

A CONSTRUCTION OF COMPACT PSEUDO-KÄHLER SOLVMANIFOLDS WITH NO KÄHLER STRUCTURES

By

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Abstract. In this paper we investigate the Hard Lefschetz property on certain compact symplectic solvmanifolds and construct compact pseudo-Kähler solvmanifolds which do not have the Hard Lefschetz property. We also construct holomorphic symplectic structures, hypercomplex structures and pseudo-hyperkähler structures on certain compact solvmanifolds.

Introduction

Let (M^{2m}, ω) be a compact symplectic manifold. We say that (M^{2m}, ω) has the Hard Lefschetz property, if the Lefschetz mapping $L^k : H_{DR}^{m-k}(M) \rightarrow H_{DR}^{m+k}(M)$ defined by $L^k([\alpha]) = [\alpha \wedge \omega^k]$ is an isomorphism for any $k \leq m$. It is well known that the Hard Lefschetz property is a necessary condition for the existence of a Kähler structure. Benson and Gordon [2] proved that non-toral compact nilmanifolds do not have the Hard Lefschetz property. They also conjecture the following:

BENSON-GORDON CONJECTURE [3]. *Let G be a simply-connected completely solvable Lie group and Γ a lattice of G . Then G/Γ has a Kähler structure if and only if it is a torus.*

Moreover, since a hyperelliptic surface has a Kähler structure and a structure of solvmanifold (not completely solvable solvmanifold), there exists the following generalized conjecture (see [6] or [12]): A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus, which has also a structure of complex torus bundle over a complex torus. A solvable Lie algebra \mathfrak{g}

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is called completely solvable if $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. By investigating the properties of the Lefschetz mapping, Benson and Gordon [3] have several necessary conditions for the existence of a Kähler structure. On the other hand, de Andrés, Fernández, de León and Mencía [1] have constructed examples of 6-dimensional non-toral compact pseudo-Kähler solvmanifolds which have the Hard Lefschetz property (See Example 5.1). We do not know whether any of these solvmanifolds admit Kähler structures. Ibáñez [14] has constructed 6-dimensional pseudo-Kähler nilmanifolds. Kodaira-Thurston manifold, which is a compact 4-dimensional nilmanifold, also admits a pseudo-Kähler structure (see [5]).

In the previous paper [21], we constructed completely solvable Lie groups which have a lattice. Let A_i, B_j be the matrices given by

$$A_i = \sum_{k=1}^m a_i^k (E_{2k-1, 2k-1} - E_{2k, 2k}) \quad i = 1, \dots, l,$$

$$B_j = \sum_{k < h} b_j^{kh} (E_{2k-1, 2h-1} + E_{2k, 2h}) \quad j = 1, \dots, n,$$

where $a_i^k, b_j^{kh} \in \mathbf{Q}$ and $E_{i,j}$ is a matrix unit. We assume that $[A_i, B_j] = [B_i, B_j] = 0$. We define a map

$$\varphi_* : \mathbf{R}^{n+l} \rightarrow \text{End}(\mathbf{R}^{2m})$$

by

$$\varphi_*(t_1, \dots, t_l, x_1, \dots, x_n) = \sum_{i=1}^l t_i A_i + \sum_{i=1}^n x_i B_i.$$

Let $\varphi(\mathbf{t}, \mathbf{x}) = \exp(\varphi_*(\mathbf{t}, \mathbf{x}))$ and we define a group structure of $\mathbf{R}^{n+l} \times \mathbf{R}^{2m}$ by

$$(\mathbf{t}_1, \mathbf{x}_1, \mathbf{y}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{y}_2) = (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2)$$

for $\mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i \in \mathbf{R}^n$ and $\mathbf{y}_i \in \mathbf{R}^{2m}$. We denote the Lie group $(\mathbf{R}^{n+l} \times \mathbf{R}^{2m}, *)$ by $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$.

In the previous paper [21], we proved the following:

PROPOSITION 1. *A Lie group $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ is a completely solvable Lie group which has a lattice Γ .*

The main purpose of this paper is to investigate the properties of the Lefschetz mapping on the compact symplectic solvmanifolds constructed in

Proposition 1 and to construct examples of compact pseudo-Kähler solvmanifolds without the Hard Lefschetz property.

In section 2, 3 and 4 we always assume that for each k , there exists an i such that $a_i^k \neq 0$ and $l+n$ are even numbers. A solvable Lie group $G = \mathbf{R}^{l+n} \ltimes_{\varphi} \mathbf{R}^{2m}$ constructed above is called A -type if $B_j = 0$ for each j . In section 4 we prove the following:

THEOREM 2. *Let $M = G/\Gamma$ be a compact solvmanifold constructed in Proposition 1 and assume that M has a symplectic structure. Then M has the Hard Lefschetz property if and only if M is a compact A -type solvmanifold.*

PROPOSITION 3. *The minimal model of a compact A -type solvmanifold $M = G/\Gamma$ is formal.*

It is known that formality is also a necessary condition for the existence of a Kähler structure and it is conjectured that if a closed symplectic manifold has the Hard Lefschetz property, then its minimal model is formal (see Tralle [19]). In the paper [1], de Andrés, Fernández, de León and Mencía proved that the minimal models of 6-dimensional compact A -type solvmanifolds are formal.

Next, let $\varphi(\mathbf{t}, \mathbf{x})$ ($\mathbf{t} \in \mathbf{R}^l, \mathbf{x} \in \mathbf{R}^n$) be an automorphism of \mathbf{R}^{2m} constructed above. We consider a solvable Lie group $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$, where $\tilde{\varphi}(\mathbf{t}, \mathbf{x}) = \varphi(\mathbf{t}, \mathbf{x}) \oplus \varphi(\mathbf{t}, \mathbf{x})$, that is, the group structure of \tilde{G} is defined by

$$\begin{aligned} & (s_1, t_1, x_1, r_1, y_1, z_1) * (s_2, t_2, x_2, r_2, y_2, z_2) \\ &= (s_1 + s_2, t_1 + t_2, x_1 + x_2, r_1 + r_2, y_1 + \varphi(t_1, x_1)y_2, z_1 + \varphi(t_1, x_1)z_2) \end{aligned}$$

for $s_i, t_i \in \mathbf{R}^l$, $x_i, r_i \in \mathbf{R}^n$ and $y_i, z_i \in \mathbf{R}^{2m}$. Then the matrix form of \tilde{G} is given by

$$\tilde{G} = \left\{ \left(\begin{array}{cccccc} \varphi(\mathbf{t}, \mathbf{x}) & 0 & 0 & 0 & 0 & 0 & \mathbf{y} \\ 0 & \varphi(\mathbf{t}, \mathbf{x}) & 0 & 0 & 0 & 0 & \mathbf{z} \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{x} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{t} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{r} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{s} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{array} \right) \mid \mathbf{s}, \mathbf{t} \in \mathbf{R}^l, \mathbf{r}, \mathbf{x} \in \mathbf{R}^n, \mathbf{y}, \mathbf{z} \in \mathbf{R}^{2m} \right\}.$$

Note that \tilde{G} is a completely solvable Lie group which has a lattice.

In section 6 we prove the following:

PROPOSITION 4. *A solvable Lie group $\tilde{G} = \mathbf{R}^{2n+2l} \rtimes_{\tilde{\varphi}} \mathbf{R}^{4m}$ has a left invariant complex structure.*

PROPOSITION 5. *If a solvable Lie group $G = \mathbf{R}^{n+l} \rtimes_{\varphi} \mathbf{R}^{2m}$ has a symplectic structure, then $\tilde{G} = \mathbf{R}^{2n+2l} \rtimes_{\tilde{\varphi}} \mathbf{R}^{4m}$ has a pseudo-Kähler structure.*

Using Theorem 2 and Proposition 5, we can construct compact pseudo-Kähler solvmanifolds which do not have the Hard Lefschetz property.

Consider the direct product $G' = \tilde{G} \times \mathbf{C}^{n+l}$. Note that G' also has a lattice and a complex structure. Let M^{2n} be a $2n$ -dimensional complex manifold. A holomorphic 2-form $\Omega \in \Omega^{2,0}(M)$ is called a holomorphic symplectic structure on M if it satisfies $d\Omega = 0$ and $\Omega^n \neq 0$ at each point of M . Todorov conjectured that any holomorphic symplectic manifold admits a Kähler structure (See [4], [8]). However, Guan has constructed non-simply-connected holomorphic symplectic non-Kähler manifolds and simply-connected holomorphic symplectic non-Kähler manifolds ([8], [9], [10]). He also consider a deformation of holomorphic symplectic manifolds. However the examples of compact holomorphic symplectic non-Kähler manifolds are not so much (In the non-compact case, many examples are known, say, complex cotangent bundle $M = \bigwedge^{1,0} N$ of a complex manifold N). We prove the following:

PROPOSITION 6. *If a solvable Lie group $G = \mathbf{R}^{n+l} \rtimes_{\varphi} \mathbf{R}^{2m}$ has a left G -invariant symplectic form, then $G'/\Gamma' = \tilde{G}/\tilde{\Gamma} \times \mathbf{C}^{n+l}/\Gamma$ has a holomorphic symplectic structure.*

In section 6, we also construct hypercomplex structures on certain compact solvmanifolds. We give some examples in section 5 and 7. In section 8 and 9, we construct solvable Lie groups with parameterized lattices and holomorphic symplectic structures. As a consequence, we get families of compact holomorphic symplectic non-Kähler solvmanifolds.

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1. Definitions and Nomizu-Hattori Theorem

Let (M, ω) be a compact symplectic manifold and $\Omega^k(M)$ the space of all differential k -forms. We define a linear mapping $L : \Omega^k(M) \rightarrow \Omega^{k+2}(M)$ by

$L(\alpha) = \alpha \wedge \omega$. Since ω is closed, we have $Ld = dL$. Hence, the mapping L induces a linear mapping $L : H_{DR}^k(M) \rightarrow H_{DR}^{k+2}(M)$ by $L([\alpha]) = [L(\alpha)]$.

DEFINITION 1.1. Let (M^{2m}, ω) be a compact symplectic manifold.

- (1) If the Lefschetz mapping $L^{m-1} : H_{DR}^1(M) \rightarrow H_{DR}^{2m-1}(M)$ is an isomorphism, then (M^{2m}, ω) is called a Lefschetz manifold.
- (2) If the Lefschetz mapping $L^k : H_{DR}^{m-k}(M) \rightarrow H_{DR}^{m+k}(M)$ is an isomorphism for any $k \leq m$, then we say that (M^{2m}, ω) has the Hard Lefschetz property.

Note that compact Kähler manifolds have the Hard Lefschetz property.

Let \mathfrak{g} be a Lie algebra and put $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i]$. A Lie algebra \mathfrak{g} is called solvable if $\mathfrak{g}_{r+1} = (0)$ for some r . A Lie group G is called solvable if its Lie algebra \mathfrak{g} is solvable.

DEFINITION 1.2. A solvable Lie algebra \mathfrak{g} is called completely solvable if $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. A solvable Lie group G is called completely solvable if its Lie algebra is completely solvable.

By a compact solvmanifold G/Γ , we mean a right coset space of G modulo Γ , where G is a simply-connected completely solvable Lie group and Γ a lattice, that is, a discrete co-compact subgroup of G .

We denote the Lie algebra of G by \mathfrak{g} . We identify $\bigwedge^* \mathfrak{g}^*$ with the space of all left G -invariant forms on G/Γ . Then Hattori [13] proved the following:

NOMIZU-HATTORI THEOREM. *The inclusion map $i : \bigwedge^k \mathfrak{g}^* \rightarrow \Omega^k(G/\Gamma)$ induces an isomorphism $H^k(\mathfrak{g}) \rightarrow H_{DR}^k(G/\Gamma)$ for each k .*

Let $(G/\Gamma, \omega)$ be a compact symplectic solvmanifold. By Nomizu-Hattori Theorem, there exists a left G -invariant closed 2-form ω_0 on G/Γ such that $\omega - \omega_0 = d\gamma$. Note that ω_0 is also a symplectic structure. Therefore we may assume that a symplectic structure on $M = G/\Gamma$ is left G -invariant to investigate the Hard Lefschetz property.

2. Closed Forms on Certain Solvable Lie Algebras

In this section we consider left G -invariant closed forms on G constructed in Proposition 1.

The Lie algebra \mathfrak{g} of G constructed in Proposition 1 can be written as follows.

$$\mathfrak{g} = \text{span}\{A_1, \dots, A_l, B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}$$

with

$$\begin{aligned} [A_i, Y_{2k-1}] &= a_i^k Y_{2k-1}, & [A_i, Y_{2k}] &= -a_i^k Y_{2k}, \\ [B_j, Y_{2h-1}] &= \sum_{k < h} b_j^{kh} Y_{2k-1}, & [B_j, Y_{2h}] &= \sum_{k < h} b_j^{kh} Y_{2k} \end{aligned} \quad (2.1)$$

for $i = 1, \dots, l$, $j = 1, \dots, n$ and $1 \leq k < h \leq m$. We assume that for each k , there exists an i such that $a_i^k \neq 0$. Let $\{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, \omega_1, \dots, \omega_{2m}\}$ be the dual basis corresponding to $\{A_1, \dots, A_l, B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}$. We write $\omega_{k_1} \wedge \dots \wedge \omega_{k_p}$ simply as ω_K and set $\#K = p$ for $K = (k_1, \dots, k_p)$. Note that $d\omega_K$ can be written as follows:

$$d\omega_K = - \sum_{i=1}^l a_i^K \alpha_i \wedge \omega_K - \sum_{j=1}^n \sum_H b_j^{KH} \beta_j \wedge \omega_H.$$

LEMMA 2.1 ([21]). *Let $\gamma = \sum_{IJK} c_{IJK} \alpha_I \wedge \beta_J \wedge \omega_K$ be a closed form such that $\#I + \#J$ and $\#K$ are constant. If for each K , there exists an i such that $a_i^K \neq 0$, then γ is an exact form.*

PROOF. See [21]. \square

We set

$$\mathfrak{a} = \text{span}\{A_1, \dots, A_l\},$$

$$\mathfrak{b} = \text{span}\{B_1, \dots, B_n\},$$

$$\mathfrak{m} = \text{span}\{Y_1, \dots, Y_{2m}\}.$$

For simplicity, we denote $\bigwedge^i(\mathfrak{a} \times \mathfrak{b})^* \wedge \bigwedge^j \mathfrak{m}^*$ by $\bigwedge^{i,j}$.

LEMMA 2.2 ([21]).

(1) *If $\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2} \in Z^2(\mathfrak{g})$, where $\alpha_{i,j} \in \bigwedge^{i,j}$, then $d\alpha_{2,0} = d\alpha_{1,1} = d\alpha_{0,2} = 0$.*

(2) $\bigwedge^{1,1} \cap Z^2(\mathfrak{g}) \subset B^2(\mathfrak{g})$.

PROOF. Since

$$d\omega_{2k-1} = -\sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h},$$

we have

$$\begin{aligned} \bigwedge^{0,2} &\xrightarrow{d} \bigwedge^{1,2}, \\ \bigwedge^{2,0} &\xrightarrow{d} 0, \\ \bigwedge^{1,1} &\xrightarrow{d} \bigwedge^{2,1}. \end{aligned}$$

Since we assume that for each k , there exists an i such that $a_i^k \neq 0$, we have Lemma 2.2 using Lemma 2.1. \square

3. Closed Forms on Nilpotent Lie Algebras

We use the same notations as in section 2.

By Lemma 2.2, we may assume that a symplectic structure ω on a solvable Lie group G constructed above is an element of $\bigwedge^{2,0} + \bigwedge^{0,2}$ to study the Hard Lefschetz property. Thus we write $\omega = \omega_{2,0} + \omega_{0,2}$, where $\omega_{2,0} \in \bigwedge^{2,0}$, $\omega_{0,2} \in \bigwedge^{0,2}$. Note that $\omega_{2,0}$ and $\omega_{0,2}$ are symplectic structures on $\mathfrak{a} \times \mathfrak{b}$, \mathfrak{m} respectively.

Let \mathfrak{n} be a Lie algebra. Put $\mathfrak{n}^{(0)} = \mathfrak{n}$ and $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ for $i \geq 0$. We say that the Lie algebra \mathfrak{n} is $(r+1)$ -step nilpotent if $\mathfrak{n}^{(r)} \neq (0)$ and $\mathfrak{n}^{(r+1)} = (0)$. A Lie group N is called $(r+1)$ -step nilpotent if its Lie algebra \mathfrak{n} is $(r+1)$ -step nilpotent.

Note that $\mathfrak{n} = \mathfrak{b} \ltimes \mathfrak{m}$ is a nilpotent Lie algebra and $\omega_{0,2}$ can be considered as a closed form on the simply-connected nilpotent Lie group N corresponding to \mathfrak{n} . Thus we consider left N -invariant closed forms on a nilpotent Lie group N .

Let \mathfrak{n} be an $(r+1)$ -step nilpotent Lie algebra. Consider the descending central series $\{\mathfrak{n}^{(i)}\}$ of \mathfrak{n} . Let $\mathfrak{u}^{(i)}$ be a vector subspace of $\mathfrak{n}^{(i)}$ such that

$$\mathfrak{n}^{(i)} = \mathfrak{n}^{(i+1)} + \mathfrak{u}^{(i)}$$

for $i = 0, 1, \dots, r-1$ and define $n_i = \dim \mathfrak{u}^{(i)}$. For simplicity, let $\bigwedge^{i_0} \mathfrak{u}^{(0)*} \wedge \dots \wedge \bigwedge^{i_r} \mathfrak{u}^{(r)*} = \bigwedge^{i_0, \dots, i_r}$. Then

$$\bigwedge^s \mathfrak{n}^* = \sum_{i_0 + \dots + i_r = s} \bigwedge^{i_0, \dots, i_r}.$$

For an $(r+1)$ -step nilpotent Lie algebra \mathfrak{n} , we have the following:

LEMMA 3.1 ([2]). *Any closed 2-form $\sigma \in \bigwedge^2 \mathfrak{n}^*$ belongs to $\bigwedge^{1,0,\dots,0,1} + \sum \bigwedge^{i_0,\dots,i_{r-1},0}$.*

Let ζ_1, \dots, ζ_r be a basis of $\bigwedge^{0,\dots,0,1}$. By Lemma 3.1, a left N -invariant symplectic form ω on a nilpotent Lie group N can be written as

$$\omega = \gamma_1 \wedge \zeta_1 + \dots + \gamma_r \wedge \zeta_r \quad \text{modulo} \quad \sum \bigwedge^{i_0,\dots,i_{r-1},0},$$

where $\gamma_1, \dots, \gamma_r$ are elements of $\bigwedge^{1,0,\dots,0}$. Since ω is non-degenerate, $\gamma_1, \dots, \gamma_r$ are linearly independent and we extend these to a basis

$$\gamma_1, \dots, \gamma_r, \dots, \gamma_{n_0}$$

of $\bigwedge^{1,0,\dots,0}$.

LEMMA 3.2 ([2]). *Let \mathfrak{n} be an $(r+1)$ -step nilpotent Lie algebra of dimension $2m$. Then we have*

- (1) $\bigwedge^{2m-1}(\mathfrak{n}^*) = \mathcal{Z}^{2m-1}(\mathfrak{n})$,
- (2) $\sum \bigwedge^{n_0, i_1, \dots, i_r} = \mathcal{B}^{2m-1}(\mathfrak{n})$.

4. The Lefschetz Mapping on Certain Compact Symplectic Solvmanifolds

In this section we prove Theorem 2 and Proposition 3. We assume that a symplectic structure ω is left G -invariant. We use the same notations as in sections 2 and 3.

Put

$$\mathfrak{m}_1 = \text{span}\{Y_1, Y_3, \dots, Y_{2m-1}\},$$

$$\mathfrak{m}_2 = \text{span}\{Y_2, Y_4, \dots, Y_{2m}\},$$

$$\mathfrak{m} = \text{span}\{Y_1, Y_2, \dots, Y_{2m}\}.$$

Then Lie algebras $\mathfrak{n}_1 = \mathfrak{b} \ltimes \mathfrak{m}_1$, $\mathfrak{n}_2 = \mathfrak{b} \ltimes \mathfrak{m}_2$ and $\mathfrak{n} = \mathfrak{b} \ltimes \mathfrak{m}$ are $(r+1)$ -step nilpotent.

Since \mathfrak{n}_1 is a nilpotent Lie algebra, there exists a basis

$$\mathfrak{n}_1^* = \text{span}\{\beta_1, \dots, \beta_n, \zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(r)}, \dots, \zeta_{n_r}^{(r)}\},$$

which satisfies for each $\rho = 0, \dots, r-1$,

$$\begin{aligned} d_{\mathfrak{n}_1} \zeta_k^{(\rho+1)} &\in \wedge \{\beta_1, \dots, \beta_n, \zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(\rho)}, \dots, \zeta_{n_\rho}^{(\rho)}\}, \\ d_{\mathfrak{n}_1} \beta_j &= d_{\mathfrak{n}_1} \zeta_k^{(0)} = 0, \end{aligned}$$

where $d_{\mathfrak{n}_1}$ is the exterior differential on $\wedge^* \mathfrak{n}_1$ (cf. [2, the proof of Lemma 2.1]).

Put

$$\mathfrak{u}_1^{(\rho)*} = \text{span}\{\zeta_1^{(\rho)}, \dots, \zeta_{n_\rho}^{(\rho)}\}.$$

Then we have $\mathfrak{n}_1^* = \mathfrak{b}^* \oplus \mathfrak{u}_1^{(0)*} \oplus \dots \oplus \mathfrak{u}_1^{(r)*}$. For simplicity, let

$$\wedge^j \mathfrak{b}^* \wedge \wedge^{i_0} \mathfrak{u}_1^{(0)*} \wedge \dots \wedge \wedge^{i_r} \mathfrak{u}_1^{(r)*} = \wedge^{(j, i_0), i_1, \dots, i_r} \mathfrak{n}_1^*.$$

Since $\{\beta_1, \dots, \beta_n, \omega_1, \omega_3, \dots, \omega_{2m-1}\}$ is also a basis of \mathfrak{n}_1^* , we can write $\zeta_k^{(\rho)}$ as

$$\zeta_k^{(\rho)} = \sum_h c_{kh}^{(\rho)} \omega_{2h-1}.$$

Then we define 1-forms $\eta_k^{(\rho)}$ by $\eta_k^{(\rho)} = \sum_h c_{kh}^{(\rho)} \omega_{2h}$. It is obvious from (2.1) that

$$d_{\mathfrak{n}_2} \eta_i^{(\rho+1)} \in \wedge \{\beta_1, \dots, \beta_n, \eta_1^{(0)}, \dots, \eta_{n_0}^{(0)}, \dots, \eta_1^{(\rho)}, \dots, \eta_{n_\rho}^{(\rho)}\}.$$

Now consider $\zeta_k^{(\rho)}, \eta_k^{(\rho)}$ as left G -invariant 1-forms on G . Then we see

$$\begin{aligned} d\zeta_k^{(\rho)} &= -\sum_{i=1}^l a_{ik}^{(\rho)} \alpha_i \wedge \zeta_k^{(\rho)} + \sum_{j=1}^n \sum_{h=1}^{n_0+\dots+n_{\rho-1}} b_{kjh}^{(\rho)} \beta_j \wedge \zeta_h, \\ d\eta_k^{(\rho)} &= \sum_{i=1}^l a_{ik}^{(\rho)} \alpha_i \wedge \eta_k^{(\rho)} + \sum_{j=1}^n \sum_{h=1}^{n_0+\dots+n_{\rho-1}} b_{kjh}^{(\rho)} \beta_j \wedge \eta_h, \end{aligned} \tag{4.1}$$

where

$$\{\zeta_1, \dots, \zeta_m\} = \{\zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(r)}, \dots, \zeta_{n_r}^{(r)}\}.$$

Similarly, we write

$$\begin{aligned} \wedge^j \mathfrak{b}^* \wedge \wedge^{i_0} \mathfrak{u}_2^{(0)*} \wedge \dots \wedge \wedge^{i_r} \mathfrak{u}_2^{(r)*} &= \wedge^{(j, i_0), i_1, \dots, i_r} \mathfrak{n}_2^*, \\ \wedge^j \mathfrak{b}^* \wedge \wedge^{i_0} \mathfrak{u}^{(0)*} \wedge \dots \wedge \wedge^{i_r} \mathfrak{u}^{(r)*} &= \wedge^{(j, i_0), i_1, \dots, i_r} \mathfrak{n}^*, \end{aligned}$$

where $\mathfrak{u}^{(\rho)*} = \mathfrak{u}_1^{(\rho)*} + \mathfrak{u}_2^{(\rho)*}$.

THEOREM 4.1. *Let $M = G/\Gamma$ be a compact solvmanifold constructed in Proposition 1 and assume that M has a symplectic structure. Then M has the Hard Lefschetz property if and only if M is a compact A -type solvmanifold.*

PROOF. By Lemma 2.2 and Lemma 3.1, $\omega_{0,2}$ can be written as

$$\omega_{0,2} = \gamma_1 \wedge \zeta_1^{(r)} + \cdots + \gamma_{n_r} \wedge \zeta_{n_r}^{(r)} + \lambda_1 \wedge \eta_1^{(r)} + \cdots + \lambda_{n_r} \wedge \eta_{n_r}^{(r)} + \tau,$$

where $\gamma_k, \lambda_k \in \bigwedge^{(0,1),0,\dots,0} \mathfrak{n}^*$ and $\tau \in \bigwedge^{(0,i_0),i_1,\dots,i_{r-1},0} \mathfrak{n}^*$. It is obvious from (4.1) that $\gamma_k \wedge \lambda_k$ is a non-exact closed 2-form for each $k = 1, \dots, n_r$ (Note that $d_{\mathfrak{n}} \bigwedge^{(0,1),0,\dots,0} \mathfrak{n}^* = 0$). Then

$$\begin{aligned} \gamma_k \wedge \lambda_k &\xrightarrow{L^{(1/2)(n+l)+m-2}} a_1 \cdot \omega_{2,0}^{(1/2)(n+l)} \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \\ &\quad \wedge \eta_1^{(0)} \wedge \eta_2^{(0)} \wedge \cdots \wedge \hat{\eta}_k^{(r)} \wedge \cdots \wedge \eta_{n_r}^{(r)} \\ &= a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \\ &\quad \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \eta_1^{(0)} \wedge \eta_2^{(0)} \wedge \cdots \wedge \hat{\eta}_k^{(r)} \wedge \cdots \wedge \eta_{n_r}^{(r)} \\ &= a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge d_{\mathfrak{n}_2} \theta \\ &= a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge d\theta \\ &= (-1)^{m+l-1} a_2 \cdot d(\alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge \theta), \end{aligned}$$

where $a_1, a_2 \in \mathbf{R}$, $\theta \in \bigwedge \mathfrak{n}_2^*$ and $d_{\mathfrak{n}_2}$ is the exterior differential on $\bigwedge \mathfrak{n}_2^*$. The second equality holds by Lemma 3.2. The third and fourth equalities hold by the following fact:

$$\begin{aligned} d\zeta_k^{(r)} &= - \sum_{i=1}^n a_{ik}^{(r)} \alpha_i \wedge \zeta_k^{(r)} + \sum \bigwedge^{(1,i_0),i_1,\dots,i_{r-1},0} \mathfrak{n}_1^*, \\ d\eta_k^{(r)} &= \sum_{i=1}^n a_{ik}^{(r)} \alpha_i \wedge \eta_k^{(r)} + \sum \bigwedge^{(1,i_0),i_1,\dots,i_{r-1},0} \mathfrak{n}_2^*. \end{aligned}$$

Then

$$L^{(1/2)(n+l)+m-2} : H^2(\mathfrak{g}) \rightarrow H^{n+l+2m-2}(\mathfrak{g})$$

is not an isomorphism if M is not a compact A -type solvmanifold.

Conversely, let M be a compact A -type solvmanifold. Since $d\omega_K = -\sum_{i=1}^l a_i^K \alpha_i \wedge \omega_K$, if $\sum_{\#I+\#K=p+q=r} c_{IK} \alpha_I \wedge \omega_K$ is a closed form, then $\sum_{\#I=p} c_{IK} \alpha_I \wedge \omega_K$ is also a closed form. Moreover, it is obvious that if $d\omega_K = 0$,

then $\sum_{\#I=p} c_{IK} \alpha_I \wedge \omega_K$ is a non-exact closed form. By Lemma 2.1, if $d\omega_K \neq 0$, then a closed form $\sum_{\#I=p} c_{IK} \alpha_I \wedge \omega_K$ is exact. Then for each de Rham cohomology class, we can choose a representation $\alpha = \sum_{I,K} c_{IK} \alpha_I \wedge \omega_K$ such that $d\omega_K = 0$.

On the other hand, we can assume that a symplectic form ω on M can be written as

$$\omega = \omega_{2,0} + \sum_{k,h} P_{kh} \omega_k \wedge \omega_h,$$

where $\omega_{2,0}$ is a non-degenerate closed form on $\bigwedge^{2,0}$. Since $d\omega_K = -\sum_{i=1}^l a_i^K \alpha_i \wedge \omega_K$, $\omega_k \wedge \omega_h$ is closed for each k, h such that $P_{kh} \neq 0$. Then we have

$$L^k \alpha = \sum_{I',K'} c_{I'K'} \alpha_{I'} \wedge \omega_{K'} \quad d\omega_{K'} = 0,$$

which implies $L^k \alpha$ is not exact by the above argument. Then A -type has the Hard Lefschetz property. \square

REMARK. In the paper [21], we showed that a compact symplectic solvmanifold constructed in Proposition 1 is a compact Lefschetz manifold.

Using the notion of differential graded algebra (or, briefly, D.G.A.), we define the minimal model of M . $\mathcal{A} = (\mathcal{A}, d)$ is called a D.G.A. if \mathcal{A} is a graded algebra $\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}^i$ with the commutativity $a \cdot b = (-1)^{pq} b \cdot a$ for $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$ and d an antiderivation of degree 1 as follows:

$$d^2 = 0,$$

$$d(a \cdot b) = da \cdot b + (-1)^p a \cdot db$$

for $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$.

DEFINITION 4.2. Let \mathcal{A}, \mathcal{B} be D.G.A., \mathcal{B} is a Hirsch extension of degree n of \mathcal{A} , if \mathcal{B} is of the following form:

$$\mathcal{B} = \mathcal{A} \otimes \bigwedge_n \langle x_1, \dots, x_k \rangle$$

$$\deg x_i = n, \quad dx_i \in \mathcal{A} \quad \text{for } i = 1, \dots, k,$$

where $\bigwedge_n \langle x_1, \dots, x_k \rangle$ is the free graded commutative algebra with unit generated by $\{x_1, \dots, x_k\}$.

DEFINITION 4.3. A D.G.A. \mathcal{A} is said to be minimal if \mathcal{A} satisfies the following:

- (i) $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_i$, where $\mathcal{A}_0 = \mathbf{R}$ and \mathcal{A}_{i+1} is a Hirsch extension of \mathcal{A}_i for $i \geq 0$.
- (ii) $dx \in \mathcal{A}_+ \cdot \mathcal{A}_+$, where $x \in \mathcal{A}$ and $\mathcal{A}_+ = \bigoplus_{i \geq 1} \mathcal{A}^i$.

DEFINITION 4.4. Let $(\mathcal{M}, d_{\mathcal{M}})$, $(\mathcal{A}, d_{\mathcal{A}})$ be D.G.A.. $(\mathcal{M}, d_{\mathcal{M}})$ is called a model for $(\mathcal{A}, d_{\mathcal{A}})$ if there exists a D.G.A.-morphism

$$p : (\mathcal{M}, d_{\mathcal{M}}) \rightarrow (\mathcal{A}, d_{\mathcal{A}})$$

which induces an isomorphism on cohomology. Moreover, if $(\mathcal{M}, d_{\mathcal{M}})$ is minimal, then $(\mathcal{M}, d_{\mathcal{M}})$ is called a minimal model for $(\mathcal{A}, d_{\mathcal{A}})$.

By the minimal model of M , we mean the minimal model of de Rham cohomology complex $(\Omega^*(M), d)$ of M .

DEFINITION 4.5. A manifold M is called formal if $(\Omega^*(M), d)$ and $(H_{DR}^*(M), d=0)$ have the same minimal model.

PROPOSITION 4.6. *The minimal model of a compact A-type solvmanifold is formal.*

PROOF. We define a mapping of cochain complex $f : (H^*(\mathfrak{g}), d=0) \rightarrow (\bigwedge^*(\mathfrak{g}^*), d)$ by

$$\left[\sum_{\#I + \#K = p+q=r} c_{IK} \alpha_I \wedge \omega_K \right] \xrightarrow{f} \sum_{\#I + \#K = p+q=r} c_{IK} \alpha_I \wedge \omega_K,$$

where each ω_K is closed. It is obvious from the proof of Theorem 4.1 that the mapping is multiplicative, that is, f satisfies $f([a] \wedge [b]) = f([a]) \wedge f([b])$. Then the minimal model of A-type is formal (See [7], p. 158 and [1]). \square

5. Examples Related to the Hard Lefschetz Property

EXAMPLE 5.1 ([1]). We consider the following matrices:

$$A = \sum_{k=1}^m a(E_{2k-1, 2k-1} - E_{2k, 2k})$$

$$B = 0.$$

We denote by \mathfrak{g} the Lie algebra constructed by using A and B in Proposition 1. By the proof of Theorem 4.1, it is easy to verify that

$$H^{2q-1}(\mathfrak{g}) = \text{span}\{[\alpha \wedge \zeta_I \wedge \eta_J], [\beta \wedge \zeta_I \wedge \eta_J] \ (\#I = \#J = q-1)\},$$

$$H^{2q}(\mathfrak{g}) = \text{span}\{[\alpha \wedge \beta \wedge \zeta_I \wedge \eta_J] \ (\#I = \#J = q-1), [\zeta_I \wedge \eta_J] \ (\#I = \#J = q)\},$$

where $\zeta_I = \omega_{2i_1-1} \wedge \cdots \wedge \omega_{2i_r-1}$ for $I = (i_1, \dots, i_r)$ and $\eta_J = \omega_{2j_1} \wedge \cdots \wedge \omega_{2j_r}$ for $J = (j_1, \dots, j_r)$. In particular, we see that the odd betti numbers $b_{2i-1}(M)$ are even and $b_i(M) \geq b_{i-2}(M)$ ($i \leq m+1$). $M(a) = \mathbf{R}^2 \ltimes \mathbf{R}^{2m}/\Gamma$ has a symplectic structure. For example,

$$\omega = \alpha \wedge \beta + \omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m}.$$

By Theorem 4.1, $M(a)$ has the Hard Lefschetz property for any symplectic structure. Moreover, if $M = \mathbf{R}^2 \ltimes \mathbf{R}^{4m}/\Gamma$, then M admits a pseudo-Kähler structure (See Section 6 and Example 7.1).

EXAMPLE 5.2. We consider the following automorphism:

$$\varphi(t_1, t_2, x_1, x_2) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where

$$P_i = \begin{pmatrix} e^{t_i} & 0 & x_i e^{t_i} & 0 \\ 0 & e^{-t_i} & 0 & x_i e^{-t_i} \\ 0 & 0 & e^{t_i} & 0 \\ 0 & 0 & 0 & e^{-t_i} \end{pmatrix}$$

for $i = 1, 2$. Then $G = \mathbf{R}^4 \ltimes_{\varphi} \mathbf{R}^8$ has a symplectic structure. For example,

$$\omega = \alpha_1 \wedge \alpha_2 + \beta_1 \wedge \beta_2 + \omega_1 \wedge \omega_4 - \omega_3 \wedge \omega_2 + \omega_5 \wedge \omega_8 - \omega_7 \wedge \omega_6.$$

Now $\omega_3 \wedge \omega_4$, $\omega_7 \wedge \omega_8$ are non-exact closed 2-forms. As in Theorem 4.1, we see

$$\begin{aligned} \omega_3 \wedge \omega_4 &\xrightarrow{L_{\omega}^4} a \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8 \\ &= \pm a \cdot d(\alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8), \\ \omega_7 \wedge \omega_8 &\xrightarrow{L_{\omega}^4} b \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_7 \wedge \omega_8 \\ &= \pm b \cdot d(\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8). \end{aligned}$$

Similarly, G/Γ does not have the Hard Lefschetz property for any symplectic structure.

6. A Construction of Compact Holomorphic Symplectic Solvmanifolds

In this section we construct pseudo-Kähler Lie groups and holomorphic symplectic Lie groups from certain Lie groups. As an application, we have Propositions 4, 5 and 6. We also construct a compact solvmanifold which have a hypercomplex structure and a pseudo-hyperkähler structure.

DEFINITION 6.1. Let M be a complex manifold of dimension $2m$. A holomorphic symplectic structure is a closed holomorphic 2-form Ω on M of maximal rank, i.e. $\Omega^m \neq 0$ at each point of M .

DEFINITION 6.2. Let M be a manifold. A set of complex structures $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ which satisfies $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$ is called a hypercomplex structure. Let (M, g) be a pseudo-Riemannian manifold which carries a hypercomplex structure $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. Then M is called a pseudo-hyperkähler manifold if $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are pseudo-Kähler forms with respect to \mathbf{I} , \mathbf{J} and \mathbf{K} respectively, where $\omega_{\mathbf{I}}(X, Y) = g(\mathbf{I}X, Y)$, $\omega_{\mathbf{J}}(X, Y) = g(\mathbf{J}X, Y)$ and $\omega_{\mathbf{K}}(X, Y) = g(\mathbf{K}X, Y)$.

We consider the following Lie algebra over \mathbf{R} :

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b},$$

where \mathfrak{a} is abelian and \mathfrak{b} is an ideal. Assume that

$$\mathfrak{a} = \text{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1\},$$

$$\mathfrak{b} = \text{span}_{\mathbf{R}}\{V_1^1, \dots, V_q^1\}.$$

Consider the complexification $\mathfrak{g}^{\mathbf{C}}$. Since $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, ${}_{\mathbf{R}}(\mathfrak{g}^{\mathbf{C}})$ has the following basis:

$${}_{\mathbf{R}}(\mathfrak{g}^{\mathbf{C}}) = \text{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, \sqrt{-1}U_1^1, \dots, \sqrt{-1}U_p^1, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\},$$

where $V_j^2 = \sqrt{-1}V_j^1$. Let \mathfrak{h} be the following Lie subalgebra of \mathfrak{g} :

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{b} + \sqrt{-1}\mathfrak{b} = \text{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}.$$

Consider a direct product

$$\mathfrak{h} \times \mathbf{R}^p = \text{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, U_1^2, \dots, U_p^2, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}.$$

We define a complex structure on $\mathfrak{h} \times \mathbf{R}^p$ by the following:

$$\begin{cases} \mathbf{I}U_i^1 = U_i^2 & (\mathbf{I}U_i^2 = -U_i^1) & i = 1, \dots, p \\ \mathbf{I}V_j^1 = V_j^2 & (\mathbf{I}V_j^2 = -V_j^1) & j = 1, \dots, q \end{cases}$$

Note that $\mathfrak{h} \times \mathbf{R}^p$ is a Lie algebra. We use the notation $\Psi_{\mathbf{I}}(\mathfrak{g}) = (\mathfrak{h} \times \mathbf{R}^p, \mathbf{I})$ and let $\Psi_{\mathbf{I}}(G)$ be the simply-connected Lie group corresponding to $\Psi_{\mathbf{I}}(\mathfrak{g})$. Then we have the following:

PROPOSITION 6.3. \mathbf{I} is integrable on $\Psi_{\mathbf{I}}(G)$.

PROOF. We show that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes. By definition of the almost complex structure \mathbf{I} and $[\mathfrak{a}, \mathfrak{a}] = 0$, it is obvious that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes except for the case when $X = U_i^2$, $Y = V_j^1$ or V_j^2 . Let $X = U_i^2$, $Y = V_j^1$. Then

$$\begin{aligned} N_{\mathbf{I}}(U_i^2, V_j^1) &= \mathbf{I}[\mathbf{I}U_i^2, V_j^1] - [\mathbf{I}U_i^2, \mathbf{I}V_j^1] \\ &= -\mathbf{I}[U_i^1, V_j^1] + [U_i^1, \mathbf{I}V_j^1] \\ &= -\sqrt{-1}[U_i^1, V_j^1] + [U_i^1, \sqrt{-1}V_j^1] = 0. \end{aligned}$$

Note that $\mathbf{I}[U_i^1, V_j^1]$ and $[U_i^1, \mathbf{I}V_j^1]$ can be considered as elements of $\mathfrak{g}^{\mathbb{C}}$. The other case is similar and hence omitted. \square

Let $\{\xi_1^1, \dots, \xi_p^1, \xi_1^2, \dots, \xi_p^2, \omega_1^1, \dots, \omega_q^1, \omega_1^2, \dots, \omega_q^2\}$ be the dual basis of $\{U_1^1, \dots, U_p^1, U_1^2, \dots, U_p^2, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}$. Thus as a basis for $(1, 0)$ -type we can take

$$\begin{cases} \mu_i = \xi_i^1 + \sqrt{-1}\xi_i^2 & i = 1, \dots, p \\ \lambda_j = \omega_j^1 + \sqrt{-1}\omega_j^2 & j = 1, \dots, q \end{cases}$$

THEOREM 6.4. If \mathfrak{b} has a non-degenerate 2-form which is closed on \mathfrak{g} , then $\Psi_{\mathbf{I}}(G) \times \mathbf{C}^p$ has a holomorphic symplectic structure. Moreover, if $[\mathfrak{b}, \mathfrak{b}] = 0$, then the solvable Lie group $\Psi_{\mathbf{I}}(G)$ has a pseudo-Kähler structure.

PROOF. Let $\omega_{\mathfrak{b}} = \sum_{k < h} P_{kh} \omega_k^1 \wedge \omega_h^1$ be a non-degenerate 2-form which is closed on \mathfrak{g} . It is obvious that if $\tau = \sum_{k < h} P_{kh} (\lambda_k \wedge \lambda_h + \bar{\lambda}_k \wedge \bar{\lambda}_h) = 2 \sum_{k < h} P_{kh} (\omega_k^1 \wedge \omega_h^1 - \omega_k^2 \wedge \omega_h^2)$ is a closed 2-form, then $\sum_{k < h} P_{kh} \lambda_k \wedge \lambda_h$ is also closed. Since $d\omega_{\mathfrak{b}} = 0$ and $\tau(X, \mathbf{I}Y) = 0$ for $X, Y \in \mathfrak{g} \subset \mathfrak{h}$, it is easy to check

that $d\tau(X, Y, Z) = d\tau(\mathbf{I}X, \mathbf{I}Y, \mathbf{I}Z) = d\tau(\mathbf{I}X, Y, Z) = 0$ for $X, Y, Z \in \mathfrak{g}$. Since $\tau(X, Y) = -\tau(\mathbf{I}X, \mathbf{I}Y) = \omega_{\mathfrak{b}}(X, Y)$ for $X, Y \in \mathfrak{g}$, we see

$$\begin{aligned} d\tau(\mathbf{I}X, \mathbf{I}Y, Z) &= -\tau([\mathbf{I}X, \mathbf{I}Y], Z) + \tau([\mathbf{I}X, Z], \mathbf{I}Y) - \tau([\mathbf{I}Y, Z], \mathbf{I}X) \\ &= +\tau([X, Y], Z) - \tau([X, Z], Y) + \tau([Y, Z], X) \\ &= +\omega_{\mathfrak{b}}([X, Y], Z) - \omega_{\mathfrak{b}}([X, Z], Y) + \omega_{\mathfrak{b}}([Y, Z], X) \\ &= -d\omega_{\mathfrak{b}}(X, Y, Z) = 0, \end{aligned}$$

where $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$. If $X \in \mathfrak{a}$ or $Y \in \mathfrak{a}$, then it is obvious that $d\tau(\mathbf{I}X, \mathbf{I}Y, Z) = 0$. Thus $\sum_{k < h} P_{kh} \lambda_k \wedge \lambda_h$ is closed. Hence

$$\Omega = \sum_{i=1}^p \mu_i \wedge \mu'_i + \sum_{k < h} P_{kh} \lambda_k \wedge \lambda_h,$$

where $\{\mu'_i\}_{i=1, \dots, p}$ is a basis of $\Omega^{1,0}(\mathbf{C}^p)$, is a holomorphic symplectic structure on $\Psi_{\mathbf{I}}(G) \times \mathbf{C}^p$.

Next assume that $[\mathfrak{b}, \mathfrak{b}] = 0$ and consider $\theta = \sum_{k < h} P_{kh} (\lambda_k \wedge \bar{\lambda}_h + \bar{\lambda}_k \wedge \lambda_h) = 2 \sum_{k < h} P_{kh} (\omega_k^1 \wedge \omega_h^1 + \omega_k^2 \wedge \omega_h^2)$. Note that $\theta(X, Y) = \theta(\mathbf{I}X, \mathbf{I}Y) = \omega_{\mathfrak{b}}(X, Y)$ and $\theta(\mathbf{I}X, Y) = 0$ for $X, Y \in \mathfrak{g} \subset \mathfrak{h}$. Since $\theta([X, Y], Z) = \omega_{\mathfrak{b}}([X, Y], Z) = 0$ for $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$, we have

$$\begin{aligned} d\theta(\mathbf{I}X, \mathbf{I}Y, Z) &= -\theta([\mathbf{I}X, \mathbf{I}Y], Z) + \theta([\mathbf{I}X, Z], \mathbf{I}Y) - \theta([\mathbf{I}Y, Z], \mathbf{I}X) \\ &= -\omega_{\mathfrak{b}}([X, Y], Z) + \omega_{\mathfrak{b}}([X, Z], Y) - \omega_{\mathfrak{b}}([Y, Z], X) \\ &= d\omega_{\mathfrak{b}}(X, Y, Z) = 0, \end{aligned}$$

where $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$. If $X \in \mathfrak{a}$ or $Y \in \mathfrak{a}$, then it is obvious that $d\theta(\mathbf{I}X, \mathbf{I}Y, Z) = 0$. The other cases are similar to the case of a holomorphic symplectic structure. Thus θ is closed. Hence,

$$\omega = \sqrt{-1} \sum_{i=1}^p \mu_i \wedge \bar{\mu}_i + \sum_{k < h} P_{kh} (\lambda_k \wedge \bar{\lambda}_h + \bar{\lambda}_k \wedge \lambda_h)$$

is a pseudo-Kähler form on $(\Psi_{\mathbf{I}}(G), \mathbf{I})$. \square

REMARK. The signature of the pseudo-Kähler metric constructed above is $(p+q, q)$.

Let $[\mathfrak{b}, \mathfrak{b}] = 0$ and consider $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$. Then the solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$ can be written as follows.

$$\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g})) = \text{span}_{\mathbb{R}}\{U_1^1, \dots, U_p^1, \dots, U_1^4, \dots, U_p^4, V_1^1, \dots, V_q^1, \dots, V_1^4, \dots, V_q^4\},$$

where the bracket products are

$$[U_i^1, V_j^h] = \sum_{k=1}^q c_{ij}^k V_k^h$$

for $i = 1, \dots, p, j = 1, \dots, q, h = 1, 2, 3, 4$.

Then we have the following:

PROPOSITION 6.5. *The simply-connected solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$ corresponding to $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$ has a hypercomplex structure.*

PROOF. Let $\{W_1^h, \dots, W_{p+q}^h\} = \{U_1^h, \dots, U_p^h, V_1^h, \dots, V_q^h\}$ for each $h = 1, 2, 3, 4$. We define almost complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$ which satisfy $\mathbf{I}\mathbf{J} = -\mathbf{J}\mathbf{I} = \mathbf{K}$ by

$$\begin{cases} \mathbf{I}W_i^1 = W_i^2, \\ \mathbf{I}W_i^4 = W_i^3, \end{cases} \quad \begin{cases} \mathbf{J}W_i^1 = W_i^3, \\ \mathbf{J}W_i^2 = W_i^4, \end{cases} \quad \begin{cases} \mathbf{K}W_i^4 = W_i^1, \\ \mathbf{K}W_i^2 = W_i^3. \end{cases}$$

It is easy to check that the Nijenhuis tensor $N(X, Y)$ vanish for each $\mathbf{I}, \mathbf{J}, \mathbf{K}$. By the construction, \mathbf{J} is integrable on $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$. Thus we only check the case of \mathbf{I} and $X = U_i^h, Y = V_j^h$. Let $X = U_i^1, Y = V_j^4$. We see

$$\begin{aligned} N_{\mathbf{I}}(U_i^1, V_j^4) &= [U_i^1, V_j^4] + \mathbf{I}[U_i^1, V_j^4] + \mathbf{I}[U_i^1, \mathbf{I}V_j^4] - \mathbf{I}[\mathbf{I}U_i^1, V_j^4] \\ &= [U_i^1, V_j^4] + \mathbf{I}[U_i^1, V_j^3] \\ &= \sum c_{ij}^k V_k^4 + \mathbf{I} \sum c_{ij}^k V_k^3 \\ &= \sum c_{ij}^k V_k^4 - \sum c_{ij}^k V_k^4 = 0. \end{aligned}$$

The other cases are similar and hence omitted. Then $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ is a hypercomplex structure on $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$. \square

Let $\{\zeta_i^h, \omega_j^h\}_{i,j,h}$ be the dual basis of $\{U_i^h, V_j^h\}_{i,j,h}$. Then we have the following:

THEOREM 6.6. *If \mathfrak{b} has a non-degenerate 2-form which is closed on \mathfrak{g} , then the solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$ has a pseudo-hyperkähler structure.*

PROOF. Let $\omega_b = \sum_{k,h} P_{kh} \omega_k \wedge \omega_h$, where $P_{kh} = -P_{hk}$, be a non-degenerate 2-form which is closed on \mathfrak{g} . Consider the following pseudo-Riemannian metric of signature $(4p + 2q, 2q)$:

$$g = \sum_{i=1}^4 \sum_{k=1}^p \xi_k^i \otimes \xi_k^i + \sum P_{kh} (\omega_k^1 \otimes \omega_h^2 + \omega_k^3 \otimes \omega_h^4) - \sum P_{hk} (\omega_h^2 \otimes \omega_k^1 + \omega_h^4 \otimes \omega_k^3)$$

Then $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are pseudo-Kähler forms with respect to \mathbf{I} , \mathbf{J} and \mathbf{K} . By a straightforward computation, we see

$$\omega_{\mathbf{I}} = 2 \sum_{k=1}^p (\xi_k^1 \wedge \xi_k^2 - \xi_k^3 \wedge \xi_k^4) - \sum P_{kh} (\omega_k^1 \wedge \omega_h^1 + \omega_k^2 \wedge \omega_h^2 - \omega_k^3 \wedge \omega_h^3 - \omega_k^4 \wedge \omega_h^4),$$

$$\omega_{\mathbf{J}} = 2 \sum_{k=1}^p (\xi_k^1 \wedge \xi_k^3 + \xi_k^2 \wedge \xi_k^4) + 2 \sum P_{kh} (\omega_k^1 \wedge \omega_h^4 - \omega_k^3 \wedge \omega_h^2),$$

$$\omega_{\mathbf{K}} = -2 \sum_{k=1}^p (\xi_k^1 \wedge \xi_k^4 - \xi_k^2 \wedge \xi_k^3) + 2 \sum P_{kh} (\omega_k^1 \wedge \omega_h^3 + \omega_k^4 \wedge \omega_h^2).$$

Moreover we see $\sum_{k,h} P_{kh} \omega_k \wedge \omega_h \xrightarrow{d} -\sum_{k,h,i,j} (P_{jh} c_{ik}^j + P_{kj} c_{ih}^j) \xi_i \wedge \omega_k \wedge \omega_h$. Hence, $2 \sum_j (P_{jh} c_{ik}^j + P_{kj} c_{ih}^j) = 0$. Since $\sum_{k,h} P_{kh} \omega_k^s \wedge \omega_h^t \xrightarrow{d} -\sum_{k,h,i,j} (P_{jh} c_{ik}^j + P_{kj} c_{ih}^j) \xi_i \wedge \omega_k^s \wedge \omega_h^t$, we see that $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are closed. \square

REMARK. Let $(M, g, \mathbf{I}, \mathbf{J}, \mathbf{K})$ be a pseudo-hyperkähler manifold. Then the complex 2-form $\omega_{\mathbf{J}} + \sqrt{-1} \omega_{\mathbf{K}}$ is a holomorphic symplectic structure on (M, \mathbf{I}) . In the above case, we have the following holomorphic symplectic structure on (M, \mathbf{J}) :

$$\Omega = -\omega_{\mathbf{I}} + \sqrt{-1} \omega_{\mathbf{K}} = 2 \sum_{k=1}^p \mu_k^{1,3} \wedge \mu_k^{2,4} + \sum_{k,h} P_{kh} (\lambda_k^{1,3} \wedge \lambda_h^{1,3} + \lambda_k^{2,4} \wedge \lambda_h^{2,4}),$$

where $\mu_k^{i,j} = \xi_k^i + \sqrt{-1} \xi_k^j$, $\lambda_k^{i,j} = \omega_k^i + \sqrt{-1} \omega_k^j$. Note that (M, \mathbf{J}) has other holomorphic symplectic structures the cohomology classes of which are different from the cohomology class of Ω . For example, by the proof of Theorem 6.4,

$$\Omega' = 2 \sum_{k=1}^p \mu_k^{1,3} \wedge \mu_k^{2,4} + \sum_{k,h} P_{kh} \lambda_k^{1,3} \wedge \lambda_k^{2,4}$$

is also a holomorphic symplectic structure on (M, \mathbf{J}) .

Let $\varphi(\mathbf{t}, \mathbf{x})$ ($\mathbf{t} \in \mathbf{R}^l, \mathbf{x} \in \mathbf{R}^n$) be an automorphism of \mathbf{R}^{2m} constructed in Proposition 1. Consider a solvable Lie group $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$, where $\tilde{\varphi}(\mathbf{t}, \mathbf{x}) = \varphi(\mathbf{t}, \mathbf{x}) \oplus \varphi(\mathbf{t}, \mathbf{x})$, that is, the group structure of \tilde{G} is defined by

$$\begin{aligned} & (\mathbf{t}_1, \mathbf{x}_1, \mathbf{s}_1, \mathbf{r}_1, \mathbf{y}_1, \mathbf{z}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{s}_2, \mathbf{r}_2, \mathbf{y}_2, \mathbf{z}_2) \\ &= (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{s}_1 + \mathbf{s}_2, \mathbf{r}_1 + \mathbf{r}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2, \mathbf{z}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{z}_2) \end{aligned}$$

for $\mathbf{s}_i, \mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i, \mathbf{r}_i \in \mathbf{R}^n$ and $\mathbf{y}_i, \mathbf{z}_i \in \mathbf{R}^{2m}$. The Lie algebra $\tilde{\mathfrak{g}}$ of \tilde{G} is

$$\tilde{\mathfrak{g}} = \{A_i, B_j, U_i, V_j, Y_k, Z_k\}_{i=1, \dots, l, j=1, \dots, n, k=1, \dots, 2m},$$

where the bracket products are

$$\begin{aligned} [A_i, Y_{2k-1}] &= a_i^k Y_{2h-1}, & [A_i, Z_{2k-1}] &= a_i^k Z_{2k-1}, \\ [A_i, Y_{2k}] &= -a_i^k Y_{2h}, & [A_i, Z_{2k}] &= -a_i^k Z_{2k}, \\ [B_j, Y_{2h-1}] &= \sum_{k < h} b_j^{kh} Y_{2k-1}, & [B_j, Z_{2h-1}] &= \sum_{k < h} b_j^{kh} Z_{2k-1}, \\ [B_j, Y_{2h}] &= \sum_{k < h} b_j^{kh} Y_{2k}, & [B_j, Z_{2h}] &= \sum_{k < h} b_j^{kh} Z_{2k}, \end{aligned}$$

for $i = 1, \dots, l$, $j = 1, \dots, n$ and $1 \leq k < h \leq m$ and the other brackets are zero.

We denote by $\{\alpha_i^1, \beta_j^1, \alpha_i^2, \beta_j^2, \omega_k^1, \omega_k^2\}$ the dual basis of $\{A_i, B_j, U_i, V_j, Y_k, Z_k\}$.

Let us consider the following Lie algebra and its decomposition:

DECOMPOSITION. 1:

$$\begin{aligned} \mathfrak{g} &= \text{span}\{A_i, B_j, Y_k\}, \\ \mathfrak{a} &= \text{span}\{A_i, B_j\}, \\ \mathfrak{b} &= \text{span}\{Y_1, \dots, Y_{2m}\}. \end{aligned}$$

Then $\tilde{\mathfrak{g}} = \Psi_1(\mathfrak{g})$. Thus we have Proposition 4, 5 and 6 by Proposition 6.3 and Theorem 6.4.

Indeed, we define an almost complex structure \mathbf{I} by

$$\begin{cases} \mathbf{I}A_i = U_i & i = 1, \dots, l \\ \mathbf{I}B_j = V_j & j = 1, \dots, n \\ \mathbf{I}Y_k = Z_k & k = 1, \dots, 2m \end{cases}$$

By Proposition 6.3, we see that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes and a basis for (1,0)-type forms is given by

$$\begin{cases} \mu_i = \alpha_i^1 + \sqrt{-1}\alpha_i^2 & i = 1, \dots, l \\ \nu_j = \beta_j^1 + \sqrt{-1}\beta_j^2 & j = 1, \dots, n \\ \lambda_k = \omega_k^1 + \sqrt{-1}\omega_k^2 & k = 1, \dots, 2m \end{cases}$$

In particular, if \mathfrak{g} is not A -type, then $M = \Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ has a pseudo-Kähler structure with respect to which M does not have the Hard Lefschetz property.

REMARK. If G is A -type, then the Frölicher spectral sequence $\{E_r(\tilde{\mathfrak{g}})\}$ satisfies $E_1(\tilde{\mathfrak{g}}) \simeq E_\infty(\tilde{\mathfrak{g}})$. In particular, $\dim H^r(\tilde{\mathfrak{g}}) = \sum_{p+q=r} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}^{\mathbb{C}})$.

Indeed, by a straightforward computation, we see

$$\begin{aligned} \bar{\partial}(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) &= \sum a_i^{K_1 K_2} \bar{\mu}_i \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2} \\ \partial(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) &= \sum a_i^{K_1 K_2} \mu_i \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}, \end{aligned}$$

which implies that if $\lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ is $\bar{\partial}$ -closed, then ∂ -closed. Put $\mu_{I\bar{J}} = \mu_I \wedge \bar{\mu}_J$. Let $\gamma = \sum_I c_{IJK_1 K_2} \mu_{I\bar{J}} \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ be a $\bar{\partial}$ -closed form such that for each K_1, K_2 , there exists an i such that $a_i^{K_1 K_2} \neq 0$. Similarly to Lemma 2.1, we can check that γ is $\bar{\partial}$ -exact. Thus for each $\bar{\partial}$ -cohomology class, we can choose a representation $\sum c_{IJK_1 K_2} \mu_{I\bar{J}} \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ such that $\bar{\partial}(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) = 0$. Then we have $H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}) = \overline{H_{\bar{\partial}}^{q,p}(\tilde{\mathfrak{g}}^{\mathbb{C}})}$ and $\partial : H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}^{\mathbb{C}}) \rightarrow H_{\bar{\partial}}^{p+1,q}(\tilde{\mathfrak{g}}^{\mathbb{C}})$ is the zero-mapping by the above argument. Hence $E_1(\tilde{\mathfrak{g}}) \simeq E_\infty(\tilde{\mathfrak{g}})$ (See [1]).

Let $\mathfrak{g} = \text{span}\{A_i, B_j, Y_k\}$ be a solvable Lie algebra constructed in Proposition 1 and consider the following decomposition:

DECOMPOSITION. 2:

$$\begin{aligned} \mathfrak{a} &= \text{span}\{A_1, \dots, A_l\}, \\ \mathfrak{b} &= \text{span}\{B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}. \end{aligned}$$

Since $\mathfrak{a}, \mathfrak{b}$ satisfy the condition in Proposition 6.3, we can construct a solvable Lie algebra $\Psi_{\mathbf{I}}(\mathfrak{g})$. Since $\text{span}\{Y_1, Y_2, \mathbf{I}Y_1, \mathbf{I}Y_2, \dots, \mathbf{I}Y_{2k-1}, \mathbf{I}Y_{2k}\}$ is an ideal of $\Psi_{\mathbf{I}}(\mathfrak{g})$, $\Psi_{\mathbf{I}}(G)$ also has a lattice. We show that $\Psi_{\mathbf{I}}(G)$ has no left $\Psi_{\mathbf{I}}(G)$ -invariant pseudo-Kähler structures with respect to \mathbf{I} except the case of A -type. For simplicity we use the following notation:

$$d\omega_k = - \sum_i A_k^i \alpha_i \wedge \omega_k - \sum_{j,h} B_k^{jh} \beta_j \wedge \omega_h.$$

Hence, $A_{2k-1}^i = -A_{2k}^i = a_i^k$, $B_{2k-1}^{j2h-1} = B_{2k}^{j2h} = b_j^{kh}$. By a straightforward computation, we see

$$d\lambda_k = d(\omega_k^1 + \sqrt{-1}\omega_k^2) = -\frac{1}{2} \sum_{k,i} A_k^i (\mu_i + \bar{\mu}_i) \wedge \lambda_k - \sum_{j,h} B_k^{jh} v_j \wedge \lambda_h.$$

PROPOSITION 6.7. *If a compact solvmanifold $\Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ constructed from the decomposition 2 has a left $\Psi_{\mathbf{I}}(G)$ -invariant pseudo-Kähler structure, then G is A -type.*

PROOF. By Stokes' theorem and the assumption of the coefficients A_k^i , if there exists a left $\Psi_{\mathbf{I}}(G)$ -invariant pseudo-Kähler structure, then there exists a $\bar{\partial}$ -closed 2-form $\sum Q^{kh} \lambda_k \wedge \bar{\lambda}_h$ of maximal rank; i.e. the matrix $Q = (Q^{kh})$ is non-degenerate. Thus

$$\begin{aligned} 0 &= \bar{\partial} \sum Q^{kh} \lambda_k \wedge \bar{\lambda}_h \\ &= -\frac{1}{2} \sum_{k,h,i} (A_k^i + A_h^i) \bar{\mu}_i \wedge \lambda_k \wedge \bar{\lambda}_h - \sum_{k,h,j,i} Q^{kh} B_h^{ji} \lambda_k \wedge \bar{v}_j \wedge \bar{\lambda}_i. \end{aligned}$$

Hence, $\sum_h Q^{kh} B_h^{ji} = 0$. By the non-degeneracy of $Q = (Q^{kh})$ it implies that $B_h^{ji} = 0$ for each i, j, h . \square

By this proposition, we can construct a compact holomorphic symplectic solvmanifold $\Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ with no left $\Psi_{\mathbf{I}}(G)$ -invariant pseudo-Kähler structures with respect to \mathbf{I} .

REMARK. Let $(N/\Gamma, \omega)$ be a non-toral compact symplectic nilmanifold. Then a compact complex nilmanifold $(\tilde{N}/\tilde{\Gamma}, \mathbf{I})$, where \tilde{N} is the simply-connected nilpotent Lie group corresponding to a complex nilpotent Lie algebra $(\mathfrak{R}(\mathfrak{n}^{\mathbf{C}}), \mathbf{I})$, has a holomorphic symplectic structure. However, $(\tilde{N}/\tilde{\Gamma}, \mathbf{I})$ has no pseudo-Kähler structures with respect to \mathbf{I} (See [9] and [18, Theorem 1]).

7. Examples of Compact Holomorphic Symplectic Solvmanifolds

EXAMPLE 7.1 ([1]). We consider the following automorphism of \mathbf{R}^2 :

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Note that

$$G_3 = \left\{ \left(\begin{array}{cccc} e^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & y_2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| y_1, y_2, t \in \mathbf{R} \right\}$$

has a $\bigwedge^{0,2}$ -type form $\omega_1 \wedge \omega_2$ with rank 2. As in Proposition 4, we have the following solvable Lie group which has a lattice:

$$\tilde{G} = \mathbf{R}^2 \times_{\tilde{\varphi}} \mathbf{R}^4 = \left\{ \left(\begin{array}{cccccc} e^t & 0 & 0 & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & 0 & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| s, t, y_1, y_2, z_1, z_2 \in \mathbf{R} \right\}.$$

By Theorem 4.1 and Theorem 6.4, $\tilde{G}/\tilde{\Gamma}$ is a compact pseudo-Kähler manifold which has the Hard Lefschetz property. By Theorem 6.4, $\tilde{G}/\tilde{\Gamma} \times \mathbf{C}/\Gamma$ has a holomorphic symplectic structure.

EXAMPLE 7.2. Let \mathfrak{g} be the following Lie algebra:

$$\mathfrak{g} = \text{span}\{A, B, Y_1, Y_2, Y_3, Y_4\},$$

where the bracket products are

$$[A, Y_1] = Y_1, \quad [A, Y_2] = -Y_2,$$

$$[A, Y_3] = Y_3, \quad [A, Y_4] = -Y_4,$$

$$[B, Y_3] = Y_1, \quad [B, Y_4] = Y_2.$$

Consider the following decomposition:

$$\mathfrak{a} = \text{span}\{A, B\},$$

$$\mathfrak{b} = \text{span}\{Y_1, Y_2, Y_3, Y_4\}.$$

By Theorem 6.4, $\Psi_1(G)$ has a holomorphic symplectic structure and a lattice. Moreover, by Theorem 6.6, $M^{24} = \Psi_{\mathbf{J}}(\Psi_1(G))/\Gamma$ has a pseudo-hyperkähler structure.

Next consider

$$\mathfrak{t} = \mathfrak{g} \oplus \mathfrak{g} = \text{span}\{A, B, Y_1, \dots, Y_4, A', B', Y'_1, \dots, Y'_4\}.$$

Let $\{\alpha, \beta, \omega_1, \dots, \omega_4, \alpha', \beta', \omega'_1, \dots, \omega'_4\}$ be the dual basis corresponding to $\{A, B, Y_1, \dots, Y_4, A', B', Y'_1, \dots, Y'_4\}$. Consider the following decomposition:

$$\mathfrak{a} = \text{span}\{A, A'\},$$

$$\mathfrak{b} = \text{span}\{B, B', Y_1, \dots, Y_4, Y'_1, \dots, Y'_4\}.$$

By a straightforward computation, we see that \mathfrak{b} has the following non-degenerate closed 2-form:

$$\omega_{\mathfrak{b}} = \beta \wedge \beta' + \sum_{k=0}^1 (-1)^k (\omega_{2k+1} \wedge \omega_{4-2k} + \omega'_{2k+1} \wedge \omega'_{4-2k}).$$

By Theorem 6.4 and Proposition 6.7, $\Psi_1(K)/\Gamma_{\Psi_1(K)}$ is a compact symplectic solvmanifold with no left $\Psi_1(K)$ -invariant pseudo-Kähler structures with respect to \mathbf{I} .

REMARK. It is easy to check that $\Psi_1(K)/\Gamma_{\Psi_1(K)}$ is a total space which has non-toral symplectic solvmanifolds as fiber and base space. Moreover, $\Psi_1(K)/\Gamma_{\Psi_1(K)}$ has a compatible symplectic structure. Indeed, consider the following Lie subalgebras:

$$\mathfrak{n}_1 = \text{span}\{B, \mathbf{I}B, Y_1, \dots, Y_4, \mathbf{I}Y_1, \dots, \mathbf{I}Y_4\},$$

$$\mathfrak{n}_2 = \text{span}\{B', \mathbf{I}B', Y'_1, \dots, Y'_4, \mathbf{I}Y'_1, \dots, \mathbf{I}Y'_4\},$$

$$\mathfrak{t} = \text{span}\{A, \mathbf{I}A, A', \mathbf{I}A'\}.$$

\mathfrak{n}_1 and $\mathfrak{t} \times \mathfrak{n}_2$ have non-degenerate 2-forms which are closed on $\Psi_1(K)/\Gamma_{\Psi_1(K)}$. Consider a symplectic fiber bundle $\pi_1 : (T \times N_1)/(\Gamma_T \times \Gamma_{N_1}) \rightarrow T/\Gamma_T$ and a mapping $\pi_2 : (T \times N_2)/(\Gamma_T \times \Gamma_{N_2}) \rightarrow T/\Gamma_T$, where T, N_1, N_2 are simply-connected Lie groups corresponding to $\mathfrak{t}, \mathfrak{n}_1, \mathfrak{n}_2$ and $\Gamma_T, \Gamma_{N_1}, \Gamma_{N_2}$ are its lattices. Then the induced fiber bundle $\pi_2^{-1}((T \times N_1)/(\Gamma_T \times \Gamma_{N_1})) = \Psi_1(K)/\Gamma_{\Psi_1(K)}$ is desired.

8. A Construction of Solvable Lie Group with a Parameterized Lattice

In this section we consider some complexification of a solvable Lie group $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ constructed in Proposition 1, each of which has a parameterized lattice.

Let $\tilde{G}_{3,3}$ be the simply-connected solvable Lie group defined by

$$\tilde{G}_{3,3} = \left\{ \left(\begin{array}{cccc} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| w_1, w_2, z \in \mathbf{C} \right\}.$$

Note that $\tilde{G}_{3,3}$ may be described as the semi-direct product $\mathbf{C}^1 \ltimes_{\varphi_3} \mathbf{C}^2$, where

$$\varphi_3(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Let $B \in SL(2, \mathbf{Z})$ be a unimodular matrix with distinct real eigenvalues, say, $\lambda, 1/\lambda$ (it's not necessary that λ is positive). Take $t_0 = \text{Log } \lambda$, i.e., $e^{t_0} = \lambda$. Then there exists a matrix $P \in GL(2, \mathbf{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let

$$\tilde{L}_1 = \mathbf{Z}[t_0, \sqrt{-1}\pi] = \{t_0k + \sqrt{-1}\pi \cdot h \mid k, h \in \mathbf{Z}\},$$

$$\tilde{L}_2 = \left\{ P \begin{pmatrix} \mu \\ \nu \end{pmatrix} \middle| \mu, \nu \in \mathbf{Z}[\sqrt{-1}] \right\},$$

and put $\tilde{\Gamma}_3 = \tilde{L}_1 \ltimes_{\varphi_3} \tilde{L}_2$. Since

$$\begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \begin{pmatrix} e^{\sqrt{-1}y} & 0 \\ 0 & e^{-\sqrt{-1}y} \end{pmatrix},$$

where $z = x + \sqrt{-1}y$, $\tilde{\Gamma}_3$ is a lattice of $\tilde{G}_{3,3}$. Similarly, the following solvable Lie groups have lattices:

$$\tilde{G}_{3,4} = \left\{ \left(\begin{array}{cccc} e^{\bar{z}} & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| w_1, w_2, z \in \mathbf{C} \right\},$$

$$\tilde{G}_{3,5} = \left\{ \left(\begin{array}{cccc} e^z & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| w_1, w_2, z \in \mathbf{C} \right\}.$$

We define mappings $\varphi_{i,*} : \mathbf{C}^{n+l} \rightarrow \text{End}(\mathbf{C}^{2m})$ by the following:

$$\varphi_{1,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l \frac{1}{2}(z_i + \bar{z}_i)A_i + \sum_{j=1}^n \frac{1}{2}(x_j + \bar{x}_j)B_j,$$

$$\varphi_{2,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l \frac{1}{2}(z_i + \bar{z}_i)A_i + \sum_{j=1}^n x_j B_j,$$

$$\varphi_{3,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l z_i A_i + \sum_{j=1}^n x_j B_j,$$

$$\varphi_{4,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l \bar{z}_i A_i + \sum_{j=1}^n x_j B_j,$$

$$\varphi_{5,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l (z_i A_i^{\text{odd}} - \bar{z}_i A_i^{\text{even}}) + \sum_{j=1}^n \frac{1}{2}(x_j + \bar{x}_j)B_j,$$

$$\varphi_{6,*}(\mathbf{z}, \mathbf{x}) = \sum_{i=1}^l (z_i A_i^{\text{odd}} - \bar{z}_i A_i^{\text{even}}) + \sum_{j=1}^n x_j B_j,$$

where

$$A_i^{\text{odd}} = \sum_{k=1}^m a_i^k E_{2k-1, 2k-1}, \quad A_i^{\text{even}} = \sum_{k=1}^m a_i^k E_{2k, 2k} \quad i = 1, \dots, l.$$

Let $\varphi_i(\mathbf{z}, \mathbf{x}) = \exp(\varphi_{i,*}(\mathbf{z}, \mathbf{x}))$ and we define group structures on $\mathbf{C}^{n+l} \times \mathbf{C}^{2m}$ by

$$(\mathbf{z}_1, \mathbf{x}_1, \mathbf{w}_1) *_i (\mathbf{z}_2, \mathbf{x}_2, \mathbf{w}_2) = (\mathbf{z}_1 + \mathbf{z}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{w}_1 + \varphi_i(\mathbf{z}_1, \mathbf{x}_1)\mathbf{w}_2)$$

for $\mathbf{z}_i \in \mathbf{C}^l$, $\mathbf{x}_i \in \mathbf{C}^n$ and $\mathbf{w}_i \in \mathbf{C}^{2m}$.

We denote the Lie group $(\mathbf{C}^{n+l} \times \mathbf{C}^{2m}, *_i)$ by $\tilde{G}_i = \mathbf{C}^{n+l} \ltimes_{\varphi_i} \mathbf{C}^{2m}$. We call that \tilde{G}_i is the complexification of $G = (\mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}, *)$ of type i .

We denote by $\alpha_i, \beta_j, \omega_k$ the left G -invariant 1-forms on $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ such that

$$(\alpha_i)_e = (dt_i)_e, \quad (\beta_j)_e = (dx_j)_e, \quad (\omega_k)_e = (dy_k)_e.$$

We denote $\bigwedge^i \{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n\} \wedge \bigwedge^j \{\omega_1, \dots, \omega_{2m}\}$ by $\bigwedge^{i,j}$. Moreover, for each $\eta = 1, \dots, 6$, we denote by $\tilde{\alpha}_{i,\eta}, \tilde{\beta}_{j,\eta}, \tilde{\omega}_{k,\eta}$ the left \tilde{G}_η -invariant $(1, 0)$ -forms on \tilde{G}_η such that

$$(\tilde{\alpha}_{i,\eta})_e = (dz_i)_e, \quad (\tilde{\beta}_{j,\eta})_e = (dx_j)_e, \quad (\tilde{\omega}_{k,\eta})_e = (dw_k)_e.$$

If there exists no possibility of confusion, we write $\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\omega}_k$ for $\tilde{\alpha}_{i,\eta}, \tilde{\beta}_{j,\eta}, \tilde{\omega}_{k,\eta}$ respectively. For simplicity, we put $\{\lambda_1, \dots, \lambda_{n+l}\} = \{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n\}$ and $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n+l}\} = \{\tilde{\alpha}_1, \dots, \tilde{\beta}_n\}$.

PROPOSITION 8.1. *For each i , the solvable Lie group $\tilde{G}_i = \mathbf{C}^{n+l} \times_{\varphi_i} \mathbf{C}^{2m}$ has a parameterized lattice.*

PROOF. We construct a co-compact lattice of \tilde{G}_1 . Let τ be a complex number such that $\text{Im } \tau > 0$ and p_i, q_j ($i = 1, \dots, l, j = 1, \dots, n$) non-zero purely imaginary numbers. Let $\mathbf{Z}[\tau] = \{k + \tau h \mid k, h \in \mathbf{Z}\}$. We put

$$\begin{aligned} \tilde{L}_{1,A}(\mathbf{p}) &= at_0\mathbf{Z}[p_1] \times \cdots \times at_0\mathbf{Z}[p_l], \\ \tilde{L}_{1,B}(\mathbf{q}) &= a^{m-1}(m-1)!\mathbf{Z}[q_1] \times \cdots \times a^{m-1}(m-1)!\mathbf{Z}[q_n], \\ \tilde{L}_2(\tau) &= \left\{ P \begin{pmatrix} \mu_1 \\ v_1 \end{pmatrix} \middle| \mu_1, v_1 \in \mathbf{Z}[\tau] \right\} \times \cdots \times \left\{ P \begin{pmatrix} \mu_m \\ v_m \end{pmatrix} \middle| \mu_m, v_m \in \mathbf{Z}[\tau] \right\}, \end{aligned}$$

where a is the least common multiple for denominators of a_i^k, b_j^{kh} . Then $\tilde{\Gamma}_1 = (\tilde{L}_{1,A}(\mathbf{p}) \times \tilde{L}_{1,B}(\mathbf{q})) \times_{\varphi_1} \tilde{L}_2(\tau)$ is a lattice of \tilde{G}_1 which has some parameters. Similarly, \tilde{G}_i has a lattice which has some parameters. \square

REMARKS.

- (i) The cases 1 and 2 correspond to the decompositions 1 and 2 respectively.
- (ii) More generally, if $\mathfrak{b}_0 = \text{span}_{\mathbf{Q}}\{B_1, \dots, B_n\}$ is a nilpotent Lie algebra over \mathbf{Q} , then we have that $\mathbf{C}^{n+l} \times_{\varphi_i} \mathbf{C}^{2m}$ admits a lattice (cf. Raghunathan [17]; Theorem 2.12 of Chapter II).
- (iii) We can apply the complexification of type 1 to other solvable Lie groups. For example,

$$\tilde{G} = \left\{ \left(\begin{array}{cccc} \cos(z + \bar{z}) & \cos(z + \bar{z}) & 0 & w_1 \\ -\sin(z + \bar{z}) & \sin(z + \bar{z}) & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| w_1, w_2, z \in \mathbf{C} \right\}$$

has a lattice.

- (iv) The author thinks it's not trivial that the existence of a lattice of complexified solvable Lie group (See Guan [9, Example of Section 4]).
- (v) If we assume that $z, x, w \in \mathbf{H}$, then we have a complex solvmanifold which admits an almost hypercomplex structure some of which are integrable.

9. Holomorphic Symplectic Structure on \tilde{G}_i

In this section we consider holomorphic symplectic structures and pseudo-Kähler structures on \tilde{G}_i . Let $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ be a completely solvable Lie group constructed in Proposition 1. In this section we always assume that for each k there exists an i such that $a_i^k \neq 0$ and there exists a j such that $B_j \neq 0$. Moreover, when we consider $\varphi_{5,*}, \varphi_{6,*}$, we always assume that for each i the signature of a_i^k is constant. For simplicity, we always assume that $n+l$ are even.

By Lemma 2.1 and 2.2, we have the following:

LEMMA 9.1. *Let G/Γ be a compact symplectic solvmanifold constructed in Proposition 1. If G/Γ has a symplectic structure, then there exists a left G -invariant symplectic structure ω which is an element of $\wedge^{2,0} + \wedge^{0,2}$.*

PROPOSITION 9.2. *If G/Γ has a symplectic structure, then $\tilde{G}_i/\tilde{\Gamma}_i$ ($i = 1, 2, 3, 4$) has a holomorphic symplectic structure.*

PROOF. By Lemma 9.1, there exists the following symplectic structure:

$$\omega = \sum P_{kh} \lambda_k \wedge \lambda_h + \sum Q_{kh} \omega_k \wedge \omega_h,$$

where $P_{kh}, Q_{kh} \in \mathbf{R}$. Then

$$\Omega = \sum P_{kh} \tilde{\lambda}_k \wedge \tilde{\lambda}_h + \sum Q_{kh} \tilde{\omega}_k \wedge \tilde{\omega}_h$$

is a holomorphic symplectic structure on $\tilde{G}_i/\tilde{\Gamma}_i$ for $i = 1, 2, 3, 4$. In the case of $\varphi_{1,*}$, since

$$d\omega_{2k-1} = - \sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h},$$

we have

$$d\tilde{\omega}_{2k-1} = -\frac{1}{2} \sum_i a_i^k (\tilde{\alpha}_i + \bar{\alpha}_i) \wedge \tilde{\omega}_{2k-1} - \frac{1}{2} \sum_{k < h} \sum_{j=1}^n b_j^{kh} (\tilde{\beta}_j + \bar{\beta}_j) \wedge \tilde{\omega}_{2h-1},$$

$$d\tilde{\omega}_{2k} = \frac{1}{2} \sum_i a_i^k (\tilde{\alpha}_i + \bar{\alpha}_i) \wedge \tilde{\omega}_{2k} - \frac{1}{2} \sum_{k < h} \sum_{j=1}^n b_j^{kh} (\tilde{\beta}_j + \bar{\beta}_j) \wedge \tilde{\omega}_{2h}.$$

By considering $\tilde{\alpha}_i + \bar{\alpha}_i$ and $\tilde{\beta}_j + \bar{\beta}_j$ as single terms, we see that $\sum Q_{kh} \tilde{\omega}_k \wedge \bar{\omega}_h$ is closed. The other cases are similar and hence omitted. \square

PROPOSITION 9.3. *If G/Γ has a symplectic structure, then $\tilde{G}_i/\tilde{\Gamma}_i$ ($i = 1, 5$) has a pseudo-Kähler structure.*

PROOF. Consider the case of $\varphi_{5,*}$. By our assumption and Lemma 9.1, there exists the following symplectic structure on G :

$$\omega = \sum P_{kh} \lambda_k \wedge \lambda_h + \sum Q_{kh} \omega_{2k-1} \wedge \omega_{2h},$$

where $P_{kh}, Q_{kh} \in \mathbf{R}$. Since

$$d\omega_{2k-1} = - \sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k < h} \sum_{j=1}^n b_j^{kh} \beta_j \wedge \omega_{2h},$$

we have

$$d\tilde{\omega}_{2k-1} = - \sum_i a_i^k \tilde{\alpha}_i \wedge \tilde{\omega}_{2k-1} - \frac{1}{2} \sum_{k < h} \sum_{j=1}^n b_j^{kh} (\tilde{\beta}_j + \bar{\beta}_j) \wedge \tilde{\omega}_{2h-1},$$

$$d\bar{\omega}_{2k} = \sum_i a_i^k \tilde{\alpha}_i \wedge \bar{\omega}_{2k} - \frac{1}{2} \sum_{k < h} \sum_{j=1}^n b_j^{kh} (\tilde{\beta}_j + \bar{\beta}_j) \wedge \bar{\omega}_{2h}.$$

Similarly to the proof of Proposition 9.2, $\sum Q_{kh} \tilde{\omega}_{2k-1} \wedge \bar{\omega}_{2h}$ is a closed $(1, 1)$ -form. \square

By the same argument in the proof of Proposition 6.7 we see that $\tilde{G}_i/\tilde{\Gamma}_i$ ($i = 3, 4, 6$) has no left \tilde{G}_i -invariant pseudo-Kähler structures. Moreover, by a straightforward computation, we see that $\tilde{G}_i/\tilde{\Gamma}_i$ ($i = 5, 6$) has no left \tilde{G}_i -invariant holomorphic symplectic structures.

Table 9.1. Left \tilde{G}_i -invariant structure on $\tilde{G}_i/\tilde{\Gamma}_i$

type	holomorphic symplectic	pseudo-Kähler
1	yes	yes
2	yes	no
3	yes	no
4	yes	no
5	no	yes
6	no	no

10. An Application: A Simple Deformation of Holomorphic Symplectic Manifolds

In this section we consider a simple deformation of compact holomorphic symplectic solvmanifolds constructed in section 9. Consider the following solvable Lie group:

$$\tilde{G} = \tilde{G}_{3,1} \times T_{\mathbb{C}}^1 = \left\{ \left(z, P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \middle| z, w_1, w_2 \in \mathbb{C} \right\} \times T_{\mathbb{C}}^1,$$

where

$$\tilde{G}_{3,1} = \left\{ \left(\begin{pmatrix} e^{(1/2)(z+\bar{z})} & 0 & 0 & w_1 \\ 0 & e^{-(1/2)(z+\bar{z})} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbb{C} \right) \right\},$$

$$T_{\mathbb{C}}^1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{C} \right\} / \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \middle| \mu \in \mathbb{Z}[\sqrt{-1}] \right\}.$$

\tilde{G} has holomorphic symplectic structures, for example, $\Omega = \tilde{\alpha} \wedge \tilde{\beta} + \tilde{\omega}_1 \wedge \tilde{\omega}_2$. Put $B = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$. Let $\varpi : \tilde{G} \times B \rightarrow B$ be the natural projection. Consider the group of automorphisms of $\tilde{G} \times B$ defined as follows.

$$K_p = \left\{ \begin{array}{l} g_{khm_1n_1m_2n_2} : \left(z, P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right), x, \tau \\ \rightarrow \left(z + pk + h, PB^h \begin{pmatrix} w_1 + m_1\tau + n_1 \\ w_2 + m_2\tau + n_2 \end{pmatrix} \right), x, \tau \end{array} \right\}.$$

K_p acts properly discontinuously without fix points. Therefore $\mathcal{M}_p = \tilde{G} \times B / K_p$ is a complex manifold. Since the projection $\varpi : \tilde{G} \times B \rightarrow B$ commutes with $g_{khm_1n_1m_2n_2}$, it induces a holomorphic map ϖ of \mathcal{M} on B . By a straightforward computation, we see $\varpi^{-1}(\tau) = \tilde{G}_{3,1} / \tilde{\Gamma}(p, \tau) \times T_{\mathbb{C}}^1$. Thus $\tilde{G}_{3,1} / \tilde{\Gamma}(p, \tau) \times T_{\mathbb{C}}^1$ and $\tilde{G}_{3,1} / \tilde{\Gamma}(p, \tau')$ are diffeomorphic. Consider the natural projection $\pi_{\tilde{G}} : \tilde{G} \times B \rightarrow \tilde{G}$ and a left \tilde{G} -invariant holomorphic symplectic structure Ω . Since $\pi_{\tilde{G}}^* \Omega$ is K_p -invariant, i.e., $g_{khm_1n_1m_2n_2}^* \pi_{\tilde{G}}^* \Omega = \pi_{\tilde{G}}^* \Omega$, $\pi_{\tilde{G}}^* \Omega$ induces a form on $\mathcal{M}_p = \tilde{G} \times B / K_p$.

REMARK. $\bigcup_{p,\tau} \tilde{G}_{3,1} / \tilde{\Gamma}(p, \tau) \times T_{\mathbb{C}}^1$ is a differentiable family.

Let (G^{2m}, I) be a solvable Lie group with left G -invariant complex structure I and Ω a left G -invariant holomorphic structure on (G, I) . Let Γ_{τ} be a lattice of G

which has parameter τ . Consider compact holomorphic symplectic solvmanifolds $(M_\tau = G/\Gamma_\tau, I_\tau, \Omega_\tau)$, where I_τ, Ω_τ are complex structures and holomorphic structures induced from I and Ω respectively. We define a volume form $dVol_\tau$ by $dVol_\tau = \Omega_\tau \wedge \cdots \wedge \Omega_\tau \wedge \bar{\Omega}_\tau \wedge \cdots \wedge \bar{\Omega}_\tau$. Moreover we define $Vol_\tau(M_\tau) = \int_{M_\tau} dVol_\tau$. Note that $Vol_\tau(M_\tau)$ can be considered as the volume of fundamental region on G . Then we have the following:

LEMMA 10.1. *If there exists a diffeomorphism $f_{\tau\tau'} : M_\tau \rightarrow M_{\tau'}$ such that $f_{\tau\tau'}^* \Omega_{\tau'} = \Omega_\tau$, then*

$$Vol_\tau(M_\tau) = Vol_{\tau'}(M_{\tau'}) = Vol_\tau(M_{\tau'}).$$

PROOF. By our assumption, we have

$$\begin{aligned} Vol_\tau(M_\tau) &= \int_{M_\tau} dVol_\tau = \int_{M_\tau} f_{\tau\tau'}^* dVol_{\tau'} = \int_{M_{\tau'}} dVol_{\tau'} \\ &= \int_{M_{\tau'}} dVol_\tau = Vol_\tau(M_{\tau'}). \quad \square \end{aligned}$$

In the above case, since we have $\alpha \wedge \omega_1 \wedge \omega_2 = dt \wedge dy_1 \wedge dy_2$, we consider $Vol_\tau(M_\tau)$ as the volume of a fundamental region on \mathbf{R}^6 . Hence if $\text{Im } \tau \neq \text{Im } \tau'$, then there exists no diffeomorphisms $f_{\tau\tau'} : (\tilde{G}_{3,1}/\tilde{\Gamma}(p, \tau) \times T_{\mathbf{C}}^1, \Omega_\tau) \rightarrow (\tilde{G}_{3,1}/\tilde{\Gamma}(p, \tau') \times T_{\mathbf{C}}^1, \Omega_{\tau'})$ such that $f_{\tau\tau'}^* \Omega_{\tau'} = \Omega_\tau$.

By applying the above argument to the complexification of type i of a symplectic solvable Lie group in Proposition 1, we get families of compact holomorphic symplectic non-Kähler solvmanifolds.

REMARK. We can apply the above argument to the case of pseudo-Kähler structures.

References

- [1] de Andrés, L. C., Fernández, M., de León, M. and Mencía, J. J., Some six dimensional compact symplectic and complex solvmanifolds, *Rendiconti di Matematica, Serie VII*, Vol. 12 (1992), 59–67.
- [2] Benson, C. and Gordon, C., Kähler and Symplectic structures on nilmanifolds, *Topology* 27 (1988), 513–518.
- [3] ———, Kähler structures on solvmanifolds, *Proc. Amer. Math. Soc.* 108 (4) (1990), 971–980.
- [4] Besse, A. L., *Einstein manifolds*, Springer-Verlag, 1987.
- [5] Fernández, M., Ibáñez, R. and de León, M., On a Brylinski conjecture for compact symplectic manifolds, *Quaternionic structures in mathematics and physics*, SISSA, Trieste, 119–126, 1994.

- [6] Fernández, M., de León, M. and Saralegui, M., A six dimensional compact symplectic solvmanifold without Kähler structures, *Osaka J. Math.* **33** (1996), 19–35.
- [7] Griffiths, P. and Morgan, P., Rational homotopy theory and differential forms, Birkhäuser, Boston, Progress in Math. Vol. **16**, 1981.
- [8] Guan, D., Examples of compact holomorphic symplectic manifolds which admit no Kähler structure, *Geometry and analysis on complex manifolds*, World Sci. Publishing, River Edge, NJ, 63–74, 1994.
- [9] ———, Examples of compact holomorphic symplectic manifolds which are not Kählerian III, *Internat. J. Math.* **6** (1995), 709–718.
- [10] ———, Examples of compact holomorphic symplectic manifolds which are not Kählerian II, *Invent. Math.* **121** (1995), 135–145.
- [11] ———, A splitting theorem for compact complex homogeneous space with a symplectic structure, *Geom. Dedicata.* **63** (1996), 217–225.
- [12] Hasegawa, K., A class of compact Kählerian solvmanifolds and a general conjecture, *Geom. Dedicata.* **78** (1999), 253–258.
- [13] Hattori, A., Spectral sequence in the de Rham cohomology of fibre bundles, *J. Fac. Sci. Uni. Tokyo, Sect. 1*, **8** (1960), 289–331.
- [14] Ibáñez, R., Coeffective-Dolbeault cohomology of compact indefinite Kähler manifolds, *Osaka J. Math.* **34** (1997), 553–571.
- [15] McDuff, D. and Salamon, D., *Introduction to symplectic topology*, Oxford University Press Inc., New York, 1998.
- [16] Nomizu, K., On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59** (1954), 531–538.
- [17] Raghunathan, M. S., *Discrete subgroups of Lie groups*, Springer-Verlag, Berlin, 1972.
- [18] Sakane, Y., On compact complex parallelisable solvmanifolds, *Osaka J. Math.* **13** (1976), 187–212.
- [19] Tralle, A., A note solvable Lie groups without lattices and the Félix-Thomas Models of fibration, [arXiv:math.DG/0009105](https://arxiv.org/abs/math/0009105) 11 Sep 2000.
- [20] Tralle, A. and Oprea, J., *Symplectic Manifolds with no Kähler Structure*, Springer-Verlag, Berlin, 1997.
- [21] Yamada, T., Examples of compact Lefschetz solvmanifolds, *Tokyo J. Math.* **25** (2002), 261–283.

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