

DIFFERENTIAL GEOMETRY OF
MICROLINEAR FRÖLICHER SPACES II

Hirokazu Nishimura
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki, 305-8571, JAPAN

Abstract: In this paper, as the second in our series of papers on differential geometry of microlinear Frölicher spaces, we study differential forms. The principal result is that the exterior differentiation is uniquely determined geometrically, just as div (ergence) and rot (ation) are uniquely determined geometrically or physically in classical vector calculus. This infinitesimal characterization of exterior differentiation has been completely missing in orthodox differential geometry.

AMS Subject Classification: 58A03

Key Words: microlinearity, synthetic differential geometry, Frölicher space, transversal limit diagram, Topos theory, infinite-dimensional differential geometry, Cahiers Topos, Cartesian closed category, Weil algebra, Weil functor, nilpotent infinitesimal

1. Introduction

Vector analysis is indispensable in studying electromagnetism and fluid mechanics. The central notions of vector analysis, namely grad , div and rot , were introduced infinitesimally as physically and geometrically meaningful operations. Indeed, their physical or geometrical meanings determine grad , div and rot uniquely and unequivocally. We know well that the operations grad , div and rot are the precursors of the exterior differentiation in the modern theory of differential forms.

In a standard course on the theory of differential forms, the exterior differentiation is introduced by decree as a highly formal operation without paying little attention to its geometric meaning. Many mathematicians believe naïvely

that the proof of Stokes' theorem is easy or even trivial once the theorem is formulated adequately. We agree completely that its standard proof is very easy, not to say that it is trivial, but we must insist that the infinitesimal characterization of the exterior differentiation, which lies at the core of Stokes' theorem, is not so easy to establish. It is the infinitesimal Stokes' theorem that underlies the standard (i.e., local or global) Stokes' theorem. In other words, once the infinitesimal Stokes' theorem, which is no other than the infinitesimal characterization of the exterior differentiation, is established, the proof of the standard Stokes' theorem is highly trivial. This is the whole story of Stokes' theorem from a conceptual viewpoint, and its infinitesimal part, which is essential to the whole story, has been completely missing in orthodox differential geometry.

Although nilpotent infinitesimals are invisible in our standard universe of mathematics, Weil functors are still meaningful there. The notion of micro-linearity, which is essential to synthetic differential geometry and is defined in another universe of mathematics, can be externalized by using Weil functors, as we have discussed in [19]. The principal objective in this paper is to present the infinitesimal story of Stokes' theorem to orthodox differential geometers without getting involved seriously in synthetic differential geometry at all. The reader is strongly recommended to read Nishimura's [17] as a heuristic introduction to the subject discussed here.

2. Preliminaries

2.1. Frölicher Spaces

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the *underlying set theory* for infinite-dimensional differential geometry in a sense. A Frölicher space is an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of mappings from the set \mathbf{R} of real numbers to the underlying set (called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from \mathbf{R} to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. What is most important

among many nice properties about the category **FS** of Frölicher spaces and smooth mappings is that it is cartesian closed, while neither the category of finite-dimensional smooth manifolds nor that of infinite-dimensional smooth manifolds modelled after any infinite-dimensional vector spaces such as Hilbert spaces, Banach spaces, Fréchet spaces or the like is so at all. For a standard reference on Frölicher spaces the reader is referred to [6].

2.2. Weil Algebras and Infinitesimal Objects

The notion of a *Weil algebra* was introduced by Weil himself in [22]. We denote by **W** the category of Weil algebras. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. By way of example, the Weil algebra $\mathbf{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbf{R}[X]$ of an indeterminate X modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra $\mathbf{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities, each Weil algebra yields its corresponding *Weil functor* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [10], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modelled on convenient vector spaces in §31 of [11].

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [12], while he or she is referred to [9] for the model theory of SDG vigorously constructed by Dubuc [2] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write W_D and W_{D_2} in place of $\mathbf{R}[X]/(X^2)$ and $\mathbf{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order

nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbf{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbf{R} \mid d^3 = 0\}$$

We will write $W_{d \in D_2 \mapsto d^2 \in D}$ in place of the homomorphism of Weil algebras $\mathbf{R}[X]/(X^2) \rightarrow \mathbf{R}[X]/(X^3)$ induced by the homomorphism $X \rightarrow X^2$ of the polynomial ring $\mathbf{R}[X]$ to itself. Such locutions are justifiable, because the category \mathbf{W} of Weil algebras in the real world and the category of infinitesimal objects in the shade are dual to each other. To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [12], even if he or she is not interested in SDG at all.

We need to fix notation and terminology for simplicial objects, which form an important subclass of infinitesimal objects. *Simplicial objects* are infinitesimal objects of the form

$$\begin{aligned} D^n\{\mathfrak{p}\} \\ = \{(d_1, \dots, d_n) \in D^n \mid d_{i_1} \dots d_{i_k} = 0 \ (\forall (i_1, \dots, i_k) \in \mathfrak{p})\} \end{aligned}$$

where \mathfrak{p} is a finite set of finite sequences (i_1, \dots, i_k) of natural numbers between 1 and n , including the endpoints, with $i_1 < \dots < i_k$. If \mathfrak{p} is empty, $D^n\{\mathfrak{p}\}$ is D^n itself. If \mathfrak{p} consists of all the binary sequences, then $D^n\{\mathfrak{p}\}$ represents $D(n)$ in the standard terminology of SDG. Given two simplicial objects $D^m\{\mathfrak{p}\}$ and $D^n\{\mathfrak{q}\}$, we define a simplicial object $D^{m+n}\{\mathfrak{p} \oplus \mathfrak{q}\}$ to be

$$D^{m+n}\{\mathfrak{p} \oplus \mathfrak{q}\}$$

where

$$\begin{aligned} \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{p} \cup \{(j_1 + m, \dots, j_k + m) \mid (j_1, \dots, j_k) \in \mathfrak{q}\} \\ \cup \{(i, j + m) \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

Since the operation \oplus is associative, we can combine any finite number of simplicial objects by \oplus without bothering about how to insert parentheses. Given morphisms of simplicial objects $\Phi_i : D^{m_i}\{\mathfrak{p}_i\} \rightarrow D^m\{\mathfrak{p}\}$ ($1 \leq i \leq n$), there exists a unique morphism of simplicial objects $\Phi : D^{m_1}\{\mathfrak{p}_1\} \oplus \dots \oplus D^{m_n}\{\mathfrak{p}_n\} \rightarrow D^m\{\mathfrak{p}\}$ whose restriction to $D^{m_i}\{\mathfrak{p}_i\}$ coincides with Φ_i for each i . We denote this Φ by $\Phi_1 \oplus \dots \oplus \Phi_n$.

2.3. Microlinearity

In [18] we have discussed how to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W , another Frölicher space $X \otimes W$ called the *Weil prolongation of X with respect to W* , which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \rightarrow \mathbf{FS}$, and then to show that the functor $\cdot \otimes W : \mathbf{FS} \rightarrow \mathbf{FS}$ is product-preserving for any Weil algebra W . Weil prolongations are well-known as *Weil functors* for finite-dimensional and infinite-dimensional smooth manifolds in orthodox differential geometry, as we have already touched upon in the preceding subsection. There is a canonical projection from $X \otimes W$ to X , and we denote the inverse image of x under the canonical projection by $(X \otimes W)_x$ for any $x \in X$.

The central object of study in SDG is *microlinear* spaces. Although the notion of a manifold (=a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined absolutely on the genuinely infinitesimal level. For the historical account of microlinearity, the reader is referred to §§2.4 of [12] or Appendix D of [9]. To get an adequately restricted cartesian closed subcategory of Frölicher spaces, we have emancipated microlinearity from within a well-adapted model of SDG to Frölicher spaces in the real world in [19]. Recall that a Frölicher space X is called *microlinear* providing that any finite limit diagram \mathcal{D} in \mathbf{W} yields a limit diagram $X \otimes \mathcal{D}$ in \mathbf{FS} , where $X \otimes \mathcal{D}$ is obtained from \mathcal{D} by putting $X \otimes$ to the left of every object and every morphism in \mathcal{D} . As we have discussed there, all convenient vector spaces are microlinear, so that all C^∞ -manifolds in the sense of [11] (cf. Section 27) are also microlinear.

We have no reason to hold that all Frölicher spaces credit Weil prolongations as exponentiation by infinitesimal objects in the shade. Therefore we need a notion which distinguishes Frölicher spaces that do so from those that do not. A Frölicher space X is called *Weil exponentiable* if

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \quad (1)$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2 . If $Y = 1$, then (1) degenerates into

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2 \quad (2)$$

If $W_1 = \mathbf{R}$, then (1) degenerates into

$$(X \otimes W_2)^Y = X^Y \otimes W_2 \quad (3)$$

We have shown in [18] that all convenient vector spaces are Weil exponentiable, so that all C^∞ -manifolds in the sense of [11] (cf. Section 27) are Weil

exponentiable.

We have demonstrated in [19] that all Frölicher spaces that are microlinear and Weil exponentiable form a cartesian closed category. In the sequel M is assumed to be such a Frölicher space.

3. Euclidean Vector Spaces

In this paper we will always mean a preconvenient vector space simply by a *vector space*. We will choose and fix a vector space \mathbb{E} in this sense throughout this section. It is evident that

Lemma 1. *The vector space structure of \mathbb{E} naturally gives rise to that of $\mathbb{E} \otimes W$ for any Weil algebra W .*

Proof. This follows readily from the bifunctionality of \otimes and the fact that the functor $\cdot \otimes W : \mathbf{FS} \rightarrow \mathbf{FS}$ is product-preserving. \square

Lemma 2. *The vector space structure of $(\mathbb{E} \otimes W_D)_0$ as the tangent space discussed in our previous paper coincides with that induced by the vector space structure on $\mathbb{E} \otimes W_D$ in the preceding lemma.*

Proof. We write $+_D$ for the addition in the former vector structure, while we write $+_{\mathbb{E}}$ for the addition in \mathbb{E} as well as for the addition in $(\mathbb{E} \otimes W_D)_0$ induced by that in Lemma 1. Given $t_1, t_2 \in (\mathbb{E} \otimes W_D)_0$, let us consider

$$(+_{\mathbb{E}} \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_1 \in D})(t_1), (\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_2 \in D})(t_2))$$

It is easy to see that

$$\begin{aligned} & (\text{id}_{\mathbb{E}} \otimes W_{d \in D \rightarrow (d, 0) \in D(2)}) \\ & ((+_{\mathbb{E}} \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_1 \in D})(t_1), (\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_2 \in D})(t_2))) \\ & = (+_{\mathbb{E}} \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{d \in D \rightarrow (d, 0) \in D(2)}) \circ (\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_1 \in D})(t_1), \\ & (\text{id}_{\mathbb{E}} \otimes W_{d \in D \rightarrow (d, 0) \in D(2)}) \circ (\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_2 \in D})(t_2)) \\ & \text{[by the bifunctionality of } \otimes \text{]} \\ & = (+_{\mathbb{E}} \otimes \text{id}_{W_{D(2)}})(t_1, (\text{id}_{\mathbb{E}} \otimes W_{d \in D \rightarrow 0 \in D})(t_2)) \\ & = t_1 \end{aligned}$$

By the same token, it is also easy to see that

$$\begin{aligned} & (\text{id}_{\mathbb{E}} \otimes W_{d \in D \rightarrow (0, d) \in D(2)}) \\ & ((+_{\mathbb{E}} \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_1 \in D})(t_1), (\text{id}_{\mathbb{E}} \otimes W_{(d_1, d_2) \in D(2) \rightarrow d_2 \in D})(t_2))) \end{aligned}$$

$$= t_2$$

Therefore we have

$$t_1 +_D t_2$$

$$= (\text{id}_{\mathbb{E}} \otimes W_{d \in D \mapsto (d,d) \in D(2)})$$

$$((+_E \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{(d_1,d_2) \in D(2) \mapsto d_1 \in D})(t_1), (\text{id}_{\mathbb{E}} \otimes W_{(d_1,d_2) \in D(2) \mapsto d_2 \in D})(t_2)))$$

$$= (+_E \otimes \text{id}_{W_{D(2)}})((\text{id}_{\mathbb{E}} \otimes W_{d \in D \mapsto (d,d) \in D(2)}) \circ (\text{id}_{\mathbb{E}} \otimes W_{(d_1,d_2) \in D(2) \mapsto d_1 \in D})(t_1),$$

$$(\text{id}_{\mathbb{E}} \otimes W_{d \in D \mapsto (d,d) \in D(2)}) \circ (\text{id}_{\mathbb{E}} \otimes W_{(d_1,d_2) \in D(2) \mapsto d_2 \in D})(t_2))$$

[by the bifunctionality of \otimes]

$$= t_1 +_E t_2$$

□

It is evident that

Proposition 3. *The following conditions on the vector space \mathbb{E} are equivalent:*

1. *The canonical mapping $\mathbf{i}_{\mathbb{E}} : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E} \otimes W_D$ induced by the mapping*

$$(\mathbf{a}, \mathbf{b}) \in \mathbb{E} \times \mathbb{E} \mapsto (x \in \mathbb{R} \mapsto \mathbf{a} + x\mathbf{b} \in \mathbb{E}) \in \mathbb{E}^{\mathbb{R}}$$

is bijective;

2. *The Kock-Lawvere axiom holds in the sense that, for any $t \in (\mathbb{E} \otimes W_D)_0$, there exists a unique $\mathbf{a} \in \mathbb{E}$ with*

$$t = \mathbf{i}_{\mathbb{E}}(\mathbf{0}, \mathbf{a})$$

Definition 4. The vector space \mathbb{E} is called *Euclidean* providing that one of the above equivalent conditions holds.

Proposition 5. *If \mathbb{E} is a Euclidean vector space, then so is \mathbb{E}^X for any Frölicher space X .*

Proof. We will check the first condition in Proposition 3. We have

$$\begin{aligned} \mathbb{E}^X \otimes W_D &= (\mathbb{E} \otimes W_D)^X \\ &= (\mathbb{E} \times \mathbb{E})^X \\ &= \mathbb{E}^X \times \mathbb{E}^X \end{aligned}$$

so that we have the desired conclusion. □

Corollary 6. *The category of Euclidean vector spaces and smooth mappings is cartesian closed.*

Proposition 7. *If \mathbb{E} is a Euclidean vector space, then so is $\mathbb{E} \otimes W$ for any Weil algebra W .*

Proof. We will check the first condition in Proposition 2. We have

$$\begin{aligned}
& (\mathbb{E} \otimes W) \otimes W_D \\
&= \mathbb{E} \otimes (W \otimes_{\infty} W_D) \\
&= \mathbb{E} \otimes (W_D \otimes_{\infty} W) \\
&= (\mathbb{E} \otimes W_D) \otimes W \\
&= (\mathbb{E} \times \mathbb{E}) \otimes W \\
&= (\mathbb{E} \otimes W) \times (\mathbb{E} \otimes W) \\
& \text{[since the functor } \cdot \otimes W \text{ is product-preserving]}
\end{aligned}$$

so that we have the desired conclusion. \square

Remark 8. Let $x \in M$. Given $t \in (M \otimes W_D)_x$, we note that $\mathbf{i}_{(M \otimes W_D)_x}(\mathbf{0}, t) \in (M \otimes W_D)_x \otimes W_D$ can be regarded as an element of $(M \otimes W_D) \otimes W_D = M \otimes W_{D^2}$, which is no other than

$$(\text{id}_M \otimes W_{(d_1, d_2) \in D^2 \rightarrow d_1 d_2 \in D})(t)$$

We note in passing that

Proposition 9. *Convenient vector spaces are Euclidean.*

Proof. The reader is referred to §2 of [7]. \square

4. Differential Forms

Let \mathbb{E} be a Euclidean vector space which is microlinear and Weil exponentiable.

Definition 10. Given a smooth mapping $\omega : M \otimes W_{D^n} \rightarrow \mathbb{E}$ and a natural number i with $1 \leq i \leq n$, we say that ω is homogeneous at the i -th position providing that we have

$$\omega(\alpha \cdot_i \gamma) = \alpha \omega(\gamma)$$

for any $\gamma \in M \otimes W_{D^n}$ and any $\alpha \in \mathbb{R}$, where $\alpha \cdot_i \gamma$ is defined by

$$\alpha \cdot_i \gamma = \left(\text{id}_M \otimes W_{\left(\begin{smallmatrix} \alpha \\ \alpha_i \end{smallmatrix} \right)_{D^n}} \right) (\gamma)$$

with the putative mapping $\left(\alpha \cdot\right)_{D^n}^i : D^n \rightarrow D^n$ being

$$(d_1, \dots, d_n) \in D^n \mapsto (d_1, \dots, d_{i-1}, \alpha d_i, d_{i+1}, \dots, d_n) \in D^n$$

Notation 11. Given $\gamma \in M \otimes W_{D^n}$ and a natural number i with $1 \leq i \leq n$, we denote

$$(\text{id}_M \otimes W_{(d_1, \dots, d_{n-1}) \in D^{n-1} \mapsto (d_1, \dots, d_{i-1}, 0, d_i, \dots, d_{n-1}) \in D^n})(\gamma)$$

by $\gamma \upharpoonright_{D^{n-1}}^i$.

Notation 12. Given $\eta \in M \otimes W_{D^{n-1}}$ and a natural number i with $1 \leq i \leq n$, we denote

$$\{\gamma \in M \otimes W_{D^n} \mid \gamma \upharpoonright_{D^{n-1}}^i = \eta\}$$

by $(M \otimes W_{D^n})_\eta^i$.

Notation 13. The putative mapping $\partial_i^n : D^n \rightarrow D^n$ is

$$(d_1, \dots, d_n) \in D^n \mapsto (d_1, \dots, d_{i-1}, d_n, d_i, \dots, d_{n-1}) \in D^n$$

Remark 14. By the natural identification

$$M \otimes W_{D^n} = (M \otimes W_{D^{n-1}}) \otimes W_D$$

the space $(M \otimes W_{D^n})_\eta^n$ is a Euclidean vector space. Under the bijective mapping $\text{id}_M \otimes W_{\partial_i^n}$, the spaces $(M \otimes W_{D^n})_\eta^i$ and $(M \otimes W_{D^n})_\eta^n$ can be identified, so that the former is also a Euclidean vector space for any natural number i with $1 \leq i \leq n$. Given $\gamma_1, \gamma_2 \in M \otimes W_{D^n}$ with

$$\gamma_1 \upharpoonright_{D^{n-1}}^i = \gamma_2 \upharpoonright_{D^{n-1}}^i = \eta$$

we denote the addition of γ_1 and γ_2 in $(M \otimes W_{D^n})_\eta^i$ by $\gamma_1 \upharpoonright_i^i \gamma_2$.

Proposition 15. A smooth mapping $\omega : M \otimes W_{D^n} \rightarrow \mathbb{E}$ which is homogeneous at the i -th position is linear at the i -th position as well in the sense that

$$\omega \left(\gamma_1 \upharpoonright_i^i \gamma_2 \right) = \omega(\gamma_1) + \omega(\gamma_2)$$

for any $\gamma_1, \gamma_2 \in M \otimes W_{D^n}$ with

$$\gamma_1 \upharpoonright_{D^{n-1}}^i = \gamma_2 \upharpoonright_{D^{n-1}}^i$$

Proof. The reader is referred to *Proposition 10* in §1.2 of [12]. □

Definition 16. A differential n -form ω on M with values in \mathbb{E} is a smooth mapping

$$\omega : M \otimes W_{D^n} \rightarrow \mathbb{E}$$

pursuant to the following conditions:

1. ω is n -homogeneous in the sense that it is homogeneous at the i -th position for any natural number i with $1 \leq i \leq n$.
2. ω is alternating in the sense that

$$\omega(\gamma^\sigma) = \epsilon_\sigma \omega(\gamma)$$

for any permutation σ of $1, \dots, n$, where γ^σ is defined by

$$\gamma^\sigma = (\text{id}_M \otimes W_{\sigma_{D^n}})(\gamma)$$

with the putative mapping $\sigma_{D^n} : D^n \rightarrow D^n$ being

$$(d_1, \dots, d_n) \in D^n \mapsto (d_{\sigma(1)}, \dots, d_{\sigma(n)}) \in D^n$$

In case that M is a convenient vector space \mathbb{F} , we have a more traditional notion of a differential n -form.

Definition 17. A differential n -form _{c} ω on \mathbb{F} with values in \mathbb{E} is a smooth mapping from \mathbb{F} to $\mathbf{L}_{\text{alt}}^n(\mathbb{F}; \mathbb{E})$, where $\mathbf{L}_{\text{alt}}^n(\mathbb{F}; \mathbb{E})$ denotes the space of smooth mappings from the direct product of n copies of \mathbb{F} to \mathbb{E} which are n -linear and alternating.

Proposition 18. In case that M is a convenient vector space \mathbb{F} , we assign, to each differential n -form _{c} ω on \mathbb{F} with values in \mathbb{E} , the mapping $\tilde{\omega} : \mathbb{F} \otimes W_{D^n} \rightarrow \mathbb{E}$ with

$$\tilde{\omega}(\gamma) = \omega(\mathbf{e}_1^\gamma, \dots, \mathbf{e}_n^\gamma)$$

for any $\gamma \in \mathbb{F} \otimes W_{D^n}$, where

$$\mathbf{i}_{\mathbb{F}}(\pi(\gamma), \mathbf{e}_i^\gamma) = (\text{id}_{\mathbb{F}} \otimes W_{i^n})(\gamma) \quad (1 \leq i \leq n)$$

The assignment gives a mapping from the totality of differential n -forms _{c} on \mathbb{F} with values in \mathbb{E} to that of differential n -forms on \mathbb{F} with values in \mathbb{E} . The mapping is bijective.

Proof. The discussion in Proposition 6 of §4.1 in [12] can be reformulated easily for our general and abstract context. We should use Proposition 0.3.9 of [16] in place of Proposition 7 in §3.4 of [12]. The details can safely be left to the reader, but we note in passing that the inverse assignment of a differential n -form _{c} $\underline{\varrho}$ on \mathbb{F} with values in \mathbb{E} to each differential n -form ϱ on \mathbb{F} with values in \mathbb{E} goes as

$$\underline{\varrho}_{\mathbf{x}}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \varrho(\mathbf{i}_{\mathbb{F}}(\mathbf{x}; \mathbf{a}_1, \dots, \mathbf{a}_n))$$

where $\mathbf{i}_{\mathbb{F}}(\mathbf{x}; \mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{F} \otimes W_{D^n}$ is the canonical mapping induced by the

mapping

$$(r_1, \dots, r_n) \in \mathbb{R}^n \mapsto \mathbf{x} + r_1 \mathbf{a}_1 + \dots + r_n \mathbf{a}_n \in \mathbb{F}$$

□

5. The Exterior Differentiation

Let us begin this section with two definitions.

Definition 19. Given $\gamma \in \mathbb{E} \otimes W_{D^n}$ and a natural number i with $1 \leq i \leq n$, we say that it is homogeneous at the i -th position provided that we have

$$\left(\text{id}_{\mathbb{E}} \otimes W_{\binom{\alpha \cdot}{i}_{D^n}} \right) (\gamma) = (\alpha_{\mathbb{E}} \otimes \text{id}_{W_{D^n}}) (\gamma)$$

for any $\alpha \in \mathbb{R}$, where the putative mapping $\binom{\alpha \cdot}{i}_{D^n} : D^n \rightarrow D^n$ is

$$(d_1, \dots, d_n) \in D^n \mapsto (d_1, \dots, d_{i-1}, \alpha d_i, d_{i+1}, \dots, d_n) \in D^n$$

and $\alpha_{\mathbb{E}}$ on the right-hand side of the equation stands for the multiplication by the scalar α . We say that γ is *n-homogeneous* provided that it is homogeneous at the i -th position for any natural number i with $1 \leq i \leq n$.

Definition 20. Given a differential n -form ω on M with values in \mathbb{E} and $\gamma \in M \otimes W_{D^n}$, we define

$$\int_{\gamma} \omega \in \mathbb{E} \otimes W_{D^n}$$

as the value of the mapping

$$\mathbb{E} \times \mathbb{E} \cong \mathbb{E} \otimes W_D \quad \begin{matrix} \text{id}_{\mathbb{E}} \otimes W_{(d_1, \dots, d_n) \in D^n \mapsto d_1 \dots d_n \in D} \\ \rightarrow \end{matrix} \quad \mathbb{E} \otimes W_{D^n}$$

at $(\mathbf{0}, \omega(\gamma)) \in \mathbb{E} \times \mathbb{E}$.

It is easy to see that

Proposition 21. *The above mapping*

$$\int_{\cdot} \omega : M \otimes W_{D^n} \rightarrow \mathbb{E} \otimes W_{D^n} \tag{4}$$

is subject to the following two conditions:

1. The mapping is a differential n -form with values in the vector space $\mathbb{E} \otimes$

W_{D^n} ;

2. The values of the mapping are all n -homogeneous.

Proof. Since the mapping

$$\text{id}_{\mathbb{E}} \otimes W_{(d_1, \dots, d_n) \in D^n \mapsto d_1 \dots d_n \in D} : \mathbb{E} \otimes W_D \rightarrow \mathbb{E} \otimes W_{D^n}$$

preserves the linear structure, we can see readily that the mapping (4) satisfies the first condition. Since $(\mathbf{0}, \omega(\gamma))$, regarded as an element of $\mathbb{E} \otimes W_D$, is 1-homogeneous by Lemma 2, and since the following diagram of putative mappings

$$\begin{array}{ccc} & (d_1, \dots, d_n) \in D^n \mapsto d_1 \dots d_n \in D & \\ & D^n \qquad \qquad \qquad \rightarrow \qquad \qquad \qquad D & \\ \left(\begin{array}{c} \alpha \\ \cdot \\ i \end{array} \right)_{D^n} & \uparrow & \uparrow \alpha_D \\ & D^n \qquad \qquad \qquad \rightarrow \qquad \qquad \qquad D & \\ & (d_1, \dots, d_n) \in D^n \mapsto d_1 \dots d_n \in D & \end{array}$$

commutes, we are sure that the mapping (4) satisfies the second condition. This completes the proof. \square

What is really surprising, we have its converse.

Theorem 22. *If a smooth mapping $\phi : M \otimes W_{D^n} \rightarrow \mathbb{E} \otimes W_{D^n}$ abides by the two conditions in Proposition 21, then there exists a unique differential n -form ω with*

$$\phi(\gamma) = \int_{\gamma} \omega$$

for any $\gamma \in M \otimes W_{D^n}$.

Proof. The limit diagram of Weil algebras

$$\begin{array}{ccccccc} & & & & W_{i_1} & & \\ & & & & \rightarrow & & \\ & & & & \vdots & & \\ W_D & \rightarrow & W_{D^n} & \rightarrow & W_{D^{n-1}} & & \\ & & & & \vdots & & \\ & & & & W_{i_n} & & \\ & & & & \rightarrow & & \\ & & & & W_{0_{n-1}^n} & & \\ & & & & \rightarrow & & \end{array}$$

gives rise to the limit diagram of Frölicher spaces

$$\begin{array}{ccc}
 & \text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{m_n}} & \\
 \mathbb{E}^{M \otimes W_{D^n}} \otimes W_D & \rightarrow & \\
 & \text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{i_1}} & \\
 & \rightarrow & \\
 & \vdots & \\
 \mathbb{E}^{M \otimes W_{D^n}} \otimes W_{D^n} & \rightarrow & \mathbb{E}^{M \otimes W_{D^n}} \otimes W_{D^{n-1}} \\
 & \vdots & \\
 & \text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{i_n}} & \\
 & \rightarrow & \\
 & \text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{0_{n-1}^n}} & \\
 & \rightarrow &
 \end{array}$$

because of the microlinearity of $\mathbb{E}^{M \otimes W_{D^n}}$, where the putative mappings $i_j : D^{n-1} \rightarrow D^n$ ($1 \leq j \leq n$) are

$$(d_1, \dots, d_{n-1}) \in D^{n-1} \mapsto (d_1, \dots, d_{j-1}, 0, d_j, \dots, d_{n-1}) \in D^n$$

while the putative mapping $0_{n-1}^n : D^{n-1} \rightarrow D^n$ is

$$(d_1, \dots, d_{n-1}) \in D^{n-1} \mapsto (0, \dots, 0) \in D^n$$

Since ϕ , regarded as an element of $\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{D^n} (= (\mathbb{E} \otimes W_{D^n})^{M \otimes W_{D^n}})$, is n -homogeneous, it is easy to see that

$$(\text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{i_1}})(\phi) = \dots = (\text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{i_n}})(\phi) = (\text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{0_{n-1}^n}})(\phi)$$

Then the above limit diagram of Frölicher spaces guarantees that there exists a unique $\psi \in \mathbb{E}^{M \otimes W_{D^n}} \otimes W_D$ with

$$\phi = (\text{id}_{\mathbb{E}^{M \otimes W_{D^n}} \otimes W_{m_n}})(\psi)$$

Since $\mathbb{E}^{M \otimes W_{D^n}}$ is Euclidean by Proposition 5 and $\psi \in (\mathbb{E}^{M \otimes W_{D^n}} \otimes W_D)_0$, there exists a unique $\omega \in \mathbb{E}^{M \otimes W_{D^n}}$ with

$$\psi = \mathbf{i}_{\mathbb{E}^{M \otimes W_{D^n}}}(\mathbf{0}, \omega)$$

Then it is easy to see that ω is a differential n -form with values in \mathbb{E} such that

$$\phi(\gamma) = \int_{\gamma} \omega$$

for any $\gamma \in M \otimes W_{D^n}$. This completes the proof. \square

Definition 23. Given a differential n -form ω on M with values in \mathbb{E} , we

define a mapping $\int_{\partial_i} \omega : M \otimes W_{D^{n+1}} \rightarrow \mathbb{E} \otimes W_{D^{n+1}}$ to be

$$M \otimes W_{D^{n+1}} \cong \begin{matrix} \text{id}_M \otimes W_{\partial_i^{n+1}} \\ \cong \\ M \otimes W_{D^{n+1}} \end{matrix} \cong (M \otimes W_{D^n}) \otimes W_D$$

$$\begin{aligned} (\int \omega) \otimes \text{id}_{W_D} \\ \rightarrow (\mathbb{E} \otimes W_{D^n}) \otimes W_D \cong \mathbb{E} \otimes W_{D^{n+1}} \end{aligned}$$

Theorem 24. *Given a differential n -form ω on M with values in \mathbb{E} , there exists a unique differential $(n+1)$ -form $\mathbf{d}\omega$ on M with values in \mathbb{E} with*

$$\int_{\gamma} \mathbf{d}\omega = \sum (-1)^{i+1} \mathbf{D}_0 \int_{\partial_i \gamma} \omega \quad (5)$$

for any $\gamma \in M \otimes W_{D^{n+1}}$, where $\mathbf{D}_0 \int_{\partial_i \gamma} \omega$ denotes the mapping

$$\int_{\partial_i \gamma} \omega - (\text{id}_{M \otimes W_{D^n}} \otimes W_{d \in D \mapsto 0 \in D}) \left(\int_{\partial_i \gamma} \omega \right)$$

Proof. By Theorem 22 it suffices to verify that the right-hand side of (5) abides by the two conditions in Proposition 21, which goes as follows:

1. We would like to show that $\mathbf{D}_0 \int_{\partial_i \gamma} \omega \in \mathbb{E} \otimes W_{D^{n+1}}$ is $(n+1)$ -homogeneous ($1 \leq i \leq n+1$). Since both $\int_{\partial_i \gamma} \omega$ and $(\text{id}_{M \otimes W_{D^n}} \otimes W_{d \in D \mapsto 0 \in D})(\int_{\partial_i \gamma} \omega)$ are homogeneous at the j -th component, $\mathbf{D}_0 \int_{\partial_i \gamma} \omega$ is homogeneous at the j -th component for $j \neq i$. That $\mathbf{D}_0 \int_{\partial_i \gamma} \omega$ is homogeneous also at the i -th component follows from the fact that $\mathbb{E} \otimes W_{D^n}$ is Euclidean.

2. We would like to show that the mapping

$$\mathbf{D}_0 \int_{\partial_i} \omega : M \otimes W_{D^{n+1}} \rightarrow \mathbb{E} \otimes W_{D^{n+1}}$$

is $(n+1)$ -homogeneous ($1 \leq i \leq n+1$). For $j < i$, it is easy to see that

$$\begin{aligned} & \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \circ \left(\text{id}_M \otimes W_{\binom{\alpha_j}{D^{n+1}}} \right) \\ &= \left(\left(\text{id}_M \otimes W_{\binom{\alpha_j}{D^n}} \right) \otimes \text{id}_{W_D} \right) \circ \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \end{aligned}$$

while, for $j > i$, it is also easy to see that

$$\begin{aligned} & \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \circ \left(\text{id}_M \otimes W_{\left(\alpha_j \right)_{D^{n+1}}} \right) \\ &= \left(\left(\text{id}_M \otimes W_{\left(\alpha_{j-1} \right)_{D^n}} \right) \otimes \text{id}_{W_D} \right) \circ \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \end{aligned}$$

Therefore, for $j \neq i$, that

$$\begin{aligned} & \left(\mathbf{D}_0 \int_{\partial_i} \omega \right) \circ \left(\text{id}_M \otimes W_{\left(\alpha_j \right)_{D^{n+1}}} \right) \\ &= \left(\alpha_{\mathbb{E}} \otimes \text{id}_{W_{D^{n+1}}} \right) \circ \left(\mathbf{D}_0 \int_{\partial_i} \omega \right) \end{aligned}$$

follows directly from the assumption that the mapping

$$\int \omega : M \otimes W_{D^n} \rightarrow \mathbb{E} \otimes W_{D^n}$$

is n -homogeneous. It remains to show that

$$\begin{aligned} & \left(\mathbf{D}_0 \int_{\partial_i} \omega \right) \circ \left(\text{id}_M \otimes W_{\left(\alpha_i \right)_{D^{n+1}}} \right) \\ &= \left(\alpha_{\mathbb{E}} \otimes \text{id}_{W_{D^{n+1}}} \right) \circ \left(\mathbf{D}_0 \int_{\partial_i} \omega \right) \end{aligned}$$

which follows readily from

$$\begin{aligned} & \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \circ \left(\text{id}_M \otimes W_{\left(\alpha_i \right)_{D^{n+1}}} \right) \\ &= \left(\text{id}_{M \otimes W_{D^n}} \otimes \alpha_D \right) \circ \left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \end{aligned}$$

and the Euclideaness of $\mathbb{E} \otimes W_{D^n}$.

3. Let σ be a permutation of $1, \dots, n+1$. We would like to show that

$$\begin{aligned} & \left(\sum (-1)^{i+1} \mathbf{D}_0 \int_{\partial_i \gamma} \omega \right) \circ \left(\text{id}_M \otimes W_{\sigma_{D^{n+1}}} \right) \\ &= \varepsilon_\sigma \sum (-1)^{i+1} \mathbf{D}_0 \int_{\partial_i \gamma} \omega \end{aligned} \tag{6}$$

We notice that

$$\left(\text{id}_M \otimes W_{\partial_i^{n+1}} \right) \circ \left(\text{id}_M \otimes W_{\sigma_{D^{n+1}}} \right)$$

$$= \left(\text{id}_M \otimes W_{(\tau_i^\sigma)_{D^n}} \right) \circ \left(\text{id}_M \otimes W_{\partial_{\sigma^{-1}(i)}^{n+1}} \right)$$

where τ_i^σ is the permutation of $1, \dots, n$ with

$$\begin{aligned} \tau_i^\sigma(1) &= \sigma(1), \dots, \tau_i^\sigma(\sigma^{-1}(i) - 1) = \sigma(\sigma^{-1}(i) - 1), \\ \tau_i^\sigma(\sigma^{-1}(i)) &= \sigma(\sigma^{-1}(i) + 1), \dots, \tau_i^\sigma(n) = \sigma(n + 1) \end{aligned}$$

We notice also that

$$\begin{aligned} & \left(\int \omega \right) \circ \left(\text{id}_M \otimes W_{(\tau_i^\sigma)_{D^n}} \right) \\ &= \varepsilon_{\tau_i^\sigma} \int \omega \end{aligned}$$

and

$$\varepsilon_{\tau_i^\sigma} = (-1)^{\sigma^{-1}(i)-i} \varepsilon_\sigma$$

Therefore (6) follows. □

References

- [1] R.W.R. Darling, *Differential Forms and Connections*, Cambridge University Press, Cambridge (1994).
- [2] E. Dubuc, Sur les modèles de la géométrie différentielle synthétique, *Cahiers de Topologie et Géométrie Différentielle*, **20** (1979), 231-279.
- [3] Alfred Frölicher, W. Bucher, Calculus in vector spaces without norm, *Lecture Notes in Mathematics*, **30** (1966), Springer-Verlag, BerlinHeidelberg.
- [4] Alfred Frölicher, Smooth structures, *Lecture Notes in Mathematics*, **962** (1982), 69-81, Springer-Verlag, Berlin and Heidelberg.
- [5] Alfred Frölicher, Cartesian closed categories and analysis of smooth maps, *Lecture Notes in Mathematics*, **1174** (1986), 43-51, Springer-Verlag, Berlin-Heidelberg.
- [6] Alfred Frölicher, Andreas Kriegl, *Linear Spaces and Differentiation Theory*, John Wiley and Sons, Chichester (1988).
- [7] Anders Kock, Convenient vector spaces embed into the Cahiers topos, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, **27** (1986), 3-17.

- [8] Anders Kock, Gonzalo E. Reyes, Corrigendum and addenda to the paper "Convenient vector spaces embed ...", *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, **28** (1987), 99-110.
- [9] Anders Kock, *Synthetic Differential Geometry*, Second Edition, Cambridge University Press, Cambridge (2006).
- [10] Ivan Kolář, Peter W. Michor, Jan Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin-Heidelberg (1993).
- [11] Andreas Kriegl, Peter W. Michor, *The Convenient Setting of Global Analysis*, American Mathematical Society, Rhode Island (1997).
- [12] René Lavendhomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer Academic Publishers, Dordrecht (1996).
- [13] Saunders MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York (1971).
- [14] Saunders MacLane, Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York (1992).
- [15] Ieke Moerdijk, Gonzalo E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer-Verlag, New York (1991).
- [16] Hirokazu Nishimura, Higher-order preconnections in synthetic differential geometry of jet bundles, *Beiträge zur Algebra und Geometrie*, **45** (2004), 677-696.
- [17] Hirokazu Nishimura, Synthetic vector analysis III -from vector analysis to differential forms-, *Far East Journal of Mathematical Sciences*, **32** (2009), 335-346.
- [18] Hirokazu Nishimura, A much larger class of Frölicher spaces that that of convenient vector spaces may embed into the Cahiers topos, *Far East Journal of Mathematical Sciences*, **35** (2009), 211-223.
- [19] Hirokazu Nishimura, Microlinearity in Frölicher spaces -beyond the regnant philosophy of manifolds-, *International Journal of Pure and Applied Mathematics*, **60** (2010), 15-24.
- [20] Hirokazu Nishimura, Differential geometry of microlinear Frölicher spaces I, *ArXiv*: 1002 3978.

- [21] Horst Schubert, *Categories*, Springer-Verlag, Berlin and Heidelberg (1972).
- [22] André Weil, Théorie des points proches sur les variétés différentiables, *Colloques Internationaux du Centre National de la Recherche Scientifique*, Strassbourg (1953), 111-117.