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BIFURCATION OF THE KOLMOGOROV FLOW WITH AN EXTERNAL FRICTION

By

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Abstract. We consider Kolmogorov's problem of stationary flows in a thin layer with a bottom friction. Using the Lyapunov-Schmidt theory, we construct bifurcating solutions of this problem in the case where the linearized equations have simple eigenvalues. Compared with the previous paper [7], where we did not take account of the bottom friction, we find some interesting properties of bifurcation curves.

1. Introduction

The Kolmogorov flow, a plane periodic flow of an incompressible fluid under the action of a spatially periodic external force, has been conceived of only as a convenient object for theoretical investigations since it has been proposed in 1959. But later in 1979, a laboratory experiment by means of electrolyte through magnetic forcing was carried out in order to mimic the Kolmogorov flow (see its outline in [9, 2]). The results of their experiments were found in some aspects to be in good qualitative agreement with the previous theories described in [8, 4], but in other aspects, there were some serious disagreement caused by the friction on the bottom of the layer. Bondarenko and his group asserted that we should take account of the influence of the friction in order to investigate a motion in a thin layer. Accordingly, they built a modified model of the Kolmogorov flow to allow for the bottom friction.

The corresponding equations in stationary case take the form:

$$(1.1)\qquad \begin{cases} uu_x + vu_y = -P_x + \nu\Delta u - \kappa u + \gamma \sin y, \\ uv_x + vv_y = -P_y + \nu\Delta v - \kappa v, \\ u_x + v_y = 0. \end{cases}$$

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where (u, v), P, ν, γ mean velocity vector, pressure, the kinematic viscosity and the amplitude of the external force, respectively. And κ means the coefficient of the bottom friction which can be defined by $\kappa \equiv 2\nu/h^2$ with h, the depth of the fluid layer. As in the case of the original Kolmogorov problem, let the system of solutions $V(x, y) = {}^t(u(x, y), v(x, y))$ and $P(x, y)$ satisfy

$$(1.2) \qquad \begin{cases} V(x, y) = V(x + 2\pi/\alpha, y) = V(x, y + 2\pi), \\ P(x, y) = P(x + 2\pi/\alpha, y) = P(x, y + 2\pi), \\ \iint_D V(x, y)\, dxdy = 0, \quad \iint_D P(x, y)\, dxdy = 0, \end{cases}$$

where $D = \{(x, y) : |x| \leq \pi/\alpha, |y| \leq \pi\}$.

Introducing the stream function $\psi(x, y)$, we reproduce the velocity as $(u, v) = (\psi_y, -\psi_x)$. As the pressure is known to be determined by the velocity, we eliminate P and replace ψ with $\gamma\nu^{-1}\psi$. Then we reduce the problem (1.1–2) to:

$$(1.3) \qquad \lambda J(\Delta\psi, \psi) = \Delta^2\psi - \zeta\Delta\psi + \cos y, \quad J(f, g) \equiv f_x g_y - f_y g_x,$$

$$(1.4) \qquad \begin{cases} \psi(x, y) = \psi(x + 2\pi/\alpha, y) = \psi(x, y + 2\pi), \\ \iint_D \psi(x, y)\, dxdy = 0, \end{cases}$$

where $\lambda \equiv \gamma/\nu^2$ and $\zeta \equiv \kappa/\nu = 2/h^2$.

We first note that $\psi_0(x, y) \equiv -(1 + \zeta)^{-1} \cos y$ satisfies (1.3–4) for any $\lambda > 0$ and $\zeta \geq 0$. We call ψ_0 a basic solution. The velocity field of the basic solution is given by $(u_0, v_0) = (\gamma\nu^{-1}(1 + \zeta)^{-1} \sin y, 0)$, which represents a shear flow parallel to the x-axis.

Defining φ by $\varphi \equiv \psi - \psi_0$, we write (1.3) as follows:

$$(1.5) \qquad f(\lambda, \varphi) \equiv \{\Delta^2 - \zeta\Delta - \lambda(1 + \zeta)^{-1} \sin y(\Delta + I)\partial_x\}\varphi - \lambda J(\Delta\varphi, \varphi) = 0,$$

where I is the identity operator. $\varphi \equiv 0$ corresponds to the basic solution for all λ and ζ. Dropping the non-linear term $\lambda J(\Delta\varphi, \varphi)$ in (1.5), we obtain a linearized equation $f_\varphi(\lambda, 0)\varphi = 0$.

It is well known in general that, if $\lambda = \lambda_0$ is a bifurcation point, the linearized equation has at least one non-trivial solution. As we consider later, if $\alpha \geq 1$, $f_\varphi(\lambda, 0)\varphi = 0$ has only trivial solution for any λ and ζ. Namely, there is no bifurcation point in $\alpha \geq 1$.

When we fix $\alpha \in (0, 1)$ and $\zeta \in [0, \infty)$ and let $r \in N$ satisfy $r\alpha < 1 \leq (r + 1)\alpha$, there exists $\lambda = \lambda_k$ for each $k \in \{\pm 1, \ldots, \pm r\}$ such that $f_\varphi(\lambda_k, 0) = 0$ has non-trivial solutions. We state the properties of λ_k as follows:

REMARK. Let $K_\alpha \equiv \{1, \ldots, r\}$. Then, each λ_k $(k \in K_\alpha)$ satisfies the following properties:

(i) $\lambda_{-k} = \lambda_k$,
(ii) For $r \geq 2$, it is possible for some $k, k' \in K_\alpha$ $(k \neq k')$ to satisfy $\lambda_k = \lambda_{k'}$.

Taking account of this remark, we later assume that φ has the symmetry as follows:

$$\varphi(-x, -y) = \varphi(x, y).$$

Remark (ii) is one of the important properties caused of the bottom friction which leads $\dim \ker f_\varphi(\lambda_k, 0) = 2$ for some k. In this paper, we treat only the case where $\dim \ker f_\varphi(\lambda_k, 0) = 1$, although we will consider the another case in our forthcoming paper.

Here is our main result:

THEOREM 1.1. *We fix* $\alpha \in (0, 1)$ *and* $\zeta \in [0, \infty)$. *Let* $r \in N$ *satisfy* $r\alpha < 1 \leq (r+1)\alpha$. *Then, for each* $k \in K_\alpha \equiv \{1, \dots, r\}$, *there exists a bifurcation point* $\lambda = \lambda_k$ *with* $\dim \ker f_\varphi(\lambda_k, 0) = 1$, *and we can construct one parameter family of non-trivial solutions of* (1.5) *in a neighborhood of* $(\lambda_k, 0)$:

$$(\lambda, \varphi) = (\mu(s), \varphi(s)), \quad |s| < 1,$$

where $\mu(0) = \lambda_k$, $\varphi(0) = 0$ *and* $\mu_s(0) = 0$. *Moreover,* $\mu_{ss}(0) > 0$ *is obtained for each* $\zeta \geq 0$ *when* $k\alpha$ *is close to one.*

In Section 2, we solve the linearized equation and obtain a function $\lambda = \lambda(\beta, \zeta)$ defined on $\beta \in (0, 1)$ and $\zeta \in [0, \infty)$. In Section 3, using the Lyapunov-Shmidt theory, we prove that bifurcation points of (1.5) are given by $\lambda_k \equiv \lambda(k\alpha, \zeta)$. In Section 4, we find some properties of the curve $(\lambda, \varphi) = (\mu(s), \varphi(s))$.

Hereafter, we consider $\varphi(s)$ in the space $X \equiv H_0^4 = \{\varphi \in H^4; \iint_D \varphi \, dxdy = 0\}$ with $\varphi(-x, -y) = \varphi(x, y)$ equipped with the inner product

$$(\varphi, \varphi)_X \equiv (\Delta^2 \varphi, \Delta^2 \varphi)_{L^2} < \infty.$$

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2. The Linearized Equation

In order to investigate bifurcations from the basic solution $\varphi \equiv 0$, we have to solve the following linearized eigenvalue problem for fixed α and ζ:

$$f_\varphi(\lambda, 0)\varphi \equiv \{\Delta^2 - \zeta\Delta - \lambda(1+\zeta)^{-1} \sin y(\Delta + I)\partial_x\}\varphi = 0, \tag{2.1}$$

where λ is called eigenvalue if (2.1) has a solution $\varphi \neq 0$.

$\varphi \in H_0^4$ is expanded in the Fourier series as

$$\varphi = \sum_{m,n} c_{m,n} e^{i(m\alpha x+ny)}, \quad \sum_{m,n} (m^2\alpha^2 + n^2)^4 |c_{m,n}|^2 < +\infty, \quad c_{0,0} = 0,$$

where the summation is taken over all the pairs of integers but $(m,n) = (0,0)$, and $c_{0,0} = 0$ follows from $\iint_D \varphi \, dxdy = 0$.

For an arbitrarily fixed m, coefficients $c_{m,n}$ satisfy the following system of infinite linear equations:

$$(m^2\alpha^2 + n^2)(m^2\alpha^2 + n^2 + \zeta)c_{m,n} + \frac{\lambda m\alpha}{2(1+\zeta)}\{m^2\alpha^2 + (n-1)^2 - 1\}c_{m,n-1}$$
$$- \frac{\lambda m\alpha}{2(1+\zeta)}\{m^2\alpha^2 + (n+1)^2 - 1\}c_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \dots.$$

We easily see that $c_{0,n} = 0$ for any n. We can prove that non-trivial solutions $c_{m,n}$ which satisfy $c_{m,n} \to 0$ as $|n| \to \infty$ cannot be found when $|m\alpha| = 1$ (this can be seen from (2.3′) below). For $m \neq 0$ and $|m\alpha| \neq 1$, the above equations are simply written as

$$a_{m,n} b_{m,n} + b_{m,n-1} - b_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \dots, \tag{2.2}$$

where

$$a_{m,n} \equiv \frac{2(1+\zeta)(m^2\alpha^2 + n^2)(m^2\alpha^2 + n^2 + \zeta)}{\lambda m\alpha(m^2\alpha^2 + n^2 - 1)}, \quad b_{m,n} \equiv (m^2\alpha^2 + n^2 - 1)c_{m,n}.$$

Let us seek non-trivial solutions of the system (2.2) such that $b_{m,n} \to 0$ as $|n| \to \infty$. We note that any non-trivial $\{b_{m,n}\}_n$ which satisfies this condition is non-zero for n as seen in [1, 2, 7] (this was proved in [1, 2, 7] in the case of $\zeta = 0$, but the same result for (2.2) follows immediately). Therefore, we can define $\rho \equiv b_{m,n}/b_{m,n-1}$ for all n and rewrite (2.2) as follows:

$$a_{m,n} + \frac{1}{\rho_{m,n}} - \rho_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \dots, \tag{2.2′}$$

Here $\lim_{n\to+\infty} \rho_{m,n} = \lim_{n\to-\infty} 1/\rho_{m,n} = 0$ is required for each m. Once we fix $\rho_{m,1}$, then other $\rho_{m,n}$ are determined uniquely. We now introduce $\rho_{m,n}^+$ and $\rho_{m,n}^-$ respectively by

$$\rho_{m,n}^+ \equiv \frac{-1}{a_{m,n}}\Big| + \frac{1}{a_{m,n+1}}\Big| + \cdots, \quad \rho_{m,n}^- \equiv a_{m,n-1} + \frac{1}{a_{m,n-2}}\Big| + \cdots,$$

where the right hand sides of both formulas represent infinite continued fractions (as for the convergence of continued fractions, see [7]). $\lim_{n\to+\infty}\rho^+_{m,n} = \lim_{n\to-\infty} 1/\rho^-_{m,n} = 0$ follows from $\lim_{n\to\pm\infty} a_{m,n} = \infty$. If we put $\rho_{m,1} = \rho^+_{m,1}$ (resp. $\rho_{m,1} = \rho^-_{m,1}$), (2.2′) yields $\rho_{m,n} = \rho^+_{m,n}$ (resp. $\rho_{m,n} = \rho^-_{m,n}$) for any n. Otherwise, we have $\lim_{n\to+\infty}|\rho_{m,n}| = \infty$ (resp. $\lim_{n\to-\infty}|\rho_{m,n}|^{-1} = \infty$). Therefore, the solutions $\rho_{m,n}$ in (2.2′) exist if and only if $\rho^+_{m,1} = \rho^-_{m,1} \equiv \rho_{m,1}$. This relation is equivalent to

(2.3)
$$-\frac{a_{m,0}}{2} = \frac{1}{a_{m,1}}\Big|+\frac{1}{a_{m,2}}\Big|+\cdots.$$

Let us put $\beta \equiv m\alpha$ and write $a_{m,n}$ as a_n for simplicity. We denote the right hand side of (2.3) by $G(\lambda,\beta,\zeta)$ and rewrite (2.3):

(2.3′)
$$\frac{(1+\zeta)\beta(\beta^2+\zeta)}{\lambda(1-\beta^2)} = G(\lambda,\beta,\zeta).$$

We state properties of (2.3′) in the following proposition.

Proposition 2.1. *For the equation* (2.3′), *we obtain the following results*:
(1) *If* $|\beta| > 1$ *and* $\zeta \geq 0$, (2.3′) *has no positive solution* λ.
(2) *If* $0 < |\beta| < 1$, *there exists a function* $\lambda(\beta,\zeta)$ *which satisfies* $\lambda(\beta,\zeta) = \lambda(-\beta,\zeta)$ *and the following properties for positive* β:
 (i) (2.3′) *has a solution if and only if* $\lambda = \lambda(\beta,\zeta)$;
 (ii) $\lim_{\beta\to 0}\lambda(\beta,\zeta) = \lim_{\beta\to 1}\lambda(\beta,\zeta) = +\infty$ *for any* $\zeta > 0$. *As for the case where* $\zeta = 0$, *we have* $\lim_{\beta\to 0}\lambda(\beta,0) = \sqrt{2}$ *and* $\lim_{\beta\to 1}\lambda(\beta,0) = +\infty$;
 (iii) $\lambda(\beta,\zeta)$ *is a strictly monotone increasing function of* $\zeta > 0$ *for fixed* β.

Proof. We multiply λ to (2.3′) so that the left hand side of this equation is independent of λ:

$$\frac{(1+\zeta)\beta(\beta^2+\zeta)}{1-\beta^2} = \lambda G(\lambda,\beta,\zeta).$$

The statement (1) is obtained from the fact that the both sides of this equation have different signs for $|\beta| > 1$, $\zeta \geq 0$ and $\lambda > 0$.

We obtain $\lambda(\beta,\zeta) = \lambda(-\beta,\zeta)$ because (2.3′) is invariant with respect to the transformation $\beta \to -\beta$. Hereafter, we consider only in the case where $\beta > 0$.

Let us prove (2)-(i). Since $0 < G(\lambda,\beta,\zeta) < a_1^{-1} \to 0$ as $\lambda \to 0$, it holds that

$$\frac{(1+\zeta)\beta(\beta^2+\zeta)}{\lambda(1-\beta^2)} > G(\lambda,\beta,\zeta)$$

for small λ. We next prove that the converse inequality holds true for large λ. To this end, it is sufficient to establish that $\lambda G(\lambda,\beta,\zeta)$ tends to $+\infty$ as $\lambda \to +\infty$. We first see

$$G(\lambda,\beta,\zeta) > \frac{1 \rfloor}{a_1} + \frac{1 \rfloor}{a_2} + \cdots + \frac{1}{a_{2n}} = \frac{a_2 + a_4 + \cdots + a_{2n} + O(\lambda^{-2})}{1 + O(\lambda^{-2})}.$$

We write λa_n as a'_n for simplicity. We note that each a'_n does not depend on λ. By virtue of the following fact

$$\lambda G(\lambda,\beta,\zeta) > \frac{\lambda(a_2 + a_4 + \cdots + a_{2n}) + \lambda O(\lambda^{-2})}{1 + O(\lambda^{-2})} = \frac{\sum_{k=1}^{n} a'_{2k} + O(\lambda^{-1})}{1 + O(\lambda^{-2})},$$

we have $\liminf_{\lambda\to\infty} \lambda G(\lambda,\beta,\zeta) \geq \sum_{k=1}^{n} a'_{2k}$. The right hand side of this inequality tends to $+\infty$ as $n \to +\infty$. Accordingly, $\lambda G(\lambda,\beta,\zeta)$ diverges as $\lambda \to +\infty$, which means that $G(\lambda,\beta,\zeta)$ is greater than the left hand side of (2.3′) for sufficiently large λ. Therefore, by virtue of the intermediate value theorem, (2.3′) has a positive solution $\lambda \equiv \lambda(\beta,\zeta)$.

The uniqueness of the solution follows from the fact that $\lambda G(\lambda,\beta,\zeta)$ is monotone increasing in λ. Actually, we have

$$\lambda G(\lambda,\beta,\zeta) = \frac{1 \rfloor}{a_1/\lambda} + \frac{1 \rfloor}{\lambda a_2} + \frac{1 \rfloor}{a_3/\lambda} + \cdots.$$

When λ increases, a_{2n-1}/λ tend to zero and λa_{2n} do not change. Hence, $\lambda G(\lambda,\beta,\zeta)$ increases. Thus (i) is proved.

As for (ii), since it follows from (2.3′)

$$0 < \lambda G(\lambda,\beta,\zeta) < \frac{\lambda}{a_1} = \frac{\lambda^2\beta^3}{2(\beta^2+1)(\beta^2+1+\zeta)},$$

we have

$$\lambda^2 > \frac{2(\beta^2+\zeta)(\beta^2+1)(\beta^2+1+\zeta)}{\beta^2(1-\beta^2)}. \tag{2.4}$$

For fixed $\zeta > 0$, the right hand side of (2.4) diverges when β tends to zero or one. The case where $\zeta = 0$ was already proved in [7]. Therefore, (ii) follows.

As for (iii), we rewrite (2.3′) in the following form

$$1 = \frac{\lambda(1-\beta^2)}{(1+\zeta)\beta(\beta^2+\zeta)} G(\lambda,\beta,\zeta) \equiv \frac{1 \rfloor}{\tilde{a}_1} + \frac{1 \rfloor}{\tilde{a}_2} + \cdots, \tag{2.5}$$

where

$$\tilde{a}_{2n} \equiv \frac{\lambda(1-\beta^2)}{(1+\zeta)\beta(\beta^2+\zeta)} a_{2n}, \quad \tilde{a}_{2n-1} \equiv \frac{(1+\zeta)\beta(\beta^2+\zeta)}{\lambda(1-\beta^2)} a_{2n-1}.$$

The even terms of the continued fraction (2.5)

$$\tilde{a}_{2n} = \frac{2(1-\beta^2)(\beta^2+4n^2)}{\beta^2(\beta^2+4n^2-1)} \cdot \frac{\beta^2+4n^2+\zeta}{\beta^2+\zeta}$$

decrease with $\zeta > 0$. Then, since the continued fraction of (2.5) is actually a constant, at least one odd term of (2.5)

$$\tilde{a}_j = \frac{2(\beta^2+j^2)}{(1-\beta^2)(\beta^2+j^2-1)} \cdot \frac{(1+\zeta)^2(\beta^2+\zeta)(\beta^2+j^2+\zeta)}{\lambda^2}, \quad j = 2n-1$$

must decrease with ζ. As $(1+\zeta)^2(\beta^2+\zeta)(\beta^2+j^2+\zeta)$ increases with ζ, we see that λ^2 increases with ζ. This completes the proof of the proposition. □

This proposition shows that there is no solution λ in (2.3) if $\alpha > 1$. (2.3) has a solution $\lambda = \lambda(k\alpha, \zeta) \equiv \lambda_k (= \lambda_k)$ where k belongs to

$$k \in K_\alpha \equiv \{1, 2, \ldots, r; r \in N, r\alpha < 1 \leq (r+1)\alpha\}.$$

We see, from the shape of $\lambda(\beta, \zeta)$, that there exist β, β' $(0 < \beta, \beta' < 1)$ which satisfy $\lambda(\beta, \zeta) = \lambda(\beta', \zeta)$. This property leads to the description of the remark in the previous section.

The solutions $b_{k,n}$ of (2.2) are given by

$$\text{(2.6)} \qquad b_{k,n} \equiv \begin{cases} \prod_{i=1}^{n} \rho_{k,i} & \text{for } n > 0, \\ 1 & \text{for } n = 0, \\ (-1)^n \prod_{i=1}^{-n} \rho_{k,i} & \text{for } n < 0, \end{cases}$$

where $k \in K_\alpha$ and

$$\rho_{k,i} = \frac{-1 \rfloor}{a_{k,i}} + \frac{1 \rfloor}{a_{k,i+1}} + \cdots, \quad i \geq 1.$$

For each λ_k, the set of the non-trivial solutions of (2.1) is given as follows:

$$\text{(2.7.1)} \qquad \ker f_\varphi(\lambda_k, 0) = \{s_1\varphi_k + s_2\varphi_{-k}; s_i \in R, i = 1, 2\},$$

or

$$\text{(2.7.2)} \quad \ker f_\varphi(\lambda_k, 0) = \{s_1\varphi_k + s_2\varphi_{-k} + s_3\varphi_{k'} + s_4\varphi_{-k'}; s_i \in R, i = 1, 2, 3, 4\},$$

where $\varphi_m \equiv \sum_n c_{m,n} e^{i(m\alpha x + ny)}$, $c_{m,n} = (m^2\alpha^2 + n^2 - 1)^{-1} b_{m,n}$ and $k'(\neq k)$ satisfies $\lambda_k = \lambda_{k'}$.

Here we note that $\overline{c_{m,n}} = c_{-m,-n}$ since φ is real-valued, and also the assumption $\varphi(-x, -y) = \varphi(x, y)$ leads $c_{-m,-n} = c_{m,n}$. By this consideration, we can rewrite (2.7.1) and (2.7.2) as follows:

(2.7.1′) $$\ker f_\varphi(\lambda_k, 0) = \{\varphi = s\varphi^{(k)}; s \in R\},$$

(2.7.2′) $$\ker f_\varphi(\lambda_k, 0) = \{\varphi = s_1\varphi^{(k)} + s_2\varphi^{(k')}; s_i \in R, i = 1, 2\},$$

where $\varphi^{(m)} \equiv \sum_{n=-\infty}^{\infty} c_{m,n} \cos(m\alpha x + ny)$.

Similarly, we seek non-trivial solutions Φ of the conjugate equation of (2.1):

(2.8) $$f_\varphi(\lambda, 0)^*\Phi = \{\Delta^2 - \zeta\Delta + \lambda(1+\zeta)^{-1}(\Delta + I) \sin y\partial_x\}\Phi = 0,$$

in the form $\Phi(x, y) = \sum_{m,n} d_{m,n} e^{i(m\alpha x + ny)}$. We see $d_{0,n} = 0$ for all n and $d_{m,n} = 0$ for $|m\alpha| = 1$. As for other m, we have the following equations of $d_{m,n}$:

$$a_{m,n}d_{m,n} - d_{m,n-1} + d_{m,n+1} = 0.$$

Putting $b'_{m,n} \equiv (-1)^n d_{m,n}$, the above equations can be written as

$$a_{m,n}b'_{m,n} + b'_{m,n-1} - b'_{m,n+1} = 0,$$

Then, we apply the same argument as that in (2.2) and obtain the set of non-trivial solutions of (2.8) for $k \in K_\alpha$:

(2.9.1) $$\ker f_\varphi(\lambda_k, 0)^* = \{t_1\Phi_k + t_2\Phi_{-k}; t_i \in R, i = 1, 2\},$$

or

(2.9.2) $$\ker f_\varphi(\lambda_k, 0)^* = \{t_1\Phi_k + t_2\Phi_{-k} + t_3\Phi_{k'} + t_4\Phi_{-k'}; t_i \in R, i = 1, 2, 3, 4\},$$

where $\Phi_m \equiv \sum_n d_{m,n} e^{i(m\alpha x + ny)}$, $d_{m,n} = (-1)^n b_{m,n}$, and $k'(\neq k)$ satisfies $\lambda_k = \lambda_{k'}$.

Since Φ is real-valued and satisfies $\Phi(-x, -y) = \Phi(x, y)$, (2.9.1) and (2.9.2) can be written respectively as follows:

(2.9.1′) $$\ker f_\varphi(\lambda_k, 0)^* = \{\Phi = s\Phi^{(k)}; s \in R\},$$

(2.9.2′) $$\ker f_\varphi(\lambda_k, 0)^* = \{\Phi = s_1\Phi^{(k)} + s_2\Phi^{(k')}; s_i \in R, i = 1, 2\},$$

where $\Phi^{(m)} \equiv \sum_{n=-\infty}^{\infty} c_{m,n} \cos(m\alpha x + ny)$.

In this paper, we consider only (2.7.1′) and (2.9.1′) as $\ker f_\varphi(\lambda_k, 0)$ and $\ker f_\varphi(\lambda_k, 0)^*$ respectively.

3. Existence of Bifurcation Points

For $\alpha \in (0, 1)$ and $\zeta \in [0, \infty)$, the linearized equation (2.1) has non-trivial solutions if and only if λ is equal to the values λ_k given in the previous section.

Using the method of Lyapunov-Schmidt, we prove that each λ_k is the bifurcation point of (1.5).

Suppose $\varphi \in X$ and $\omega \in Y \equiv L_0^2$. $g \in L_0^2$ means $g \in L^2$ and $\iint_D g\, dxdy = 0$. We decompose them orthogonally by:

$$\varphi = \varphi_1 + \varphi_2, \quad \varphi_1 \in X_1,\ \varphi_2 \in X_2,$$

$$\omega = \omega_1 + \omega_2, \quad \omega_1 \in Y_1,\ \omega_2 \in Y_2.$$

where X_i and Y_i $(i = 1,2)$ are defined as follows: $X_1 = \ker f_\varphi(\lambda_k, 0)$, $Y_2 = Rf_\varphi(\lambda_k, 0)$ and X_2, Y_1 are the orthogonal complements of X_1, Y_2, respectively.

We first note that

$$Y_1 = \ker f_\varphi^*(\lambda_k, 0). \tag{3.1}$$

The proof of this may be safely omitted, since it is clear at least formally and a rigorous proof is standard.

We denote the projection to Y_1 of Y by P. $Q \equiv I - P$ means the projection to Y_2. Corresponding to the above decomposition, we have a system of the following two equations which is equivalent to (1.5):

$$\begin{cases} Qf(\lambda, \varphi_1 + \varphi_2) = 0 \quad \text{in } Y_2, & \cdots (3.2) \\ Pf(\lambda, \varphi_1 + \varphi_2) = 0 \quad \text{in } Y_1. & \cdots (3.3) \end{cases}$$

Hereafter, we seek the set of solutions (λ, φ) which depends on one parameter $s \in (-1, 1)$ as follows: $(\lambda, \varphi) = (\mu(s), \varphi_1(s) + \varphi_2(s))$. We suppose that $\mu(s) \in R$ satisfies $\mu(0) = \lambda_k$. We put $\varphi_1(s) = s\varphi^{(k)}$ where $\varphi^{(k)}$ is a non-trivial solution of (2.1) given in (2.7.1′). Now we look for $\lambda = \mu(s)$ and $\varphi_2(s) \in X_2$.

First, let us consider (3.2). Put $Qf(\lambda, \varphi_1 + \varphi_2) \equiv g(\tau, \varphi_2)$ with $\tau \equiv (\lambda, s)$. Then, $g(\tau, 0) = 0$ follows from $f(\lambda, 0) = 0$ for all λ. We see from its definition that $g_{\varphi_2}(\tau_k, 0)$ with $\tau_k \equiv (\lambda_k, 0)$ is a bijective mapping from X_2 to Y_2. By virtue of the implicit function theorem, there exists a function $\psi(\tau)$ which satisfies $g(\tau, \psi(\tau)) = 0$ and $\psi(\tau_k) = 0$ in the neighborhood of $(\tau_k, 0)$. We shall determine $\psi = \psi(\tau)$ more precisely. In (3.2) with $\varphi_1 = s\varphi^{(k)}$ and $\varphi_2 = \psi$, ψ satisfies the following equation:

$$H[\psi] - \tilde{L}[s\varphi^{(k)} + \psi] - \lambda J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0,$$

where $H \equiv g_{\varphi_2}(\tau_k, 0)$, $\tilde{L} \equiv (\lambda - \lambda_k)(1 + \zeta)^{-1} \sin y(\Delta + I)\partial_x$. Since H is a bijective mapping from X_2 to Y_2, there exists the inverse mapping H^{-1} and it holds that

$$\psi - H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1} J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0.$$

We define a sequence of functions $\{\psi_n\}$ $(n = 0, 1, 2, \ldots)$ as follows:

$$\psi_0 = 0, \quad \psi_n \equiv H^{-1}\tilde{L}[s\varphi^{(k)} + \psi_{n-1}] - \lambda H^{-1}J(\Delta(s\varphi^{(k)} + \psi_{n-1}), s\varphi^{(k)} + \psi_{n-1}).$$

Let us show that $\{\psi_n\}$ is a Cauchy sequence in the neighborhood of $s = 0$. The non-linear term can be omitted since it becomes $O(s^2)$. Choosing λ such as $|\lambda - \lambda_k| \leq 4^{-1}\|H^{-1}\|^{-1}$, we have $\|\psi_1\| = O(s)$ and $\|\psi_2 - \psi_1\| \leq 2^{-1}\|\psi_1\|$. Similarly, $\|\psi_{n+1} - \psi_n\| \leq 2^{-n}\|\psi_1\|$ follows. Then $\{\psi_n\}$ is a Cauchy sequence and converges to a limit $\psi = \psi(\lambda, s)$ which belongs to X_2 with $\psi(\lambda, 0) = 0$ and for small s

$$\psi = H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1}J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi). \tag{3.4}$$

In order to show that each λ_k is a bifurcation point, we have to prove the existence of the solution $\mu(s)$ of (3.3) with $\mu(0) = \lambda_k$. Substituting $\varphi_2 = \psi(\tau)$ into the left hand side of (3.3) and putting

$$Pf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) \equiv h(\lambda, s),$$

we define

$$\chi(\lambda, s) \equiv \begin{cases} h(\lambda, s)/s, & \text{for } s \neq 0, \\ h_s(\lambda, 0), & \text{for } s = 0. \end{cases}$$

We note that $h(\lambda, 0) \equiv 0$ holds for all λ, and the continuity of χ follows from that of h_s. The reason why we define $\chi(\lambda, s)$ is that we cannot apply the implicit function theorem to $h(\lambda, s)$ because $h_\lambda(\lambda, 0) = 0$ follows from $\psi(\lambda, 0) = 0$ for all λ. And from $h_s(\lambda, s) = Pf_\varphi(\lambda, s\varphi^{(k)} + \psi(\lambda, s))[\varphi^{(k)} + \psi_s(\lambda, s)]$, we have $h_s(\lambda, 0) = Pf_\varphi(\lambda, 0)[\varphi^{(k)} + \psi_s(\lambda, 0)]$. Now we verify $\psi_s(\lambda_k, 0) = 0$ which we will use later. Differentiating $Qf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) = 0$ by s and putting $(\lambda, s) = (\lambda_k, 0)$, we have $Qf_\varphi(\lambda_k, 0)[\psi_s(\lambda_k, 0)] = 0$. Since $Qf_\varphi(\lambda_k, 0)$ is a bijective mapping from X_2 to Y_2, $\psi_s(\lambda_k, 0) = 0$ is verified.

When we consider that $X_1 = \ker f_\varphi(\lambda_k, 0)$ is given in (2.7.1′), $Y_1 = \ker f_\varphi^*(\lambda_k, 0)$ takes the form (2.9.1′). Then $\chi(\lambda, s) = 0$ is equivalent to the following equation:

$$\chi^{(1)}(\lambda, s) \equiv (\chi(\lambda, s), \Phi^{(k)})_{L^2} = 0, \tag{3.5}$$

where $\Phi^{(k)} \in Y_1$. We seek a solution λ of (3.5). Differentiating (3.5) by λ, we have

$$\begin{aligned} \chi_\lambda^{(1)}(\lambda_k, 0) &= \left(\lim_{\Delta\lambda \to 0} \frac{\chi(\lambda_k + \Delta\lambda, 0) - \chi(\lambda_k, 0)}{\Delta\lambda}, \Phi^{(k)}\right)_{L^2} \\ &= (Pf_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], \Phi^{(k)})_{L^2} = (f_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], P^*\Phi^{(k)})_{L^2} \\ &= (f_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], P\Phi^{(k)})_{L^2} \\ &= s(-(1 + \zeta)^{-1} \sin y(\Delta + I)\partial_x\varphi^{(k)}, \Phi^{(k)})_{L^2}. \end{aligned}$$

Let us show

$$(3.6) \qquad (-(1+\zeta)^{-1}\sin y(\Delta+I)\partial_x\varphi^{(k)}, \Phi^{(k)})_{L^2} > 0.$$

Since $\varphi^{(k)}$ is a solution of (2.1), we have

$$-(1+\zeta)^{-1}\sin y(\Delta+I)\partial_x\varphi^{(k)} = \lambda_k^{-1}(-\Delta^2+\zeta\Delta)\varphi^{(k)}.$$

$\varphi^{(k)} = \sum_n c_{k,n}\cos(k\alpha x+ny)$ and $\Phi^{(k)} = \sum_n d_{k,n}\cos(k\alpha x+ny) = \sum_n (-1)^n \cdot (k^2\alpha^2+n^2-1)c_{k,n}\cos(k\alpha x+ny)$ yield

$$((-\Delta^2+\zeta\Delta)\varphi^{(k)}, \Phi^{(k)})_{L^2} \equiv 2^{-1}|D|\sum_{n=-\infty}^{\infty}(-1)^{n+1}\tilde{c}_{k,n},$$

where $\tilde{c}_{k,n} \equiv (k^2\alpha^2+n^2)(k^2\alpha^2+n^2+\zeta)(k^2\alpha^2+n^2-1)c_{k,n}^2$. Meanwhile, we can verify $\sum_n \tilde{c}_{k,n} = 0$. In fact, multiplying the both side of $f_\varphi(\lambda_k, 0)\varphi^{(k)} = 0$ by $(\Delta+I)\varphi^{(k)}$ and integrating over D, we obtain

$$\begin{aligned} 0 = &\iint_D (\Delta+I)\varphi^{(k)}(\Delta^2-\zeta\Delta)\varphi^{(k)}\,dxdy \\ &- \lambda_k(1+\zeta)^{-1}\iint_D (\Delta+I)\varphi^{(k)}\sin y(\Delta+I)\partial_x\varphi^{(k)}\,dxdy. \end{aligned}$$

The second term of the right hand side of this equation vanishes. Then, we have

$$\iint_D (\Delta+I)\varphi^{(k)}(\Delta^2-\zeta\Delta)\varphi^{(k)}\,dxdy = -2^{-1}|D|\sum_{n=-\infty}^{\infty}\tilde{c}_{k,n} = 0.$$

From $\sum_n \tilde{c}_{k,n} = 0$ and $\tilde{c}_{k,-n} = \tilde{c}_{k,n}$, we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty}(-1)^{n+1}\tilde{c}_{k,n} &= -\tilde{c}_{k,0} + 2\sum_{m=1,3,5,\dots}\tilde{c}_{k,m} - 2\sum_{m=2,4,6,\dots}\tilde{c}_{k,m} \\ &= 4\sum_{m=1,3,5,\dots}\tilde{c}_{k,m} > 0. \end{aligned}$$

Therefore, we obtain (3.6), namely, $\chi_\lambda^{(1)}(\lambda_k, 0) \neq 0$ if $s \neq 0$. By virtue of the implicit function theorem, there exists a function $\lambda = \mu(s)$ which satisfies $\chi^{(1)}(\mu(s), s) = 0$ and $\mu(0) = \lambda_k$. This means that the solution of (1.5) is given by

$$(\lambda, \varphi) = (\mu(s), \varphi(s)) = (\mu(s), s\varphi^{(k)} + \psi(\mu(s), s)).$$

4. Properties of the Bifurcation Curve

Following the method in [7], we shall consider the convex property of $\lambda = \mu(s)$ with regard to s. Namely, we prove that $\mu_{ss}(0) > 0$. We rewrite $f(\mu(s), \varphi(s)) = 0$ as

$$T\varphi(s) = \tilde{\lambda}(s)(1+\zeta)^{-1} \sin y(\Delta + I)\partial_x \varphi(s) + \mu(s) J(\Delta\varphi(s), \varphi(s)), \tag{4.1}$$

where $T \equiv f_\varphi(\lambda_k, 0)$ and $\tilde{\lambda}(s) \equiv \mu(s) - \lambda_k$. Let us differentiate (4.1) by s:

$$\begin{aligned}
T\varphi_s(s) &= \frac{\tilde{\lambda}_s(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi(s) + \frac{\tilde{\lambda}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_s(s) \\
&\quad + \mu_s(s) J(\Delta\varphi(s), \varphi(s)) + \mu(s) J(\Delta\varphi(s), \varphi(s))_s; \\
T\varphi_{ss}(s) &= \frac{\tilde{\lambda}_{ss}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi(s) + \frac{2\tilde{\lambda}_s(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_s(s) \\
&\quad + \frac{\tilde{\lambda}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_{ss}(s) + \mu_{ss}(s) J(\Delta\varphi(s), \varphi(s)) \\
&\quad + 2\mu_s(s) J(\Delta\varphi(s), \varphi(s))_s + \mu(s) J(\Delta\varphi(s), \varphi(s))_{ss}; \\
\varphi_s(s) &= \varphi^{(k)} + \psi_\lambda(\mu(s), s)\mu_s(s) + \psi_s(\mu(s), s).
\end{aligned}$$

$\varphi(0) = 0$ and $\tilde{\lambda}(0) = 0$ yield

$$T\varphi_{ss}(0) = \frac{2\mu_s(0)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi^{(k)} + 2\lambda_k J(\Delta\varphi^{(k)}, \varphi^{(k)}). \tag{4.2}$$

If we take the L^2 inner-product with $\Phi^{(k)} \in \ker T^*$, (4.2) becomes

$$0 = 2\mu_s(0)(1+\zeta)^{-1}(\sin y(\Delta + I)\partial_x \varphi^{(k)}, \Phi^{(k)})_{L^2},$$

and $T\varphi^{(k)} = 0$ implies

$$0 = 2\mu_s(0)\lambda_k^{-1}((\Delta^2 - \zeta\Delta)\varphi^{(k)}, \Phi^{(k)})_{L^2}.$$

Since we already have shown in (3.6) that

$$((\Delta^2 - \zeta\Delta)\varphi^{(k)}, \Phi^{(k)})_{L^2} < 0, \tag{4.3}$$

we obtain $\mu_s(0) = 0$.

Differentiating (4.1) once more and putting $s = 0$:

$$\begin{aligned} T\varphi_{sss}(0) &= 3\mu_{ss}(0)(1+\zeta)^{-1}\sin y(\Delta+I)\partial_x\varphi^{(k)} \\ &\quad + 3\lambda_k\{J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0))\} \\ &= 3\mu_{ss}(0)\lambda_k^{-1}(\Delta^2-\zeta\Delta)\varphi^{(k)} \\ &\quad + 3\lambda_k\{J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0))\}, \end{aligned}$$

and taking the L^2 inner-product with $\Phi^{(k)} \in \ker T^*$, we have

$$\begin{aligned} 0 &= (T\varphi_{sss}(0),\Phi^{(k)})_{L^2} \\ &= 3\mu_{ss}(0)\lambda_k^{-1}((\Delta^2-\zeta\Delta)\varphi^{(k)},\Phi^{(k)})_{L^2} \\ &\quad + 3\lambda_k(J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0)),\Phi^{(k)})_{L^2}, \end{aligned}$$

namely,

$$\mu_{ss}(0) = \frac{-\lambda_k^2}{((\Delta^2-\zeta\Delta)\varphi^{(k)},\Phi^{(k)})_{L^2}}(J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0)),\Phi^{(k)})_{L^2}.$$

(4.3) informs us that the sign of $\mu_{ss}(0)$ is equal to that of

$$\iint_D \{J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0))\}\Phi^{(k)}\,dxdy. \tag{4.4}$$

Since $\varphi_{ss}(0) = \psi_{ss}(\lambda_k, 0)$ is obtained by

$$T\varphi_{ss}(0) = 2\lambda_k J(\Delta\varphi^{(k)},\varphi^{(k)}), \tag{4.5}$$

we have

$$\begin{aligned} &\iint_D \{J(\Delta\varphi_{ss}(0),\varphi^{(k)}) + J(\Delta\varphi^{(k)},\varphi_{ss}(0))\}\Phi^{(k)}\,dxdy \equiv D_1 + D_2, \\ &D_1 \equiv \iint_D \{J(\Delta Z_1,\varphi^{(k)}) + J(\Delta\varphi^{(k)},Z_1)\}\Phi^{(k)}\,dxdy, \\ &D_2 \equiv \iint_D \{J(\Delta Z_2,\varphi^{(k)}) + J(\Delta\varphi^{(k)},Z_2)\}\Phi^{(k)}\,dxdy, \end{aligned}$$

where Z_1, Z_2 are functions extended respectively by $\cos \ell y$, $\cos(2k\alpha x + \ell y)$ (we define them precisely later in Appendix). As for D_1 and D_2, the following proposition holds true:

PROPOSITION 4.1. *For each fixed $\zeta \geq 0$, $D_1 > |D_2|$ holds if $k\alpha$ is close to one.*

This proposition means that $\mu_{ss}(0) > 0$ holds if $k\alpha \in (0,1)$ is sufficiently close to one. Thus, Theorem 1.1 is proved.

5. Appendix

This appendix gives the proof of the proposition asserted in the previous section.

PROOF OF PROPOSITION 4.1. The outline of this proof is based on the previous paper (see Section 4 and 5 of [7]). First, we introduce a proposition which can be proved similarly to [7].

PROPOSITION 5.1. *The solution of* (4.5) *takes the following form*:

$$\varphi_{ss}(0) = {}^t\boldsymbol{w}^{(0)}\boldsymbol{\Lambda c}(0) + {}^t\boldsymbol{w}^{(2k)}\boldsymbol{DEc}(2k\alpha) \equiv Z_1 + Z_2, \tag{5.1}$$

$$Z_1 \equiv {}^t\boldsymbol{w}^{(0)}\boldsymbol{\Lambda c}(0), \quad Z_2 \equiv {}^t\boldsymbol{w}^{(2k)}\boldsymbol{DEc}(2k\alpha).$$

Here $\boldsymbol{c}(0)$, $\boldsymbol{c}(2k\alpha)$, $\boldsymbol{w}^{(0)}$ and $\boldsymbol{w}^{(2k)}$ are column vectors with the following ℓ-th components:

$$(\boldsymbol{c}(0))_\ell = \cos \ell y, \quad (\boldsymbol{c}(2k\alpha))_\ell = \cos(2k\alpha x + \ell y),$$

$$(\boldsymbol{w}^{(0)})_\ell = \lambda_k k\alpha\ell\, {}^t\boldsymbol{\varphi}^{(k)}\boldsymbol{K}_1 S^\ell \boldsymbol{\varphi}^{(k)},$$

$$(\boldsymbol{w}^{(2k)})_\ell = \lambda_k k\alpha\, {}^t\boldsymbol{\varphi}^{(k)}\boldsymbol{K}_1(2\boldsymbol{N} - \ell\boldsymbol{I})RS^\ell\boldsymbol{\varphi}^{(k)},$$

where $\boldsymbol{\varphi}^{(k)}$ is a column vector corresponding to the Fourier coefficients of $\varphi^{(k)}$ with n-th component $\varphi_n = (k^2\alpha^2 + n^2 - 1)^{-1} b_{k,n}$ ($b_{k,n}$ is defined by (2.6)), *$\boldsymbol{K}_1$ and $\boldsymbol{N}$ are diagonal matrices with n-th elements $-k_n \equiv -(k^2\alpha^2 + n^2)$ and $(\boldsymbol{N})_n = n$ respectively. S^ℓ and R are matrices with (i,j) elements as follows*:

$$(S^\ell)_{i,j} = \begin{cases} 1 & \text{for } j - i = \ell, \\ 0 & \text{otherwise}, \end{cases} \quad (R)_{i,j} = \begin{cases} 1 & \text{for } i + j = 0, \\ 0 & \text{otherwise}. \end{cases}$$

$\boldsymbol{\Lambda}$ and $\boldsymbol{E}$ are diagonal matrices with n-th elements

$$\boldsymbol{\Lambda}_n = \begin{cases} (n^4 + \zeta n^2)^{-1} & \text{for } n \neq 0, \\ 0 & \text{for } n = 0, \end{cases} \quad \boldsymbol{E}_n = \frac{1+\zeta}{\lambda_k k\alpha(4k^2\alpha^2 + n^2 - 1)},$$

and $\boldsymbol{D} = (\cdots \boldsymbol{d}^{(m)} \cdots)$ *is a matrix where* $\boldsymbol{d}^{(m)}$ *are column vectors with n-th component* $\boldsymbol{d}_n^{(m)}$ *as follows*:

$$\boldsymbol{d}_n^{(m)} = \begin{cases} \left(\prod_{i=m+1}^{n} \eta_i^+\right) N_{m+1}^{-1} & \text{for } n > m, \\ N_{m+1}^{-1} & \text{for } n = m, \\ \left(\prod_{i=n+1}^{m} \eta_i^-\right)^{-1} N_{m+1}^{-1} & \text{for } n < m, \end{cases}$$

where

$$\eta_n^+ \equiv \frac{1}{a_n'}\Big| + \frac{1}{a_{n+1}'}\Big| + \cdots,$$

$$\eta_n^- \equiv -a_{n-1}' + \frac{-1}{a_{n-2}'}\Big| + \cdots,$$

$$N_{m+1} \equiv \eta_{m+1}^+ - \eta_{m+1}^-,$$

$$a_n' \equiv \frac{(1+\zeta)(4k^2\alpha^2+n^2)(4k^2\alpha^2+n^2+\zeta)}{\lambda_k k\alpha(4k^2\alpha^2+n^2-1)}.$$

We can write D_1 in the previous section as

$$D_1 = \lambda_k k^2 \alpha \pi^2 \sum_{\ell=1,3,5,\ldots} (w_\ell + w_{-\ell}) \left\{ A_{\ell 0}\Phi_0 + 2\sum_{m=1}^{\infty} A_{\ell m}\Phi_m \right\},$$

where

$$w_\ell \equiv {}^t\boldsymbol{\varphi}^{(k)} \boldsymbol{K}_1 \ell S^\ell \boldsymbol{\varphi}^{(k)} = \ell \sum_{j=-\infty}^{\infty} (-k_j)\varphi_j\varphi_{\ell+j}, \quad \Phi_m \equiv (\boldsymbol{\Phi}^{(k)})_m = (-1)^m b_{k,n},$$

$$\begin{aligned} A_{\ell m} &= [\boldsymbol{\Lambda N}\{\boldsymbol{N}^2(I-R)S^m + (I-R)S^m\boldsymbol{K}_1\}\boldsymbol{\varphi}^{(k)}]_\ell \\ &= \{(\boldsymbol{\Lambda N})_\ell k_{-\ell+m} - (\boldsymbol{\Lambda N}^3)_\ell\}\varphi_{-\ell+m} + \{(\boldsymbol{\Lambda N}^3)_\ell - (\boldsymbol{\Lambda N})_\ell k_{\ell+m}\}\varphi_{\ell+m}. \end{aligned}$$

Here we use the following facts: $A_{\ell m} = A_{(-\ell)m}$, $\varphi_{-m} = (-1)^m\varphi_m$, $\Phi_{-m} = (-1)^m\Phi_m$ and $k_{-j} = k_j$. We also employ the following lemma in [7].

Lemma 5.2. *For matrices* S, N *and* R, *we obtain*
(i) $S^\ell N - NS^\ell = \ell S^\ell$, $\ell \in N$,
(ii) $NRS^\ell - RS^\ell N = (2N - \ell I)RS^\ell$, $\ell \in N$.

Let us consider each terms of D_1. Since it holds that $(\boldsymbol{\Lambda N})_\ell > 0$, $\ell^2 - k_\ell = -k^2\alpha^2 < 0$, $\varphi_\ell < 0$ for odd $\ell \geq 1$ and $\Phi_0 = 1$, we have

$$A_{\ell 0}\Phi_0 = 2(\boldsymbol{\Lambda N})_\ell(\ell^2 - k_\ell)\varphi_\ell > 0.$$

The terms which contain φ_0 are dominant throughout this proof, since only $\varphi_0 = (k^2\alpha^2 - 1)^{-1} < 0$ diverges if $k\alpha$ tends to one. In $\sum_{m=1}^{\infty} A_{\ell m}\Phi_m$, we see that the corresponding term $(\Lambda N)_\ell(-\ell^2 + k^2\alpha^2)\varphi_0\Phi_\ell > 0$ diverges to $+\infty$ if $k\alpha$ tends to one and the sum of other terms is uniformly bounded. On the other hand, in

$$w_\ell + w_{-\ell} = \ell\left\{2\ell^2\varphi_\ell\varphi_0 + \sum_{i\neq 0,-\ell}(k_{\ell+i} - k_i)\varphi_i\varphi_{\ell+i}\right\}, \quad \ell = 1, 3, 5, \dots,$$

if $k\alpha$ tends to one, then only the first term $2\ell^3\varphi_\ell\varphi_0 > 0$ diverges to $+\infty$ and the other terms remain bounded. Therefore, $w_\ell + w_{-\ell} > 0$ holds and, then, we obtain $D_1 > 0$.

Similarly, D_2 can be written as

$$D_2 = \lambda_k k^2\alpha\pi^2 \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \tilde{w}_\ell \tilde{A}_{\ell m}\Phi_m,$$

$$\tilde{w}_\ell \equiv {}^t\varphi^{(k)}K_1(2N - \ell I)RS^\ell\varphi^{(k)} = \sum_{j=-\infty}^{\infty} k_j(\ell - 2j)\varphi_j\varphi_{\ell-j},$$

$$\tilde{A}_{\ell m} \equiv \{DE(K_2S^{-m} - S^{-m}K_1)(N - mI)\varphi^{(k)}\}_\ell$$

$$= \sum_{j=-\infty}^{\infty} d_\ell^{(j)}E_j(-\tilde{k}_j + k_{j-m})(-2m + j)\varphi_{j-m},$$

where K_2 is a diagonal matrix with n-th element $-\tilde{k}_j \equiv -(4k^2\alpha^2 + j^2)$. In this case, the terms which contain φ_0^2 are dominant. The corresponding terms in D_2 are given by

$$(A) \qquad \lambda_k k^2\alpha\pi^2 \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} -\ell^3\varphi_\ell m d_\ell^{(m)}E_m(3k^2\alpha^2 + m^2)\Phi_m\varphi_0^2,$$

while in D_1, the terms which contain φ_0^2 are

$$(B) \qquad \lambda_k k^2\alpha\pi^2 \sum_{n=1,3,5,\dots} 4n^2(k^2\alpha^2 - n^2)(n^2 + \zeta)^{-1}\varphi_n\Phi_n\varphi_0^2.$$

We compare (A) and (B) in the case where $k\alpha$ tends to one. Neglecting $\lambda_k k^2\alpha\pi^2\varphi_0^2 > 0$ involved in both terms and putting $k\alpha = 1$, we have

$$\tilde{D}_1 \equiv 4\sum_{n=1,3,5,\dots,}(n^2 - 1)(n^2 + \zeta)^{-1}b_{k,n}^2,$$

$$\tilde{D}_2 \equiv (1 + \zeta)\sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^{m+1}\ell m\lambda_k^{-1}d_\ell^{(m)}b_{k,\ell}b_{k,m}.$$

We already knew that $\tilde{D}_1 > 0$, and we can estimate $\tilde{D}_2$ as follows:

$$|\tilde{D}_2| < (1+\zeta)\sum_{\ell=1}^{\infty}|b_{k,\ell}|^2(C_1\ell M^{-1} + C_2\ell^2 M^{-28/15}).$$

Here C_1 and C_2 are suitable constants and $M \equiv [\sqrt{\lambda_k}] + 1$ where $[x]$ is the integral part of x (as for the estimation, we use the same method in Section 5.2 of [7]). For each fixed $\zeta \geq 0$, the right hand side is smaller than $\tilde{D}_1$ mainly since λ_k goes to ∞ as $k\alpha$ is close to one. Therefore, the proposition is proved.

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