TSUKUBA J. MATH.
Vol. 29 No. 1 (2005), 147–171

# FINITE JET DETERMINATION FOR BIHOLOMORPHISMS OF REAL-ANALYTIC HYPERSURFACES IN $\mathbf{C}^N$

By

Gabriela PUTINAR

**Abstract.** We prove that for real-analytic hypersurfaces $M$ in $\mathbf{C}^N$, finite jet determination for biholomorphisms of $M$ holds under minimal assumptions on the geometry of $M$, in both the infinite type and in the finite type cases. In the finite type case, we extend from $N = 2$ to arbitrary $N$ a result of Ebenfelt, Lamel and Zaitsev on 2-jet determination of such biholomorphisms.

## 1. Introduction

The question of finite determination of (local) $CR$ automorphisms of generic real submanifolds in $\mathbf{C}^N$ by their jets of a sufficiently high order has been studied under various assumptions on the geometry of the manifold (see for instance [1], [2], [7], [9] and the references there).

For $M$ a $CR$ submanifold in $\mathbf{C}^N$ with $p \in M$, let $\mathrm{Aut}(M, p)$ denote the *stability group* of $M$ at $p$, i.e. the group of all germs of biholomorphisms $H : \mathbf{C}^N \to \mathbf{C}^N$ such that $H(M) \subset M$ and $H(p) = p$. Such biholomorphisms are in particular $CR$ automorphisms and we shall call them "automorphisms" of $M$ for short. (In the case of real-analytic hypersurfaces that are essentially-finite, $\mathrm{Aut}(M, p)$ coincides with the group of $CR$ automorphisms of $M$ that fix $p$; [1].)

A reference result ([1]) is that finite determination holds (at the generic point) for automorphisms of real-analytic generic submanifolds that are holomorphically non-degenerate and of finite type at some point.

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Mathematics Subject Classification (2000). 32H02, 32V40.
Keywords. CR manifold, real-analytic hypersurface, stability group, jet determination.
Received July 22, 2003.
Revised November 5, 2003.

Further, as was shown in [3], holomorphic non-degeneracy is a necessary condition for finite determination of automorphisms to hold, even if the manifold and the biholomorphisms are assumed to be real-analytic or formal.

On the other hand, the condition of finite type is by no means a necessary condition in general; in the infinite-type case, it is still possible to describe non-degeneracy conditions for the CR geometry of real hypersurfaces in $\mathbf{C}^N$ under which typical results hold (such as e.g. the Schwarz principle; or finite determination by jets of maps that extend holomorphically to one side). Such conditions are: weakly essential, infinite type non-degenerate, etc. (see [8], [12], [10]).

From this perspective, it was proved in [9] that for $N = 2$, i.e. for (real-analytic) 3-manifolds in $\mathbf{C}^2$, Levi non-flatness is a necessary and *sufficient* condition for finite determination of CR automorphisms to hold. In order to eliminate the finite type assumption, the authors prove first that finite determination holds along the 0-Segre set (which is the set on which finite type no longer holds). This is combined with the method of complete singular systems developed in [7], [8], by which one obtains also finite determination of automorphisms in the characteristic direction (transversal to the 0-Segre set).

In this paper we extend these finite determination results for 3-manifolds in $\mathbf{C}^2$, to the case of automorphisms of (real-analytic) hypersurfaces in $\mathbf{C}^N$. In the infinite-type case, we assume that a certain finiteness assumption holds (F.A. for short, see Section 3); for 3-manifolds in $\mathbf{C}^2$ this condition coincides with weakly-essential and with Levi non-flatness; for $N \geq 2$ it neither implies nor is implied by weakly-essential, while it still allows for arbitrarily large infinite type. We show (as a consequence to Proposition 1, Section 2) that F.A. is invariant under any change of normal coordinates. In the infinite-type case, our main result (Theorem 2, Section 5) states that, under these assumptions, there exists a sufficiently large $k$ such that finite determination by $k$-jets at $p$ holds for all automorphisms that fix $p$.

Our proofs are parallel to those of [9]. In fact, in the characteristic direction we use the above-mentioned method of complete singular systems in its initial setting, that of weakly-essential hypersurfaces in $\mathbf{C}^N$. The only technical difficulties in the infinite type case in $\mathbf{C}^N$ are related to the finite determination along the 0-Segre set (Theorem 1, Section 3). Using the finiteness assumption F.A., we reduce this question to that of solving an equation (given by a Weierstrass polynomial) for each of the components of a biholomorphism $H$. We then show that for large $k$, all biholomorphisms with given $k$-jet must satisfy (componentwise) an equation given by a single irreducible component of the zero variety of the corresponding Weierstrass polynomial, and that the latter must have degree one. From this, finite determination along the 0-Segre set is straightforward.

In the case of hypersurfaces of *finite type*, a remarkable result proved in [9] is that if $M \subset \mathbf{C}^2$ is a real-analytic real hypersurface of finite type at $p \in M$, then the elements in $\mathrm{Aut}(M, p)$ are determined by their jets of length two.

To extend this result to higher dimensions, we make certain assumptions on the geometry of the hypersurface, that are stronger versions of the condition F.A. mentioned above and therefore imply holomorphic non-degeneracy; these assumptions are minimal, in some sense (see Example, Section 7). Also, for $M \subset \mathbf{C}^2$ (a real hypersurface) these conditions reduce to assuming finite type only. With these assumptions, and using computations as in [9] and [4] for points outside the 0-Segre set, we prove (Theorem 4, Section 7) 2-jet determination for $\mathrm{Aut}(M, p)$, where $M$ is a real-analytic hypersurface of arbitrary dimension.

The main technical result in the finite type case is a result on determination by one-jets of sections in analytic sets, provided that the initial homogeneous polynomials of the sections satisfy a certain generic condition (Theorem 3, Section 6).

**Acknowledgment.** The author would like to thank P. Ebenfelt for stimulating discussions on the subject.

Notations. Besides the standard notation for partial derivatives using multi-indices in $\mathbf{N}^n$, we shall use the following notation.

For an arbitrary integer $l \geq 1$, we define the sets $\Lambda_l = \{1, \ldots, n\}^l$, $\Lambda = \bigcup_{l \geq 1} \Lambda_l$. We write $|\mathbf{k}| = l$ if $\mathbf{k} \in \Lambda_l$.

We can then write partial derivatives of a function $f$ of $n$ variables $z_1, \ldots, z_n$, using subscripts, such that for a (multi)-index $\mathbf{k} = (k_1, \ldots, k_l) \in \Lambda_l$, $f_{z^{\mathbf{k}}}$ denotes the partial derivative $\partial^l f / \partial z_{k_1} \cdots \partial z_{k_l}$.

If $l = 1$, it is convenient to use instead the notation $f_{z_k}$, $k = 1, \ldots, n$, for the first-order partial derivatives of $f$.

For a complex function $f$, the function $\bar{f}$ is defined by $\bar{f}(z) := \overline{f(\bar{z})}$.

We denote by $in(f)$ the initial homogeneous polynomial of the Taylor expansion at 0 of $f$.

For $f \not\equiv 0$, the degree of this polynomial is by definition the order $v(f)$ of $f$.

The notations $o(z^k)$ (resp. $O(z^k)$) are reserved for quantities of order $> k$, (resp. $= k$.)

As usual, by $j_0^l(F)$ we denote the jet of length $l$ at the point 0 of the (vector) function $F$.

By $\mathbf{C}\{z_1, \dots, z_n\}$ we denote the ring of convergent series at 0 in $\mathbf{C}^n$.

## 2. Invariants and Identities in Normal Coordinates

This section is parallel with the computations in [9] in the infinite type case and extends the invariants defined there from $n = 1$ to arbitrary $n \geq 1$.

Let $M \subset \mathbf{C}^{n+1}$ be a real-analytic hypersurface with $p \in M$. We shall choose normal coordinates $(z, w) \in \mathbf{C}^n \times \mathbf{C}$ for $M$ at $p$; i.e. $(z, w)$ vanish at $p$ and $M$ is written (in complex form) near $p = 0$ as

$$M : w = Q(z, \bar{z}, \bar{w}), \tag{1}$$

where $Q(z, \chi, \tau)$ is a holomorphic function in a neighborhood of 0 in $\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$, satisfying

$$Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau. \tag{2}$$

Recall [1] that a holomorphic function $Q$ defines a real-analytic hypersurface if and only if it satisfies the reality condition

$$Q(z, \chi, \bar{Q}(\chi, z, \tau)) \equiv \tau. \tag{3}$$

Further, let us define for $\mathbf{k} \in \Lambda$, $\mu \in \mathbf{N}$, the analytic function

$$q_{\mathbf{k}\mu}(\chi) := Q_{z^{\mathbf{k}}\tau^{\mu}}(0, \chi, 0). \tag{4}$$

Note that the normality of the coordinates implies

$$q_{\mathbf{k}\mu}(0) = 0 \quad \text{for all } \mathbf{k} \in \Lambda. \tag{5}$$

Using the functions $q_{\mathbf{k}\mu}$, we define a set of invariants for $M$ as follows. First, let us define

$$m_0 := \min\{m \in \mathbf{Z}_+ \mid \exists \mathbf{k} \in \Lambda, \mu \in \mathbf{N}, \text{ such that } m = |\mathbf{k}| + \mu, q_{\mathbf{k}\mu}(\chi) \not\equiv 0\}. \tag{6}$$

We shall assume in what follows that the set defining $m_0$ is non-empty, i.e. that $m_0$ is finite. This is equivalent to assuming that $M$ is not Levi-flat at $p$.

In order to define the other invariants, we consider as in [9] the order $\prec$ on pairs $(a, m) \in \mathbf{N}^2$ defined by $(a, m) \prec (b, n)$ iff, either $a + m < b + n$ or, if $a + m = b + n$, then $m < n$.

For multi-indices $(\mathbf{k}, \mu) \in \Lambda \times \mathbf{N}$ this induces a partial order (denoted still $\prec$) defined by $(\mathbf{k}, \mu) \prec (\mathbf{m}, \nu)$ iff $(|\mathbf{k}|, \mu) \prec (|\mathbf{m}|, \nu)$.

Note that the set of multi-indices $(\mathbf{k}, \mu)$ in $\Lambda \times \mathbf{N}$ that are minimal with respect to $\prec$, and are such that $q_{\mathbf{k}\mu} \not\equiv 0$, is a subset of $A \times \{\mu_0\}$, with the notation $A = \Lambda_{\nu_0}$, for a unique $\mu_0$ and for $\nu_0 = m_0 - \mu_0$.

Let us next define $\mathscr{I} = \mathscr{I}_M(0)$ to be the ideal generated by the set

$$(q_{\mathbf{k}\mu_0})_{\mathbf{k} \in \Lambda_{\nu_0}}, \tag{7}$$

in the ring $\mathbf{C}\{\chi\} = \mathbf{C}\{\chi_1, \dots, \chi_n\}$ (where $\chi$ is the vector $(\chi_1, \dots, \chi_n)$).

With these definitions, we have the following

PROPOSITION 1. *For $M \subset \mathbf{C}^{n+1}$ a non-Levi-flat real-analytic hypersurface and $p \in M$, the integers $m_0$, $\mu_0$ (therefore also $\nu_0$) are invariant under changes of normal coordinates on $\mathbf{C}^{n+1}$ that fix $p$.*

*The ideal $\mathscr{I}_M(0)$ transforms under a change of normal coordinates given by $(z', w') = H(z, w)$, with $H = (F, G)$ by*

$$\mathscr{I}_M(0) = \bar{F}(\chi, 0)^* \mathscr{I}'_M(0), \tag{8}$$

*with $*$ denoting pull-back of functions.*

Before entering the proof of Proposition 1, let us consider another real-analytic hypersurface $M'$ in $\mathbf{C}^{n+1}$, with $p' \in M'$, and defining equation given by a holomorphic function $Q'(z', \chi', \tau')$ in some normal coordinates $(z', w')$ on $\mathbf{C}^{n+1}$, and let $H = (F, G)$, where $F = (F^1, \dots, F^n)$, be an arbitrary biholomorphism that takes $(M, p)$ to $(M', p')$. The condition that $H(M) \subset M'$ becomes the following (main) identity:

$$G(z, Q(z, \chi, \tau)) \equiv Q'(F(z, Q(z, \chi, \tau)), \bar{F}(\chi, \tau), \bar{G}(\chi, \tau)), \tag{9}$$

for $(z, \chi, \tau) \in \mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$ in a neighborhood of 0.

We shall use this identity repeatedly together with the chain rule for derivatives $z^{\mathbf{k}} \tau^{\mu}$ at $(z, \tau) = 0$ in order to obtain several other identities in $\chi$.

First, note that by setting $\chi = \tau = 0$, we obtain

$$G(z, 0) \equiv 0. \tag{10}$$

Since $H$ is a biholomorphism, from (10) it also follows that $\mathrm{Jac}_z(\bar{F})(0) \neq 0$ and $G_w(0) \neq 0$.

By differentiating with respect to $\tau$ at $\tau = 0$ in (9) and using normality of the coordinates, we obtain (at $z = 0$) the following expression for $\bar{G}_w(\chi, 0)$:

$$\bar{G}_w(\chi, 0) \equiv G_w(0) - \sum\nolimits_{j=1}^{n} Q'_{z_j}(0, \bar{F}(\chi, 0), 0) \cdot F^j_w(0). \tag{11}$$

In particular if $m_0 \geq 2$, we see that $\bar{G}_w(\chi, 0) \equiv G_w(0)$.

If in (9) we take the partial derivative at $(z,\tau)=0$ relative to $z^{\mathbf{k}}\tau^{\mu}$, for some $\mathbf{k}\in\Lambda$, $\mu\in\mathbf{N}$, and if $l=|\mathbf{k}|$, we obtain the following identity in $\chi$:

$$G_{z^{\mathbf{k}}w^{\mu}}(0)+G_w(0)q_{\mathbf{k}\mu}(\chi)+\Psi(\chi)\equiv\bar{G}_w(\chi,o)^{\mu}\cdot\sum\nolimits_{|\mathbf{j}|=l}q'_{\mathbf{j}\mu}(\bar{F}(\chi,0))\cdot P_{\mathbf{jk}}+\Psi'(\chi), \tag{12}$$

where for $\mathbf{k},\mathbf{j}\in\Lambda_l$, $P_{\mathbf{jk}}$ is defined by

$$P_{\mathbf{jk}}=P_{\mathbf{jk}}(\chi)=\prod\nolimits_{m=1}^{l}(F^{j_m}_{z_{k_m}}(0)+F^{j_m}_{w}(0)Q_{z_{k_m}}(0,\chi,0)) \tag{13}$$

and $\Psi$ (resp. $\Psi'$) is a sum of terms each of which has a factor $q_{\mathbf{m}\nu}$ (resp. $q'_{\mathbf{m}\nu}$), with $(\mathbf{m},\nu)\prec(\mathbf{k},\mu)$.

Note that for $m_0\geq 2$, $P$ is in fact independent of the variable $\chi$. In this case, by definition, $P$ is the $l$-th tensor power of the matrix $F_z(0)$, (w.r.t. lexicographic order for indices in $\Lambda_l$), and therefore is non-singular, for every $H$. For $m_0=1$, we see in the same way that $P(0)$ is non-singular, therefore also that $P(\chi)$ is non-singular at every sufficiently small $\chi$.

Let us write

$$q^{H}_{\mathbf{k}\mu}(\chi):=\sum\nolimits_{|\mathbf{j}|=|\mathbf{k}|}q'_{\mathbf{j}\mu}(\bar{F}(\chi,0))\cdot P_{\mathbf{jk}}(\chi) \tag{14}$$

for part of the expression appearing in the right hand term of (12). Note that if $H$ is the identity, then $q^H\equiv q'$.

We can rewrite definition (14) in matrix form as

$$q^H(\chi)\equiv P(\chi)^T\cdot q'(\bar{F}(\chi,0)), \tag{15}$$

where

$$q^H=(q^H_{\mathbf{k}\mu})_{\mathbf{k}\in\Lambda_{\nu_0}},\quad q'=(q'_{\mathbf{k}\mu})_{\mathbf{k}\in\Lambda_{\nu_0}} \tag{16}$$

are column vectors in $\mathbf{C}^{n+1}$ (and in (15) we omitted writing the dependence in $\mu$).

With the above computations, we are now ready to prove Proposition 1.

PROOF. Let $M'=M$ as sets (and also $p=p'$), but with coordinates $(z',w')$ and corresponding functions $q'_{\mathbf{k}\mu}(\chi')$ for $M'$, and let $H$ be the biholomorphism $H:\mathbf{C}^{n+1}\to\mathbf{C}^{n+1}$ taking $M$ to $M'$ and $p$ to $p'$, that changes the coordinates on $\mathbf{C}^{n+1}$ from $(z,w)$ to $(z',w')$.

Let $\mu'_0$, $\nu'_0$, $\mathscr{I}'$ be the corresponding invariants for $M'$. In the identity (12) at $\mathbf{k}'\in\Lambda_{\nu'_0}$ and $\mu'_0$, the remainders $\Psi$ and $\Psi'$ are identically 0. Since by the same identity (at $\chi = 0$), we have $G_{z^{\mathbf{k}'}\tau^{\mu_0'}}(0) = 0$, (12) becomes (apart from a non-zero factor) a vectorial equality of the form $q = q^H$, with notations as in (16) above.

Since as we have seen, $P$ is nonsingular, from $q' \not\equiv 0$ it follows that $q \not\equiv 0$.

In the same identity (12) at an arbitrary multi-index $(\mathbf{m}', \mu')$ preceding $(\mathbf{k}', \mu_0')$ in the order $\prec$, we see that the right-hand term is zero, therefore $q_{\mathbf{k}'\mu'}(\chi) \equiv 0$.

Together, these two facts show that $m_0 = m_0'$ and $\mu_0 = \mu_0'$, therefore $\nu_0 = \nu_0'$. From the relation $q = q^H$, we also have $\mathscr{I} \subset \bar{F}(\chi, 0)^* \mathscr{I}'$. Finally, reverting $(z, w)$ with $(z', w')$ (or using the fact that $P$ is non-singular) we see that this inclusion is in fact an equality. $\square$

Using the above proof, we can now write the identity (12) for an arbitrary $H \in \mathrm{Aut}(M, 0)$ and fixed normal coordinates $(z, w)$ in a simplified form. For $\mathbf{k} \in \Lambda_{\nu_0}$, we have $G_{z^{\mathbf{k}} w^{\mu_0}}(0) = 0$ and $\Psi(\chi) \equiv 0$, $\Psi'(\chi) \equiv 0$. Further, using (11) if $m_0 \geq 2$ and the fact that if $m_0 = 1$, we must have $\mu_0 = 0$, we get that $\bar{G}_w(\chi, 0)^{\mu_0} \equiv G_w(0)^{\mu_0}$ in both cases $m_0 = 1$ and $m_0 \geq 2$. Therefore (12) becomes

$$q_{\mathbf{k}\mu_0}(\chi) \equiv q^H_{\mathbf{k}\mu_0}(\chi) \cdot G_w(0)^{\mu_0 - 1}, \tag{17}$$

for arbitrary $\mathbf{k} \in \Lambda_{\nu_0}$, or in vectorial notation,

$$q(\chi) \equiv q^H(\chi) \cdot G_w(0)^{\mu_0 - 1}. \tag{18}$$

Let us continue to derive some further identities from (9).

By differentiating (9) with respect to $z^{\mathbf{m}} \tau^{\mu_0 + k}$ at $(z, \tau) = 0$, where $|\mathbf{m}| = \nu_0$ and $k \geq 1$, we obtain equations for $\bar{F}_{w^k}(\chi, 0)$ and $\bar{G}_{w^k}(\chi, 0)$ in terms of lower order derivatives. Precisely, we have

$$\begin{aligned} &G_{z^{\mathbf{m}} w^{\mu_0 + k}}(0) + \Psi_1(\chi, j_0^{k+1} G) \\ &\quad \equiv \sum_{j=1}^n \partial_{\chi_j'} q^H_{\mathbf{m}\mu_0}(\chi) \cdot \bar{F}^j_{w^k}(\chi, 0) \cdot \bar{G}_w(\chi, 0)^{\mu_0} \\ &\qquad + \Psi_2(\chi, \bar{F}(\chi, 0), j_0^{k+1} F, (\bar{F}_{w^r}(\chi, 0))_{r \leq k-1}, (\bar{G}_{w^s}(\chi, 0))_{s \leq k+1}), \end{aligned} \tag{19}$$

where the functions $\Psi_1(\chi, \Lambda)$ and $\Psi_2(\chi, \chi', \Lambda_1, \Lambda_2, \Lambda_3)$ are polynomials in $\Lambda$ and $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ respectively, with holomorphic coefficients in $\chi$, $\chi'$. Moreover, if $m_0 = 1$, the term $\bar{G}_{w^{k+1}}(\chi, 0)$ does not appear in $\Psi_2$, because in this case $\tau_0 = 0$, i.e. we differentiate (9) once in $z$ and $k$ times in $\tau$. Note also that applying (19) at $\chi = 0$ shows that its first term $G_{z^{\mathbf{m}} w^{\mu_0 + k}}(0)$ depends on the $(k+1)$-jet of $H$ only.

Thus the identity (19) is an equation for (the vector) $\bar{F}_{w^k}(\chi, 0)$ in terms of lower order such derivatives for $\bar{F}$ and of order at most $k+1$ for $\bar{G}$ if $m_0 \geq 2$ and at most $k$ if $m_0 = 1$.

Further, we can obtain an expression for $\bar{G}_{w^s}(\chi,0)$ in terms of lower such derivatives by differentiating the main identity (repeatedly) in $\tau$ at $(z,\tau)=0$. Namely, we have, for every $s\geq 2$:

$$G_{w^s}(0) \equiv \sum_{j=1}^{n} Q'_{z_j}(0,\bar{F}(\chi,0),0)\cdot F^j_{w^s}(0) + \bar{G}_{w^s}(\chi,0)$$
$$+\sum_{j,l=1}^{n} \partial_{\chi'_l} Q'_{z_j}(0,\bar{F}(\chi,0),0)\cdot \bar{F}^j_{w^{s-1}}(\chi,0)\cdot F^l_w(0) + \Psi_3, \qquad (20)$$

where

$$\Psi_3 = \Psi_3(\bar{F}(\chi,0),(F_{w^i}(0))_{i\leq s-1},(\bar{F}_{w^r}(\chi,0))_{r\leq s-2},(\bar{G}_{w^j}(\chi,0))_{j\leq s-1}), \qquad (21)$$

is such that $\Psi_3(\chi',\Lambda_1,\Lambda_2,\Lambda_3)$ is a polynomial in $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ with holomorphic coefficients in $\chi'$. For $m_0\geq 2$, the $Q'_{z_j}(0,\bar{F}(\chi,0),0)$ terms are $\equiv 0$, so there is no term involving $\bar{F}_{w^{s-1}}$. Combining the two equations (19) and (20), we see that in both cases $m_0=1$ and $m_0\geq 2$ it is possible to obtain an equation for $\bar{F}_{w^k}(\chi,0)$ in terms of lower such derivatives which depends on $j_0^{k+1}(H)$, and similarly for $\bar{G}_{w^s}(\chi,0)$, for every $k,s\geq 1$. (For $s=1$, the identity (11) gives such an expression for $\bar{G}$.)

## 3. Jet Determination along the 0-Segre Set

In this section we shall use the identities obtained in Section 2 to prove finite determination along the 0-Segre set $\{(z,0)\,|\,z\in\mathbf{C}^n\}$ for germs of automorphisms at 0 of a real-analytic hypersurface $M\subset\mathbf{C}^{n+1}$ by their jets at 0 of a sufficiently high order.

With the notations of Section 2, we make the following *finiteness assumption* (F.A. at 0):

$$\mathrm{rk}(\partial_{\chi_j} q_{\mathbf{k}\mu_0}(\chi))_{|\mathbf{k}|=\nu_0,\, j=1,\dots,n} = n, \qquad (22)$$

which means that at the generic point $\chi$ near 0, the maximal rank $n$ is attained. Note that from relation (18) of Section 2, it follows that F.A. is invariant under normal changes of coordinates at 0.

As an aside, let us note that condition F.A. is implied, using e.g. Theorem 5.1.37 of [1], by the following condition on the ideal $\mathscr{I}_M(0)$ of $\mathbf{C}\{\chi\}$ defined by (7) in Section 2:

$$\mathrm{codim}\,\mathscr{I}_M(0) := \dim_{\mathbf{C}} \mathbf{C}\{\chi\}/\mathscr{I}_M(0) < \infty, \qquad (23)$$

By Proposition 1, condition (23) is also invariant under changes of normal coordinates for $M$ at 0.

Also, it is obvious that for $n = 1$, condition F.A. is equivalent with *Levi non-flatness.*

Our main technical result is the following extension from $n = 1$ to arbitrary $n \geq 1$ of Theorem 3.1 in [9] (which assumes Levi non-flatness).

THEOREM 1. *Let $M \subset \mathbf{C}^{n+1}$ be a real-analytic hypersurface and let $(z, w)$ be normal coordinates at $p = 0 \in M$. Assume that the finiteness condition F.A. holds for $M$ at $p$. Then for every integer $k \geq 0$, there exists a sufficiently large integer $l = l(k)$ such that for every automorphism $H$ of $(M, p)$, the derivative $H_{w^k}(z, 0)$ is determined by $j_0^l(H)$.*

For given $M$ and normal coordinates $(z, w) \in \mathbf{C}^{n+1}$ at $p$, let us fix an $l$-jet of an automorphism $H$ of $(M, 0)$, where $l$ is large enough (to be determined). Let us call this the "data" of our problem. From the above theorem it follows also that all derivatives $H_{z^{\mathbf{m}} w^k}(z, 0)$ are determined from the data only, with $l = l(\mathbf{m}, k)$; in particular this shows that for given $k$, there exists a sufficiently large $l = l(k)$ such that the jet $j_Z^k(H)$ of $H$ along the 0-Segre set is determined by $j_0^l(H)$, i.e. by the data only.

For the proof of Theorem 1, we shall need the following two lemmas.

LEMMA 1. *Let $P(\chi, \zeta) \in \mathbf{C}\{\chi\}[\zeta]$, where $(\chi, \zeta) \in \mathbf{C}^n \times \mathbf{C}$, be an irreducible Weierstrass polynomial. If there exists a holomorphic root $\Phi = \Phi(\chi)$ of $P$ for $\chi$ in a neighborhood of $0 \in \mathbf{C}^n$, then $deg(P) = 1$.*

PROOF. Let $W$ be the germ of subvariety of $\mathbf{C}^{n+1}$ consisting of the zeros of $P$ and let $\pi : \mathbf{C}^{n+1} \to \mathbf{C}^n$ denote the projection on the first coordinates. The map $\tilde{\Phi} = (id, \Phi) : (\mathbf{C}^n, 0) \to (W, 0)$ is such that $\pi_{|W} \circ \tilde{\Phi} = \text{id}$. Let $i : W \to \mathbf{C}^{n+1}$ denote the inclusion. Since $\pi \circ i \circ \tilde{\Phi} = \text{id}$, $d_0(i \circ \tilde{\Phi})$ is injective, therefore the set $X = \tilde{\Phi}\ (\mathbf{C}^n)$ is (the germ of) a complex manifold at 0. Using $X \subset W$, $\dim(X) = \dim(W) = n$ and $W$ irreducible, we obtain (e.g. [13], ch. 3, Prop. 7) that $W = X$ (as germs at 0). Further, since $\pi_{|W} \circ \tilde{\Phi} = \text{id}$, and $W$ is a manifold, we see that $\pi_{|W}$ is a submersion on a neighborhood of 0. Its fibres above a sufficiently small neighborhood of 0 must have the same cardinality, which must be one (the cardinality of the fibre above 0). But for points where the discriminant $Dis(P) \neq 0$, (an open set since $P$ is irreducible), this cardinality equals $\deg(P)$, therefore $\deg(P) = 1$. $\square$

We shall use Lemma 1 to prove the following

LEMMA 2. *Let $P(\chi,\zeta)$ be a Weierstrass polynomial with $(\chi,\zeta)\in\mathbf{C}^n\times\mathbf{C}$. Then there exists an integer $l$ (depending on $P$ only) such that every germ of a holomorphic root of $P$ is determined by its $l$-jet at* 0.

PROOF. Let $P(\chi,\zeta)=(\zeta-a_1(\chi))^{k_1}\cdot\ldots\cdot(\zeta-a_m(\chi))^{k_m}\cdot R(\chi,\zeta)$ be the decomposition of $P$ into irreducible degree one factors, where $a_j(\chi)$ are distinct holomorphic functions near $0\in\mathbf{C}^n$, and $R$ has only irreducible factors of higher degree. If $m\geq 2$, choose $l$ large enough such that $j_0^l(a_s)\neq j_0^l(a_t)$ for $s\neq t$, $s,t=1,\ldots,m$. If $m=1$, one may choose any $l\geq 0$.

Let $\Phi$, $\Psi$ be holomorphic roots near 0 of $P$ having the same $l$-jet at 0 and let $\tilde{\Phi}=(id,\Phi)$, $\tilde{\Psi}=(id,\Psi)$.

Since $\tilde{\Phi}$ (resp. $\tilde{\Psi}$) maps $\mathbf{C}^n$ to irreducible varieties, it follows that $\Phi$, $\Psi$ are each roots of some irreducible factors of $P$; by Lemma 1, these factors are of degree one, therefore $\Phi=a_s$ and $\Psi=a_t$ for some $t,s\in\{1,\ldots,m\}$. By the choice of $l$ we must have $s=t$, therefore $\Phi=\Psi$. $\square$

We are now ready to prove Theorem 1.

PROOF. We use induction on $k$. For $k=0$, we want to show that $H(z,0)$ is determined by the data only. By (10), we need only prove this for $\bar{F}(\chi,0)$ instead. By (18), $\zeta=\bar{F}(\chi,0)$ is a simultaneous root of the holomorphic functions of $2n$ variables

$$q(\chi)-P(\chi)^T q(\zeta)\cdot G_w(0)^{\mu_0-1}, \tag{24}$$

in the matricial notation of (15) and (16), with $\mu=\mu_0$. Note that by the finiteness assumption F.A. and because the matrix $P$ is non-singular, the ideal in $\mathbf{C}\{\zeta\}$ generated by the restrictions of the above functions to some generic $\chi_0$ near 0 has finite codimension. We may thereby apply a theorem (see e.g. Theorem 5.3.9 of [1]) which reduces simultaneous zeros to a normal form, to obtain $n$ Weierstrass polynomials $P_j(\chi,\zeta_j)\in\mathbf{C}\{\chi-\chi_0\}[\zeta_j]$, such that $\zeta_j=\bar{F}^j(\chi,0)$ satisfies $P_j(\chi,\zeta_j)\equiv 0$, for $j=1,\ldots,n$ and $\chi$ near $\chi_0$. Moreover, the polynomials $P_j$ may be chosen to depend on the data only. This reduces the question of finite determination along the 0-th Segre set to the case of $\bar{F}^j(\chi,0)$, for every fixed $j$. By Lemma 2 for the polynomial $P_j$ and for $l=l_0$ sufficiently large, we obtain that $\bar{F}^j(\chi,0)$ is determined (for $\chi$ in an open set, therefore by the identity principle, everywhere) from the data only, which proves the case $k=0$.

For the inductive step $k-1\to k$, where $k\geq 1$, we shall use the identities (19) and (20) which together show that the map $\zeta=(\zeta_1,\ldots,\zeta_n)$, defined by

$\zeta_j = \bar{F}^j_{w^k}(\chi, 0)$, $j = 1, \dots, n$, is a simultaneous root of a set of equations of the form

$$\sum_{j=1}^{n} \partial_{\chi_j} q^H_{\mathbf{k}\mu_0}(\chi) \cdot \zeta_j \equiv \Psi_{\mathbf{k}\mu_0}(\chi), \tag{25}$$

for every $\mathbf{k} \in \Lambda_{\nu_0}$, where $\Psi_{\mathbf{k}\mu_0}(\chi)$ depends on $j_0^{k+1}(H)$ and on the lower order derivatives $(\bar{F}_{w^r}(\chi, 0))_{r<k}$ and $(\bar{G}_{w^s}(\chi, 0))_{s<k}$, therefore by induction on the data only.

From condition F.A., it follows that there exists a subset $B \subset \Lambda_{\nu_0}$ consisting of $n$ multi-indices such that $\det(\partial_{\chi_j} q^H_{\mathbf{m}\mu_0}(\chi))_{\mathbf{m} \in B, j=1,\dots,n} \not\equiv 0$. This implies that for $\chi$ outside a proper complex subvariety, we may use Cramer's rule to solve the linear system corresponding to (25), where the set $A$ is replaced by $B$ and where $\chi$ is a parameter.

We obtain that (for $\chi$ generic, therefore by the identity principle for all $\chi$ near 0), $\bar{F}^j_{w^k}(\chi, 0)$ is determined from the data (with jet-length $= \max(l_0, k+1)$), for every $j = 1, \dots, n$. By induction using the identity (20), it follows that the same is true for $\bar{G}_{w^k}(\chi, 0)$ therefore for $H_{w^k}(\chi, 0)$, thus proving the inductive step and the theorem. $\square$

## 4. Relation between F.A. and other Finiteness Assumptions

In this Section we compare the finiteness assumption F.A. and the invariants $m_0$, $\mu_0$ (defined in Sections 2 and 3) with various other finiteness conditions and invariants. We show that (in terms of a real-valued defining function $\Phi$ for $M$ in normal coordinates near 0) if

$$\Phi_{\tau^\mu}(z, \chi, 0) \equiv 0 \quad \text{for all } \mu < \mu_0, \tag{26}$$

then F.A. implies $\mu_0$-infinite $n$-nondegenerate (as defined in [8]), at $\chi_0$ in the 0-Segre set, generic near 0; from this it follows [10], [8], that we also have in the complement of a real-analytic subvariety of the 0-Segre set, $\mu_0$-infinite type 2; recall that we are assuming $\mu_0$ finite (Section 2). These remarks will be needed in Section 5 for the proof of Theorem 2.

Further, it is proved in [10] that the subset of the 0-Segre set consisting of weakly essential points is open; and also that if $M$ is weakly essential then $M$ is essentially finite at all points in the complement of the 0-th Segre set, therefore finitely non-degenerate at the generic point on $M$.

Let us briefly recall the definitions at this point. In terms of a real-valued defining function

$$M : \mathrm{Im}\, w = \Phi(z, \bar{z}, \mathrm{Re}\, w), \tag{27}$$

satisfying the normality conditions

$$\Phi(z, 0, s) \equiv \Phi(0, \bar{z}, s) \equiv 0, \tag{28}$$

and with $w = s + it$, the *m-infinite type* of $M$ along $s = 0$ is the smallest integer s.t. $\Phi_{s^m}(z, \chi, 0) \not\equiv 0$. In fact, for infinite type one assumes $m \geq 1$, since $m = 0$ corresponds to finite type hypersurfaces.

Further, if $M$ is of $m$-infinite type, $M$ is called *weakly-essential* (or *m-weakly-essential*) at 0, if the ideal generated by the collection $(\Phi_{z^{\mathbf{k}} s^m}(0, \chi, 0))_{\mathbf{k} \in \Lambda}$ is of finite codimension in $\mathbf{C}\{\chi\}$.

Besides, $M$ is said to be of *m-infinite type* $r \geq 2$ if there exist multi-indices $\mathbf{k}$, $\mathbf{l}$ satisfying $|\mathbf{k}| + |\mathbf{l}| = r$ and $\Phi_{z^{\mathbf{k}} \chi^{\mathbf{l}} \tau^m}(0) \neq 0$, and if $r$ is minimal with this property.

Also, $M$ is said to be *m-infinite l-nondegenerate* at 0 if

$$\det(\partial_{\chi_j} \Phi_{z^{\mathbf{k}} s^m}(0))_{j=1,\dots,n,\, \mathbf{k} \in B} \neq 0, \tag{29}$$

for some subset $B$ of the set of multi-indices $\mathbf{k}$ such that $|\mathbf{k}| \leq l$.

It is known that the above integers $m$ and $r$ for hypersurfaces of infinite type are invariant under change of normal coordinates ([7], [12]).

Because the (complexification of the real) defining function $\Phi$ as in (27) and the corresponding (complex) defining function $Q$ considered in Sections 2 and 3 are related by

$$Q(z, \chi, \tau) - \tau - 2i\Phi(z, \chi, (Q(z, \chi, \tau) + \tau)/2) \equiv 0, \tag{30}$$

it follows by the normality of the coordinates and the chain rule that (up to non-zero constant factors)

$$Q_{z^{\mathbf{k}} \tau^{\mu_0}}(0, \chi, 0) \equiv \Phi_{z^{\mathbf{k}} s^{\mu_0}}(0, \chi, 0), \tag{31}$$

for all multi-indices $\mathbf{k} \in \Lambda_{\nu_0}$.

Therefore in the definitions of the invariants of Section 2 we may replace $Q$ with $\Phi$. In particular it is clear from the definitions that F.A. implies that at the generic point near 0 in the 0-Segre set, we have $m$-infinite type for some $m \leq \mu_0$ and that if we assume (26), i.e. that $\mu_0 = m$, then F.A. implies $\mu_0$-infinite $n$-non-degenerate at the generic point as above, which in turn implies holomorphically non-degenerate at some point, therefore everywhere in a neighborhood of 0. (Of course, that F.A. implies holomorphic-nondegeneracy can be checked directly, cf. [1].)

Before ending this section, let us illustrate with some examples.

EXAMPLE 1. For $n = 1$, condition F.A. coincides with weakly essential and with Levi non-flatness. But for $n \geq 2$ there exist weakly essential hypersurfaces that do not satisfy F.A.

For instance, let the (real, as in (27)) defining function for $M \subset \mathbf{C}^3$ be given by

$$\Phi(z, \chi, \tau) \equiv \tau^{\mu}(z_1^l \chi_1^l + z_2^k \chi_2^k),$$

for some integers $k, l \geq 2$. Then $M$ is weakly essentially finite with $m = \mu$ and if $k \neq l$, $M$ does not satisfy F.A. at 0.

EXAMPLE 2. To show that F.A. does not imply weakly-essential in general, let us fix integers $1 \leq s < r$, $0 \leq \mu_0 < m_0$, and let $M \subset \mathbf{C}^3$ be the hypersurface defined (in real form) by

$$\Phi(z, \chi, \tau) \equiv \tau^{\mu_0}((z_1\chi_1)^{m_0-\mu_0} + (z_2\chi_2)^{m_0-\mu_0}) + \tau^{\mu_0-s}(z_1\chi_1)^{m_0-\mu_0+r}.$$

Then $\mu_0$, $m_0$ are the invariants of Section 2 and F.A. holds, but $M$ is not weakly-essential.

EXAMPLE 3. It is easy to construct examples of families of hypersurfaces that satisfy F.A. and are also $m$-weakly-essential; in fact, for every fixed $m$ and every ideal of finite codimension $\mathscr{I} \subset \mathbf{C}\{\chi\}$ that is generated by monomials of the same degree, there exists a continuous family of hypersurfaces $M$ such that $\mathscr{I} = \mathscr{I}_M(0)$ and $m = \mu_0$.

Namely, if $(\phi_i)_{i=1,\dots,r}$ is a set of linearly independent monomial generators of $\mathscr{I}$, such a family is given by hypersurfaces $M$ defined (in real form) by

$$\Phi(z, \chi, \tau) \equiv \tau^m \sum_{i=1}^{r} c_i \phi_i(z) \cdot \bar{\phi}_i(\chi),$$

with $(c_i)_{i=1,\dots,r}$ arbitrary real non-zero constants.

## 5. Finite Determination of Automorphisms

In this section we prove a theorem on finite determination of automorphisms at a point $p$ of a real-analytic hypersurface $M$ by their $k$-jets at $p$ for some sufficiently large $k$. We shall use Theorem 1 together with the results in [7] and [8] on singular complete systems in the characteristic direction, and shall assume that $M$ is $\mu_0$-infinite type and satisfies F.A. at $p$, (in particular it is holomorphically-nondegenerate). Our main result in the infinite-type case is the following

THEOREM 2. *Let $M \subset \mathbf{C}^{n+1}$ be a real-analytic hypersurface which is $\mu_0$-infinite type and satisfies F.A. at $p \in M$.*

*Then there exists a sufficiently large integer $k = k_M(p)$ such that every automorphism of $(M, p)$ is determined by its $k$-jet at $p$.*

The proof follows closely that of Theorem 1.1 in [9], but we include it for completeness.

PROOF. Let us fix the $k$-jet at $p$ (with $k$ to be determined) of an automorphism $H^1$ such that $H^1(p) = p$, and let $H^2$ be a variable automorphism having the same $k$-jet at $p$.

Since, by the remarks in the beginning of Section 4, F.A. together with (26) implies that there is an integer $l$ such that $M$ is $l$-non-degenerate $\mu_0$-infinite type 2 at the generic point on the 0-th Segre set, and since by Theorem 1, the $k'$-jet at $p$ determines the $k$-jet along the 0-Segre set if $k'$ is large enough, we may assume, by replacing $p$ with the generic point and $k'$ with $k$, that $M$ is $l$-nondegenerate $\mu_0$-infinite type 2 at $p$.

At $p$, for any fixed normal coordinates $(z, w)$, let us write in real coordinates, $w = s + it$, $x = z \in \mathbf{R}^{2n}$, and let $(x, s)$ be the parameter on $M$.

Then for the CR diffeomorphisms $h^i = H^i_{|M}$, $i = 1, 2$, (we may omit the superscripts and write $h$ to refer to both $h^1$ and $h^2$ in the sequel) we may apply Theorem 2.1 of [8] from which it follows that a certain map $U_h$ (that will be described below) satisfies one and the same (for $i = 1, 2$) real-analytic singular O.D.E. in the variable $s$ and parameter $x$.

The construction for $U_h$ is as follows. By Theorem 2.1 of [8] we have first, for $h = (f, g)$, with $f = (f^1, \dots, f^n)$ in the real coordinates above, that $g$ satisfies a singular O.D.E.

$$s^m \partial_s g = v_h \cdot g^m + r, \tag{32}$$

where $v_h$ is a $C^\infty$ function (since $M$ is type 2 at $p$) and $r$ is a remainder depending (real-analytically) on $s^m \partial_s f$. Next, if we form $u_h = (\partial_x h, s^m \partial_s f, v_h)$, then by the same theorem, all the derivatives of $u_h$ of order $2l + 1$ depend real-analytically on (and therefore are determined by) its derivatives of order $\leq 2l$. From this it follows, using (32), that the map $U_h = ((u_{h,\mathbf{m}})_{0 \leq |\mathbf{m}| \leq 2l}, h)$ satisfies a real-analytic equation of the form $s^m \partial_s U_h = R(U_h, s)$.

Next, for the function $\tilde{U}_h = \tilde{U}_h(x, s) \equiv U_h(x, s) - U_h(x, 0)$, a similar equation holds (by Theorem 1) and we may apply Theorem 5.1 on singular O.D.E.s with a parameter from [9], since the initial condition $\tilde{U}_h(x, 0) \equiv 0$ is satisfied. It follows that there exists a sufficiently high order $k'$ such that, for every fixed value of the parameter $x$, the solution $\tilde{U}_h(x,.)$ is uniquely determined from its $k'$-jet at 0, which (as we shall see) is the same for both $h^1$ and $h^2$. Once we know this latter fact, our theorem follows, since $h$ is a component of the map $U_h$. (Note also that the same $k'$ can be used for all $H^1$, $H^2$ since we may reduce this question to the case of identity and $H = H^2 \circ (H^1)^{-1}$.)

Thus it only remains to check that $\partial_{s^r} U_{h^i}(x,0)$ is the same for $i = 1,2$, and for all $r \leq k'$. By Theorem 1, for fixed $k_1$, the jet $j_0^{k_1}H$ along the 0-Segre set is determined from $j_0^k H$, if $k$ is sufficiently large. Next we choose $k_1$ such that along $s = 0$, the $k_1$-jet of $H$ determines the $k'$-jet of $U_h$. This is possible with $k_1 = k' + 2l + 1$, since by definition $U_h$ depends on the derivatives of order $\leq 2l + 1$ of $h$. (It also depends on $v_h$ which by the equation (32) for $g$, is uniquely determined by $s^m \partial_s h$, therefore its derivatives of order $\leq 2l$ that appear in $U_h$ are reduced to the above dependence of order $2l + 1$.) □

## 6. One-jet Determination for Sections in Analytic Sets

In sections 2 and 3 we have encountered analytic sets $Z$ of the form

$$Z = \{(\chi,\zeta) \in \mathbf{C}^n \times \mathbf{C}^n \mid q^{(\alpha)}(\zeta) = q^{(\alpha)}(\chi), \alpha \in A\}. \tag{33}$$

where the $q^{(\alpha)}$'s are analytic and 0 at the origin, and the indexing set $A$ is finite.

Namely, if the 1-jet at 0 of a biholomorphism $H = (F,G) \in Aut(M,p)$ coincides with that of the identity map, then condition (18) on the (vector) functions $q$ is a set of equations defining a set $Z$ as above.

Recall that in the proof of Theorem 1, assuming F.A., we used precisely such equations in order to determine $\zeta = \bar{F}(\chi,0)$, and therefore $H$ along the 0-Segre set, from the jet of $H$ of a sufficiently high length.

In the finite type case (Section 7) where *two* sets of the form above appear, the following theorem will be used to prove 2-jet sufficiency for elements in $\mathrm{Aut}(M,p)$, under assumptions on the non-vanishing of corresponding Jacobians along the diagonal $n$-plane in $\mathbf{C}^n \times \mathbf{C}^n$.

THEOREM 3. *Let $q^{(\alpha)} : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ be germs of analytic functions such that $q^{(\alpha)}(0,0) = 0$, where $\alpha \in A$, with $A$ a finite set. Let $p : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ be the projection on the first factor and let the analytic set $Z = \{(\chi,\zeta) \in \mathbf{C}^n \times \mathbf{C}^n \mid q^{(\alpha)}(\chi,\zeta) = 0, \alpha \in A\}$ be such that $Z \supseteq \pi$, where $\pi$ denotes a non-vertical n-plane in $\mathbf{C}^n \times \mathbf{C}^n$.*

*Assume that there exists $B \subset A$, $|B| = n$, such that*

$$Jac_{\zeta}(in(q^{(\alpha)}))_{\alpha\in B|\pi} \not\equiv 0. \tag{34}$$

*Then every analytic section* $\Phi$ *in* $Z$ *(i.e.* $\Phi : \mathbf{C}^n \to Z$ *such that* $p\circ\Phi = id$*) for which the tangent space* $T_0(\Phi) = \pi$*, must coincide with the section* $\sigma : \mathbf{C}^n \to \pi$ *such that* $p\circ\sigma = id$*.*

REMARKS. 1) The above theorem is in the spirit of Lemma 13.2 of [5]. In addition, the condition on the *initial* homogeneous polynomials of the defining functions $(q^{(\alpha)})_\alpha$ allows for *one*-jet determination of sections of $Z$.

2) Theorem 3 may be regarded as a version of the implicit function theorem. For $n = 1$, taking $\pi$ to be the $\chi$-axis, we are given a function $q(\chi,\zeta)$ which is zero along the $\chi$-axis and is such that $\partial(in(q))/\partial\zeta(\chi,0) \not\equiv 0$ for $\chi \neq 0$, although possibly $\partial(in(q))/\partial\zeta(0,0) = 0$. Then by Theorem 3, $\Phi(\chi) \equiv (\chi,0)$ is the only section in the zero set of $q$ which is tangent to the $\chi$-axis at 0.

3) The non-vanishing assumption (34) in Theorem 3 cannot be replaced by the weaker assumption $Jac_{\zeta}(q^{(\alpha)})_{\alpha\in B|\pi} \not\equiv 0$, for some $B \subset A$, $|B| = n$.

For example, let $n = 1$, $\pi = \{\zeta = 0\}$ and let $q(\chi,\zeta) \equiv \zeta(\zeta-\chi^2)$. Then $\partial q/\partial\zeta(\chi,0) \equiv -\chi^2 \not\equiv 0$, while $\partial(in(q))/\partial\zeta(\chi,0) \equiv 0$, and the conclusion of Theorem 3 does not hold, since the zero set of $q$ has two distinct sections $\Phi_1(\chi) \equiv (\chi,0)$ and $\Phi_2(\chi) \equiv (\chi,\chi^2)$ which are tangent to $\pi$ at 0.

PROOF. By a linear change of coordinates on $\mathbf{C}^n \times \mathbf{C}^n$ (note that this does not affect condition (34)), we may assume that $\pi = \{\zeta = 0\}$. For every $\alpha \in A$ we may assume $v(q^{(\alpha)}) \geq 1$. Since $Z \supseteq \pi$, we may write

$$q^{(\alpha)}(\chi,\zeta) \equiv \sum_{j=1}^{n} \zeta_j \cdot d_j^{\alpha}(\chi,\zeta), \tag{35}$$

where the function $d_j^{\alpha}$ is analytic in all its variables and by the Taylor formula is related to the $\zeta$-derivative of $q^{(\alpha)}$ by

$$d_j^{\alpha}(\chi,0) \equiv \partial q^{(\alpha)}/\partial\zeta_j(\chi,0). \tag{36}$$

If the section $\Phi$ in $Z$ is defined by $\Phi = (id,\zeta)$, with $\zeta = \zeta(\chi)$, then

$$\sum_{j=1}^{n} \zeta_j(\chi) \cdot d_j^{\alpha}(\chi,\zeta(\chi)) \equiv 0, \tag{37}$$

for every $\alpha \in A$.

Let us denote $\Delta := det(d_j^{\alpha})_{\alpha\in B, j=1,\dots,n}$, a function of the variables $(\chi,\zeta)$. The rest of the proof will consist in proving that, for a section $\Phi$ as in the statement of the theorem, i.e. such that its component $\zeta$ satisfies $\zeta(\chi) = o(\chi)$, $\Delta(\chi,\zeta(\chi)) \not\equiv 0$;

from this it will follow that for fixed $\chi$ generic, the homogeneous linear system (37) has the unique solution $\zeta_j(\chi) \equiv 0$, $j = 1, \dots, n$, i.e. the uniqueness assertion of the theorem.

From the relation (36) obtained above, we have

$$\Delta(\chi, 0) \equiv det(d_j^\alpha(\chi, 0))_{\alpha \in B, j=1,\dots,n} \equiv Jac_\zeta(q^{(\alpha)})(\chi, 0)_{\alpha \in B}, \tag{38}$$

and because by assumption $Jac_\zeta(in(q^{(\alpha)}))_{\alpha \in B}(\chi, 0) \not\equiv 0$, $\Delta(\chi, 0)$ has finite order equal with

$$m_0 := \sum\nolimits_{\alpha \in B}(v(q^{(\alpha)}) - 1). \tag{39}$$

In particular $\Delta(\chi, 0) \not\equiv 0$ and therefore $\Delta(\chi, \zeta)$ is also non-zero and of a possibly smaller order, say $m_0'$.

On the other hand, by the construction of $\Delta(\chi, \zeta)$ using the finite differences $d_j^\alpha$ of the $q^{(\alpha)}$'s, its order is $\geq m_0$. Therefore $m_0' = m_0$, so that

$$v(\Delta(\chi, \zeta)) = v(\Delta(\chi, 0)) \tag{40}$$

Let now $\zeta = \zeta(\chi)$ be the second component of a section $\Phi$ as in the statement of the Theorem. We claim that $\Delta(\chi, \zeta(\chi)) \not\equiv 0$. Indeed, by (40) we can write

$$\Delta(\chi, \zeta) \equiv \sum\nolimits_{m=m_0}^{\infty} \Delta_m(\chi, \zeta), \tag{41}$$

where for every $m$, $\Delta_m$ is a (possibly zero) homogeneous polynomial of degree $m$ in the variables $(\chi, \zeta)$ and $\Delta_{m_0}(\chi, 0) \not\equiv 0$.

Since $\zeta(\chi) = o(\chi)$, it follows from (41) that

$$\Delta(\chi, \zeta(\chi)) \equiv \Delta_{m_0}(\chi, \zeta(\chi)) + o(m_0) \equiv \Delta_{m_0}(\chi, 0) + o(m_0) \not\equiv 0, \tag{42}$$

which is what was left to prove. $\square$

The following elementary lemma describing the (non-)vanishing of the Jacobian of homogeneous polynomials will be used in Section 7.

Let us first make some notations. For fixed $m \geq 1$, let $N = C_{m+n-1}^{n-1}$ be the dimension of the space of homogeneous polynomials in $\mathbf{C}[z_1, \dots, z_n]$ of degree $m$.

If $g_1, \dots, g_n \in \mathbf{C}[z_1, \dots, z_n]$, $n \geq 2$, are homogeneous polynomials of the same degree $m \geq 1$, given by $g_j(z) = \sum_{|\alpha|=m} c_{j\alpha} z^\alpha$, $j = 1, \dots, n$, let $C = (c_{j\alpha})_{|\alpha|=m, 1 \leq j \leq n}$ be their *coefficient matrix*, an $n \times N$-matrix with constant coefficients.

We consider also the $n \times N$-matrix with coefficients in $\mathbf{C}[z_1, \dots, z_n]$ given by $W = (w_{j\alpha})$, where

$$w_{j\alpha} = w_{j\alpha}(z) \equiv \alpha_j \cdot z^\alpha, \tag{43}$$

for $1 \le j \le n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ such that $|\alpha| = m$.

LEMMA 3. a) *Let* $g_1, \dots, g_n \in \mathbf{C}[z_1, \dots, z_n]$, $n \ge 2$, *be homogeneous polynomials of the same degree* $m \ge 1$, *with coefficient matrix* $C$.

*Then* $Jac(g_1, \dots, g_n) \not\equiv 0 \Leftrightarrow det(C \cdot W^T) \not\equiv 0$.

*For* $n = 2$, *these conditions are further equivalent with* $rk\ C = 2$.

b) *Let* $p_1, \dots, p_n \in \mathbf{C}[z_1, \dots, z_n]$ *be non-constant homogeneous polynomials of arbitary degrees. Then* $Jac(p_1, \dots, p_n) \not\equiv 0 \Leftrightarrow$ *the polynomials* $g_j := p_j^{r_j}$, $j = 1, \dots, n$ *for some (any) integers* $r_1, \dots, r_n$ *such that* $g_1, \dots, g_n$ *have the same degree, are such that their coefficient matrix satisfies the condition at* a).

*For* $n = 2$, $Jac(p_1, p_2) \equiv 0$ *if and only if* $p_1$ *and* $p_2$ *satisfy a polynomial relation of the form*

$$p_2^{r_2} = K \cdot p_1^{r_1}, \tag{44}$$

*for some integers* $r_1, r_2 \in \mathbf{N}$ *and non-zero constant* $K \in \mathbf{C}$.

REMARK. If $n \ge 3$, condition rk $C = n$ does not imply the non-vanishing of the Jacobian, i.e. linear independence (of homogeneous polynomials of the same degree) does not imply functional independence. An example of this is given by the polynomials $g_1 = z_1^2$, $g_2 = z_2^2$ and $g_3 = (z_1 + z_2)^2$ in $\mathbf{C}[z_1, z_2, z_3]$.

PROOF. a) Since

$$\text{grad}(g_j)(z) \equiv \left(z_1^{-1}\left(\sum\nolimits_{|\alpha|=m} c_{j\alpha}\alpha_1 z^\alpha\right), \dots, z_n^{-1}\left(\sum\nolimits_{|\alpha|=m} c_{j\alpha}\alpha_n z^\alpha\right)\right), \tag{45}$$

we have

$$\text{Jac}(g_1, \dots, g_n) = (z_1 \cdots z_n)^{-1}\, det\left(\sum\nolimits_{|\alpha|=m} c_{j\alpha} w_{l\alpha}\right)_{1 \le j,l \le n}, \tag{46}$$

from which the assertion for general $n$ follows. The assertion for $n = 2$ follows from this by direct computation with $C$ or by the following argument. Assume $\text{Jac}(g_1, g_2) \equiv 0$. Then the rank of the Jacobi matrix is one (since the $g_i$ are non-constant), and by the constant rank theorem, there exists (locally) an analytic function $\phi : \mathbf{C} \to \mathbf{C}$ such that $g_2(z) = \phi(g_1(z))$, for $z$ near some generic $z^0 \in \mathbf{C}^2$. Deshomogenizing, we have one-variable polynomials $G_1$, $G_2$ such that $G_i \circ \pi = g_i$, $i = 1, 2$, with $\pi : \mathbf{C}^2 \to \mathbf{P}^1(\mathbf{C})$ the canonical projection. Comparing degrees in the corresponding identity $G_2(z) = \phi(G_1(z))$ between one-variable functions, we see that (locally) $\phi(z) = cz$, for some non-zero constant $c$ and the assertion follows.

b) This follows from a) using

$$Jac(p_1^{r_1}, \dots, p_n^{r_n}) = (r_1 p_1^{r_1-1}) \cdots (r_n p_n^{r_n-1}) \cdot Jac(p_1, \dots, p_n). \qquad \square$$

## 7. Sufficiency of 2-jets in the Finite Type Case

In this section we shall assume that the real-analytic hypersurface $M \subset \mathbf{C}^{n+1}$ is of *finite type* at the point $p = 0$. In this case one can derive [9] more information from the defining function $Q$, using computations with

$$Q_{z_j}(z, \chi, \bar{Q}(\chi, z, 0)), \quad j = 1, \dots, n,$$

besides the computations with $Q_{z^{\mathbf{k}}}(0, \chi, 0)$ that were used in the infinite type case.

From this and the results of Section 6 on analytic sections, we prove (Theorem 4) that the elements of $Aut(M, p)$ are determined by their 2-jets at $p$, under some generic assumptions on $M$. By Lemma 3, these assumptions (conditions i) and ii) of Theorem 4) are slightly stronger than condition F.A. (or actually its corresponding version in the finite type case); therefore they imply holomorphic non-degeneracy, and even for $n = 1$ they do not imply finite non-degeneracy. Moreover, in the case $n = 2$, an example below [9] (of a finitely non-degenerate hypersurface that does not satisfy i)) shows that these assumptions are minimal, while if $n = 1$, our result extends Theorem 4.1 of [9], which shows that finite type implies 2-jet determination for elements in $Aut(M, p)$.

As in Section 2, let $M$ be given in normal coordinates at 0 by (1). We define ([9]) the invariant $\beta_0$ (which is finite iff $M$ is of finite type at 0) by

$$\beta_0 = \min\{\mathbf{b} \in \mathbf{N}^n \mid Q_{z^{\mathbf{b}}}(0, \chi, 0) \not\equiv 0\}. \tag{47}$$

As a side note, $\beta_0$ is only slightly related to the invariants of Section 2. Namely, if $\beta_0 < \infty$ then $m_0 < \infty$ and in fact $m_0 \le \beta_0$, and it is easy to see that equality does not hold in general. If $m_0 < \infty$ and $\mu_0 = 0$, then $m_0 = \beta_0$, therefore finite type holds, but conversely finite type does not imply $\mu_0 = 0$.

For this section, it is convenient to use the functions $r^{(\mathbf{b})}(\chi)$, for $\beta \in \mathbf{N}^n$, defined by expanding $Q$ at $\tau = 0$ w.r.t the variables $z$:

$$Q(z, \chi, 0) \equiv \sum\nolimits_{\mathbf{b} \in \mathbf{N}^n} r^{(\mathbf{b})}(\chi) z^{\mathbf{b}}. \tag{48}$$

These are related with the functions defined in Section 1, by $r^{(\mathbf{b})}(\chi) = q_{\mathbf{k}(\mathbf{b})0}(\chi)/\mathbf{b}!$, where $\mathbf{k}(\mathbf{b})$ denotes the index in $\Lambda$ that defines the same derivative as $\mathbf{b} \in \mathbf{N}^n$.

For $j = 1, \dots, n$, and $z^0 \in \mathbf{C}^n$, we denote by

$$P_j = P_j^{(z^0)}(\chi) \equiv \sum\nolimits_{|\mathbf{b}|=\beta_0} \partial_{z_j} \bar{r}^{(\mathbf{b})}(z^0)\chi^{\mathbf{b}}, \tag{49}$$

a homogeneous polynomial of degree $\beta_0$ in $(\chi_1, \dots, \chi_n)$.

THEOREM 4. *Let $M \subset \mathbf{C}^{n+1}$, $n \geq 1$, be a real-analytic hypersurface which is of finite type at a point $p = 0 \in M$ and let $(z, w) \in \mathbf{C}^{n+1}$ be normal coordinates for $M$ at 0.*

*Assume that*

i) *there exists $B \subset \{|\mathbf{b}| = \beta_0\}$, $|B| = n$, such that*

$$Jac(in(r^{(\mathbf{b})})_{\mathbf{b} \in B}) \not\equiv 0 \tag{50}$$

*and*

ii) *for $z^0$ small, generic in $\mathbf{C}^n$,*

$$Jac(P_1^{(z^0)}, \dots, P_n^{(z^0)}) \not\equiv 0. \tag{51}$$

*Then every element in $Aut(M, p)$ is determined by its 2-jet at $p$.*

REMARK. By a proof similar to that of Proposition 1, (using derivatives in $z^{\mathbf{b}}$ only, where $\mathbf{b} \in \mathbf{N}^n$, $|\mathbf{b}| = \beta_0$), it follows that $\beta_0$ is invariant under change of normal coordinates at 0 and that the functions $(r_{\mathbf{b}}(\chi)_{|\mathbf{b}|=\beta_0})$ transform formally as (18) where we set $\mu_0 = 0$.

From this (using the transformation law for the Jacobian under change of coordinates), it is easy to see that condition i) of Theorem 4 is invariant under change of normal coordinates.

Further, condition ii) is also invariant. Indeed, if $z' = F(z, w)$, $w' = G(z, w)$, where $H = (F, G) \in Aut(M, 0)$ defines a normal change of coordinates at 0, taking first-order derivatives in $z$ in the main identity (9) at $\tau = \bar{Q}(\chi, z, 0)$, and using the reality condition (3), we obtain, as in the proof below (and without assuming $j_0^2(H) = j_0^2(id)$), relation (55) for $\zeta = \bar{F}(\chi, \bar{Q}(\chi, z^0, 0))$, which by formula (63) gives the following transformation law for the polynomials $(P_j)_{1 \leq j \leq n}$, assuming they are all not identically zero:

$$\begin{aligned}(P_1^{(z^0)}(\chi), &\dots, P_n^{(z^0)}(\chi)) \\ &\equiv G_w(z^0, 0)^{-1}\, Jac_z\, F(z^0, 0)^T \cdot (P_1'^{(F(z^0,0))}(\zeta), \dots, P_n'^{(F(z^0,0))}(\zeta)),\end{aligned} \tag{52}$$

where $\zeta = \bar{F}(\chi, \bar{Q}(\chi, z^0, 0))$ ranges over a neighborhood of $0 \in \mathbf{C}^n$, since for $z^0 = 0$, $\zeta = \bar{F}(\chi, 0)$ is an automorphism.

It follows that

$$Jac(P_1^{(z^0)}, \dots, P_n^{(z^0)}) \not\equiv 0 \Leftrightarrow Jac(P_1'^{(F(z^0,0))}, \dots, P_n'^{(F(z^0,0))}) \not\equiv 0 \tag{53}$$

and from this, the invariance of condition ii), since $z'^0 = F(z^0, 0)$ is generic near $0 \in \mathbf{C}^n$.

Proof. First, note that it suffices to prove that if $H \in Aut(M, 0)$ is such that $j_0^2(H) = j_0^2(id)$ then $H = id$. (Indeed, if $H_1$, $H_2$ are such that $j_0^2(H_1) = j_0^2(H_2)$, then $j_0^2(H_2 \circ H_1^{-1}) = j_0^2(id)$ and the general case follows.)

With the functions $(r^{(\mathbf{b})})_{|\mathbf{b}|=\beta_0}$ (resp. $(P_j)_{j=1,\dots,n}$), we shall associate (as in Section 6) analytic sets $Z$ (resp. $Y^{(z^0)}$) as follows.

Let

$$Z = \{(\chi, \zeta) \in \mathbf{C}^n \times \mathbf{C}^n \mid r^{(\mathbf{b})}(\chi) = r^{(\mathbf{b})}(\zeta), |\mathbf{b}| = \beta_0\}. \tag{54}$$

Note that (by identities (15) and (18) of Section 2 with $\nu_0$ replaced by $\beta_0$ and $\mu_0$ with 0), the set $Z$ is a set of equations that are necessarily satisfied by $\zeta = \bar{F}(\chi, 0)$ for all elements $H = (F, G) \in Aut(M, 0)$ for which $j_0^1 H = j_0^1(id)$.

Next, we define the set

$$\begin{aligned} Y^{(z^0)} = \{(\chi, \zeta) \in \mathbf{C}^n \times \mathbf{C}^n \mid &grad_z\, Q(z^0, \chi, \bar{Q}(\chi, z^0, 0)) \\ &- G_w(z^0, 0)^{-1} M(z^0, \chi)^T \cdot grad_z\, Q(F(z^0, 0), \zeta, \bar{Q}(\zeta, F(z^0, 0), 0)) = 0\}, \end{aligned} \tag{55}$$

for small, generic $z^0$. Here $grad_z$ is a column vector, while $M = M(z^0, \chi)$ denotes an $n \times n$ matrix with $\mathbf{C}$-valued entries $M_{kj}$ defined by

$$M_{kj} = M_{kj}(\chi) \equiv F_{z_j}^k(z^0, 0) + F_w^k(z^0, 0) \cdot Q_{z_j}(z^0, \chi, \bar{Q}(\chi, z^0, 0)). \tag{56}$$

The set $Y^{(z^0)}$ is a set of equations satisfied by $\zeta(\chi) \equiv \bar{F}(\chi, \bar{Q}(\chi, z^0, 0))$. Indeed, by taking derivatives w.r.t $z_j$ at $\tau = \bar{Q}(\chi, z^0, 0)$ in (9) one sees that $(\chi, \zeta(\chi)) \in Y^{(z^0)}$.

From assumption i) it follows that condition (34) of Theorem 3 is fulfilled for $Z$ along the diagonal $n$-plane $\pi = \{(\chi, \chi) \mid \chi \in \mathbf{C}^n\}$, while obviously $Z$ depends on the 1-data only. Therefore the map $\chi \to \zeta(\chi) = \bar{F}(\chi, 0)$ is 1-determined, i.e. $\bar{F}(\chi, 0) \equiv \chi$.

From this it follows also that the set $Y^{(z^0)}$ depends on the 2-data of the problem only. To see this, by construction, $Y^{(z^0)}$ depends on (parameters) $G_w(z^0, 0)$, $F_{z_j}^k(z^0, 0)$ and $F_w^k(z^0, 0)$, $1 \le k \le n$. Of these, the first two parameters are obviously those of the identity. Indeed since $F(z, 0)$ is 1-determined, so are its $z$-derivatives; by formula (11), $G_w(z, 0)$ is also 1-determined.

Further, to show that $F_w(z^0, 0)$ is 2-determined, let us replace the variable $z^0$ with $\chi$ and let us re-inspect the main identity (9) and—at stage (I)—its derivatives with respect to $z$ of order $\beta_0$.

If in the latter—at stage (II)—we take one further derivative with respect to $\tau$, and apply this at $(z, \tau) = 0$, we get terms involving $F_w(\chi, 0)$ as follows.

At stage (I), the LHT and the RHT consist of main terms with factors $z$-derivatives of $Q$ of order $\beta_0$, and remaining terms.

In the LHT, at stage (II), the $\tau$-derivatives of the main terms give the 2-jet dependence, while some of the remainder terms may have non-zero $\tau$-derivative (e.g. if $m_0 < \beta_0$, $\mu_0 = 1$). But, if at stage (I) a term contains a product of lower $z$-derivatives of $Q$, this will vanish at stage (II), by the Leibniz rule and the minimality of $\beta_0$. Thus the only surviving terms are those that at stage (I) do not contain products of $z$-derivatives of $Q$, but (by the chain rule applied to (9)) these can only contain factors of the form $G_{w^k z^{\mathbf{b}}}$, with $k \leq 1$ (and some $\mathbf{b}$) which as we have noted above are 1-determined.

In the RHT, at stage (II), the main terms produce the 2-jet dependence and the factors for $F_w^k(\chi, 0)$, which coincide with those of (19) taking formally $\mu_0 = 0$. For the remainder terms, the $\chi$-derivatives are zero by the minimality of $\beta_0$; the $z$- and respectively $\tau$-derivatives that survive depend on the 1-jet of $H$.

In total, $(F_w^k(\chi, 0))_{1 \leq k \leq n}$ is determined by the Cramer formula in terms of the 2-data, if

$$rk(\partial_{\chi_j} r^{(\mathbf{b})}(\chi))_{|\mathbf{b}| = \beta_0, 1 \leq j \leq n} = n, \tag{57}$$

i.e. (substituting back $\chi$ with $z^0$), $F_w(z^0, 0)$ is 2-determined if for $z^0$ generic the coefficient matrix of the polynomials $P_j^{(z^0)}$, $j = 1, \ldots, n$, is of maximal rank. This clearly holds by either one of the assumptions i) or ii).

Let us check next that assumption ii) implies that condition (34) of Theorem 3 holds also for the set $Y^{(z^0)}$ along the diagonal $n$-plane in $\mathbf{C}^n \times \mathbf{C}^n$. Differentiating the reality condition (3) for $Q$ at $\tau = 0$ with respect to $z_j$, we have, for $\mathbf{b} \in \mathbf{N}^n$,

$$Q_{z_j}(z, \chi, \bar{Q}(\chi, z, 0)) \equiv -Q_\tau(z, \chi, \bar{Q}(\chi, z, 0)) \cdot \bar{Q}_{z_j}(\chi, z, 0). \tag{58}$$

From (48), we have

$$r^{(\mathbf{b})}(\chi) \equiv Q_{z^{\mathbf{b}}}(0, \chi, 0)/\mathbf{b}!, \tag{59}$$

therefore

$$\bar{r}^{(\mathbf{b})}(z) \equiv \bar{Q}_{\chi^{\mathbf{b}}}(0, z, 0)/\mathbf{b}! \tag{60}$$

and

$$\partial_{z_j}\bar{r}^{(\mathbf{b})}(z) \equiv \partial_{z_j}\bar{Q}_{\chi^{\mathbf{b}}}(0,z,0)/\mathbf{b}!. \tag{61}$$

The latter implies

$$\bar{Q}_{z_j}(\chi,z,0) \equiv \sum_{\mathbf{b}\in\mathbf{N}^n}\bar{Q}_{z_j\chi^{\mathbf{b}}}(0,z,0)/\mathbf{b}!\cdot\chi^{\mathbf{b}} \equiv \sum_{|\mathbf{b}|=\beta_0}\partial_{z_j}\bar{r}^{(\mathbf{b})}(z)\chi^{\mathbf{b}} + o(|\chi|^{\beta_0}). \tag{62}$$

Combining with (58) and using $Q_\tau(0,\chi,0)\equiv 1$, we obtain, since by ii) $P_j^{(z^0)} \not\equiv 0$ for $z^0$ generic,

$$in_\chi(Q_{z_j})(z^0,\chi,\bar{Q}(\chi,z^0,0)) \equiv -\sum_{|\mathbf{b}|=\beta_0}\partial_{z_j}\bar{r}^{(\mathbf{b})}(z^0)\chi^{\mathbf{b}} \equiv -P_j^{(z^0)}(\chi). \tag{63}$$

Using this identity, assumption ii) becomes

$$Jac(in_\chi(Q_{z_j}(z^0,\chi,\bar{Q}(\chi,z^0,0))))_{1\le j\le n} \not\equiv 0, \tag{64}$$

for $z^0$ generic, which is condition (34) for the set $Y^{(z^0)}$ relative to the diagonal $n$-plane.

We may therefore apply Theorem 3, from which it follows that the map $\chi \to \zeta(\chi) = \bar{F}(\chi,\bar{Q}(\chi,z^0,0))$ is 2-determined.

We next show that, for fixed generic points $z^0$ and $\chi^0 \in \mathbf{C}^n$, setting $\tau^0 = \bar{Q}(\chi^0,z^0,0)$, the equation

$$\tau = \bar{Q}(\chi,z,0) \tag{65}$$

has a solution $z^1 = z^1(\chi,\tau)$ near $z^0$, for $(\chi,\tau)\in\mathbf{C}^n\times\mathbf{C}$ near $(\chi^0,\tau^0)$.

For this, by the finite type assumption we may choose $j$, $1\le j\le n$, and $z^0$, $\chi^0$ generic such that

$$\bar{Q}_{z_j}(\chi^0,z^0,0) \neq 0. \tag{66}$$

With $\tau^0$ defined as above, by condition (66), we may use the implicit function theorem to solve the equation $\tau = \bar{Q}(\chi,(z_j,z'),0)$ for $z_j = z_j(\chi,\tau,z')\in\mathbf{C}$ (where $z'\in\mathbf{C}^{n-1}$) such that $z_j(\chi^0,\tau^0,z'^0) = z_j^0$. Then the solution $z^1$ is defined by $z^1(\chi,\tau) \equiv (z_j(\chi,\tau,z'^0),z'^0)$ for $(\chi,\tau)$ near $(\chi^0,\tau^0)$.

Moreover, since $z^1$ is near $z^0$, which is generic, we have $Jac(P_1^{(z^1)},\dots,P_n^{(z^1)}) \not\equiv 0$.

Finally, this implies 2-determination for elements in $\mathrm{Aut}(M,0)$. Indeed, by ii) with respect to $z^1 = z^1(\chi,\tau)$, the map $\chi \to \bar{F}(\chi,\bar{Q}(\chi,z^1(\chi,\tau),0))$ is 2-determined, i.e., by (65), $\bar{F}(\chi,\tau)$ is determined for $(\chi,\tau)$ in an open set in $\mathbf{C}^n\times\mathbf{C}$. By the identity principle, this determines $\bar{F}$. Similarly, using (9) at $\tau = 0$ and the solution $z^1(\chi, \tau)$ of equation (65), we have that $\bar{G}$ is 2-determined, therefore $H$ itself is 2-determined. □

In the case $n = 2$ we obtain the following corollary.

COROLLARY. *Let $M \subset \mathbf{C}^3$ be a real-analytic hypersurface of finite type at $p = 0 \in M$.*

*Assume that for some multi-indices $\mathbf{b}_1, \mathbf{b}_2 \in \mathbf{N}^2$ of length $\beta_0$, $in(r^{(\mathbf{b}_1)})$ and $in(r^{(\mathbf{b}_2)})$ are not in the polynomial relation* (44). *Then the elements in $Aut(M, p)$ are determined by their 2-jets at $p$.*

PROOF. By Lemma 3, for $n = 2$ the assumption of the corollary is equivalent with condition i) of Theorem 4, which also implies $Jac(r^{(\mathbf{b}_1)}, r^{(\mathbf{b}_2)}) \not\equiv 0$. From this it follows that condition ii) holds. Indeed by Lemma 3 we need only check that the coefficient matrix of $(P_1, P_2)$ has rank 2. By (49), two columns of this matrix have determinant $= Jac(\bar{r}^{(\mathbf{b}_1)}, \bar{r}^{(\mathbf{b}_2)})(z^{(0)})$, which is non-zero for $z^{(0)}$ generic. □

The following example shows that even if the hypersurface $M$ is finitely non-degenerate at $p$, therefore finite determination by jets for elements in $Aut(M, p)$ holds, one cannot obtain a universal bound for the length of jets.

EXAMPLE ([9]). Let $l \geq 2$ be an arbitrary integer, and let $M_l$ be the real-analytic hypersurface in $\mathbf{C}^3$ defined, in coordinates $(z_1, z_2, w)$ on $\mathbf{C}^3$, by

$$M_l = \{Im\ w = |z_1|^2 + Re(z_1^l \bar{z}_2)\}, \tag{67}$$

which is 2-non-degenerate (therefore of finite type) at 0.

Then for $a \in \mathbf{C}$, the map $H_a : \mathbf{C}^3 \to \mathbf{C}^3$ given by

$$H_a(z_1, z_2, w) \equiv (z_1, z_2 + iaz_1^l, w) \tag{68}$$

induces an element in $Aut(M, 0)$ such that $j_0^{l-1}(H_a) = j_0^{l-1}(id)$, but $H_a \neq id$ for $a \not\equiv 0$.

Note that the assumption of Corollary 2 is not satisfied for this example, Indeed, $Q(z, \chi, 0) \equiv 2iz_1\chi_1 + i(z_1^l\chi_2 + \chi_1^l z_2)$, $\beta_0 = 1$ and $r^{(1,0)}(\chi) \equiv 2i\chi_1$, $r^{(0,1)}(\chi) \equiv i\chi_1^l$, are in the polynomial relation (44). In particular, this example shows that the assumptions of the corollary are minimal.

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Mathematics Department
University of California at Santa Barbara
e-mail address: gputinar@att.net