REMARKS ON HYPERBOLIC SYSTEMS OF FIRST ORDER WITH CONSTANT COEFFICIENT CHARACTERISTIC POLYNOMIALS

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Abstract

In this paper we shall deal with hyperbolic systems of first order with constant coefficient characteristic polynomials and give a necessary and sufficient condition for the Cauchy problem to be C^∞ well-posed under the maximal rank condition (see the condition (R) below).

1. Introduction

The Cauchy problem for hyperbolic operators has been studied by many authors, and necessary and sufficient conditions for C^∞ well-posedness have been obtained in restricted situations (see, e.g., [19], [6], [4], [22], [12], [26], [20], [10], [11], [17] and [21]). We think that it is meaningfully to obtain necessary and sufficient conditions for C^∞ well-posedness in various restricted situations as a step forward in the study of C^∞ well-posedness.

In [26] we proved that the Cauchy problem for a single higher order operator $P(x, D)$ with constant coefficient hyperbolic principal part is C^∞ well-posed (in $\mathbb{R}^n$) and has the finite propagation property if and only if $P(x, \xi)$ is hyperbolic in the sense of Gårding for each $x \in \mathbb{R}^n$. We should note that the sufficiency was proved by Dunn [5] and that the condition of the finite propagation property can be removed by applying the arguments in this paper. We shall attempt to extend the result to hyperbolic systems of first order whose characteristic polynomials have constant coefficients, and give a necessary and sufficient condition for C^∞ well-posedness under the maximal rank condition in this paper.

Concerning the necessity of C^∞ well-posedness for hyperbolic systems of first order, Benvenuti, Bernardi and Bove defined invariantly "determinants" of the systems under the maximal rank condition in [1] and gave necessary conditins of Ivrii-Petkov type by means of "determinants." The assumption on the rank was removed by Bove and Nishitani [3]. The necessary conditions obtained in [1] is not sufficient ones for the hyperbolic systems treated here. For hyperbolic systems with constant multiplicities a necessary and sufficient condition is obtained under the maximal rank condition (see

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[22] and [12]). The maximal rank condition considerably simplifies the problems. One of our aim is to illustrate how to prove the sufficiency of C^∞ well-posedness of the Cauchy problem for general hyperbolic systems satisfying the maximal rank condition by applying the results in [14]. Another aim is to show how to prove the necessity by transforming microlocally the systems. A part of the results here was announced in [30] without proof.

Let $A_1(x), \ldots, A_n(x), L_0(x) \in C^\infty(\mathbb{R}^n; M_m(\mathbb{C}))$, and put

$$L_1(x, \xi) = \sum_{j=1}^n \xi_j A_j(x), \quad L(x, \xi) = L_1(x, \xi) + L_0(x),$$

where m is an integer ≥ 2, $M_{m,m'}(R)$ denotes the collection of all $m \times m'$ matrices with their entries in R, $M_m(R) = M_{m,m}(R)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $C^\infty(\mathbb{R}^n; V)$ denotes the collection of C^∞ functions defined on $\mathbb{R}^n$ with their values in V. For $t \in \mathbb{R}$ we consider the Cauchy problem

$$(\text{CP})_t \qquad \begin{cases} L(x, D)u(x) = f(x) & \text{in} \quad \mathbb{R}^n, \\ \text{supp } u \subset \{x \in \mathbb{R}^n; x_1 \geq t\} \end{cases}$$

in the C^∞ (or $\mathscr{D}'$) category, where $D = (D_1, \ldots, D_n) = -i(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $f = {}^t(f_1, \ldots, f_m)$ satisfies supp $f \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$. Assume that
(H) $A_1(x) = I_m$, $\det L_1(x, \xi)$ does not depend on x and $p(\xi) \equiv \det L_1(x, \xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \ldots, 0) \in \mathbb{R}^n$, i.e., $p(\xi - i\vartheta) \neq 0$ for any $\xi \in \mathbb{R}^n$, where I_m denotes the identity matrix of order m.

We also assume that $L(x, D)$ satisfies the maximal rank condition, i.e.,
(R) rank $L_1(x, \xi) = m - 1$ for any $(x, \xi) \in \mathbb{R}^n \times S^{n-1}$ with $dp(\xi) = 0$, where $S^{n-1} = \{\xi \in \mathbb{R}^n; |\xi| = 1\}$.

Under the assumption (R) we shall reduce the Cauchy problem $(\text{CP})_t$ to that of single higher order operators in a microlocal sense. Let $x^0 \in \mathbb{R}^n$ and $\xi^{0\prime} = (\xi_2^0, \ldots, \xi_n^0) \in S^{n-2}$. By a linear coordinate transformation of $x' = (x_2, \ldots, x_n)$ we may assume that $\xi^{0\prime} = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}$. Write

$$p(\xi^0 + \lambda \vartheta) = \prod_{j=1}^r (\lambda + \lambda_j)^{m_j}, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_r,$$

where $\xi^0 = (0, \ldots, 0, 1) \in \mathbb{R}^n$. Then there are an open neighborhood V of x^0, an open conic neighborhood $\mathscr{C}'$ of $\xi^{0\prime}$ in $\mathbb{R}^{n-1} \setminus \{0\}$, $S(x, \xi') \in C^\infty(\mathbb{R}^n \times \mathscr{C}'; M_m(\mathbb{C}))$ and $A^j(x, \xi') \in C^\infty(V \times \mathscr{C}'; M_{m_j}(\mathbb{C}))$ $(1 \leq j \leq r)$ such that $\det S(x, \xi') \neq 0$ for $(x, \xi') \in V \times \mathscr{C}'$, the entries of $S(x, \xi')$ are (positively) homogeneous of degree 0, $(A^j(x, \xi^{0\prime}) - \lambda_j I_{m_j})^{m_j} = 0$ and

$$S(x, \xi')^{-1} A(x, \xi') S(x, \xi') = \text{diag}(A^1(x, \xi'), \ldots, A^r(x, \xi')) \quad \text{in} \quad V \times \mathscr{C}',$$

where $\xi' = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}$, $A(x, \xi') = \sum_{j=2}^{n} \xi_j A_j(x)$ and

$$\mathrm{diag}(A^1, \ldots, A^r) = \begin{pmatrix} A^1 & & 0 \\ & \ddots & \\ 0 & & A^r \end{pmatrix}$$

(see Lemma 2.2 below). We can assume without loss of generality that $A_2(x), \ldots,$ $A_n(x), L_0(x) \in \mathcal{B}(\mathbb{R}^n; M_m(\mathbb{C}))$, i.e., $\sup_{x \in \mathbb{R}^n} \left(\sum_{j=2}^{n} |D^\alpha A_j(x)| + |D^\alpha L_0(x)| \right) \leq C_\alpha$ for every $\alpha \in (\mathbb{Z}_+)^n$, where $|A|$ denotes the matrix norm of A and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ $(= \{0, 1, 2, \ldots\})$. We say that a symbol $a(x, \xi)$ belongs to $S_{1,0}^\kappa$ if $a(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ and $\left| a_{(\beta)}^{(\alpha)}(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{\kappa - |\alpha|}$ for any $(x, \xi) \in T^*\mathbb{R}^n$ $(\simeq \mathbb{R}^n \times \mathbb{R}^n)$ and $\alpha, \beta \in (\mathbb{Z}_+)^n$, where $\kappa \in \mathbb{R}$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $a_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta a(x, \xi)$. Similarly, we say that $a(x, \xi') \in S_{1,0}^\kappa(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ if $a(x, \xi') \in C^\infty(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ and $\left| a_{(\beta)}^{(\alpha')}(x, \xi') \right| \leq C_{\alpha', \beta} \langle \xi' \rangle^{\kappa - |\alpha'|}$ for any $(x, \xi') \in \mathbb{R} \times T^*\mathbb{R}^{n-1}$ $(\simeq \mathbb{R}^n \times \mathbb{R}^{n-1})$ and $\alpha' = (\alpha_2, \ldots, \alpha_n) \in (\mathbb{Z}_+)^{n-1}$ and $\beta \in (\mathbb{Z}_+)^n$, where $a_{(\beta)}^{(\alpha')}(x, \xi') = \partial_{\xi'}^{\alpha'} D_x^\beta a(x, \xi')$. Moreover, we say that a symbol $a(x, \xi)$ belongs to $\mathscr{S}_{1,0}^{k,l}$ if $a(x, \xi) = \sum_{j=0}^{[k]} a_j(x, \xi') \xi_1^j$ and the $a_j(x, \xi')$ are classical symbols and $a_j(x, \xi') \in S_{1,0}^{k+l-j}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$, where $k, l \in \mathbb{R}$, $[k]$ denotes the largest integer $\leq k$ and $\mathscr{S}_{1,0}^{k,l} = \{0\}$ if $k < 0$. We write $\mathscr{S}_{1,0}^k = \mathscr{S}_{1,0}^{k,0}$, $\mathscr{S}_{1,0}^{k,-\infty} = \bigcap_{l \in \mathbb{R}} \mathscr{S}_{1,0}^{k,l}$ and $\mathscr{S}_{1,0}^\infty = \bigcup_{k \geq 0} \mathscr{S}_{1,0}^k$. By $\mathscr{L}_{1,0}^{k,l}$ we denote the set of pseudodifferential operators whose symbols belong to $\mathscr{S}_{1,0}^{k,l}$. We also write $\mathscr{L}_{1,0}^k = \mathscr{L}_{1,0}^{k,0}$. We need the following block-diagonalization (see, e.g., §3.3 of [18] and, also, Lemma 2.3 below): There are classical symbols $S_\pm(x, \xi') \in M_m(\mathscr{S}_{1,0}^0)$, $\widetilde{A}^\mu(x, \xi') \in M_{m_\mu}(\mathscr{S}_{1,0}^{0,1})$ and $C^\mu(x, \xi') \in M_{m_\mu}(\mathscr{S}_{1,0}^0)$ $(1 \leq \mu \leq r)$ and $Q(x, \xi') \in M_m(\mathscr{S}_{1,0}^{0,-1})$ such that

$$S_+(x, \xi') = S(x, \xi'), \quad \widetilde{A}^\mu(x, \xi') = A^\mu(x, \xi')$$

$$\text{for} \quad (x, \xi') \in V \times \mathscr{C}' \quad \text{with} \quad |\xi'| \geq 1 \quad \text{and} \quad 1 \leq \mu \leq r,$$

$$S_-(x, D')S_+(x, D') \equiv I \bmod \mathscr{L}_{1,0}^{0,-\infty} \quad \text{in} \quad V \times \mathscr{C}',$$

$$S_-(x, D')L(x, D)S_+(x, D')(I + Q(x, D'))$$

$$\equiv (I + Q(x, D'))\left\{ D_1 I + \mathrm{diag}\left(\widetilde{A}^1(x, D') + C^1(x, D'), \ldots, \widetilde{A}^r(x, D') + C^r(x, D') \right) \right\}$$

$$\bmod \mathscr{L}_{1,0}^{1,-\infty} \qquad \qquad \text{in} \quad V \times \mathscr{C}',$$

with modifications of V and $\mathscr{C}'$ if necessary, where I denotes the identity operator and $D' = (D_2, \ldots, D_n)$. Here $A(x, D) \equiv B(x, D) \bmod \mathscr{L}_{1,0}^{k,-\infty}$ in $V \times \mathscr{C}'$ means that $(A(x, \xi) - B(x, \xi))\psi(x, \xi') \in M_m(\mathscr{S}_{1,0}^{k,-\infty})$ for any $\psi(x, \xi') \in \mathscr{S}_{1,0}^0$ with $\mathrm{supp}\, \psi \subset V \times \mathscr{C}'$. We fix $\mu \in \{1, 2, \ldots, r\}$. Then it follows from the assumption (R) that there is $T_{\mu,1} \in$

$M_{m_\mu}(\mathbb{C})$ such that $\det T_{\mu,1} \neq 0$ and

$$T_{\mu,1}^{-1} A^\mu(x^0, \xi^{0\prime}) T_{\mu,1} = \begin{pmatrix} \lambda_\mu & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_\mu \end{pmatrix}.$$

Putting

$$T_{\mu,2} = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \in M_{m_\mu}(\mathbb{C}),$$

we have

$$L^\mu(x,\xi) \equiv T_{\mu,1}^{-1}\big(\xi_1 I_{m_\mu} + \widetilde{A}^\mu(x,\xi') + C^\mu(x,\xi')\big) T_{\mu,1} T_{\mu,2}$$

(1.1)
$$= \begin{pmatrix} \xi_n & 0 & & & \xi_1 + \lambda_\mu \xi_n \\ \xi_1 + \lambda_\mu \xi_n & \xi_n & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & \xi_n & 0 \\ & & & \xi_1 + \lambda_\mu \xi_n & 0 \end{pmatrix} + \widetilde{L}^\mu(x,\xi'),$$

$$\widetilde{L}^\mu(x,\xi') = \widetilde{L}_1^\mu(x,\xi') + \widetilde{L}_0^\mu(x,\xi'), \quad \widetilde{L}_1^\mu(x^0,\xi^{0\prime}) = 0,$$

where $\widetilde{L}_1^\mu(x,\xi') \in \mathscr{S}_{1,0}^{0,1}$, $\widetilde{L}_0^\mu(x,\xi') \in \mathscr{S}_{1,0}^0$ and $\widetilde{L}_1^\mu(x,\xi')$ is (positively) homogeneous of degree 1 for $|\xi'| \geq 1$. We may assume that $\xi_n + \widetilde{L}_{1,j,j}^\mu(x,\xi') \neq 0$ for $1 \leq j \leq m_\mu$, $x \in V$ and $\xi' \in \mathscr{C}'$ with $|\xi'| \geq 1$, modifying V and $\mathscr{C}'$ if necessary, where $\widetilde{L}_1^\mu(x,\xi') = \big(\widetilde{L}_{1,j,k}^\mu(x,\xi')\big)$. Now we perform the following elementary transformations on $L^\mu(x,D)$ in turn from $j = 1$ to $j = m_\mu - 1$: (i) We multiply the j-th column on the right by a pseudodifferential operator in $\mathscr{L}_{1,0}^0$ and add it to the k-th column to annihilate the (j,k)-th entry modulo $\mathscr{L}_{1,0}^{0,-\infty}$ in $V \times \mathscr{C}'$ ($j < k \leq m_\mu - 1$). (ii) We multiply the j-th column on the right by an operator in $\mathscr{L}_{1,0}^{j,-j}$ and add it to the m_μ-th column to annihilate the (j, m_μ)-th entry modulo $\mathscr{L}_{1,0}^{j,-\infty}$ in $V \times \mathscr{C}'$. (iii) We multiply the k-th row on the left by an operator in $\mathscr{L}_{1,0}^0$ and add it to the $(j + 1)$-th row to eliminate terms containing D_1 in the $(j + 1, k - 1)$-th entry ($j + 2 \leq k \leq m_\mu$). As a result we

have a matrix-valued operator $\widetilde{\mathscr{L}}^\mu(x, D)$ satisfying the following:

(1.2)
$$\widetilde{\mathscr{L}}^\mu(x, D) \equiv \left(\widetilde{\mathscr{L}}^\mu_{j,k}(x, D)\right)$$
$$= \begin{pmatrix} D_n + \tilde{l}^\mu_1(x, D') & & & 0 \\ & \ddots & & \\ & & D_n + \tilde{l}^\mu_{m_\mu - 1}(x, D') & \\ * & & & \tilde{l}^\mu(x, D) \end{pmatrix} + \widetilde{\mathscr{R}}^\mu(x, D)$$

in $V \times \mathscr{C}'$, $\tilde{l}^\mu_j(x, \xi') \in \mathscr{S}^{0,1}_{1,0}$, $\tilde{l}^{\mu,0}_j(x^0, \xi^{0\prime}) = 0$ and $\widetilde{\mathscr{L}}^\mu_{j+1,j}(x, \xi) \in \mathscr{S}^1_{1,0}$ $(1 \le j \le m_\mu - 1)$, $\widetilde{\mathscr{L}}^\mu_{j,k}(x, \xi) \equiv \widetilde{\mathscr{L}}^\mu_{j,k}(x, \xi') \in \mathscr{S}^{0,1}_{1,0}$ $(3 \le j \le m_\mu, \ 1 \le k \le j - 2)$, $\tilde{l}^\mu(x, \xi) \in \mathscr{S}^{m_\mu, -m_\mu + 1}_{1,0}$, $\widetilde{\mathscr{R}}^\mu_{j,k}(x, \xi) \in \mathscr{S}^{0,-\infty}_{1,0}$ if $k \le m_\mu - 1$, and $\widetilde{\mathscr{R}}^\mu_{j,m_\mu}(x, \xi) \in \mathscr{S}^{m_\mu, -\infty}_{1,0}$, where $\tilde{l}^{\mu,0}_j(x, \xi')$ denotes the principal symbol of $\tilde{l}^\mu_j(x, \xi')$ and $\widetilde{\mathscr{R}}^\mu(x, \xi) = \left(\widetilde{\mathscr{R}}^\mu_{j,k}(x, \xi)\right)$. Here $A(x, D) = B(x, D)$ in $V \times \mathscr{C}'$ means that $A(x, \xi) = B(x, \xi)$ for $(x, \xi') \in V \times \mathscr{C}'$. Next we multiply $\widetilde{\mathscr{L}}^\mu(x, D)$ on the right by $\operatorname{diag}\left(k^\mu_1(x, D'), \ldots, k^\mu_{m_\mu - 1}(x, D'), k^\mu(x, D')\right)$, where $k^\mu_j(x, D')$ is a parametrix of $D_n + \tilde{l}^\mu_j(x, D')$ in $V \times \mathscr{C}'$ $(1 \le j \le m_\mu - 1)$ and $k^\mu(x, D') = (-1)^{m_\mu - 1}\left(D_n + \tilde{l}^\mu_1(x, D')\right) \cdots \left(D_n + \tilde{l}^\mu_{m_\mu - 1}(x, D')\right)$. Finally we perform elementary transformations for the rows to annihilate the off-diagonal entries in $V \times \mathscr{C}'$, and we have a matrix-valued operator $\mathscr{L}^\mu(x, D) \in M_{m_\mu}\left(\mathscr{L}^{m_\mu}_{1,0}\right)$ satisfying the following:

$$\mathscr{L}^\mu(x, \xi) = \operatorname{diag}(1, \ldots, 1, l^\mu(x, \xi)) + \mathscr{R}^\mu(x, \xi) \quad \text{in} \quad V \times \mathscr{C}',$$

(1.3)
$$l^\mu(x, \xi) = (\xi_1 + \lambda_\mu \xi_n)^{m_\mu} + \sum_{k=1}^{m_\mu} l^\mu_k(x, \xi')(\xi_1 + \lambda_\mu \xi_n)^{m_\mu - k} \in \mathscr{S}^{m_\mu}_{1,0},$$

$$l^{\mu,0}(x^0, \xi_1, 0, \ldots, 0, \xi_n) = (\xi_1 + \lambda_\mu \xi_n)^{m_\mu},$$

$\mathscr{R}^\mu_{j,k}(x, \xi) \in \mathscr{S}^{j-1,-\infty}_{1,0}$ if $k \le m_\mu - 1$, and $\mathscr{R}^\mu_{j,m_\mu}(x, \xi) \in \mathscr{S}^{m_\mu + j - 1, -\infty}_{1,0}$, where $\mathscr{R}^\mu(x, \xi) = \left(\mathscr{R}^\mu_{j,k}(x, \xi)\right)$ and $l^{\mu,0}(x, \xi)$ denotes the principal symbol of $l^\mu(x, \xi)$ (see Lemma 2.4 below). We write $V = V(x^0, \xi^{0\prime})$ and $\mathscr{C}' = \mathscr{C}'(x^0, \xi^{0\prime})$ since V and $\mathscr{C}'$ depend on x^0 and $\xi^{0\prime}$. Note that $l^{\mu,0}(x, \xi) \equiv l^{\mu,0}(\xi)$ and $p(\xi) = l^{1,0}(\xi) \cdots l^{r,0}(\xi)$ in $V \times \mathscr{C}'$. We define

$$Q(x, \xi; x^0, \xi^{0\prime}) = l^1(x, \xi) \cdots l^r(x, \xi) - p(\xi)$$

for $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ and $(x, \xi) \in V(x^0, \xi^{0\prime}) \times \mathbb{R} \times \mathscr{C}'(x^0, \xi^{0\prime})$, and impose the following condition (L) for sufficiency of C^∞ well-posedness:
(L) For each $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ there is $C > 0$ such that

(1.4)
$$|Q(x, \xi; x^0, \xi^{0\prime})| \le C|p(\xi - i\vartheta)|$$

for $x \in V(x^0, \xi^{0\prime})$ with $x_1 \ge 0$ and $\xi \in \mathbb{R} \times \mathscr{C}'(x^0, \xi^{0\prime})$.

The condition (L) is equivalent to the following condition (L)$'$:
(L)$'$ For each $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ and $x \in V(x^0, \xi^{0\prime})$ with $x_1 \geq 0$ there is $C > 0$ such that

$$|Q(x, \xi; x^0, \xi^{0\prime})| \leq C|p(\xi - i\vartheta)| \quad \text{for} \quad \xi \in \mathbb{R} \times \mathscr{C}'(x^0, \xi^{0\prime}).$$

Indeed, it is obvious that the condition (L) implies the condition (L)$'$. By Lemma 4.3 below the condition (L)$'$ implies the condition (L), with modifications of $V(x^0, \xi^{0\prime})$ and $\mathscr{C}'(x^0, \xi^{0\prime})$ if necessary. Note that (1.4) is always satisfied if $(dp)(\xi_1, \xi^{0\prime}) \neq 0$ for any $\xi_1 \in \mathbb{R}$. We say that the Cauchy problem (CP)$_t$ is C^∞ well-posed if the following two conditions are satisfied:
(E) For any $f \in C^\infty(\mathbb{R}^n; \mathbb{C}^m)$ with supp $f \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$ there is $u \in C^\infty(\mathbb{R}^n; \mathbb{C}^m)$ satisfying (CP)$_t$.
(U) If $s > t$, $u \in C^\infty(\mathbb{R}^n; \mathbb{C}^m)$, supp $u \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$ and supp $L(x, D)u \subset \{x \in \mathbb{R}^n; x_1 \geq s\}$, then supp $u \subset \{x \in \mathbb{R}^n; x_1 \geq s\}$.

Theorem 1.1. *Assume that the conditions* (H) *and* (R) *are satisfied.*
(i) *We assume that the condition* (L) *is satisfied. Let* $t > 0$, *and let* $f \in \mathscr{D}'$ *satisfy* supp $f \subset \{x \in \mathbb{R}^n; x_1 \geq t\}$. *Then the Cauchy problem* (CP)$_t$ *has a solution* $u \in \mathscr{D}'$. *If* $x^0 \in \mathbb{R}^n$, $u \in \mathscr{D}'$ *satisfies* (CP)$_t$ *and* $f = 0$ *near* $\{x^0\} - \Gamma(p, \vartheta)^*$, *then* $x^0 \notin$ supp u, *where* $\Gamma(p, \vartheta)$ *denotes the connected component of the set* $\{\xi \in \mathbb{R}^n; p(\xi) \neq 0\}$ *which contains* ϑ, *and* $\Gamma^* = \left\{x \in \mathbb{R}^n; x \cdot \xi \ (\equiv \sum_{j=1}^n x_j\xi_j) \geq 0 \text{ for any } \xi \in \Gamma\right\}$. *Moreover, if* $u \in \mathscr{D}'$ *satisfies* (CP)$_t$ *and* $f \in C^\infty(\mathbb{R}^n; \mathbb{C}^m)$, *then* $u \in C^\infty(\mathbb{R}^n; \mathbb{C}^m)$, *i.e., the Cauchy problem* (CP)$_t$ *is* C^∞ *well-posed.*
(ii) *If the Cauchy problem* (CP)$_t$ *is* C^∞ *well-posed for any* $t > 0$, *then the condition* (L) *is satisfied.*

REMARK.   (i)   It is possible to reduce the operator $L(x, D)$ to the operator of the form (2.6) below in different ways from the proof of Lemma 2.4. Then Theorem 1.1 is still valid.
(ii) Under the conditions (H) and (R) the condition (L) is a necessary and sufficient condition for (CP)$_t$ to be C^∞ well-posed for any $t > 0$. Therefore, the condition (L) does not depend on the reduction procedure.
(iii) The Cauchy problem for a single higher order operator $P(x, D)$ with constant coefficient hyperbolic principal part is C^∞ well-posed (in $\mathbb{R}^n$) if and only if $P(x, \xi)$ is hyperbolic with respect to ϑ for each $x \in \mathbb{R}^n$ (see [5] and [26]), which is equivalent to the condition that for every $x \in \mathbb{R}^n$ there is $C > 0$ satisfying $|P(x, \xi)|/|p(\xi - i\vartheta)| \leq C$ for $\xi \in \mathbb{R}^n$, where $p(\xi)$ is the principal part of $P(x, \xi)$ (see Svensson [23]).

The remainder of this paper is organized as follows. In §2 we shall reduce $L(x, D)$ to an operator diag$(1, \ldots, 1, l^1(x, D), \ldots, l^r(x, D))$ in a microlocal sense, by multiplying $L(x, D)$ on the both sides by invertible matrix-valued operators in $M_m(\mathscr{L}_{1,0}^{m-1})$. In

the proof of the assertion (ii) of Theorem 1.1 we shall use the Tarski-Seidenberg theorem. So we have to prove that the Tarski-Seidenberg theorem can be applicable to the $l^\mu(x, \xi)$. In §3 we shall prove the assertion (i) of Theorem 1.1, applying the results in [14]. In doing so, some simple modifications are necessary. The assertion (ii) of Theorem 1.1 will be proved in §4, applying the arguments in Ivrii-Petkov [10] and [26]. We shall modify the arguments in [9] to remove the assumption on the finite propagation property (see, also, [16]). Some examples and remarks will be given in §5.

2. Reduction to a simple form

We begin with several definitions.

DEFINITION 2.1.   (i) The subset $\mathscr{A}$ of $\mathbb{R}^N$ is said to be a semi-algebraic set if $\mathscr{A}$ is a finite union of finite intersections of sets defined by a real polynomial equation or inequality.

(ii) Let V be an open subset of $\mathbb{R}^n$ and $\mathscr{C}'$ be an open conic semi-algebraic subset of $\mathbb{R}^{n-1} \setminus \{0\}$, and let $f(x, \xi)$ be a symbol in $C^\infty(V \times \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}')))$ such that $f(x, \lambda\xi) = \lambda^l f(x, \xi)$ for $(x, \xi) \in V \times \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}'))$ and $\lambda \in \mathbb{R} \setminus \{0\}$, where $l \in \mathbb{Z}$ and $-\mathscr{C}' = \{-\xi' \in \mathbb{R}^{n-1}; \xi' \in \mathscr{C}'\}$. We say that $f(x, \xi)$ is an SA-symbol in $V \times \mathscr{C}'$ if there are $N \in \mathbb{N}$, a semi-algebraic set $\mathscr{A}$ in $\mathbb{R}^{N+n-1}$ and a polynomial $F(x, \xi, \lambda)$ of (ξ, λ) $(\equiv (\xi, \lambda_1, \ldots, \lambda_N))$ and a polynomial $G(x, \xi', \lambda)$ of (ξ', λ), whose coefficients belong to $C^\infty(V)$, satisfying the following:

(1) The projection $\pi \equiv \pi_{N,n-1} \colon \mathscr{A} \ni (\xi', \lambda) \mapsto \xi' \in \mathbb{R}^{n-1}$ is injective, $\pi(\mathscr{A}) = \mathscr{C}' \cup (-\mathscr{C}')$, and the $\Lambda_j(\xi')$ are real analytic and homogeneous in $\mathscr{C}' \cup (-\mathscr{C}')$, where $\pi^{-1}(\xi') = (\xi', \Lambda_1(\xi'), \ldots, \Lambda_N(\xi'))$.

(2) $F(x, \xi_1, \pi^{-1}(\xi'))$ and $G(x, \pi^{-1}(\xi'))$ are homogeneous in $\xi \in \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}'))$ and $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$, respectively.

(3) $G(x, \pi^{-1}(\xi')) \neq 0$ and $f(x, \xi) = G(x, \pi^{-1}(\xi'))^{-1} F(x, \xi_1, \pi^{-1}(\xi'))$ for $x \in V$ and $\xi \in \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}'))$.

(iii) Let V be an open subset of $\mathbb{R}^n$ and $\mathscr{C}'$ be an open conic semi-algebraic subset of $\mathbb{R}^{n-1} \setminus \{0\}$, and let $f(x, \xi)$ be a symbol in $\mathscr{S}_{1,0}^{k,l}$ which has an asymptotic expansion of the form $f(x, \xi) \sim \sum_{j=0}^\infty f^j(x, \xi)$, where $k \in \mathbb{Z}_+$, $l \in \mathbb{Z}$ and $f^j(x, \xi)$ is positively homogeneous of degree $k + l - j$ $(j \in \mathbb{Z}_+)$. We say that $f(x, \xi)$ is an S-symbol in $V \times \mathscr{C}'$ if for each $\alpha, \beta \in (\mathbb{Z}_+)^n$ and $j \in \mathbb{Z}_+ f_{(\beta)}^{j(\alpha)}(x, \xi)$ is an SA-symbol in $V \times \mathscr{C}'$.

Let $x^0 \in \mathbb{R}^n$ and $\xi^{0'} = (\xi_2^0, \ldots, \xi_n^0) \in S^{n-2}$. By a linear coordinate transformation of x' we may assume that $\xi^{0'} = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}$. Write

$$p(\xi^0 + \lambda \vartheta) \ (\equiv p(\lambda, \xi^{0'}))$$

$$= \prod_{j=1}^r (\lambda + \lambda_j)^{m_j}, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_r$$

as in §1, and

$$p(\lambda, \xi') = \prod_{j=1}^{m} (\lambda + \tau_j(\xi')), \quad \tau_1(\xi) \le \tau_2(\xi') \le \cdots \le \tau_m(\xi').$$

We choose $\delta_j > 0$ $(1 \le j \le r)$ and an open neighborhood U' of $\xi^{0\prime}$ so that $\{\lambda \in \mathbb{C}; |\lambda + \lambda_j| \le \delta_j\}$ $(1 \le j \le r)$ are mutually disjoint and $\tau_k(\xi') \in \{\lambda \in \mathbb{R}; |\lambda + \lambda_j| \le \delta_j/2\}$ for $1 \le j \le r$, $m_1 + \cdots + m_{j-1} + 1 \le k \le m_1 + \cdots + m_j$ and $\xi' \in U'$. Put

$$P_j(x, \xi') = (2\pi i)^{-1} \oint_{|\lambda + \lambda_j| = \delta_j} L_1(x, \lambda, \xi')^{-1} \, d\lambda$$

for $(x, \xi') \in \mathbb{R}^n \times U'$ and $1 \le j \le r$. Then we have rank $P_j(\xi') = m_j$ and we can choose an open neighborhood V of x^0 and m_j column vectors $P_j^1(x, \xi'), \ldots, P_j^{m_j}(x, \xi')$ of $P_j(x, \xi')$ so that $P_j^k(x, \xi')$ $(1 \le k \le m_j)$ are linearly independent for each $(x, \xi') \in V \times U'$, modifying U' if necessary. We put

$$S(x, \xi') = \left(P_1^1(x, \xi'), \ldots, P_1^{m_1}(x, \xi'), P_2^1(x, \xi'), \ldots, P_r^{m_r}(x, \xi') \right).$$

Then $S(x, \xi')$ can be defined for $(x, \xi') \in \mathbb{R}^n \times (\mathscr{C}' \cup (-\mathscr{C}'))$ by homogeneity, i.e., $S(x, \xi') = S(x, \pm\xi'/|\xi'|)$ if $(x, \xi') \in \mathbb{R}^n \times (\pm\mathscr{C}')$, where $\mathscr{C}' = \{\lambda\xi' \in \mathbb{R}^{n-1}; \xi' \in U', |\xi'| = 1$ and $\lambda > 0\}$. We may assume that $\mathscr{C}'$ is semi-algebraic. Since

$$R_{j,k,l}(\xi') \equiv (2\pi i)^{-1} \oint_{|\lambda + \lambda_j| = \delta_j} \frac{\lambda^k}{p(\lambda, \xi')^l} \, d\lambda$$

is a rational function of fundamental symmetric functions of $\{\tau_{m_1+\cdots+m_{\mu-1}+\nu}(\xi')\}_{\nu=1,\ldots,m_\mu}$ $(1 \le \mu \le r)$ for $1 \le j \le r$, $k \in \mathbb{Z}_+$ and $l \in \mathbb{N}$, the entries of $S_{(\beta)}^{(\alpha)}(x, \xi')$ are SA-symbols in $V \times \mathscr{C}'$ for $\alpha, \beta \in (\mathbb{Z}_+)^n$. Indeed, for $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$ the $R_{j,k,l}(\xi')$ are analytic and expressed as rational functions of $(\zeta_1, \ldots, \zeta_m)$, where $(\zeta_1, \ldots, \zeta_m)$ is defined by the following system of real polynomial equations and inequalities for $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$:

$$\tau_1 \le \tau_2 \le \cdots \le \tau_m, \quad p(\xi) = \prod_{j=1}^{m} (\xi_1 + \tau_j),$$

$$\prod_{j=1}^{m} (\xi_1 + \tau_{m_1+\cdots+m_{\mu-1}+j}) = \xi_1^{m_\mu} + \sum_{j=1}^{m_\mu} \zeta_{m_1+\cdots+m_{\mu-1}+j} \xi_1^{m_\mu - j} \quad (1 \le \mu \le r).$$

Therefore, we have the following

Lemma 2.2. *There are an open neighborhood V of x^0 in $\mathbb{R}^n$, an open conic semi-algebaraic subset $\mathscr{C}'$ of $\mathbb{R}^{n-1} \setminus \{0\}$, $S(x, \xi') \in C^\infty(\mathbb{R}^n \times (\mathscr{C}' \cup (-\mathscr{C}')); M_m(\mathbb{C}))$ and*

$A^j(x, \xi') \in C^\infty(V \times \mathscr{C}'; M_{m_j}(\mathbb{C}))$ $(1 \le j \le r)$ *such that* $S(x, \xi')$ *and* $A^j(x, \xi')$ *are homogeneous of degree* 0 *and* 1, *respectively,* $\det S(x, \xi') \ne 0$ *for* $(x, \xi') \in V \times \mathscr{C}'$, *the entries of* $S(x, \xi')$ *and the* $A^j(x, \xi')$ *are SA-symbols in* $V \times \mathscr{C}'$, $(A^j(x, \xi^{0\prime}) - \lambda_j I_{m_j})^{m_j} = 0$ *and*

$$S(x, \xi')^{-1} A(x, \xi') S(x, \xi') = \mathrm{diag}(A^1(x, \xi'), \dots, A^r(x, \xi'))$$

for $(x, \xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$.

Modifying V and $\mathscr{C}'$, if necessary, we can construct $S_\pm(x, \xi') \in M_m\big(\mathscr{S}_{1,0}^0\big)$ so that $S_+(x, \xi') = S(x, \xi')$ for $(x, \xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ with $|\xi'| \ge 1$ and $S_-(x, D') \times S_+(x, D') \equiv I \bmod \mathscr{L}_{1,0}^{0,-\infty}$ in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$. It is easy to see that the entries of $S_\pm(x, \xi')$ are S-symbols in $V \times \mathscr{C}'$. We can assume without loss of generality that $A_2(x), \dots,$ $A_n(x), L_0(x) \in \mathscr{B}(\mathbb{R}^n; M_m(\mathbb{C}))$. Although the following lemma is well-known, for completeness we give the proof (see, e.g., [18]).

Lemma 2.3. *There are symbols* $Q(x, \xi') \in M_m\big(\mathscr{S}_{1,0}^{0,-1}\big)$ *and* $\widetilde{A}^\mu(x, \xi') \in M_{m_\mu}\big(\mathscr{S}_{1,0}^{0,1}\big)$ *and* $C^\mu(x, \xi') \in M_{m_\mu}\big(\mathscr{S}_{1,0}^0\big)$ $(1 \le \mu \le r)$ *such that the entries of* $Q(x, \xi')$, *the* $\widetilde{A}^\mu(x, \xi')$ *and the* $C^\mu(x, \xi')$ *are S-symbols in* $V \times \mathscr{C}'$ *and*

(2.1)

$$\widetilde{A}^\mu(x, \xi') = A^\mu(x, \xi') \quad for \quad (x, \xi') \in V \times \mathscr{C}' \quad with \quad |\xi'| \ge 1 \quad and \quad 1 \le \mu \le r,$$

$$S_-(x, D') L(x, D) S_+(x, D')(I + Q(x, D'))$$

$$\equiv (I + Q(x, D')) \times \big\{ D_1 I + \mathrm{diag}\big(\widetilde{A}^1(x, D') + C^1(x, D'), \dots, \widetilde{A}^r(x, D') + C^r(x, D')\big) \big\}$$

$$\bmod \mathscr{L}_{1,0}^{1,-\infty} \qquad in \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')).$$

Proof. Write

(2.2) $$Q(x, \xi') \sim \sum_{j=1}^{\infty} Q^j(x, \xi'), \quad C^\mu(x, \xi') \sim \sum_{j=0}^{\infty} C^{\mu, j}(x, \xi')$$

(asymptotic expansions), where $Q^j(x, \xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_m(\mathbb{C})))$ and $C^{\mu, j}(x, \xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_{m_\mu}(\mathbb{C})))$ are positively homogeneous of degree $-j$ and $T^*\mathbb{R}^{n-1} \setminus 0 \simeq \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus \{0\})$. We also write

$$\widetilde{A}(x, D') = S_-(x, D') A(x, D') S_+(x, D') \in \mathscr{L}_{1,0}^{0,1},$$

$$\widetilde{A}(x, \xi') \sim \sum_{j=-1}^{\infty} \widetilde{A}_j(x, \xi') \quad \text{(asymptotic expansion)},$$

where $\widetilde{A}_j(x, \xi')$ $(\in C^\infty(\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0); M_m(\mathbb{C})))$ is positively homogeneous of degree $-j$. By Lemma 2.2 we have

$$\widetilde{A}_{-1}(x, \xi') = \mathrm{diag}(A^1(x, \xi'), \dots, A^r(x, \xi'))$$

for $(x, \xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$. Choose $\widetilde{A}^\mu(x, \xi') \in M_{m_\mu}\left(\mathscr{S}_{1,0}^{0,1}\right)$ $(1 \le \mu \le r)$ so that the $\widetilde{A}^\mu(x, \xi')$ are positively homogeneous for $|\xi'| \ge 1$ and

$$\operatorname{diag}\left(\widetilde{A}^1(x, \xi'), \ldots, \widetilde{A}^r(x, \xi')\right) = \widetilde{A}_{-1}(x, \xi')$$

for $(x, \xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ with $|\xi'| \ge 1$. We put

$$\begin{aligned}
B(x, D') = {}& S_-(x, D')S_{+(e_1)}(x, D') + \widetilde{A}(x, D') \\
& - \operatorname{diag}\left(\widetilde{A}^1(x, D'), \ldots, \widetilde{A}^r(x, D')\right) + S_-(x, D')L_0(x)S_+(x, D').
\end{aligned}$$

and write

$$B(x, \xi') \sim \sum_{j=-1}^{\infty} B^j(x, \xi'),$$

where $e_1 = (1, 0, \ldots, 0) \in (\mathbb{Z}_+)^n$ and $B^j(x, \xi')$ is positively homogeneous of degree $-j$ $(j \in \mathbb{Z}_+)$. It is easy to see that $B^{-1}(x, \xi') = 0$ in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ and that the entries of $B(x, \xi')$ are S-symbols in $V \times \mathscr{C}'$. If (2.1) holds, then $\{Q^j(x, \xi')\}$ and $\{C^{\mu,j}(x, \xi')\}$ satisfy

$$
\begin{aligned}
(2.3) \quad & B^l(x, \xi') + Q^l_{(e_1)}(x, \xi') \\
& + \sum_{\substack{j+|\alpha'|=l+1 \\ j \ge 1}} \frac{1}{\alpha'!} \operatorname{diag}(A^{1(\alpha')}(x, \xi'), \ldots, A^{r(\alpha')}(x, \xi'))Q^j_{(\alpha')}(x, \xi') \\
& + \sum_{\substack{j+k+|\alpha'|=l \\ j \ge 1, k \ge 0}} \frac{1}{\alpha'!} B^{k(\alpha')}(x, \xi')Q^j_{(\alpha')}(x, \xi') \\
& = \operatorname{diag}(C^{1,l}(x, \xi'), \ldots, C^{r,l}(x, \xi')) \\
& + \sum_{\substack{j+|\alpha'|=l+1 \\ j \ge 1}} \frac{1}{\alpha'!} Q^{j(\alpha')}(x, \xi') \operatorname{diag}\left(A^1_{(\alpha')}(x, \xi'), \ldots, A^r_{(\alpha')}(x, \xi')\right) \\
& + \sum_{\substack{j+k+|\alpha'|=l \\ j \ge 1, k \ge 0}} \frac{1}{\alpha'!} Q^{j(\alpha')}(x, \xi') \operatorname{diag}\left(C^{1,k}_{(\alpha')}(x, \xi'), \ldots, C^{r,k}_{(\alpha')}(x, \xi')\right)
\end{aligned}
$$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ $(l = 0, 1, 2, \ldots)$, where $Q^0(x, \xi') \equiv 0$. If $A_{\mu,\nu}$ is an $m_\mu \times m_\nu$ matrix $(1 \le \mu, \nu \le r)$ and $A = (A_{\mu,\nu})_{\substack{\mu \downarrow 1, \ldots, r \\ \nu \to 1, \ldots, r}}$, we define $\mathbb{B}_{\mu,\nu}(A) = A_{\mu,\nu}$. We choose $Q(x, \xi')$

so that $\mathbb{B}_{\mu,\mu}(Q(x, \xi')) = 0$ for $1 \leq \mu \leq r$. Then it follows from (2.3) that

$$C^{\mu,l}(x, \xi') = \mathbb{B}_{\mu,\mu}(B^l(x, \xi'))$$

(2.4)
$$+ \sum_{\substack{j+k+|\alpha'|=l \\ j\geq 1,\, k\geq 0}} \sum_{\gamma=1}^{r} \frac{1}{\alpha'!} \mathbb{B}_{\mu,\gamma}(B^{k(\alpha')}(x, \xi'))\mathbb{B}_{\gamma,\mu}\big(Q^j_{(\alpha')}(x, \xi')\big),$$

$$\mathbb{B}_{\mu,\nu}(Q^{l+1}(x, \xi'))A^\nu(x, \xi') - A^\mu(x, \xi')\mathbb{B}_{\mu,\nu}(Q^{l+1}(x, \xi'))$$

$$= \mathbb{B}_{\mu,\nu}(B^l(x, \xi')) + \mathbb{B}_{\mu,\nu}\big(Q^l_{(e_1)}(x, \xi')\big)$$

$$+ \sum_{\substack{j+|\alpha'|=l+1 \\ 1\leq j\leq l}} \frac{1}{\alpha'!} A^{\mu(\alpha')}(x, \xi')\mathbb{B}_{\mu,\nu}\big(Q^j_{(\alpha')}(x, \xi')\big)$$

(2.5)
$$+ \sum_{\substack{j+k+|\alpha'|=l \\ j\geq 1,\, k\geq 0}} \sum_{\gamma=1}^{r} \frac{1}{\alpha'!} \mathbb{B}_{\mu,\gamma}(B^{k(\alpha')}(x, \xi'))\mathbb{B}_{\gamma,\nu}\big(Q^j_{(\alpha')}(x, \xi')\big)$$

$$- \sum_{\substack{j+|\alpha'|=l+1 \\ 1\leq j\leq l}} \frac{1}{\alpha'!} \mathbb{B}_{\mu,\nu}(Q^{j(\alpha')}(x, \xi'))A^\nu_{(\alpha')}(x, \xi')$$

$$- \sum_{\substack{j+k+|\alpha'|=l \\ j\geq 1,\, k\geq 0}} \frac{1}{\alpha'!} \mathbb{B}_{\mu,\nu}(Q^{j(\alpha')}(x, \xi'))C^{\nu,k}_{(\alpha')}(x, \xi')$$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ for $l = 0, 1, 2, \ldots$ and $1 \leq \mu, \nu \leq r$ with $\nu \neq \mu$. Let $A \equiv (a_{i,k}) \in M_{n_1, n_2}(\mathbb{C})$ and $B \equiv (b_{j,l}) \in M_{l_1, l_2}(\mathbb{C})$. Then the Kronecker product $A \otimes B \equiv (c_{\lambda,\mu}) \in M_{n_1 l_1, n_2 l_2}(\mathbb{C})$ is defined by $c_{\lambda,\mu} = a_{i,k}b_{j,l}$ for $\lambda = (i, j)$ and $\mu = (k, l)$, where $\lambda \in \{(i, j) \in \mathbb{N}^2; 1 \leq i \leq n_1$ and $1 \leq j \leq l_1\}$ and $\mu \in \{(k, l) \in \mathbb{N}^2; 1 \leq k \leq n_2$ and $1 \leq l \leq l_2\}$ are properly arranged. Assume that $n_1 = n_2$, $l_1 = l_2$ and the repeated eigenvalues of A and B are $\{\lambda_j\}_{1 \leq j \leq n_1}$ and $\{\tau_k\}_{1 \leq k \leq l_1}$, respectively. Then the eigenvalues of $I_{l_1} \otimes {}^tA - B \otimes I_{n_1} \equiv (h_{\lambda,\mu})$ are $\{\lambda_j - \tau_k\}_{1 \leq j \leq n_1, 1 \leq k \leq l_1}$. For $X = (x_{i,j})$, $F = (f_{i,j}) \in M_{l_1, n_1}(\mathbb{C})$ the equation $XA - BX = F$ is equivalent to $\sum_\mu h_{\lambda,\mu}x_\mu = f_\lambda$ for $\lambda = (i, j)$ with $1 \leq i \leq l_1$ and $1 \leq j \leq n_1$. In particular, the equation is uniquely solvable in X if $\lambda_j \neq \tau_k$ for $1 \leq j \leq n_1$ and $1 \leq k \leq l_1$. The eigenvalues of $A^\nu(x, \xi')$ are different from those of $A^\mu(x, \xi')$ for $\mu \neq \nu$ and $(x, \xi') \in V \times \mathscr{C}'$. Therefore, by (2.4) and (2.5) we can determine $\{Q^j(x, \xi')\}$ and $\{C^{\mu,j}(x, \xi')\}$ in $V \times \mathscr{C}'$, inductively. Modifying the symbols outside $\{(x, \xi') \in V \times (\mathscr{C}' \cup (-\mathscr{C}')); |\xi'| \geq 1\}$, we can choose $Q(x, \xi')$ and $C^\mu(x, \xi')$ $(1 \leq \mu \leq r)$ so that (2.1) and (2.2) are valid. It is easy to see that the entries of $Q(x, \xi')$ and the $C^\mu(x, xi')$ are S-symbols in $V \times \mathscr{C}'$. $\quad\square$

As stated in §1, using elementary transformations we have the following

Lemma 2.4. *With modifications of V and $\mathscr{C}'$ if necessary, there are symbols* $N_1(x, \xi) \in M_m\big(\mathscr{S}^{m-1,-m+1}_{1,0}\big)$, $N_2(x, \xi) \equiv (N_{2,j,k}(x, \xi)) \in M_m\big(\mathscr{S}^{m-1}_{1,0}\big)$ *and* $l^j(x, \xi) \in \mathscr{S}^{m_j}_{1,0}$

$(1 \leq j \leq r)$ *satisfying the following*:
(i) $N_{2,j,k}(x,\xi) \equiv N_{2,j,k}(x,\xi') \in \mathscr{S}_{1,0}^{0,-1}$ *for* $1 \leq k \leq m-r$, $N_{2,j,j}(x,\xi) \equiv N_{2,j,j}(x,\xi')$ *for*
$m - r < j \leq m$, $N_{2,j,m-r+k}(x,\xi) \in \mathscr{S}_{1,0}^{m_k-1}$ *for* $1 \leq k \leq r$, *and* $N_1(x,D)$ *and* $N_2(x,D)$
have parametrices in $M_m\big(\mathscr{L}_{1,0}^{m-1}\big)$.
(ii) $l^j(x,\xi)$ *can be written in the form*

$$l^j(x,\xi) = (\xi_1 + \lambda_j \xi_n)^{m_j} + \sum_{k=1}^{m_j} l_k^j(x,\xi')(\xi_1 + \lambda_j \xi_n)^{m_j-k},$$

$l_k^j(x,\xi') \in \mathscr{S}_{1,0}^{0,k}$ $(1 \leq j \leq r,\ 1 \leq k \leq m_j)$, *The* $l_k^j(x,\xi')$ *are S-symbols in* $V \times \mathscr{C}'$,
$l_k^{j,0}(x,\xi') \equiv l_k^{j,0}(\xi')$ *and* $l_k^{j,0}(\xi^{0\prime}) = 0$, *where* $l_k^{j,0}(x,\xi')$ *denotes the principal symbol of*
$l_k^j(x,\xi')$.
(iii)

(2.6) $N_1(x,D)L(x,D)N_2(x,D) = \mathrm{diag}(1,\ldots,1,l^1(x,D),\ldots,l^r(x,D)) + R(x,D)$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, *where* $R(x,\xi) \in M_m\big(\mathscr{S}_{1,0}^{2m-1,-\infty}\big)$.

   Proof.   We fix $\mu \in \{1,2,\ldots,r\}$. As stated in §1, there are symbols $\widetilde{N}_\nu(x,\xi) \equiv$
$\big(\widetilde{N}_{\nu,j,k}(x,\xi)\big) \in M_{m_\mu}\big(\mathscr{S}_{1,0}^{m_\mu-1}\big)$ $(\nu = 1,2)$ satisfying the following: (i) $\widetilde{N}_{\nu,j,j}(x,\xi) = 1$,
$\widetilde{N}_{\nu,j,k}(x,\xi) = 0$ $(j > k)$, $\widetilde{N}_{1,j,k}(x,\xi) \equiv \widetilde{N}_{1,j,k}(x,\xi') \in \mathscr{S}_{1,0}^0$ $(j < k \leq m_\mu)$, $\widetilde{N}_{2,j,k}(x,\xi) \equiv$
$\widetilde{N}_{2,j,k}(x,\xi') \in \mathscr{S}_{1,0}^0$ for $j < k < m_\mu$ and $\widetilde{N}_{2,j,m_\mu}(x,\xi) \in \mathscr{S}_{1,0}^{m_\mu-1,-m_\mu+1}$. (ii) $\widetilde{N}_1(x,D)$
$(= \widetilde{N}_1(x,D'))$ and $\widetilde{N}_2(x,D)$ have the inverses in $M_{m_\mu}\big(\mathscr{L}_{1,0}^0\big)$ and $M_{m_\mu}\big(\mathscr{L}_{1,0}^{m_\mu-1}\big)$, respec-
tively. (iii)

$$\widetilde{N}_1(x,D)L^\mu(x,D)\widetilde{N}_2(x,D) = \widetilde{\mathscr{L}}^\mu(x,D) \quad \text{in} \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')),$$

where $L^\mu(x,\xi)$ is as in (1.1) and $\widetilde{\mathscr{L}}^\mu(x,\xi) \equiv \big(\widetilde{\mathscr{L}}_{j,k}^\mu(x,\xi)\big)$ has the same form as in (1.2)
which is valid in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$. We note that $f(x,\xi) + g(x,\xi)$ and $f(x,\xi)g(x,\xi)$ are
S-symbols in $V \times \mathscr{C}'$ if $f(x,\xi)$ and $g(x,\xi)$ are S-symbols in $V \times \mathscr{C}'$. Let $h(x,\xi') \in \mathscr{S}_{1,0}^{0,1}$
be an S-symbol in $V \times \mathscr{C}'$ such that $h^0(x^0,\xi^{0\prime}) = 0$, where $h^0(x,\xi')$ denotes the principal
symbol of $h(x,\xi')$. Then there is a symbol $k(x,\xi') \in \mathscr{S}_{1,0}^{0,-1}$ such that $k(x,\xi')$ is an
S-symbol in $V \times \mathscr{C}'$ and $k(x,D')$ is a parametrix of $(D_n + h(x,D'))$ in $V \times \mathscr{C}'$, with
modifications of V and $\mathscr{C}'$ if necessary. Therefore, $\widetilde{N}_{\nu,j,k}(x,\xi)$ and $\widetilde{\mathscr{L}}_{j,k}^\mu(x,\xi)$ $(\nu = 1,2,$
$1 \leq j,k \leq m_\mu)$ are S-symbols in $V \times \mathscr{C}'$. It follows from the reduction procedure that

$$\tilde{l}^\mu(x,D) - (-1)^{m_\mu-1} k_{m_\mu-1}^\mu(x,D') \cdots k_2^\mu(x,D') k_1^\mu(x,D') D_1^{m_\mu} \in \mathscr{L}_{1,0}^{m_\mu-1,1}$$

in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, i.e., $\big(\tilde{l}^\mu(x,D) - (-1)^{m_\mu-1} k_{m_\mu-1}^\mu(x,D') \cdots k_1^\mu(x,D') D_1^{m_\mu}\big) \times \psi(x,D') \in$
$\mathscr{L}_{1,0}^{m_\mu-1,1}$ for any $\psi(x,\xi') \in \mathscr{S}_{1,0}^0$ with $\mathrm{supp}\,\psi \subset V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, where $k_j^\mu(x,D')$ is

a parametrix of $D_n + \tilde{l}_j^\mu(x, D')$ in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$ $(1 \leq j \leq m_\mu - 1)$. Now we multiply $\widetilde{\mathscr{L}}^\mu(x, D)$ on the right by $\mathrm{diag}\big(k_1^\mu(x, D'), \ldots, k_{m_\mu-1}^\mu(x, D'), k^\mu(x, D')\big)$ and, then, on the left by operators corresponding to elementary transformations for the rows to annihilate the off-diagonal entries in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, where $k^\mu(x, D') = (-1)^{m_\mu - 1}\big(D_n + \tilde{l}_1^\mu(x, D')\big) \cdots \big(D_n + \tilde{l}_{m_\mu-1}^\mu(x, D')\big)$. Finally we obtain $\mathscr{L}^\mu(x, D)$ of the form (1.3) in $V \times (\mathscr{C}' \cup (-\mathscr{C}'))$, i.e., there are $N_\nu^\mu(x, \xi) \equiv \big(N_{\nu,j,k}^\mu(x,\xi)\big) \in M_m\big(\mathscr{S}_{1,0}^{m_\mu-1}\big)$ $(\nu = 1, 2)$ satisfying the following: (i) $N_{2,j,j}^\mu(x, \xi) = k_j^\mu(x, \xi')$ $(1 \leq j \leq m_\mu - 1)$, $N_{2,m_\mu,m_\mu}^\mu(x, \xi) = k^\mu(x, \xi')$, $N_{2,j,k}^\mu(x, \xi) = 0$ $(j > k)$, $N_{1,j,k}^\mu(x, \xi) \in \mathscr{S}_{1,0}^{j-1,-j+1}$, $N_{2,j,k}^\mu(x, \xi) \equiv N_{2,j,k}^\mu(x, \xi') \in \mathscr{S}_{1,0}^{0,-1}$ $(k < m_\mu)$, $N_{2,j,m_\mu}^\mu(x, \xi) \in \mathscr{S}_{1,0}^{m_\mu-1}$, and the $N_{\nu,j,k}^\mu(x, \xi)$ are S-symbols in $V \times \mathscr{C}'$. (ii) $N_1^\mu(x, D)$ has the inverse in $M_{m_\mu}\big(\mathscr{L}_{1,0}^{m_\mu-1}\big)$ and $N_2^\mu(x, D)$ has a parametrix in $M_{m_\mu}\big(\mathscr{L}_{1,0}^{m_\mu-1}\big)$. (iii)

$$N_1^\mu(x, D)L^\mu(x, D)N_2^\mu(x, D) = \mathscr{L}^\mu(x, D) \quad \text{in} \quad V \times (\mathscr{C}' \cup (-\mathscr{C}')).$$

This proves the lemma. $\qquad\qquad\square$

Let $q(x, \xi)$ be an SA-symbol in $V \times \mathscr{C}'$, and assume that

$$(2.7) \qquad \sup_{\xi \in \mathbb{R} \times \mathscr{C}', |\xi'| \geq 1} \left| \frac{q(x, \xi)}{p(\xi - i\vartheta)} \right| < \infty \quad \text{for any} \quad x \in V.$$

By definition there are $N \in \mathbb{N}$, a semi-algebraic subset $\mathscr{A}$ of $\mathbb{R}^{N+n-1}$ and a polynomial $F(x, \xi, \lambda)$ of (ξ, λ) $(\equiv (\xi, \lambda_1, \ldots, \lambda_N))$ and a polynomial $G(x, \xi', \lambda)$ of (ξ', λ), whose coefficients belong to $C^\infty(V)$, satisfying the conditions (1)–(3) of Definition 2.1 (ii) with $f(x, \xi)$ replaced by $q(x, \xi)$. Write

$$F(x, \xi, \lambda) = \sum_{|\alpha|+|\hat{\beta}| \leq M} a_{\alpha,\hat{\beta}}(x)\xi^\alpha \lambda^{\hat{\beta}}, \quad a_{\alpha,\hat{\beta}}(x) \in C^\infty(V),$$

where $M \in \mathbb{Z}_+$, $\hat{\beta} = (\beta_1, \ldots, \beta_N) \in (\mathbb{Z}_+)^N$ and $\lambda^{\hat{\beta}} = \lambda_1^{\beta_1} \cdots \lambda_N^{\beta_N}$. We put

$$\mathscr{B} = \left\{ \boldsymbol{b} = (b_{\alpha,\hat{\beta}}; |\alpha| + |\hat{\beta}| \leq M) \in \mathbb{C}^{N'}; \right.$$

$$\left. \sup_{\xi \in \mathbb{R} \times \mathscr{C}', |\xi'| \geq 1} |\xi'|^{-\kappa} \left| \sum_{|\alpha|+|\hat{\beta}| \leq M} \frac{b_{\alpha,\hat{\beta}}\xi^\alpha \Lambda(\xi')^{\hat{\beta}}}{p(\xi - i\vartheta)} \right| < \infty \right\},$$

where $N' = \binom{M+N-n}{M}$, $\Lambda(\xi') = (\Lambda_1(\xi'), \ldots, \Lambda_N(\xi'))$ is as in the condition (1) of Definition 2.1 (ii) and $G(x, \xi', \Lambda(\xi'))$ is homogeneous of degree κ in $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$.

Then $\mathscr{B}$ is a subspace of $\mathbb{C}^{N'}$. Let $\{\boldsymbol{b}^j\}_{j=1,\ldots,l}$ be a basis of $\mathscr{B}$. From (2.7) we have $\boldsymbol{a}(x) \equiv \big(a_{\alpha,\hat{\beta}}(x); |\alpha| + |\hat{\beta}| \leq M\big) \in \mathscr{B}$. Therefore, there are $a_j(x) \in C^\infty(V)$ $(1 \leq j \leq l)$ such that $\boldsymbol{a}(x) = \sum_{j=1}^{l} a_j(x)\boldsymbol{b}^j$. This gives the following

Lemma 2.5. *Under the above assumptions there are* $l \in \mathbb{N}$, *SA-symbols* $q_j(\xi)$ $(1 \leq j \leq l)$ *in* $V \times \mathscr{C}'$ *(or* $\mathbb{R}^n \times \mathscr{C}'$*),* $a_j(x) \in C^\infty(V)$ $(1 \leq j \leq l)$, $\kappa \in \mathbb{Z}$ *and* $e(x,\xi') \in$ $C^\infty(V \times (\mathscr{C}' \cup (-\mathscr{C}')))$ *such that* $e(x,\xi')$ *is homogeneous of degree* κ *in* $\xi' \in \mathscr{C}' \cup (-\mathscr{C}')$, $e(x,\xi') \neq 0$ *for* $(x,\xi') \in V \times \mathscr{C}'$ *and*

$$\sup_{\xi \in \mathbb{R} \times \mathscr{C}', |\xi'| \geq 1} \left| \frac{q_j(\xi)}{p(\xi - i\vartheta)} \right| |\xi'|^{-\kappa} < \infty \quad (1 \leq j \leq l),$$

$$q(x,\xi) = \sum_{j=1}^{l} a_j(x) e(x,\xi')^{-1} q_j(\xi).$$

3. Proof of the assertion (i) of Theorem 1.1

In this section we shall prove the assertion (i) of Theorem 1.1, applying Theorem 1.4 of [14]. We use the notations in [14]. The conditions on $\mathscr{N}^1(x, D; z^{0\prime})$, $\mathscr{N}^2(x, D; z^{0\prime})$ and $R(x, D; z^{0\prime})$ in (1.3) of [14] are slightly different from the conditions satisfied by $N_1(x, D)$, $N_2(x, D)$ and $R(x, D)$ in (2.6). Therefore, we must show that Theorem 1.4 of [14] is still valid if, instead of the conditions there, $\mathscr{N}^1(x, D; z^{0\prime}) \equiv \big(\mathscr{N}^1_{j,k}(x, D; z^{0\prime})\big)$, $\mathscr{N}^2(x, D; z^{0\prime}) \equiv \big(\mathscr{N}^2_{j,k}(x, D; z^{0\prime})\big)$ and $R(x, D; z^{0\prime}) \equiv (R_{j,k}(x, D; z^{0\prime}))$ satisfy the following:

(N-1) There is $m', m'', \bar{m} \in \mathbb{Z}_+$ and $\mathscr{M}(x, \xi; z^{0\prime}) \equiv (\mathscr{M}_{j,k}(x, \xi; z^{0\prime}))$ such that $\mathscr{N}^1_{j,k}(x, \xi; z^{0\prime}) \in \mathscr{S}^{l_{0,j} - m_k + m', -m'}_{1,0}$, $\mathscr{M}_{j,k}(x, \xi; z^{0\prime}) \in \mathscr{S}^{m_j - l_{0,k} + m'', -m''}_{1,0}$ and $\mathscr{M}(x, D; z^{0\prime})$ is a microlocal parametrix of $\mathscr{N}^1(x, D; z^{0\prime})$ $(\mathrm{mod}\ \mathscr{L}^{\bar{m}, -\infty}_{1,0})$ in a conic neighborhood $\mathscr{C}'(z^{0\prime})$ of $z^{0\prime}$.

(N-2) $\mathscr{N}^2_{j,k}(x, \xi; z^{0\prime}) \in \mathscr{S}^{-n_j - l_{v,k} + m', -m'}_{1,0}$ and $R(x, \xi; z^{0\prime}) \in \mathscr{S}^{l_{0,j} - l_{v,k} + m', -\infty}_{1,0}$.

In particular, (N-1) implies that the parametrix $\mathscr{M}(x, D; z^{0\prime})$ of $\mathscr{N}^1(x, D; z^{0\prime})$ is a differential operator of x_1. We assume that the above conditions (N-1) and (N-2) are satisfied, and we shall modify the proof of Lemmas 4.11, 4.13 and 4.14 in [14]. By (N-2) we have

$$\left| \mathscr{R}^{(\alpha)}_{j,k(\beta)}(x, \xi; z^{0\prime}; \gamma) \right| \leq C_{M,\alpha,\beta} \langle \xi \rangle_\gamma^{l_{0,k} - l_{v,j} + m'} \langle \xi' \rangle_\gamma^{-M}$$

for $M \in \mathbb{Z}_+$. Therefore, Lemma 4.11 of [14] is valid. We put $\widetilde{\mathscr{N}}^\mu(x, \xi; z^{0\prime}; \gamma) = {}^t\mathscr{N}^\mu(x, \xi; z^{0\prime})(1 - \Theta_{\gamma/8}(\xi'))$ $(\mu = 1, 2)$ and $\widetilde{\mathscr{M}}(x, \xi; z^{0\prime}; \gamma) = {}^t\mathscr{M}(x, \xi; z^{0\prime})(1 - \Theta_{\gamma/8}(\xi'))$, where ${}^t\mathscr{N}^\mu(x, D; z^{0\prime})$ and ${}^t\mathscr{N}^\mu(x, \xi; z^{0\prime})$ denote the transposed operator of $\mathscr{N}^\mu(x, D; z^{0\prime})$ and the symbol of ${}^t\mathscr{N}^\mu(x, D; z^{0\prime})$, respectively. It follows from (N-1) that for any $\Psi(x, \xi') \in S^0_{1,0}(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ with $\mathrm{supp}\,\Psi(x, \xi') \subset -\mathscr{C}'(z^{0\prime})$ there is $r_\psi(x, \xi; \gamma) \equiv$

$(r_{\Psi,j,k}(x,\xi;\gamma)) \in M_m\big(\mathscr{S}_{1,0}^{\bar{m},-\infty}\big)$ satisfying

$$\widetilde{\mathscr{N}}^1(x,D;z^{0\prime};\gamma).\widetilde{\mathscr{M}}(x,D;z^{0\prime};\gamma)\Psi_\gamma(x,D') = \Psi_\gamma(x,D')I_m + r_\psi(x,D;\gamma),$$

$$\big|r_{\Psi,j,k(\beta)}^{(\alpha)}(x,\xi;\gamma)\big| \le C_{\alpha,\beta,M}\,\langle\xi\rangle_\gamma^{\bar{m}-\alpha_1}\langle\xi'\rangle_\gamma^{-M} \quad (M \in \mathbb{Z}_+),$$

where $\Psi_\gamma(x,\xi') = (1 - \Theta_{\gamma/2}(\xi'))\Psi(x,\xi')$. Instead of ${}^t\widetilde{\mathscr{N}}^1(x,D;z^{0\prime};b;\gamma)^{-1}$ we simply use $\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma) \equiv \big(\widetilde{\mathscr{M}}_{j,k}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big)$. Note that

$$\big|\widetilde{\mathscr{M}}_{j,k(\beta)}^{(\alpha)}(x,\xi+i\gamma\vartheta;z^{0\prime};\gamma)\big| \le C_{\alpha,\beta}\langle\xi\rangle_\gamma^{m_k-l_{0,j}+m''-\alpha_1}\langle\xi'\rangle_\gamma^{-m''-|\alpha'|}.$$

The estimates in Lemma 4.13 of [14] are replaced by

$$\big\|\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)v\big\|_{+,\gamma,\{l_{0,j}\},(l,s)} \le C\,\|v\|_{+,\gamma,\{m_j\},(l+m'',s-m'')},$$

$$\big\|\Psi_1(x,D')\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\Psi_2(x,D')v\big\|_{+,\gamma,\{l_{0,j}\},(l,s)}$$

$$\le C_M\|v\|_{+,\gamma,\{m_j\},(l+m'',-M)} \quad (M \in \mathbb{Z}_+)$$

if $\Psi_j(x,\xi') \in S_{1,0}^0(\mathbb{R} \times T^*\mathbb{R}^{n-1})$ $(j=1,2)$ satisfy $\text{supp }\Psi_1 \cap \text{supp }\Psi_2 = \emptyset$ and $\big|\Psi_{j(\beta)}^{(\alpha')}(x,\xi')\big| \le C_{\alpha',\beta}\langle\xi'\rangle_\gamma^{-|\alpha'|}$. The proof of the above estimates is obvious while the proof of Lemma 4.13 of [14] is not so simple. Instead of (4.70) and (4.71) of [14], we have

$$\psi_\gamma(x,D')u = \widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big(\exp[\widehat{\Lambda}']\big)(x,D')v - r_\psi(x,D+i\gamma\vartheta;\gamma)u$$

$$- \widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\tilde{r}(x,D').\widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')u.$$

$${}^t\mathscr{L}_{\widehat{\Lambda}'}(x,D+i\gamma\vartheta;z^{0\prime})v$$

$$= \big(\exp[-\widehat{\Lambda}']\big)(x,D')\big\{\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')^tL(x,D+i\gamma\vartheta)u$$

$$+ \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)[{}^tL(x,D+i\gamma\vartheta),\psi_\gamma(x,D')I_N]u$$

$$+ \big({}^t\mathscr{L}(x,D+i\gamma\vartheta;z^{0\prime}) - \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\,{}^tL(x,D+i\gamma\vartheta)$$

$$\times \widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\big)(1+\tilde{r}(x,D'))$$

$$\times \widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')u$$

$$+ \widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\,{}^tL(x,D+i\gamma\vartheta)$$

$$\times \big(\widetilde{\mathscr{N}}^1(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\tilde{r}(x,D')$$

$$\times \widetilde{\mathscr{M}}(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D') + r_\psi(x,D+i\gamma\vartheta;\gamma)\big)u\big\},$$

respectively. Hence (4.72)–(4.74) in [14] are replaced by

$$\big\|\big(\exp[-\widehat{\Lambda}']\big)(x,D').\widetilde{\mathscr{N}}^2(x,D+i\gamma\vartheta;z^{0\prime};\gamma)\psi_\gamma(x,D')\,{}^tL(x,D+i\gamma\vartheta)u\big\|_{+,\gamma,\{l_{v,j}\},(l,0)}$$

$$\le C_{a',l}\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+m',a'(\delta(z^{0\prime})+c(z^{0\prime}))-m')},$$

$$\left\|\left(\exp\left[-\widehat{\Lambda}'\right]\right)(x,D')\widetilde{\mathcal{N}}^2(x,D+i\gamma\vartheta;z^{0'};\gamma)[{}^tL(x,D+i\gamma\vartheta),\psi_\gamma(x,D')I_N]u\right\|_{+,\gamma,\{l_{v,j}\},(l,0)}$$

$$\leq C_{a'}\|u\|_{+,\gamma,\{m_j\},(l+m',-2a'\delta(z^{0'})-1-m')},$$

$$\left\|\left(\exp\left[-\widehat{\Lambda}'\right]\right)(x,D')\left({}^t\mathcal{L}(x,D+i\gamma\vartheta;z^{0'})-\widetilde{\mathcal{N}}^2(x,D+i\gamma\vartheta;z^{0'};\gamma)\right.\right.$$
$$\left.\times{}^tL(x,D+i\gamma\vartheta)\widetilde{\mathcal{N}}^1(x,D+i\gamma\vartheta;z^{0'};\gamma)\right)(1+\tilde{r}(x,D'))$$
$$\times(1-\check{\psi}_\gamma(x,D'))\widetilde{\mathcal{M}}(x,D+i\gamma\vartheta;z^{0'};\gamma)\psi_\gamma(x,D')u\Big\|_{+,\gamma,\{l_{v,j}\},(l,0)}$$

$$\leq C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+2m'+m'',-M)}\quad(M\in\mathbb{Z}_+),$$

respectively. $\widetilde{\mathcal{R}}(x,\xi;z^{0'})\equiv\left(\widetilde{\mathcal{R}}_{i,j}(x,\xi;z^{0'})\right)$ defined in [14], with obvious modifications, satisfies

$$\left|\widetilde{\mathcal{R}}_{i,j(\beta)}^{(\alpha)}(x,\xi;z^{0'})\right|\leq C_{M,\alpha,\beta}\langle\xi\rangle_\gamma^{-l_{v,i}+l_{0,j}+2m'-\alpha_1}\langle\xi'\rangle_\gamma^{-M}.$$

Instead of (4.75) and (4.76) in [14], we have

$$\left\|\left(\exp\left[-\widehat{\Lambda}'\right]\right)(x,D')\big\{\widetilde{\mathcal{R}}(x,D;z^{0'})\widetilde{\mathcal{M}}(x,D+i\gamma\vartheta;z^{0'};\gamma)\psi_\gamma(x,D')\right.$$
$$+\widetilde{\mathcal{N}}^2(x,D+i\gamma\vartheta;z^{0'};\gamma)\,{}^tL(x,D+i\gamma\vartheta)$$
$$\times\left(\widetilde{\mathcal{N}}^1(x,D+i\gamma\vartheta;z^{0'};\gamma)\tilde{r}(x,D')\right.$$
$$\left.\times\widetilde{\mathcal{M}}(x,D+i\gamma\vartheta;z^{0'};\gamma)\psi_\gamma(x,D')+r_\psi(x,D+i\gamma\vartheta;\gamma)\right)\big\}u\Big\|_{+,\gamma,\{l_{v,j}\},(l,0)}$$

$$\leq C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+\bar{m}',-M)}$$

$$\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}$$
$$\leq C\left\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l'+m',a'\delta'(z^{0'})-m')}+\|u\|_{+,\gamma,\{m_j\},(l+\tilde{l}',-s)}\right\},$$

respectively, where $\bar{m}'=\max\{2m'+m'',m'+\bar{m}+\max_j m_j-\min_j m_j\}$, $\tilde{l}'=\max\{l'+2m'+m'',l'+\bar{m}',l''+m''\}$ and $s\leq 2a'\delta(z^{0'})+1+m'$. Modifying l' if necessary, instead of (4.76)–(4.78) in [14] we have

$$\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}$$
$$\leq C\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l',a'\delta'(z^{0'})-m')}+\|u\|_{+,\gamma,\{m_j\},(l+l',-s)}\},$$
$$\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l-m'-(3a'\delta(z^{0'})/2-m')_+,0)}$$
$$\leq\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l-m',-3a'\delta(z^{0'})/2+m')}$$
$$\leq C_{a',l}\|v\|_{+,\gamma,\{l_{0,j}\},(l,0)}+C_{a',l,M}\|u\|_{+,\gamma,\{m_j\},(l+\bar{m}'-2m',-M)},$$
$$\|\chi_\gamma(x,D')u\|_{+,\gamma,\{m_j\},(l,0)}$$
$$\leq C\{\|{}^tL(x,D+i\gamma\vartheta)u\|_{+,\gamma,\{-n_j\},(l+l_1,0)}+\|u\|_{+,\gamma,\{m_j\},(l+l_2,-l_2-1)}\},$$

respectively, where $s\leq 2a'\delta(z^{0'})+1+m'$, $l_1=l'+(3a'\delta(z^{0'})/2-m')_++a'\delta'(z^{0'})$, $l_2=\max\{\tilde{l}'+m',\bar{m}'-m'\}+(3a'\delta(z^{0'})/2-m')_+$, $a'=\max\{a_0\delta(z^{0'}),m',2(\tilde{l}'-m'),2(\bar{m}-$

$3m'$)$\big\}\big/\delta(z^{0\prime})$ and $a_+ = \max\{a, 0\}$. With these modifications, Lemma 4.14 and the arguments after Lemma 4.14 in §4 of [14] are valid and Theorem 1.4 of [14] is valid under the conditions (N-1) and (N-2) on $\mathscr{N}^1(x, D; z^{0\prime})$, $\mathscr{N}^2(x, D; z^{0\prime})$ and $R(x, D; z^{0\prime})$.

   Now we return to the proof of the assertions (i) of Theorem 1.1. We fix $t > 0$. We shall show that for every $z^{0\prime} = (x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ with $x_1^0 = t$ the condition (E-2)$_{z^{0\prime},1}$ of [14] is satisfied. We should remark that $\nu(z^{0\prime}) = 1$ in our case. Let $\Theta(s)$ be a function in $C_0^\infty(\mathbb{R})$ such that $0 \le \Theta(s) \le 1$, $\Theta(s) = 1$ if $|s| \le 1$, and supp $\Theta \subset (-2, 2)$. Let $\varphi_1(\xi')$ and $\varphi_2(x, \xi')$ be symbols in $\mathscr{S}_{1,0}^0$ such that $0 \le \varphi_1(\xi'), \varphi_2(x, \xi') \le 1$, $\varphi_1(\xi')$ and $\varphi_2(x, \xi')$ are positively homogeneous of degree 0 for $|\xi'| \ge 1$, $\varphi_1(\xi') = 1$ in a conic neighborhood of supp φ_2 if $|\xi'| \ge 1$, supp $\varphi_1 \subset \{\xi' \in \mathscr{C}'; |\xi'| \ge 1/4\}$, $\varphi_2(x, \xi') = 1$ in a conic neighborhood of $(x^0, \xi^{0\prime})$ if $|\xi'| \ge 1$, and supp $\varphi_2 \subset \{(x, \xi') \in V \times \mathscr{C}'; |\xi'| \ge 1/2\}$, where V and $\mathscr{C}'$ are a neighborhood of x^0 and a conic neighborhood of $\xi^{0\prime}$, respectively, as in Lemma 2.4. We put

$$P_j(x, \xi)$$

$$= p_j(\xi) + \Theta\left(\frac{2|x - x^0|}{t}\right)\varphi_2(x, \xi') \sum_{k=1}^{m_j}\left(l_{j,k}(x, \xi') - l_{j,k}^0(\xi')\right)(\xi_1 + \lambda_j \xi_n)^{m_j - k},$$

$$p_j(\xi) = (\xi_1 + \lambda_j \varphi_1(\xi')\xi_n)^{m_j} + \sum_{k=1}^{m_j} l_{j,k}^0(\xi')\varphi_1(\xi')^k(\xi_1 + \lambda_j \varphi_1(\xi')\xi_n)^{m_j - k}$$

$(1 \le j \le r)$ and $\tilde{p}(\xi) = \prod_{j=1}^r p_j(\xi)$, where the λ_j and the $l_{j,k}(x, \xi')$ are as in Lemma 2.4. We note that $p(\xi) = \tilde{p}(\xi)$ if $\varphi_2(x, \xi') \ne 0$ and that $p_j(\xi)$ is hyperbolic with respect to ϑ. Put

$$\widetilde{Q}(x, \xi) = \prod_{j=1}^r P_j(x, \xi) - \tilde{p}(\xi).$$

Then the condition (L) implies that there is $C > 0$ satisfying

(3.1) $$\left|\widetilde{Q}(x, \xi)\right| \le C|\tilde{p}(\xi - i\vartheta)| \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and, therefore,

$$|P_j(x, \xi) - p_j(\xi)| \le C|p_j(\xi - i\vartheta)| \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

So, in order to verify (E-2)$_{z^{0\prime},1}$, it suffices to prove (E-2)$_{z^{0\prime},1}$ with $\mathscr{L}^1(x, \xi; z^{0\prime})$ replaced by $P_j(x, \xi)$ for $1 \le j \le r$. Fix j so that $1 \le j \le r$. It is obvious that the conditions (i)–(v) of (E-2)$_{z^{0\prime},1}$ are satisfied with a conic neighborhood $\mathscr{C}'(z^{0\prime})$ of $z^{0\prime}$ in $\mathbb{R} \times (T^*\mathbb{R}^{n-1} \setminus 0)$, $\Omega(z^{0\prime}) = \{x \in \mathbb{R}^n; |x - x^0| < t\}$ and $\kappa = m_j/(m_j - 1)$. Let $z^1 = (x^1, \xi^1)$ be a point in $\overline{\Omega(z^{0\prime})} \times S^{n-1}$ such that $(dp_j)(\xi^1) = 0$. Since we can easily obtain microlocal *a priori* estimates for $P_j(x, D) = D_1^{m_j}$, we can assume that $\xi^{1\prime} \in \mathscr{C}'$ (see, e.g.,

Lemma 2.10 of [14]). Define the localization polynomial $p_{j,\xi}(\eta)$ of p_j at ξ by

$$p_j(\xi + s\eta) = s^\mu (p_{j,\xi}(\eta) + o(1)) \quad \text{as} \quad s \downarrow 0, \quad p_{j,\xi}(\eta) \not\equiv 0 \quad \text{in} \quad \eta.$$

Then $p_{j,\xi}(\eta)$ is hyperbolic with respect to ϑ. (see, e.g., [7]). Let $t_\pm(x, \xi)$ and $t(x, \xi)$ are positively homogeneous of degree 0, $(\nabla_x t)(z^1) \in \Gamma(p_{j,\xi^1}, \vartheta)$ and

$$t_\pm(x, \xi) = x_1 - x_1^1 \pm |x - x^1|^2 \pm \left| \frac{\xi}{|\xi|} \mp \xi^1 \right|^2 \quad \text{near} \quad z^1.$$

Now we can prove that $P_j(x, D - i\gamma\vartheta)$ satisfies the condition $\big(E; z^1, \mathscr{C}_1, \mathscr{C}_2, \{at_+(x, \xi) + a't(x, \xi) \log\langle\xi'\rangle / \log\langle\xi\rangle\}_{a \geq a_0, a' \geq a_0'}, m/(m-1), \infty\big)$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are conic neighborhoods of z^1 satisfying $\mathscr{C}_1 \cap \mathbb{R}^n \times S^{n-1} \Subset \mathscr{C}_2$, i.e., there are $\psi_k(x, \xi) \in C^\infty(T^*\mathbb{R}^n \setminus 0)$ ($k = 1, 2$), $l_k \in \mathbb{R}$ ($1 \leq k \leq 4$) such that the $\psi_k(x, \xi)$ are positively homogeneous of degree 0, $\mathrm{supp}\,\psi_2 \cap \mathbb{R}^n \times S^{n-1} \Subset \mathscr{C}_2$, $\psi_k(x, \xi) = 1$ in a conic neighborhood of $\mathscr{C}_1$ ($k = 1, 2$) and for any $a \geq 1$, $a' \geq 1$ and $b \in \mathbb{R}$ there are $\gamma_0 \geq 1$, $K \geq 1$ and $C > 0$ satisfying

$$\big\| \langle D \rangle_\gamma^{l_1} u \big\| \leq C \big\{ \big\| \langle D \rangle_\gamma^{l_2} P_{j,\Lambda}(x, D; \gamma) u \big\|$$
$$+ \big\| \langle D \rangle_\gamma^{l_3} (1 - \psi_{1,h}(x, D)) u \big\| + \big\| \langle D \rangle_\gamma^{l_4} \Theta_h(D) u \big\| \big\}$$

if $u \in H^\infty(\mathbb{R}^n)$, $\gamma \geq \gamma_0$, $h = K\gamma^{m/(m-1)}$ and

$$\Lambda(x, \xi) \ (\equiv \Lambda_{a,a',b,h}(x, \xi))$$
$$= \{(at_+(x, \xi) - b) \log\langle\xi\rangle + a't(x, \xi) \log\langle\xi'\rangle\}(1 - \Theta_{h/4}(\xi))\psi_2(x, \xi),$$

where $\Theta_h(\xi) = \Theta(|\xi|/h)$, $\psi_{1,h}(x, \xi) = (1 - \Theta_{h/2}(\xi))\psi_1(x, \xi)$ and $P_{j,\Lambda}(x, D; \gamma) = (e^{-\Lambda})(x, D)P_j(x, D - i\gamma\vartheta)(e^\Lambda)(x, D)$. Here $A \Subset B$ means that the closure $\overline{A}$ of A is compact and included in the interior $\mathring{B}$ of B, and $\|f\| = \left(\int_{\mathbb{R}^n} |f(x)|^2 \, dx\right)^{1/2}$. Indeed, it follows from (3.1), Lemma 2.5 and Lemma 2.7 of [28] that $P_j(x, D)$ satisfies the condition $(\text{A-2})_{z^1}$ of [15] (see, also, Lemma 2.1 of [15]). Similarly, we can prove that ${}^t P_j(x, D + i\gamma\vartheta)$ satisfies the condition $\big(E; (x^1, -\xi^1), \check{\mathscr{C}}_1, \check{\mathscr{C}}_2, \{-at_-(x, \xi) - a't(x, -\xi') \times \log\langle\xi'\rangle / \log\langle\xi\rangle\}_{a \geq a_0, a' \geq a_0'}, m/(m-1), \infty\big)$, where $\check{\mathscr{C}}_k = \{(x, -\xi); (x, \xi) \in \mathscr{C}_k\}$ and ${}^t P_j(x, \xi) = \sigma({}^t P_j(x, D))$. Here $\sigma(a(x, D))$ $(= \sigma(a(x, D))(x, \xi))$ denotes the symbol of $a(x, D)$. Therefore, the condition $(\text{E-2})_{z^{0'},1}$ of [14] is satisfied for every $z^{0'} = (x^0, \xi^{0'}) \in \mathbb{R}^n \times S^{n-2}$ with $x^0 = t$. By the same argument we can prove that the condition $(\text{E-1})_{z^0}$ of [14] is satisfied for every $z^0 = (x^0, \xi^0) \in \mathbb{R}^n \times S^{n-1}$ with $x_1^0 > 0$ and $(dp)(\xi^0) = 0$. For the condition (U) of [14] we need the following

Lemma 3.1. *Let $N \in \mathbb{N}$ and $\mathscr{C}$ be an open conic subset of $T^*\mathbb{R}^n \setminus 0$. Assume that*

$$L(x, D) = M_1(x, D)l(x, D)M_2(x, D) + R(x, D) \quad \text{in} \quad \mathscr{C},$$

where $M_k(x, \xi) \in M_m\big(S_{1,0}^{m-1}\big)$ $(k = 1, 2)$, $l(x, \xi) \in M_m\big(S_{1,0}^m\big)$, $R(x, \xi) \in M_m\big(S_{1,0}^{m-N}\big)$ and the entries of $M_k(x, \xi)$ $(k = 1, 2)$, $l(x, \xi)$ and $R(x, \xi)$ are classical symbols. Here $A(x, D) = B(x, D)$ in $\mathscr{C}$ means that $A(x, \xi) = B(x, \xi)$ in $\mathscr{C}$. Let $\zeta(x) \in C^\infty(\mathbb{R}^n)$ satisfy $\partial_k \zeta(x) \in \mathscr{B}(\mathbb{R}^n)$ $(1 \le k \le n)$. Then there are $r_{N,k}(x, \xi) \in M_m\big(S_{1,0}^{3m-2-N}\big)$ $(0 \le k \le 3(N-1))$ such that

$$e^{-\gamma\zeta(x)}L(x, D)e^{\gamma\zeta(x)} \ (= L(x, D) + \gamma L_1(-i\nabla\zeta(x)))$$

(3.2)

$$= M_{1,N-1}(x, D; \gamma)l_{N-1}(x, D; \gamma)M_{2,N-1}(x, D; \gamma) + \sum_{k=0}^{3(N-1)} \gamma^k r_{N,k}(x, D)$$

in $\mathscr{C}$, where

$$M_{\mu,N-1}(x, \xi; \gamma) = \sum_{|\alpha| \le N-1} \frac{M_\mu^{(\alpha)}(x, \xi)\omega_\alpha(\gamma\zeta(x))}{\alpha!} \quad (\mu = 1, 2),$$

$$l_{N-1}(x, \xi; \gamma) \sum_{|\alpha| \le N-1} \frac{l^{(\alpha)}(x, \xi)\omega_\alpha(\gamma\zeta(x))}{\alpha!}$$

and $\omega_\alpha(\gamma\zeta(x)) = e^{-\gamma\zeta(x)}D^\alpha e^{\gamma\zeta(x)}$.

Proof. We can write

$$\sigma(e^{-\gamma\zeta(x)}M_\mu(x, D)e^{\gamma\zeta(x)})(x, \xi) = M_{\mu,N-1}(x, \xi; \gamma) + \widetilde{M}_{\mu,N}(x, \xi; \gamma) \quad (\mu = 1, 2),$$

$$\sigma(e^{-\gamma\zeta(x)}l(x, D)e^{\gamma\zeta(x)})(x, \xi) = l_{N-1}(x, \xi; \gamma) + \widetilde{l}_N(x, \xi; \gamma),$$

where $\widetilde{M}_{\mu,N}(x, \xi; \gamma) \in M_m\big(S_{1,0}^{m-1-N}\big)$ $(\mu = 1, 2)$ and $\widetilde{l}_N(x, \xi; \gamma) \in M_m\big(S_{1,0}^{m-N}\big)$. So we have

$$L(x, D) + \gamma L_1(-i\nabla\zeta(x))$$

$$= M_{1,N-1}(x, D; \gamma)l_{N-1}(x, D; \gamma)M_{2,N-1}(x, D; \gamma) + \widetilde{R}_N(x, D; \gamma)$$

$$= \sum_{k=0}^{3(N-1)} \gamma^k (A_k(x, D) + B_k(x, D)) + \widetilde{R}_N(x, D; \gamma) \quad \text{in} \quad \mathscr{C},$$

$$A_k(x, \xi) \sim \sum_{\mu=0}^\infty A_k^\mu(x, \xi) \quad \text{if} \quad k \le N-1,$$

where $\widetilde{R}_N(x, \xi; \gamma) \in M_m\big(S_{1,0}^{3m-2-N}\big)$, the $A_k^\mu(x, \xi)$ are positively homogeneous of degree $3m - 2 - k - \mu$, $A_k(x, \xi) \equiv 0$ for $N \le k \le 3(N-1)$ and $B_k(x, \xi) \in M_m\big(S_{1,0}^{3m-2-N}\big)$. So, modifying the $A_k(x, \xi)$ for $|\xi| \le 1$, we may assume that $A_0(x, \xi) = L(x, \xi)$ in $\mathscr{C}$, $A_1(x, \xi) = L_1(-i\nabla\zeta(x))$ in $\mathscr{C}$, and $A_k(x, \xi) = 0$ in $\mathscr{C}$ if $2 \le k \le 3(N-1)$. This gives

$$\widetilde{R}(x, \xi; \gamma) = -\sum_{k=0}^{3(N-1)} \gamma^k B_k(x, \xi) \quad \text{in} \quad \mathscr{C}.$$

Taking $r_{N,k}(x, \xi) = -B_k(x, \xi)$ we have (3.2).   □

Applying the same arguments as in §3 of [15], we can prove that the condition (U) of [14] is satisfied. Then the assertion (i) of Theorem 1.1 easily follows from Theorem 1.4 of [14].

4.   Proof of the assertion (ii) of Theorem 1.1

In this section we assume that the Cauchy problem $(CP)_t$ is C^∞ well-posed for any $t > 0$. We can assume without loss of generality that $L(x, \xi) \in M_m\big(\mathscr{S}^1_{1,0}\big)$. For $\tau \in \mathbb{R}$, $p \in \mathbb{Z}_+$, $q \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^n)$ we define

$$\|u\|_{\tau,-,(p,q)} = \sum_{|\alpha| \leq p} \|\langle D' \rangle^q D^\alpha u\|_{\tau,-},$$

$$\|u\|_{\tau,-} = \left(\int_{x_1 < \tau} |u(x)|^2 \, dx \right)^{1/2}.$$

From Banach's closed graph theorem or the Baire category theorem we have the following lemma (see, e.g., [10] and [27]).

Lemma 4.1.   *Let $t > 0$. Then for every compact subset K of $\{x \in \mathbb{R}^n; x_1 > t\}$ and $p \in \mathbb{Z}_+$ there are $C \equiv C_{p,K} > 0$ and $q \in \mathbb{Z}_+$ such that*

$$\|u\|_{\tau,-,(p,0)} \leq C \|L(x, D)u\|_{\tau,-,(q,0)}$$

for any $\tau > t$ and $u \in C^\infty_0(\mathbb{R}^n; \mathbb{C}^m)$ with $\operatorname{supp} u \subset K$.

Let $t > 0$, $x^0 \in \mathbb{R}^n$ and $\xi^{0'} \in S^{n-2}$ such that $x^0_1 = t$, and let $V \equiv V(x^0, \xi^{0'})$ and $\mathscr{C}' \equiv \mathscr{C}'(x^0, \xi^{0'})$ be as in the condition (L) and Lemmas 2.2 and 2.4. Moreover, let r, m_j $(1 \leq j \leq r)$, $N_k(x, \xi)$ $(k = 1, 2)$, $l^j(x, \xi)$ $(1 \leq j \leq r)$ and $R(x, \xi)$ be as in Lemma 2.4. Choose an open subset V_0 of V, an open conic subset $\mathscr{C}'_0$ of $\mathscr{C}'$ and $\chi \in C^\infty_0(V)$ and $\Psi(\xi') \in C^\infty(\mathbb{R}^{n-1})$ so that $V_0 \Subset V$, $\mathscr{C}'_0 \cap S^{n-2} \Subset \mathscr{C}'$, $\chi(x) = 1$ in V_0, $\Psi(\xi')$ is positively homogeneous of degree 0 for $|\xi'| \geq 1$, $\Psi(\xi) = 1$ for $\xi' \in \mathscr{C}'_0$ with $|\xi'| \geq 1$ and $\operatorname{supp} \Psi \subset \{\xi' \in \mathscr{C}'; |\xi'| \geq 1/2\}$. We put $\Psi_\pm(\xi') = \Psi(\pm\xi')$. Then from Lemma 2.4 we have

(4.1)
$$N_1(x, D)L(x, D)N_2(x, D)\Psi_\pm(D')(\chi(x)u(x))$$
$$= l(x, D)\Psi_\pm(D')(\chi(x)u(x)) + R_\pm(x, D)u,$$

where $l(x, \xi) = \operatorname{diag}(1, \ldots, 1, l^1(x, \xi), \ldots, l^r(x, \xi))$ and $R_\pm(x, \xi) \in M_m\big(\mathscr{S}^{2m-1,-\infty}_{1,0}\big)$.

Lemma 4.2. *Let $t > 0$. Then for every compact subset K of $\{x \in V_0; x_1 > t\}$ there are $C_{K,M} > 0$ $(M \in \mathbb{Z}_+)$ and $p_j \in \mathbb{Z}_+$ $(1 \le j \le 3)$ such that*

$$\max_{x_1 \le \tau} |u(x)| \le C_{K,M}(\|l(x, D)u\|_{\tau,-,(p_1,0)}$$

$$+ \|u\|_{\tau,-,(p_2,-M)} + \|(1 - \Psi_\pm(D'))u\|_{\tau,-,(p_3,0)})$$

for any $\tau > t$, $u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^m)$ with $\operatorname{supp} u \subset K$ and $M \in \mathbb{Z}_+$.

Proof. Let $\tau > t$, and let K and K_j $(j = 1, 2)$ be compact subsets of $\{x \in V_0; x_1 > t\}$ satisfying $K \Subset K_1 \Subset K_2$. We choose $\chi_j \in C^\infty(\mathbb{R}^n)$ $(j = 1, 2)$ so that $\chi_1(x) = 1$ near K, $\operatorname{supp} \chi_1 \subset K_1$, $\chi_2(x) = 1$ near K_1 and $\operatorname{supp} \chi_2 \subset K_2$. Let $u \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^m)$ satisfy $\operatorname{supp} u \subset K$. Then by (4.1) we have

(4.2)
$$N_1(x, D)L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u = l(x, D)\Psi_\pm(D')u + R_{1,\pm}(x, D)u,$$

$$R_{1,\pm}(x, D) = N_1(x, D)L(x, D)(\chi_2(x) - 1)N_2(x, D)\Psi_\pm(D')\chi_1(x)$$

$$+ R_\pm(x, D) \in \mathscr{L}_{1,0}^{2m-1,-\infty}.$$

Let $p \in \mathbb{Z}_+$ satisfy $p \ge m - 1$. From Lemma 4.1 it follows that there are $C > 0$ and $q \in \mathbb{Z}_+$ satisfying

(4.3)
$$\|\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(p,0)}$$

$$\le C\|L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(q,0)}.$$

Let $M_j(x, D) \in \mathscr{L}_{1,0}^{m-1}$ be a parametrix of $N_j(x, D)$ $(j = 1, 2)$. Then we have

$$L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u$$

$$= M_1(x, D)N_1(x, D)L(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u + R_{2,\pm}(x, D)u,$$

$$R_{2,\pm}(x, \xi) \in \mathscr{L}_{1,0}^{3m-2,-\infty}.$$

This, together with (4.2) and (4.3), yields

(4.4)
$$\|\chi_2(x)N_2(x, D)\Psi_\pm(D')u\|_{\tau,-,(p,0)}$$

$$\le C_M(\|l(x, D)\Psi_\pm(D')u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(q+3m-2,-M)})$$

$$\le C'_M(\|l(x, D)u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(q+3m-2,-M)}$$

$$+ \|(1 - \Psi_\pm(D'))u\|_{\tau,-,(q+2m-1,0)})$$

$(M \in \mathbb{Z}_+)$. On the other hand, we have

(4.5)
$$u = \Psi_\pm(D')u + (1 - \Psi_\pm(D'))u$$

$$= M_2(x, D)\chi_2(x)N_2(x, D)\Psi_\pm(D')u + (1 - \Psi_\pm(D'))u + R_{3,\pm}(x, D)u,$$

where $R_{3,\pm}(x, \xi) \in \mathscr{S}_{1,0}^{2m-2,-\infty}$. From (4.4) and (4.5) we have

$$\|u\|_{\tau,-,(p-m+1,0)} \leq C_M(\|l(x,D)u\|_{\tau,-,(q+m-1,0)} + \|u\|_{\tau,-,(p_2,-M)}$$
$$+ \|(1 - \Psi_\pm(D'))u\|_{\tau,-,(p_3,0)}),$$

where $M \in \mathbb{Z}_+$, $p_2 = \max\{q+3m-2, p+m-1\}$ and $p_3 = \max\{q+2m-1, p-m+1\}$. This proves the lemma. □

Let $x^0 \in \mathbb{R}^n$ and $\xi^{0\prime} \in S^{n-2}$, and let V and $\mathscr{C}'$ be an open neighborhood of x^0 and an open conic neighborhood of $\xi^{0\prime}$, respectively, such that $\mathscr{C}'$ is semi-algebraic. Moreover, let $q(x, \xi)$ be an SA-symbol in $V \times \mathscr{C}'$ such that

$$q(x, \lambda\xi) = \lambda^\mu q(x, \xi) \quad \text{for} \quad (x, \xi) \in V \times \mathbb{R} \times (\mathscr{C}' \cup (-\mathscr{C}')) \quad \text{and} \quad \lambda \in \mathbb{R} \setminus \{0\},$$

where $\mu \in \mathbb{N}$. For a conic subset Γ of $\mathbb{R} \times \mathscr{C}'$ and a fixed $x \in V$ we define

$$n(p, q(x, \cdot); \Gamma) = \inf\left\{\rho \in \mathbb{R}; \sup_{\xi \in \Gamma, |\xi'| \geq 1} \langle\xi\rangle^{-\rho}\left|\frac{q(x, \xi)}{p(\xi - i\vartheta)}\right| < \infty\right\}.$$

Let $\eta^0 \in \mathbb{R} \times \mathscr{C}' \cap S^{n-1}$, and put

$$N_{\eta^0} = \left\{\hat{\eta}(s) = s^{-1}\left(\eta^0 + \sum_{j=1}^\infty s^{j/l}\eta^j\right), \eta^j \in \mathbb{R}^n,\right.$$

$$\left. l \in \mathbb{N} \text{ and } \hat{\eta}(s) \text{ is convergent for } 0 < s \ll 1\right\}.$$

For $x \in V$ and $\hat{\eta} \in N_{\eta^0}$ we define $n(x; \hat{\eta}) \equiv n(p, q(x, \cdot); \hat{\eta}) \in \mathbb{R}$ by

$$\left|\frac{q(x, \hat{\eta}(s))}{p(\hat{\eta}(s) - i\vartheta)}\right| = s^{-n(x;\hat{\eta})}(c(x, \hat{\eta}) + o(1)) \quad \text{as} \quad s \downarrow 0,$$

where $c(x; \hat{\eta}) \neq 0$. Here we have defined $n(x; \hat{\eta}) = -\infty$ if $q(x, \hat{\eta}(s)) \equiv 0$ in s. Moreover, we define

$$n(p, q(x, \cdot); \eta^0) = \sup\{n(p, q(x, \cdot); \hat{\eta}); \hat{\eta} \in N_{\eta^0}\}.$$

Lemma 4.3. (i) *For a fixed $x \in V$ there is $c_0 > 0$ such that $\Gamma(c) \subset \mathbb{R} \times \mathscr{C}'$ and $n(p, q(x, \cdot); \eta^0) = n(p, q(x, \cdot); \Gamma(c))$ for any $c > 0$ with $c \leq c_0$, where $\Gamma(c) = \{\xi \in \mathbb{R}^n \setminus \{0\}; |\xi/|\xi| - \eta^0| \leq c\}$.*
(ii) *Let $c > 0$ satisfy $\Gamma(c) \subset \mathbb{R} \times \mathscr{C}'$, where $\Gamma(c)$ is as in the assertion* (i). *Then $n(p, q(x, \cdot); \Gamma(c)) \in \mathbb{Q}$ for $x \in V$, and for a fixed $x \in V$ there is $C > 0$ satisfying*

$$\left|\frac{q(x, \xi)}{p(\xi - i\vartheta)}\right| \leq C\langle\xi\rangle^{n(p,q(x,\cdot);\Gamma(c))} \quad \text{for} \quad \xi \in \Gamma(c).$$

(iii) *The set* $\{n(p,q(x,\cdot);\xi);(x,\xi) \in V \times \mathbb{R} \times \mathscr{C}'$ *and* $|\xi| = 1\}$ *is finite, with a modification of* $\mathscr{C}'$ *if necessary.*

(iv) *Let* $\hat{\eta} \in N_{\eta^0}$. *Then for any open subset* U *of* V *there is an open subset* U_0 *of* U *such that*

$$\max_{x \in U} n(p, q(x, \cdot); \hat{\eta}) = n(p, q(y, \cdot); \hat{\eta}) \quad for \quad y \in U_0.$$

(v) *Assume that* $n(p,q(x,\cdot);\xi) \leq 0$ *for every* $x \in K$ *and* $\xi \in \omega$, *where* K *is a compact subset of* V *and* ω *is a compact subset of* $\mathbb{R} \times \mathscr{C}' \cap S^{n-1}$. *Then there is* $C > 0$ *such that*

$$(4.6) \qquad \left| \frac{q(x,\xi)}{p(\xi - i\vartheta)} \right| \leq C \quad if \quad x \in K, \quad \frac{\xi}{|\xi|} \in \omega \quad and \quad |\xi'| \geq 1.$$

Proof.   The assertion (i) can be proved by the same arguments as in the proof of Lemma 1.2.3 of [26] (see, also, the proof of Lemma 1.2 of [23]). Let us prove the assertion (ii). Put

$$\mathscr{E} = \{(\phi, \xi, r) \in \mathbb{R}^{n+2}; \phi|p(\xi - i\vartheta)|^2 = |q(x,\xi)|^2,$$
$$|\xi|^2 = r^2, \ r \geq 1 \text{ and } |\xi - r\eta^0|^2 \leq c^2 r^2\},$$

where $x \in V$ is fixed. By assumption $\mathscr{E}$ is a semi-algebraic set. It follows from Corollary A.2.6 of [8] that there are $a \in \mathbb{Q}$ and $A \in \mathbb{C} \setminus \{0\}$ such that

$$\sup\{\phi \colon \text{there is } \xi \in \mathbb{R}^n \text{ satisfying } (\phi, \xi, r) \in \mathscr{E}\} = r^a(A + o(1)) \quad \text{as} \quad r \to \infty.$$

This gives $n(p, q(x, \cdot); \Gamma(c)) = a/2$ and proves the assertion (ii). We omit the proof of the assertion (iii) since we will not use it in this paper and it can be proved by the same arguments as in the beginning of §2.1 of [26] and in the proof of Lemma 1.1.3 of [26]. Next let us prove the assertion (iv). By definition there are $N \in \mathbb{N}$, a semi-algebraic set $\mathscr{A}$ in $\mathbb{R}^{N+n-1}$ and a polynomial $F(x, \xi, \lambda)$ of (ξ, λ) $(\equiv (\xi, \lambda_1, \ldots, \lambda_N))$ and a polynomial $G(x, \xi', \lambda)$ of (ξ', λ), whose coefficients belong to $C^\infty(V)$, satisfying the conditions (1)–(3) of Definition 2.1 (ii) with $f(x, \xi)$ replaced by $q(x, \xi)$. Put

$$\tilde{q}(x, \xi) = F(x, \xi_1, \pi^{-1}(\xi'))|\xi'|^{-\kappa},$$

where π is as in the condition (1) of Definition 2.1 (ii) and $G(x, \pi^{-1}(\xi'))$ is homogeneous of degree κ. Let $\hat{\eta} \in N_{\eta^0}$, and let U be an open subset of V. It is obvious that there are $c_0 \in \mathbb{C} \setminus \{0\}$ and $\mu_0 \in \mathbb{Q}$ such that $p(\hat{\eta}(s) - i\vartheta) = s^{-\mu_0}(c_0 + o(1))$ as $s \downarrow 0$. We may assume that $\tilde{q}(x, \hat{\eta}(s)) \not\equiv 0$ in $(x, s) \in U \times (0, s_0]$, where $0 < s_0 \ll 1$. Then there are $c(x) \in C^\infty(U)$ and $\mu_1 \in \mathbb{Q}$ such that $c(x) \not\equiv 0$ in U and $\tilde{q}(x, \hat{\eta}(s)) = s^{-\mu_1}(c(x) + o(1))$ as $s \downarrow 0$. Therefore, we have $n(p, \tilde{q}(x, \cdot); \hat{\eta}) \leq \mu_1 - \mu_0$ for $x \in U$, and $n(p, \tilde{q}(x, \cdot); \hat{\eta}) = \mu_1 - \mu_0$ for $x \in U$ with $c(x) \neq 0$. Since $n(p, q(x, \cdot); \hat{\eta}) = n(p, \tilde{q}(x, \cdot); \hat{\eta})$, this proves the assertion (iv). The assertion (v) easily follows from the assertions (i) and (ii) and Lemma 2.5.   □

Now suppose that the condition (L) is not satisfied. Then it follows from Lemma 4.3 that there are $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$, $(y^0, \eta^0) \in V \times (\mathbb{R} \times \mathscr{C}' \cap S^{n-1})$ and $\hat{\eta}(s) \equiv s^{-1} \sum_{j=0}^{\infty} s^{j/l} \eta^j \in N_{\eta^0}$ such that $y_1^0 \geq 0$ and

$$\max_{1 \leq j \leq m-1} n(p, Q^j(y^0, \cdot); \hat{\eta}) > 0,$$

where $V \equiv V(x^0, \xi^{0\prime})$, $\mathscr{C}' \equiv \mathscr{C}'(x^0, \xi^{0\prime})$ and $Q(x, \xi) \equiv Q(x, \xi; x^0, \xi^{0\prime})$ are as in the condition (L) and Lemmas 2.2 and 2.4, $Q^j(x, \xi)$ is positively homogeneous of degree $m - j$ and $Q(x, \xi) \sim \sum_{j=1}^{\infty} Q^j(x, \xi)$. By the assertion (iv) of Lemma 4.3 we may assume that $y_1^0 > 0$ and

$$\sigma \ (= \sigma(\hat{\eta}))$$

$$\equiv \max_{\substack{1 \leq j \leq m-1 \\ x \in V, \, x_1 \geq 0}} \frac{n_+(p, Q^j(x, \cdot); \hat{\eta})}{j + n_+(p, Q^j(x, \cdot); \hat{\eta})}$$

$$= \max_{1 \leq j \leq m-1} \frac{n_+(p, Q^j(y^0, \cdot); \hat{\eta})}{j + n_+(p, Q^j(x^0, \cdot); \hat{\eta})},$$

where $n_+(p, q; \hat{\eta}) = \max\{n(p, q; \hat{\eta}), 0\}$. We note that $0 < \sigma < 1$. By translation we may assume that $y^0 = 0$. Then there is $t_0 > 0$ such that the Cauchy problem $(CP)_t$ is C^∞ well-posed for any $t > -t_0$. Let $l^\mu(x, \xi)$ $(1 \leq \mu \leq r)$ be as in Lemma 2.4, and write $l^\mu(x, \xi) \sim \sum_{j=0}^{\infty} l^{\mu, j}(x, \xi)$ $(1 \leq \mu \leq r)$, where $l^{\mu, j}(x, \xi)$ is positively homogeneous of degree $m_\mu - j$ $(j \in \mathbb{Z}_+)$. We may also assume that $l^{1,0}(\eta^0) = 0$. Then we have $l^{\mu, 0}(\eta^0) \neq 0$ for $2 \leq \mu \leq r$. Recall that

$$(4.7) \qquad Q^j(x, \hat{\eta}(s) - i\vartheta) = \sum_{j_k \geq 0, \, j_1 + \cdots + j_r = j} l^{1, j_1}(x, \hat{\eta}(s) - i\vartheta) \cdots l^{r, j_r}(x, \hat{\eta}(s) - i\vartheta)$$

$(j \in \mathbb{N})$.

Lemma 4.4. *Modifying $\hat{\eta} \in N_{\eta^0}$ if necessary, we have*

$$(4.8) \qquad \begin{aligned} \sigma &= \max_{\substack{x \in V, \, x_1 \geq -t_0 \\ \xi \in \mathbb{R}^n, \, j \geq 1}} \frac{n_+(l^{1,0}, l^{1,j}(x, \cdot); \hat{\eta} + \xi)}{j + n_+(l^{1,0}, l^{1,j}(x, \cdot); \hat{\eta} + \xi)} \\ &= \max_{1 \leq j \leq m-1} \frac{n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})}{j + n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})}, \end{aligned}$$

and

$$(4.9) \qquad\qquad n(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}) \geq n(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}_a^\lambda)$$

if $\sigma = n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta})/(j + n_+(l^{1,0}, l^{1,j}(0, \cdot); \hat{\eta}))$, $0 \leq \lambda < 1$, $a \in \mathbb{R}$ and $1 \leq j \leq m_1 - 1$, where $\hat{\eta}_a^\lambda(s) = s^{\lambda/(1-\lambda)} \hat{\eta}(s^{1/(1-\lambda)}) + a\vartheta$.

Proof.   From (4.7) we have

$$n(p, Q^j(x, \cdot); \hat{\eta}) \leq \max_{1 \leq k \leq j} (n(l^{1,0}, l^{1,k}(x, \cdot); \hat{\eta}) + k - j)$$

$(1 \leq j \leq m - 1)$. This yields

$$\sigma \leq \max_{1 \leq j \leq m-1} \max_{1 \leq k \leq j} \frac{n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})}$$

$$\leq \max_{1 \leq k \leq m_1 - 1} \frac{n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})} \equiv \hat{\sigma},$$

since $n(l^{1,0}, l^{1,k}(x, \cdot); \hat{\eta}) \leq 0$ for $k \geq m_1$. Put

$$k_0 = \max \left\{ k \in \mathbb{N}; k \leq m_1 - 1 \text{ and } \frac{n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})}{k + n_+(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta})} = \hat{\sigma} \right\}.$$

Then there is $c > 0$ such that

$$\left| \frac{Q^{k_0}(0, \hat{\eta}(s) - i\vartheta)}{p(\hat{\eta}(s) - i\vartheta)} \right| \geq \left| \frac{l^{1,k_0}(0, \hat{\eta}(s) - i\vartheta)}{l^{1,0}(\hat{\eta}(s) - i\vartheta)} \right|$$

$$- \sum_{\substack{0 \leq \mu_1 < k_0 \\ \mu_1 + \cdots + \mu_r = k_0}} \prod_{\nu=1}^{r} \left| \frac{l^{\nu,\mu_\nu}(0, \hat{\eta}(s) - i\vartheta)}{l^{\nu,0}(\hat{\eta}(s) - i\vartheta)} \right|$$

$$\geq cs^{-k_0 \hat{\sigma}/(1-\hat{\sigma})} \quad \text{for} \quad 0 < s \ll 1,$$

since $n(l^{1,0}, l^{1,k_0}(0, \cdot); \hat{\eta}) = k_0\hat{\sigma}/(1 - \hat{\sigma}) > k\hat{\sigma}/(1 - \hat{\sigma}) - (k_0 - k) \geq n(l^{1,0}, l^{1,k}(0, \cdot); \hat{\eta}) - (k_0 - k)$ for $0 \leq k < k_0$. This gives (4.8). (4.9) can be proved by the same argument as in the proof of Lemma 1.2.4 of [26]. □

Let $\sigma_j > 0$ $(j = 1, 2)$ satisfy $0 < \sigma_j < \sigma$, and define

$$l(y, \eta; s) = l(s^{\sigma_1} y_1, s^{\sigma_2} y', s^{-\sigma_1} \eta_1, s^{-\sigma_2} \eta').$$

We shall determine σ_1 and σ_2 later. It is obvious that

$$(l(x, D_x)u)(s^{\sigma_1} y_1, s^{\sigma_2} y') = l(y, D_y; s)u_s(y),$$

where $u_s(y) = u(s^{\sigma_1} y_1, s^{\sigma_2} y')$. The following lemma easily follows from Lemma 4.2.

Lemma 4.5. *For any open bounded neighborhood W of the origin in $\mathbb{R}^n$ there are $s(W) > 0$, $C_M \equiv C_M(W) > 0$ $(M \in \mathbb{Z}_+)$ and $p_j \in \mathbb{Z}_+$ $(j = 1, 2, 3)$ such that $W \Subset$*

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$\{x \in \mathbb{R}^n; (s(W)^{\sigma_1}x_1, s(W)^{\sigma_2}x') \in V \ \text{and} \ s(W)^{\sigma_1}x_1 > -t_0\}$ and

(4.10)

$$|u(0)| \leq s^{a(\sigma_1,\sigma_2)} C_M \max_{y_1 \leq 0} \left\{ \sum_{|\alpha| \leq p_1} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|} \left| D_y^\alpha l^1(y, D_y; s)u(y) \right| \right.$$

$$+ \sum_{|\alpha| \leq p_2} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|} \| \langle s^{-\sigma_2} D_{y'} \rangle^{-M} D_y^\alpha u(y_1, y') \|(y_1)$$

$$\left. + \sum_{|\alpha| \leq p_3} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|} \| (1 - \Psi_\pm(D_{y'}; s)) D_y^\alpha u(y_1, y') \|(y_1) \right\}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}\, u \subset W$, $0 < s \leq s(W)$ and $M \in \mathbb{Z}_+$, where $a(\sigma_1, \sigma_2) = (\sigma_1 + (n-1)\sigma_2)/2$, $\Psi_\pm(\eta'; s) = \Psi_\pm(s^{-\sigma_2}\eta')$ and $\|f(y)\|(y_1) = \left(\int |f(y_1, y')|^2 \, dy' \right)^{1/2}$.

Let $N \in \mathbb{N}$ and $\gamma \in \mathbb{R} \setminus \{0\}$, and put

$$P_N(x, \xi; s; \gamma) = \sum_{j=0}^{N-1} \sum_{|\alpha| \leq n(N-j;\sigma)} \frac{s^{-m_1+j+(1-\sigma)|\alpha|} \gamma^{-m_1+j+|\alpha|} l^{1,j(\alpha)}(x, s\eta(s))\xi^\alpha}{\alpha!},$$

$$R^N(x, \xi; s; \gamma) = l^1(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - P_N(x, \xi; s; \gamma),$$

where $\eta(s) = s^{-\sigma}\hat{\eta}(s^{1-\sigma})$ and $n(N; \sigma) = [N/(1 - \sigma)]$. Then $R^N(x, \xi; s; \gamma) \equiv \sum_{j=0}^{m_1} R^{N,j}(x, \xi'; s; \gamma)\xi_1^j$ satisfies

(4.11)

$$\left| R_{(\beta)}^{N,j(\alpha')}(x, \xi'; s; \gamma) \right|$$
$$\leq C_{N,\alpha',\beta}(\gamma)\{s^{-\sigma j - \sigma|\alpha'|}(s^{-1} + s^{-\sigma}|\xi'|)^{m_1-j}$$
$$+ s^{-m_1}\langle\xi'\rangle^{-j-|\alpha'|}(1 + s^N\langle\xi'\rangle^{N/(1-\sigma)})\}.$$

Choose $c_0 > 0$ and $s_0 > 0$ so that $\eta^{0\prime} + \xi' \in \mathscr{C}'$ for $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| \leq c_0$ and $\left| \sum_{j=1}^\infty s^{(1-\sigma)j/l}\eta^j \right| \leq c_0/4$ for $0 < s \leq s_0$. For $< s \leq s_0$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| \leq c_0|\gamma^{-1}s^{\sigma-1}|/4$ we write

$$R^N(x, \xi; s; \gamma) = R_1^N(x, \xi; s; \gamma) + R_2^N(x, \xi; s; \gamma),$$

where

$$R_1^N(x, \xi; s; \gamma) = l^1(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - \sum_{j=0}^{N-1} l^{1,j}(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi),$$

$$R_2^N(x, \xi; s; \gamma) = \sum_{j=0}^{N-1} l^{1,j}(x, \gamma^{-1}\eta(s) + s^{-\sigma}\xi) - P_N(x, \xi; s; \gamma).$$

Then it is easy to see that

$$(4.12) \qquad \left| R_{1(\beta)}^{N,j(\alpha')}(x, \xi'; s; \gamma) \right| \leq C_{N,\alpha',\beta}(\gamma) s^{N-m_1+(1-\sigma)(j+|\alpha'|)},$$

$$(4.13) \qquad \left| R_{2(\beta)}^{N,j(\alpha')}(x, \xi'; s; \gamma) \right| \leq C_{N,\alpha',\beta}(\gamma) s^{N-m_1} \langle \xi' \rangle^{N/(1-\sigma)+1}$$

if $|\xi'| \leq c_0 |\gamma^{-1} s^{\sigma-1}|/4$, $0 < s \leq s_0$ and $0 \leq j \leq m_1$, where $R_k^N(x, \xi; s; \gamma) = \sum_{j=0}^{m_1} R_k^{N,j}(x, \xi'; s; \gamma) \xi_1^j$ $(k = 1, 2)$. Write

$$l^{1,j(\alpha)}(x, s\eta(s)) = s^{\mu_{j,\alpha}} (C_{j,\alpha}(x) + o(1)) \quad \text{as} \quad s \downarrow 0, \quad C_{j,\alpha}(x) \not\equiv 0$$

for $x \in V$ with $x_1 \geq -t_0$, if $l^{1,j(\alpha)}(x, s\eta(s)) \not\equiv 0$ in (x,s) for $x \in V$ with $x_1 \geq -t_0$. Note that $C_{0,\alpha}(x) \equiv C_{0,\alpha}$. We put $C_{j,\alpha}(x) \equiv 0$ and $\mu_{j,\alpha} = \infty$ if $l^{1,j(\alpha)}(x, s\eta(s)) \equiv 0$ in (x,s) for $x \in V$ with $x_1 \geq -t_0$. Moreover, we put $\mu_j = \max_\alpha(m_1 - j - \mu_{j,\alpha} - (1-\sigma)|\alpha|)$. Then we can write

$$l^{1,0}(\hat{\eta}(s) + \xi) = s^{-(\mu_0 - \sigma m_1)/(1-\sigma)} (c_0(\xi) + o(1)),$$

$$l^{1,j}(x, \hat{\eta}(s) + \xi) = s^{-(\mu_j - \sigma(m_1 - j))/(1-\sigma)} (c_j(x, \xi) + o(1)),$$

$$\text{if} \quad j \geq 1, \quad \mu_j > -\infty, \quad (x, \xi) \in V \times \mathbb{R}^n \quad \text{and} \quad x_1 \geq -t_0$$

as $s \downarrow 0$, where $c_0(\xi) \not\equiv 0$ and $c_j(x, \xi) \not\equiv 0$ in $(x, \xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$. Then we have

$$c_0(\xi) = \sum_{m_1 - \mu_{0,\alpha} - (1-\sigma)|\alpha| = \mu_0} \frac{C_{0,\alpha} \xi^\alpha}{\alpha!},$$

$$c_j(x, \xi) = \sum_{m_1 - j - \mu_{j,\alpha} - (1-\sigma)|\alpha| = \mu_j} \frac{C_{j,\alpha}(x) \xi^\alpha}{\alpha!}$$

for $j \geq 1$ with $\mu_j > -\infty$ and $(x, \xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$. From (4.8) it follow that

$$0 = \max_{\substack{x \in V, x_1 \geq -t_0 \\ \xi \in \mathbb{R}^n, j \geq 1}} \left\{ n(l^{1,0}, l^{1,j}(x, \cdot); \hat{\eta} + \xi) - \frac{\sigma j}{1 - \sigma} \right\}$$

$$= \max_{1 \leq j < m_1, \mu_j > -\infty} \frac{\mu_j - \mu_0}{1 - \sigma}.$$

This yields

$$\mu_0 = \max_{1 \leq j \leq m_1 - 1} \mu_j \quad \left(= \max_{j \geq 1} \mu_j \right).$$

Put

$$T_\gamma(x, \xi) = \gamma^{-m_1} c_0(\gamma \xi) + \sum_{\substack{1 \leq j \leq m_1 - 1 \\ \mu_j = \mu_0}} \gamma^{-m_1 + j} c_j(x, \gamma \xi)$$

for $(x, \xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$. By assumption we have $T_\gamma(0, \xi) \not\equiv 0$ in ξ. Choose $N \in \mathbb{N}$ so that $N > (1 - \sigma)m_1$, which yields $N > m_1 - \mu_0$. Then there are a positive constant $\delta_0 \in \mathbb{Q}$ and a polynomial $r_N(x, \xi; s; \gamma)$ of (ξ, γ) such that

$$(4.14) \qquad P_N(x, \xi; s; \gamma) = s^{-\mu_0}(T_\gamma(x, \xi) + s^{\delta_0} r_N(x, \xi; s; \gamma)),$$

$$(4.15) \qquad |r_{N,\alpha(\beta)}(x; s; \gamma)| \leq C_{N,\alpha,\beta}(\gamma)s^{((1-\sigma)|\alpha|+\mu_0-m_1-\delta_0)_+}$$

for $(x, \xi) \in V \times \mathbb{R}^n$ with $x_1 \geq -t_0$, $0 < s \leq s_0$ and $\alpha, \beta \in (\mathbb{Z}_+)^n$ with $|\alpha| \leq [N/(1 - \sigma)]$, where $r_N(x, \xi; s; \gamma) = \sum_{|\alpha| \leq [N/(1-\sigma)],\, \alpha_1 \leq m_1} r_{N,\alpha}(x; s; \gamma)\xi^\alpha$. From the arguments as in the proof of Theorem 1.2.5 of [26] there are $\zeta^{0\prime} \in \mathbb{R}^{n-1}$, a neighborhood U of $\zeta^{0\prime}$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{N}$ such that the equation $T_\gamma(0, \tau, \eta') = 0$ in τ has a root with negative imaginary part for $\eta' \in U$ and its multiplicity is equal to q for $\eta' \in U$ (see, also, §§2.2 and 2.3 of [26]). We fix γ as above. Then there are real analytic functions $\tau(\eta')$ and $\widetilde{T}(\tau, \eta')$ defined for $\eta' \in U$ such that $\widetilde{T}(\tau, \eta')$ is a polynomial of τ, $\widetilde{T}(\tau(\eta'), \eta') \neq 0$, $\operatorname{Im} \tau(\eta') < 0$ and $T_\gamma(0, \tau, \eta') = (\tau - \tau(\eta'))^q \widetilde{T}(\tau, \eta')$ for $\eta' \in U$. Let $\varphi(x)$ be a solution of

$$\frac{\partial \varphi}{\partial x_1} = \tau(\nabla_{x'}\varphi(x)), \quad \varphi(0, x') = x' \cdot \zeta^{0\prime} + i|x'|^2$$

in a neighborhood V_1 of 0 in V. We choose σ_0, σ_1 and σ_2 so that $0 < \sigma_0 < \sigma/3$, $\sigma_1 = \sigma - \sigma_0$ and $\sigma_2 = \sigma - 2\sigma_0$. We shall impose further conditions on σ_0. It follows from §3 of Chapter VI of [24] that

$$T_\gamma(0, s^{3\sigma_0}D)(\exp[is^{-3\sigma_0}\varphi(x)]u(x))$$
$$= \exp[is^{-3\sigma_0}\varphi(x)] \sum_{|\alpha| \geq q} T_\gamma^{(\alpha)}(0, \nabla\varphi(x))\mathfrak{N}_\alpha(x, D; s)u(x),$$

where $\Phi(x, y) = \varphi(y) - \varphi(x) - (y - x) \cdot \nabla\varphi(x)$ and

$$\mathfrak{N}_\alpha(x, D; s)u(x) = \frac{1}{\alpha!}s^{3\sigma_0|\alpha|}\big[D_y^\alpha(\exp[is^{-3\sigma_0}\Phi(x, y)]u(y))\big]_{y=x}$$

(see, also, Lemma 3.1 of [2]). It is easy to see that

$$\mathfrak{N}_\alpha(x, \xi; s) = \frac{s^{3\sigma_0|\alpha|}\xi^\alpha}{\alpha!} + \sum_{\beta < \alpha} s^{3\sigma_0|\beta|}b_{\alpha,\beta}(x; s)\xi^\beta,$$

$$|b_{\alpha,\beta(\beta^1)}(x; s)| \leq C_{\alpha,\beta,\beta^1}s^{3\sigma_0(|\alpha|-|\beta|-[(|\alpha|-|\beta|)/2])},$$

$$b_{\alpha,\beta}(x; s) \equiv 0 \quad \text{if} \quad |\alpha| - |\beta| = 1 \quad \text{and} \quad \beta < \alpha$$

for $x \in V_1$. Put $\varphi(y; s) = \varphi(s^{2\sigma_0} y_1, s^{\sigma_0} y')$. Then a simple calculation yields

$$T_\gamma(0, s^{\sigma_0} D_{y_1}, s^{2\sigma_0} D_{y'})(\exp[is^{-3\sigma_0}\varphi(y; s)]u(y))$$

$$= \left[T_\gamma(0, s^{3\sigma_0} D_x)(\exp[is^{-3\sigma_0}\varphi(x)]u(s^{-2\sigma_0} x_1, s^{-\sigma_0} x'))\right]_{x_1 = s^{2\sigma_0} y_1, \, x' = s^{\sigma_0} y'}$$

$$= \exp[is^{-3\sigma_0}\varphi(y; s)] \sum_{|\alpha| \geq q} T_\gamma^{(\alpha)}(0, (\nabla\varphi)(s^{2\sigma_0} y_1, s^{\sigma_0} y'))$$

$$\times \left\{ s^{\sigma_0(3|\alpha|-2\alpha_1-|\alpha'|)} \frac{D_y^\alpha u(y)}{\alpha!} \right.$$

(4.16)
$$\left. + \sum_{\beta < \alpha} s^{\sigma_0(3|\beta|-2\beta_1-|\beta'|)} b_{\alpha,\beta}(s^{2\sigma_0} y_1, s^{\sigma_0} y'; s) D_y^\beta u(y) \right\}$$

$$= s^{\sigma_0 q} \exp[is^{-3\sigma_0}\varphi(y; s)] \left\{ T_\gamma^{(qe_1)}(0, \tau(\zeta^{0\prime}), \zeta^{0\prime}) \frac{D_{y_1}^q u(y)}{q!} \right.$$

$$\left. + s^{\sigma_0} \sum_{|\alpha| \leq q_0} c_\alpha(y; s) D_y^\alpha u(y) \right\},$$

$$|c_{\alpha(\beta)}(y; s)| \leq C_{\alpha,\beta}$$

for $y \in \mathbb{R}^n$ with $(s^{2\sigma_0} y_1, s^{\sigma_0} y') \in V_1$, where $q_0 = \deg_\xi T_\gamma(0, \xi)$. Put

$$E(y; s) = \exp[i\gamma^{-1}(s^{\sigma_1} y_1 \eta_1(s) + s^{\sigma_2} y' \cdot \eta'(s))],$$

where $\eta(s) = (\eta_1(s), \eta'(s))$. Let W be an open bounded neighborhood of 0 in $\mathbb{R}^n$. Then we have

$$E(y; s)^{-1} l^1(y, D_y; s)(E(y; s) \exp[is^{-3\sigma_0}\varphi(y; s)]u(y))$$

$$= l^1(y, \gamma^{-1}s^{\sigma_1}\eta_1(s) + D_{y_1}, \gamma^{-1}s^{\sigma_2}\eta'(s) + D_{y'}; s)(\exp[is^{-3\sigma_0}\varphi(y; s)]u(y))$$

(4.17)
$$= \{P_N(s^{\sigma_1} y_1, s^{\sigma_2} y', s^{\sigma_0} D_{y_1}, s^{2\sigma_0} D_{y'}; s; \gamma)$$

$$+ R^N(s^{\sigma_1} y_1, s^{\sigma_2} y', s^{\sigma_0} D_{y_1}, s^{2\sigma_0} D_{y'}; s; \gamma)\}(\exp[is^{-3\sigma_0}\varphi(y; s)]u(y))$$

for $u \in C_0^\infty(W)$ and $0 < s \leq s(W)$, where $s(W)$ is a positive constant satisfying $\{(s^{2\sigma_0} y_1, s^{\sigma_0} y'); y \in W, \, 0 < s \leq s(W)\} \subset V_1 \cap \{x \in \mathbb{R}^n; x_1 \geq -t_0\}$. Now we take $\sigma_0 = \min\{\delta_0/(1+q), \sigma/(3+q)\}$. From (4.14)–(4.16) we have

$$P_N(s^{\sigma_1} y_1, s^{\sigma_2} y', s^{\sigma_0} D_{y_1}, s^{2\sigma_0} D_{y'}; s; \gamma)(\exp[is^{-3\sigma_0}\varphi(y; s)]u(y))$$

$$= s^{-\mu_0 + \sigma_0 q} \exp[is^{-3\sigma_0}\varphi(y; s)]\{a_0 D_{y_1}^q u(y) - s^{\sigma_0} H_N(y, D_y; s)u(y)\},$$

where $a_0 = T_\gamma^{(qe_1)}(0, \tau(\zeta^{0\prime}), \zeta^{0\prime})/q!$ $(= \tilde{T}(\tau(\zeta^{0\prime}), \zeta^{0\prime}) \neq 0)$ and $H_N(y, \eta; s) \equiv \sum_{|\alpha| \leq [N/(1-\sigma)], \, \alpha_1 \leq m_1} H_{N,\alpha}(y; s)\eta^\alpha$ satisfies

$$|H_{N,\alpha,(\beta)}(y; s)| \leq C_{N,\alpha,\beta} \quad \text{for} \quad y \in W \quad \text{and} \quad 0 < s \leq s(W).$$

We define $\{u_{N,j}(y;s)\}_{0 \le j \le [N/\sigma_0]}$ by

$$
\begin{cases}
u_{N,0}(y;s) = 1, \\
D_{y_1}^q u_{N,j}(y;s) = a_0^{-1} H_N(y, D_y; s) u_{N,j-1}(y;s), \\
D_1^k u_{N,j}(0, y'; s) = 0 \quad (0 \le k \le q - 1), \\
\left(1 \le j \le \left[\dfrac{N}{\sigma_0}\right]\right),
\end{cases}
$$

for $y \in W$ and $0 < s \le s(W)$. Note that

$$(4.18) \qquad \left| D_y^\alpha u_{N,j}(y;s) \right| \le C_{N,\alpha}$$

for $y \in W$, $0 < s \le s(W)$, $0 \le j \le [N/\sigma_0]$ and $\alpha \in (\mathbb{Z}_+)^n$. It is easy to see that

$$(4.19) \qquad \varphi(y;s) = s^{\sigma_0} y' \cdot \zeta^{0\prime} + i s^{2\sigma_0} |y'|^2 + s^{2\sigma_0} \tau(\zeta^{0\prime}) y_1 + O(s^{3\sigma_0}) \quad \text{as} \quad s \downarrow 0,$$

$$(4.20) \qquad \operatorname{Im} s^{-3\sigma_0} \varphi(y;s) \ge \frac{s^{-\sigma_0}(\operatorname{Im} \tau(\zeta^{0\prime}) y_1 + |y'|^2)}{2}$$

for $y \in W$ with $y_1 \le 0$ and $0 < s \le s(W)$, modifying $s(W)$ if necessary. Let $\chi(x)$ be a function in $C_0^\infty(W)$ such that $\chi(x) = 1$ near 0, and put

$$u_N(y;s) = \sum_{j=0}^{[N/\sigma_0]} s^{\sigma_0 j} u_{N,j}(y;s) \chi(y).$$

Then, by standard arguments we have

$$(4.21) \qquad \max_{y_1 \le 0, |\alpha| \le p_1} s^{-\sigma_1 \alpha_1 - \sigma_2 |\alpha'|} \left| D_y^\alpha \{ E(y;s) P_N(s^{\sigma_1} y_1, s^{\sigma_2} y', s^{\sigma_0} D_{y_1}, s^{2\sigma_0} D_{y'}; s; \gamma) \right.$$
$$\left. \times (\exp[i s^{-3\sigma_0} \varphi(y;s)] u_N(y;s)) \} \right| \le C_N s^{N - \mu_0 + \sigma_0 q - p_1}.$$

From (4.18)–(4.20) it follows that for $k, l \in \mathbb{Z}_+$ and $y_1 \le 0$

$$(4.22) \qquad \left| D_1^k \mathscr{F}_{y'} [\exp[i s^{-3\sigma_0} \varphi(y;s)] u_N(y;s)](\eta') \right| \le C_{N,k,l} s^{-\sigma_0 k} \langle s^{2\sigma_0} \eta' \rangle^{-l},$$

where $\mathscr{F}_{y'}[u](\eta') = \int_{\mathbb{R}^{n-1}} e^{-iy' \cdot \eta'} u(y) \, dy'$. Indeed, we have

$$
\left| D_1^k (\eta')^{\alpha'} \mathscr{F}_{y'} [\exp[i s^{-3\sigma_0} \varphi(y;s)] w(y)](\eta') \right|
$$
$$
\le \int \left| D_1^k D_{y'}^{\alpha'} \exp[i s^{-3\sigma_0} \varphi(y;s)] w(y) \, dy' \right|
$$
$$
\le C_{k,\alpha'}(w) s^{-\sigma_0 k - 2\sigma_0 |\alpha'|} \quad \text{for} \quad w \in C_0^\infty(\mathbb{R}^n).
$$

Choose $\psi(\xi') \in C_0^\infty(\mathbb{R}^{n-1})$ so that $\psi(\xi') = 1$ for $|\xi'| \leq c_0/(8|\gamma|)$ and $\psi(\xi') = 0$ for $|\xi'| \geq c_0/(4|\gamma|)$. From (4.11)–(4.13) we have

$$\left| D_y^\beta R^{N,j}(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{2\sigma_0}\eta'; s; \gamma)\psi(s^{1-\sigma_2}\eta') \right|$$
$$\leq C_{N,\beta} s^{N-m_1}\langle s^{2\sigma_0}\eta'\rangle^{N/(1-\sigma)+1},$$
$$\left| D_y^\beta R^{N,j}(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{2\sigma_0}\eta'; s; \gamma)(1 - \psi(s^{1-\sigma_2}\eta')) \right|$$
$$\leq C_{N,\beta} s^{N-m_1}\langle s^{2\sigma_0}\eta'\rangle^{m_1-j+N/(1-\sigma)},$$

since $Cs^{1-\sigma}\langle s^{2\sigma_0}\eta'\rangle \geq 1$ if $1 - \psi(s^{1-\sigma_2}\eta') \neq 0$. This, together with (4.22), yields

(4.23)
$$\max_{y_1\leq 0, |\alpha|\leq p_1} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\left| D_y^\alpha\{E(y; s)R^N(s^{\sigma_1}y_1, s^{\sigma_2}y', s^{\sigma_0}D_{y_1}, s^{2\sigma_0}D_{y'}; s; \gamma)\right.$$
$$\left. \times (\exp[is^{-3\sigma_0}\varphi(y; s)]u_N(y; s))\}\right| \leq C_{N,p_1} s^{N-m_1-2\sigma_0 n-p_1},$$

since $\langle s^{2\sigma_0}\eta'\rangle^{-n} \leq s^{-2\sigma_0 n}\langle\eta'\rangle^{-n}$ if $0 < s \leq 1$. It is obvious that

$$\langle s^{-\sigma_2}D_{y'}\rangle^{-M} D_y^\alpha(E(y; s)\exp[is^{-3\sigma_0}\varphi(y; s)]u_N(y; s))$$
$$= E(y; s)(\gamma^{-1}s^{\sigma_1}\eta_1(s) + D_{y_1})^{\alpha_1}\langle\gamma^{-1}\eta'(s) + s^{-\sigma_2}D_{y'}\rangle^{-M}$$
$$\times (\gamma^{-1}s^{\sigma_2}\eta'(s) + D_{y'})^{\alpha'}(\exp[is^{-3\sigma_0}\varphi(y; s)]u_N(y; s)),$$
$$|\langle\gamma^{-1}\eta'(s) + s^{-\sigma_2}\eta'\rangle^{-M}(\gamma^{-1}s^{\sigma_2}\eta'(s) + \eta')^{\alpha'}\psi(s^{1-\sigma_2}\eta')|$$
$$\leq C_{M,\alpha'}s^{\sigma_2|\alpha'|+M-|\alpha'|},$$
$$|\langle\gamma^{-1}\eta'(s) + s^{-\sigma_2}\eta'\rangle^{-M}(\gamma^{-1}s^{\sigma_2}\eta'(s) + \eta')^{\alpha'}(1 - \psi(s^{1-\sigma_2}\eta'))|$$
$$\leq C_l s^{\sigma_2|\alpha'|+(1-\sigma)l}\langle s^{2\sigma_0}\eta'\rangle^l \quad (l \in \mathbb{Z}_+)$$

if $M \geq |\alpha'|$. So, (4.22) gives

(4.24)
$$\max_{y_1\leq 0, |\alpha|\leq p_2} s^{-\sigma_1\alpha_1-\sigma_2|\alpha'|}\left\| \langle s^{-\sigma_2}D_{y'}\rangle^{-M} D_y^\alpha \right.$$
$$\left. \times (E(y; s)\exp[is^{-3\sigma_0}\varphi(y; s)]u_N(y; s))\right\|(y_1) \leq C_{N,M,p_2}s$$

if $M \geq p_2 + \sigma_0 n + 1$. Put $\varepsilon(\gamma) = +$ if $\gamma > 0$ and $\varepsilon(\gamma) = -$ if $\gamma < 0$. We modify $\mathscr{C}_0'$ so that $\eta^{0'} \in \mathscr{C}_0'$. Then $\Psi(\xi') = 1$ if ξ' belongs to a conic neighborhood of $\eta^{0'}$ and $|\xi'| \geq 1$. Since $\sigma = \sigma_2 + 2\sigma_0$, and $1 \leq Cs^{1-\sigma_2}\langle\eta'\rangle$ and $\langle s^{2\sigma_0}\eta'\rangle^{-1} \leq Cs^{-2\sigma_0}\langle\eta'\rangle^{-1}$ if $1 - \Psi_{\varepsilon(\gamma)}(\gamma^{-1}\eta'(s) + s^{-\sigma_2}\eta') \neq 0$, by (4.22) we have

$$\left| D_1^k(1 - \Psi_{\varepsilon(\gamma)}(\gamma^{-1}\eta'(s) + s^{-\sigma_2}\eta'))(\gamma^{-1}s^{\sigma_2}\eta'(s) + \eta')^{\alpha'}\right.$$
$$\left. \times \mathscr{F}_{y'}[\exp[is^{-3\sigma_0}\varphi(y; s)]u_N(y; s)](\eta')\right|$$
$$\leq C_{N,k,l}s^{-\sigma_0 k+(1-\sigma)l-2\sigma_0(|\alpha'|+n/2)}\langle\eta'\rangle^{-n/2}$$

for $y_1 \leq 0$ and $l \in \mathbb{Z}_+$. Therefore, we have

$$(4.25) \quad \max_{y_1 \leq 0, |\alpha| \leq p_3} s^{-\sigma_1 \alpha_1 - \sigma_2 |\alpha'|} \|(1 - \Psi_{\varepsilon(\gamma)}(D_{y'}; s))$$
$$\times (E(y; s) \exp[i s^{-3\sigma_0} \varphi(y; s)] u_N(y; s)) \|(y_1) \leq C_{N, p_3} s.$$

By (4.17) the estimates (4.21) and (4.23)–(4.25) with $N \geq \max\{\mu_0 + p_1 + 1 - \sigma_0 q, m_1 + 2\sigma_0 n + p_1 + 1\}$ and $M \geq p_2 + \sigma_0 n + 1$ contradict (4.10), since $E(0; s) \exp[i s^{-3\sigma_0} \varphi(0; s)] \times u_N(0; s) = 1$. This proves the assertion (ii) of Theorem 1.1.

5. Some remarks and examples

First we remark that we can obtain the same results on propagation of singularities for operators satisfying the conditions (H), (R) and (L) as given in the colloray of Theorem 3.1 of [25], combining the arguments in this paper with results in [13].

In the case where the characterisic polynomial depends on x, we can also prove C^∞ well-posedness of the Cauchy problem under the maximal rank condition if for each $(x^0, \xi^{0\prime}) \in \mathbb{R}^n \times S^{n-2}$ the reduced operators $l^\mu(x, D)$ satisfy microlocal *a priori* estimates in [14], where the $l^\mu(x, D)$ are defined as in the form of (2.6).

EXAMPLE 5.1.   Let $n = 3$, and let

$$L(x, \xi) = \begin{pmatrix} \xi_1 + b(x) & \xi_3 & -\xi_2 \\ a(x) & \xi_1 + \xi_2 & \xi_3 \\ 0 & -a(x) & \xi_1 \end{pmatrix}.$$

It is easy to see that $L(x, D)$ satisfies the conditions (H) and (R). For each $\xi^{0\prime} \in S^1$ with $\xi_3^0 \neq 0$ there are a conic neighborhood $\mathscr{C}'(\xi^{0\prime})$ of $\xi^{0\prime}$, $N_k(x, \xi) \in M_3(\mathscr{S}_{1,0}^2)$ ($k = 1, 2$) and $l(x, \xi) \in \mathscr{S}_{1,0}^3$ such that the $N_k(x, D)$ have parametrices in $M_3(\mathscr{L}_{1,0}^2)$ and

$$N_1(x, D) L(x, D) N_2(x, D) \equiv \text{diag}(1, 1, l(x, D)) \bmod \mathscr{L}_{1,0}^{5, -\infty} \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}),$$

$$l(x, \xi)$$
$$\equiv \xi_1^2(\xi_1 + \xi_2) + b(x)\xi_1(\xi_1 + \xi_2) + (a(x)b(x) - D_1 a(x))\xi_3 + (D_1 b(x))(2\xi_1 + \xi_2)$$
$$+ a(x)^2 \xi_2 + (D_2 b(x))\xi_1 + (D_1 a(x))\xi_1 \xi_2 \xi_3^{-1} + (D_2 a(x))\xi_1^2 \xi_3^{-1}$$
$$- \{a(x)b(x)\xi_1(\xi_1 + \xi_2)\xi_2^2 \xi_3^{-1} + 2((D_2 b(x))\xi_2 + (D_3 b(x))\xi_3)\xi_1(\xi_1 + \xi_2)$$
$$- ((D_1 a(x))(2\xi_1 + \xi_2) - (D_2 a(x))\xi_1)\xi_1 \xi_2^2 \xi_3^{-1} - 2(D_2 a(x))\xi_1 \xi_2 \xi_3$$
$$+ (D_3 a(x))\xi_1(\xi_2^2 - \xi_3^2)\}(\xi_2^2 + \xi_3^2)^{-1} \bmod S_{1,0}^0 \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}),$$

where V is a bounded open subset of $\mathbb{R}^3$. Let $\xi^{0\prime} \in S^1$ satisfy $\xi_2^0 \neq 0$. First we can reduce $L(x, \xi)$ to

$$\begin{pmatrix} \xi_1 + b(x) - a(x)\xi_2^{-1}\xi_3 & -\xi_2^{-1}\left(\xi_2^2 + \xi_3^2\right) - a(x)\xi_2^{-3}\xi_3^3 & 0 \\ 0 & \xi_1 + a(x)\xi_2^{-1}\xi_3 & 0 \\ 0 & 0 & \xi_1 + \xi_2 \end{pmatrix}$$

mod $S_{1,0}^{-1}$ in $V \times \mathscr{C}'(\xi^{0\prime})$ (block-diagonalization). Then, by elementary transformations we can transform $L(x, D)$ to

$$\operatorname{diag}\left(1, D_1^2 + b(x)D_1, D_1 + D_2\right) \bmod S_{1,0}^0 \quad \text{in} \quad V \times \mathscr{C}'(\xi^{0\prime}).$$

Therefore, it follows from Theorem 1.1 that the Cauchy problem $(CP)_t$ for $L(x, D)$ is C^∞ well-posed for any $t > 0$ if and only if $D_1 a(x) = a(x)b(x)$ for any $x \in \mathbb{R}^3$ with $x_1 \geq 0$. On the other hand, for a fixed $x \in \mathbb{R}^3$ $\det L(x, \xi)$ is hyperbolic with respect to ϑ, i.e., $\det L(x, \xi - i\vartheta) \neq 0$ for any $\xi \in \mathbb{R}^3$, if and only if $a(x)b(x) = 0$.

EXAMPLE 5.2. Let $n = 3$, and let

$$L(x, \xi) = \begin{pmatrix} \xi_1 + a(x) & \alpha(x)\xi_2 + \xi_3 + b(x) \\ c(x) & \xi_1 + \xi_2 + d(x) \end{pmatrix}.$$

Then $p(\xi) \equiv \det L_1(x, \xi) = \xi_1(\xi_1 + \xi_2)$ and $L(x, D)$ satisfies the conditions (H) and (R). Note that $(dp)(\xi_1, \xi^{0\prime}) \neq 0$ for any $\xi_1 \in \mathbb{R}$ if $\xi_2^0 \neq 0$, where $\xi^{0\prime} = \left(\xi_2^0, \xi_3^0\right) \in S^1$. Let $x^0 \in \mathbb{R}^3$ and $\xi^{0\prime} = (0, \pm 1) \in S^1$. Then there are a neighborhood $V(x^0)$ of x^0, a conic neighborhood $\mathscr{C}'(\xi^{0\prime})$ of $\xi^{0\prime}$, $N_k(x, \xi) \in M_2\left(\mathscr{S}_{1,0}^1\right)$ $(k = 1, 2)$ and $l(x, \xi) \in \mathscr{S}_{1,0}^2$ such that the $N_k(x, D)$ have parametrices in $M_2\left(\mathscr{L}_{1,0}^1\right)$ and

$$N_1(x, D)L(x, D)N_2(x, D) \equiv \operatorname{diag}(1, l(x, D)) \bmod \mathscr{L}_{1,0}^{3,-\infty} \quad \text{in} \quad V(x^0) \times \mathscr{C}'(\xi^{0\prime}),$$

$$l(x, \xi) \equiv \xi_1(\xi_1 + \xi_2) + (a(x) + d(x))\xi_1 + a(x)\xi_2$$

$$+ (D_1\alpha(x))\xi_2(\xi_1 + \xi_2)(\xi_3 + \alpha(x)\xi_2)^{-1} - c(x)\xi_3 - \alpha(x)c(x)\xi_2$$

mod $S_{1,0}^0$ in $V(x^0) \times \mathscr{C}'(\xi^{0\prime})$. Therefore, it follows from Theorem 1.1 that the Cauchy problem $(CP)_t$ for $L(x, D)$ is C^∞ well-posed for any $t > 0$ if and only if $c(x) = 0$ for $x \in \mathbb{R}^3$ with $x_1 \geq 0$. For $x^0 \in \mathbb{R}^3$ and $\xi^{0\prime} = (0, \pm 1) \in S^1$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 + \alpha(x)\xi_2\xi_3^{-1} \end{pmatrix} L_1(x, \xi) \begin{pmatrix} 1 & 0 \\ 0 & (1 + \alpha(x)\xi_2\xi_3^{-1})^{-1} \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_3 \\ 0 & \xi_1 + \xi_2 \end{pmatrix}$$

in a conic neighborhood of $(x^0, \xi^{0\prime})$. This implies that $L(x, D)$ can be reduced microlocally to a system with constant coefficient principal part. However, we can not directly apply the results of [29] to $L(x, D)$ since the lower order terms become pseudo-differential operators.

References

[1] S. Benvenuti, E. Bernardi and A. Bove: *The Cauchy problem for hyperbolic systems with multiple characteristics*, Bull. Sci. Math. **122** (1998), 603–634.

[2] A. Bove and T. Nishitani: *Necessary conditions for the well-posedness of the Cauchy problem for hyperbolic systems*, Osaka J. Math. **39** (2002), 149–179.

[3] A. Bove and T. Nishitani: *Necessary conditions for hyperbolic systems*. II, Japan J. Math. (N.S.) **29** (2003), 357–388.

[4] J. Chazarain: *Opérateurs hyperboliques a caractéristiques de multiplicité constante*, Ann. Inst. Fourier (Grenoble) **24** (1974), 173–202.

[5] J.L. Dunn: *A sufficient condition for hyperbolicity of partial differential operators with constant coefficient principal part*, Trans. Amer. Math. Soc. **201** (1975), 315–327.

[6] H. Flaschka and G. Strang: *The correctness of the Cauchy problem*, Advances in Math. **6** (1971), 347–379.

[7] L. Hörmander: The Analysis of Linear Partial Differential Operators I, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[8] L. Hörmander: The Analysis of Linear Partial Differential Operators II, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[9] V.Ya. Ivrii: *Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey classes*, Sib. Math. J. **17** (1977), 921–931.

[10] V.Ya. Ivrii and V.M. Petkov: *Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed*, Russian Math. Surveys **29** (1974), 1–70.

[11] N. Iwasaki: *The Cauchy problem for effectively hyperbolic equations* (*general cases*), J. Math. Kyoto Univ. **25** (1985), 727–743.

[12] K. Kajitani: *Cauchy problem for nonstrictly hyperbolic systems*, Publ. Res. Inst. Math. Sci. **15** (1979), 519–550.

[13] K. Kajitani and S. Wakabayashi: *Propagation of singularities for several classes of pseudo-differential operators*, Bull. Sci. Math. **115** (1991), 397–449.

[14] K. Kajitani and S. Wakabayashi: *Microlocal a priori estimates and the Cauchy problem* I, Japan. J. Math. (N.S.) **19** (1994), 353–418.

[15] K. Kajitani and S. Wakabayashi: *Microlocal a priori estimates and the Cauchy problem* II, Japan. J. Math. (N.S.) **20** (1994), 1–71.

[16] T. Mandai: *Generalized Levi conditions for weakly hyperbolic equations—an attempt to treat the degeneracy with respect to the space variables*, Publ. Res. Inst. Math. Sci. **22** (1986), 1–23.

[17] W. Matsumoto: *Normal form of systems of partial differential and pseudo-differential operators in formal symbol classes*, J. Math. Kyoto Univ. **34** (1994), 15–40.

[18] W. Matsumoto: *Direct proof of the perfect block diagonalization of systems of pseudo-differential operators in the ultradifferentiable classes*, J. Math. Kyoto Univ. **40** (2000), 541–566.

[19] S. Mizohata and Y. Ohya: *Sur la condition de E.E. Levi concernant des équations hyperboliques*, Publ. Res. Inst. Math. Sci. Ser. A **4** (1968/1969), 511–526.

[20] T. Nishitani: *A necessary and sufficient condition for the hyperbolicity of second order equations in two independent variables*, J. Math. Kyoto Univ. **24** (1984), 91–104.

[21] T. Nishitani: *Hyperbolicity of two by two systems with two independent variables*, Comm. Partial Differential Equations **23** (1998), 1061–1110.

[22] V.M. Petkov: *The parametrix of the Cauchy problem for nonsymmetrizable hyperbolic equations with characteristics of constant multiplicity*, Trans. Moscow Math. Soc. **37** (1980), 1–47.

[23] S.L. Svensson: *Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part*, Ark. Mat. **8** (1969), 145–162.

[24] F. Trèves: Introduction to Pseudodifferential and Fourier Integral Operators. II, Plenum, New York, 1980.

[25] S. Wakabayashi: *Propagation of singularities for hyperbolic operators with constant coefficient principal part*, Tsukuba J. Math. **2** (1978), 91–107.

[26] S. Wakabayashi: *The Cauchy problem for operators with constant coefficient hyperbolic principal part and propagation of singularities*, Japan. J. Math. (N.S.) **6** (1980), 179–228.

[27] S. Wakabayashi: *A necessary conditions for the mixed problem to be C^∞ well-posed*, Comm. Partial Differential Equations **5** (1980), 1031–1064.

[28] S. Wakabayashi: *Singularities of solutions of the Cauchy problems for operators with nearly constant coefficient hyperbolic principal part*, Comm. Partial Differential Equations **8** (1983), 347–406.

[29] S. Wakabayashi: *Remarks on the Cauchy problem for hyperbolic systems of first order with constant coefficient principal part*, unpublished.

[30] S. Wakabayashi: *Remarks on the Cauchy problem for first order hyperbolic systems with constant coefficient principal part*, preprint.

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