TSUKUBA J. MATH.
Vol. 29 No. 1 (2005), 65–77

SIMILARITIES BETWEEN THE TRIGONOMETRIC FUNCTION AND THE LEMNISCATE FUNCTION FROM ARITHEMETIC VIEW POINT

By

Takuma OGAWA

Abstract. We show that the rational functions associated with the lemniscate functions and primary numbers of Gaussian integers, defined by Eisenstein, has a symmetrical relation with respect to the zeros and the poles. We also consider some polynomials associated with the trigonometric functions and odd integers, and point out a similar symmetry of the polynomials.

1. Introduction

Throughout the present paper, $\mathrm{sl}(u)$ denotes the lemniscate sine. The lemniscate functions play important roles in various branches of number theory. Here we focus on the following three aspects:

(1) A proof of the law of biquadratic reciprocity (by F. G. M. Eisenstein [1], [2], [4], [5], [10]).

Let $B = \{r_\beta \in \mathbf{Z}[i] \mid 1 \le \beta \le \frac{Nr-1}{4}, \beta \in \mathbf{Z}\}$ be a quarter-system modulo a primary prime r in $\mathbf{Z}[i]$, where r_β are suitably chosen. Then we have

$$\left(\frac{s}{r}\right)_4 = \prod\nolimits_{r_\beta \in B} \frac{\mathrm{sl}\left(s\left(\frac{2\omega r_\beta}{r}\right)\right)}{\mathrm{sl}\left(\frac{2\omega r_\beta}{r}\right)}, \quad \omega = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}},$$

where r and s are coprime.

AMS Mathemactics Subject Classification 2000: Primary 11C99, 33B99, 33E05. Secondary: 01-02, 01A55.
Key words and phrase: Abelian equation, Abelian extension, complex multiplication, lemniscate function, reciprocity law.
Date: 2002.9.24.
Received May 26, 2003.
Revised August 31, 2004.

(2) Abelian equation over $\mathbf{Z}[i]$ (by Abel [6], [7], [8]).

For a primary m in $\mathbf{Z}[i]$, the numerator of the rational function $R_m(x)$ defined by the equality

$$R_m(\mathrm{sl}(u)) = \frac{\mathrm{sl}(mu)}{\mathrm{sl}(u)}$$

is an abelian equation over $\mathbf{Z}[i]$. The above rational function was used in (1).

(3) Abelian extension of $\mathbf{Q}(i)$ (by T. Takagi [9]).

Every abelian extension of $\mathbf{Q}(i)$ is obtained by adjoining $\mathrm{sl}(\alpha)$, where α is a root of the equation $\mathrm{sl}(mu) = 0$, m being a Gaussian integer.

The above relationship among the reciprocity law, the abelian equation and the abelian extension suggests that these three may be viewed as an appearance of the same object. This view is explicitly stated by M. Takase [11], [12]. Similar phenomena have been known for the trigonometric functions:

(1′) A proof of the law of quardratic reciprocity (by Eisenstein [1], [3], [10]).

Let $A = \{a \in \mathbf{Z} \mid 1 \le a \le (p-1)/2\}$ be a half-system modulo an odd prime p in $\mathbf{Z}$, then

$$\left(\frac{q}{p}\right)_2 = \prod\nolimits_{a \in A} \frac{\sin q\left(\frac{2\pi a}{p}\right)}{\sin \frac{2\pi a}{p}}, \quad \pi = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}},$$

where p and q are coprime.

(2′) Abelian equation over $\mathbf{Z}$ (by Abel [7]).

For an odd q in $\mathbf{Z}$, the polynomial $P_q(x)$ defined by the equality

$$P_q(\sin z) = \frac{\sin qz}{\sin z}$$

is an abelian equation over $\mathbf{Z}$. Eisenstein also used the above polynomial in (1′).

(3′) Abelian extension of $\mathbf{Q}$ (Kronecker-Weber Theorem).

Every abelian extension of $\mathbf{Q}$ is contained in a field generated by $\exp(\frac{2\pi i}{n})$, n being an integer.

In the present paper, we examine the similarity between the lemniscate function and the trigonometric function in more detail, with the above phenomena being regarded as a guiding principle.

First we recall the following well-known fact. A Gaussian integer $m = a + ib \in \mathbf{Z}[i]$ $(a, b \in \mathbf{Z})$ is said to be *primary*, if $m \equiv 1 \pmod{(1+i)^3}$. If $a + ib$ is a primary integer, then $a^2 + b^2 \equiv 1 \pmod 4$ (see Lemma 3.1).

PROPOSITION 1.1 ([10, Proposition 8.2], cf. [1], [2]). *Let $m = a + ib$ be a primary Gaussian integer and let $N := a^2 + b^2$. Then there exist polynomials $V_m(X), W_m(X) \in \mathbf{Z}[i][X]$ of degree $(N-1)/4$ such that*

$$\frac{\mathrm{sl}(mu)}{\mathrm{sl}(u)} = \frac{W_m(x^4)}{V_m(x^4)}, \quad \textit{where } x = \mathrm{sl}(u).$$

Moreover, these polynomials satisfy the following two equalities:

$$W_m(x^4) = x^{N-1} V_m(x^{-4}), \quad \textit{and}$$

$$W_m(0) = \textit{the coefficient of the top degree of } V_m(x) = m.$$

The last equality of the above proposition shows that the rational function defined by

$$\frac{W_m(x^4)}{V_m(x^4)}$$

is symmetric with respect to the zeros and poles. This leads us to study the function $\mathrm{cl}(mu)/\mathrm{cl}(u)$ ($\mathrm{cl}(u)$ denotes the lemniscate cosine). It has been shown to be a rational function of $\mathrm{sl}^2(u)$ with coefficients in $\mathbf{Q}(i)$ [15, Corollary 3.2.3]. We study the symmetry of the rational function, similar to the one of the function $W_m(\mathrm{sl}^4(u))/V_m(\mathrm{sl}^4(u))$.

2. Notation and Statement of the Result

Throughout the present paper, $\mathbf{P}$ denotes the set of primary numbers and let

$$M := \{m = a + ib \in \mathbf{Z}[i] \mid a, b \in \mathbf{Z} \ a - b \equiv 1 \pmod 2\}.$$

It is easy to see that $M = \mathbf{P} + i\mathbf{P} + i^2\mathbf{P} + i^3\mathbf{P}$. Eisenstein [5, Theorem 3.3] proved that

$$\frac{\mathrm{sl}(mu)}{\mathrm{sl}(u)} \in \mathbf{Q}(i)(\mathrm{sl}^4(u)), \quad \text{for each } m \in M.$$

DEFINITION 2.1. For $m \in M$, let $R_{s,m}(x)$ and $R_{c,m}(x)$ be functions satisfying the following equalities:

$$R_{s,m}(\mathrm{sl}(u)) := \frac{\mathrm{sl}(mu)}{\mathrm{sl}(u)}, \quad R_{c,m}(\mathrm{sl}(u)) := \frac{\mathrm{cl}(mu)}{\mathrm{cl}(u)}.$$

It follows from [5, Theorem 3.3] and [15, Corollary 3.2.3] that both of $R_{s,m}(x)$ and $R_{c,m}(x)$ are rational functions. Note that for a primary m, $R_{s,m}(x) = W_m(x^4)/V_m(x^4)$, where $V_m(x)$, $W_m(x)$ are the polynomials in Proposition 1.1.

As the following result shows, the rational function $R_{c,m}(x)$ has a strong symmetry with respect to zeros and poles.

PROPOSITION 2.2. *For each $m \in M$, there exists a polynomial $K_m(X) \in \mathbf{Z}[i][X]$ such that*

$$R_{c,m}(x) = \frac{K_m(x^2)}{K_m(-x^2)}.$$

Also $R_{c,m}(x)$ and $R_{s,m}(x)$ satisfy the following equality.

THEOREM 2.3. *Let $m \in \mathbf{P}$ be a primary number. Then*

$$R_{s,m}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = R_{c,m}(x).$$

From these results, we can derive a relationship on $V_m(X)$, $W_m(X)$ and $K_m(X)$ for a primary m.

COROLLARY 2.4. *Let $m = a + ib \in \mathbf{P}$ $(a, b \in \mathbf{Z})$ be a primary number. We have the following statements*:

(1) $W_m(x^4) = x^{N-1}V_m(x^{-4})$, *where* $N = a^2 + b^2$ (*Proposition* 1.1).

(2) *Let* $y^2 = (1-x^2)/(1+x^2)$. *Then, we have*

$$\begin{cases} W_m(y^4)K_m(-x^2) = V_m(y^4)K_m(x^2), \\ W_m(x^4)K_m(-y^2) = V_m(x^4)K_m(y^2). \end{cases}$$

(3) $K_m(X)$ *can be chosen as a reciprocal polynomial.*

EXAMPLE 2.5 (see [15]). For $m = 3 - 2i \in \mathbf{P}$, the functions $R_{c,m}(x)$ and $R_{s,m}(x)$ are written as follows:

$$R_{s,m}(x) = \frac{(3-2i) + (7+4i)x^4 + (-11-10i)x^8 + x^{12}}{1 + (-11-10i)x^4 + (7+4i)x^8 + (3-2i)x^{12}},$$

$$R_{c,m}(x)$$

$$= \frac{1 - (2-6i)x^2 + (3+8i)x^4 + (12-4i)x^6 + (3+8i)x^8 - (2-6i)x^{10} + x^{12}}{1 + (2-6i)x^2 + (3+8i)x^4 - (12-4i)x^6 + (3+8i)x^8 + (2-6i)x^{10} + x^{12}}.$$

For these polynomials, $V_m(X)$, $W_m(X)$ and $K_m(X)$ in Proposition 1.1 and Proposition 2.2 have the following form:

$$V_m(X) = 1 + (-11 - 10i)X + (7 + 4i)X^2 + (3 - 2i)X^3,$$

$$W_m(X) = (3 - 2i) + (7 + 4i)X + (-11 - 10i)X^2 + X^3,$$

$$K_m(X) = 1 - (2 - 6i)X + (3 + 8i)X^2 + (12 - 4i)X^3 + (3 + 8i)X^4 - (2 - 6i)X^5 + X^6.$$

3. Preliminaries

3.1. Lemniscate functions (see [13]). The lemniscate sine $\mathrm{sl}(u)$ is the function defined by the meromorphic continuation to the whole complex plane of the inverse function of the following equation

$$u = \int_0^x \frac{dx}{\sqrt{1 - x^4}} \quad (\Leftrightarrow x := \mathrm{sl}(u)).$$

We define the constant ω by

$$\frac{\omega}{2} := \int_0^1 \frac{dx}{\sqrt{1 - x^4}},$$

and define the lemniscate cosine $\mathrm{cl}(u)$ as follows:

$$\mathrm{cl}(u) := \mathrm{sl}\left(\frac{\omega}{2} - u\right).$$

The functions $\mathrm{sl}(u)$ and $\mathrm{cl}(u)$ are elliptic functions. The fundamental periods of $\mathrm{sl}(u)$ and $\mathrm{cl}(u)$ are 2ω and $2\omega i$ respectively.

The addition formulas of $\mathrm{sl}(u)$ and $\mathrm{cl}(u)$ take the following form:

$$\mathrm{sl}(u + v) = \frac{\mathrm{sl}(u)\,\mathrm{cl}(v) + \mathrm{cl}(u)\,\mathrm{sl}(v)}{1 - \mathrm{sl}(u)\,\mathrm{sl}(v)\,\mathrm{cl}(u)\,\mathrm{cl}(v)}, \quad \mathrm{cl}(u + v) = \frac{\mathrm{cl}(u)\,\mathrm{cl}(v) - \mathrm{sl}(u)\,\mathrm{sl}(v)}{1 + \mathrm{sl}(u)\,\mathrm{sl}(v)\,\mathrm{cl}(u)\,\mathrm{cl}(v)}.$$

Also the functions $\mathrm{sl}(u)$ and $\mathrm{cl}(u)$ satisfy the following equalities:

$$\mathrm{cl}^2(u) = \frac{1 - \mathrm{sl}^2(u)}{1 + \mathrm{sl}^2(u)}, \quad \mathrm{sl}(iu) = i\,\mathrm{sl}(u), \quad \mathrm{cl}(iu) = \frac{1}{\mathrm{cl}(u)}.$$

In this paper, the addition formulas of the lemniscate functions play an important role.

3.2. Primary numbers of $\mathbf{Z}[i]$ (see [14]). Recall that a non-unit Gaussian integer $\alpha = a + ib$ $(a, b \in \mathbf{Z})$ is said to be primary if $\alpha \equiv 1 \pmod{(1+i)^3}$. In terms of a and b, the above congruence is replaced with the following form.

LEMMA 3.1. *$\alpha = a + ib \in \mathbf{Z}[i] \backslash \{1\}$ $(a, b \in \mathbf{Z})$ is primary if and only if*

$$a \equiv 1, b \equiv 0 \pmod 4, \quad or \quad a \equiv 3,\ b \equiv 2 \pmod 4.$$

We denote by $\mathbf{P}$ the set of primary numbers.

3.3. Preparation for the proof of the results. The following result corresponds to Theorem 3.3 of [5], in which it is shown that $\mathrm{sl}(mu)/\mathrm{sl}(u) \in \mathbf{C}(\mathrm{sl}^4(u))$. It is used in the proof of Proposition 2.2 and Corollary 2.4 in the next section.

LEMMA 3.2 ([15, Corollary 3.2.3]). *If $m = a + ib$ $(a, b \in \mathbf{Z}; a - b \equiv 1 \pmod 2)$, then*

$$\frac{\mathrm{cl}(mu)}{\mathrm{cl}(u)} \in \mathbf{Q}(i)(\mathrm{sl}^2(u)),$$

where $\mathbf{Q}(i)(\mathrm{sl}^2(u))$ is the field of rational functions of $\mathrm{sl}^2(u)$ over $\mathbf{Q}(i)$.

For a proof of Theorem 2.3, we prepare the following.

LEMMA 3.3. *For $p \in \mathbf{P}$, the functions $\mathrm{sl}(u)$ and $\mathrm{cl}(u)$ satisfy the following equalities:*

$$\frac{\mathrm{sl}(pu)}{\mathrm{sl}(u)} = \frac{\mathrm{cl}\left(p\left(\frac{\omega}{2} - u\right)\right)}{\mathrm{cl}\left(\frac{\omega}{2} - u\right)}, \quad \frac{\mathrm{cl}(pu)}{\mathrm{cl}(u)} = \frac{\mathrm{sl}\left(p\left(\frac{\omega}{2} - u\right)\right)}{\mathrm{sl}\left(\frac{\omega}{2} - u\right)}.$$

PROOF. These equalities immediately follow from addition formulas and Lemma 3.1. □

4. Proofs of the Results

We prove the main results by using Lemma 3.2 and Lemma 3.3.

PROOF OF PROPOSITION 2.2. By Lemma 3.2, there exist polynomials $F_m(X)$, $G_m(X) \in \mathbf{Z}[i][X]$ such that

$$R_{c,m}(x) = \frac{F_m(x^2)}{G_m(x^2)}, \quad \text{where } R_{c,m}(\mathrm{sl}(u)) = \frac{\mathrm{cl}(mu)}{\mathrm{cl}(u)}.$$

Let $x = \mathrm{sl}(u)$. By the definition of $R_{c,m}(x)$, we have

$$R_{c,m}(ix) = R_{c,m}(i\,\mathrm{sl}(u)) = R_{c,m}(\mathrm{sl}(iu)) = \frac{\mathrm{cl}(imu)}{\mathrm{cl}(iu)} = \frac{\mathrm{cl}(u)}{\mathrm{cl}(mu)} = \frac{1}{R_{c,m}(x)}.$$

From the above, we obtain the following equality

$$\frac{F_m(-x^2)}{G_m(-x^2)} = \frac{G_m(x^2)}{F_m(x^2)},$$

which implies

$$F_m(-x^2)F_m(x^2) = G_m(x^2)G_m(-x^2).$$

Hence, we have

$$F_m(x^2)(F_m(-x^2) + G_m(x^2)) = G_m(x^2)(F_m(x^2) + G_m(-x^2)).$$

From Lemma 3.2 and the definition of $G_m(x^2)$, we have $G_m(x^2) \not\equiv 0$. And it is easy to see $F_m(-x^2) + G_m(x^2) \not\equiv 0$. Hence, we obtain

$$R_{c,m}(x) = \frac{F_m(x^2)}{G_m(x^2)} = \frac{F_m(x^2) + G_m(-x^2)}{F_m(-x^2) + G_m(x^2)} = \frac{K_m(x^2)}{K_m(-x^2)},$$

where $K_m(X) := F_m(X) + G_m(-X)$. It is easy to see $K_m(X) \in \mathbf{Z}[i][X]$. □

Proof of Theorem 2.3. For $x = \mathrm{sl}(u)$, we have

$$y = \mathrm{cl}(u) = \sqrt{\frac{1-x^2}{1+x^2}}.$$

Thus, for $m \in \mathbf{P}$, we obtain the following equality by using Definition 2.1 and Lemma 3.3:

$$R_{s,m}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = R_{s,m}(y) = R_{s,m}(\mathrm{cl}(u)) = R_{s,m}\left(\mathrm{sl}\left(\frac{\omega}{2} - u\right)\right)$$

$$= \frac{\mathrm{sl}\left(m\left(\frac{\omega}{2} - u\right)\right)}{\mathrm{sl}\left(\frac{\omega}{2} - u\right)} = \frac{\mathrm{cl}(mu)}{\mathrm{cl}(u)} = R_{c,m}(x).$$

Similarly, we have

$$R_{c,m}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)=R_{c,m}(y)=R_{c,m}(\mathrm{cl}(u))=R_{c,m}\left(\mathrm{sl}\left(\frac{\omega}{2}-u\right)\right)$$

$$=\frac{\mathrm{cl}\left(m\left(\frac{\omega}{2}-u\right)\right)}{\mathrm{cl}\left(\frac{\omega}{2}-u\right)}=\frac{\mathrm{sl}(mu)}{\mathrm{sl}(u)}=R_{s,m}(x). \qquad \square$$

PROOF OF COROLLARY 2.4. The equality (1) has been stated in Proposition 1.1. The equality (2) follows immediately from Proposition 1.1, Definition 2.1, Proposition 2.2, and Theorem 2.3. It remains to show (3).

From Proposition 2.2, for $m \in \mathbf{P}$, we have

$$R_{c,m}(x)=\frac{K_m(x^2)}{K_m(-x^2)} \qquad (*)$$

and by Theorem 2.3,

$$R_{c,m}(x)=R_{s,m}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)=\frac{W_m\left(\left(\frac{1-x^2}{1+x^2}\right)^2\right)}{V_m\left(\left(\frac{1-x^2}{1+x^2}\right)^2\right)}.$$

Let $W_m(X)=a_0+a_1X+a_2X^2+\cdots+a_{(N-1)/4}X^{(N-1)/4}$. Then by Proposition 1.1, we obtain

$$R_{c,m}(x)$$

$$=\frac{a_0+a_1\left(\frac{1-x^2}{1+x^2}\right)^2+a_2\left(\frac{1-x^2}{1+x^2}\right)^4+\cdots+a_{(N-1)/4}\left(\frac{1-x^2}{1+x^2}\right)^{(N-1)/2}}{a_{(N-1)/4}+a_{(N-5)/4}\left(\frac{1-x^2}{1+x^2}\right)^2+a_{(N-9)/4}\left(\frac{1-x^2}{1+x^2}\right)^4+\cdots+a_0\left(\frac{1-x^2}{1+x^2}\right)^{(N-1)/2}}$$

$$=\frac{a_0(1+x^2)^{(N-1)/2}+a_1(1-x^2)^2(1+x^2)^{(N-5)/2}+\cdots+a_{(N-1)/4}(1-x^2)^{(N-1)/2}}{a_0(1-x^2)^{(N-1)/2}+a_1(1+x^2)^2(1-x^2)^{(N-5)/2}+\cdots+a_{(N-1)/4}(1+x^2)^{(N-1)/2}}.$$

Comparing the above equality with $(*)$, we see that $K_m(X)$ may be chosen as a reciprocal polynomial. $\square$

5. The Trigonometric Function Case

5.1. Existence for some polynomials. The perspective of M. Takase mentioned in Section 1 ([11], [12]) and the close connection between lemniscate sine and lemniscate cosine functions proved in Section 4 naturally lead to the simultaneous study of the polynomials given by $P_{s,q}(\sin z) = \sin qz/\sin z$, and $P_{c,q}(\sin z) = \cos qz/\cos z$. It is easy to see that $P_{s,q}$ and $P_{c,q}$ are indeed polynomials by using the following lemma.

LEMMA 5.1. *For an odd integer $q \geq 1$, we define $f_q(z)$ and $g_q(z)$ as follows*:

$$f_q = f_q(z) := \frac{\sin qz}{\sin z}, \quad g_q = g_q(z) := \frac{\cos qz}{\cos z}.$$

Then, for each odd integer $q \geq 5$, f_q and g_q satisfy the following recurrence formulas:

$$f_q = f_3 f_{q-2} - f_{q-2} - f_{q-4}, \quad g_q = g_3 g_{q-2} + g_{q-2} - g_{q-4}.$$

COROLLARY 5.2. *If q is an odd integer, then we have*

$$\frac{\sin qz}{\sin z}, \frac{\cos qz}{\cos z} \in \mathbf{Z}[\sin^2 z],$$

where $\mathbf{Z}[\sin^2 z]$ is the ring of the polynomial of $\sin^2 z$ over $\mathbf{Z}$.

PROOF. By using an induction, we prove Lemma 5.1 and Corollary 5.2 simultaneously. Direct calculations for f_1, f_3, and f_5 give the following equalities:

$$f_1 = 1, \quad f_3 = 3 - 4\sin^2 z, \quad f_5 = 5 - 20\sin^2 z + 16\sin^4 z.$$

Eliminating $\sin^2 z$ from these equalities, we obtain

$$f_5 = f_3 f_3 - f_3 - f_1.$$

For an inductive argument, we assume

$$f_k = f_3 f_{k-2} - f_{k-2} - f_{k-4}.$$

Now we compute f_{k+2}. By using the addition formulas of the trigonometric functions, we have

$$\begin{aligned}
f_{k+2} &= \frac{\sin(k+2)z}{\sin z} = \frac{\sin kz \cos 2z + \cos kz \sin 2z}{\sin z} \\
&= \frac{\sin kz}{\sin z}(1 - 2\sin^2 z) + \frac{\cos kz \sin 2z}{\sin z} \\
&= \frac{\sin kz}{\sin z}((3 - 4\sin^2 z) - 2 + 2\sin^2 z) + \frac{\cos kz \sin 2z}{\sin z} \\
&= \frac{\sin kz}{\sin z}(3 - 4\sin^2 z) - 2\frac{\sin kz}{\sin z} + \frac{2\sin kz \sin^2 z}{\sin z} + \frac{\cos kz \sin 2z}{\sin z} \\
&= \frac{\sin kz}{\sin z}(3 - 4\sin^2 z) - \frac{\sin kz}{\sin z} - \left(\frac{\sin kz(1 - 2\sin^2 z)}{\sin z} - \frac{\cos kz \sin 2z}{\sin z}\right) \\
&= \frac{\sin kz}{\sin z}(3 - 4\sin^2 z) - \frac{\sin kz}{\sin z} - \frac{\sin(k-2)z}{\sin z} \\
&= f_3 f_k - f_k - f_{k-2}.
\end{aligned}$$

This completes the induction and thus we obtain the first recurrence formula.

The functions g_1 and g_3 are easily computed as follows:

$$g_1 = 1, \quad g_3 = 1 - 4\sin^2 z.$$

The recurrence formula $g_q = g_3 g_{q-2} + g_{q-2} - g_{q-4}$ is proved by the same inductive argument starting with the above. This completes the proof of Lemma 5.1.

By induction and Lemma 5.1, we obtain Corollary 5.2. □

5.2. A similar symmetry of the polynomials.

DEFINITION 5.3. For an odd integer q, let $P_{s,q}(x)$ and $P_{c,q}(x)$ be functions satisfying the following equalities:

$$P_{s,q}(\sin z) := \frac{\sin qz}{\sin z}, \quad P_{c,q}(\sin z) := \frac{\cos qz}{\cos z}.$$

It follows from Corollary 5.2 that both of $P_{s,q}(x)$ and $P_{c,q}(x)$ are polynomials. Applying the same method in the proof of Theorem 2.3 in Section 4, we can get the following symmetrical equality. Comparing with Theorem 2.3, we can again recognize a similarity between the trigonometric function and the lemniscate function.

THEOREM 5.4. *For any odd integer q, we have*

$$P_{s,q}(\sqrt{1-x^2}) = (-1)^{(q-1)/2} P_{c,q}(x).$$

EXAMPLE 5.5. For $q = 13$, the polynomials $P_{s,q}(x)$ and $P_{c,q}(x)$ have the following form:

$$P_{s,q}(x) = 13 - 364x^2 + 2912x^4 - 9984x^6 + 16640x^8 - 13312x^{10} + 4096x^{12},$$

$$P_{c,q}(x) = 1 - 84x^2 + 1120x^4 - 5376x^6 + 11520x^8 - 11264x^{10} + 4096x^{12}.$$

5.3. Remark. A simultaneous study of the polynomials $P_{s,q}(x)$ and $P_{c,q}(x)$ led us to obtain Theorem 5.4. Here we demonstrate another example of the advantage of this approach. From Corollary 5.2 and Definition 5.3, we easily see that $P_{s,q}(x)$ and $P_{c,q}(x)$ are polynomials of $\mathbf{Z}[x^2]$. So, we study the polynomials in view of the mentioned above again.

DEFINITION 5.6. For an odd integer q, let $F_q(w)$ and $G_q(w)$ be functions satisfying the following equalities:

$$F_q(w) := \frac{\sin qz}{\sin z}, \quad G_q(w) := \frac{\cos qz}{\cos z}, \quad \text{where } w := \sin^2 z.$$

Note that $f_q(z) = F_q(\sin^2 z)$ and $g_q(z) = G_q(\sin^2 z)$.

THEOREM 5.7. *For an odd integer q, the polynomials $F_q(w)$ and $G_q(w)$ have the following properties*:

(1) $\deg F_q = \deg G_q = (q-1)/2$, *and the coefficients of F_q and G_q of the highest degree are equal to* $(-4)^{(q-1)/2}$.

(2) $F_q(1-w) = (-1)^{(q-1)/2}G_q(w)$.

(3) *Let q be an odd prime and* $F_q(w) = A_0 + A_1 w + A_2 w^2 + \cdots + A_{(q-3)/2}w^{(q-3)/2} + (-4)^{(q-1)/2}w^{(q-1)/2}$, $A_0, A_1, \ldots, A_{(q-3)/2} \in \mathbf{Z}$. *Then,*

$$A_0 = q, \quad \frac{A_1}{q}, \frac{A_2}{q}, \ldots, \frac{A_{(q-3)/2}}{q} \in \mathbf{Z}.$$

PROOF. It is easy to obtain (1) by using the recurrence formulas of Lemma 5.1. Moreover (2) is a rewriting of Theorem 5.4. We prove (3) below.

From Definition 5.6, we have

$$\sin qz = F_q(w)\sin z, \quad \text{where } w = \sin^2 z.$$

Differentiating the equality above, we obtain the following:

$$q\cos qz = \cos zF_q(w) + 2(\cos z)wF_q'(w).$$

Dividing the above by $\cos z$, it follows from Definition 5.6 that

$$G_q(w) = \frac{1}{q}(F_q(w) + 2wF_q'(w))$$

$$= \frac{1}{q}(A_0 + 3A_1 w + 5A_2 w^2 + 7A_3 w^3$$

$$+ \cdots + (q-2)A_{(q-3)/2} w^{(q-3)/2} + (-4)^{(q-1)/2} q w^{(q-1)/2}).$$

Since $G_q(w)$ is a polynomial in $\mathbf{Z}[w]$ and q is an odd prime in $\mathbf{Z}$, so we get the following:

$$\frac{A_0}{q}, \frac{A_1}{q}, \frac{A_2}{q}, \ldots, \frac{A_{(q-3)/2}}{q} \in \mathbf{Z}.$$

By using the recurrence formula for f_q of Lemma 5.1, we get $A_0 = q$. □

Acknowledgments

The author would like to thank Professor Kimio Watanabe, University of Tsukuba, for introducing him to this subject. Also author would like to thank the members of Watanabe seminar for many good advices and friendship, especially to Naohiro Kanesaka.

References

[1] G. Eisenstein, Application de l'algébre' à l'arithmétique transcendante, J. fur reine u. angew. Math. **29** (1845), 177–184.

[2] G. Eisenstein, Beiträge zur Theorie der elliptischen Functionen, I: Ableitung des biquadratischen Fundamentaltheorems aus der Theorie der Lemniscatenfunctionen, nebst Bemerkungen zu den Multiplications- und Transformationsformeln, J. fur reine u. angew. Math. **30** (1846), 185–210.

[3] M. Yukitaka, Entrance examination and the law of quardratic reciprocity, Mathematical communication for mathematician, No. **30** (1998), 20–23, in Japanese.

[4] T. Kubota, Some arithmetical applications of an elliptic function, J. Reine Angew. Math. **214/215** (1964), 141–145.

[5] K. Watanabe, Y. Miyagawa and T. Higuchi, A remark on the analytic proof of the law of biquadratic reciprocity, Journal of the Yokohama National University, Sec. **1**, No. 43, (1996).

[6] N. H. Abel, Recherches sur les functions elliptiques, J. fur reine u. angew. Math. Bd. **2** (1827), 101–181. Bd. **3** (1828), 160–190.

[7] N. H. Abel, Mémoire sur une classe particulière d'equations résolubles algébriquement, J. fur reine u. angew. Math. Bd. **4** (1829), 131–156.

[8] L. Kronecker, Über die algebraisch auflösbaren Gleichungen, Monatsber. kgl. Preuss. Acad. Wiss. Berlin (1853), 365–374; Werke IV, 1–11.

[9] T. Takagi, Über die im Bereiche der rationalen komplexen Zahlen Abelscher Zahlkörper, J. Coll. Sci. Tokyo **19** (1903), 1–42; Collected Papers, 13–39.

[10] F. Lemmermeyer, Reciprocity Laws; From Euler to Eisenstein, Chapter 8, Springer, (2000).
[11] M. Takase, Three aspects of the theory of complex multiplication, The intersection of history and mathematics, Sci. Network Hist. Stud. **15** (1994), 91–108.
[12] M. Takase, Gauss's Disquisitiones arithmeticae and Hilbert's 12th problem, Sugaku, Mathematical Society of Japan **54** (2002.10), 415–426, in japanese.
[13] Y. Kawada, Gauss and elliptic functions, Sophia Kokyuroku in Mathematics, No. **24** (1986), in Japanese.
[14] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, New York: Springer-Verlag, (1990).
[15] T. Ogawa, The recurrence formulas and the symmetrical relation of the rational functions defined by the lemniscate functions and primary numbers of Gaussian integers, preprint.

(Takuma Ogawa) Institute of Mathematics
University of Tsukuba
Tsukuba-shi Ibaraki
305-8571, Japan
E-mail address: tukutaku@math.tsukuba.ac.jp