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A COMPLETE SEQUENCE OF ORTHOGONAL SUBSETS IN $H^M(\boldsymbol{R}^n)$ AND A NUMERICAL APPROXIMATION FOR BOUNDARY VALUE PROBLEMS

By

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Introduction

Let us consider the boundary value problem of partial differential equations on a domain Ω in $\boldsymbol{R}^n$:

$$\text{(P)} \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = 0 & \text{on } \partial\Omega \ (j = 1, \dots, \mu), \end{cases}$$

subject to the following two conditions:

(1) the energy estimate holds for the adjoint problem in $H^M(\Omega)$,

(2) there exists a continuous map from $H^M(\Omega)$ to $H^M(\boldsymbol{R}^n)$.

In our previous work ([1]), we have discussed the existence of weak solutions in $L^2(\Omega)$ and its approximations using a basis S of $H^M(\Omega)$. The problem we address in this paper is the construction of this set S. When Ω is bounded, we take $a > 0$ large enough so that $\Omega \Subset \Omega_1 = (-a\pi, a\pi)^n$. Then, since $S = \{\exp(i\alpha \cdot x/a) \mid \alpha \in \boldsymbol{Z}^n\}$ ($\boldsymbol{Z} = \{0, \pm 1, \pm 2, \dots\}$) is a basis of $H^M(\Omega_1)$, $S|_\Omega$ is a basis of $H^M(\Omega)$, under the condion (2) (see [1]).

Therefore, our main concern is the case where Ω is unbounded. This is easily reduced to the case where $\Omega = \boldsymbol{R}^n$. In fact, by virtue of the assumption (2), if S is a basis of $H^M(\boldsymbol{R}^n)$, $S|_\Omega$ is a basis of $H^M(\Omega)$. As a preliminary to the construction of S, we introduce the notion of a complete sequence of orthogonal subsets in §0. We then construct complete sequences of orthogonal subsets $\{\Phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$ ($N \in \boldsymbol{N}$) in $L^2(\boldsymbol{R}^n)$ and $\{\phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$ ($N \in \boldsymbol{N}$) in $H^M(\boldsymbol{R}^n)$ in §1 and §2, respectively. Our ultimate aim (Theorems 3.1 and 3.2) will be proved in §3.

§0. A Complete Sequence of Orthogonal Subsets in a Hilbert Space

Let H be a Hilbert space. Let $\{S_N\}$ ($N \in \boldsymbol{N}$) be a sequence of subsets in H. Let us say that $\{S_N\}$ ($N \in \boldsymbol{N}$) is a *sequence of orthogonal subsets* in H, if

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$$S_N = \{\phi_{N,j}\ (j = 1, 2, \ldots)\}, \quad \phi_{N,j} \neq 0, \quad (\phi_{N,j}, \phi_{N,k})_H = 0 \quad (j \neq k),$$

where $(\,,\,)_H$ denotes the inner product of H. Let us say that $\{S_N\}$ $(N \in \boldsymbol{N})$ is a *complete sequence of orthogonal subsets* in H, if there exists a series $\{f_N\}$ for any $f \in H$ such that

$$f_N \in \langle S_N \rangle, \quad f_N \to f \quad \text{in } H,$$

where $\langle S \rangle$ denotes the set of linear combinations of finite elements of S.

From the definition, we have

LEMMA 0.1. *Let $\{S_N\}$ $(N \in \boldsymbol{N})$ be a complete sequence of orthogonal subsets in H, then $\langle \bigcup_{\ell=N}^{\infty} S_\ell \rangle$ is dense in H.*

LEMMA 0.2. *Let $\{S_N\}$ $(N \in \boldsymbol{N})$ be a sequence of orthogonal subsets in H. Set*

$$F_N = \sum_{j=1}^{\infty} \|\phi_{N,j}\|_H^{-2} (f, \phi_{N,j})_H \phi_{N,j}$$

for $f \in H$. Then $\{S_N\}$ $(N \in \boldsymbol{N})$ is a complete sequence of orthogonal subsets in H, iff it holds

$$F_N \to f \quad \text{in } H \ (N \to \infty).$$

PROOF. Let $\{S_N\}$ $(N \in \boldsymbol{N})$ be a complete sequence of orthogonal subsets in H, then there exists $\{f_N\}$ for $f \in H$ such that

$$f_N \in \langle S_N \rangle, \quad f_N \to f \quad \text{in } H.$$

From the definition of F_N, it holds

$$\|F_N - f\|_H \leqq \|f_N - f\|_H,$$

which means

$$F_N \to f \quad \text{in } H \ (N \to \infty).$$

Conversely, let

$$F_N = \sum_{j=1}^{\infty} \|\phi_{N,j}\|_H^{-2} (f, \phi_{N,j})_H \phi_{N,j}$$

satisfy

$$F_N \to f \quad \text{in } H \ (N \to \infty).$$

From the definition of F_N, we can define

$$f_N = \sum_{j=1}^{K(N)} \|\phi_{N,j}\|_H^{-2}(f, \phi_{N,j})_H \phi_{N,j} \in \langle S_N \rangle$$

such that

$$\|f_N - F_N\|_H < 2^{-N}.$$

Therefore, we have

$$f_N \in \langle S_N \rangle$$

and

$$\|f_N - f\|_H \leqq \|f_N - F_N\|_H + \|F_N - f\|_H \to 0. \qquad \square$$

When $\{S_N\}$ $(N = 1, 2, \ldots)$ is a complete sequence of orthogonal subsets in H, we say that $\{F_N\}$ $(N = 1, 2, \ldots)$ is a *sequence of quasi-Fourier series* of $f \in H$, corresponding to $\{S_N\}$ $(N = 1, 2, \ldots)$, where

$$F_N \doteq \sum_{j=1}^{\infty} \|\phi_{N,j}\|^{-2}(f, \phi_{N,j})_H \phi_{N,j}.$$

Let V_N be a closed subspace in H with basis S_N, then F_N is the orthogonal projection of f on V_N.

From the definition, we have

LEMMA 0.3. *Let $\{S_N\}$ $(N \in \boldsymbol{N})$ be a complete sequence of orthogonal subsets in H. Then any infinite subsequence $\{S_{N(\lambda)}\}$ $(\lambda \in \boldsymbol{N})$, satisfying $N(1) < N(2) < \cdots$ is a complete sequence of orthogonal subsets in H.*

§ 1. A Complete Sequence of Orthogonal Subsets in $L^2(\boldsymbol{R}^n)$

1.1. $\{\Phi_{N,k}\}$ in $L^2(\boldsymbol{R}^n)$ Let $\gamma \in C^\infty(\boldsymbol{R})$ satisfy

$$\gamma(t) = \begin{cases} 1 & (|t| < 1/2) \\ 0 & (|t| > 1), \end{cases}$$

and set $\gamma_A(x) = \gamma(x_1/A) \cdots \gamma(x_n/A)$ for $A \in \boldsymbol{N}$. Set

$$f_A(x) = \gamma_A(x) f(x)$$

for $f \in L^2(\boldsymbol{R}^n)$, then we have

$$f_A \in L^2(\boldsymbol{R}^n) \cap L^1(\boldsymbol{R}^n), \quad f_A \to f \quad \text{in } L^2(\boldsymbol{R}^n) \ (A \to \infty).$$

Let

$$\hat{f}_A(\xi) = \int f_A(x) e^{-ix\cdot\xi}\, dx$$

be the Fourier transform of f_A, then we have

$$\hat{f}_A \in L^2(\boldsymbol{R}^n) \cap \mathscr{B}(\boldsymbol{R}^n),$$

where

$$\mathscr{B}(\boldsymbol{R}^n) = \{f \in C^\infty(\boldsymbol{R}^n) \mid \partial_x^\nu f(x) \text{ is bounded in } \boldsymbol{R}^n \text{ for any } \nu\}.$$

Moreover, for $B \in \boldsymbol{N}$, we have

$$(2\pi)^{-n} \int_{\Omega_B} \hat{f}_A(\xi) e^{ix\cdot\xi}\, d\xi \to f_A(x) \quad (B \to \infty) \quad \text{in } L^2(\boldsymbol{R}^n) \ (\Omega_B = (-B, B)^n).$$

Set

$$g_{A,N}(\xi) = \hat{f}_A(k/N) \quad \text{if } \xi \in \Omega_{N,k},$$

where $\Omega_{N,k} = (k_1/N, (k_1+1)/N) \times \cdots \times (k_n/N, (k_n+1)/N)$. Since

$$\begin{aligned} \sup_{\xi\in\Omega_{N,k}} |g_{A,N}(\xi) - \hat{f}_A(\xi)| &= \sup_{\xi\in\Omega_{N,k}} |\hat{f}_A(k/N) - \hat{f}_A(\xi)| \\ &\leqq (1/N) \sup_\xi (|\partial_{\xi_1}\hat{f}_A(\xi)| + \cdots + |\partial_{\xi_n}\hat{f}_A(\xi)|), \end{aligned}$$

we have

$$g_{A,N}(\xi) \to \hat{f}_A(\xi) \quad (N \to \infty) \text{ (uniformly in } \boldsymbol{R}^n).$$

Hence we have

$$(2\pi)^{-n} \int_{\Omega_B} g_{A,N}(\xi) e^{ix\cdot\xi}\, d\xi \to (2\pi)^{-n} \int_{\Omega_B} \hat{f}_A(\xi) e^{ix\cdot\xi}\, d\xi \quad (N \to \infty) \quad \text{in } L^2(\boldsymbol{R}^n).$$

From the definition of $g_{A,N}(\xi)$, we have

$$(2\pi)^{-n} \int_{\Omega_B} g_{A,N}(\xi) e^{ix\cdot\xi}\, d\xi = \sum\nolimits_{-NB\leqq k_1,\dots,k_n<NB} \hat{f}_A(k/N)(2\pi)^{-n} \int_{\Omega_{N,k}} e^{ix\cdot\xi}\, d\xi.$$

Set

$$\Phi_{N,k}(x) = (2\pi)^{-n} \int_{\Omega_{N,k}} e^{ix\cdot\xi}\, d\xi,$$

then we have

$$(2\pi)^{-n}\int_{\Omega_B} g_{A,N}(\xi)e^{ix\cdot\xi}\,d\xi = \sum\nolimits_{-NB\leqq k_1,\dots,k_n<NB} \hat{f}_A(k/N)\Phi_{N,k}(x) \in S_N,$$

where $S_N = \{\Phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$.

By the way, we have

$$\begin{aligned}\Phi_{N,k}(x) &= (2\pi)^{-n}\int_{\Omega_{N,k}} e^{ix\cdot\xi}\,d\xi \\ &= (2\pi)^{-n}e^{ik\cdot x/N}\int_{\Omega_{N,0}} e^{ix\cdot\xi}\,d\xi \\ &= (2\pi N)^{-n}e^{ik\cdot x/N}\int_{\Omega_{1,0}} e^{ix\cdot\xi}\,d\xi,\end{aligned}$$

$$\int_{\Omega_{1,0}} e^{2ix\cdot\xi}\,d\xi = e^{i(x_1+\cdots+x_n)}(x_1^{-1}\sin x_1)\cdots(x_n^{-1}\sin x_n),$$

$$(\Phi_{N,k},\Phi_{N,\ell}) = 0 \quad (k \neq \ell),$$

$$\|\Phi_{N,k}\|^2 = (2\pi)^{-n}\|\hat{\Phi}_{N,k}\|^2 = (2\pi)^{-n}\int_{\Omega_{N,k}} d\xi = (2\pi N)^{-n}.$$

Hence we have

Lemma 1.1. (1) *Set*

$$\Phi_{N,k}(x) = (2\pi)^{-n}\int_{\Omega_{N,k}} e^{ix\cdot\xi}\,d\xi,$$

then

$$\Phi_{N,k}(x) = (2\pi N)^{-n}e^{ik\cdot x/N}s(x/(2N)),$$

where

$$s(x) = e^{i(x_1+\cdots+x_n)}(x_1^{-1}\sin x_1)\cdots(x_n^{-1}\sin x_n),$$

and

$$\Phi_{N,k}(x) = N^{-n}\Phi_{1,k}(x/N), \quad \Phi_{1,k}(x) = \Phi_{1,0}(x)e^{ix\cdot k}, \quad \Phi_{1,0}(x) = (2\pi)^{-n}s(x/2),$$

$$(\Phi_{N,k},\Phi_{N,\ell}) = 0 \quad (k \neq \ell), \quad \|\Phi_{N,k}\|^2 = (2\pi N)^{-n}.$$

(2) *Set* $S_N = \{\Phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$, *then* $\{S_N\}$ $(N \in \boldsymbol{N})$ *is a complete sequence of orthogonal subsets in* $L^2(\boldsymbol{R}^n)$.

Here we have from Lemma 0.2

THEOREM 1.1. *Set*

$$F_N(x)=(2\pi N)^n\sum_{k\in Z^n}(f,\Phi_{N,k})\Phi_{N,k}(x)\quad (N\in \boldsymbol{N})$$

for $f\in L^2(\boldsymbol{R}^n)$, *then it holds*

$$F_N\to f\quad in\ L^2(\boldsymbol{R}^n).$$

《$\{F_N(x)\}$ $(N\in \boldsymbol{N})$ is a sequence of quasi-Fourier series in $L^2(\boldsymbol{R}^n)$, corresponding to $\{S_N\}$ $(N\in \boldsymbol{N})$》

Let us consider

$$F_N(x)=\sum_{k\in Z^n}\|\Phi_{N,k}\|^{-2}(f,\Phi_{N,k})\Phi_{N,k}(x),$$

more precisely. Setting

$$a_{N,k}=\|\Phi_{N,k}\|^{-2}(f,\Phi_{N,k}),$$

we have

$$F_N(x)=\sum_{k\in Z^n}a_{N,k}\Phi_{N,k}(x).$$

We remark that

$$\begin{aligned}a_{N,k}&=\left\{(2\pi)^{-n}\int_{\Omega_{N,k}}d\xi\right\}^{-1}\left\{(2\pi)^{-n}\int_{\Omega_{N,k}}\hat{f}(\xi)\,d\xi\right\}\\&=\left\{\int_{\Omega_{N,k}}d\xi\right\}^{-1}\left\{\int_{\Omega_{N,k}}\hat{f}(\xi)\,d\xi\right\}\end{aligned}$$

is the integral-mean value of $\hat{f}(\xi)$ in $\Omega_{N,k}$. Moreover, we have

$$\hat{F}_N(\xi)=a_{N,k}\quad(\xi\in\Omega_{N,k}),\quad F_N(x)=(2\pi)^{-n}\int\hat{F}_N(\xi)e^{ix\xi}\,d\xi.$$

THEOREM 1.2. *Set*

$$F_N(x)=(2\pi N)^n\sum_{k\in Z^n}(f,\Phi_{N,k})\Phi_{N,k}(x)\quad (N\in \boldsymbol{N}),$$

$$a_{N,k}=\left\{\int_{\Omega_{N,k}}d\xi\right\}^{-1}\left\{\int_{\Omega_{N,k}}\hat{f}(\xi)\,d\xi\right\}$$

for $f \in L^2(\boldsymbol{R}^n)$. *Then*

$$F_N(x) = \sum_{k \in Z^n} a_{N,k}\Phi_{N,k}(x), \quad \hat{F}_N(\xi) = a_{N,k} \quad (\xi \in \Omega_{N,k}),$$

and

$$F_N(x) \to f(x) \quad \textit{in } L^2(\boldsymbol{R}^n).$$

《$\hat{F}_N(\xi)$ is a step-function approximation of $\hat{f}(\xi)$》

1.2. Analogy to trigonometrical series

(1) When the support of $f(x) \in L^2(\boldsymbol{R}^n)$ is contained in $(-\pi, \pi)^n$, we have

$$f(x) = (2\pi)^{-n} \sum_{k \in Z^n} \left\{ \int f(y) e^{-iy\cdot k}\, dy \right\} e^{ix\cdot k} \quad \text{in } L^2((-\pi, \pi)^n).$$

(2) When the support of $f(x) \in L^2(\boldsymbol{R}^n)$ is contained in $(-N\pi, N\pi)^n$, we have

$$f(x) = (2\pi N)^{-n} \sum_{k \in Z^n} \left\{ \int f(y) e^{-iy\cdot k/N}\, dy \right\} e^{ix\cdot k/N} \quad \text{in } L^2((-N\pi, N\pi)^n).$$

In other words,

《When the support of $f(x) \in L^2(\boldsymbol{R}^n)$ is contained in $(-N\pi, N\pi)^n$,

$$f(x) = \sum_{k \in Z^n} c_{N,k} e^{ix\cdot k/N} \quad \text{in } L^2((-N\pi, N\pi)^n) \text{ (Fourier series)}$$

where

$$c_{N,k} = (2\pi N)^{-n} \int f(y) \overline{e^{iy\cdot k/N}}\, dy \quad \text{(Fourier coefficient)}》$$

(3) In our case, the sequence of quasi-Fourier series of $f(x) \in L^2(\boldsymbol{R}^n)$ is written as

$$\begin{aligned} F_N(x) &= (2\pi N)^n \sum_{k \in Z^n} (f, \Phi_{N,k}) \Phi_{N,k}(x) \\ &= (2\pi N)^n \sum_{k \in Z^n} \int f(y) \{\overline{(2\pi N)^{-n} e^{iy\cdot k/N} s(y/(2N))}\}\, dy \\ &\quad \times \{(2\pi N)^{-n} e^{ix\cdot k/N} s(x/(2N))\} \\ &= (2\pi N)^{-n} \sum_{k \in Z^n} \int f(y) \{\overline{e^{iy\cdot k/N} s(y/(2N))}\}\, dy \\ &\quad \times \{e^{ix\cdot k/N} s(x/(2N))\} \quad \text{in } L^2(\boldsymbol{R}^n), \end{aligned}$$

where we remark

$$s(x/(2N)) \to 1 \quad (N \to \infty) \text{ (uniformly in a compact set).}$$

In other words,

《Let $f(x) \in L^2(\boldsymbol{R}^n)$, then we have

$$F_N(x) = \sum\nolimits_{k \in Z^n} c_{N,k}\{e^{ix\cdot k/N} s(x/(2N))\} \quad \text{in } L^2(\boldsymbol{R}^n)$$

(analogue of Fourier series),

$$F_N \to f \quad \text{in } L^2(\boldsymbol{R}^n)$$

where

$$c_{N,k} = (2\pi N)^{-n} \int f(y)\{\overline{e^{iy\cdot k/N} s(y/(2N))}\} \, dy$$

(analogue of Fourier coefficient)》

(4) Especially when the support of $f(x) \in L^2(\boldsymbol{R}^n)$ is contained in $(-N\pi, N\pi)^n$, since

$$c_{N,k} = (2\pi N)^{-n} \int f(y)\{\overline{e^{iy\cdot k/N} s(y/(2N))}\} \, dy$$

(:analogue of Fourier coefficient of f),

$$= (2\pi N)^{-n} \int \{f(y)\overline{s(y/(2N))}\}\overline{e^{iy\cdot k/N}} \, dy$$

(:Fourier coefficient of $\{f(x)\overline{s(x/(2N))}\}$),

we have from (2)

$$\sum\nolimits_{k \in Z^n} c_{N,k} e^{ix\cdot k/N} = f(x)\overline{s(x/(2N))} \quad \text{in } L^2((-N\pi, N\pi)^n).$$

Since

$$F_N(x) = \sum\nolimits_{k \in Z^n} c_{N,k}\{e^{ix\cdot k/N} s(x/(2N))\}$$
$$= \left\{\sum\nolimits_{k \in Z^n} c_{N,k} e^{ix\cdot k/N}\right\} s(x/(2N))$$

in $L^2(\boldsymbol{R}^n)$, we have

$$F_N(x) = f(x)\overline{s(x/(2N))} s(x/(2N)) \quad \text{in } L^2((-N\pi, N\pi)^n).$$

Let $\tilde{f}_N(x)$ be a periodic function with period $2N\pi$ in each variable x_j satisfying

$$\tilde{f}_N(x) = f(x)\overline{s(x/(2N))} \quad \text{in } (-N\pi, N\pi)^n,$$

then we have

$$\tilde{f}_N(x) = \sum\nolimits_{k \in Z^n} c_{N,k} e^{ix\cdot k/N} \quad \text{in } \boldsymbol{R}^n$$

and

$$F_N(x) = \tilde{f}_N(x) s(x/(2N)) \quad \text{in } \boldsymbol{R}^n.$$

Hence we have

THEOREM 1.3. *Suppose that the support of* $f(x) \in L^2(\boldsymbol{R}^n)$ *is contained in* $(-N\pi, N\pi)^n$. *Let* $\tilde{f}_N(x)$ *be a periodic function with period* $2N\pi$ *in each variable* x_j *satisfying*

$$\tilde{f}_N(x) = f(x)\overline{s(x/(2N))} \quad \textit{in } (-N\pi, N\pi)^n,$$

then we have

$$F_N(x) = \tilde{f}_N(x) s(x/(2N)) \quad \textit{in } \boldsymbol{R}^n.$$

§2. Complete Sequences of Orthogonal Subsets in $H^M(\boldsymbol{R}^n)$

2.1. $\{\Phi_{N,k}\}$ **in** $H^M(\boldsymbol{R}^n)$ In general, in the same way as in §1, $\{S_N\}$ $(N \in \boldsymbol{N})$ is a complete sequence of orthogonal subsets in $H^M(\boldsymbol{R}^n)$, where $S_N = \{\Phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$. In fact, for $f \in H^M(\boldsymbol{R}^n)$, setting

$$f_A(x) = \gamma_A(x) f(x),$$

we have

$$f_A \in H^M(\boldsymbol{R}^n) \cap L^1(\boldsymbol{R}^n), \quad f_A \to f \text{ in } H^M(\boldsymbol{R}^n) \ (A \to \infty).$$

Then we have

$$(|\xi| + 1)^M \hat{f}_A \in L^2(\boldsymbol{R}^n) \cap \mathscr{B}(\boldsymbol{R}^n)$$

and

$$(2\pi)^{-n} \int_{\Omega_B} \hat{f}_A(\xi) e^{ix\cdot\xi} \, d\xi \to f_A(x) \quad \text{in } H^M(\boldsymbol{R}^n) \ (B \to \infty).$$

Setting

$$g_{A,N}(\xi) = \hat{f}_A(k/N) \quad \text{if } \xi \in \Omega_{N,k},$$

we have

$$(2\pi)^{-n}\int_{\Omega_B} g_{A,N}(\xi)e^{ix\cdot\xi}\,d\xi = \sum\nolimits_{-NB\leqq k_1,\dots,k_n<NB}\hat{f}_A(k/N)\Phi_{N,k}(x),$$

and

$$(2\pi)^{-n}\int_{\Omega_B} g_{A,N}(\xi)e^{ix\cdot\xi}\,d\xi \to (2\pi)^{-n}\int_{\Omega_B}\hat{f}_A(\xi)e^{ix\cdot\xi}\,d\xi \quad (N\to\infty) \text{ in } H^M(\boldsymbol{R}^n).$$

Therefore, $\{S_N\}$ $(N\in\boldsymbol{N})$ is a complete sequence of orthogonal subsets in $H^M(\boldsymbol{R}^n)$. Moreover, since

$$\partial^\nu\Phi_{N,k}(x) = (2\pi)^{-n}\int_{\Omega_{N,k}}(i\xi)^\nu e^{ix\cdot\xi}\,d\xi,$$

we have

$$\begin{aligned}\|\Phi_{N,k}\|_M^2 &= (2\pi)^{-n}\int_{\Omega_{N,k}} L_M(\xi)\,d\xi\\ &= (2\pi N)^{-n}\int_{\Omega_{1,0}} L_M((\xi+k)/N)\,d\xi\\ &= (2\pi N)^{-n}\int_{\Omega_{1,0}}\sum\nolimits_{|\nu|\leqq M}((\xi_1+k_1)/N)^{2\nu_1}\cdots((\xi_n+k_n)/N)^{2\nu_n}\,d\xi\\ &= (2\pi N)^{-n}P_M(1/N,k/N),\end{aligned}$$

where

$$L_M(\xi) = \sum\nolimits_{|\nu|\leqq M}|\xi^\nu|^2$$

and $P_M(X_0,X_1,\dots,X_n)$ is a polynomial with respect to $(X_0,X_1,\dots,X_n)$ of order $2M$.

Here we have

LEMMA 2.1.

(1) *It holds*

$$(\Phi_{N,k},\Phi_{N,\ell})_M = 0 \quad (k\neq\ell), \quad \|\Phi_{N,k}\|_M^2 = (2\pi N)^{-n}P_M(1/N,k/N).$$

(2) $S_N = \{\Phi_{N,k}\,|\,k\in\boldsymbol{Z}^n\}$ $(N\in\boldsymbol{N})$ *is a complete sequence of orthogonal subsets in* $H^M(\boldsymbol{R}^n)$.

Therefore we have from Lemma 2.1 and Lemma 0.2

THEOREM 2.1. *Set*

$$F_N(x) = \sum_{k \in Z^n} \|\Phi_{N,k}\|_M^{-2}(f, \Phi_{N,k})_M \Phi_{N,k}(x) \quad (N \in \boldsymbol{N})$$

for $f \in H^M(\boldsymbol{R}^n)$, *then*

$$F_N \to f \quad \text{in } H^M(\boldsymbol{R}^n).$$

《$\{F_N(x)\}$ $(N \in \boldsymbol{N})$ is a sequence of quasi-Fourier series in $H^M(\boldsymbol{R}^n)$, corresponding to $\{S_N\}$ $(N \in \boldsymbol{N})$》

Let us consider

$$F_N(x) = \sum_{k \in Z^n} \|\Phi_{N,k}\|_M^{-2}(f, \Phi_{N,k})_M \Phi_{N,k}(x),$$

more precisely. Setting

$$a_{N,k} = \|\Phi_{N,k}\|_M^{-2}(f, \Phi_{N,k})_M,$$

we have

$$F_N(x) = \sum_{k \in Z^n} a_{N,k} \Phi_{N,k}(x).$$

We remark that

$$\begin{aligned} a_{N,k} &= \left\{(2\pi)^{-n} \int_{\Omega_{N,k}} L_M(\xi)\, d\xi\right\}^{-1} \left\{(2\pi)^{-n} \int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)\, d\xi\right\} \\ &= \left\{\int_{\Omega_{N,k}} L_M(\xi)\, d\xi\right\}^{-1} \left\{\int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)\, d\xi\right\} \end{aligned}$$

is the weighted-integral-mean value of $\hat{f}(\xi)$ in $\Omega_{N,k}$. Moreover, we have

$$\hat{F}_N(\xi) = a_{N,k} \quad (\xi \in \Omega_{N,k}), \quad F_N(x) = (2\pi)^{-n} \int \hat{F}_N(\xi) e^{ix\xi}\, d\xi.$$

Hence we have

THEOREM 2.2. *Set*

$$F_N(x) = \sum_{k \in Z^n} \|\Phi_{N,k}\|_M^{-2}(f, \Phi_{N,k})_M \Phi_{N,k}(x) \quad (N \in \boldsymbol{N}),$$

$$a_{N,k} = \left\{\int_{\Omega_{N,k}} L_M(\xi)\, d\xi\right\}^{-1} \left\{\int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)\, d\xi\right\}$$

for $f \in H^M(\boldsymbol{R}^n)$. *Then*

$$F_N(x) = \sum_{k \in Z^n} a_{N,k}\Phi_{N,k}(x), \quad \hat{F}_N(\xi) = a_{N,k} \quad (\xi \in \Omega_{N,k}),$$

and

$$F_N(x) \to f(x) \quad in \ H^M(\boldsymbol{R}^n).$$

《$\hat{F}_N(\xi)$ is a step-function approximation of $\hat{f}(\xi)$》

2.2. $\{\phi_{N,k}(x)\}$ **in** $H^M(\boldsymbol{R}^n)$ Set

$$\begin{aligned}\phi_{N,k}(x) &= L_M^{-1/2}\Phi_{N,k}(x)\\ &= (2\pi)^{-n}\int_{\Omega_{N,k}} e^{ix\cdot\xi}L_M(\xi)^{-1/2}\,d\xi,\end{aligned}$$

then we have

LEMMA 2.2. *It holds*

$$(\phi_{N,k}, \phi_{N,\ell})_M = 0 \quad (k \neq \ell), \quad \|\phi_{N,k}\|_M^2 = (2\pi N)^{-n}.$$

THEOREM 2.3. *Set*

$$\mathscr{F}_N(x) = (2\pi N)^n \sum_{k \in Z^n} (f, \phi_{N,k})_M \phi_{N,k}(x) \quad (N \in \boldsymbol{N})$$

for $f \in H^M(\boldsymbol{R}^n)$, *then*

$$\mathscr{F}_N \to f \quad in \ H^M(\boldsymbol{R}^n) \ (N \to \infty).$$

PROOF. Since $f \in H^M(\boldsymbol{R}^n)$, we have $L_M^{1/2}f \in L^2(\boldsymbol{R}^n)$. Therefore, we have from Theorem 1.1

$$G_N(x) = (2\pi N)^n \sum_{k \in Z^n} (L_M^{1/2}f, \Phi_{N,k})\Phi_{N,k}(x) \in L^2(\boldsymbol{R}^n),$$

$$\|G_N - L_M^{1/2}f\| \to 0 \quad (N \to \infty).$$

Set

$$\mathscr{F}_N(x) = L_M^{-1/2}G_N(x) \in H^M(\boldsymbol{R}^n),$$

then

$$\mathscr{F}_N(x) = (2\pi N)^n \sum_{k\in Z^n} (L_M^{1/2} f, \Phi_{N,k}) L_M^{-1/2} \Phi_{N,k}(x)$$
$$= (2\pi N)^n \sum_{k\in Z^n} (f, \phi_{N,k})_M \phi_{N,k}(x)$$

and

$$\|\mathscr{F}_N - f\|_M = \|L_M^{1/2}(\mathscr{F}_N - f)\| \to 0 \quad (N \to \infty). \qquad \square$$

Set $s_N = \{\phi_{N,k}(x) \mid k \in \boldsymbol{Z}^n\}$, then, $\{s_N\}$ $(N \in \boldsymbol{N})$ is a complete sequence of orthogonal subsets in $H^M(\boldsymbol{R}^n)$, from Lemma 0.2. In other words, $\{\mathscr{F}_N(x)\}$ $(N \in \boldsymbol{N})$ in Theorem 2.3 is a sequence of quasi-Fourier series of f in $H^M(\boldsymbol{R}^n)$, corresponding to $\{s_N\}$ $(N \in \boldsymbol{N})$.

THEOREM 2.4. *Set*

$$\mathscr{F}_N(x) = (2\pi N)^n \sum_{k\in Z^n} (f, \phi_{N,k})_M \phi_{N,k}(x) \quad (N \in \boldsymbol{N})$$

for $f \in H^M(\boldsymbol{R}^n)$, *then* $\mathscr{F}_N(x) \to f(x)$ *in* $H^M(\boldsymbol{R}^n)$ $(N \to \infty)$. *Moreover, set*

$$b_{N,k} = \left\{\int_{\Omega_{N,k}} d\xi\right\}^{-1} \left\{\int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)^{1/2}\, d\xi\right\}$$

(: *integral-mean of* $(L_M^{1/2}\hat{f})(\xi)$ *in* $\Omega_{N,k}$),

then

$$\mathscr{F}_N(x) = \sum_{k\in Z^n} b_{N,k}\phi_{N,k}(x), \quad \hat{\mathscr{F}}_N(\xi) = b_{N,k} L_M(\xi)^{-1/2} \quad (\xi \in \Omega_{N,k}).$$

《$\hat{\mathscr{F}}_N(\xi)$ is a waved-step-function approximation of $\hat{f}(\xi)$》

PROOF. Since

$$b_{N,k} = (2\pi N)^n (f, \phi_{N,k})_M$$
$$= N^n \left\{\int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)^{1/2}\, d\xi\right\}$$
$$= \left\{\int_{\Omega_{N,k}} d\xi\right\}^{-1} \left\{\int_{\Omega_{N,k}} \hat{f}(\xi) L_M(\xi)^{1/2}\, d\xi\right\}$$
$$= \left\{\int_{\Omega_{N,k}} d\xi\right\}^{-1} \left\{\int_{\Omega_{N,k}} (L_M^{1/2}\hat{f})(\xi)\, d\xi\right\},$$

we have

$$\begin{aligned}\mathscr{F}_N(x) &= (2\pi N)^n \sum\nolimits_{k\in Z^n} (f, \phi_{N,k})_M \phi_{N,k}(x) \\ &= \sum\nolimits_{k\in Z^n} b_{N,k}\phi_{N,k}(x) \\ &= (2\pi)^{-n} \sum\nolimits_{k\in Z^n} b_{N,k} \int_{\Omega_{N,k}} L_M(\xi)^{-1/2} e^{ix\cdot\xi}\, d\xi,\end{aligned}$$

that is,

$$\hat{\mathscr{F}}_N(\xi) = b_{N,k} L_M(\xi)^{-1/2} \quad (\xi \in \Omega_{N,k}). \qquad \square$$

§3. A Sequence of Orthogonal Bases

3.1. A sequence of orthogonal bases in $L^2(\boldsymbol{R}^n)$ In §1, we considered a complete sequence of orthogonal subsets $\{S_N\}$ $(N \in \boldsymbol{N})$ in $L^2(\boldsymbol{R}^n)$, where $S_N = \{\Phi_{N,k} \mid$
$k \in \boldsymbol{Z}^n\}$. Here, for simplicity, we consider a sub-sequence of $\{S_N\}$:

$$\{S_{N(\lambda)}\}, \quad N(\lambda) = 2^\lambda \quad (\lambda \in \boldsymbol{N}).$$

From Lemma 0.3, $\{S_{N(\lambda)}\}$ $(\lambda \in \boldsymbol{N})$ is also a complete sequence of orthogonal subsets in $L^2(\boldsymbol{R}^n)$. Let us construct a sequence of orthogonal bases $\{\Sigma_\lambda\}$ $(\lambda \in \boldsymbol{N})$ in $L^2(\boldsymbol{R}^n)$ satisfying $S_{N(\lambda)} \subset \Sigma_\lambda$.

First, we define fundamental functions. Set

$$\alpha(t) = \begin{cases} 1 & (0 < t < 1) \\ 0 & (\text{otherwise})\end{cases},$$

$$\alpha_k(t) = \alpha(t-k),$$

$$\alpha_{j,k}(t) = \alpha_k(2^j t) = \alpha(2^j t - k) = \alpha(2^j(t - 2^{-j}k)).$$

Set

$$A_\lambda = \{\alpha_{\lambda+j,k}(t) \mid j = 0, 1, 2, \dots, k \in \boldsymbol{Z}\}$$

for $\lambda \in \boldsymbol{N}$, then $\langle A_\lambda \rangle$ is dense in $L^2(\boldsymbol{R})$. Set

$$\beta(t) = \begin{cases} 1 & (0 < t < 1/2) \\ -1 & (1/2 < t < 1), \\ 0 & (\text{otherwise})\end{cases}$$

$$\beta_k(t) = \beta(t-k),$$

$$\beta_{j,k}(t) = \beta_k(2^j t)$$

and

$$B_\lambda = \{\alpha_{\lambda,k}(t)\ \ (k \in \boldsymbol{Z}), \beta_{\lambda+j,k}(t)\ \ (j = 0, 1, 2, \dots, k \in \boldsymbol{Z})\},$$

then B_λ is an orthogonal subset in $L^2(\boldsymbol{R})$. Moreover, since

$$\alpha_{j+1,2k}(t) = (1/2)\alpha_{j,k}(t) + (1/2)\beta_{j,k}(t),$$

$$\alpha_{j+1,2k+1}(t) = (1/2)\alpha_{j,k}(t) - (1/2)\beta_{j,k}(t),$$

we have $\langle A_\lambda \rangle = \langle B_\lambda \rangle$, therefore, $\langle B_\lambda \rangle$ is dense in $L^2(\boldsymbol{R})$. Hence B_λ is an orthogonal basis in $L^2(\boldsymbol{R})$. Set

$$\boldsymbol{J} = \{-1, 0, 1, 2, \dots\},$$

$$\beta_{(\lambda),-1,k}(t) = \alpha_{\lambda,k}(t),$$

$$\beta_{(\lambda),j,k}(t) = \beta_{\lambda+j,k}(t) \quad (j = 0, 1, 2, \dots),$$

then

$$B_\lambda = \{\beta_{(\lambda),j,k}(t)\ \ (j \in \boldsymbol{J}, k \in \boldsymbol{Z})\}.$$

Now, define

$$\begin{aligned}\hat{\Psi}_{(\lambda),j,k}(\xi) &= \hat{\Psi}_{(\lambda),(j_1,\dots,j_n),(k_1,\dots,k_n)}(\xi) \\ &= \beta_{(\lambda),j_1,k_1}(\xi_1)\cdots\beta_{(\lambda),j_n,k_n}(\xi_n) \quad \text{for } j \in \boldsymbol{J}^n \text{ and } k \in \boldsymbol{Z}^n.\end{aligned}$$

Remarking

$$\hat{\Psi}_{(\lambda),(-1,\dots,-1),k}(\xi) = \alpha_{\lambda,k_1}(\xi_1)\cdots\alpha_{\lambda,k_n}(\xi_n) = \hat{\Phi}_{N(\lambda),k}(\xi),$$

we have

LEMMA 3.1.

(1) *Set*

$$\Sigma_\lambda = \{\Psi_{(\lambda),j,k}(x) \mid j = (j_1, j_2, \dots, j_n) \in \boldsymbol{J}^n,\ k = (k_1, k_2, \dots, k_n) \in \boldsymbol{Z}^n\},$$

then Σ_λ *is an orthogonal basis in* $L^2(\boldsymbol{R}^n)$.

(2) *It holds*

$$\Psi_{(\lambda),(-1,\dots,-1),k}(x) = \Phi_{N(\lambda),k}(x), \quad \|\Psi_{(\lambda),(-1,\dots,-1),k}\| = (2\pi N(\lambda))^{-n/2}.$$

Hence we have

$$f(x) = \sum_{j \in \boldsymbol{J}^n, k \in \boldsymbol{Z}^n} \|\Psi_{(\lambda),j,k}\|^{-2}(f, \Psi_{(\lambda),j,k})\Psi_{(\lambda),j,k}(x) \quad \text{in } L^2(\boldsymbol{R}^n)$$

for $f \in L^2(\boldsymbol{R}^n)$. On the other hand, $F_{N(\lambda)}(x)$ in Theorem 1.1 is written as

$$F_{N(\lambda)}(x) = (2\pi N(\lambda))^n \sum_{k \in \boldsymbol{Z}^n} (f, \Psi_{(\lambda),(-1,\dots,-1),k}) \Psi_{(\lambda),(-1,\dots,-1),k}(x).$$

Here we have

THEOREM 3.1. *Let*

$$f(x) = \sum_{j \in \boldsymbol{J}^n, k \in \boldsymbol{Z}^n} \|\Psi_{(\lambda),j,k}\|^{-2} (f, \Psi_{(\lambda),j,k}) \Psi_{(\lambda),j,k}(x) \quad \text{in } L^2(\boldsymbol{R}^n)$$

be the Fourier series for $f \in L^2(\boldsymbol{R}^n)$, corresponding to the orthogonal basis Σ_λ. Then its sub-series

$$F_{N(\lambda)}(x) = \sum_{k \in \boldsymbol{Z}^n} \|\Psi_{(\lambda),(-1,\dots,-1),k}\|^{-2} (f, \Psi_{(\lambda),(-1,\dots,-1),k}) \Psi_{(\lambda),(-1,\dots,-1),k}(x)$$

satisfies

$$\|F_{N(\lambda)} - f\| \to 0 \quad (\lambda \to \infty).$$

A sequence of orthogonal bases in $H^M(\boldsymbol{R}^n)$ In §2, we considered a complete sequence of orthogonal subsets $\{s_N\}$ $(N \in \boldsymbol{N})$ in $H^M(\boldsymbol{R}^n)$, where $s_N = \{\phi_{N,k} \mid k \in \boldsymbol{Z}^n\}$. Here, we consider sub-sequence of $\{s_N\}$:

$$\{s_{N(\lambda)}\}, \quad N(\lambda) = 2^\lambda \quad (\lambda \in \boldsymbol{N}).$$

From Lemma 0.3, $\{s_{N(\lambda)}\}$ $(\lambda \in \boldsymbol{N})$ is also a complete sequence of orthogonal subsets in $H^M(\boldsymbol{R}^n)$. Let us construct a sequence of orthogonal bases $\{\boldsymbol{\sigma}_\lambda\}$ $(\lambda \in \boldsymbol{N})$ in $H^M(\boldsymbol{R}^n)$ satisfying $s_{N(\lambda)} \subset \boldsymbol{\sigma}_\lambda$. Set

$$\hat{\psi}_{(\lambda),j,k}(\xi) = \hat{\Psi}_{(\lambda),j,k}(\xi) L_M(\xi)^{-1/2},$$

then

$$\hat{\psi}_{(\lambda),(-1,\dots,-1),k}(\xi) = \alpha_{\lambda,k_1}(\xi_1) \cdots \alpha_{\lambda,k_n}(\xi_n) L_M(\xi)^{-1/2}$$
$$= \hat{\Phi}_{N(\lambda),k}(\xi) L_M(\xi)^{-1/2} = \hat{\phi}_{N(\lambda),k}(\xi).$$

Hence we have

LEMMA 3.2.
(1) *Set*

$$\boldsymbol{\sigma}_\lambda = \{\psi_{(\lambda),j,k}(x) \mid j = (j_1, j_2, \dots, j_n) \in \boldsymbol{J}^n, k = (k_1, k_2, \dots, k_n) \in \boldsymbol{Z}^n\},$$

then $\boldsymbol{\sigma}_\lambda$ is an orthogonal basis in $H^M(\boldsymbol{R}^n)$.

(2) *It holds*

$$\psi_{(\lambda),(-1,\dots,-1),k}(x) = \phi_{N(\lambda),k}(x), \quad \|\psi_{(\lambda),(-1,\dots,-1),k}\|_M = (2\pi N(\lambda))^{-n/2}.$$

Hence we have

$$f(x) = \sum_{j \in J^n, k \in Z^n} \|\psi_{(\lambda),j,k}\|_M^{-2} (f, \psi_{(\lambda),j,k})_M \psi_{(\lambda),j,k}(x) \quad \text{in } H^M(\boldsymbol{R}^n)$$

for $f \in H^M(\boldsymbol{R}^n)$. On the other hand, $\mathscr{F}_{N(\lambda)}(x)$ in Theorem 2.3 is written as

$$\begin{aligned}\mathscr{F}_{N(\lambda)}(x) &= (2\pi N(\lambda))^n \sum_{k \in Z^n} (f, \phi_{N(\lambda),k})_M \phi_{N(\lambda),k}(x) \\ &= \sum_{k \in Z^n} \|\psi_{(\lambda),(-1,\dots,-1),k}\|_M^{-2} (f, \psi_{(\lambda),(-1,\dots,-1),k})_M \psi_{(\lambda),(-1,\dots,-1),k}(x).\end{aligned}$$

Here we have

THEOREM 3.2. *Let*

$$f(x) = \sum_{j \in J^n, k \in Z^n} \|\psi_{(\lambda),j,k}\|_M^{-2} (f, \psi_{(\lambda),j,k})_M \psi_{(\lambda),j,k}(x) \quad \text{in } H^M(\boldsymbol{R}^n)$$

be the Fourier series for $f \in H^M(\boldsymbol{R}^n)$, corresponding to the orthogonal basis σ_λ. Then its sub-series

$$\mathscr{F}_{N(\lambda)}(x) = \sum_{k \in Z^n} \|\psi_{(\lambda),(-1,\dots,-1),k}\|_M^{-2} (f, \psi_{(\lambda),(-1,\dots,-1),k})_M \psi_{(\lambda),(-1,\dots,-1),k}(x)$$

satisfies

$$\|\mathscr{F}_{N(\lambda)} - f\|_M \to 0 \quad (\lambda \to \infty).$$

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