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ON AN ALGEBRA ASSOCIATED WITH A CIRCULAR QUIVER AND ITS PERIODIC PROJECTIVE BIMODULE RESOLUTION

By

Takahiko FURUYA

Abstract. In this paper, we describe the structure of a subalgebra $B_s^k(t)$ of a basic self-injective Nakayama algebra B_s^k, and we give a periodic projective bimodule resolution for $B_s^k(t)$.

Introduction

Let K be a field, s a positive integer and Γ the circular quiver (or cyclic quiver, oriented cycle) with s vertices $e_1, e_2, \ldots, e_s$ and s arrows $a_1, a_2, \ldots, a_s$ such that a_i starts at e_i and ends at e_{i+1}. Hence $a_i = e_{i+1}a_ie_i$ holds for each $1 \leq i \leq s$ in the path algebra $K\Gamma$, where we regard the subscripts i of e_i as modulo s. If we set $X = a_1 + a_2 + \cdots + a_s (\in K\Gamma)$, then $K\Gamma$ is the algebra generated by the elements $e_1, e_2, \ldots, e_s$, X, that is, $K\Gamma = K[e_1, e_2, \ldots, e_s, X]$. It is known that a basic self-injective Nakayama algebra over the field K is of the form $K\Gamma/J^k$, where $k \geq 2$ and J is the two-sided ideal of $K\Gamma$ generated by all arrows: $J = (X)$ (see [EH]). As in [EH], we denote $K\Gamma/J^k$ by B_s^k. In that paper, Erdmann and Holm give a periodic projective bimodule resolution of B_s^k and compute the Hochschild cohomology ring $\mathrm{HH}^*(B_s^k)$. Also similar results have been obtained by Bardzell, Locateli and Marcos ([BLM]). In this paper, we describe the structure of a subalgebra of B_s^k and give a projective bimodule resolution.

Let $B_s(t)$ be the subalgebra of $K\Gamma$ generated by the elements $e_1, e_2, \ldots, e_s$, X^t for $t \geq 1$, that is, $B_s(t) = K[e_1, e_2, \ldots, e_s, X^t]$. We define a subalgebra $B_s^k(t) = \pi(B_s(t))$, where $\pi : K\Gamma \to B_s^k$ is the natural map. If $t \geq k$ then $X^t = 0$ in B_s^k, so we assume $t < k$. Since $\mathrm{Ker}\, \pi|_{B_s(t)} = B_s(t) \cap J^k$, we have $B_s^k(t) \simeq B_s(t)/(B_s(t) \cap J^k)$. Therefore the paths whose length is a multiple of t and less than k

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give a basis of $B_s^k(t)$. Clearly $B_s^k(1) = B_s^k$ holds. In Section 1, we show that $B_s^k(t)$ is isomorphic to a direct sum of basic self-injective Nakayama algebras (Theorem 1). In Section 2, we give a periodic projective bimodule resolution of $B_s^k(t)$, which is given by means of some exact sequences of $B_s^k(t)$-bimodules (Theorem 2).

§1. The Structure of $B_s^k(t)$

Let K be a field and we fix two integers $s \geq 1$ and $k \geq 2$. If t is an integer such that $1 \leq t < k$, then we put $k = qt + r$ for some integers q and r such that $0 \leq r < t$. If $r \neq 0$ then we have

$$B_s^k(t) \simeq B_s^{(q+1)t}(t),$$

since $B_s(t) \cap J^k = B_s(t) \cap J^{(q+1)t}$. So it suffices to consider the case that k is a multiple of t: $k = qt$ for $q \geq 2$.

EXAMPLE 1.1. If $s = 6$ and $t = 10$, then we have $B_6^k(10) \simeq B_6^{30}(10)$ for all k such that $21 \leq k \leq 30$.

In this section, we will show that $B_s^{qt}(t)$ is isomorphic to a direct sum of copies of a self-injective Nakayama algebra. Let $d = \gcd(s, t)$, and we set $s = s'd$, $t = t'd$ for $s' \geq 1$, $t' \geq 1$. We regard the subscripts i of e_i as modulo s.

LEMMA 1.2. *The set* $\{X^{nt}e_{i+xt} \mid 0 \leq n < q, 1 \leq i \leq d, 0 \leq x < s'\}$ *gives a* K-*basis of* $B_s^{qt}(t)$.

PROOF. Since $\{X^{jt}e_i \mid 1 \leq i \leq s, 0 \leq j < q\}$ gives a K-basis of $B_s^{qt}(t)$, we have $\dim_K B_s^{qt}(t) = qs$. So we will show that the elements of $\{X^{nt}e_{i+xt} \mid 0 \leq n < q, 1 \leq i \leq d, 0 \leq x < s'\}$ are distinct each other. It suffices to prove that e_{i+xt} are distinct for $1 \leq i \leq d$ and $0 \leq x < s'$. If $e_{i+xt} = e_{j+yt}$ for $1 \leq i, j \leq d$ and $0 \leq x, y < s'$, then we have $i + xt \equiv j + yt \pmod{s}$. Hence, $(j - i) + (y - x)t \equiv 0 \pmod{s}$. Since $d | s$, it follows that $(j - i) + (y - x)t \equiv 0 \pmod{d}$. From $t \equiv 0 \pmod{d}$, we have $j - i \equiv 0 \pmod{d}$, which implies that $i = j$. Then we have $xt \equiv yt \pmod{s}$. Since $\gcd(s', t') = 1$, we have $y - x \equiv 0 \pmod{s'}$. Hence we have $x = y$. □

Next, we consider $B_s^{qd}(d)$. In order to distinguish the vertices and the arrows of $B_s^{qd}(d)$ from ones of $B_s^{qt}(t)$, we denote the idempotents of $B_s^{qd}(d)$ by f_i and the sum of all arrows of $B_s^{qd}(d)$ by Y.

LEMMA 1.3. *The set* $\{Y^{nd}f_{i+xd} \mid 0 \leq n < q, 1 \leq i \leq d, 0 \leq x < s'\}$ *gives a* K-*basis of* $B_s^{qd}(d)$.

PROOF. Since $\dim_K B_s^{qd}(d) = qs$, we will show that the elements of $\{Y^{nd}f_{i+xd} \mid 0 \leq n < q, 1 \leq i \leq d, 0 \leq x < s'\}$ are distinct each other. It suffices to prove that f_{i+xd} are distinct for $1 \leq i \leq d$ and $0 \leq x < s'$. If $f_{i+xd} = f_{j+yd}$ for $1 \leq i, j \leq d$ and $0 \leq x, y < s'$, then we have $i + xd \equiv j + yd \pmod{s}$. Thus $(j - i) + (y - x)d \equiv 0 \pmod{s}$. Since $d|s$, we have $(j - i) + (y - x)d \equiv 0 \pmod{d}$. So $j - i \equiv 0 \pmod{d}$ which implies $i = j$. Then we have $xd \equiv yd \pmod{s}$. So it follows that $x \equiv y \pmod{s'}$. Thus we have $x = y$. □

By Lemmas 1.2 and 1.3, we have the isomorphism of K-vector spaces

$$\Phi : B_s^{qt}(t) \to B_s^{qd}(d);\ X^{nt}e_{i+xt} \mapsto Y^{nd}f_{i+xd} \quad (0 \leq n < q, 1 \leq i \leq d, 0 \leq x < s').$$

PROPOSITION 1.4. Φ *is an isomorphism of* K-*algebras.*

PROOF. We will show that

$$\Phi((X^{rt}e_{i+xt})(X^{pt}e_{j+yt})) = \Phi(X^{rt}e_{i+xt})\Phi(X^{pt}e_{j+yt}) \tag{1.1}$$

for $X^{rt}e_{i+xt}, X^{pt}e_{j+yt} \in B_s^{qt}(t)$ where $0 \leq r, p < q$, $1 \leq i, j \leq d$ and $0 \leq x, y < s'$. Since $(X^{rt}e_{i+xt})(X^{pt}e_{j+yt}) = (X^{rt}e_{i+xt})(e_{j+(p+y)t}X^{pt}e_{j+yt})$, we consider the following two cases.

(i) Case $e_{i+xt} = e_{j+(p+y)t}$. The left hand of (1.1) equals

$$\Phi((X^{rt}e_{j+(p+y)t})(X^{pt}e_{j+yt})) = \Phi(X^{(r+p)t}e_{j+yt})$$
$$= \begin{cases} Y^{(r+p)d}f_{j+yd} & \text{if } r + p < q, \\ 0 & \text{if } r + p \geq q. \end{cases}$$

By the assumption, we have $i + xt \equiv j + (p + y)t \pmod{s}$. By the similar argument as in Lemma 1.2, we have $i = j$, so $xt \equiv (p + y)t \pmod{s}$. Hence we have $(p + y - x)t' \equiv 0 \pmod{s'}$. By $\gcd(s', t') = 1$, it follows that $p + y - x \equiv 0 \pmod{s'}$, so $xd \equiv (p + y)d \pmod{s}$. Thus we have $i + xd \equiv j + (p + y)d \pmod{s}$, which implies $f_{i+xd} = f_{j+(p+y)d}$. Then the right hand of (1.1) equals

$$(Y^{rd}f_{i+xd})(Y^{pd}f_{j+yd}) = (Y^{rd}f_{j+(p+y)d})(Y^{pd}f_{j+yd})$$
$$= \begin{cases} Y^{(r+p)d}f_{j+yd} & \text{if } r + p < q, \\ 0 & \text{if } r + p \geq q. \end{cases}$$

(ii) Case $e_{i+xt} \neq e_{j+(p+y)t}$. Since $e_{i+xt}e_{j+(p+y)t} = 0$, the left hand of (1.1) equals $\Phi(0) = 0$. We will show $f_{i+xd} \neq f_{j+(p+y)d}$. If $f_{i+xd} = f_{j+(p+y)d}$, then $i + xd \equiv j + (p+y)d \pmod s$. So we have $i = j$ by the similar argument as in Lemma 1.3. Hence $xd \equiv (p+y)d \pmod s$, which implies $xt \equiv (p+y)t \pmod s$. So $i + xt \equiv j + (p+y)t \pmod s$ which means $e_{i+xt} = e_{j+(p+y)t}$. This is contradiction. Then the right hand of (1.1) equals $(Y^{rd}f_{i+xd})(Y^{pd}f_{j+yd}) = (Y^{rd}f_{i+xd})(f_{j+(p+y)d}Y^{pd}f_{j+yd}) = 0$.

Hence (1.1) is proved. Also, by the definition of Φ, we have $\Phi(\sum_{i,x} e_{i+xt}) = \sum_{i,x} f_{i+xd}$, the identity of $B_s^{qd}(d)$, where i, x range over $1 \leq i \leq d$, $1 \leq x < s'$, respectively. Thus Φ is an isomorphism of K-algebras. □

Let

$$A_i = \bigoplus_{\substack{0 \leq n < q \\ 0 \leq x < s'}} KY^{nd}f_{i+xd}$$

for $1 \leq i \leq d$. Each A_i is a K-subspace of $B_s^{qd}(d)$, and it is easy to see that A_i is a two-sided ideal of $B_s^{qd}(d)$. Thus $B_s^{qd}(d)$ is the direct sum of all the two-sided ideals A_i:

$$B_s^{qd}(d) = \bigoplus_{i=1}^{d} A_i. \tag{1.2}$$

We will show that each A_i is isomorphic to a basic self-injective Nakayama algebra. In the following, in order to distinguish A_i from $B_{s'}^q$, we will denote the idempotents of $B_{s'}^q$ by g_i and the sum of all arrows of $B_{s'}^q$ by Z.

PROPOSITION 1.5. *There exists an isomorphism of K-algebras $A_i \simeq B_{s'}^q$ for every i $(1 \leq i \leq d)$.*

PROOF. Fix i such that $1 \leq i \leq d$. Since the set $\{Y^{nd}f_{i+xd} \mid 0 \leq n < q, 0 \leq x < s'\}$ is a K-basis of A_i and the set $\{Z^n g_x \mid 0 \leq n < q, 1 \leq x \leq s'\}$ is a K-basis of $B_{s'}^q$, the map

$$\Psi : B_{s'}^q \to A_i; \quad Z^n g_x \mapsto Y^{nd}f_{i+(x-1)d} \quad (0 \leq n < q, 1 \leq x \leq s')$$

is an isomorphism of K-vector spaces. We will show that

$$\Psi((Z^r g_x)(Z^p g_y)) = \Psi(Z^r g_x)\Psi(Z^p g_y) \tag{1.3}$$

for $0 \leq r, p < q$, $1 \leq x, y \leq s'$. Since $(Z^r g_x)(Z^p g_y) = (Z^r g_x)(g_{p+y}Z^p g_y)$, we consider the following two cases.

(i) Case $g_x = g_{p+y}$. The left hand of (1.3) equals

$$\Psi((Z^r g_{p+y})(Z^p g_y)) = \Psi(Z^{r+p} g_y) = \begin{cases} Y^{(r+p)d} f_{i+(y-1)d} & \text{if } r+p < q, \\ 0 & \text{if } r+p \geq q. \end{cases}$$

By the assumption, we have $x \equiv p + y \pmod{s'}$. Hence $(x-1)d \equiv (p+y-1)d \pmod{s}$, so $i + (x-1)d \equiv i + (p+y-1)d \pmod{s}$. Thus we obtain $f_{i+(x-1)d} = f_{i+(p+y-1)d}$. Then the right hand of (1.3) equals

$$(Y^{rd} f_{i+(x-1)d})(Y^{pd} f_{i+(y-1)d}) = (Y^{rd} f_{i+(p+y-1)d})(Y^{pd} f_{i+(y-1)d}) = \begin{cases} Y^{(r+p)d} f_{i+(y-1)d} & \text{if } r+p < q, \\ 0 & \text{if } r+p \geq q. \end{cases}$$

(ii) Case $g_x \neq g_{p+y}$. Clearly the left hand of (1.3) equals $\Psi(0) = 0$, since $g_x g_{p+y} = 0$. We will show $f_{i+(x-1)d} \neq f_{i+(p+y-1)d}$. If $f_{i+(x-1)d} = f_{i+(p+y-1)d}$, then we have $i + (x-1)d \equiv i + (p+y-1)d \pmod{s}$. Hence $(p+y-x)d \equiv 0 \pmod{s}$, which means $p + y - x \equiv 0 \pmod{s'}$. Thus $x \equiv p + y \pmod{s'}$, so we have $g_x = g_{p+y}$. This contradicts the assumption. Then the right hand of (1.3) equals $(Y^{rd} f_{i+(x-1)d})(Y^{pd} f_{i+(y-1)d}) = (Y^{rd} f_{i+(x-1)d})(f_{i+(p+y-1)d} Y^{pd} f_{i+(y-1)d}) = 0$.

Hence (1.3) is proved. Also, we have $\Psi(\sum_{x=1}^{s'} g_x) = \sum_{x=1}^{s'} f_{i+(x-1)d}$, the identity of A_i. Therefore, Ψ is an isomorphism of K-algebras. □

By the discussion in the beginning of this section, (1.2) and Propositions 1.4 and 1.5, we have the following structure theorem of $B_s^k(t)$ for any $s \geq 1$, $k \geq 2$ and t such that $1 \leq t < k$:

THEOREM 1. *Let s, t and k be integers satisfying $s \geq 1$, $k \geq 2$ and $1 \leq t < k$, and let $d := \gcd(s, t)$ and $s' := s/d$. If q is the least integer such that $k/t \leq q$, then we have the following isomorphism of K-algebras*

$$B_s^k(t) \simeq \bigoplus_{i=1}^{d} B_{s'}^q \quad (\textit{the direct sum of } d \textit{ copies of } B_{s'}^q).$$

PROOF. By the discussion in the beginning of this section, we have $B_s^k(t) \simeq B_s^{qt}(t)$. Moreover we have

$$\begin{aligned} B_s^{qt}(t) &\simeq B_s^{qd}(d) && \text{by Proposition 1.4} \\ &= \bigoplus_{i=1}^{d} A_i && \text{by (1.2)} \\ &\simeq \bigoplus_{i=1}^{d} B_{s'}^{q} && \text{by Proposition 1.5.} \end{aligned} \qquad \square$$

REMARK 1.6. If $d = \gcd(s,t) = 1$, then $B_s^k(t)$ is isomorphic to the basic self-injective Nakayama algebra B_s^q, where q is the least integer such that $k/t \leq q$.

In [EH, Section 4.2], Erdmann and Holm give a projective bimodule resolution of B_s^k of period $2 \cdot \mathrm{lcm}(k,s)/k$, where $s \geq 1$ and $k \geq 2$. From this result and Theorem 1, we have the following:

COROLLARY 1.7. *Let $s \geq 1$, $k \geq 2$ and t be integers such that $1 \leq t < k$, and let $d = \gcd(s,t)$ and $s' = s/d$. If q is the least integer such that $k/t \leq q$, then $B_s^k(t)$ has a projective bimodule resolution of period $2 \cdot \mathrm{lcm}(q,s')/q$, and the Hochschild cohomology ring $\mathrm{HH}^*(B_s^k(t))$ is isomorphic to the direct sum of d copies of the Hochschild cohomology ring $\mathrm{HH}^*(B_{s'}^q)$.*

EXAMPLE 1.8. Let $s = 28$, $k = 68$ and $t = 35$. Then it follows that $d = \gcd(28,35) = 7$, $s' = 4$ and $q = 2$. By Theorem 1, we have

$$B_{28}^{68}(35) \simeq \underbrace{B_4^2 \oplus \cdots \oplus B_4^2}_{7 \text{ copies}}.$$

Also, by Corollary 1.7, there is a projective bimodule resolution of $B_{28}^{68}(35)$ of period $2 \cdot \mathrm{lcm}(2,4)/2 = 4$. Moreover if K is a field with $\mathrm{Char}\, K = 0$, then by Theorem in [EH, Section 4.8], it follows that the even Hochschild cohomology ring $\mathrm{HH}^{ev}(B_4^2)$ is the commutative graded K-algebra with generators y_0, y_2 and y_4 modulo the ideal generated by the elements y_0, $y_2^2 - y_4 \cdot y_0$, y_2 and $y_2^2 - y_2^2 \cdot y_0$, where $\deg y_i = i$. Hence we have $\mathrm{HH}^{ev}(B_4^2) \simeq K[y_0, y_2, y_4]/(y_0, y_2^2 - y_4 \cdot y_0, y_2, y_2^2 - y_2^2 \cdot y_0) \simeq K[y_4]$. Thus $\mathrm{HH}^{ev}(B_4^2)$ is isomorphic to the polynomial ring $K[x]$. Therefore we have the following isomorphism of graded rings:

$$\mathrm{HH}^{ev}(B_{28}^{68}(35)) \simeq \underbrace{K[x] \oplus \cdots \oplus K[x]}_{7 \text{ copies}}, \quad \deg x = 4.$$

Note that $B_{28}^{k}(35) \simeq B_{28}^{68}(35)$ for all k such that $36 \leq k \leq 70$.

§2. A Periodic Projective Bimodule Resolution of $B_s^k(t)$

The projective bimodule resolution of B_s^k in [EH, Section 4.2] is given by means of the exact sequence which consists of four terms, where $s \geq 1$ and $k \geq 2$. But all maps of the exact sequence are not explicitly given there. In this section, we explicitly give the maps of the exact sequence. Since $B_s^k(1) = B_s^k$, we will consider a projective bimodule resolution of $B_s^k(t)$, in general, for integers $s \geq 1$, $k \geq 2$ and t such that $1 \leq t < k$.

Let q be the least integer such that $k/t \leq q$. Since $B_s^k(t) \simeq B_s^{qt}(t)$, we consider $B_s^{qt}(t)$. Denote $B_s^{qt}(t)$ by B, and set $qt = ns + \overline{qt}$ for some integer n and $\overline{qt}$ such that $0 \leq \overline{qt} < s$. We will denote $\otimes_K$ by $\otimes$ and the enveloping algebra $B \otimes B^{\mathrm{op}}$ of B by B^e. We regard the subscripts i of e_i as modulo s. Define left B^e-modules, equivalently B-bimodules,

$$Q_0 = \bigoplus_{i=1}^{s} Be_i \otimes e_iB, \quad Q_1 = \bigoplus_{i=1}^{s} Be_{i+t} \otimes e_iB,$$

and let $\beta : B \to B$ be the automorphism induced by the automorphism $B_s^{qt} \to B_s^{qt}$ defined by $e_i \mapsto e_{i-1}$ and $a_i \mapsto a_{i-1}$. Note that the order of β equals s. Moreover, we define a left B^e-module ${}_1B_{\beta^{-\overline{qt}}}$ as follows: ${}_1B_{\beta^{-\overline{qt}}}$ has the underlying space B, and the action on the left is the usual one. The action on the right is given by $b * x = b\beta^{-\overline{qt}}(x)$ for $b \in {}_1B_{\beta^{-\overline{qt}}}$ and $x \in B$. Here, note that $\beta^{-\overline{qt}} = \beta^{-qt}$.

Lemma 2.1. *We can define a left B^e-homomorphism $\phi : Q_1 \to Q_0$ by*

$$\phi(e_{i+t} \otimes e_i) = e_{i+t}(X^t \otimes 1 - 1 \otimes X^t)e_i \quad \textit{for } 1 \leq i \leq s.$$

And if we define a left B-homomorphism $\kappa : {}_1B_{\beta^{-\overline{qt}}} \to Q_1$ by

$$\kappa(e_i) = e_i\left(\sum_{j=0}^{q-1} X^{jt} \otimes X^{(q-j-1)t}\right)e_{i-qt} \quad \textit{for } 1 \leq i \leq s,$$

then κ is a right B-homomorphism. Thus κ is a left B^e-homomorphism.

Proof. Since $X^t e_i = a_{i+t-1} \cdots a_{i+1}a_i = e_{i+t}X^t$, it follows that $\phi(e_{i+t} \otimes e_i)$ is an element of Q_0 for each $1 \leq i \leq s$. It is clear that we can make ϕ a left B^e-homomorphism. Also, since $X^{mt}e_i = a_{i+mt-1} \cdots a_{i+1}a_i = e_{i+mt}X^{mt}$ for each $1 \leq m \leq q-1$, it follows that $\kappa(e_i)$ is an element of Q_1 for each $1 \leq i \leq s$. We will show that κ is a right B-homomorphism. Since B is generated by e_i $(1 \leq i \leq s)$

and X^t, we show that $\kappa(e_i * e_j) = \kappa(e_i)e_j$ and $\kappa(e_i * X^t) = \kappa(e_i)X^t$ for each $1 \le i \le s$ and $1 \le j \le s$. It is easy to see that the first equation holds. We will show that the second equation holds. Since $\kappa(e_i * X^t) = \kappa(e_i\beta^{-\overline{qt}}(X^t)) = \kappa(e_iX^t) = \kappa(X^te_{i-t}) = X^t\kappa(e_{i-t})$, it suffices to show that $X^t\kappa(e_{i-t}) = \kappa(e_i)X^t$ for each $1 \le i \le s$. We put $Y := X^t$ for simplicity. Here, note that $Ye_i = e_{i+t}Y$ and $Y^q = 0$.

$$
\begin{aligned}
&Y\kappa(e_{i-t}) - \kappa(e_i)Y \\
&= Ye_{i-t}\left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j-1}\right)e_{i-t-qt} - e_i\left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j-1}\right)e_{i-qt}Y \\
&= e_i\left(\sum_{j=0}^{q-1} Y^{j+1} \otimes Y^{q-j-1}\right)e_{i-t-qt} - e_i\left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j}\right)e_{i-t-qt} \\
&= e_i\left(\sum_{j=0}^{q-1}(Y^{j+1} \otimes Y^{q-j-1} - Y^j \otimes Y^{q-j})\right)e_{i-t-qt} \\
&= e_i(Y^q \otimes 1 - 1 \otimes Y^q)e_{i-t-qt} = e_i(0-0)e_{i-t-qt} = 0. \qquad \square
\end{aligned}
$$

THEOREM 2. *There exists the exact sequence of left B^e-modules:*

$$
0 \to {}_1B_{\beta^{-\overline{qt}}} \xrightarrow{\kappa} Q_1 \xrightarrow{\phi} Q_0 \xrightarrow{\pi} B \to 0, \tag{2.4}
$$

where π is the multiplication. Thus we have the periodic projective B^e-resolution of period $2 \cdot \mathrm{lcm}(s', q)/q$, where $s' = s/\gcd(s,t)$ as in Section 1.

We prepare the following lemma for the proof of Theorem 2. In the rest of this section, we put $Y := X^t$ as in the proof above. Here, we again note that $Ye_i = e_{i+t}Y$ and $Y^q = 0$.

LEMMA 2.2. *The sequence (2.4) is a complex of left B^e-modules, that is, $\pi\phi = \phi\kappa = 0$.*

PROOF. For $1 \le i \le s$, we have

$$
(\pi\phi)(e_{i+t} \otimes e_i) = \pi(e_{i+t}(Y \otimes 1 - 1 \otimes Y)e_i) = e_{i+t}(Y - Y)e_i = 0
$$

and

$$(\phi\kappa)(e_i) = \phi\left(e_i\left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j-1}\right)e_{i-qt}\right)$$

$$= e_i\left(\sum_{j=0}^{q-1} Y^j(Y \otimes 1 - 1 \otimes Y)Y^{q-j-1}\right)e_{i-qt}$$

$$= e_i\left(\sum_{j=0}^{q-1}(Y^{j+1} \otimes Y^{q-j-1} - Y^j \otimes Y^{q-j})\right)e_{i-qt} = 0. \qquad \square$$

Proof of Theorem 2. Define left B-homomorphisms $h_{-1} : B \to Q_0$, $h_0 : Q_0 \to Q_1$ and $h_1 : Q_1 \to {}_1B_{\beta^{-\bar{q}t}}$ by

$$h_{-1}(x) = x\left(\sum_{j=1}^{s} e_j \otimes e_j\right) \quad \text{for } x \in B,$$

$$h_0(e_i \otimes e_i Y^m) = \begin{cases} 0 & \text{if } m = 0, \\ -e_i\left(\sum_{j=0}^{m-1} Y^j \otimes Y^{m-j-1}\right)e_{i-mt} & \text{if } 1 \le m \le q-1, \end{cases}$$

$$h_1(e_{i+t} \otimes e_i Y^m) = \begin{cases} 0 & \text{if } 0 \le m \le q-2, \\ e_{i+t} & \text{if } m = q-1. \end{cases}$$

It is easy to see that $h_0(e_i \otimes e_i Y^m)$ is an element of Q_1 for all $1 \le i \le s$. We will show that $\{h_{-1}, h_0, h_1\}$ is a contracting homotopy of (2.4).

(a) $\pi h_{-1} = id_B$: For $x \in B$, we have

$$(\pi h_{-1})(x) = \pi\left(x\sum_{j=1}^{s} e_j \otimes e_j\right) = x\sum_{j=1}^{s} e_j = x.$$

(b) $h_{-1}\pi + \phi h_0 = id_{Q_0}$: For $1 \le i \le s$, we have

$$(h_{-1}\pi + \phi h_0)(e_i \otimes e_i) = h_{-1}(e_i) + \phi(0) = e_i\left(\sum_{j=1}^{s} e_j \otimes e_j\right) = e_i \otimes e_i.$$

Also, for $1 \le i \le s$ and $1 \le m \le q-1$, we have

$$(h_{-1}\pi + \phi h_0)(e_i \otimes e_i Y^m)$$

$$= h_{-1}(e_i Y^m) + \phi\left(-e_i\left(\sum_{j=0}^{m-1} Y^j \otimes Y^{m-j-1}\right)e_{i-mt}\right)$$

$$= Y^m e_{i-mt} \otimes e_{i-mt} - e_i \left(\sum_{j=0}^{m-1} (Y^{j+1} \otimes Y^{m-j-1} - Y^j \otimes Y^{m-j}) \right) e_{i-mt}$$

$$= e_i \otimes e_i Y^m.$$

Hence the desired equation holds.

(c) $h_0\phi + \kappa h_1 = id_{Q_1}$: For $1 \leq i \leq s$, we have

$$\begin{aligned}(h_0\phi + \kappa h_1)(e_{i+t} \otimes e_i) &= h_0(e_{i+t}(Y \otimes 1 - 1 \otimes Y)e_i) + \kappa(0) \\ &= h_0(e_{i+t} Y e_i \otimes e_i - e_{i+t} \otimes e_{i+t} Y) \\ &= e_{i+t} \otimes e_i.\end{aligned}$$

Also, for $1 \leq i \leq s$ and $1 \leq m \leq q-2$ we have

$$\begin{aligned}(h_0\phi + \kappa h_1)(e_{i+t} \otimes e_i Y^m) &= h_0(e_{i+t}(Y \otimes 1 - 1 \otimes Y)e_i Y^m) + \kappa(0) \\ &= h_0(e_{i+t} Y e_i \otimes e_i Y^m - e_{i+t} \otimes e_{i+t} Y^{m+1}) \\ &= e_{i+t} Y \left(-e_i \left(\sum_{j=0}^{m-1} Y^j \otimes Y^{m-j-1} \right) e_{i-mt} \right) \\ &\quad - \left(-e_{i+t} \left(\sum_{j=0}^{m} Y^j \otimes Y^{m-j} \right) e_{i-mt} \right) \\ &= e_{i+t} \otimes e_i Y^m.\end{aligned}$$

Moreover, for $1 \leq i \leq s$, we have

$$\begin{aligned}&(h_0\phi + \kappa h_1)(e_{i+t} \otimes e_i Y^{q-1}) \\ &= h_0(e_{i+t}(Y \otimes 1 - 1 \otimes Y)e_i Y^{q-1}) + \kappa(e_{i+t}) \\ &= h_0(e_{i+t} Y e_i \otimes e_i Y^{q-1} - e_{i+t} \otimes e_{i+t} Y^q) + \kappa(e_{i+t}) \\ &= e_{i+t} Y h_0(e_i \otimes e_i Y^{q-1}) + \kappa(e_{i+t}) \\ &= e_{i+t} Y \left(-e_i \left(\sum_{j=0}^{q-2} Y^j \otimes Y^{q-j-2} \right) e_{i-qt+t} \right) \\ &\quad + e_{i+t} \left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j-1} \right) e_{i+t-qt} \\ &= e_{i+t} \otimes e_i Y^{q-1}.\end{aligned}$$

Hence we obtain the desired equation.

(d) $h_1\kappa = id_{{}_1B_{\beta^{-\overline{qt}}}}$: For $1 \le i \le s$, we have

$$(h_1\kappa)(e_i) = h_1\left(e_i\left(\sum_{j=0}^{q-1} Y^j \otimes Y^{q-j-1}\right)e_{i-qt}\right)$$

$$= e_i\left(\sum_{j=0}^{q-1} Y^j h_1(e_{i-jt} \otimes e_{i-jt-t}Y^{q-j-1})\right) = e_i.$$

So we get the desired equation.

Consequently (2.4) is exact. Furthermore, since the order of β equals s, the order of $\beta^{-\overline{qt}} = \beta^{-qt}$ equals $s/\gcd(s, qt) = \operatorname{lcm}(s, qt)/qt = \operatorname{lcm}(s', q)/q$. Hence we have the periodic projective B^e-resolution of B of period $2 \cdot \operatorname{lcm}(s', q)/q$. □

REMARK 2.3. We get an immediate consequence that (2.4) is left B-split.

In particular, we consider the case $t = 1$, that is, B_s^k with the automorphism β. Let $C = B_s^k$ for $s \ge 1$ and $k \ge 2$, and we put

$$R_0 = \bigoplus_{i=1}^{s} Ce_i \otimes e_iC, \quad R_1 = \bigoplus_{i=1}^{s} Ce_{i+1} \otimes e_iC.$$

By setting $t = 1$ in Theorem 2, we have the following:

COROLLARY 2.4 ([EH, Section 4.2]). *There is the exact sequence of left C^e-modules*:

$$0 \to {}_1C_{\beta^{-\bar{k}}} \xrightarrow{\kappa} R_1 \xrightarrow{\phi} R_0 \xrightarrow{\pi} C \to 0,$$

where left C^e-homomorphisms ϕ and κ are given by

$$\phi(e_{i+1} \otimes e_i) = e_{i+1}(X \otimes 1 - 1 \otimes X)e_i,$$

$$\kappa(e_i) = e_i\left(\sum_{j=0}^{k-1} X^j \otimes X^{k-j-1}\right)e_{i-k} \quad \textit{for } 1 \le i \le s$$

and π is the multiplication. From this sequence, we obtain the periodic projective C^e-resolution of period $2 \cdot \operatorname{lcm}(s, k)/k$.

REMARK 2.5. In [S], the concrete form of κ is described. Also, an exact sequence similar to one in Corollary 2.4 appears in [KSS].

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Takahiko Furuya
Department of Mathematics, Tokyo University of Science
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: j1103705@ed.kagu.tus.ac.jp