6&0&4&3&2&6&0 \end{pmatrix}$$

Therefore there are three Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(7) Let $n = 7$ and $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7)$. Then X has the following four φ-orbits.

$$\begin{array}{lcllcl} X_1 & = & \{\varphi^t(2,1,4) \mid 0 \le t \le 3\} & X_2 & = & \{\varphi^t(2,1,5) \mid 0 \le t \le 35\} \\ X_3 & = & \{\varphi^t(3,1,5) \mid 0 \le t \le 35\} & X_4 & = & \{\varphi^t(5,1,7) \mid 0 \le t \le 35\} \end{array}$$

We have relations $x_2 = x_3 = x_4 = 0$ by Theorem 2.1.27. Hence we have a candidate $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$. The following Gorenstein tiled D-order corresponds to a Frobenius $\mathbb{A}$-full matrix algebra determined by $(1, 0, 0, 0)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 8 & 6 & 6 & 6 \\ 8 & 0 & 0 & 8 & 4 & 4 & 4 \\ 8 & 0 & 0 & 0 & 2 & 2 & 2 \\ 6 & 0 & 2 & 4 & 0 & 6 & 3 \\ 6 & 0 & 2 & 4 & 3 & 0 & 6 \\ 6 & 0 & 2 & 4 & 6 & 3 & 0 \end{pmatrix}$$

Therefore there are two Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

(8) Let $n = 7$ and $\sigma = (1\ 2\ 3)(4\ 5)(6\ 7)$. Then X has the following six φ-orbits

$$\begin{array}{lcllcl} X_1 & = & \{\varphi^t(2,1,4) \mid 0 \le t \le 17\} & X_2 & = & \{\varphi^t(2,1,6) \mid 0 \le t \le 17\} \\ X_3 & = & \{\varphi^t(4,1,6) \mid 0 \le t \le 17\} & X_4 & = & \{\varphi^t(4,1,7) \mid 0 \le t \le 17\} \\ X_5 & = & \{\varphi^t(6,1,4) \mid 0 \le t \le 17\} & X_6 & = & \{\varphi^t(6,1,5) \mid 0 \le t \le 17\}. \end{array}$$

We have the following relations.

$$x_1^2 = x_2^2 = x_3x_5 = x_3x_6 = x_4x_5 = x_4x_6 = 0$$

Hence we have the following six candidates of (x_3, x_4, x_5, x_6).

$$(1,0,0,0),\ (1,1,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1),\ (0,0,1,1)$$

The latter four candidates can be eliminated. The following two Gorenstein tiled D-orders have the factor algebras isomorphic to Frobenius $\mathbb{A}$-full matrix algebras determined by $(1, 0, 0, 0), (1, 1, 0, 0)$, respectively.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 2 & 2 & 2 & 2 \\ 3 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 2 & 2 & 2 & 2 \\ 3 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Therefore there are three Frobenius $\mathbb{A}$-full matrix algebras of this type, all of which have corresponding Gorenstein tiled orders.

2.2 Minor degenerations of the full matrix algebra

Let $n \geq 2$ be an integer, D a discrete valuation ring and K a field. A structure system is an n-tuple of $n \times n$ matrices over K with certain properties. A full matrix algebra with a strucure system is an n^2-dimensional K-vector space with an associative multiplication defined by a structure system. In [7] and [8], we mainly studied full matrix algebras with $(0,1)$-structure systems, that is, their components are 0 or 1, just as structure systems of factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ, and we are interested in Frobenius full matrix algebras and showed that the class of Frobenius full matrix algebras is a strictly larger class than that of the factor algebras of Gorenstein tiled orders. Then one may ask, as a next step, whether there are full matrix algebras which are not isomorphic to ones with $(0,1)$-structure systems at all. This is one of the motivations for our study. In this section, we provide such examples in Subsection 2.2.3 and 2.2.4.

The other motivation for our study is the fact that we are able to treat the class of full matrix algebras with structure systems by an elementary algebraic geometry technique and study them in a deformation theory context [13]. It turns out that, for suitable choice of structure matrix q, the algebra $\mathbb{M}_n^q(K)$ is a degeneration of the full matrix algebra $\mathbb{M}_n(K)$, see [12] and Subsection 2.2.1. So, in this section, we consider the class of full matrix algebras with structure systems as a subclass of minor degenerations of the full matrix algebra $\mathbb{M}_n(K)$, see Subsection 2.2.1 for definition. We would like to note here that we are also followinig an old idea of the skew matrix ring construction by Kupisch in [19] and [20], see also Oshiro and Rim [22].

There is also anothier motivation coming from the fact proved in [30] that, given a prime $p \geq 2$ and an algebraically closed field K of characteristic zero, any Hopf K-algebra of dimension p^2 is semisimple or is isomorphic to the Taft Hopf algebra. In connection with this result and the facts that Hopf algebras are Frobenius algebras and the Taft Hopf algebra is a Nakayama algebra, the existence of a Hopf algebra structure on a Frobenius algebra of the form $\mathbb{M}_n^q(K)$ (of dimension n^2 !), seems to be a natural problem to solve. We do not solve it here, but we shall study it in a subsequent paper. Here we only describe Nakayama algebras (Subsection 2.2.2) and Frobenius algebras (Subsection 2.2.4) of the form $\mathbb{M}_n^q(K)$ for a class of matrices q.

Subsection 2.2.1 contains basic definitons, examples and properties of minor q-degenerations $\mathbb{M}_n^q(K)$ of the full matrix K-algebra $\mathbb{M}_n(K)$. In particular, we give a criterion for the existence of a K-algebra isomorphism $\mathbb{M}_n^q(K) \simeq \mathbb{M}_n^{q'}(K)$ in terms of an action

$$* : \mathbb{G}_n(K) \times \mathbb{ST}_n(K) \longrightarrow \mathbb{ST}_n(K)$$

of an algebraic group $\mathbb{G}_n(K) = \mathcal{T}_n \ltimes S_n$ (containing the symmetric group S_n and the torus $\mathcal{T}_n$) on the algebraic K-variety $\mathbb{ST}_n(K) \subset \mathbb{M}_{n\times n^2}(K)$ of the minor constant matrices $q = [q^{(1)}|\cdots|q^{(n)}]$, see (2.7), and (2.8) . The algebras $\mathbb{M}_n^q(K)$ and their modules are investigated by means of the properties of q and by applying quivers with relations. In case the algebra is basic, the Gabriel quiver of $\mathbb{M}_n^q(K)$ is described.

A complete clasification, up to isomorphism, of basic algebras $\mathbb{M}_n^q(K)$ in case $n = 2$ and $n = 3$ is given in Subsection 2.2.3. The matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ in $\mathbb{ST}_n(K)$ such that $\mathbb{M}_n^q(K)$ is a

Nakayama algebra are described in Subsection 2.2.2, where also $(0,1)$-limits of algebras $\mathbb{M}_n^q(K)$ are studied.

Conditions for the matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ in $\mathbb{ST}_n(K)$ to be $A_q = \mathbb{M}_n^q(K)$ a Frobenius algebra are given in Subsection 2.2.4, by extending some of the Fujita's results in [7, Section 4]. All matrices q such that A_q is a Frobenius algebra and the cube $J(A_q)^3$ of the Jacobson radical $J(A_q)$ of $A_q = \mathbb{M}_n^q(K)$ is zero are described in Theorem 2.2.26. In case K is an infinite field, for each $n \geq 4$, we construct a one-parameter K-algebraic family $\{C_\mu\}_{\mu \in K^*}$ of basic pairwise non-isomorphic Frobenius K-algebras of the form $C_\mu = \mathbb{M}_n^{q_\mu}(K)$.

Finally, we show that if $A_q = \mathbb{M}_n^q(K)$ is a Frobenius algebra such that $J(A_q)^3 = 0$, then the representation type of A_q is completely determined as follows:

(i) A_q is representation-finite if and only if $n = 3$,

(ii) A_q is tame representation-finite [28, Section 14.4] if and only if $n = 4$, and

(iii) A_q is representation-wild [28, Section 14.4] if and only if $n \geq 5$,

where we assume in (ii) and in (iii) that the field K is algebraically closed.

Throughout this section K is a field and R is a ring with an identity element. We denote by $J(R)$ the Jacobson radical of R, and by $\mathrm{mod}(R)$ the category of finitely generated right R-modules. Given $n \geq 1$, we denote by $\mathbb{M}_n(R)$ the full matrix R-algebra consisting of all square $n \times n$ matrices with coefficients in R and by e_{ij} the matrix unit in $\mathbb{M}_n(K)$ with 1 on the (i,j)- entry, and zero elsewhere. We denote by $e_1, \ldots, e_n$ the standard matrix idempotents $e_{11}, \ldots, e_{mm}$ of $A_q = \mathbb{M}_n(R)$.

2.2.1 Minor constant structure matrices and minor degenerations of $\mathbb{M}_n(R)$.

Throughout, we fix an integer $n \geq 2$. We suppose that K is an arbitrary field and R is a ring with an identity element. We recall that, given a finite dimensional K-algebra A and a complete set $e_1, \ldots, e_n$ of pairwise orthogonal primitive idempotents of A, we define the *Cartan matrix* of A to be the matrix $C_A = [c_{ij}] \in \mathbb{M}_n(\mathbb{Z})$, where $c_{ij} = \dim_K e_i A e_j$. The algebra A is said to be *basic* if $e_j A \not\cong e_i A$ for $i \neq j$, and A is said to be *connected* if A is not a direct product of two K-algebras (see [1] and [2]).

Following Fujita [7], we introduce the following definition.

Definition 2.2.1. Assume that $n \geq 2$. A minor constant structure matrix of size $n \times n^2$, with coefficients in a ring R, is the n-block matrix

$$q = [q_{ij}^{(1)}|q_{ij}^{(2)}|\ldots|q_{ij}^{(n)}] \tag{2.1}$$

where $q^{(1)} = [q_{ij}^{(1)}], \ldots, q^{(n)} = [q_{ij}^{(n)}] \in \mathbb{M}_n(R)$ are $n \times n$ square matrices with coefficients in R satisfying the following two conditions

(C1) $q_{rj}^{(r)} = 1$ and $q_{jr}^{(r)} = 1$, for all $j, r \in \{1, \ldots, n\}$.

(C2) $q_{ij}^{(r)} q_{is}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$, for all $i, j, r, s \in \{1, \ldots, n\}$.

We call q *basic* if, in addition, the following condition is satisfied

(C3) $q_{jj}^{(r)} = 0$, for $r = 1, \dots, n$ and all $j \in \{1, \dots, n\}$ such that $j \neq r$,

The minor constant structure matrix q is called $(0,1)$-matrix, if each entry $q_{ij}^{(r)}$ is either 0 or 1. Throughout this section, a minor constant structure matrix will be called a structure matrix of $\mathbb{M}_n(R)$, in short. We denote by

$$\mathbb{ST}_n(R) \subseteq \mathbb{M}_{n \times n^2}(R) \tag{2.2}$$

the set of all minor constant structure matrices q of size $n \times n^2$, with coefficients in a ring R.

LEMMA 2.2.2. (a) *Let $n \geq 2$ and let $q = [q_{ij}^{(1)} | q_{ij}^{(2)} | \dots | q_{ij}^{(n)}]$ be a matrix of the form (2.1) satisfying the condition (C1). Then the equality $q_{ij}^{(r)} q_{is}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$ in (C2) holds, if $r = i$, orr $= j$, $j = i$, $j = s$, or $r = s$ and $i, j, r, s \in \{1, \dots, n\}$.*

(b) *Assume that $q = [q_{ij}^{(1)} | q_{ij}^{(2)} | \dots | q_{ij}^{(n)}]$ is a structure matrix (2.1) in $\mathbb{ST}_n(R)$.*

- (b1) $q_{jj}^{(r)} = q_{rr}^{(j)}$, *for all $j, r \in \{1, \dots, n\}$.*
- (b2) $q_{jj}^{(r)} = q_{js}^{(r)} q_{rs}^{(j)} = q_{sj}^{(r)} q_{sr}^{(j)}$, *for any triple of elements $j, r, s \in \{1, \dots, n\}$.*
- (b3) *If $q_{jj}^{(r)} \neq 0$ and $q_{ss}^{(r)} \neq 0$, then $q_{ss}^{(j)} = q_{jj}^{(s)} \neq 0$.*

(c) *If $n \geq 3$ and the matrix $q = [q_{ij}^{(1)} | q_{ij}^{(2)} | \dots | q_{ij}^{(n)}]$ is basic then, for any $i, j, r \in \{1, \dots, n\}$ $q_{ij}^{(r)} q_{ir}^{(j)} = 0$ if $j \neq r$, and $q_{rj}^{(i)} q_{ij}^{(r)} = 0$ if $i \neq j$.*

Proof. (a) Let $r = i$. Then (C1) yields $q_{rj}^{(r)} = 1$, $q_{rs}^{(r)} = 1$ and we get $q_{ij}^{(r)} q_{is}^{(j)} = q_{rj}^{(r)} = q_{rs}^{(j)} = q_{rs}^{(r)} q_{rs}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$. If $r = j$ or $j = s$, the equality $q_{ij}^{(r)} q_{is}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$ follows in a similar way.

(b) (b1) Apply (C2) with $i = j$, $s = r$ and then use (C1).

- (b2) By (C2), we have $q_{jj}^{(r)} q_{js}^{(j)} = q_{js}^{(r)} q_{rs}^{(j)}$. Since $q_{js}^{(j)} = 1$, the first equality holds. The second one follows in a similar way.
- (b3) By (C2), we have $q_{sr}^{(j)} q_{ss}^{(r)} = q_{ss}^{(j)} q_{js}^{(r)}$. Since $q_{jj}^{(r)} \neq 0$ then, according to (b1), the number $q_{sr}^{(j)}$ is non-zero and the equation yields $q_{ss}^{(j)} \neq 0$.

(c) By applying (C2) with $s = r$ we get $q_{ij}^{(r)} q_{ir}^{(j)} = q_{ir}^{(r)} q_{rr}^{(j)} = 0$, because $j \neq r$ implies $q_{rr}^{(j)} = 0$, by (C3). The equality $q_{rj}^{(i)} q_{ij}^{(r)} = 0$ follows in a similar way.

□

Now we introduce the minor q-degeneration $\mathbb{M}_n^q(R)$ of the algebra $\mathbb{M}_n(R)$.

DEFINITION 2.2.3. Let $n \geq 2$ be an integer and let $q = [q^{(1)} | \dots | q^{(n)}]$ be a minor constant structure matrix (2.1) in $\mathbb{ST}_n(R)$ with coefficients in th center of a ring R. By a q-degeneration $\mathbb{M}_n^q(R)$ of the full matrix R-algebra $\mathbb{M}_n(R)$ is defined to be the R-module $\mathbb{M}_n(R)$ equipped with the q-multiplication

$$\cdot_q : \mathbb{M}_n(R) \otimes_R \mathbb{M}_n(R) \longrightarrow \mathbb{M}_n(R)$$

that associates to any pair of matrices $\lambda' = [\lambda'_{ij}], \lambda'' = [\lambda''_{ij}] \in \mathbb{M}_n(R)$ the matrix

$$\lambda' \cdot_q \lambda'' = [\lambda_{ij}], \text{ where } \lambda_{ij} = \sum_{s=1}^{n} \lambda'_{is} q_{ij}^{(s)} \lambda''_{sj}, \tag{2.3}$$

for $i, j \in \{1, \ldots, n\}$. Throughout, we simply write $\lambda'\lambda''$ instead of $\lambda' \cdot_q \lambda''$.

A straightforward computation shows that $\mathbb{M}_n^q(R)$ is a ring and the identity matrix $E = \mathrm{diag}(1, \ldots, 1)$ of $\mathbb{M}_n(R)$ is the identity of $\mathbb{M}_n^q(R)$.

By a minor degeneration of the full matrix ring $\mathbb{M}_n(R)$ we mean a q-degeneration ring $\mathbb{M}_n^q(R)$, where $n \geq 2$ and q is a structure matrix (2.1) in $\mathbb{ST}_n(R)$.

Elementary properties of the K-algebra $\mathbb{M}_n^q(K)$ are collected in Theorem 2.2.5 below. In particular, it follows that $\mathbb{M}_n^q(K)$ is a non-semisimple basic K-algebra, if q is basic, $n \geq 2$, and K is a field.

We remark that if $q = [q^{(1)}| \ldots |q^{(n)}]$ is the matrix (2.1) with $q_{ij}^{(s)} = 1$ for all $i, j, s \in \{1, \ldots, n\}$, then the conditions (C1) and (C2) are satisfied, but the condition (C3) is not. In this case, we have $\mathbb{M}_n^q(R) = \mathbb{M}_n(R)$, because the formula (2.3) defines the usual matrix multiplication on $\mathbb{M}_n(R)$.

It turns out that, under a suitable choice of q, the algebra $\mathbb{M}_n^q(K)$ is a degeneration of $\mathbb{M}_n(K)$ in the sense of [13], if K is a field, see Example 2.2.4 and 2.2.8. We recall from [13] and [11] that given two K-algebras A_1 and A_0 (with an underlying K-space K^m) defined by the constant structure matrices μ_1 and μ_0, respectively, μ_1 and μ_0 are viewed as elements of the algebraic variety $\mathcal{A}lg(K^m)$ of associative unitary K-algebra structures on the vector space K^m. The general linear group $\mathrm{Gl}(K^m)$ acts on $\mathcal{A}lg(K^m)$ by the transport of structures, see also [18, p. 225]. An algebra A_1 is said to be a deformation of the algebra A_0 (or that A_0 is a degeneration of the algebra A_1), if μ_0 lies in the closure of the $\mathrm{Gl}(K^m)$-orbit of μ_1 in $\mathcal{A}lg(K^m)$, see [11], [12] and [18]. We note that the set $\mathbb{ST}_n(K) \subseteq \mathbb{M}_{n\times n^2}(K)$ of minor structure matrices (2.1) of size $n \times n^2$ is an algebraic K-variety. Moreover, there is a variety embedding

$$\mathbb{ST}_n(K) \subseteq \mathcal{A}lg(K^{n^2}) = \mathcal{A}lg(\mathbb{M}_n(K)) \tag{2.4}$$

defined by attaching to any minor constant structure matrix q the matrix of constants of the multiplication $\cdot_q : \mathbb{M}_n^q(K) \otimes \mathbb{M}_n^q(K) \longrightarrow \mathbb{M}_n^q(K)$ in the matrix unit basis, see (2.5) below. It is clear that $\mathbb{ST}_n(K)$ is a locally closed subset of $\mathcal{A}lg(K^{n^2})$.

In this subsection we study the basic K-algebras $\mathbb{M}_n^q(K)$ and their modules by means of quivers with relations. We recall that, given a quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$, by an oriented paths in $\mathcal{Q}$ starting from the vertex $i = i_0$ and ending at the vertex $j = i_m$ we mean a formal composition

$$\beta_1\beta_2 \cdots \beta_m \equiv (i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m)$$

of arrows $\beta_1, \ldots, \beta_m$. We denote by $K\mathcal{Q}$ the path K-algebra, that is, the K-algebra generated by all oriented paths in $\mathcal{Q}$, see [1, Chapter II], [2], [28, Chapter 14] and [32].

Now we illustrate the notion of a minor degeneration algebra by the following example.

EXAMPLE 2.2.4. Assume that $n = 2$ and R is a ring with identity. It follows from Lemma 2.2.2(b) and the conditions (C1) and (C2) in Definition 2.2.1 that $q = [q^{(1)}|q^{(2)}]$ is a structure matrix (2.1) in $\mathbb{ST}_2(R)$ if and only if q has the form $q(\mu) = \left[\begin{array}{cc|cc} 1 & 1 & \mu & 1 \\ 1 & \mu & 1 & 1 \end{array}\right]$, where $\mu = q_{22}^{(1)} = q_{11}^{(2)}$ is a scalar in R. The matrix $q = q(0) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$ is a unique basic structure matrix in $\mathbb{ST}_2(R)$.

Assume that K is a field, $q(\mu)$ is the structure matrix presented above with $\mu \in K$, and let $A(\mu) = \mathbb{M}_2^{q(\mu)}(K)$. We claim that:

- The K-algebra $A(\mu)$ is semisimple and $A(\mu) \cong A(1) = \mathbb{M}_2(K)$ if and only if $\mu \neq 0$.
- For each $\mu \in K$, $A(\mu)$ is a degeneration of the full matrix algebra $A(1) = \mathbb{M}_2(K)$.
- $A(0)$ is a non-semisimple self-injective Nakayama K-algebra of finite representation type.
- The algebra $A(0)$ admits a Hopf algebra structure (by [30]). If $\operatorname{char} K \neq 2$, then the Hopf algebra $A(0)$ is isomorphic to the Sweedler Hopf algebra, see [21, p. 8].

The first statement and the second one are easily verified. To see the third one we note that, by the multiplication rule (2.3), the Jacobson radical $J(A)$ of the K-algebra $A = A(0)$ has the form $J(A) = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} = Ke_{12} \oplus Ke_{21}$. Note also that $J(A)^2 = 0$ and $\operatorname{soc} A_A = J(A)$. Hence we easily conclude that there is a K-algebra isomorphism

$$A = A(0) \cong K\mathcal{Q}/I,$$

where $\mathcal{Q}$ is the quiver

$$\mathcal{Q}: \quad 1 \underset{\beta_{21}}{\overset{\beta_{12}}{\rightleftarrows}} 2$$

and $I = (\beta_{12}\beta_{21}, \beta_{21}\beta_{12})$ is the two-sided ideal of the path K-algebra $K\mathcal{Q}$ of $\mathcal{Q}$ generated by two zero relations $\beta_{12}\beta_{21}$ and $\beta_{21}\beta_{12}$ (see [1], [28, Chapter 14]). The K-algebra isomorphism $A(0) \cong K\mathcal{Q}/I$ is given by the formulae $e_1 \mapsto \varepsilon_1$, $e_2 \mapsto \varepsilon_2$, $e_{12} \mapsto \beta_{12}$ and $e_{21} \mapsto \beta_{21}$, where ε_1 and ε_2 are the primitive idempotents of the path algebra $K\mathcal{Q}$ defined by the stationary paths at the vertices 1 and 2. Hence easily follows that A is a non-semisimple self-injective Nakayama K-algebra of finite representation type.

We extend [7, 1.2(1)-1.3] as follows.

THEOREM 2.2.5. *Assume that K is a field, $n \geq 2$ is an integer, $q = [q^{(1)}|\ldots|q^{(n)}]$ is a minor constant structure matrix* (2.1) *in $\mathbb{ST}_n(K)$, and let $A_q = \mathbb{M}_n^q(K)$.*

(a) *A_q is an associative K-algebra such that*

$$e_{is}e_{tj} = \begin{cases} q_{ij}^{(s)} e_{ij} & \text{for } s = t, \\ 0 & \text{for } s \neq t, \end{cases} \tag{2.5}$$

and $e_i e_{ij} = e_{ij} = e_{ij}e_j$, for all $i, j, s, t \in \{1, \ldots, , n\}$, where e_{ij} is the (i, j)-matrix unit.

(b) *The standard matrix idempotents* $e_1 = e_{11}, \dots, e_n = e_{nn}$ *of the algebra* $\mathbb{M}_n(K)$ *are pairwise orthogonal primitive idempotents of the algebra* A_q. *Moreover, there is a right ideal decomposition* $A = e_1 A_q \oplus \dots \oplus e_n A_q$, *there are* K*-algebra isomorphisms* $End_A(e_j A_q) \cong e_i A_q e_i \cong K$, *for* $i = 1, \dots, n$, *and an isomorphism* $Hom_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong K e_{ij}$ *of* K*-vector spaces, for* $i \neq j$. *Moreover, there is an isomorphism* $e_i A_q \cong e_j A_q$ *of right ideals if and only if* $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$.

(c) *The algebra* A_q *is basic if and only if the matrix* q *is basic.*

(d) *If* A_q *is basic then*

 (i) A_q *is connected, the ideal* J *of* A_q *consisting of all matrices* $\lambda = [\lambda_{ij}]$ *with* $\lambda_{11} = \dots = \lambda_{nn} = 0$ *is the Jacobson radical* $J(A_q)$ *of* A_q, *and* $J(A_q)^n = 0$,

 (ii) *the group* $Gl(A_q)$ *of units of* A_q *consists of all matrices* $\lambda = [\lambda_{ij}] \in \mathbb{M}_n(K)$ *with* $\lambda_{11} \cdot \dots \cdot \lambda_{nn} \neq 0$,

 (iii) *every non-zero two-sided ideal of* A_q *is generated by a finite subset of the set* $\{e_{ij};\ i, j = 1, \dots, n\}$ *of the matrix units* e_{ij} *of* A_q, *and*

 (iv) *the global dimension of the algebra* A_q *is infinite.*

Proof. (a) The definition of the multiplication $\cdot_q$ (2.3) in $A_q = \mathbb{M}_n^q(K)$ yields the formula (2.5). Hence, in view of (C1), we get the equalities $e_i e_{ij} = e_{ij} = e_{ij} e_j$. It follows that the matrix of constants of $\mathbb{M}_n^q(K)$ in the matrix units basis $\{e_{ij}\}_{i,j}$ is obtained from $q = [q^{(1)}, \dots, q^{(n)}]$ by completing it with zeros at the remaining entries, see [23]. Moreover, the multiplication rule (2.3) yields

$$\begin{aligned} (e_{is} e_{sj}) e_{jt} &= q_{ij}^{(s)} e_{ij} e_{jt} = q_{ij}^{(s)} q_{it}^{(t)} e_{it}, \text{ and} \\ e_{is}(e_{sj} e_{jt}) &= e_{is}(q_{st}^{(j)} e_{st}) = q_{it}^{(s)} q_{st}^{(j)} e_{it}. \end{aligned}$$

Hence we easily conclude that the multiplication $\cdot_q$ in $A_q = \mathbb{M}_n^q(K)$ defined by (2.3) is associative if and only the condition (C2) is satisfied, see [23, Section 1.5]. It follows that $A_q = \mathbb{M}_n^q(K)$ is an associative K-algebra, the identity matrix $E = \operatorname{diag}(1, \dots, 1)$ of $\mathbb{M}_n(K)$ is the identity of A_q and the equalities (2.5) hold.

(b) Given a matrix $\lambda = [\lambda_{pq}] \in A_q = \mathbb{M}_n^q(K)$ and $p \leq n$, we have $\lambda = \sum_{i,j} \lambda_{ij} e_{ij}$ and, according to (2.5), we get $e_p \lambda e_p = e_p(\sum_{i,j} \lambda_{ij} e_{ij}) e_p = \lambda_{pp} q_{pp}^{(p)} q_{pp}^{(p)} e_p = \lambda_{pp} e_p$, because $q_{pp}^{(p)} = 1$. It follows that the map $e_p \lambda e_p \mapsto \lambda_{pp}$ defines a K-algebra isomorphism $e_p A_q e_p \cong K$. The K-algebra isomorphism $\operatorname{End}_{A_q}(e_p A_q) \cong e_p A_q e_p$ is given by $f \mapsto f(e_p)$. The vector space isomorphisms $\operatorname{Hom}_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong K e_{ij}$ follow in a similar way.

To prove the remaining part of (b), assume that $q_{jj}^{(i)} \neq 0$, where $i \neq j$. By Lemma 2.2.2 (a), $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$. Consider the A_q-module homomorphisms $e_i A_q \underset{e_{ij}\cdot}{\overset{e_{ji}\cdot}{\rightleftarrows}} e_j A_q$ defined as the left hand side multiplication by e_{ji} and by e_{ij}, respectively. Since $e_{ji} e_{ij} e_j = e_j q_{ii}^{(j)}$ and

$e_{ij}e_{ji}e_i = e_iq_{jj}^{(i)}$ then the right ideals e_iA_q and e_jA_q of A_q are isomorphic. Conversely, assume that there exists an isomorphism $h : e_iA_q \to e_jA_q$, and let $h(e_i) = e_ja$, where $a = \sum_{s,r} \lambda_{sr}e_{sr}$ and $\lambda_{sr} \in K$. Then $0 \neq h(e_{ij}) = h(e_ie_{ij}) = h(e_i)e_{ij} = e_jae_{ij} = \lambda_{ji}e_{ji}e_{ij} = e_j\lambda_{ji}q_{jj}^{(i)}$. In a view of Lemma 2.2.2 (a), this yields $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$.

(c) Assume that A_q is basic and suppose, to the contrary, that q is not basic, that is, $q_{jj}^{(r)} \neq 0$, for some r and $j \neq r$. Then $n \geq 2$ and by Lemma 2.2.2 (a), $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$. It follows from (c) that the right ideals e_iA_q and e_jA_q of A_q are isomorphic; contrary to the assumption that A_q is basic.

Conversely, assume that q is basic. By (b), there is a right ideal decomposition $A_q = e_1A_q \oplus \ldots \oplus e_nA_q$ and the vector space $\mathrm{Hom}_{A_q}(e_jA_q, e_iA_q)$ is non-zero, for all $i,j \in \{1,\ldots,,n\}$. It follows that A_q is connected. Moreover, a simple calculation shows that J is a two-sided ideal of A_q such that $J^n = 0$ and $A_q/J \cong K \times \ldots \times K$. Hence we conclude that $J = J(A_q)$ and the algebra A_q is basic.

(d) Assume that q is basic. The statement (i) is proved above. To prove (ii), assume that $\lambda = [\lambda_{ij}] \in \mathbb{M}_n^q(K)$. First we show that

$$\lambda \text{ is invertible in } A_q \text{ if and only if } \lambda_{11} \cdot \ldots \cdot \lambda_{nn} \neq 0.$$

To prove the sufficiency, assume that $\lambda_{11} \cdot \ldots \cdot \lambda_{nn} \neq 0$ and consider the diagonal matrix $d_\lambda := \mathrm{diag}(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn}) \in \mathbb{M}_n^q(K)$ with the coefficients $\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn}$ on the main diagonal. Now we view the matrix $d_\lambda^{-1} \cdot_q \lambda = \mathrm{diag}(\lambda_{11^{-1}}, \lambda_{22}^{-1}, \ldots, \lambda_{nn}^{-1}) \cdot_q \lambda$ in the form $d_\lambda^{-1} \cdot_q \lambda = E - \check{\lambda}$, where $\check{\lambda} \in J(A_q)$, see (i). It follows that $\check{\lambda}^n = 0$ and therefore

$$d_\lambda^{-1}\lambda \cdot_q (E + \check{\lambda} + \check{\lambda}^2 + \cdots + \check{\lambda}^{n-1}) = (E - \check{\lambda}) \cdot_q (E + \check{\lambda} + \check{\lambda}^2 + \cdots + \check{\lambda}^{n-1}) = E.$$

This shows that λ is invertible in A_q and the matrix

$$\lambda^{-1} = d_\lambda \cdot_q (E + \check{\lambda} + \check{\lambda}^2 + \cdots + \check{\lambda}^{n-1})$$

is the inverse of λ in A_q. Conversely, assume that λ is invertible in A_q and assume, to the contrary, that $\lambda_{11} \cdot \ldots \cdot \lambda_{nn} = 0$; say $\lambda_{11} = 0$. It follows from (i) that λ has the form $\lambda = \lambda_{22}e_2 + \cdots + \lambda_{nn}e_n + \check{\lambda}$, where $\check{\lambda} \in J(A_q)$ and $\check{\lambda}^n = 0$. If μ is an inverse of λ in A_q then

$$\begin{aligned} E = \lambda \cdot_q \mu &= (\lambda_{22}e_2 + \cdots + \lambda_{nn}e_n + \check{\lambda}) \cdot_q \mu \\ &= \lambda_{22}e_2 \cdot_q \mu + \cdots + \lambda_{nn}e_n \cdot_q \mu + \check{\lambda} \cdot_q \mu = c_{22}e_2 + \cdots + c_{nn}e_n + \lambda', \end{aligned}$$

where $c_{22}, \ldots, c_{nn} \in K$ and $\lambda' \in J(A_q)$. It follows that the coefficient at the $(1,1)$-entry of the matrix $c_{22}e_2 + \cdots + c_{nn}e_n + \lambda'$ is zero, and we get a contradiction. This finishies the proof of (ii).

(iii) Assume that $\mathfrak{A}$ is a non-zero two-sided ideal of A_q. If $\lambda = [\lambda_{ij}]$ is a non-zero matrix in $\mathfrak{A}$, with $\lambda_{ij} \in K$, then $\lambda = \sum_{i,j} \lambda_{ij} e_{ij}$. It follows that, given i and j such that $\lambda_{ij} \neq 0$, the element $e_i \lambda e_j = \lambda_{ij} e_{ij}$ belongs to $\mathfrak{A}$ and, consequently, the matrix unit e_{ij} belongs to $\mathfrak{A}$, because $\lambda_{ij} \neq 0$. Hence (iii) follows.

(iv) Since, by (b), $e_i A_q e_j \cong K e_{ij}$, for all $i, j \in \{1, \ldots, n\}$, then C_{A_q} has the form

$$C_{A_q} = \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}.$$

On the other hand, it is well-known that the determinat of the Cartan matrix of any K-algebra R is 1 or -1, if R is basic of finite global dimension, see [1, Chapter I]. Then (iv) follows and the proof of the theorem is complete.

□

COROLLARY 2.2.6. *If K is a field and $q = [q^{(1)}| \ldots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a structure matrix. There is a K-algebra isomorphism $\mathbb{M}_n^q(K) \cong \mathbb{M}_n(K)$ if and only if $q_{22}^{(1)} \neq 0$, $q_{33}^{(1)} \neq 0$, $\ldots$, $q_{nn}^{(1)} \neq 0$.*

Proof. Let $A_q = \mathbb{M}_n^q(K)$. We recall from Lemma 2.2.2 (b) that $q_{jj}^{(r)} = q_{rr}^{(j)}$, for all $j, r \in \{1, \ldots, n\}$. Hence, in view of Theorem 2.2.5 (b), there are isomorphisms $e_1 A_q \cong \ldots \cong e_n A_q$ of right ideals of A_q if and only if $q_{22}^{(1)} \neq 0$, $q_{33}^{(1)} \neq 0$, $\ldots$, $q_{nn}^{(1)} \neq 0$. Since $\operatorname{End} e_1 A_q \cong K$, the corollary follows. □

DEFINITION 2.2.7. (a) Given a matrix $\lambda = [\lambda_{pq}] \in \mathbb{M}_n(R)$ and a permutation $\sigma \in S_n$ of the set $\{1, \ldots, n\}$ we denote by $\sigma * \lambda = [\lambda_{pq}^{\sigma}]$ the matrix in $\mathbb{M}_n(R)$ with $\lambda_{pq}^{\sigma} = \lambda_{\sigma^{-1}(p)\,\sigma^{-1}(q)}$.

(b) Given a minor constant structure matrix $q = [q^{(1)}| \ldots |q^{(n)}]$ and $\sigma \in S_n$, we set $\sigma * q = [\sigma * q^{(\sigma^{-1}(1))}| \ldots |\sigma * q^{(\sigma^{-1}(n))}]$. We also define the *transpose* of q to be the n-block matrix $q^{tr} = q = [\, q^{(1)}| \ldots |\, q^{(n)}]$, where $q^{(j)} = [q^{(j)}]^{tr}$ is the transpose of $q^{(j)}$, for $j = 1, \ldots, n$.

It is clear that the map $(\sigma, q) \mapsto \sigma * q$ defines an action

$$* : S_n \times \mathbb{ST}_n(K) \longrightarrow \mathbb{ST}_n(K) \tag{2.6}$$

of the symmetric group S_n on the K-variety $\mathbb{ST}_n(K)$ of all minor constant structure matrices q (2.1) of size $n \times n^2$. The subsets consisting of all basic matrices and of all basic $(0,1)$-matrices are S_n-invariant.

EXAMPLE 2.2.8. A simple calculation shows that, in case $n = 3$, every matrix $q = [q^{(1)}|q^{(2)}|q^{(3)}]$

in $\mathbb{ST}_3(K)$ has one of the following three forms, up to the S_3-action,

$$q_1 = \begin{bmatrix} 1\,1\,1 & \lambda\,1\,\frac{\lambda}{\mu} & \xi\,\frac{\xi}{\nu}\;\;1 \\ 1\,\lambda\;\mu & 1\;1\;1 & \frac{\xi}{\mu}\,\frac{\lambda\xi}{\mu\nu}\,1 \\ 1\;\nu\;\xi & \frac{\lambda}{\nu}\,1\,\frac{\lambda\xi}{\mu\nu} & 1\;\;1\;\;1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1\,1\,1 & \lambda\,1\,\frac{\lambda}{\mu} & 0\,0\,1 \\ 1\,\lambda\;\mu & 1\;1\;1 & 0\,0\,1 \\ 1\;\nu\;0 & \frac{\lambda}{\nu}\,1\,0 & 1\,1\,1 \end{bmatrix},$$

$$q_3 = \begin{bmatrix} 111 & 010 & 001 \\ 10\mu & 111 & 001 \\ 1\nu 0 & 010 & 111 \end{bmatrix}, \qquad q_4 = \begin{bmatrix} 111 & 01\lambda & 001 \\ 101 & 111 & \tau 01 \\ 1\nu 0 & 010 & 111 \end{bmatrix},$$

where $\lambda, \mu, \nu, \xi, \tau \in K$; and we assume that $\mu\nu \neq 0$ in the matrices q_1 and q_2. Note that $q_2 = q_1|_{\xi=0}$ and $q_3 = q_2|_{\lambda=0}$. It follows Corollary 2.2.6 that, if $\lambda\xi \neq 0$ then $A_1 = \mathbb{M}_3^{q_1}(K)$ is isomorphic to $\mathbb{M}_n(K)$, because $(q_1)_{22}^{(1)} = \lambda \neq 0$ and $(q_1)_{33}^{(1)} = \xi \neq 0$. Note also $A_2 = \mathbb{M}_3^{q_2}(K)$ is Morita equivalent to the algebra $A(0) = \mathbb{M}_2^{q(0)}(K)$ of Example 2.2.4. Indeed, by Theorem 2.2.5 (b) there is an isomorphism $e_1A_2 \cong e_2A_1$, because $(q_2)_{11}^{(2)} = (q_2)_{22}^{(1)} = \lambda \neq 0$. Moreover, the right ideals e_1A_2 and e_3A_2 are not isomorphic, because $(q_2)_{33}^{(1)} = 0$.

The following simple result is very useful.

Lemma 2.2.9. *Let $n \geq 2$ and let $q = [q^{(1)}|\ldots|q^{(n)}]$ be a basic structure matrix (2.1) in $\mathbb{ST}_n(K)$, with coefficients in a ring R. Let $\mathbb{M}_n^q(R)$ be the q-degeneration of $\mathbb{M}_n(R)$.*

(a) *The transpose $q^{tr} = \; q = [\; q^{(1)}|\ldots|\; q^{(n)}]$ of q is a basic structure matrix in $\mathbb{ST}_n(K)$ and the K-linear map $\mathbb{M}_n^q(R) \to \mathbb{M}_n^{q^{tr}}(R)$, defined by $\lambda \mapsto \lambda^{tr}$, is an R-algebra anti-isomorphism, that is, it defines an R-algebra isomorphism $(\mathbb{M}_n^q(R))^{op} \cong \mathbb{M}_n^{q^{tr}}(R)$.*

(b) *If $\sigma \in S_n$ is a permutation of the set $\{1,\ldots,n\}$ then $\sigma * q = [\sigma * q^{(\sigma(1))}|\ldots|\sigma * q^{(\sigma(n))}]$ is a basic structure matrix in $\mathbb{ST}_n(K)$ and the map $\lambda \mapsto \sigma * \lambda$ defines the R-algebra isomorphism $\mathbb{M}_n^q(R) \cong \mathbb{M}_n^{\sigma * q}(R)$ such that $e_{ij} \mapsto e_{\sigma(i)\,\sigma(j)}$, for all i and j.*

Proof. The proof of (a) and (b) is straightforward, and is left to the reader. □

Now we extend the action $* : S_n \times \mathbb{ST}_n(K) \longrightarrow \mathbb{ST}_n(K)$ of the symmetric group S_n to an action of the following semidirect product algebraic group

$$\mathbb{G}_n(K) = \mathcal{T}_n \ltimes S_n \tag{2.7}$$

containing S_n, where $\mathcal{T}_n \ltimes S_n = \mathcal{T}_n \times S_n$ is the Cartesian product,

$$\mathcal{T}_n = \{T = [t_{ij}] \in \mathbb{M}_n(K);\ t_{11} = \ldots = t_{nn} = 1 \text{ and } t_{ij} \neq 0, \text{for all } i,j\}$$

is viewed as a group with the coordinate-wise multiplication $[t_{ij}]\cdot[t'_{ij}] = [t_{ij}t'_{ij}]$ and the multiplication in $\mathbb{G}_n(K)$ is defined by the formula $(T,\sigma)\cdot(T',\sigma') = (T\cdot(\sigma * T'),\ \sigma\sigma')$, for $T, T' \in \mathcal{T}_n$ and $\sigma, \sigma' \in S_n$.

It is clear that $\mathcal{T}_n$ is isomorphic to the (n^2-n)-dimensional K-torus $T_{n^2-n}(K) = K^* \times K^* \times \ldots \times K^*$ (the product of $n^2 - n$ copies of the multiplicative group $K^* = K \setminus \{0\}$ of K). We define an action

$$* : \mathbb{G}_n(K) \times \mathbb{ST}_n(K) \longrightarrow \mathbb{ST}_n(K) \tag{2.8}$$

by the formula $(T,\sigma)*q = [\, q^{(1)}| \ldots |\, q^{(n)}]$, where $T=[t_{ij}] \in \mathcal{T}_n$, $\sigma \in S_n$, and $q^{(r)} = [\, q^{(r)}_{ij}] \in \mathbb{M}_n(K)$ is defined by $q^{(r)}_{ij} = q^{(\sigma^{-1}(r))}_{\sigma^{-1}(i)\,\sigma^{-1}(j)} \cdot t^{-1}_{ir} t_{ij} t^{-1}_{rj}$, for $i,j,r \in \{1,\ldots,n\}$.

The following result shows that the $\mathbb{G}_n(K)$-orbits classify the isomorhism classes of the basic algebras $\mathbb{M}^q_n(K)$ of dimension n^2.

THEOREM 2.2.10. *Assume that K is a field and that $n \geq 2$ is an integer.*

(a) *The map (2.8) is an action of the algebraic group $\mathbb{G}_n(K)$ (2.7) on the algebraic K-variety $\mathbb{ST}_n(K)$ of minor structure matrices $q = [q^{(1)}|\ldots|q^{(n)}]$ (2.1). The subvariety of $\mathbb{ST}_n(K)$ consisting of the basic matrices is $\mathbb{G}_n(K)$-invariant.*

(b) *Given two basic structure matrices $q = [q^{(1)}|\ldots|q^{(n)}]$ and $q' = [q'^{(1)}|\ldots|q'^{(n)}]$ in $\mathbb{ST}_n(K)$, the following statements are equivalent.*

 (b1) *The K-algebras $\mathbb{M}^q_n(K)$ and $\mathbb{M}^{q'}_n(K)$ are isomorphic.*

 (b2) *The matrices q and q' belong to the same $\mathbb{G}_n(K)$-orbit.*

 (b3) *There exist a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and a square matrix $T = [t_{ij}] \in \mathbb{M}_n(K)$ such that*

 - $t_{11} = \ldots = t_{nn} = 1$,
 - $t_{ij} \neq 0$, *for all* $i,j \in \{1,\ldots,n\}$, *and*
 - $t_{ir} \cdot {q'}^{(r)}_{ij} \cdot t_{rj} = q^{(\sigma(r))}_{\sigma(i)\,\sigma(j)} \cdot t_{ij}$, *for all* $i,r,j \in \{1,\ldots,n\}$.

Proof. (a) The proof is straightforward and we leave it to the reader.

(b) A simple calculation shows that q' belongs to the $\mathbb{G}_n(K)$-orbit of q if and only if there exist a permutation $\sigma \in S_n$ and a matrix $T = [t_{ij}] \in \mathbb{M}_n(K)$ such that the conditions stated in (b3) are satisfied. Consequently, the statements (b2) and (b3) are equivalent.

(b3)⇒(b1) Suppose that $T = [t_{ij}] \in \mathbb{M}_n(K)$ and $\sigma \in S_n$, are such that the conditions stated in (b3) are satisfied. Then the map $e_{\sigma(i)\,\sigma(j)} \mapsto t_{ij}e_{ij}$ defines a K-algebra isomorphism $\mathbb{M}^q_n(K) \cong \mathbb{M}^{q'}_n(K)$.

(b1)⇒(b3) Assume that there is an K-algebra isomorphism $h : \mathbb{M}^q_n(K) \to \mathbb{M}^{q'}_n(K)$. The elements $h(e_1),\ldots,h(e_n)$ are primitive orthogonal idempotents of $\mathbb{M}^{q'}_n(R)$ such that $1 = h(e_1)+\ldots+h(e_n)$. By [5, Theorem 3.4.1], there exist a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and an invertible element $B \in \mathbb{M}^{q'}_n(K)$ such that $e_j = B\cdot h(e_{\sigma(j)})\cdot B^{-1}$, for $j = 1,\ldots,n$. Hence we conclude that there exists a K-algebra isomorphism $h' : \mathbb{M}^q_n(K) \to \mathbb{M}^{q'}_n(K)$ such that $e_1 = h'(e_{\sigma(1)}),\ldots,e_n = h'(e_{\sigma(n)})$. Since $h'(e_{\sigma(i)\,\sigma(j)}) = h'(e_{\sigma(i)} \cdot e_{\sigma(i)\,\sigma(j)} \cdot e_{\sigma(j)}) = e_i \cdot h'(e_{\sigma(i)\,\sigma(j)}) \cdot e_j$, then there exists a non-zero element $t_{ij} \in K^*$ such that $h'(e_{\sigma(i)\sigma(j)}) = t_{ij}e_{ij}$, for $i,j \in \{1,\ldots,n\}$. It is clear that $t_{11} = \ldots = t_{nn} = 1$. Moreover, the equality $h'(e_{\sigma(i)\,\sigma(r)} \cdot e_{\sigma(r)\,\sigma(j)}) = h'(e_{\sigma(i)\,\sigma(r)}) \cdot h'(e_{\sigma(r)\,\sigma(j)})$ yields $q^{(\sigma(r))}_{\sigma(i)\,\sigma(j)} t_{ij} = t_{ir} {q'}^{(r)}_{ij} t_{rj}$, for all $i,r,j \in \{1,\ldots,n\}$. Consequently, the matrix $T = [t_{ij}] \in \mathbb{M}_n(K)$ satisfies the conditions stated in (b3) and (T,σ) is an element of the group $\mathbb{G}_n(K)$. This completes the proof. □

As an immediate consequence of Theorem 2.2.10 we get the following isomorphism criterion.

COROLLARY 2.2.11. *Let K be a field, $n \geq 2$, and let $q = [q^{(1)}| \dots |q^{(n)}]$, $q' = [q'^{(1)}| \dots |q'^{(n)}]$ be basic structure $(0,1)$-matrices (2.1) in $\mathbb{ST}_n(K)$. The K-algebras $\mathbb{M}_n^q(K)$ and $\mathbb{M}_n^{q'}(K)$ are isomorphic if and only if q and q' are in the same S_n-orbit, that is, there exist a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $q^{(\sigma(r))}_{\sigma(i)\,\sigma(j)} = q'^{(r)}_{ij}$, for all $i, r, j \in \{1, \dots, n\}$.*

Proof. In this case the matrix $T = [t_{ij}] \in \mathcal{T}_n(K)$ required in Theorem 2.2.10 (b) has $t_{ij} = 1$, for all i and j. $\square$

Following P. Gabriel [10], we associate to any basic and connected finite dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$, *the Gabriel quiver* $\mathcal{Q}(A) = (\mathcal{Q}(A)_0, \mathcal{Q}(A)_1)$ as follows. The set $\mathcal{Q}(A)_0 = \{1, 2, \dots, n\}$ is the set of points of $\mathcal{Q}(A)$, its elements are in bijective correspondence with the idempotents $e_1, e_2, \dots, e_n$. Given two points $i, j \in \mathcal{Q}(A)_0$, the arrows $\beta : i \to j$ in $\mathcal{Q}(A)_1$ are in bijective correspondence with the vectors in a basis of the K-vector space $e_i[J(A)/J(A)^2]e_j$, see [1, Chapter II].

COROLLARY 2.2.12. *Let $n \geq 2$ and let $q = [q^{(1)}| \dots |q^{(n)}]$ be a basic minor constant structure matrix (2.1) in $\mathbb{ST}_n(K)$. Let $A_q = \mathbb{M}_n^q(K)$ be the q-degeneration K-algebra of $\mathbb{M}_n(K)$ and let $\mathcal{Q}(A_q) = (\mathcal{Q}(A_q)_0, \mathcal{Q}(A_q)_1)$ be the Gabriel quiver of A. Then the following statements hold.*

(a) $\mathcal{Q}(A_q)_0 = \{1, \dots, n\}$.

(b) *Given $i, j \in \mathcal{Q}(A_q)_0$, there exists an arrow $i \to j$ in $\mathcal{Q}(A_q)_1$ if and only if $i \neq j$ and $q^{(r)}_{ij} = 0$, for all $r \notin \{i, j\}$. In this case, there is a unique arrow $\beta_{ij} : i \to j$ that corresponds to the coset $\overline{e}_{ij} \in e_i[J(A_q)/J(A_q)^2]e_j$ of the matrix unit e_{ij}.*

(c) *The quiver $\mathcal{Q}(A_q)$ is connected and has no loops.*

Proof. (a) It follows from Theorem 2.2.5 that the algebra $A_q = \mathbb{M}_n^q(K)$ is basic and $A_q/J(A_q) \cong K\overline{e}_1 \oplus \ldots \oplus K\overline{e}_n$. The points of the quiver $\mathcal{Q}(A_q)$ correspond to the primitive idempotents $e_1, \dots, e_n$ of A, and (a) follows.

(b) It follows from Theorem 2.2.5 that, given two primitive idempotents e_i and e_j, we have $\mathrm{Hom}_{A_q}(e_j A_q, e_i A_q) \cong K$, if $i = j$, and $\mathrm{Hom}_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong e_{ij}K$, if $i \neq j$. Hence we get $e_i J(A_q) e_i = 0$, that is, the quiver $\mathcal{Q}(A_q)$ has no loops. If $i \neq j$, we get $e_i J(A_q) e_j \cong e_{ij}K$ and therefore $e_i J(A_q)^2 e_j = e_i J(A) e_j$ if and only if there is an $s \in \{1, \dots, n\} \setminus \{i, j\}$ such that $e_{ij} = \mu e_{is} e_{sj}$, for some non-zero $\mu \in K$. Since $e_{is} e_{sj} = q^{(s)}_{ij} e_{ij}$, then $0 \neq \overline{e}_{ij} \in e_i[J(A_q)/J(A_q)^2]e_j$ if and only if $q^{(s)}_{ij} = 0$, for all $s \notin \{i, j\}$. Hence (b) follows.

(c) By Theorem 2.2.5 (e), the algebra A_q is connected. Hence we conclude that the quiver $\mathcal{Q}(A_q)$ is connected (see [1, Corollary II.3.4]). Since, by (C3), $q^{(r)}_{jj} = 0$, for $r = 1, \dots, n$ and all $j \in \{1, \dots, n\}$ such that $j \neq r$ then, according to (b), the quiver $\mathcal{Q}(A)$ has no loops.
This finishes the proof. $\square$

Now assume that $A = \mathbb{M}_n^q(K)$ is a minor degeneration of the algebra $\mathbb{M}_n(K)$, where $q = [q^{(1)}|\ldots|q^{(n)}]$. Let I be a non-empty subset of $\{1,\ldots,n\}$. Assume that $s = |I|$ is the cardinality of I and $I = \{i_1,\ldots,i_s\}$. Define q_I to be the s-block matrix

$$q_I = [q_I^{(i_1)}|\ldots|q_I^{(i_s)}] \tag{2.9}$$

obtained from q by the restriction to I, that is, each matrix $q_I^{(i_s)} \in \mathbb{M}_s(K)$ is obtained from $q^{(i_s)} \in \mathbb{M}_n(K)$ by deleting the j-th row and the j-th column, for all $j \notin I$. It is clear that q_I is a basic structure matrix of size $s \times s^2$. We set

$$A_I = \mathbb{M}_s^{q_I}(K).$$

Let $e_I = \sum_{j\in I} e_j = e_{i_1} + \ldots + e_{i_s}$, where are the standard primitive idempotents of A. Then e_I is an idempotent of $A = \mathbb{M}_n^q(K)$ and there is a K-algebra isomorphism

$$e_I A e_I = e_I \mathbb{M}_n^q(K) e_I \cong \mathbb{M}_s^{q_I}(K) = A_I \tag{2.10}$$

given by associating to any matrix $e_I \lambda e_I \in e_I \mathbb{M}_n^q(K) e_I$ the restriction of $\lambda = [\lambda_{ij}] \in \mathbb{M}_n^q(K)$ to $I = \{i_1,\ldots,i_s\}$.

Now we define three additive K-linear covariant functors

$$\operatorname{mod}\ \mathbb{M}_s^{q_I}(K) \underset{\operatorname{res}_I}{\overset{T_I, L_I}{\rightleftarrows}} \operatorname{mod}\ \mathbb{M}_n^q(K) \tag{2.11}$$

by the formulae $\operatorname{res}_I(-) = (-)e_I$, $T_I(-) = - \otimes_{e_I A e_I} e_I A$, $L_I(-) = \operatorname{Hom}_{e_I A e_I}(Ae_I, -)$, where $A = \mathbb{M}_n^q(K)$. If $f : X \to X'$ is a homomorphism of A-modules, we define a homomorphism of $\mathbb{M}_s^{q_I}(K)$-modules $\operatorname{res}_I(f) : \operatorname{res}_I(X) \to \operatorname{res}_I(X')$ by the formula $xe_I \mapsto f(x)e_I$, that is, $\operatorname{res}_I(f)$ is the restriction of f to the subspace Xe_I of X, see [1, Section I.6] and [28, Section 17.5].

The following result is very useful in applications.

THEOREM 2.2.13. *Suppose that $A = \mathbb{M}_n^q(K)$ and $A_I = \mathbb{M}_I^{q_I}(K)$ are as above. Then there is a K-algebra isomorphism $A_I \cong e_I A e_I$ described above and the functors T_I, L_I (2.11) associated to I satisfy the following conditions.*

(a) *T_I and L_I are full and faithful K-linear functors such that $res_I \circ T_I \cong \mathrm{id} \cong res_I \circ L_I$, the functor L_I is right adjoint to res_I and T_I is left adjoint to res_I.*

(b) *The restriction functor res_I is exact, T_I is right exact and L_I is left exact.*

(c) *The functors T_I and L_I preserve indecomposability, T_I carries projectives to projectives and L_I carries injectives to injectives.*

(d) *An A-module X is in the category $Im T_I$ if and only if there is an exact sequence $P_1 \xrightarrow{h} P_0 \longrightarrow X \longrightarrow 0$, where P_1 and P_0 are direct sums of summands of $e_I A = e_{i_1} A \oplus \ldots \oplus e_{i_s} A$.*

Proof. Apply [1, Theorem I.6.8] and [28, Section 17.5], and the arguments used there. The details are left to the reader. □

COROLLARY 2.2.14. *Suppose that $A = \mathbb{M}_n^q(K)$ and $A_I = \mathbb{M}_I^{q_I}(K)$ are as above.*

(a) *If A is representation-finite, then A_I is also representation-finite.*

(b) *If $\overline{K} = K$ and A is representation-tame, then A_I is also representation-tame* [28, Section 14.4], [33, Chapter XIX].

(c) *If $\overline{K} = K$ and A_I is representation-wild, then A is representation-wild* [28, Section 14.2], [33, Chapter XIX].

Proof. (a) Assume that A is representation-finite and consider the fully faithful functor T_I : $\operatorname{mod} A_I \longrightarrow \operatorname{mod} A$, see (2.11) and Theorem 2.2.13. Since T_I carries indecomposable A_I-modules to indecomposable indecomposable A-modules, and nonisomorphic A_I-modules to nonisomorphic A_I-modules, then (a) follows.

(b) Assume that the field K is algebraically closed and A that is representation-tame. Fix a dimension $d \in \mathbb{N}$ and consider the functors T_I and res_I presented in (2.11).

First we show that, given a module X in $\operatorname{mod} A_I$ with $\dim_K X = d$, the K-dimension of the A-module $T_I(X)$ is not geater than $\overline{d} = d \cdot p_I$, where $p_I = \max\{\dim_K e_i A;\ i \in I\}$. To see this we note that the A_I-projective cover of X has the form $\bigoplus_{i\in I}(e_i A_I)^{d_i} \longrightarrow X \longrightarrow 0$, where $d_i = \dim_K(\operatorname{top} X)e_i \leq d$. By Theorem 2.2.13, the functor T_I is right exact and there is an A-module isomorphism $T_I(e_i A_I) \cong e_i A$, for all $i \in I$. It follows that T_I induces an epimorphism $\bigoplus_{i\in I}(e_i A)^{d_i} \longrightarrow T_I(X) \longrightarrow 0$ of right A-modules. Hence we get

$$\dim_K T_I(X) \leq \dim_K \bigoplus_{i\in I}(e_i A)^{d_i} \leq \sum_{i\in I}(d_i \cdot \dim_K e_i A) \leq (\sum_{i\in I} d_i) \cdot p_I \leq d \cdot p_I = \overline{d},$$

and our claim follows.

Since the algebra A is representation-tame then, given the K-dimension $\overline{d} = d \cdot p_I$, there exist a non-zero polynomial $h \in K[t]$ and a family of K-linear functors

$$(-) \otimes_S N^{(1)}, \ldots, (-) \otimes_S N^{(r)} : \operatorname{ind}(\operatorname{mod} S) \longrightarrow \operatorname{mod}\ A$$

where $S = K[t, h^{-1}]$, $N^{(1)}, \ldots, N^{(r)}$ are S-A-bimodules satisfying the following two conditions:

(T0) The left S-modules ${}_S N^{(1)}, \ldots, {}_S N^{(r)}$ are finitely generated and free.

(T1) All but finitely many indecomposable modules in $\operatorname{mod}\ A$ of K-dimension $\leq \overline{d}$ are isomorphic to modules in $\operatorname{Im}(-) \otimes_S N^{(1)} \cup \cdots \cup \operatorname{Im}(-) \otimes_S N^{(r)}$, see [28, Section 14.4] and [33, Chapter XIX].

Here $\operatorname{ind}(\operatorname{mod} S)$ is the category of indecomposable S-modules of finite dimension. Consider the restricted S-A_I-bimodules $\operatorname{res}_I N^{(1)} = N^{(1)} e_I, \ldots,\ \operatorname{res}_I N^{(r)} = N^{(r)} e_I$. It is clear that the S-module $\operatorname{res}_I N^{(j)}$ is finitely generated and free, for each j, because the functor res_I is exact. Now, if X is an indecomposable a module in $\operatorname{mod}\ A_I$ with $\dim_K X = d$ then, according to Theorem 2.2.13 and our claim above, the A-module $T_I(X)$ is indecomposable and $\dim_K T_I(X) \leq \overline{d}$. It follows that there exists an S-module N in ind(mod S) such that $T_I(X) \cong N \otimes_S N^{(j)} = N \otimes_S N^{(j)} e_I$, for some $j \leq r$. In view of Theorem 2.2.13 (a), we get A_I-module isomorphisms

$$X \cong \operatorname{res}_I(T_I(X)) \cong \operatorname{res}_I(N \otimes_S N^{(j)}) = (N \otimes_S N^{(j)})e_I \cong N \otimes_S (N^{(j)})e_I \cong N \otimes_S \operatorname{res}_I N^{(j)}.$$

This shows that the algebra A_I is representation-tame.

(c) Assume that the field K is algebraically closed and that the algebra A_I is representation-wild. By the tame-wild dichotomy [4], [28, Theorem 14.14],[33] A_I is not representation-tame. It follows from (b), that the algebra A is not representation-tame. Hence, A is representation-wild, by the tame-wild dichotomy. □

Corollary 2.2.15. *Assume that K is a field, $q = [q^{(1)}|\dots|q^{(n)}]$ is a minor constant matrix in $\mathbb{ST}_n(K)$ and let $I = \{i_1, \dots, i_s\}$ be a maximal subset of $\{1, \dots, n\}$ such that $q^{(r)}_{jj} = 0$, whenever $j, r \in I$ and $j \neq r$. Then the minor constant matrix q_I in $\mathbb{ST}_s(K)$ is basic, the K-algebra $\mathbb{M}^{q_I}_s(K)$ is basic and is Morita equivalent to the algebra $\mathbb{M}^q_n(K)$.*

Proof. Let $A = \mathbb{M}^q_n(K)$ and suppose that $I = \{i_1, \dots, i_s\}$ satisfies the maximality conditions. It follows that the constant matrix q_I is basic and, in view of Theorem 2.2.5, the K-algebra A is basic and $e_jA \not\cong e_rA$, for all $j, r \in I$ such that $j \neq r$. By the maximality of I, given $r \notin I$ there exists $j \in I$ such that $q^{(r)}_{jj} \neq 0$. Since $q^{(r)}_{jj} = q^{(j)}_{rr}$, by Lemma 2.2.2, then $e_rA \cong e_jA$, see Theorem 2.2.5. Consequently, for each $r \in \{1, \dots, n\}$ there is $i_r \in I$ such that $e_rA \cong e_{i_r}A$ and the modules $e_{i_1}A, \dots, e_{i_r}A$ are pairwise non-isomorphic. In view of Theorem 2.2.13 (d), it follows that the functor $T_I : \operatorname{Mod} \mathbb{M}^{q_I}_s(K) \longrightarrow \operatorname{Mod} \mathbb{M}^q_n(K)$ is dense. Since, according to Theorem 2.2.13 (a), the functor T_I is fully faithful then it is an equivalence of categories. This shows that the K-algebras $\mathbb{M}^{q_I}_s(K)$ and $\mathbb{M}^q_n(K)$ are Morita equivalent. □

2.2.2 $(0,1)$-limits and Nakayama algebras.

Throughout this subsection the following definition is of importance.

Definition 2.2.16. Let $A_q = \mathbb{M}^q_n(K)$ be a minor degeneration algebra of $\mathbb{M}_n(K)$ with a structure matrix $q = [q^{(1)}|\dots|q^{(n)}]$, where $q^{(s)} = [q^{(s)}_{ij}]$.

(a) We define a $(0,1)$*-limit of* q to be the structure $(0,1)$-matrix $\overline{q} = [\overline{q}^{(1)}|\dots|\overline{q}^{(n)}]$, where the matrix $\overline{q}^{(s)} = [\overline{q}^{(s)}_{ij}]$ is defined by the formulae

$$\overline{q}^{(s)}_{ij} = \begin{cases} 1 & \text{if } q^{(s)}_{ij} \neq 0, \\ 0 & \text{if } q^{(s)}_{ij} = 0. \end{cases}$$

(b) The algebra $\overline{A}_q = A_{\overline{q}} = \mathbb{M}^{\overline{q}}_n(K)$ is called *the* $(0,1)$*-limit of* $A_q = \mathbb{M}^q_n(K)$.

We recall that a finite dimensional K-algebra A is a Frobenius algebra if there exists a K-linear map $\psi : A \to K$ such that $\operatorname{Ker}\psi$ does not contain non-zero right (or left) ideals of A, see [37]. It is clear that a basic K-algebra A is Frobenius if and only if A is self-injective, see [35].

PROPOSITION 2.2.17. *Assume that K is a field, $A_q = \mathbb{M}_n^q(K)$ is a basic minor degeneration of $\mathbb{M}_n(K)$ and $\overline{A}_q = \mathbb{M}_n^{\overline{q}}(K)$ is the $(0,1)$-limit of A_q.*

(a) *A vector K-subspace $\mathfrak{A}$ of $\mathbb{M}_n(K)$ is a two-sided ideal of A_q if and only if $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. In particular, $J(A_q)^s = J(\overline{A}_q)^s$, for each $s \geq 1$.*

(b) *The Gabriel quivers of A_q and $\overline{A}_q$ coincide.*

(c) *Assume that the field K is algebraically closed and $\{A_{q_\mu}\}_{\mu \in K}$ is an algebraic family [18] of minor degenerations $A_{q_\mu} = \mathbb{M}_n^{q_\mu}(K)$ of $\mathbb{M}_n(K)$ such that $A_{q_0} = \overline{A}_q$ and almost all algebras A_{q_μ} are isomorphic. If the algebra $\overline{A}_q$ is representation-finite (resp. representation-tame) then A_{q_μ} is representation-finite (resp. representation-tame), for almost all structure matrices q_μ.*

Proof. (a) Let $\mathfrak{A}$ be a non-zero vector K-subspace of $\mathbb{M}_n(K)$. Suppose that $\mathfrak{A}$ is a two-sided ideal of A_q. It follows from Theorem 2.9(e) that $\mathfrak{A}$ is generated by a finite set of the matrix units e_{ij} of A_q. We show that $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. Denote by $\cdot'$ and $\cdot''$ the multiplication in A_q and in $\overline{A}_q$, respectively.

Since the matrix units form a K-basis of $\overline{A}_q$, it is sufficient to show that $e_{st} \cdot'' e_{ij} \in \mathfrak{A}$ and $e_{ij} \cdot'' e_{rp} \in \mathfrak{A}$, for any $e_{ij} \in \mathfrak{A}$ and any $e_{st}, e_{rp} \in \overline{A}_q$. Recall that $e_{st} \cdot'' e_{ij} = 0$, for $t \neq i$, and $e_{ij} \cdot'' e_{rp} = 0$, for $j \neq r$. Therefore, we can assume that $t = i$ and $j = r$. In this case, we get

$$e_{si} \cdot'' e_{ij} = \overline{q}_{sj}^{(i)} e_{sj} = \begin{cases} e_{sj} & \text{if } \overline{q}_{sj}^{(i)} \neq 0, \\ 0 & \text{if } \overline{q}_{sj}^{(i)} = 0. \end{cases}$$

Assume that $\overline{q}_{sj}^{(i)} \neq 0$, that is, $\overline{q}_{sj}^{(i)} = 1$. Then $q_{sj}^{(i)} \neq 0$ and the element $e_{si} \cdot' e_{ij} = q_{sj}^{(i)} e_{sj}$ belongs to $\mathfrak{A}$, because $\mathfrak{A}$ is a two-sided ideal of A_q. It follows that $e_{sj} = e_{si} \cdot'' e_{ij} \in \mathfrak{A}$. Similarly, we show that $e_{ij} \cdot'' e_{jp} \in \mathfrak{A}$. Consequently, $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. The same type of arguments shows that $\mathfrak{A}$ is a two-sided ideal of A_q, if $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. This finishes the proof of the first statement in (a). The second one follows from the first one by applying it to $\mathfrak{A} = J(A_q)^s$.

(b) Since $J(A_q) = J(\overline{A}_q)$ and $J(A_q)^2 = J(\overline{A}_q)^2$, then

$$e_i[J(A_q)/J(A_q)^2]e_j = e_i[J(\overline{A}_q)/J(\overline{A}_q)^2]e_j,$$

for all i, j, and hence $\mathcal{Q}(A_q) = \mathcal{Q}(\overline{A}_q)$.

(c) Since, according to [11], the algebras of finite representation type define an open subset in $\mathcal{A}lg(K^{n^2})$, then almost all algebras A_{q_μ} are of finite representation type if so is $\overline{A}_q = A_{q_0}$, see also [18, Chapter III]. Further, according to Geiss [12], the tameness of $\overline{A}_q = A_{q_0}$ implies the tameness of A_{q_μ}, for almost all structure matrices q_μ. Hence (c) follows and the proof is complete. □

We recall that a finite dimensional K-algebra A is said to be a *Nakayama algebra*, if for every primitive idempotent $e \in A$, the left ideal Ae has a unique composition series and the right ideal eA has a unique composition series.

Now we describe the minor degenerations of $\mathbb{M}_n(K)$ that are Nakayama algebras.

THEOREM 2.2.18. *Assume that $n \geq 2$ and $q = [q^{(1)}|\dots|q^{(n)}]$ is a basic structure matrix (2.1) of size $n \times n^2$. Let $\overline{q} = [\overline{q}^{(1)}|\dots|\overline{q}^{(n)}]$ be the $(0,1)$-limit of q, let $A_q = \mathbb{M}_n^q(K)$ and $\overline{A}_q = \mathbb{M}_n^{\overline{q}}(K)$. The following four conditions are equivalent.*

(a) *A_q is a self-injective Nakayama K-algebra.*

(b) *A_q is a Nakayama K-algebra.*

(c) *There exist a K-algebra isomorphism $A_q \cong \overline{A}_q$ and a permutation $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ such that the matrix $\sigma * \overline{q} = [\, \overline{q}^{(1)} | \dots | \, \overline{q}^{(n)}]$ has the form*

$$\sigma * \overline{q} = \left[\begin{array}{ccccc|c|c|c|cccccc} 1 & 1 & \dots & 1 & 1 & & & & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & & & & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \dots & \overline{q}^{(r)} & \dots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \ddots & 0 & 0 & & & & 1 & 1 & \ddots & 0 & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 & & & & 1 & 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 & & & & 1 & 1 & \dots & 1 & 1 & 1 \end{array}\right]$$

and, for each $r \in \{1, \dots, n\}$, $\overline{q}^{(r)}$ is the matrix

$$\widehat{\overline{q}}^{(r)} = \left[\begin{array}{ccccccccccc} & & & & & \overset{r}{\downarrow} & & & & & \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \ddots & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ & & & & & 1 & 0 & \dots & 0 & 0 & 0 \\ & & & & & 1 & 1 & \ddots & 0 & 0 & 0 \\ & & 0 & & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ & & & & & 1 & 1 & \dots & 1 & 0 & 0 \\ & & & & & 1 & 1 & \dots & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \\ \\ \leftarrow r \\ \\ \\ \\ \\ \\ \end{array}$$

(d) *There exist K-algebra isomorphisms $A_q \cong \overline{A}_q \cong KQ/I$, where KQ is the path K-algebra of the quiver*

$$\mathcal{Q} : \quad \begin{array}{ccccccc} 1 & \overset{\beta_1}{\text{———}} & 2 & \overset{\beta_2}{\text{—}} & \cdots & \overset{\beta_{n-4}}{\text{—}} & n-3 \\ \beta_n \Big| & & & & & & \Big| \beta_{n-3} \\ n & \overset{\beta_{n-1}}{\text{———}} & n-1 & & \overset{\beta_{n-2}}{\text{———}} & & n-2 \end{array}$$

and $I = (\omega_1, \dots, \omega_n)$ is the two-sided ideal of KQ generated by n zero relations $\omega_1, \dots, \omega_n$, where $\omega_j = \beta_j \beta_{j+1} \dots \beta_n \beta_1 \dots \beta_{j-1}$, for $j = 1, \dots, n$ (see [1], [28, Chapter 14]*).*

If any of the conditions (a)-(d) *holds then $soc(A_q) = J(A_q)^{n-1}$ and $A_q = \mathbb{M}_n^q(K)$ is of finite representation type.*

Proof. The implication (a)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(c) Assume that $A_q = \mathbb{M}_n^q(K)$ is a Nakayama algebra. Since A_q is connected, then the Gabriel quiver $\mathcal{Q}(A_q)$ of A_q is either an oriented cycle or $\mathcal{Q}(A_q)$ is of the form

$$(*) \qquad j_1 \longrightarrow j_2 \longrightarrow \dots \longrightarrow j_n,$$

and has no oriented cycle, see [2] and [1, Chapter 5]. Since, according to Theorem 2.2.5 (c), there is a non-zero A_q-module homomorphism $e_iA_q \to e_jA_q$, for all $i, j \in \{1, \dots, n\}$, then the second form $(*)$ of $\mathcal{Q}(A_q)$ is excluded. Consequently, there is a permutation σ of the set $\{1, \dots, n\}$ such that the Gabriel quiver of the algebra $\mathbb{M}_n^{\sigma * q}(K)$ is the cycle $\mathcal{Q}$ presented in (d). By Corollary 2.2.12, this implies that $(\sigma * q^{(\sigma^{-1}(r))})_{j\,j+1} = 0$, for all $r = 1, \dots, n$ and $j \neq r$.

It follows from [1, Proposition IV.3.8] that $A_q \cong \mathbb{M}_n^{\sigma * q}(K) \cong K\mathcal{Q}/R_{\mathcal{Q}}^s$, for some $s \geq 2$, where $R_{\mathcal{Q}} = (\beta_1, \dots, \beta_n)$ is the two-sided ideal of the path K-algebra $K\mathcal{Q}$ of $\mathcal{Q}$ generated by the arrows $\beta_1, \dots, \beta_n$. Since $\dim_K A_q = n^2$, it follows that $s = n$. Similarly, there is a K-algebra isomorphism $\overline{A}_q \cong \mathbb{M}_n^{\sigma * \overline{q}}(K) \cong K\mathcal{Q}/R_{\mathcal{Q}}^n$. Hence we easily conclude that the matrix $\sigma * \overline{q}$ has the form required in (c).

(c)⇒(d) Assume that $A_q \cong \overline{A}_q$ and σ is a permutation of the set $\{1, \dots, n\}$ such that the matrix $\sigma * \overline{q} = [\,\overline{q}^{(1)} | \dots | \,\overline{q}^{(n)}]$ has the form shown in (c).

By Lemma 2.2.9, there is a K-algebra isomorphism $\mathbb{M}_n^{\overline{q}}(K) \cong \mathbb{M}_n^{\sigma * \overline{q}}(K)$. On the other hand, by Corollary 2.2.12, the Gabriel quiver of the algebra $\mathbb{M}_n^{\sigma * \overline{q}}(K)$ is the quiver $\mathcal{Q}$ shown in (c). Now we define a K-linear map

$$\varphi : \mathbb{M}_n^{\sigma * \overline{q}}(K) \longrightarrow K\mathcal{Q}/I$$

as follows. First we note that, by the form of $\sigma * \overline{q}$, each matrix units e_{ij} of $\mathbb{M}_n^{\sigma * \overline{q}}(K)$ is the composition of some of the matrix units $e_{1\,2}, \dots, e_{n-1\,n}, e_{n\,1}$. Consider the correspondences $e_j \mapsto \eta_j$, $e_{n\,1} \mapsto \beta_n$ and $e_{j\,j+1} \mapsto \beta_j$, for $j = 1, \dots, n-1$, where η_j is the stationary path at j. It is easy to see that the correspondences extend to the K-algebra homomorphism $\varphi : \mathbb{M}_n^{\sigma * \overline{q}}(K) \longrightarrow K\mathcal{Q}/I$. Since $I = (\omega_1, \dots, \omega_n)$, then φ is surjective and $\dim_K K\mathcal{Q}/I = \dim_K \mathbb{M}_n^{\sigma * \overline{q}}(K) = n^2$. It then follows that φ is bijective.

The implication (d)⇒(a) and the final statement of the corollary are well-known facts and can be found in [1, Chapter 5]. This finishes the proof. □

2.2.3 Basic minor degenerations of small dimensions.

In this subsection we study in details basic minor degenerations $A_q = \mathbb{M}_n^q(K)$ of $\mathbb{M}_n(K)$ for $n = 3$, and some examples of such algebras for $n = 4$, and $n = 6$, by means of their bound quiver presentations of the form $A_q \cong K\mathcal{Q}/\Omega$, where $\mathcal{Q}$ is the Gabriel quiver of A_q and Ω is an admissible ideal of the path K-algebra $K\mathcal{Q}$ of $\mathcal{Q}$. We recall that, up to S_3-action, the constant matrices $q = [q^{(1)}|q^{(2)}|q^{(3)}]$ in $\mathbb{ST}_3(K)$ are described in Example 2.2.8.

THEOREM 2.2.19. *Assume that $n = 3$ and let $A_q = \mathbb{M}_3^q(K)$ be a basic minor degeneration of $\mathbb{M}_3(K)$.*

(a) *The K-algebra $A_q = \mathbb{M}_3^q(K)$ is isomorphic to its $(0,1)$-limit $\overline{A}_q = \mathbb{M}_3^{\overline{q}}(K)$.*

(b) *Any basic minor degeneration $A_q = \mathbb{M}_3^q(K)$ of $\mathbb{M}_3(K)$ is isomorphic to one of the five basic minor degeneration K-algebras*

$$A_{q_1} = \mathbb{M}_3^{q_1}(K), \quad A_{q_2} = \mathbb{M}_3^{q_2}(K), \quad A_{q_3} = \mathbb{M}_3^{q_3}(K), \quad A_{q_4} = \mathbb{M}_3^{q_4}(K), \quad A_{q_5} = \mathbb{M}_3^{q_5}(K)$$

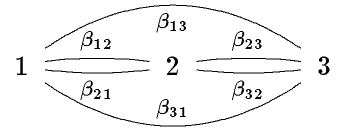
defined by the following structure $(0,1)$*-matrices in* $\mathbb{ST}_3(K)$

$$q_1 = \begin{matrix} 111 & 010 & 001 \\ 100 & 111 & 001 \\ 100 & 010 & 111 \end{matrix}, \quad q_2 = \begin{matrix} 111 & 010 & 001 \\ 101 & 111 & 001 \\ 100 & 010 & 111 \end{matrix}, \quad q_3 = \begin{matrix} 111 & 011 & 001 \\ 100 & 111 & 001 \\ 110 & 010 & 111 \end{matrix},$$

$$q_4 = \begin{matrix} 111 & 010 & 001 \\ 101 & 111 & 001 \\ 110 & 010 & 111 \end{matrix}, \quad q_5 = \begin{matrix} 111 & 010 & 011 \\ 101 & 111 & 001 \\ 100 & 110 & 111 \end{matrix}.$$

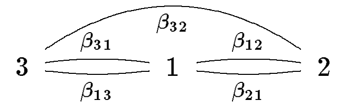
(c) *The algebras* A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} *and* A_{q_5} *are pairwise non-isomorphic, self-dual, and special biserial. The algebra* A_{q_5} *is self-injective, but the algebras* A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} *are not. The algebra* A_{q_1} *is tame of infinite representation type, and the algebras* A_{q_2}, A_{q_3}, A_{q_4}, A_{q_5} *are of finite representation type, see* [34], *compare with* [29]. *There exist* K*-algebra isomorphisms*

(c1) $A_{q_1} \cong K\mathcal{Q}^{(1)}/\Omega^{(1)}$, *where* $\mathcal{Q}^{(1)}$:



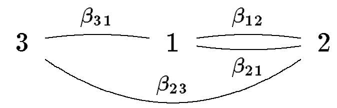
and the ideal $\Omega^{(1)}$ *of the path algebra* $K\mathcal{Q}^{(1)}$ *is generated by all zero relations* $\beta\gamma$, *with* $\beta, \gamma \in \mathcal{Q}_1^{(1)}$.

(c2) $A_{q_2} \cong K\mathcal{Q}^{(2)}/\Omega^{(2)}$, *where* $\mathcal{Q}^{(2)}$:



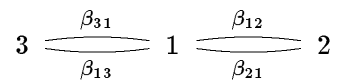
and the ideal $\Omega^{(2)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{31}\beta_{12}$, $\beta_{32}\beta_{21}$, $\beta_{13}\beta_{32}$.

(c3) $A_{q_3} \cong K\mathcal{Q}^{(3)}/\Omega^{(3)}$, *where* $\mathcal{Q}^{(3)}$:



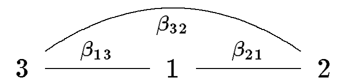
and the ideal $\Omega^{(3)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{23}\beta_{31}$, $\beta_{31}\beta_{12}$.

(c4) $A_{q_4} \cong K\mathcal{Q}^{(4)}/\Omega^{(4)}$, *where* $\mathcal{Q}^{(4)}$:



and the ideal $\Omega^{(4)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$.

(c5) $A_{q_5} \cong K\mathcal{Q}^{(5)}/\Omega^{(5)}$, *where* $\mathcal{Q}^{(5)}$:



and the ideal $\Omega^{(5)}$ *is generated by the zero relations* $\beta_{21}\beta_{13}\beta_{32}$, $\beta_{13}\beta_{32}\beta_{21}$, $\beta_{32}\beta_{21}\beta_{13}$.

Proof. (a) Let $\overline{A}_q$ be the $(0,1)$-limit of A_q. We define a K-linear map $\varphi : \overline{A_q} \to A_q$ by setting

$$\varphi(e_{ij}) = \begin{cases} q_{ij}^{(k)} e_{ij}, & \text{if } q_{ij}^{(k)} \neq 0, \text{ for } k \neq i, j, \\ e_{ij}, & \text{otherwise,} \end{cases}$$

for distinct $i, j \in \{1, 2, 3\}$, and we set $\varphi(e_{ii}) = e_{ii}$ for $i = 1, 2, 3$. Denote by $\cdot'$ and $\cdot''$ the multiplication in A_q and in $\overline{A}_q$, respectively.

To show that $\varphi : \overline{A}_q \to A_q$ is a K-algebra isomorphism, it is sufficient to prove that $\varphi(e_{ir} \cdot'' e_{rj}) = \varphi(e_{ik}) \cdot' \varphi(e_{rj})$, for all i, r, j.

First, we consider the case when $i, r, j \in \{1, 2, 3\}$ are pairwise different and $q_{ij}^{(r)} \neq 0$. It follows from Lemma 2.2.2 that $q_{ir}^{(j)} = q_{rj}^{(i)} = 0$, so that $\varphi(e_{ir}) = e_{ir}$ and $\varphi(e_{rj}) = e_{rj}$. Hence

$$\varphi(e_{ir} \cdot'' e_{rj}) = \varphi(e_{ij}) = q_{ij}^{(r)} e_{ij} = e_{ir} \cdot' e_{rj} = \varphi(e_{ir}) \cdot' \varphi(e_{rj}),$$

and we are done. The proof in remaining cases is analogous and it is left to the reader.

(b) In view of (a), Theorem 2.2.10 and Corollary 2.2.11, it is sufficient to classify the S_3-orbits of all basic structure $(0,1)$-matrices in $\mathbb{ST}_3(K)$ with respect to the action of the symmetric group S_3 defined in Definition 2.2.7.

Note that, by Lemma 2.2.2 (c), the product of any successive pair of $q_{23}^{(1)}$, $q_{21}^{(3)}$, $q_{31}^{(2)}$, $q_{32}^{(1)}$, $q_{12}^{(3)}$, $q_{13}^{(2)}$, $q_{23}^{(1)}$ is zero. Hence we conclude that here are precisely five S_3-orbits of basic $(0,1)$-matrices in $\mathbb{ST}_3(K)$ and they are represented by the five constant matrices q_1, q_2, q_3, q_4, q_5 listed in (b). The remaining statement in (b) easily follows from the quiver descrition of the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} given in (c). On the other hand, this also follows from Theorem 2.2.26 proved in the next subsection.

(c) Since the constant matrices q_1, q_2, q_3, q_4, q_5 belongs to different S_3-orbits then, according to Corollary 2.2.11, the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} are pairwise non-isomorphic.

Note also that, in the notation of Definition 2.2.7, we have $q_1^{tr} = q_1$, ${}_{(2,3)} * q_2^{tr} = q_2$, ${}_{(1,3)} * q_3^{tr} = q_3$, $q_4^{tr} = q_4$ and ${}_{(1,3)} * q_5^{tr} = q_5$. It follows from Lemma 2.2.9 (a) that $A_s^{op} \cong A_s$, for $s = 1, \ldots, 5$, that is, the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} are self-dual.

By Corollary 2.2.12, the Gabriel quivers of the algebras $A_1, \ldots, A_5$ are just the quivers listed in (c1)–(c5). It is easy to check that, for each $s \in \{1, 2, 3, 4, 5\}$, the correspondences $\varepsilon_j \mapsto e_j$ and $\beta_{ij} \mapsto e_{ij}$ define a K-algebra surjection $K\mathcal{Q}^{(s)}/\Omega^{(s)} \longrightarrow A_{q_s}$, where ε_j is the primitive idempotent of the path algebra $K\mathcal{Q}^{(s)}$ defined by the stationary path at the vertex j, for every $j \in \mathcal{Q}_0^{(s)}$. Since $\dim_K K\mathcal{Q}^{(s)}/\Omega^{(s)} = \dim_K A_{q_s} = 9$, the surjection is an isomorphism of K-algebras.

It follows from the shape of $\mathcal{Q}^{(s)}$ and $\Omega^{(s)}$ that $K\mathcal{Q}^{(s)}/\Omega^{(s)} \cong A_{q_s}$ is a special biserial algebra, that is,

(a) any vertex of $\mathcal{Q}^{(s)}$ is a starting point of at most two arrows and is an end point of at most two arrows.

(b) given an arrow $\beta : i \to j$ in $\mathcal{Q}^{(s)}$ there is at most one arrow $\alpha : s \to i$ and at most one arrow $\gamma : j \to r$ in $\mathcal{Q}^{(s)}$ such that $\alpha\beta \notin \Omega^{(s)}$ and $\beta\gamma \notin \Omega^{(s)}$, see [34].

We recall that any special biserial algebra is representation-tame, see [3, 5.2]. Note that for $s = 1$, there is a cyclic walk

$$1 \xrightarrow{\beta_{13}} 3 \xleftarrow{\beta_{23}} 2 \xrightarrow{\beta_{21}} 1 \xleftarrow{\beta_{31}} 3 \xrightarrow{\beta_{32}} 2 \xleftarrow{\beta_{12}} 1$$

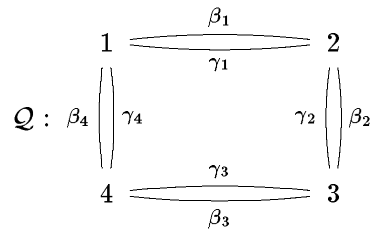
of the quiver $\mathcal{Q}^{(1)}$ and according to the finite representation type criterion in [34], the algebra A_{q_1} is of infinite representation type. Similarly, by looking at the walks of each of the quivers $\mathcal{Q}^{(2)}$, $\mathcal{Q}^{(3)}$, $\mathcal{Q}^{(4)}$, $\mathcal{Q}^{(5)}$; and by applying the finite representation type criterion in [34], we conclude that the algebra A_{q_s} is of finite representation type, for $s = 2, 3, 4, 5$. This finishes the proof. $\square$

It follows from Theorem 2.2.19, that for $n = 3$, each basic minor degeneration $A_q = \mathbb{M}_3^q(K)$ of $\mathbb{M}_3(K)$ is special biserial and A_q is isomorphic to its $(0,1)$-limit algebra $A_{\overline{q}}$. We show below and in Subsection 2.2.4 that this facts do not hold, for each $n \geq 4$.

EXAMPLE 2.2.20. Assume that $n = 4$ and $A_q = \mathbb{M}_4^q(K)$ is a basic minor degeneration of $\mathbb{M}_4(K)$ given by the following structure matrix

$$q = \begin{bmatrix} 1111 & 0110 & 0010 & 0011 \\ 1001 & 1111 & 0011 & 0001 \\ 1000 & 1100 & 1111 & 1001 \\ 1100 & 0100 & 0110 & 1111 \end{bmatrix} \in \mathbb{ST}_4(K)$$

One can show that A_q is isomorphic to the bound quiver K-algebra $K\mathcal{Q}/\Omega$ (see [1]), where $\mathcal{Q}$ is the quiver



and Ω is the two-sided ideal of the path K-algebra $K\mathcal{Q}$ of $\mathcal{Q}$ generated by the following relations:

- $\beta_j\gamma_j$ and $\gamma_j\beta_j$, for $j = 1, 2, 3, 4$,
- $\delta_1\delta_2\delta_3$, if the arrows δ_1, δ_2, δ_3 form a path of length 3,
- $\beta_1\beta_2 - \gamma_4\gamma_3$, $\beta_2\beta_3 - \gamma_1\gamma_4$,
- $\gamma_2\gamma_1 - \beta_3\beta_4$, $\gamma_3\gamma_2 - \beta_4\beta_1$.

It follows that $A_q \cong K\mathcal{Q}/\Omega$ is a special biserial algebra and hence it is representation-tame, see [3, 5.2]. Note that there is a cyclic walk

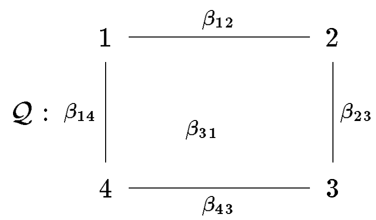
$$1 \xrightarrow{\delta_1} 4 \xleftarrow{\beta_3} 3 \xrightarrow{\gamma_2} 2 \xleftarrow{\beta_1} 1$$

of the quiver $\mathcal{Q}$ and, according to the finite representation type criterion in [34], the algebra A_q is of infinite representation type. Since $_{(2,3)} * q^{tr} = q$ then, by Lemma 2.2.9, $A_q^{op} \cong A_q$ and therefore the algebra A_q is self-dual. Note also that $J(A_q)^3 = 0$ and $\operatorname{soc} A_q = J(A_q)^2 = Ke_{13} + Ke_{31} + Ke_{24} + Ke_{42}$.

EXAMPLE 2.2.21. Assume that $n = 4$ and $B_q = \mathbb{M}_4^q(K)$ is a basic minor degeneration of $\mathbb{M}_4(K)$ given by the following structure matrix

$$q = \begin{bmatrix} 1111 & 0110 & 0010 & 0011 \\ 1001 & 1111 & 1011 & 0001 \\ 1101 & 0100 & 1111 & 0001 \\ 1100 & 0100 & 1110 & 1111 \end{bmatrix} \in \mathbb{ST}_4(K),$$

see [7, (2.4)]. One can show that B_q is isomorphic to the bound quiver K-algebra $K\mathcal{Q}/\Omega$ (see [1]), where $\mathcal{Q}$ is the quiver



and Ω is the two-sided ideal of the path K-algebra $K\mathcal{Q}$ of $\mathcal{Q}$ generated by the following relations:

- $\beta_{12}\beta_{23} = \beta_{14}\beta_{43}$,
- $\beta_{12}\beta_{23}\beta_{31},\ \beta_{31}\beta_{12}\beta_{23}, \beta_{23}\beta_{31}\beta_{12}, \beta_{43}\beta_{31}\beta_{14}, \beta_{14}\beta_{43}\beta_{31}$.

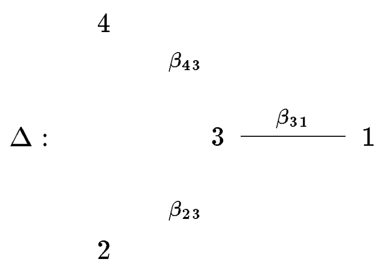
It follows that $J(B_q)^4 = 0$ and $J(B_q)^3 = Ke_{24} \oplus Ke_{42} = K\overline{\beta_{23}\beta_{31}\beta_{14}} \oplus K\overline{\beta_{43}\beta_{31}\beta_{12}}$. Since $\overline{\beta_{31}\beta_{12}} \neq 0$ and $\overline{\beta_{31}\beta_{14}} \neq 0$, then the algebra B_q is not special biserial. Note also that B_q is a self-dual algebra, because $(1,3) * q^{tr} = q$ and Lemma 2.2.9 yields $B_q^{op} \cong B_q$.

The algebra B_q is not self-injective and the injective dimension inj.dimB_q of B_q equals one. Indeed, there are isomorphisms $e_1B_q \simeq D(B_qe_3)$, $e_2B_q \simeq D(B_qe_4)$, $e_4B_q \simeq D(B_qe_2)$ and that there is non-split exact sequence $0 \to e_3B_q \to e_2B_q \oplus e_4B_q \to D(B_qe_1) \to 0$, where $D(-) = \mathrm{Hom}_K(-, K)$. Hence we get inj.dim$B_q = 1$. Note also that the algebra B_q is isomorphic to the quotient algebra $\Lambda/\pi\Lambda$ of the tiled R-order

$$\Lambda = \begin{bmatrix} R & R & R & R \\ \boldsymbol{\pi} & R & R & \boldsymbol{\pi} \\ \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} \\ \boldsymbol{\pi} & \boldsymbol{\pi} & R & R \end{bmatrix},$$

where $R = K[[t]]$ the power series K-algebra and $\boldsymbol{\pi} = t \cdot K[[t]]$. We can easily compute that gl.dim$\Lambda = 2$. Hence we get inj.dimB_q = inj.dim$\Lambda - 1 = 1$, see [24, Theorem 2.10]. Finally, we show that B_q is representation-finite.

To prove this, we denote by $R = K\Delta$ the path algebra of the Dynkin subquiver



of type $\mathbb{D}_4$ of $\mathcal{Q}$. Denote by $\sigma : R \to R$ the K-algebra automorphism of R given by the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ of the vertices of Δ. Let ${}_\sigma D(R)_R$ be the vector space $D(R) = \mathrm{Hom}_K(R, K)$ is viewed as an R-R-bimodule, with the left R-module structure induced by th automorphism

$\sigma : R \to R$. It follows [31, Corollary 4 and Remark 2] that the trivial extension $C = R \ltimes_\sigma D(R)$ is non-symmetric selfinjective K-algebra of finite representation type. One can show that $\dim_K C = 18$ and, by applying [31, Theorem 2 and Proposition 1], the number of pairwise non-isomorphic indecomposable C-modules equals 24. The Gabriel quiver of C coincides with the quiver $\mathcal{Q}$ of the algebra B_q of dimension 16 and there is K-algebra surjection $\varepsilon : C \to B_q$, with $\mathrm{Ker}\varepsilon = \mathrm{soc}I(1) \oplus \mathrm{soc}I(3)$, where $I(1) = D(B_q e_1)$ and $I(3) = D(B_q e_3)$ are the indecomposable injective C-modules at the vertices q and 3 of $\mathcal{Q}$. It follows that the algebra B_q is representation-finite. One can show, as in [31, Example 2 and 3], that the Auslander-Reiten quiver $\Gamma(\mathrm{mod}B_q)$ of B_q has a shape of a Möbius band consisting of 22 indecomposable modules, see also [31, Remark 2].

EXAMPLE 2.2.22. Assume that $n = 6$ and consider the one-parameter family of basic minor degeneration K-algebras $A_{q_\mu} = \mathbb{M}_6^{q_\mu}(K)$, where $\mu \in K$ and

$$q_\mu = \begin{bmatrix} 111111 & 010000 & 011000 & 010100 & 011110 & 011101 \\ 100000 & 111111 & 001000 & 000100 & 001110 & 001101 \\ 100111 & 010111 & 111111 & 000100 & 000110 & 000101 \\ 101011 & 011011 & 001000 & 111111 & 001010 & 001001 \\ 100000 & 010000 & \mu 11000 & 110100 & 111111 & 000001 \\ 100010 & 010010 & 111010 & 110110 & 000010 & 111111 \end{bmatrix}.$$

Note that, If K is infinite, then the family $\{A_\mu\}_{\mu \in K \setminus \{0,1\}}$ is infinite, because $A_\mu \cong A_\gamma$ if and only $\mu = \gamma$, for $\mu, \gamma \in K \setminus \{0, 1\}$ (apply Theorem 2.2.10). One can show that each of the algebra A_μ is representation-wild and not self-injective (the right ideals $e_2 A_\mu$ and $e_5 A_\mu$ are not injective, by [7, Proposition 2.3] and [9, Lemma 2.3]).

We show in Subsection 2.2.4 that the set of the isomorphism classes of self-injective algebras $A_q = \mathbb{M}_n^q(K)$ is infinite, for each $n \geq 4$.

2.2.4 Frobenius basic minor degenerations of $\mathbb{M}_n(K)$

In this subsection we study basic minor q-degenerations of $\mathbb{M}_n(K)$ that are Frobenius K-algebras, where K is a field. We start by a description of the socle $\mathrm{soc}\, A_A$ of such an algebra $A = \mathbb{M}_n^q(K)$. In particular we show that $A = \mathbb{M}_n^q(K)$ is a Frobenius K-algebra if and only if its $(0,1)$-limit algebra $\overline{A} = \mathbb{M}_n^{\overline{q}}(K)$ is a Frobenius K-algebra.

PROPOSITION 2.2.23. *Assume that $n \geq 2$, q is a basic structure matrix (2.1) in $\mathbb{ST}_n(K)$ and $\overline{q}$ is the $(0,1)$-limit of q. Let $A = \mathbb{M}_n^q(K)$ and $\overline{A} = \mathbb{M}_n^{\overline{q}}(K)$ be the corresponding basic minor degenerations of $\mathbb{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and $\overline{A}$.*

(a) *Given $j \in \{1, \ldots, \}$, a right ideal S of A is simple if and only if S has the form $S = e_{js}K$, where e_{js} is a matrix unit such that $s \neq j$ and $q_{jr}^{(s)} = 0$, for all $r \neq s$.*

(b) *Given $j \in \{1, \ldots, \}$, $soc(e_j A) = \sum_{s \in U_j} e_{js}K$, where*

$$U_j = \{s;\ q_{jr}^{(s)} = 0,\ \textit{for all } r \neq s\} = \{s;\ s \neq j \textit{ and } e_{is} \cdot_q J(A) = 0, \} \subset \{1, \ldots, n\}.$$

(c) *If S and S' are two different simple submodules of e_jA, then $S \not\cong S'$.*

(d) *The socle $soc(A_A)$ of the right A-module A is a two-sided ideal of A of the form*

$$soc(A_A) = \{x \in J(A);\ \ x \cdot_q J(A) = 0\} = \sum_{s \in U_j} e_{js}K,$$

that is, the sum runs through all pairs $(j,s) \in \{1,\dots,n\} \times U_j$ such that $j \neq s$.

(e) *$soc(A_A) = soc(\overline{A}_{\overline{A}})$ and $soc(e_jA) = soc(e_j\overline{A})$, for all $j \in \{1,\dots,n\}$*

Proof. Since q is a basic matrix then, according to Theorem 2.2.5 (d), the algebra $A = \mathbb{M}_n^q(K)$ is basic and the projective right ideals $e_1A,\dots,e_nA$ of A are pairwise non-isomorphic.

(a) Assume that $S \subset e_jA$ is a simple right ideal of A. Then $S \neq 0$ and S contains a non-zero elelment $s = e_j \cdot_q \sum_{i,r} e_{ir}\lambda_{ir} = \sum_{r=1}^{n} e_{jr}\lambda_{jr}$, where $\lambda_{jr} \in K$ and some λ_{js} is non-zero. It follows that $s \cdot_q e_s = e_{js}\lambda_{js}$ belongs to S, and therefore $S = e_{is}A$. The module S is simple if and only if $S \cdot_q J(A) = 0$, or equivalently, if and only if $e_{js} \cdot_q e_{sr} = q_{jr}^{(s)} e_{jr} = 0$, for all $r \neq s$, because $J(A) = \sum_{s \neq r} e_{sr}K$, by Theorem 2.2.5. Hence $S = e_{js}K \simeq e_sA/w_sJ(A)$ and (a) follows.

The statement (b) is a consequence of (a).

(c) Assume that $S = e_{js}K$ and $S' = e_{js'}K$ are two different simple submodules of e_jA and assume, to the contrary, that there is an R-module isomorphism $\varphi : S \to S'$. It follows that $0 \neq \varphi(e_{js}) = \varphi(e_{js} \cdot_q e_s) = \varphi(e_{js}) \cdot_q e_s = \lambda e_{js'} \cdot_q e_s$, for some $\lambda \in K \setminus \{0\}$. Hence in a view of (2.5) we get $s = s'$ and $S = S'$, a contradiction.

(d) Since $\mathrm{soc}(A_A) = \mathrm{soc}(e_1A) \oplus \cdots \oplus \mathrm{soc}(e_nA)$ then (b) yields $\mathrm{soc}(A_A) = \sum_{s \in U_j} e_{js}K$, that is, $\mathrm{soc}(A_A)$ is spanned by all matrix units $e_{js} \in J(A)$ such that $j \neq s$ and $e_{js} \cdot_q J(A) = 0$. Hence (d) follows.

(e) By Theorem 2.2.5, $J(A) = J(\overline{A})$. Then (e) immediately follows from (b) and (d); and the proof is complete. □

REMARK 2.2.24. Assume that $A = \mathbb{M}_n^q(K)$ is basic. Let $m \geq 1$ be such that $J(A)^m = 0$ and $J(A)^{m-1} \neq 0$. It is clear that $J(A)^{m-1} \subseteq \mathrm{soc}(A_A)$, however the equality does not hold in general. For this consider the algebra $A = A_{q_4} = \mathbb{M}_3^{q_4}(K)$ of Theorem 2.2.19 (c4). In this case $m = 3$, $J(A)^2 = e_{32}K + e_{23}K$, $\mathrm{soc}(A_A) = J(A)^2 + e_{13}K + e_{12}K \neq J(A)^2$. Note also that $\mathrm{soc}({}_AA) = J(A)^2 + e_{31}K + e_{21}K \neq J(A)^2$ and hence $\mathrm{soc}({}_AA) \neq \mathrm{soc}(A_A)$.

We recall that a basic finite dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$, is a Frobenius algebra if and only if each projective module e_jA has a simple socle and $\mathrm{soc}(e_iA) \not\cong \mathrm{soc}(e_jA)$, for all $i \neq j$. In this case, there is a permutation σ of the set $\{1,\dots,n\}$, called the Nakayama permutation, such that $\mathrm{soc}(e_jA) \cong \mathrm{top}(e_{\sigma(j)}A)$, see [5]. If A is a Frobenius algebra then (see [37, Theorem 2.4.3] and [35])

$$\mathrm{soc}({}_AA) = \mathrm{soc}(A_A) := \mathrm{soc}(A).$$

Now, following Fujita [7, Lemma 4.2], we give neccessary and sufficient conditions for a basic minor structure matrix q in $\mathbb{ST}_n(K)$ to be the K-algebra $\mathbb{M}_n^q(K)$ Frobenius. In paritcular, we remove the assumption on $(0,1)$-matrices made in [7, Lemma 4.2].

THEOREM 2.2.25. *Assume that $n \geq 2$, q is a basic structure matrix* (2.1) *in $\mathbb{ST}_n(K)$ and $\overline{q}$ is the $(0,1)$-limit of q. Let $A = \mathbb{M}_n^q(K)$ and $\overline{A} = \mathbb{M}_n^{\overline{q}}(K)$ be the corresponding basic minor degenerations of $\mathbb{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and $\overline{A}$. The following seven conditions are equivalent.*

(a) *A is a Frobenius K-algebra.*

(a′) *$\overline{A}$ is a Frobenius K-algebra.*

(b) *For each $j \in \{1, \ldots, n\}$, $\dim_K soc(e_j A) = 1$, and the right simple ideals $soc(e_1 A), \ldots, soc(e_n A)$ of A are pairwise non-isomorphic.*

(c) *$\dim_K soc(A_A) = n$, and the right ideals $e_1(soc\, A_A), \ldots, e_n(soc\, A_A)$ of A are pairwise non-isomorphic.*

(d) *The block matrix $q \in \mathbb{ST}_n(K)$ satisfies the following two conditions:*

(d1) *For every $j \in \{1, \ldots, n\}$ there exists a unique $s \neq j$ such that $q_{jr}^{(s)} = 0$, for all $r \neq s$.*

(d2) *Given $i, j, s \in \{1, \ldots, n\}$ such that $i \neq j$ and $s \notin \{i, j\}$, there exists an $r \in \{1, \ldots, n\}$ such that $r \neq s$ and $q_{ir}^{(s)} \neq 0$ or $q_{jr}^{(s)} \neq 0$.*

(e) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the block matrix $q \in \mathbb{ST}_n(K)$ satisfies the following condition :*

(e1) *Given $s, j \in \{1, \ldots, n\}$, the equality $q_{jr}^{(s)} = 0$ holds for all $r \neq s$ if and only if $s = \sigma(j)$.*

(f) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the block matrix $q \in \mathbb{ST}_n(K)$ satisfies the following condition :*

(f1) $q_{j\sigma(j)}^{(s)} \neq 0$, *for any $j, r \in \{1, \ldots, n\}$.*

In this case σ is the Nakayama permutation of A and $soc(e_j A) = K e_{j\sigma(j)}$.

If A is a Frobenius algebra and $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is an in (f) *then:*

(i) *the Frobenius structure of $A = \mathbb{M}_n^q(K)$ is given by the K-linear map $\psi : A_q \to K$ defined by the formula*

$$\psi(e_{js}) = \begin{cases} 1 & \text{if } q_{jr}^{(s)} = 0, \text{ for all } r \neq s, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *any indecomposable module M in mod A is projective, or $M \cdot_q soc(A) = 0$, that is, M is a module over the quotient algebra $A/soc(A)$.*

Proof. Since q is a basic matrix then the algebra $A = \mathbb{M}_n^q(K)$ is basic, by Theorem 2.2.5 (c). Hence the projective right ideals $e_1 A, \ldots, e_n A$ of A are pairwise non-isomorphic,

It follows from [5] that $A = \mathbb{M}_n^q(K)$ is a Frobenius algebra if and only if each projective module $e_j A$ has a simple socle and $\mathrm{soc}(e_i A) \not\cong \mathrm{soc}(e_j A)$, for $i \neq j$. Since simple A-modules are one-dimensional and $e_j(\mathrm{soc}\, A) = \mathrm{soc}(e_j A)$, then the conditions (a), (b) and (c) are equivalent.

Now we prove that the conditions (b) and (d) are equivalent. We recall from Proposition 2.2.23, that the module $S_j = \mathrm{soc}(e_j A)$ is simple if and only if there exists a unique s such that $s \neq j$, $S_j = e_{js}K$, and $e_{js} \cdot_q J(A) = 0$ and $S_j \simeq e_s/e_s J(A)$. Since $J(A) = \sum\limits_{s \neq r} e_{sr}K$, then the equality $e_{js} \cdot_q J(A) = 0$ holds if and only if $q_{jr}^{(s)} = 0$, for all $r \neq s$, that is, if (d1) holds.

Assume that (d1) holds and $S_j = e_{js}K$, $S_i = e_{iu}K$ are two simple right submodules of A, where $s \neq j$ and $u \neq i$. Then $e_{js} \cdot_q e_{sr} = 0$ and $e_{iu} \cdot_q e_{ur'} = 0$, for all $r \neq s$ and $r' \neq u$, or equivalently, $q_{jr}^{(s)} = 0$ and $q_{jr'}^{(u)} = 0$, for all $r \neq s$ and $r' \neq u$. Hence we easily conclude that the right simple ideals $\mathrm{soc}(e_1 A), \dots, \mathrm{soc}(e_n A)$ of A are pairwise non-isomorphic if and only if the condition (d2) holds.

Since, obviously, the conditions (d) and (e) are equivalent then the conditions (a), (b), (c), (d), and (e) are equivalent. Note that σ is the Nakayama permutation of A.

The conditions (a) and (a$'$) are equivalent, because (d) holds for q if and only if (d) holds for $\overline{q}$.

Now we prove the implication (f)$\Rightarrow$(e) by showing that the condition (f1) implies (e1). To see it, we note that, if the condition (f1) holds and $s, j \in \{1, \dots, n\}$, are such that the equality $q_{jr}^{(s)} = 0$ holds for all $r \neq s$ then $s = \sigma(j)$. Conversely, if $s = \sigma(j)$ then Lemma 2.2.2 (c) yields $q_{j\sigma(j)}^{(r)} q_{jr}^{(\sigma(j))} = 0$, for all $r \neq s = \sigma(j)$. Hence by (f1), we have $q_{jr}^{(\sigma(j))} = 0$, for all $r \neq s$ and $j \in \{1, \dots, n\}$.

It remains to prove the implication (e)$\Rightarrow$(f) holds. Assume that $A = \mathbb{M}_n^q(K)$ is a Frobenius algebra with Nakayama permutation σ. It follows that, for each $j \in \{1, \dots, n\}$, there is an isomorphism $e_j A \simeq D(Ae_{\sigma(j)})$. Since the representation matrix (see [7]) of the right ideal of $e_j A$ with respect to K-basis $\{e_{j1}, \dots, e_{jn}\}$ of $e_j A$ is the matrix $(q_{js}^{(r)})_{j,s}$ then, according to [9, Lemma 2.3 (ii)], we have $q_{j\sigma(j)}^{(r)} \neq 0$, for all $r \in \{1, \dots, n\}$, and (f) follows.

To finish the proof, assume that $A = \mathbb{M}_n^q(K)$ is a Frobenius algebra and let $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ be as in (f). For the proof of the statement (i), it is enough to show that $\mathrm{Ker}\,\psi$ does not contain a non-zero right ideal of A. Assume, to the contrary, that $\mathrm{Ker}\psi$ contains a non-zero right ideal aA, where $a = \sum_{i,j=1}^n a_{ij} e_{ij}$ and $a_{ij} \in K$. Since a is non-zero then $a_{rs} \neq 0$, for some $r, s \in \{1, \dots, n\}$. It follows that

$$\psi(a \cdot_q e_{s\sigma(r)}) = \psi(\sum_{i=1}^n a_{is} e_{is} \cdot_q e_{s\sigma(r)}) = \sum_{i=1}^n a_{is} q_{i\sigma(r)}^{(s)} \psi(e_{i\sigma(i)}) = a_{rs} q_{r\sigma(r)}^{(s)} \neq 0,$$

and we get a contradiction $a \cdot_q e_{s\sigma(r)} \in aA \subset \mathrm{Ker}\psi$. By Proposition 2.2.23, the socle $\mathrm{soc}(A)$ is spanned by all e_{ij} such that $i \neq j$ and $e_{is} \cdot_q J(R) = 0$, that is, $e_{is} \cdot_q e_{sr} = q^{(s)} e_{ir} = 0$, for all $r \neq s$. If I is a non-zero right ideal of A then I contains some e_{is} with the above property, and therefore $\psi(e_{is}) = 1$, that is, $\mathrm{Ker}\,\psi$ does not contain I.

Now we prove (ii) by applying the arguments given in [15]. Assume that M is an indecomposable module in mod A such that $M \cdot_q \mathrm{soc}(A) \neq 0$. Let S be a simple submodule of $M \cdot_q \mathrm{soc}(A)$ and let $P = E(S)$ be the injective envelope of S. Since A is Frobenius then P is indecomposable projective. By the injectivity of P, there is $f \in \mathrm{Hom}_A(M, P)$ such that the restriction of f to S is the embedding $S \hookrightarrow P$. We recall that P has a unique maximal submodule $\mathrm{rad}\, P = P \cdot_q J(A)$. Note that $\mathrm{Im}\, f$ is not contained in $\mathrm{rad}\, P$, because the inclusions $S \subseteq P$, $S \subseteq M \cdot_q \mathrm{soc}(A)$ and

$\operatorname{Im} f \subseteq \operatorname{rad} P$ imply $0 \neq f(S) \subseteq f(M \cdot_q \operatorname{soc}(A)) = f(M) \cdot_q \operatorname{soc}(A) \subseteq P \cdot_q J(A) \cdot_q \operatorname{soc}(A) = 0$; and we get a contradiction. It follows that $\operatorname{Im} f + \operatorname{rad} P = P$, and the Nakayama lemma yields $\operatorname{Im} f = P$. By the projectivity of P, the homomorphism f is bijective, because M is indecomposable. Consequently, the module M is projective. This finishes the proof of the theorem. □

Now we give simple description of all basic minor structure matrices q in $\mathbb{ST}_n(K)$ such that the K-algebra $A_q = \mathbb{M}_n^q(K)$ Frobenius and $J(A_q)^3 = 0$. To formulate it we associate to a given a permutation σ of the set $\{1, \ldots, n\}$, where $n \geq 3$, the block matrix

$$q(\sigma) = [q(\sigma)^{(1)}| \ldots |q(\sigma)^{(n)}] \tag{2.12}$$

defined in [7, Lemma 4.4] by the formulae

$$q(\sigma)_{ij}^{(r)} = \begin{cases} 1 & \text{if } r \in \{i, j\}, \quad \text{or } \ j = \sigma(i), \\ 0 & \text{otherwise.} \end{cases}$$

for all $i, j, r \in \{1, \ldots, n\}$. It is easy to check that the block matrix $q(\sigma)$ is a basic structure $(0,1)$-matrix in $\mathbb{ST}_n(K)$, see [7, Theorem 4.4] and [8, Corollary 1.8].

THEOREM 2.2.26. *Assume that $n \geq 2$, q is a basic structure matrix (2.1) in $\mathbb{ST}_n(K)$ and $\overline{q}$ is the $(0,1)$-limit of q. Let $A = \mathbb{M}_n^q(K)$ and $\overline{A} = \mathbb{M}_n^{\overline{q}}(K)$ be the corresponding basic minor degenerations of $\mathbb{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and of $\overline{A}$. The following conditions are equivalent.*

(a) *A is a Frobenius K-algebra and $J(A)^3 = 0$.*

(a′) *$\overline{A}$ is a Frobenius K-algebra and $J(\overline{A})^3 = 0$.*

(b) *Either $n = 2$ and $A = \mathbb{M}_2^q(K)$ is the Nakayama algebra $A(0)$ of Example 2.2.4, or $n \geq 3$ and A is a Frobenius K-algebra such that $J(A)^2 = soc(A)$.*

(c) *Either $n = 2$ and $q = q(0) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$, or $n \geq 3$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the block matrix $q \in \mathbb{ST}_n(K)$ satisfies the following condition:*

$$q_{ij}^{(r)} \neq 0 \quad \text{if and only if} \quad r \in \{i, j\} \ \text{or} \ j = \sigma(i).$$

(d) *Either $n = 2$ and $q = q(0) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$, or $n \geq 3$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$ and the $(0,1)$-limit $\overline{q} \in \mathbb{ST}_n(K)$ of the block matrix q has the form $\overline{q} = q(\sigma)$ (2.12).*

In this case σ is the Nakayama permutation of A and of $\overline{A}$. Moreover, $A/J(A)^2 \cong \overline{A}/J(\overline{A})^2$.

Proof. Since q is a basic matrix and $n \geq 2$ then the algebra $A = \mathbb{M}_n^q(K)$ is basic, non-semisimple, and the projective right ideals $e_1 A, \ldots, e_n A$ of A are pairwise non-isomorphic by Theorem 2.2.5 (d).

(a)⇒(b) Assume that $J(A)^3 = 0$ and that the algebra $A = \mathbb{M}_n^q(K)$ is Frobenius. It follows from Proposition 2.2.23 that, for each $j \in \{1, \ldots, n\}$, the simple module $S_j = \operatorname{soc}(e_j A)$ has the form $S_j = e_{j\sigma(j)} K$, where $\sigma \in S_n$ is the Nakayama permutation of A. Note that if $e_{j\sigma(j)} \in J(A) \setminus J(A)^2$

then, by the description of the simple ideals given in Proposition 2.2.23, $e_jA = e_jK + e_{j\sigma(j)}K$ is of dimension two. It follows that $n = \dim_K e_jA = 2$. Consequently, if $J(A)^2 = 0$ then $n = 2$ and $A = A(0)$ is the Nakayama algebra of Example 2.2.4. Moreover, if $n \geq 3$ then $J(A)^2 \neq 0$ and $e_{j\sigma(j)} \in J(A)^2$, for every j. It follows that $\mathrm{soc}(A) = \mathrm{soc}(e_1A) \oplus \ldots \oplus \mathrm{soc}(e_nA) \subseteq J(A)^2$. Since $J(A)^3 = 0$, then $\mathrm{soc}(A) \supseteq J(A)^2$ and we get the equality $\mathrm{soc}(A) = J(A)^2$.

(b)⇒(a) If $n = 2$ and $A = A(0)$ is the Nakayama algebra of Example 2.2.4, then A is a non-semisimple Frobenius algebra such that $J(A)^2 = 0$. If $n \geq 3$ and $J(A)^2 = \mathrm{soc}(A)$ then $J(A)^3 = J(A)\mathrm{soc}(A) = 0$, and (a) follows.

(b)⇒(c) In case $n = 2$, the matrix q has the form $q = q(0) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$, see Example 2.2.4.

Assume that $n \geq 3$, $J(A)^2 = \mathrm{soc}(A)$ and that the algebra $A = \mathbb{M}_n^q(K)$ is Frobenius. Take for $\sigma \in S_n$ the Nakayama permutation of A. It follows from Proposition 2.2.23 that, for each $j \in \{1, \ldots, n\}$, the simple submodule $S_j = \mathrm{soc}(e_jA)$ of e_jA has the form $S_j = e_{j\sigma(j)}K$, where $e_{j\sigma(j)} \in e_jJ(A)^2$. Since $J(A)^3 = 0$ then the condition (d1) of Proposition 2.2.23 (with $s = \sigma(j)$), together with the condition (d2), implies the condition required in (c) for $n \geq 3$.

The implication (c)⇒(d) easily follows from the definition of the $(0,1)$-limit $\overline{q}$ of q and of the block matrix $q(\sigma)$ associated to σ.

(d)⇒(a) If $n = 2$ and $q = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$, then $A = \mathbb{M}_2^q(K)$ is the Nakayama algebra of Example 2.2.4. Hence A is a Frobenius algebra such that $J(A)^2 = 0$.

Assume that $n \geq 3$ and there exists a permutation $\sigma \in S_n$ such that $\overline{q} = q(\sigma)$ and $\sigma(j) \neq j$, for all $j = 1, \ldots, n$. Let $\overline{A} = \mathbb{M}_n^{\overline{q}}(K)$ be the $(0,1)$-limit of A.

It is clear that, for each $j \in \{1, \ldots, n\}$, the module $S_j = \mathrm{soc}(e_j\overline{A}) = e_{j\sigma(j)}K$ is simple and $S_j \cong S_i$ if and only if $j = i$. It follows that $\overline{A}$ is a Frobenius algebra and, according to Theorem 2.2.25, the algebra A is Frobenius. Since $n \geq 3$ and $q_{j\,s}^{(r)} = 0$ if and only if $r \notin \{s, j\}$ and $s \neq \sigma(j)$, then $J(\overline{A})^2 = \sum_{j=1}^{n} e_{j\sigma(j)}K$ and $J(\overline{A})^3 = J(A)^3 = 0$, see Proposition 2.2.17. Hence (a) follows.

Since the conditions (a) and (a′) are equivalent, by Theorem 2.2.25 and Proposition 2.2.17, then the proof is complete. □

Following Gabriel [10] we associate to a basic algebra $A = e_1A \oplus \ldots \oplus e_nA$ the separated quiver $\mathcal{Q}^s(A) = (\mathcal{Q}^s(A)_0, \mathcal{Q}^s(A)_1)$ of A with the set of points $\mathcal{Q}^s(A)_0 = \{1, \ldots, n, 1', \ldots, n'\}$. There is an arrow $\beta'_{ij} : i \to j'$ in $\mathcal{Q}^s(A)_1$ if and only if there is an arrow $\beta_{ij} : i \to j$ in the quiver $\mathcal{Q}(A)$ of A, see Corollary 2.2.12.

COROLLARY 2.2.27. *Assume that* $n \geq 3$, q *is a basic structure matrix* (2.1) *in* $\mathbb{ST}_n(K)$ *such that* $A_q = \mathbb{M}_n^q(K)$ *is a Frobenius algebra and* $J(A)^3 = 0$.

(a) *The algebra* A_q *is of finite representation type if and only if* $n = 3$.

(b) *Assume that the field* K *is algebraically closed. Then* A_q *is tame of infinite representation type if and only if* $n = 4$.

(c) *If* $n \geq 5$ *and the field* K *is algebraically closed then the algebra* A_q *is of wild representation type.*

Proof. Since A_q is a Frobenius algebra and $J(A)^3 = 0$ then, according to Theorem 2.2.26, either $n = 2$ and A_q is the Nakayama algebra of Example 2.2.4, or $n \geq 3$ and $\mathrm{soc}(A_q) = J(A_q)^2$. Assume that $n \geq 3$. By Theorem 2.2.25, any indecomposable non-projective A_q-module is a module over the quotient algebra $B_q = A_q/J(A_q)^2$. It follows that A_q is representation-finite (resp. representation-tame) if and only if so is B_q.

Since $J(B_q)^2 = 0$, then by Gabriel [10], B_q is representation-finite if and only if the separated quiver $\mathcal{Q}^s(B_q)$ is a disjoint union of Dynkin quivers, and B_q is representation-tame if and only if the separated quiver $\mathcal{Q}^s(B_q)$ is a disjoint union of Dynkin quivers and Euclidean quivers. Moreover, B_q is representation-infinite if and only if $\mathcal{Q}^s(A_q)$ contains a subquiver isomorphic to an Euclidean quiver.

It follows from Theorem 2.2.19 that in case $n = 3$, up to isomorphism, the only Frobenius algebra A_q is the Nakayama algebra A_{q_5} of Theorem 2.2.19. Obviously, A_{q_5} is of finite representation type.

Assume that $n \geq 4$. Since A_q is a Frobenius algebra then $\overline{A}_q = \mathbb{M}_n^{\overline{q}}(K)$ is also a Frobenius algebra and, according to Theorem 2.2.26, the $(0,1)$-limit $\overline{q}$ of q has the form $\overline{q} = q(\sigma)$, where $\sigma \in S_n$ is the Nakayama permutation of A_q. It follows from Corollary 2.2.12 and Theorem 2.2.26 (c) that there is an arrow $i \to j$ in $\mathcal{Q}(A_q)$ if and only if $i \neq j$ and $j \neq \sigma(i)$.

Now assume that $n = 4$. By the observation made above and the definition of the separated quiver $\mathcal{Q}^s(A_q) = \mathcal{Q}^s(\overline{A}_q)$, we conclude that $\mathcal{Q}^s(A_q)$ is the Euclidean quiver

$$\begin{array}{cccccc} 1 & & \sigma^3(1) & & \sigma^2(1) & & \sigma(1) \\ \\ \\ & \sigma^2(1)' & & \sigma(1)' & & 1' & & \sigma^3(1)' \end{array}$$

of type $\tilde{\mathbb{A}}_7$ It follows that the algebra A_q is of infinite representation type, and A_q is tame if K is algebraically closed.

Finally assume that $n \geq 5$. It is easy to see that $\mathcal{Q}^s(A_q)$ contains a wild subquiver of the form

$$\begin{array}{ccc} \sigma(1) & \sigma^2(1) & \sigma^3(1) \\ \Big| & \Big| & \\ \sigma^4(1)' & 1' & \end{array}$$

It follows that A_q is representation-wild, and the proof is complete. □

THEOREM 2.2.28. *Assume that K is a field and $n \geq 4$. Given $\mu \in K^* = K \setminus \{0\}$, we define the matrix $q_\mu = [q_\mu^{(1)}| \dots |q_\mu^{(n)}]$ of the form (2.1) in $\mathbb{ST}_n(K)$ by the formulae*

$$(q_\mu)_{ij}^{(r)} = \begin{cases} \mu & \text{if } r = 1,\ i = 2,\ j = 3, \\ 1 & \text{if } (i,r,j) \neq (2,1,3) \text{ and } r \in \{i,j\}, \quad \text{or } j = i+1 \text{ (modulo } n \text{)} \\ 0 & \text{otherwise,} \end{cases}$$

for all $i, j, r \in \{1, \ldots, n\}$.

(a) *For each* $\mu \in K^*$, q_μ *is a basic matrix in* $\mathbb{ST}_n(K)$ *such that* $C_{q_\mu} = \mathbb{M}_n^{q_\mu}(K)$ *is a basic Frobenius* K*-algebra with* $J(C_\mu)^3 = 0$ *and with the Nakayama permutation* $\sigma = (1, 2, \ldots, n)$.

(b) *If* $\mu, \nu \in K^*$ *are such that* $\mu \neq \nu$ *and* $\mu \neq \nu^{-1}$, *then* $C_\mu \not\cong C_\nu$.

(c) *If the field* K *is algebraically closed and* $n = 4$, *each of the algebras* C_μ *is tame of infinite representation type.*

(d) *If the field* K *is algebraically closed and* $n \geq 5$, *each of the algebras* C_μ *is of wild representation type.*

Proof. (a) Fix $n \geq 4$ and set $q_{ij}^{(r)} = (q_\mu)_{ij}^{(r)}$, for simplicity of the notation. It is clear that the matrix $q_\mu = [q_\mu^{(1)}| \ldots |q_\mu^{(n)}]$ satisfies the conditions (C1) and (C3) of Definition 2.2.1. To prove that q_μ satisfies the condition (C2), we denote by $\mathcal{I}$ the set of all triples (i, r, j) such that $1 \leq i, r, j \leq n$, and $r \in \{i, j\}$ or $j = i+1$ modulo n. First we recall from [8, Proposition 1.7 (1)] that $(i, r, j), (i, j, s) \in \mathcal{I}$ if and only if $(i, r, s), (r, j, s) \in \mathcal{I}$. It follows that $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$ if and only if $q_{is}^{(r)} q_{rs}^{(j)} \neq 0$, whenever $1 \leq i, j, r, s \leq n$. The verification of (C2) splits into several cases.

1° Assume that $(i, r, j, s) = (2, 1, 3, s)$ and $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$. Then $(2, 3, s) \in \mathcal{I}$. It follows that $s = 3$ and we get $q_{23}^{(1)} q_{23}^{(3)} = \mu = q_{23}^{(1)} q_{13}^{(3)}$.

2° Assume that $(i, r, j, s) = (2, r, 1, 3)$ and $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$. Then $(2, r, 1) \in \mathcal{I}$ and therefore $r = 1$ or $r = 2$. In either case we have $q_{21}^{(r)} q_{23}^{(1)} = \mu = q_{23}^{(r)} q_{r3}^{(1)}$.

3° Assume that $(i, r, j, s) = (2, 1, j, 3)$ and $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$. Then $(1, j, 3) \in \mathcal{I}$ and therefore $j = 1$ or $j = 2$. In either case we have $q_{2j}^{(1)} q_{23}^{(j)} = \mu = q_{23}^{(1)} q_{13}^{(j)}$.

4° Assume that $(i, r, j, s) = (i, 2, 1, 3)$ and $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$. Then $(i, 2, 3) \in \mathcal{I}$ and therefore $i = 2$. Then we get $q_{21}^{(2)} q_{23}^{(1)} = \mu = q_{23}^{(2)} q_{23}^{(1)}$.

5° Assume that $(2, 1, 3) \notin \{(i, r, j), (i, j, s), (i, r, s), (r, j, s), (i, r, j)\}$ and $q_{ij}^{(r)} q_{is}^{(j)} \neq 0$. Then $q_{ij}^{(r)} q_{is}^{(j)} = 1 = q_{is}^{(r)} q_{rs}^{(j)}$.

This shows that the matrix $q_\mu = [q_\mu^{(1)}| \ldots |q_\mu^{(n)}]$ satisfies the conditions (C2) and, consequently, q_μ is a basic matrix in $\mathbb{ST}_n(K)$. By Theorem 2.2.26, the minor q_μ-deformation $C_{q_\mu} = \mathbb{M}_n^{q_\mu}(K)$ is a basic Frobenius K-algebra with Nakayama permutation $\sigma = (1, 2, \ldots, n)$.

(b) Assume that $\mu, \nu \in K^*$ are such that $\mu \neq \nu$ and $\mu \neq \nu^{-1}$. Without loss of generality, we may suppose that $\nu \neq 1$. For simplicity of the notation, we set $q_{ij}^{(r)} = (q_\mu)_{ij}^{(r)}$ and $p_{ij}^{(r)} = (q_\nu)_{ij}^{(r)}$.

Suppose, to the contrary, that there is a K-algebra isomorphism $C_\mu \cong C_\nu$. By Theorem 2.2.10, the matrices q_μ and q_ν belong to the same $\mathbb{G}_n(K)$-orbit, that is, there exist a permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and a square matrix $T = [t_{ij}] \in \mathbb{M}_n(K)$ such that

- $t_{11} = \ldots = t_{nn} = 1$,
- $t_{ij} \neq 0$, for all $i, j \in \{1, \ldots, n\}$, and
- $t_{ir} p_{ij}^{(r)} t_{rj} = q_{\tau(i)\,\tau(j)}^{(\tau(r))} t_{ij}$, for all $i, r, j \in \{1, \ldots, n\}$.

We set $d_{ij}^{(r)} := q_{\tau(i)\,\tau(j)}^{(\tau(r))}$, for short, and let $\sigma = (1, 2, \ldots, n)$ be the cyclic permutation of $\{1, 2, \ldots, n\}$. Then

$$\prod_{i=1}^{n} (d_{i\,\sigma(i)}^{(\sigma^2(i))} t_{i\,\sigma(i)})(p_{i\,\sigma(i)}^{(\sigma^{-1}(i))} t_{i\,\sigma^{-1}(i)} t_{\sigma^{-1}(i)\,\sigma(i)}) = \prod_{i=1}^{n} (p_{i\,\sigma(i)}^{(\sigma^2(i))} t_{i\,\sigma^2(i)} t_{\sigma^2(i)\,\sigma(i)})(d_{i\,\sigma(i)}^{(\sigma^{-1}(i))} t_{i\,\sigma(i)})$$

and hence we get

$$\prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^2(i))} \cdot \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^{-1}(i))} = \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^2(i))} \cdot \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$$

Since $n \geq 4$ and $\sigma = (1, 2, \ldots, n)$, then $p_{i\,\sigma(i)}^{(\sigma^2(i))} = 1$ for all $i = 1, \ldots, n$. Hence, in view of the equality $\nu = \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$, we get

$$(*) \qquad \nu \cdot \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^2(i))} = \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}.$$

Since $t_{ir} p_{i\,\sigma(i)}^{(r)} t_{r\,\sigma(i)} \neq 0$ then $d_{i\,\sigma(i)}^{(r)} = q_{\tau(i)\,\tau(\,\sigma(i))}^{(\tau(r))} \in \{1, \mu\}$, for $1 \leq r \leq n$. Note that $\mu \neq 1$, because the equality $\mu = 1$ yields $\nu = 1$, contrary to our assumption $\nu \neq 1$. Further, note that there is at most one $i \in \{1, \ldots, n\}$ such that $\mu = d_{i\,\sigma(i)}^{(\sigma^2(i))} = d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$. On the other hand, since $n \geq 4$ and $\sigma = (1, 2, \ldots, n)$, there is no such an i such that $\mu = d_{i\,\sigma(i)}^{(\sigma^2(i))} = d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$. Then $\nu \neq 1$ and the equality yield $\mu\nu = 1$ or $\mu = \nu$, contrary to the assumption that $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.

Since the statements (c) and (d) follow from Corollary 2.2.27, the proof of the theorem is complete. □

COROLLARY 2.2.29. *Assume that K is an infinite field. Then for each $n \geq 4$ there is a one-parameter K-algebraic family $\{C_\mu\}_{\mu \in K^*}$ of basic Frobenius K-algebras of the form $C_\mu = \mathbb{M}_n^{q_\mu}(K)$ such that $\sigma = (1, 2, \ldots, n)$ is the Nakayama permutation of C_μ and $C_\mu \not\cong C_\nu$, if $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.*

Proof. Apply Theorem 2.2.28. □

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