D-branes in OSp invariant closed string field theory

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Abstract

We study D-branes in the OSp invariant closed string field theory.

The OSp invariant string field theory is a covariantized version of the light-cone gauge string field theory. It has been proposed that the BRST invariant sector of this theory leads to a unitary S-matrix. We define the BRST invariant observables corresponding to on-shell asymptotic states. From the correlation functions of these observables, one can obtain the unitary S-matrix.

After these preparations, we construct the states in which D-branes are excited in the formulation of the OSp invariant closed string field theory. We calculate the disk amplitudes using these states and show that they coincide with the disk amplitudes in usual string theory.

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Chapter 1

Introduction

Superstring theory is a candidate for the theory of everything. This theory is expected to be a consistent theory of quantum gravity and can include all the physics of the standard model of the elementary particles. In order to explain the real world using this theory, we need to choose the vacuum corresponding to our world and discuss the theory around it. However, it is known that superstring theory has an enormous number of vacua and one can construct a consistent theory around each of them. We should decide which is the true vacuum of superstring theory. For this purpose, perturbative treatment of superstrings is unsatisfactory. We need to know the nonperturbative aspects of superstring theory.

Many proposals for nonperturbative formulations of strings have been made. Here we concentrate on one of them, string field theory. It is formulated as a second quantized theory of strings, and not only reproduces the results of the first quantized theory but also includes off-shell and nonperturbative physics. While several superstring field theories have been proposed, they involve unsolved difficult problems. For this reason, in this thesis, we consider only the bosonic string field theories as a first step.

Since it is difficult to find the true vacuum of string theory, let us consider an easier question: how can one describe some known vacua in the formulation of string field theories? We treat the flat Dp-branes in string field theories. Dp-branes were discovered in first quantized string theory. These are defined as $(p+1)$-dimensional hypersurfaces in spacetime on which open strings can end and they can emit and absorb closed strings. They have a definite tension. We would like to study the D-branes in bosonic string field theories.

In bosonic open string field theory, Sen conjectured that Dp-branes with $p < 25$ are described as unstable lump solutions [1] and this was tested in many papers starting with [2][3][4]. On the other hand, in the context of closed string field theories, although several attempts have been made [5][6][7], D-branes have not been studied so much. What we would like to study in this thesis is how one can realize D-branes in bosonic closed string field theory.

One of the reasons why D-branes have not been considered so much in closed string field theories is the difficulties in treating the closed string field theories. Although some closed string field theories have been constructed, almost all of them are not useful for our purpose. The most complete closed string field theory is the light-cone gauge string field theory [8][9][10]. Since this is a gauge fixed theory, the Hilbert space does not have negative norm states and this theory is manifestly unitary. Moreover, while it does not have manifest Lorentz invariance, it has been shown that the S-matrix elements of this theory are Lorentz covariant [11]. Other string field theories should be constructed so as to reproduce the S-matrix elements of the light-cone gauge string field theory. The most famous covariant closed string field theory may be the non-polynomial closed string field theory [12][13]. This theory reproduces the light-cone gauge results and have the gauge invariance. However, it has very complicated interaction terms and is very difficult to treat. String field theories which are covariant and have a simpler interaction were also proposed. One of them is HIKKO closed string field theory [14].[1] This theory has only three-point interactions. However, string field has unphysical parameter "string-length" and this causes unphysical divergence in loop-level. Here we consider another covariant theory, the OSp invariant closed string field theory [15] (See also [16][17][18][19].). The OSp invariant closed string field theory also has unphysical parameters, but they do not cause any divergence.

The OSp invariant string field theory is a covariantized version of the light-cone gauge string field theory. This theory was constructed by adding two bosonic directions and two fermionic directions to the light-cone gauge string field theory. It was expected that the BRST invariant sector of this theory is equivalent to the light-cone gauge string field theory via Parisi-Sourlas mechanism [23]. Since an extra time variable exists in the formulation, the OSp invariant string field theory should be considered as something like stochastic quantization. Indeed, the action of this theory looks different from the usual one but rather like stochastic quantization.

In this thesis, we will study the following two things. Firstly, we will explain that the OSp invariant string field theory provides the correct unitary S-matrix elements. In order to show the unitarity of this theory, we must specify the normalization of the external states. In previous works [16][18], S-matrix elements were calculated for on-shell physical states. Since the kinetic term of the action is not similar to the usual formulation, it is difficult to fix the normalization of these states. We will consider the BRST invariant observables instead, and fix the normalization using the two-point correlation functions. In [24], we have shown that the general S-matrix elements can be derived from the correlation functions of these observables and they coincide with those of the light-cone gauge string field theory. The proof is abstract. In this thesis, instead we will evaluate some S-matrix elements explicitly and show that they agree with those in the light-cone gauge string field theory.

Secondly, we will consider how D-branes are described in the OSp invariant closed string field theory. In perturbative closed string theories, one should deal with worldsheets without boundaries. In the presence of D-branes, we should consider worldsheets with boundaries. In a second quantized closed string theory, D-branes should be described as an object which have the effect of generating boundaries on the worldsheet. We will consider a state in the second quantized Hilbert space corresponding to such an object. Imposing the BRST invariance of the state, we can fix the form of the state.[2] Such states can be considered as states in which D-branes or ghost D-branes [25] are excited. Thus we refer to these states as D-brane states. We will calculate the disk amplitudes using the D-brane states and show that the disk amplitudes of bosonic string theory in the presence of D-branes and ghost D-branes are reproduced.

[1]While this theory has the unphysical divergence in loop-level, one can consider D-branes in the context of this theory. In [5], the authors introduced D-branes as a source term of the action of the HIKKO closed string field theory. However, they cannot determine the tension of the D-branes.

[2]This construction is very similar to the construction of the creation operators of D-branes [20][21] in a field theoretical description for the noncritical strings [22]. In that case, the Virasoro constraint was imposed.

The organization of this thesis is as follows.

In chapter 2, after reviewing the light-cone gauge string field theory in section 2.1, we will explain the formulation of the OSp invariant string field theory in section 2.2. We will see that this theory has the BRST invariance. We will also see that the kinetic term of the action of this theory looks like that of the stochastic quantization.

In chapter 3, we will construct the observables of the OSp invariant string field theory corresponding to on-shell asymptotic states. Although the kinetic term of the action has unusual form, the two-point functions of these observables yield usual propagators. We will also calculate some S-matrix elements explicitly and show that they coincide with the S-matrix elements of the light-cone gauge string field theory.

In first quantized closed string theory, D-branes can be seen as the source of closed strings. The emission and absorption of closed strings can be described using the boundary states. In chapter 4, we will introduce the boundary states and study their properties in the OSp invariant string field theory. We will first define the boundary states corresponding to the flat Dp-branes in section 4.1. Since the norm of the boundary states is infinite, we will introduce a BRST invariant regularization. In section 4.2 and 4.3, we will study overlaps of the boundary states with three-string vertices. We will obtain the idempotency relations [7] of the boundary states. These results will be used in chapter 5 to construct the D-brane states.

In chapter 5, we will study D-brane states. In section 5.1, we will construct D-brane states. Imposing the condition that the states are BRST invariant in the leading order of regularization parameter ϵ, we can fix the form of the states. In section 5.2, we will calculate disk amplitudes using these states and show that disk amplitudes in the presence of D-branes and ghost D-branes are reproduced including the normalizations.

Chapter 6 is devoted to discussions.

In appendix A, we will explain how to construct the string vertices. The prescription explained here will be used in chapter 2 to define the three-string vertex and in section 4.2 and 4.3 to calculate the overlaps of the boundary states with three-string vertices.

In appendix B and C, we will present the details of calculations needed in chapter 2 and 4, respectively.

This thesis is based on my papers [24][26][27] which have been done in collaboration with N. Ishibashi and K. Murakami. In this thesis, we set the string slope parameter $\alpha' = 2$.

Chapter 2

OSp Invariant String Field Theory

The string field theory we consider in this thesis is the OSp invariant string field theory [15]. This theory can be constructed from the light-cone gauge string field theory [8][9][10]. In this chapter, we review the formulation of this theory.

2.1 Review of Light-Cone Gauge String Field Theory

Since the OSp invariant string field theory can be constructed from the light-cone gauge string field theory, we briefly review it.

2.1.1 Light-cone gauge string field theory

In the light-cone gauge, the single-closed string states are specified by using their length $\alpha = 2p^+$, their transverse momenta $p^i = \alpha_0^i = \tilde{\alpha}_0^i$ $(i = 1, \ldots, 24)$ and their transverse oscillators $\alpha_n^i, \tilde{\alpha}_n^i$ $(n \neq 0, i = 1, \ldots, 24)$. The oscillators α_n^i and $\tilde{\alpha}_n^i$ are the coefficients of transverse string coordinate $X^i(\tau, \sigma)$ $(i = 1, \ldots, 24)$:

$$X^i(\tau, \sigma) = x^i - 2ip^i\tau + i\sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^i e^{-n(\tau+i\sigma)} + \tilde{\alpha}_n^i e^{-n(\tau-i\sigma)}\right) \ . \tag{2.1}$$

Here (τ, σ) denote the coordinate of the cylinder worldsheet. We use the Fock representation of the oscillators and the wave function for the zero modes. We take the wave function to be a function of α and p^i, i.e. we take the momentum representation for the zero modes. In this representation, the vacuum state $|0\rangle^{\mathrm{LC}}$ satisfies

$$\alpha_n^i|0\rangle^{\mathrm{LC}} = \tilde{\alpha}_n^i|0\rangle^{\mathrm{LC}} = i\frac{\partial}{\partial p^i}|0\rangle^{\mathrm{LC}} = \frac{\partial}{\partial \alpha}|0\rangle^{\mathrm{LC}} = 0 \qquad \text{for } n > 0 \ . \tag{2.2}$$

The integration measure for the zero modes of the r-th string is defined as

$$dr_{\mathrm{LC}} \equiv \frac{\alpha_r d\alpha_r}{2}\frac{d^{24}p_r}{(2\pi)^{25}} \ . \tag{2.3}$$

The string field $|\Phi(t)\rangle^{\mathrm{LC}}$ are defined as a state in the closed string Hilbert space. We can expand it as

$$|\Phi(t)\rangle^{\mathrm{LC}} = T(t, p^i, \alpha)|0\rangle^{\mathrm{LC}} + h_{ij}(t, p^i, \alpha)\alpha_{-1}^i\tilde{\alpha}_{-1}^j|0\rangle^{\mathrm{LC}} + \ldots \ , \tag{2.4}$$

where $t = x^+$ denotes the proper time and the coefficients functions $T(t, p^i, \alpha)$ and $h_{ij}(t, p^i, \alpha)$ are the spacetime fields corresponding to tachyon and graviton.

The reflector

To describe the BPZ conjugate of the string field, we use the reflector. The reflector ${}^{\mathrm{LC}}\langle R(1,2)|$ is given by

$$
{}^{\mathrm{LC}}\langle R(1,2)| = \delta_{\mathrm{LC}}(1,2)\, {}^{\mathrm{LC}}_{12}\langle 0|\, e^{E_{\mathrm{LC}}(1,2)}\, \frac{1}{\alpha_1}\ , \tag{2.5}
$$

where

$$
\begin{aligned}
{}^{\mathrm{LC}}_{12}\langle 0| &= {}^{\mathrm{LC}}_{1}\langle 0|\, {}^{\mathrm{LC}}_{2}\langle 0|\ , \\
E_{\mathrm{LC}}(1,2) &= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{24} \left(\alpha_n^{i(1)} \alpha_n^{i(2)} + \tilde{\alpha}_n^{i(1)} \tilde{\alpha}_n^{i(2)} \right)\ , \\
\delta_{\mathrm{LC}}(1,2) &= 2\delta(\alpha_1+\alpha_2)(2\pi)^{25}\delta^{24}(p_1+p_2)\ .
\end{aligned} \tag{2.6}
$$

We also introduce

$$
|R(1,2)\rangle^{\mathrm{LC}} = \delta_{\mathrm{LC}}(1,2) \frac{1}{\alpha_1} e^{E^{\dagger}_{\mathrm{LC}}(1,2)} |0\rangle^{\mathrm{LC}}_{12}\ . \tag{2.7}
$$

The reflector ${}^{\mathrm{LC}}\langle R(1,2)|$ satisfies

$$
\begin{aligned}
{}^{\mathrm{LC}}\langle R(1,2)|\,(\alpha_1+\alpha_2) = 0\ , &\qquad {}^{\mathrm{LC}}\langle R(1,2)|\left(x^{i(1)} - x^{i(2)}\right) = 0\ , \\
{}^{\mathrm{LC}}\langle R(1,2)|\left(\alpha_n^{i(1)} + \alpha_{-n}^{i(2)}\right) = 0\ , &\qquad {}^{\mathrm{LC}}\langle R(1,2)|\left(\tilde{\alpha}_n^{i(1)} + \tilde{\alpha}_{-n}^{i(2)}\right) = 0
\end{aligned} \tag{2.8}
$$

for $\forall n \in \mathbb{Z}$. This yields

$$
{}^{\mathrm{LC}}\langle R(1,2)|\left(L_n^{\mathrm{LC}(1)} - L_{-n}^{\mathrm{LC}(2)}\right) = 0\ , \quad {}^{\mathrm{LC}}\langle R(1,2)|\left(\tilde{L}_n^{\mathrm{LC}(1)} - \tilde{L}_{-n}^{\mathrm{LC}(2)}\right) = 0 \tag{2.9}
$$

for $\forall n \in \mathbb{Z}$. Here L_n^{LC} and $\tilde{L}_n^{\mathrm{LC}}$ are the Virasoro generators defined as

$$
L_n^{\mathrm{LC}} = \frac{1}{2} \sum_m \sum_{i=1}^{24} {}^{\circ}_{\circ} \alpha^i_{n+m} \alpha^i_{-m} {}^{\circ}_{\circ}\ , \qquad \tilde{L}_n^{\mathrm{LC}} = \frac{1}{2} \sum_m \sum_{i=1}^{24} {}^{\circ}_{\circ} \tilde{\alpha}^i_{n+m} \tilde{\alpha}^i_{-m} {}^{\circ}_{\circ}\ , \tag{2.10}
$$

where ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ means the normal ordering of the oscillators in which the non-negative modes should be moved to the right of the negative modes.

Using the reflector, we define the BPZ conjugate ${}^{\mathrm{LC}}\langle \Phi|$ of the string field $|\Phi\rangle^{\mathrm{LC}}$ as

$$
{}^{\mathrm{LC}}_{2}\langle \Phi| = \int d1_{\mathrm{LC}}\, {}^{\mathrm{LC}}\langle R(1,2)|\, \Phi\rangle^{\mathrm{LC}}_1\ . \tag{2.11}
$$

From the definitions, we have

$$
{}^{\mathrm{LC}}\langle R(1,2)| = -{}^{\mathrm{LC}}\langle R(2,1)|\ , \qquad \int d1_{\mathrm{LC}}\, {}^{\mathrm{LC}}_{1}\langle \Phi | R(1,2)\rangle^{\mathrm{LC}} = |\Phi\rangle^{\mathrm{LC}}_2\ . \tag{2.12}
$$

The three-string vertex

The action of the light-cone gauge closed string field theory has the three-string interaction term. To describe this term, we use the three-string vertex. One can construct it by using the method we will explain in appendix A. The three-string vertex ${}^{\mathrm{LC}}\langle V_3^0(1,2,3)|$ is given by

$$\begin{aligned}
{}^{\mathrm{LC}}\langle V_3^0(1,2,3)| &= {}^{\mathrm{LC}}\langle v_3^0(1,2,3)|\mathcal{P}_{123}^{\mathrm{LC}}\ , \\
{}^{\mathrm{LC}}\langle v_3^0(1,2,3)| &= \delta_{\mathrm{LC}}(1,2,3){}_{123}^{\mathrm{LC}}\langle 0|e^{E_{\mathrm{LC}}(1,2,3)}\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3}\ ,
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
{}_{123}^{\mathrm{LC}}\langle 0| &= {}_1^{\mathrm{LC}}\langle 0|\,{}_2^{\mathrm{LC}}\langle 0|\,{}_3^{\mathrm{LC}}\langle 0|\ , \\
\mathcal{P}_{123}^{\mathrm{LC}} &= \mathcal{P}_1^{\mathrm{LC}}\mathcal{P}_2^{\mathrm{LC}}\mathcal{P}_3^{\mathrm{LC}}\ , \qquad \mathcal{P}_r^{\mathrm{LC}} = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta\left(L_0^{\mathrm{LC}(r)}-\tilde{L}_0^{\mathrm{LC}(r)}\right)}\ , \\
\delta_{\mathrm{LC}}(1,2,3) &= 2\delta\left(\sum_{s=1}^{3}\alpha_s\right)(2\pi)^{25}\delta^{24}\left(\sum_{r=1}^{3}p_r\right)\ , \\
E_{\mathrm{LC}}(1,2,3) &= \frac{1}{2}\sum_{n,m\geq 0}\sum_{r,s}\sum_{i=1}^{24}\left(\bar{N}_{nm}^{rs}\alpha_n^{i(r)}\alpha_m^{i(s)}+(\bar{N}_{nm}^{rs})^*\,\tilde{\alpha}_n^{i(r)}\tilde{\alpha}_m^{i(s)}\right)\ , \\
\mu(1,2,3) &= \exp\left(-\hat{\tau}_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\right)\ , \qquad \hat{\tau}_0 = \sum_{r=1}^{3}\alpha_r\ln|\alpha_r|\ .
\end{aligned} \tag{2.14}$$

Here the Neumann coefficients $\bar{N}_{nm}^{rs}$ are defined in eq.(A.12). Nevertheless the Neumann coefficients $\bar{N}_{nm}^{rs}$ depend on the choice of Z_1, Z_2, Z_3, the vertex ${}^{\mathrm{LC}}\langle V_3^0(1,2,3)|$ is independent of it. This includes that the three-string vertex is invariant under any permutation of string $1,2,3$. If we choose $Z_1=1, Z_2=0, Z_3=\infty$, the Neumann coefficients are

$$\begin{aligned}
\bar{N}_{nm}^{rs} &= -\alpha_1\alpha_2\alpha_3\left(\frac{\alpha_r}{n}+\frac{\alpha_s}{m}\right)^{-1}\bar{N}_n^r\bar{N}_m^s \qquad (n,m\geq 1)\ , \\
\bar{N}_{n0}^{rs} &= -c_s\frac{\alpha_1\alpha_2}{\alpha_s}\bar{N}_n^r \qquad (c_1=1, c_2=-1, c_3=0) \qquad (n\geq 1)\ , \\
\bar{N}_{00}^{rs} &= \hat{\tau}_0\left(\frac{\delta_{rs}}{\alpha_r}-\frac{\delta_{r3}}{\alpha_3}-\frac{\delta_{s3}}{\alpha_3}\right)\ , \\
\bar{N}_n^r &= \frac{1}{\alpha_r}f_n\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)e^{\frac{n\hat{\tau}_0}{\alpha_r}} \qquad (\alpha_4=\alpha_1)\ , \\
f_n(x) &= \frac{\Gamma(nx)}{n!\Gamma(nx-n+1)}\ ,
\end{aligned} \tag{2.15}$$

where $\hat{\tau}_0$ is defined in eq.(2.14).

String field action

The action of the light-cone gauge string field theory is

$$S^{\mathrm{LC}} = \int dt\Bigg[\frac{1}{2}\int d1_{\mathrm{LC}}d2_{\mathrm{LC}}\,{}^{\mathrm{LC}}\langle R(1,2)|\Phi\rangle_1^{\mathrm{LC}}\left(i\frac{\partial}{\partial t}-\frac{L_0^{\mathrm{LC}(2)}+\tilde{L}_0^{\mathrm{LC}(2)}-2}{\alpha_2}\right)|\Phi\rangle_2^{\mathrm{LC}}$$

$$+\frac{2g}{3}\int d1_{\mathrm{LC}}d2_{\mathrm{LC}}d3_{\mathrm{LC}}\ {}^{\mathrm{LC}}\langle V_3^0(1,2,3)|\Phi\rangle_1^{\mathrm{LC}}|\Phi\rangle_2^{\mathrm{LC}}|\Phi\rangle_3^{\mathrm{LC}}\Bigg] \ , \tag{2.16}$$

where t denotes the proper time and g denotes the coupling constant for strings. In this thesis, we take $g > 0$. We require that the string field $|\Phi\rangle^{\mathrm{LC}}$ satisfies the level matching condition

$$\mathcal{P}^{\mathrm{LC}}|\Phi\rangle^{\mathrm{LC}} = |\Phi\rangle^{\mathrm{LC}} \ , \tag{2.17}$$

and the reality condition

$$\left(|\Phi\rangle^{\mathrm{LC}}\right)^\dagger = {}^{\mathrm{LC}}\langle\Phi| \ , \tag{2.18}$$

where the left hand side is the hermitian conjugate of $|\Phi\rangle^{\mathrm{LC}}$ and the right hand side is the BPZ conjugate of $|\Phi\rangle^{\mathrm{LC}}$.

By using the expansion of the string field eq.(2.4), one can easily find that the kinetic term of the action (2.16) is usual form.

2.1.2 Lorentz invariance

The light-cone gauge string field theory is manifestly unitary, but its Lorentz invariance is not manifest and needs a proof. The invariance of the on-shell S-matrix elements was first proved by Mandelstam [11]. In [28][29], the full transformation of the string fields were given and the invariance of the action under this transformation and the closure of the algebra were proved. In this subsection, we briefly review their proof. Explicit calculations will be given in Appendix B.

The nontrivial Lorentz transformation of the string field $|\Phi\rangle^{\mathrm{LC}}$ is given in [28][29] as

$$\begin{aligned}\mathcal{M}^{i-}|\Phi\rangle_4^{\mathrm{LC}} &= M^{i-(4)}|\Phi\rangle_4^{\mathrm{LC}} \\ &\quad +g\int d1_{\mathrm{LC}}d2_{\mathrm{LC}}d3_{\mathrm{LC}}\ {}^{\mathrm{LC}}\langle V_3^0(1,2,3);X^i|\Phi\rangle_1^{\mathrm{LC}}|\Phi\rangle_2^{\mathrm{LC}}|R(3,4)\rangle^{\mathrm{LC}} \ ,\end{aligned} \tag{2.19}$$

where M^{i-} denotes the Lorentz generator for free string theory[1]

$$\begin{aligned}M^{i-} &= \frac{x^i}{2\alpha}(L_0^{\mathrm{LC}}+\tilde{L}_0^{\mathrm{LC}}-2)+ip^i\frac{\partial}{\partial\alpha} \\ &\quad -\frac{i}{\alpha}\sum_{n=1}^{\infty}\left(\frac{\alpha_{-n}^iL_n^{\mathrm{LC}}-L_{-n}^{\mathrm{LC}}\alpha_n^i}{n}+\frac{\tilde{\alpha}_{-n}^i\tilde{L}_n^{\mathrm{LC}}-\tilde{L}_{-n}^{\mathrm{LC}}\tilde{\alpha}_n^i}{n}\right) \ ,\end{aligned} \tag{2.20}$$

and

$${}^{\mathrm{LC}}\langle V_3^0(1,2,3);X^i| = {}^{\mathrm{LC}}\langle v_3^0(1,2,3)|X^i(\rho_I,\bar{\rho}_I)\mathcal{P}_{123}^{\mathrm{LC}} \ . \tag{2.21}$$

Here ρ_I denotes the ρ coordinate for the interaction point of the three strings.

The action of the light-cone gauge string field theory eq.(2.16) is invariant under the Lorentz transformation eq.(2.19):

$$\mathcal{M}^{i-}S^{\mathrm{LC}} = 0 \ . \tag{2.22}$$

[1] One may think that the first line in eq.(2.20) is not hermite. However it is hermite under the integration measure eq.(2.3).

One can show it, by using the following identities:

$$
\begin{aligned}
&{}^{\mathrm{LC}}\langle R(1,2)| \left(M^{i-(1)} + M^{i-(2)}\right) = 0 \ , \\
&{}^{\mathrm{LC}}\langle V_3^0(1,2,3); X^i| \sum_{r=1}^{3} \frac{L_0^{\mathrm{LC}(r)} + \tilde{L}_0^{\mathrm{LC}(r)} - 2}{2\alpha_r} = {}^{\mathrm{LC}}\langle V_3^0(1,2,3)| \sum_{r=1}^{3} M^{i-(r)} \ , \\
&\int d1_{\mathrm{LC}} d2_{\mathrm{LC}} d3_{\mathrm{LC}} d4_{\mathrm{LC}} d5_{\mathrm{LC}} d6_{\mathrm{LC}}\, {}^{\mathrm{LC}}\langle V_3^0(1,2,5); X^i|\, {}^{\mathrm{LC}}\langle V_3^0(3,4,6)| \\
&\qquad\qquad \times |R(5,6)\rangle^{\mathrm{LC}} |\Phi\rangle_1^{\mathrm{LC}} |\Phi\rangle_2^{\mathrm{LC}} |\Phi\rangle_3^{\mathrm{LC}} |\Phi\rangle_4^{\mathrm{LC}} = 0 \ .
\end{aligned} \tag{2.23}
$$

The first equation can be easily proved by using the relations eqs.(2.8) and (2.9). The other two equations will be shown in appendix B.

We should show the closure of the Lorentz algebra. The nontrivial one is

$$
[\mathcal{M}^{i-}, \mathcal{M}^{j-}] = 0 \ . \tag{2.24}
$$

One can show this equation from the following relations:

$$
\begin{aligned}
&[M^{i-}, M^{j-}] = 0 \ , \\
&{}^{\mathrm{LC}}\langle V_3^0(1,2,3); X^i| \sum_{r=1}^{3} M^{j-(r)} - {}^{\mathrm{LC}}\langle V_3^0(1,2,3); X^j| \sum_{r=1}^{3} M^{i-(r)} = 0 \ , \\
&\int d2_{\mathrm{LC}} d3_{\mathrm{LC}} d4_{\mathrm{LC}} d5_{\mathrm{LC}} d6_{\mathrm{LC}} \Big[{}^{\mathrm{LC}}\langle V_3^0(1,2,5); X^i|\, {}^{\mathrm{LC}}\langle V_3^0(3,4,6); X^j| \\
&\quad - {}^{\mathrm{LC}}\langle V_3^0(1,2,5); X^j|\, {}^{\mathrm{LC}}\langle V_3^0(3,4,6); X^i| \Big] |R(5,6)\rangle^{\mathrm{LC}} |\Phi\rangle_2^{\mathrm{LC}} |\Phi\rangle_3^{\mathrm{LC}} |\Phi\rangle_4^{\mathrm{LC}} = 0 \ .
\end{aligned} \tag{2.25}
$$

The first equation[2] is shown in first quantized theory [30]. We will show the other two equations in appendix B.

2.1.3 Canonical quantization

We can decompose the string field as

$$
|\Phi\rangle^{\mathrm{LC}} = |\psi\rangle^{\mathrm{LC}} + |\bar{\psi}\rangle^{\mathrm{LC}} \ , \tag{2.26}
$$

where $|\psi\rangle^{\mathrm{LC}}$ is the part with positive α and $|\bar{\psi}\rangle^{\mathrm{LC}}$ is the one with negative α. The kinetic term of eq.(2.16) can be rewritten as

$$
\begin{aligned}
S_K^{\mathrm{LC}} &= \frac{1}{2} \int dt \int d1_{\mathrm{LC}} d2_{\mathrm{LC}}\, {}^{\mathrm{LC}}\langle R(1,2)|\Phi\rangle_1^{\mathrm{LC}} \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - 2}{\alpha_2} \right) |\Phi\rangle_2^{\mathrm{LC}} \\
&= \int dt \int d1_{\mathrm{LC}} d2_{\mathrm{LC}}\, {}^{\mathrm{LC}}\langle R(1,2)|\bar{\psi}\rangle_1^{\mathrm{LC}} \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - 2}{\alpha_2} \right) |\psi\rangle_2^{\mathrm{LC}} \\
&= \int dt \int d2_{\mathrm{LC}}\, {}_2^{\mathrm{LC}}\langle\bar{\psi}| \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - 2}{\alpha_2} \right) |\psi\rangle_2^{\mathrm{LC}} \ .
\end{aligned} \tag{2.27}
$$

[2] In this equation, we have neglected terms proportional to $L_0^{\mathrm{LC}} - \tilde{L}_0^{\mathrm{LC}}$. They equal to zero for the states which satisfy the level matching condition.

From this equation, we obtain the canonical commutation relation

$$\left[|\psi\rangle_r^{\mathrm{LC}}\,,\ {}^{\mathrm{LC}}_{\ \ s}\langle\bar{\psi}|\right] = I^{\mathrm{LC}}(r,s)\ , \qquad I^{\mathrm{LC}}(r,s) = \int du_{\mathrm{LC}}\,{}^{\mathrm{LC}}\langle R(s,u)|R(u,r)\rangle^{\mathrm{LC}}\ . \tag{2.28}$$

Here $I^{\mathrm{LC}}(r,s)$ serves as the identity operator. In fact, the following relations are hold for an arbitrary string field $|\Psi\rangle^{\mathrm{LC}}$,

$$\int ds_{\mathrm{LC}}\, I^{\mathrm{LC}}(r,s)|\Psi\rangle_s^{\mathrm{LC}} = |\Psi\rangle_r^{\mathrm{LC}}\ , \qquad \int dr_{\mathrm{LC}}\,{}^{\mathrm{LC}}_{\ \ r}\langle\Psi|I^{\mathrm{LC}}(r,s) = {}^{\mathrm{LC}}_{\ \ s}\langle\Psi|\ . \tag{2.29}$$

The first equation in (2.28) is equivalent to

$$\left[|\psi\rangle_r^{\mathrm{LC}}, |\bar{\psi}\rangle_s^{\mathrm{LC}}\right] = |R(r,s)\rangle^{\mathrm{LC}}\ . \tag{2.30}$$

From the hermiticity defined in eq.(2.18), one can deduce that ${}^{\mathrm{LC}}\langle\psi|$ and ${}^{\mathrm{LC}}\langle\bar{\psi}|$ are hermitian conjugate to $|\bar{\psi}\rangle^{\mathrm{LC}}$ and $|\psi\rangle^{\mathrm{LC}}$, respectively. We identify $|\psi\rangle^{\mathrm{LC}}$ with the annihilation mode and $|\bar{\psi}\rangle^{\mathrm{LC}}$ with the creation mode. Accordingly we define the vacuum state $|0\rangle\!\rangle^{\mathrm{LC}}$ in the second quantization as

$$|\psi\rangle^{\mathrm{LC}}|0\rangle\!\rangle^{\mathrm{LC}} = 0\ , \quad {}^{\mathrm{LC}}\langle\!\langle 0|{}^{\mathrm{LC}}\langle\bar{\psi}| = 0\ . \tag{2.31}$$

2.2 OSp Invariant String Field Theory

2.2.1 OSp extension

The OSp invariant string field theory can be constructed from replacing the $O(24)$ transverse vector X^i $(i = 1,\dots,24)$ in the light-cone gauge string field theory by the $OSp(26|2)$ vector $X^M = \left(X^\mu, X^C = C, X^{\bar{C}} = \bar{C}\right)$. Here $X^\mu = (X^i, X^{25}, X^{26})$ are Grassmann even and the ghost fields C and $\bar{C}$ are Grassmann odd. The metric of the $OSp(26|2)$ vector space is

$$\eta_{MN} = \begin{array}{c|c} & \begin{array}{cc} & C \quad \bar{C} \end{array} \\ \begin{array}{c} \\ \\ C \\ \bar{C} \end{array} & \left(\begin{array}{c|cc} \delta_{\mu\nu} & & \\ \hline & 0 & -i \\ & i & 0 \end{array}\right) \end{array} = \eta^{MN}\ . \tag{2.32}$$

As we will see, X^μ $(\mu = 1, 2, \dots, 26)$ can be regarded as a coordinate of 26 dimensional Euclidean space. In this meaning, this theory is covariant.

By using the metric (2.32), we can write down the Euclidean worldsheet action

$$S = \frac{1}{8\pi}\int d\tau d\sigma \partial_a X^M \partial^a X^N \eta_{MN}\ , \tag{2.33}$$

where (τ,σ) denote the coordinates on the cylinder worldsheet. We define the mode expansion of $X^M(\tau,\sigma)$ as

$$X^M(\tau,\sigma) = x^M - 2ip^M\tau + i\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^M e^{-n(\tau+i\sigma)} + \tilde{\alpha}_n^M e^{-n(\tau-i\sigma)}\right)\ . \tag{2.34}$$

The nonvanishing canonical commutation relations are

$$[x^N, p^M\} = i\eta^{NM} \ , \quad [\alpha_n^N, \alpha_m^M\} = n\eta^{NM}\delta_{n+m,0} \ , \quad [\tilde{\alpha}_n^N, \tilde{\alpha}_m^M\} = n\eta^{NM}\delta_{n+m,0} \tag{2.35}$$

for $n, m \neq 0$, where the graded commutator $[A, B\}$ denotes the anti-commutator when A and B are both fermionic operators and the commutator otherwise. We also use

$$x^M \equiv (x^\mu, C_0, \bar{C}_0) \ , \qquad \alpha_0^M = \tilde{\alpha}_0^M = p^M \equiv (p^\mu, -\pi_0, \bar{\pi}_0) \ . \tag{2.36}$$

The Hilbert space for the string consists of the Fock space of the oscillators and the wave function for the zero modes. Again, we take the momentum representation for the zero modes $(\alpha, p^\mu, \pi_0, \bar{\pi}_0)$. The vacuum state $|0\rangle$ in the first quantization is defined by

$$\begin{aligned} &\alpha_n^M|0\rangle = \tilde{\alpha}_n^M|0\rangle = 0 \qquad \text{for } n > 0 \ , \\ &x^M|0\rangle = i\eta^{MN}\frac{\partial}{\partial p^N}|0\rangle = 0 \ , \quad \frac{\partial}{\partial\alpha}|0\rangle = 0 \ . \end{aligned} \tag{2.37}$$

The integration measure for the zero modes of the r-th string is now defined as

$$dr \equiv \frac{\alpha_r d\alpha_r}{2}\frac{d^{26}p_r}{(2\pi)^{26}}\, id\bar{\pi}_0^{(r)}d\pi_0^{(r)} \ . \tag{2.38}$$

Various quantities appearing in the OSp invariant string field theory can be defined by the OSp extension from the corresponding objects in the light-cone gauge string field theory.

The reflector

The reflector in the OSp invariant string field theory is defined as

$$\langle R(1,2)| = \delta(1,2)\ {}_{12}\langle 0|\, e^{E(1,2)}\,\frac{1}{\alpha_1} \ , \tag{2.39}$$

where

$$\begin{aligned} {}_{12}\langle 0| &= {}_1\langle 0|{}_2\langle 0| \ , \\ E(1,2) &= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n^{N(1)}\alpha_n^{M(2)} + \tilde{\alpha}_n^{N(1)}\tilde{\alpha}_n^{M(2)}\right)\eta_{NM} \ , \\ \delta(1,2) &= 2\delta(\alpha_1+\alpha_2)(2\pi)^{26}\delta^{26}(p_1+p_2)i(\bar{\pi}_0^{(1)}+\bar{\pi}_0^{(2)})(\pi_0^{(1)}+\pi_0^{(2)}) \ . \end{aligned} \tag{2.40}$$

We also introduce

$$|R(1,2)\rangle = \delta(1,2)\frac{1}{\alpha_1}e^{E^\dagger(1,2)}|0\rangle_{12} \ . \tag{2.41}$$

The reflector $\langle R(1,2)|$ satisfies the similar relations to eqs.(2.8) and (2.9):

$$\begin{aligned} \langle R(1,2)|\,(\alpha_1+\alpha_2) = 0 \ , &\qquad \langle R(1,2)|\left(x^{M(1)} - x^{M(2)}\right) = 0 \ , \\ \langle R(1,2)|\left(\alpha_n^{M(1)} + \alpha_{-n}^{M(2)}\right) = 0 \ , &\qquad \langle R(1,2)|\left(\tilde{\alpha}_n^{M(1)} + \tilde{\alpha}_{-n}^{M(2)}\right) = 0 \ , \\ \langle R(1,2)|\left(L_n^{(1)} - L_{-n}^{(2)}\right) = 0 \ , &\qquad \langle R(1,2)|\left(\tilde{L}_n^{(1)} - \tilde{L}_{-n}^{(2)}\right) = 0 \qquad \text{for } \forall n \in \mathbb{Z} \ , \end{aligned} \tag{2.42}$$

where L_n and $\tilde{L}_n$ ($n \in \mathbb{Z}$) are the Virasoro generators defined as

$$L_n \equiv \frac{1}{2}\sum_m {}^\circ_\circ\alpha^N_{n+m}\alpha^M_{-m}{}^\circ_\circ\eta_{NM} \ , \qquad \tilde{L}_n \equiv \frac{1}{2}\sum_m {}^\circ_\circ\tilde{\alpha}^N_{n+m}\tilde{\alpha}^M_{-m}{}^\circ_\circ\eta_{NM} \ . \tag{2.43}$$

Here ${}^\circ_\circ \ {}^\circ_\circ$ means the normal ordering of the oscillators. $|R(1,2)\rangle$ satisfies similar identities.

The BPZ conjugate $\langle\Phi|$ of $|\Phi\rangle$ is defined as

$${}_2\langle\Phi| = \int d1\, \langle R(1,2)|\Phi\rangle_1 \ . \tag{2.44}$$

From the definitions, we have

$$\int d1d2\, \langle R(1,2)|\Phi\rangle_1|\Psi\rangle_2 = -(-1)^{|\Phi||\Psi|}\int d1d2\, \langle R(1,2)|\Psi\rangle_1|\Phi\rangle_2 \ , \tag{2.45}$$

and

$$\int d1\, {}_1\langle\Phi|R(1,2)\rangle = |\Phi\rangle_2 \ , \tag{2.46}$$

where $(-1)^{|\Phi|}$ denotes the Grassmann parity of the string field Φ.

The three-string vertex

The three-string vertex in the OSp invariant string field theory is given by

$$\begin{aligned}\langle V_3^0(1,2,3)| &= \langle v_3^0(1,2,3)|\mathcal{P}_{123} \ , \\ \langle v_3^0(1,2,3)| &= \delta(1,2,3)\ {}_{123}\langle 0|e^{E(1,2,3)}\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3} \ ,\end{aligned} \tag{2.47}$$

where $\mu(1,2,3)$ is defined in eq.(2.14) and

$$\begin{aligned}{}_{123}\langle 0| &= {}_1\langle 0|\,{}_2\langle 0|\,{}_3\langle 0| \ , \\ \mathcal{P}_{123} &= \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 \ , \quad \mathcal{P}_r = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta\left(L_0^{(r)}-\tilde{L}_0^{(r)}\right)} \ , \\ \delta(1,2,3) &= 2\delta\left(\sum_{s=1}^3\alpha_s\right)(2\pi)^{26}\delta^{26}\left(\sum_{r=1}^3 p_r\right) i\left(\sum_{r'=1}^3\bar{\pi}_0^{(r')}\right)\left(\sum_{s'=1}^3\pi_0^{(s')}\right) \ , \\ E(1,2,3) &= \frac{1}{2}\sum_{n,m\geq 0}\sum_{r,s}\bar{N}^{rs}_{nm}\left(\alpha_n^{N(r)}\alpha_m^{M(s)}+\tilde{\alpha}_n^{N(r)}\tilde{\alpha}_m^{M(s)}\right)\eta_{NM} \ .\end{aligned} \tag{2.48}$$

Here the Neumann coefficients $\bar{N}^{rs}_{nm}$ are defined in eq.(A.12).

String field action

The action of the OSp invariant string field theory is given as

$$\begin{aligned}S = \int dt\,\Bigg[\frac{1}{2}\int d1d2\ \langle R(1,2)\,|\Phi\rangle_1\left(i\frac{\partial}{\partial t}-\frac{L_0^{(2)}+\tilde{L}_0^{(2)}-2}{\alpha_2}\right)|\Phi\rangle_2 \\ +\frac{2g}{3}\int d1d2d3\,\langle V_3^0(1,2,3)|\Phi\rangle_1|\Phi\rangle_2|\Phi\rangle_3\Bigg] \ ,\end{aligned} \tag{2.49}$$

where t denotes the proper time and $g > 0$ denotes the coupling constant for strings. The string field $|\Phi\rangle$ is taken to be Grassmann even and subject to the level matching condition $\mathcal{P}|\Phi\rangle = |\Phi\rangle$ and the reality condition

$$(|\Phi\rangle)^\dagger = \langle\Phi| \ , \tag{2.50}$$

where $(|\Phi\rangle)^\dagger$ denotes the hermitian conjugate of $|\Phi\rangle$, and $\langle\Phi|$ denotes the BPZ conjugate of $|\Phi\rangle$ defined in eq.(2.44).

At first sight, the action eq.(2.49) looks similar to that of the light-cone gauge string field theory (2.16). Actually they are very different. To see this, let us expand the string field $|\Phi\rangle$ in terms of π_0 and $\bar{\pi}_0$ as

$$|\Phi\rangle = |\bar{\phi}\rangle + i\pi_0|\bar{\chi}\rangle + i\bar{\pi}_0|\chi\rangle + i\pi_0\bar{\pi}_0|\phi\rangle \ . \tag{2.51}$$

Substituting this into the kinetic term of the action eq.(2.49), we have

$$\frac{1}{2}\int dt\, d\alpha \frac{d^{26}p}{(2\pi)^{26}}\Big[\langle\bar{\phi}|\bar{\phi}\rangle + \langle\bar{\phi}|\left(-i\alpha\partial_t + p^2 + 2N - 2\right)|\phi\rangle + i\langle\bar{\chi}|\left(-i\alpha\partial_t + p^2 + 2N - 2\right)|\chi\rangle\Big], \tag{2.52}$$

where N is the level operator. The interaction terms are at most quadratic in $\bar{\phi}$. Therefore $\bar{\phi}$ can be regarded as an auxiliary field and integrated out. If one integrates $\bar{\phi}$ out, the kinetic term for ϕ looks quite different from that of the usual field theory. It rather looks similar to that of the stochastic quantization.

2.2.2 BRST invariance

Since we add the coordinates to construct the OSp invariant string field theory, the Hilbert space of the OSp invariant string field theory is larger than that of the light-cone gauge string field theory. Since what we would like to do is to construct the theory from which we can obtain the same S-matrix elements as the light-cone gauge string field theory, we need to limit the Hilbert space. In doing so, we would like to preserve the covariance of the directions X^μ $(\mu = 1, 2, \dots, 26)$. In order to do so, let us identify the "Lorentz transformation" $\mathcal{M}^{C-}$ with the BRST transformation $\delta_{\rm B}$ [31][32]:

$$\delta_{\rm B}|\Phi\rangle = \mathcal{M}^{C-}|\Phi\rangle = Q_{\rm B}|\Phi\rangle + g|\Phi * \Phi\rangle \ , \tag{2.53}$$

where the BRST charge $Q_{\rm B}$ is defined as

$$\begin{aligned} Q_{\rm B} = M^{C-} \ &= \ \frac{C_0}{2\alpha}(L_0 + \tilde{L}_0 - 2) - i\pi_0\frac{\partial}{\partial\alpha} \\ &\quad - \frac{i}{\alpha}\sum_{n=1}^{\infty}\left(\frac{\alpha^C_{-n}L_n - L_{-n}\alpha^C_n}{n} + \frac{\tilde{\alpha}^C_{-n}\tilde{L}_n - \tilde{L}_{-n}\tilde{\alpha}^C_n}{n}\right) \ , \end{aligned} \tag{2.54}$$

and the $*$-product $\Phi * \Psi$ of two arbitrary closed string fields Φ and Ψ is defined as

$$|\Phi * \Psi\rangle_4 = \int d1d2d3\, \langle V_3(1,2,3)|\Phi\rangle_1|\Psi\rangle_2|R(3,4)\rangle \ . \tag{2.55}$$

Here the vertex $\langle V_3(1,2,3)|$ is

$$\langle V_3(1,2,3)| = \langle v_3^0(1,2,3)|C(\rho_I, \bar{\rho}_I)\mathcal{P}_{123} \ , \tag{2.56}$$

where ρ_I denotes the interaction point of the three strings and the vertex $\langle v_3^0(1,2,3)|$ is defined in (2.47). As we will see in the next chapter, the invariant sector under the BRST transformation (2.53) reproduces the light-cone gauge string field theory.

The nilpotency of the BRST transformation (2.53) is assured by the nilpotency of Q_{B} and[3]

$$\langle V_3(1,2,3)| \sum_{r=1}^{3} Q_{\mathrm{B}}^{(r)} = 0 \ ,$$

$$\begin{aligned} \int d5d6 \langle V_3(1,2,5)|\langle V_3(3,4,6)|R(5,6)\rangle & \\ + \int d5d6 \langle V_3(2,3,5)|\langle V_3(1,4,6)|R(5,6)\rangle & \\ + \int d5d6 \langle V_3(3,1,5)|\langle V_3(2,4,6)|R(5,6)\rangle &= 0 \ . \end{aligned} \tag{2.57}$$

The second equation is known as the Jacobi identity. These equations will be shown in appendix B.

The action (2.49) is invariant under the BRST transformation (2.53). The proof is the same as that of the Lorentz invariance of the action of the light-cone gauge string field theory. Another way to show it is by using the nilpotency of the BRST transformation (2.53) and the fact that the action (2.49) can be expressed as

$$S = \int dt \left[\frac{1}{2} \int d1d2 \langle R(1,2)|\Phi\rangle_1 i \frac{\partial}{\partial t} |\Phi\rangle_2 + \delta_{\mathrm{B}} \left(\int d1d2 \langle R(1,2)|\Phi\rangle_1 \bar{\pi}_0^{(2)} |\Phi\rangle_2 \right) \right] \ . \tag{2.58}$$

Here we have used the relations:

$$\{Q_{\mathrm{B}}, \bar{\pi}_0\} = \frac{L_0 + \tilde{L}_0 - 2}{2\alpha} \ , \tag{2.59}$$

$$\langle V_3(1,2,3)| \left(\sum_{r=1}^{3} \bar{\pi}_0^{(r)} \right) = \langle V_3^0(1,2,3)| \ . \tag{2.60}$$

2.2.3 Canonical quantization

Since the action (2.49) and the formulation of the OSp invariant string field theory are quite similar to those of the light-cone gauge string field theory, we can perform the canonical quantization in an analogous way. We can decompose the string field as

$$|\Phi\rangle = |\psi\rangle + |\bar{\psi}\rangle \ , \tag{2.61}$$

where $|\psi\rangle$ is the part with positive α and $|\bar{\psi}\rangle$ is the one with negative α. From the kinetic term of eq.(2.49), we can see that they satisfy the canonical commutation relation

$$\left[|\psi\rangle_r , |\bar{\psi}\rangle_s \right] = |R(r,s)\rangle \ . \tag{2.62}$$

From the hermiticity defined in eq.(2.50), one can deduce that $\langle\psi|$ and $\langle\bar{\psi}|$ are hermitian conjugate to $|\bar{\psi}\rangle$ and $|\psi\rangle$, respectively. We identify $|\psi\rangle$ with the annihilation mode and $|\bar{\psi}\rangle$ with the creation mode. Accordingly we define the vacuum state $|0\rangle\rangle$ in the second quantization as

$$|\psi\rangle|0\rangle\rangle = 0 \ , \quad \langle\langle 0|\langle\bar{\psi}| = 0 \ . \tag{2.63}$$

[3]To be precise, Q_{B} is nilpotent modulo terms vanishing under the level matching condition.

Chapter 3

Observables and S-matrix Elements in OSp Invariant String Field Theory

In the OSp invariant string field theory, we consider BRST invariant objects as physical quantities. The S-matrix elements are defined for BRST invariant on-shell asymptotic states. In this chapter, we will first construct the BRST invariant observables for these states. We will see that they are in one-to-one correspondence to the on-shell physical states of usual string theory. Then we will calculate some correlation functions of these observables and show that the S-matrix elements derived from them coincide with those of the light-cone gauge string field theory. We will also evaluate the low energy effective action for later use.

3.1 BRST Invariant Observables

3.1.1 BRST invariant observables

In order to deal with the BRST invariant asymptotic states of the OSp invariant string field theory, we define the BRST invariant observables corresponding to them. They are of the form

$$\mathcal{O} = \langle\ |\Phi\rangle\ . \tag{3.1}$$

Here $|\ \rangle$ is a first quantized string state and the inner product should be considered as including the integrations in the zero-mode part. The BRST transformation of this quantity is given as

$$\delta_{\mathrm{B}}\mathcal{O} = \langle\ |\Big(Q_{\mathrm{B}}|\Phi\rangle + g|\Phi * \Phi\rangle\Big)\ . \tag{3.2}$$

In the discussion of the asymptotic sates, the second term in the transformation can be ignored. Therefore for BRST invariant states, we should impose the condition

$$\langle\ |Q_{\mathrm{B}}|\Phi\rangle = 0\ , \tag{3.3}$$

which implies

$$Q_{\mathrm{B}}|\ \rangle = 0\ . \tag{3.4}$$

For a BRST exact state $|\ \rangle = Q_{\mathrm{B}}|\ \rangle'$,

$$\mathcal{O} \simeq \delta_{\mathrm{B}}{}'\langle\ |\Phi\rangle\ , \tag{3.5}$$

up to multi-string contribution. Therefore $|\ \rangle$ should be chosen from a nontrivial cohomology class of Q_{B}.

BRST cohomology of $Q_{\rm B}$

In order to obtain the BRST cohomology of the operator $Q_{\rm B}$, it is convenient to relate $Q_{\rm B}$ to usual Kato-Ogawa's BRST operator [33]. To do so, we identify the worldsheet variable $(C,\bar{C})$ with the diffeomorphism (b,c) ghosts as

$$\begin{aligned} &\alpha^C_n = -in\alpha c_n \ , \quad \tilde{\alpha}^C_n = -in\alpha\tilde{c}_n \ ; \quad \alpha^{\bar{C}}_n = \frac{1}{\alpha}b_n \ , \quad \tilde{\alpha}^{\bar{C}}_n = \frac{1}{\alpha}\tilde{b}_n \ , \\ &C_0 = 2\alpha c_0^+ \ , \quad \bar{\pi}_0 = \frac{1}{2\alpha}b_0^+ \ , \end{aligned} \tag{3.6}$$

for $n \neq 0$. Indeed with these identifications, the BRST operator $Q_{\rm B}$ becomes almost the same as Kato-Ogawa's BRST operator:

$$\begin{aligned} Q_{\rm B} &= Q_{\rm B}^{\rm KO} - i\frac{\pi_0}{\alpha}\left(\alpha\left.\frac{\partial}{\partial\alpha}\right|_{C,\bar{C}} + N_{\rm FP}\right) \\ &= Q_{\rm B}^{\rm KO} - i\frac{\pi_0}{\alpha}\left(\alpha\left.\frac{\partial}{\partial\alpha}\right|_{b,c} + 1\right) \ , \end{aligned} \tag{3.7}$$

where $N_{\rm FP}$ is defined as

$$N_{\rm FP} = \sum_{n=1}^{\infty}\left(c_{-n}b_n - b_{-n}c_n + \tilde{c}_{-n}\tilde{b}_n - \tilde{b}_{-n}\tilde{c}_n\right) + \pi_0\frac{\partial}{\partial\pi_0} - \bar{\pi}_0\frac{\partial}{\partial\bar{\pi}_0} + 1 \ , \tag{3.8}$$

$\left.\frac{\partial}{\partial\alpha}\right|_{C,\bar{C}}$ denotes the α-derivative with $\alpha^C_n, \alpha^{\bar{C}}_n, \tilde{\alpha}^C_n, \tilde{\alpha}^{\bar{C}}_n$ $(n \in \mathbb{Z})$ kept fixed and $\left.\frac{\partial}{\partial\alpha}\right|_{b,c}$ denotes the α-derivative with $b_0^+, c_0^+, b_n, c_n, \tilde{b}_n, \tilde{c}_n$ $(n \neq 0)$ kept fixed. $Q_{\rm B}^{\rm KO}$ is Kato-Ogawa's BRST operator with b_0^- omitted. From the identification eq.(3.6), one can find that the OSp invariant theory is the usual covariantly quantized theory with extra variables π_0, α. Then the first-quantized Hilbert space of the OSp invariant theory is the tensor product of that of the usual covariant string theory and that of π_0, α.

Now let us consider a BRST closed state $|\ \rangle$. Just like the BRST operator, we expand $|\ \rangle$ in π_0 as follows:

$$|\ \rangle = |1\rangle + \pi_0|2\rangle \ , \tag{3.9}$$

where the states $|1\rangle$ and $|2\rangle$ are independent of π_0. In this notation, the condition that $|\ \rangle$ is $Q_{\rm B}$-closed becomes

$$Q_{\rm B}^{\rm KO}|1\rangle = 0 \ , \tag{3.10}$$

$$Q_{\rm B}^{\rm KO}|2\rangle = \mathcal{D}_\alpha|1\rangle \ , \tag{3.11}$$

where

$$\mathcal{D}_\alpha \equiv -i\frac{1}{\alpha}\left(\alpha\left.\frac{\partial}{\partial\alpha}\right|_{b,c} + 1\right) \ . \tag{3.12}$$

Since we know the BRST cohomology of $Q_{\rm B}^{\rm KO}$, solutions to eq.(3.10) can be easily found to be a linear combination of states of the form $f(\alpha)|{\rm phys}\rangle$ and $g(\alpha)Q_{\rm B}^{\rm KO}|\ \rangle'$, where $|{\rm phys}\rangle$ denotes a state in a nontrivial cohomology class of $Q_{\rm B}^{\rm KO}$ and $f(\alpha), g(\alpha)$ are arbitrary functions of α. Substituting these into eq.(3.11), one can see $f(\alpha) = \frac{1}{\alpha}$ and $|2\rangle$ should be a linear combination of the solutions to

$$Q_{\rm B}^{\rm KO}|2\rangle = (\mathcal{D}_\alpha g(\alpha)) Q_{\rm B}^{\rm KO}|\ \rangle' \ . \tag{3.13}$$

Solutions to this equation can also be easily found and eventually we see that the BRST closed state $|\ \rangle$ should be a linear combination of the states of the form

$$\frac{1}{\alpha}|{\rm phys}\rangle \ , \tag{3.14}$$

and

$$\pi_0 h(\alpha)|{\rm phys}\rangle \ , \tag{3.15}$$

up to $Q_{\rm B}$ exact states. Here $h(\alpha)$ is an arbitrary function of α.

For $|{\rm phys}\rangle$, one can choose the states of the form

$$|0\rangle_{b,c} \otimes |{\rm primary}; k\rangle_X \ , \tag{3.16}$$

or

$$b_0^+|0\rangle_{b,c} \otimes |{\rm primary}; k\rangle_X \ , \tag{3.17}$$

where $|0\rangle_{b,c}$ is the oscillator vacuum for the (b, c) ghosts satisfying $c_0^+|0\rangle_{b,c} = 0$[1], and k^μ is the momentum eigenvalue of the state $|{\rm primary}; k\rangle_X$:

$$|{\rm primary}; k\rangle_X = |\overline{\rm primary}\rangle_X (2\pi)^{26}\delta^{26}(p - k) \ . \tag{3.18}$$

Here $|\overline{\rm primary}\rangle_X$ denotes the oscillator part of a physical state in the old covariant quantization (i.e. it satisfies the Virasoro conditions.). We normalize it as

$${}_X\langle\overline{\rm primary}|\overline{\rm primary}\rangle_X = 1 \ . \tag{3.19}$$

By using the original variables of OSp theory, the BRST closed states can be written as a linear combination of the states of the following forms

$$\begin{aligned} &\frac{1}{\alpha}|0\rangle_{C,\bar{C}} \otimes |{\rm primary}; k\rangle_X \ , \qquad \bar{\pi}_0|0\rangle_{C,\bar{C}} \otimes |{\rm primary}; k\rangle_X \ , \\ &\pi_0 h'(\alpha)|0\rangle_{C,\bar{C}} \otimes |{\rm primary}; k\rangle_X \ , \qquad \pi_0\bar{\pi}_0 h''(\alpha)|0\rangle_{C,\bar{C}} \otimes |{\rm primary}; k\rangle_X \ , \end{aligned} \tag{3.20}$$

up to $Q_{\rm B}$ exact states. Here $|0\rangle_{C,\bar{C}}$ is the oscillator vacuum (2.37) for the $C, \bar{C}$ sector and $h'(\alpha), h''(\alpha)$ are arbitrary functions of α.

α-dependence

The wave functions for the OSp invariant string field theory should satisfy appropriate boundary conditions. Especially one should be careful about the dependence on the zero-mode α, since the integration measure (2.38) has an unusual form for α. Treating the regions $\alpha > 0$ and $\alpha < 0$ separately, let us introduce a real variable ω as

$$\alpha = \pm e^\omega. \tag{3.21}$$

[1]Notice that b_0^-, c_0^- are omitted.

If we express the wave functions using the original variables in the OSp invariant string field theory, the α dependent part of the wave functions should be of the form

$$e^{-\omega} f(\omega) \ , \tag{3.22}$$

where $f(\omega)$ is a delta function normalizable function with respect to the norm

$$\|f\|^2 \equiv \int_{-\infty}^{\infty} d\omega |f(\omega)|^2 \ . \tag{3.23}$$

We can take $e^{i\omega x}$ ($x \in \mathbb{R}$) as a basis for such wave functions. It is straightforward to show that under such conditions Q_{B} and M^{+-} are hermitian.

Now let us take this condition into account and further restrict the form of the BRST closed states in eq.(3.20). They should be of the form

$$\begin{aligned} &\frac{1}{\alpha}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \ , \\ &\frac{\pi_0}{\alpha} e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \ , \\ &\frac{\pi_0\bar{\pi}_0}{\alpha} e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \ , \end{aligned} \tag{3.24}$$

but the second one is BRST exact

$$\frac{\pi_0}{\alpha} e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X = Q_{\mathrm{B}} \left(\frac{1}{x-i} \, e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \right) \ , \tag{3.25}$$

and the last one is BRST exact if $x \neq 0$

$$\frac{\pi_0\bar{\pi}_0}{\alpha} e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X = Q_{\mathrm{B}} \left(\frac{\bar{\pi}_0}{x} \, e^{i\omega x}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \right) \ . \tag{3.26}$$

Therefore we have shown that the BRST closed states can be written as a linear combination of the states of the form

$$\frac{1}{\alpha}|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \ , \tag{3.27}$$

and

$$\frac{1}{\alpha}\pi_0\bar{\pi}_0|0\rangle_{C,\bar{C}} \otimes |\text{primary}; k\rangle_X \ , \tag{3.28}$$

up to Q_{B} exact states.

Observables

As was explained below eq.(2.52), the string field $|\Phi\rangle$ of the OSp invariant string field theory has the auxiliary modes $\bar{\phi}$. For constructing the observables, we discard the case when $\langle\ |\Phi\rangle$ is an auxiliary mode. Then $|\ \rangle$ should be of the form eq.(3.27). Making the zero-mode integral explicit, one can describe the observables constructed above as

$$\mathcal{O}(t,k) = \int dr \, \frac{1}{\alpha_r} \, {}_r\Big({}_X\langle \text{primary}; k| \otimes {}_{C,\bar{C}}\langle 0| \Big) |\Phi(t)\rangle_r \ , \tag{3.29}$$

where ${}_X\langle\text{primary};k|\otimes{}_{C,\bar{C}}\langle 0|$ denotes the BPZ conjugate of $|0\rangle_{C,\bar{C}}\otimes|\text{primary};k\rangle_X$.[2] The mass M of the particle corresponding to the operator $\mathcal{O}(t,k)$ can be read off from the relation

$$\left(L_0+\tilde{L}_0-2\right)|0\rangle_{C,\bar{C}}\otimes|\text{primary};k\rangle_X=\left(k^2+2i\pi_0\bar{\pi}_0+M^2\right)|0\rangle_{C,\bar{C}}\otimes|\text{primary};k\rangle_X\ . \quad (3.31)$$

$\mathcal{O}(t,k)$ is BRST exact unless $k^2+M^2=0$.

Since we consider correlation functions, the primary states introduced here are off-shell in general, i.e. $k^2+M^2\neq 0$. For later use, we introduce the on-shell primary states $|\text{primary};\mathbf{k}\rangle_X=|\text{primary};k\rangle_X|_{k^2+M^2=0}$, where $\mathbf{k}$ denotes the spatial 25-momentum.

As we can see from the expression eq.(2.59), the Hamiltonian on the worldsheet $\frac{L_0+\tilde{L}_0-2}{\alpha}$ is a BRST exact operator. Thus $\mathcal{O}(t+\delta t,k)$ and $\mathcal{O}(t,k)$ are BRST equivalent and one can obtain the same correlation functions from each of them.

3.1.2 Free propagator

We would like to study BRST invariant asymptotic states using the observables constructed above. Once the auxiliary field $\bar{\phi}$ is integrated out, the action eq.(2.52) no longer possesses the kinetic term similar to that of the usual field theory action. Therefore it may seem unlikely that these observables correspond to usual particle states. However, as we will show, the free propagators corresponding to these operators yield propagators for particles propagating in 26 dimensions.

Let us consider the observables $\mathcal{O}_r(t_r,k_r)$ $(r=1,2)$ which are of the form eq.(3.29) corresponding to a common primary state, i.e. $|\overline{\text{primary}}\rangle_X\equiv|\overline{\text{primary}_1}\rangle_X=|\overline{\text{primary}_2}\rangle_X$ and $M\equiv M_1=M_2$. We define the two-point function

$$\langle\!\langle\mathcal{O}_1(t_1,k_1)\mathcal{O}_2(t_2,k_2)\rangle\!\rangle\equiv\langle\!\langle 0|\mathrm{T}\,\mathcal{O}_1(t_1,k_1)\mathcal{O}_2(t_2,k_2)|0\rangle\!\rangle\ , \quad (3.32)$$

where $|0\rangle\!\rangle$ denotes the vacuum in the second quantization defined in eq.(2.63) and T denotes the time ordering of the operator. We consider the case $t_1>t_2$.[3] By using the canonical commutation relations (2.62), we can rewrite the lowest order contribution to the two-point function as

$$\begin{aligned}&\int_0^\infty\frac{d1}{\alpha_1}\ {}_1\Big({}_X\langle\text{primary};k_1|\otimes{}_{C,\bar{C}}\langle 0|\,e^{-i\frac{t_1}{\alpha_1}(p_1^2+2i\pi_0^{(1)}\bar{\pi}_0^{(1)}+M^2-i\epsilon)}\Big)\\&\quad\times\int_{-\infty}^0\frac{d2}{\alpha_2}\ {}_2\Big({}_X\langle\text{primary};k_2|\otimes{}_{C,\bar{C}}\langle 0|\,e^{-i\frac{t_2}{\alpha_2}(p_2^2+2i\pi_0^{(2)}\bar{\pi}_0^{(2)}+M^2-i\epsilon)}\Big)|R(1,2)\rangle\ ,\end{aligned} \quad (3.33)$$

where the limits of the zero-mode integration dr represent the integration regions of the string length α_r. After the integration over the zero-modes for string 2, we have

$$=\int_0^\infty\frac{d1}{\alpha_1^2}\,e^{-i\frac{t_1-t_2}{\alpha_1}(k_1^2+2i\pi_0^{(1)}\bar{\pi}_0^{(1)}+M^2-i\epsilon)}\,(2\pi)^{26}\delta^{26}(k_1+k_2)(2\pi)^{26}\delta^{26}(p_1-k_1)$$

[2]Here k_μ is the momentum eigenvalue of the state $|\text{primary};k\rangle_X$:

$${}_X\langle\text{primary};k|=(2\pi)^{26}\delta^{26}(p+k)\,{}_X\langle\overline{\text{primary}}|\ . \quad (3.30)$$

[3]We can obtain the same results for the case $t_2>t_1$.

$$
\begin{aligned}
&= \int_0^\infty d\alpha_1 \frac{i(t_1-t_2)}{\alpha_1^2}\, e^{-i\frac{t_1-t_2}{\alpha_1}(k_1^2+M^2-i\epsilon)}\,(2\pi)^{26}\delta^{26}(k_1+k_2) \\
&= i\int_0^\infty dT\, e^{-iT(k_1^2+M^2-i\epsilon)}\,(2\pi)^{26}\delta^{26}(k_1+k_2) \\
&= \frac{(2\pi)^{26}\delta^{26}(k_1+k_2)}{k_1^2+M^2}\ ,
\end{aligned}
\tag{3.34}
$$

where we introduced $T \equiv \frac{t_1-t_2}{\alpha_1}$. Therefore we obtain

$$
\langle\!\langle \mathcal{O}_1(t_1,k_1)\mathcal{O}_2(t_2,k_2)\rangle\!\rangle_{\text{free}} = \frac{(2\pi)^{26}\delta^{26}(k_1+k_2)}{k_1^2+M^2}\ . \tag{3.35}
$$

This coincides with the Euclidean propagator for a particle with mass M. Thus we have shown that although the string field action possesses an unusual form, modes corresponding to the operators $\mathcal{O}(t,k)$ yield usual propagators. $\mathcal{O}(t,k)$ are in one-to-one correspondence to the states with physical polarizations in usual string theory.

Now that we identify the modes of Φ corresponding to the particle states in string theory, we can construct the asymptotic states using them. Wick rotating as $x^{26} \to x^0 = -ix^{26}$, we can canonically quantize the theory considering x^0 as time. Since the free propagator corresponding to $\mathcal{O}(t,k)$ coincides with that for a particle with mass M, it is straightforward to define properly normalized asymptotic states using these operators.

3.2 Correlation Functions and S-matrix Elements

We have constructed the BRST invariant observables for on-shell asymptotic states. Now let us consider N-point correlation functions ($N \geq 3$) for these observables which are defined as

$$
\left\langle\!\!\left\langle \prod_{r=1}^{N} \mathcal{O}_r(t_r,k_r) \right\rangle\!\!\right\rangle \equiv \langle\!\langle 0|\,\mathrm{T}\prod_{r=1}^{N}\mathcal{O}_r(t_r,k_r)\,|0\rangle\!\rangle\ . \tag{3.36}
$$

Since these correlation functions are independent of the extra time variables t_r, they can be considered as the correlation functions in the 26 dimensional Euclidean space. Eq.(3.36) behaves as

$$
\sim \prod_{r=1}^{N}\left(\frac{1}{k_r^2+M_r^2}\right) C + \text{less singular terms}\ \ , \tag{3.37}
$$

around $k_r^2+M_r^2 \sim 0$ [24]. By Wick rotating the pole residue, we can derive the S-matrix elements. We can prove that the S-matrix elements derived from this procedure coincide with those of the light-cone gauge string field theory. The proof is given in [24]. In this section, instead of giving a general proof, we will calculate some S-matrix elements explicitly. We also write down the space-time low energy effective action of the OSp invariant string field theory, which will be used to check the normalization and the sign of the amplitudes involving D-branes.

3.2.1 Three-point S-matrix elements

First, we consider the tree level three-point S-matrix elements as the simplest example. By using the canonical commutation relation (2.62), the lowest order contribution of the three-point correlation function for the observables $\mathcal{O}_r(t_r,k_r)$ $(r=1,2,3)$ $(t_1>t_2>t_3)$ with mass

M_r is evaluated as

$$
\begin{aligned}
&\left\langle\!\!\left\langle \mathcal{O}_1(t_1,k_1)\mathcal{O}_2(t_2,k_2)\mathcal{O}_3(t_3,k_3)\right\rangle\!\!\right\rangle \\
&\quad = 4ig\left[\int_{t_3}^{t_2} dT \prod_{s=1}^{2}\left(-\int_{-\infty}^{0}\frac{ds}{\alpha_s}\right)\int_0^{\infty}\frac{d3}{\alpha_3} + \int_{t_2}^{t_1} dT\left(-\int_{-\infty}^{0}\frac{d1}{\alpha_1}\right)\prod_{s=2}^{3}\left(\int_0^{\infty}\frac{ds}{\alpha_s}\right)\right] \\
&\qquad\times \langle V_3^0(1,2,3)|\prod_{r=1}^{3}\left[e^{-i\frac{|T-t_r|}{|\alpha_r|}\left(p_r^2+M_r^2+2i\pi_0^{(r)}\bar{\pi}_0^{(r)}\right)}\left(|0\rangle_{C,\bar{C}}\otimes|\mathrm{primary}_r;k_r\rangle_X\right)_r\right]. \qquad (3.38)
\end{aligned}
$$

Here T is the proper time of the interaction Hamiltonian, and the first and second terms are corresponding to the diagram depicted in Figs. 3.1 and 3.2, respectively. In order to obtain

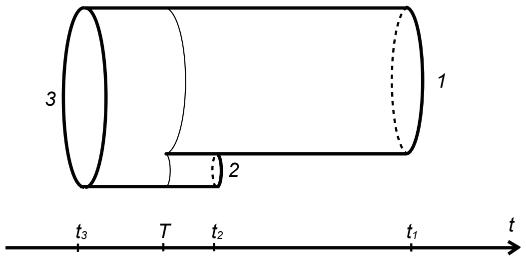


Figure 3.1: The string diagram corresponding to $t_3 < T < t_2$.

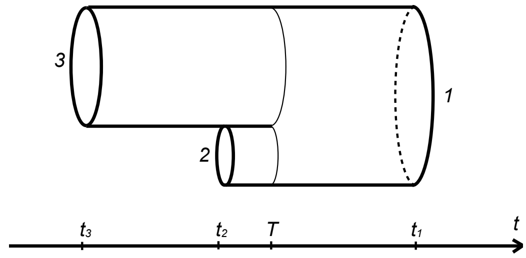


Figure 3.2: The string diagram corresponding to $t_2 < T < t_1$.

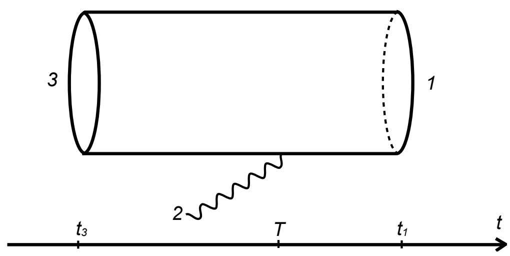


Figure 3.3: The string diagram in the limit $\alpha_2 \to 0$. A wavy line denotes the insertion of the vertex operator $\mathcal{V}_2(\mathbf{k}_2)$ corresponding to the state $|\mathrm{primary}_2;\mathbf{k}_2\rangle_X$.

the S-matrix elements, we need to look for the on-shell poles for the external momenta. The singular behavior at $k_2^2 + M_2^2 = 0$ comes from the region $\alpha_2 \sim 0$ in the integration over α_2. Therefore we should consider the limit $\alpha_2 \to 0$ in the three-string vertex $\langle V_3^0(1,2,3)|$ (See also Fig. 3.3.). In this limit, the three-string vertex behaves as

$$
\begin{aligned}
&\langle V_3^0(1,2,3)|e^{-i\frac{|T|}{|\alpha_2|}\left(p_2^2+M_2^2+2i\pi_0^{(2)}\bar{\pi}_0^{(2)}\right)}\left(|0\rangle_{C,\bar{C}}\otimes|\mathrm{primary}_2;k_2\rangle_X\right)_2 \\
\sim\;& \langle R(1,3)|\mathcal{V}_2^{(3)}(\mathbf{k}_2)\mathcal{P}_{13}\frac{1}{\alpha_2\alpha_3}\frac{|T|}{|\alpha_2|}2\pi_0^{(2)}\bar{\pi}_0^{(2)}(2\pi)^{26}\delta^{26}(p_2-k_2)e^{-i\frac{|T|}{|\alpha_2|}\left(k_2^2+M_2^2\right)}\,, \qquad (3.39)
\end{aligned}
$$

where $\mathcal{V}_2^{(3)}(\mathbf{k}_2)$ denotes the vertex operator corresponding to the primary state $|\mathrm{primary}_2;\mathbf{k}_2\rangle_X$ on the mass-shell and is made from the oscillating modes for the string 3. Here we have used the formula

$$(2\pi)^{26}\delta^{26}(p_1+k_2+p_3)=(2\pi)^{26}\delta^{26}(p_1+p_3)e^{ik_2\cdot x_3}\ . \tag{3.40}$$

Integrating over the zero-modes for the string 2, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{d2}{\alpha_2}\langle V_3^0(1,2,3)|e^{-i\frac{|T|}{|\alpha_2|}\left(p_2^2+M_2^2+2i\pi_0^{(2)}\bar{\pi}_0^{(2)}\right)}\Big(|0\rangle_{C,\bar{C}}\otimes|\mathrm{primary}_2;k_2\rangle\Big)_2 \\
\sim\ & \frac{1}{k_2^2+M_2^2}\langle R(1,3)|\mathcal{V}_2^{(3)}(\mathbf{k}_2)\mathcal{P}_{13}\frac{1}{\alpha_3}\ ,
\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}
&-\int_{-\infty}^0 \frac{d2}{\alpha_2}\langle V_3^0(1,2,3)|e^{-i\frac{|T|}{|\alpha_2|}\left(p_2^2+M_2^2+2i\pi_0^{(2)}\bar{\pi}_0^{(2)}\right)}\Big(|0\rangle_{C,\bar{C}}\otimes|\mathrm{primary}_2;k_2\rangle\Big)_2 \\
\sim\ & \frac{1}{k_2^2+M_2^2}\langle R(1,3)|\mathcal{V}_2^{(3)}(\mathbf{k}_2)\mathcal{P}_{13}\frac{1}{\alpha_3}\ .
\end{aligned} \tag{3.42}$$

By using eqs.(3.41) and (3.42), we can obtain the singular behavior of the three-point correlation function at $k_2^2+M_2^2=0$ as

$$\begin{aligned}
&\Big\langle\!\!\Big\langle \mathcal{O}_1(t_1,k_1)\mathcal{O}_2(t_2,k_2)\mathcal{O}_3(t_3,k_3)\Big\rangle\!\!\Big\rangle \\
&\sim \frac{1}{k_2^2+M_2^2}\,4ig\int_{t_3}^{t_1}dT\int_0^\infty\frac{d3}{\alpha_3^3}e^{-i\frac{t_1-T}{\alpha_3}\left(k_1^2+M_1^2+2i\pi_0^{(3)}\bar{\pi}_0^{(3)}\right)}e^{-i\frac{T-t_3}{\alpha_3}\left(k_3^2+M_3^2+2i\pi_0^{(3)}\bar{\pi}_0^{(3)}\right)} \\
&\qquad\times{}_3\Big({}_X\langle\mathrm{primary}_1;k_1|\Big)\mathcal{V}_2^{(3)}(\mathbf{k}_2)\Big(|\mathrm{primary}_3;k_3\rangle_X\Big)_3\ .
\end{aligned} \tag{3.43}$$

After the integration over $\pi_0^{(3)}$ and $\bar{\pi}_0^{(3)}$, it becomes

$$\begin{aligned}
&= \frac{1}{k_2^2+M_2^2}\,(-4g)\int_{t_3}^{t_1}dT\int_0^\infty d\alpha_3\frac{t_1-t_2}{\alpha_3^3}e^{-i\frac{t_1-T}{\alpha_3}\left(k_1^2+M_1^2\right)}e^{-i\frac{T-t_3}{\alpha_3}\left(k_3^2+M_3^2\right)} \\
&\qquad\times\int\frac{d^{26}p_3}{(2\pi)^{26}}\,{}_3\Big({}_X\langle\mathrm{primary}_1;k_1|\Big)\mathcal{V}_2^{(3)}(\mathbf{k}_2)\Big(|\mathrm{primary}_3;k_3\rangle_X\Big)_3 \\
&= \frac{1}{k_2^2+M_2^2}(-4g)\int_0^\infty dT'\int_0^\infty dT''e^{-iT'\left(k_1^2+M_1^2\right)}e^{-iT''\left(k_3^2+M_3^2\right)} \\
&\qquad\qquad\times\int\frac{d^{26}p}{(2\pi)^{26}}\,{}_X\langle\mathrm{primary}_1;k_1|\mathcal{V}_2(\mathbf{k}_2)|\mathrm{primary}_3;k_3\rangle_X \\
&= \prod_{r=1}^3\left(\frac{1}{k_r^2+M_r^2}\right)4g\int\frac{d^{26}p}{(2\pi)^{26}}\,{}_X\langle\mathrm{primary}_1;\mathbf{k}_1|\mathcal{V}_2(\mathbf{k}_2)|\mathrm{primary}_3;\mathbf{k}_3\rangle_X\ .
\end{aligned} \tag{3.44}$$

In the second equality, we have changed the integration variables from T and α_3 to $T'=\frac{t_1-T}{\alpha_3}$ and $T''=\frac{T-t_3}{\alpha_3}$. Carrying out the Wick rotation to make the space-time signature Lorentzian, we can see that the lowest order contribution to the S-matrix element for this process is

$$S=4ig\int\frac{d^{26}p}{(2\pi)^{26}}\,{}_X\langle\mathrm{primary}_1;\mathbf{k}_1|\mathcal{V}_2(\mathbf{k}_2)|\mathrm{primary}_3;\mathbf{k}_3\rangle_X\ . \tag{3.45}$$

This coincides with the results of the light-cone gauge string field theory.

Spacetime effective action

In chapter 5, we will discuss the normalization and the sign of the disk amplitudes. In doing so, we need the space-time low energy effective action for the tachyon $T(x)$ and the graviton $h_{\mu\nu}(x)$. Let us calculate the S-matrix elements for processes involving only tachyons and gravitons. The primary states corresponding to these particles are

$$|\mathrm{primary}_r; k_r\rangle_X = \begin{cases} |0\rangle_X \,(2\pi)^{26}\delta^{26}(p-k_r) & \text{for the tachyon} \\ e_{r,\mu\nu}(k_r)\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0\rangle_X \,(2\pi)^{26}\delta^{26}(p-k_r) & \text{for the graviton} \end{cases}, \tag{3.46}$$

where $|0\rangle_X$ denotes the Fock vacuum for the X^μ sector and $e_{r,\mu\nu}(k_r)$ denotes the polarization of the asymptotic graviton state with momentum $k_{r,\mu}$. The polarization $e_{r,\mu\nu}(k_r)$ satisfies the following relations:

$$e_{r,\mu\nu} = e_{r,\nu\mu}\ , \quad \eta^{\mu\nu}e_{r,\mu\nu} = 0\ , \quad k^\mu_r e_{r,\mu\nu} = 0\ , \quad e_{r,\mu\nu}\, e^{\mu\nu}_r = 1\ . \tag{3.47}$$

The vertex operators appearing in eq.(3.43) are

$$\mathcal{V}_r(\mathbf{k}_r) = \,\mathop{\circ}\limits^{\circ}\, e^{ik_{r,\mu}X^\mu}(0)\,\mathop{\circ}\limits^{\circ}\, \tag{3.48}$$

for the tachyon and

$$\begin{aligned} \mathcal{V}_r(\mathbf{k}_r) &= -e_{r,\mu\nu}(k_r)\,\mathop{\circ}\limits^{\circ}\,\partial X^\mu\bar{\partial}X^\nu e^{ik_{r,\lambda}X^\lambda}(0)\,\mathop{\circ}\limits^{\circ}\, \\ &= e_{r,\mu\nu}(k_r)\,\mathop{\circ}\limits^{\circ}\,\left(p^\mu + \sum_{n\neq 0}\alpha^\mu_n\right)\left(p^\nu + \sum_{m\neq 0}\tilde{\alpha}^\nu_m\right) e^{ik_{r,\lambda}X^\lambda(0)}\,\mathop{\circ}\limits^{\circ}\, \end{aligned} \tag{3.49}$$

for the graviton. In these equations, $\mathop{\circ}\limits^{\circ}\ \mathop{\circ}\limits^{\circ}$ denotes the normal ordering of the oscillators and 0 in the arguments of the operators indicates the origin $(\tau,\sigma)=(0,0)$ of the worldsheet.

Plugging eqs.(3.46), (3.48) and (3.49) into eq.(3.45), we obtain three-point S-matrix elements for tachyons and gravitons:

$$\begin{aligned} S_{TTT} &= 4ig\,(2\pi)^{26}\delta^{26}(k_1+k_2+k_3)\ , \\ S_{TTh} &= ig\, e_{3,\mu\nu}k^\mu_{12}k^\nu_{12}\,(2\pi)^{26}\delta^{26}\,(k_1+k_2+k_3)\ , \\ S_{hhh} &= ig\, e_{1,\mu\nu}e_{2,\alpha\beta}e_{3,\gamma\delta}\,T^{\mu\alpha\gamma}T^{\nu\beta\delta}\,(2\pi)^{26}\delta^{26}(k_1+k_2+k_3)\ , \end{aligned} \tag{3.50}$$

where the subscripts T and h denote the tachyon and the graviton respectively and

$$\begin{aligned} k^\mu_{rs} &= k^\mu_r - k^\mu_s\ , \\ T^{\mu\alpha\gamma} &= \eta^{\mu\alpha}k^\gamma_{12} + \eta^{\alpha\gamma}k^\mu_{23} + \eta^{\gamma\mu}k^\alpha_{31} + \frac{1}{4}k^\mu_{23}k^\alpha_{31}k^\gamma_{12}\ . \end{aligned} \tag{3.51}$$

Eq.(3.50) coincide with the results in the light-cone gauge string field theory.

We can reproduce the results obtained in eq.(3.50) from the following space-time effective action for the metric $G_{\mu\nu}(x)$ and the tachyon field $T(x)$,

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^{26}x\sqrt{-G}R + \int d^{26}x\sqrt{-G}\left(-\frac{1}{2}G^{\mu\nu}\partial_\mu T\partial_\nu T + T^2 + \frac{2g}{3}T^3\right) \\ +\text{higher derivative terms}\ , \end{aligned} \tag{3.52}$$

by expanding the metric $G_{\mu\nu}(x)$ around the flat metric $\eta_{\mu\nu}$ as

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(x)\ . \tag{3.53}$$

We find that the gravitational coupling constant κ is related to the string coupling g as

$$\kappa = 2g\ . \tag{3.54}$$

3.2.2 Four-point S-matrix elements

Next, we evaluate the tree-level four point S-matrix elements. In order to do so, we should calculate the correlation function for the observables $\mathcal{O}_r(t_r, k_r)$ $(r = 1, \ldots, 4)$ $(t_1 > t_2 > t_3 > t_4)$ with mass M_r:

$$\left\langle\!\!\left\langle \mathcal{O}_1(t_1, k_1)\mathcal{O}_2(t_2, k_2)\mathcal{O}_3(t_3, k_3)\mathcal{O}_4(t_4, k_4) \right\rangle\!\!\right\rangle . \tag{3.55}$$

Again, the pole at $k_2^2 + M_2^2 = 0$ comes from the region $\alpha_2 \sim 0$, and the external state for the string 2 is represented as the insertion of the corresponding vertex operator. In the limit $\alpha_2 \to 0$, the diagrams have the form depicted in Figs. 3.4-3.9.

Now we should be looking for the singular behavior at $k_3^2 + M_3^2 = 0$. It comes from the region $\alpha_3 \sim 0$ in the integration over α_3. In the limit $\alpha_3 \to 0$, the diagrams Figs. 3.5 and 3.8 do not contribute to the correlation function. This can be seen as follows: in this limit, these diagrams have the intermediate state which propagating an infinitely long distance.[4] Since we consider the case that the intermediate momentum $(k_2+k_3)^\mu$ is off-shell, the contributions from these diagrams become to zero.

Collecting the contributions from the other diagrams, we obtain

$$\begin{aligned}
&\left\langle\!\!\left\langle \mathcal{O}_1(t_1, k_1)\mathcal{O}_2(t_2, k_2)\mathcal{O}_3(t_3, k_3)\mathcal{O}_4(t_4, k_4) \right\rangle\!\!\right\rangle \\
&\sim \frac{1}{k_2^2+M_2^2}\frac{1}{k_3^2+M_3^2}(-16g^2)\int_0^{2\pi}\frac{d\theta}{2\pi}\int_{t_4}^{t_1} dT_2 \int_{T_2}^{t_1} dT_1 \int_0^\infty \frac{d4}{\alpha_4^4} \\
&\qquad \times e^{-i\frac{t_1-T_1}{\alpha_4}\left(k_1^2+M_1^2+2i\pi_0^{(4)}\bar{\pi}_0^{(4)}\right)} e^{-i\frac{T_2-t_4}{\alpha_4}\left(k_4^2+M_4^2+2i\pi_0^{(4)}\bar{\pi}_0^{(4)}\right)} \\
&\qquad \times \Bigg[{}_4\Big({}_X\langle \text{primary}_1; k_1| \otimes {}_{C,\bar{C}}\langle 0|\Big) \mathcal{V}_2^{(4)}(\mathbf{k}_2) e^{-i\frac{T_1-T_2}{\alpha_4}\left(L_0^{(4)}+\tilde{L}_0^{(4)}-2\right)} \\
&\qquad\qquad \times e^{i\theta\left(L_0^{(4)}-\tilde{L}_0^{(4)}\right)} \mathcal{V}_3^{(4)}(\mathbf{k}_3) \Big(|0\rangle_{C,\bar{C}} \otimes |\text{primary}_4; k_4\rangle_X\Big)_4 \\
&\qquad\quad + {}_4\Big({}_X\langle \text{primary}_1; k_1| \otimes {}_{C,\bar{C}}\langle 0|\Big) \mathcal{V}_3^{(4)}(\mathbf{k}_3) e^{-i\frac{T_1-T_2}{\alpha_4}\left(L_0^{(4)}+\tilde{L}_0^{(4)}-2\right)} \\
&\qquad\qquad \times e^{i\theta\left(L_0^{(4)}-\tilde{L}_0^{(4)}\right)} \mathcal{V}_2^{(4)}(\mathbf{k}_2) \Big(|0\rangle_{C,\bar{C}} \otimes |\text{primary}_4; k_4\rangle_X\Big)_4 \Bigg] ,
\end{aligned} \tag{3.56}$$

where the first and second terms are the contributions from the diagram corresponding to Figs. 3.10 and 3.11 respectively and the integration over θ comes from the level matching projection for the intermediate state. Here we have used eqs.(3.41) and (3.42). After the integration over $\pi_0^{(4)}$ and $\bar{\pi}_0^{(4)}$, eq.(3.56) becomes

$$\begin{aligned}
&= \frac{1}{k_2^2+M_2^2}\frac{1}{k_3^2+M_3^2}(-16g^2)\int_0^{2\pi}\frac{d\theta}{2\pi}\int_{t_4}^{t_1} dT_1 \int_{t_4}^{T_1} dT_2 \int_0^\infty d\alpha_4 \frac{i(t_1-t_4)}{\alpha_4^4} \\
&\qquad \times e^{-i\frac{t_1-T_1}{\alpha_4}\left(k_1^2+M_1^2\right)} e^{-i\frac{T_2-t_4}{\alpha_4}\left(k_3^2+M_3^2\right)} \\
&\qquad \times \int \frac{d^{26}p}{(2\pi)^{26}} \Bigg[{}_X\langle \text{primary}_1; k_1| \mathcal{V}_2(\mathbf{k}_2) e^{-i\frac{T_1-T_2}{\alpha_4}\left(L_0^X+\tilde{L}_0^X-2\right)} \\
&\qquad\qquad \times e^{i\theta\left(L_0^X-\tilde{L}_0^X\right)} \mathcal{V}_3(\mathbf{k}_3) |\text{primary}_4; k_4\rangle_X + (2 \leftrightarrow 3) \Bigg]
\end{aligned}$$

[4] If we consider $\alpha_3 \sim 0$ and $T_2 \sim T_1$ at the same time in the integration over the moduli, these diagrams may contribute to the correlation function. This region of the moduli space is considered in other diagrams.

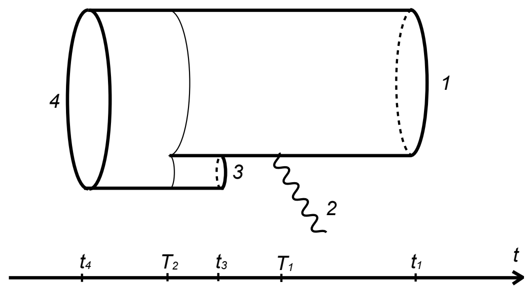


Figure 3.4: One of the string diagrams corresponding to $\alpha_3 < 0$.

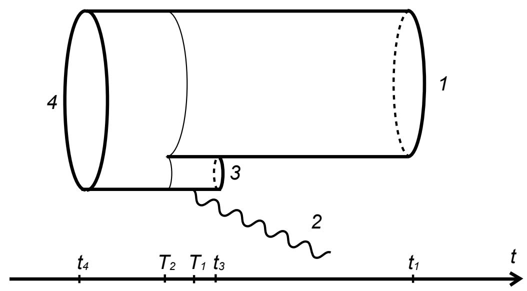


Figure 3.5: One of the string diagrams corresponding to $\alpha_3 < 0$.

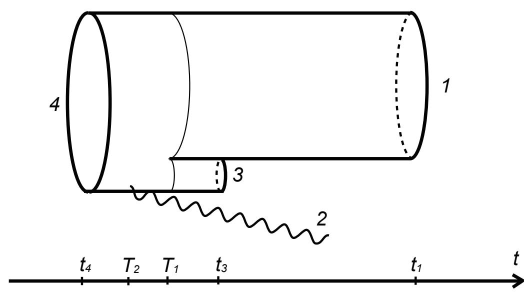


Figure 3.6: One of the string diagrams corresponding to $\alpha_3 < 0$.

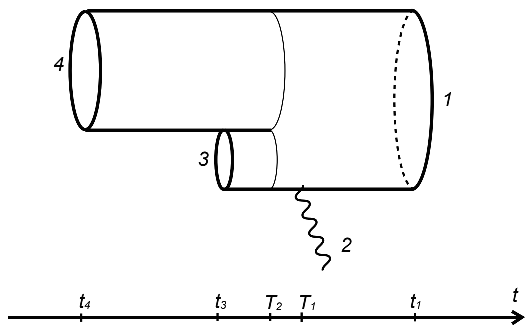


Figure 3.7: One of the string diagrams corresponding to $\alpha_3 > 0$.

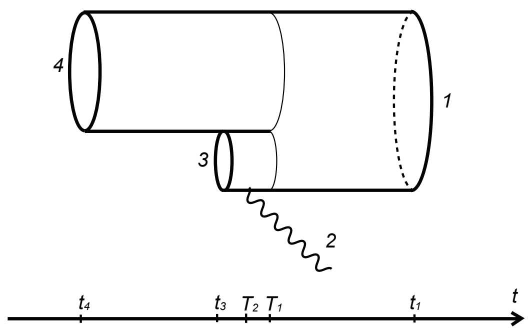


Figure 3.8: One of the string diagrams corresponding to $\alpha_3 > 0$.

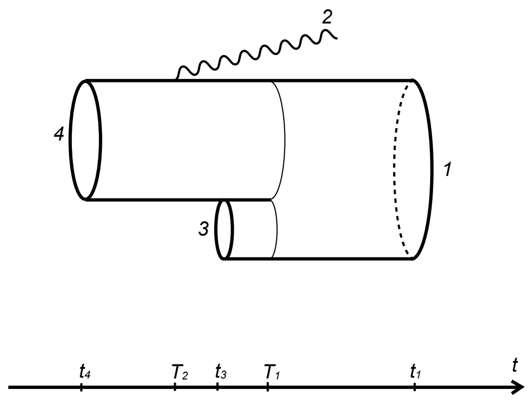


Figure 3.9: One of the string diagrams corresponding to $\alpha_3 > 0$.

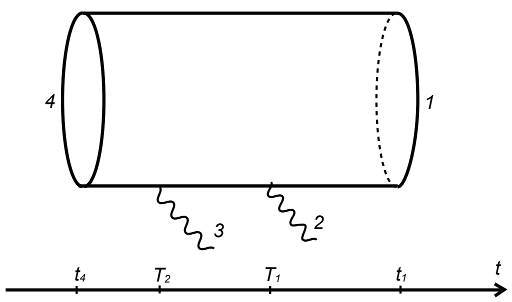


Figure 3.10: One of the string diagrams in the limit $\alpha_2, \alpha_3 \to 0$.

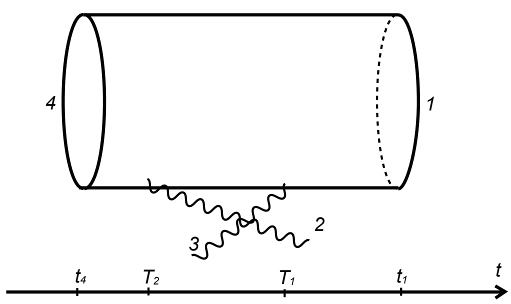


Figure 3.11: One of the string diagrams in the limit $\alpha_2, \alpha_3 \to 0$.

$$
\begin{aligned}
&= \frac{1}{k_2^2+M_2^2}\frac{1}{k_3^2+M_3^2}(-16ig^2)\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^\infty dT'\int_0^\infty dT''\int_0^\infty dT''' \\
&\qquad \times e^{-iT'\left(k_1^2+M_1^2\right)}e^{-iT'''\left(k_4^2+M_4^2\right)} \\
&\qquad \times \int\frac{d^{26}p}{(2\pi)^{26}}\Big[{}_X\langle\text{primary}_1;k_1|\mathcal{V}_2(\mathbf{k}_2)e^{-iT''\left(L_0^X+\tilde{L}_0^X-2\right)} \\
&\qquad\qquad \times e^{i\theta\left(L_0^X-\tilde{L}_0^X\right)}\mathcal{V}_3(\mathbf{k}_3)|\text{primary}_4;k_4\rangle_X+(2\leftrightarrow 3)\Big] \\
&= \prod_{r=1}^4\left(\frac{1}{k_r^2+M_r^2}\right)16ig^2\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^\infty dT'' \\
&\qquad \times \int\frac{d^{26}p}{(2\pi)^{26}}\Big[{}_X\langle\text{primary}_1;k_1|\mathcal{V}_2(\mathbf{k}_2)e^{-iT''\left(L_0^X+\tilde{L}_0^X-2\right)} \\
&\qquad\qquad \times e^{i\theta\left(L_0^X-\tilde{L}_0^X\right)}\mathcal{V}_3(\mathbf{k}_3)|\text{primary}_4;k_4\rangle_X+(2\leftrightarrow 3)\Big]\,, \qquad (3.57)
\end{aligned}
$$

where L_0^X and $\tilde{L}_0^X$ are the zero-modes of the Virasoro generators in the X^μ sector:

$$
L_0^X=\frac{1}{2}p^2+\sum_{\mu=1}^{26}\sum_{n=1}^{\infty}\alpha_{-n}^\mu\alpha_{n\mu}\,,\quad \tilde{L}_0^X=\frac{1}{2}p^2+\sum_{\mu=1}^{26}\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^\mu\tilde{\alpha}_{n\mu}\,. \qquad (3.58)
$$

In eq.(3.57), we have changed the integration variables from α_4, T_1 and T_2 to $T'=\frac{t_1-T_1}{\alpha_4}, T''=\frac{T_1-T_2}{\alpha_4}$ and $T'''=\frac{T_2-t_4}{\alpha_4}$. Performing the Wick rotation, we can see that the lowest order contribution to the S-matrix element for this process is

$$
\begin{aligned}
S &= -16g^2\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^\infty dT''\int\frac{d^{26}p}{(2\pi)^{26}}\Big[{}_X\langle\text{primary}_1;k_1|\mathcal{V}_2(\mathbf{k}_2)e^{-iT''\left(L_0^X+\tilde{L}_0^X-2\right)} \\
&\qquad \times e^{i\theta\left(L_0^X-\tilde{L}_0^X\right)}\mathcal{V}_3(\mathbf{k}_3)|\text{primary}_4;k_4\rangle_X+(2\leftrightarrow 3)\Big]\,. \qquad (3.59)
\end{aligned}
$$

This is the same as the lowest order contribution to the four-point S-matrix element of the light-cone gauge string field theory.

It is straightforward to generalize the above calculations for the other tree-level S-matrix elements. One can find that they coincide with the results of the light-cone gauge string field theory. In the loop level, although we need more complicated calculations, we can obtain the same S-matrix elements as the light-cone gauge string field theory.

Chapter 4

Boundary States in OSp Invariant String Field Theory

D-branes can be considered as a source of closed strings. In first quantized closed string theory, the emission and absorption of closed strings are represented by boundary states. In order to construct D-brane states in the OSp invariant closed string field theory, we should study the boundary states in the OSp invariant theory. In this thesis, we will consider the simplest boundary states associated with flat Dp-brane.

In this chapter, after defining the boundary states in the OSp invariant string theory, we will study the overlap of three-string vertex with them. We will obtain idempotency equations [7] satisfied by the boundary states in the OSp invariant string field theory. The results will be used to construct D-brane states in the next chapter.

4.1 Boundary States for Flat Dp-brane

In this section, we will define the boundary states $|B_0\rangle$ for flat Dp-brane that extends in the X^μ ($\mu = 26, 1, \ldots, p$) directions and is located at $X^i = 0$ ($i = p+1, \ldots, 25$). We denote these directions by X^μ ($\mu \in \mathrm{N}$) and X^i ($i \in \mathrm{D}$), respectively. To avoid the singularity of the norm of $|B_0\rangle$, we will introduce BRST invariant regularization.

The boundary state $|B_0\rangle$ satisfy the following boundary conditions,

$$\begin{aligned} &\partial_\tau X^\mu(\tau=0,\sigma)|B_0\rangle = 0\ , \quad X^i(\tau=0,\sigma)|B_0\rangle = 0\ , \\ &C(\tau=0,\sigma)|B_0\rangle = \bar{C}(\tau=0,\sigma)|B_0\rangle = 0\ . \end{aligned} \tag{4.1}$$

In terms of the oscillation modes, these conditions read

$$\begin{aligned} &x^i\,|B_0\rangle = 0\ , \qquad C_0\,|B_0\rangle = 0\ , \qquad \bar{C}_0\,|B_0\rangle = 0\ , \qquad (\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu)\,|B_0\rangle = 0\ , \\ &\left(\alpha_n^i - \tilde{\alpha}_{-n}^i\right)|B_0\rangle = 0\ , \qquad \left(\alpha_n^C - \tilde{\alpha}_{-n}^C\right)|B_0\rangle = 0\ , \qquad \left(\alpha_n^{\bar{C}} - \tilde{\alpha}_{-n}^{\bar{C}}\right)|B_0\rangle = 0 \end{aligned} \tag{4.2}$$

for $\forall n \in \mathbb{Z}$.[1] They coincide with the usual boundary conditions for the b, c ghosts assuming eq.(3.6). To satisfy the boundary conditions (4.2), we define the boundary state $|B_0\rangle$ as

$$|B_0\rangle \equiv \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}^N\tilde{\alpha}_{-n}^M D_{NM}\right]|0\rangle(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p)\ , \tag{4.3}$$

[1] While the $n = 0$ case of the second line of eq.(4.2) is not derived from eq.(4.1), it holds automatically by definition of α_0^M and $\tilde{\alpha}_0^M$.

where $\delta_{\mathrm{N}}^{p+1}(p)$ denotes the delta function of the momentum in the directions along the Dp-brane defined as $\delta_{\mathrm{N}}^{p+1}(p) = \prod_{\mu\in\mathrm{N}} \delta(p_\mu)$, and D_{NM} denotes

$$D_{NM} = D^{NM} = \begin{array}{cc} & \begin{array}{cccc} \phantom{\delta_{\mu\nu}} & \phantom{-\delta_{ij}} & C & \bar{C} \end{array} \\ \begin{array}{c} \\ \\ C \\ \bar{C} \end{array} & \begin{pmatrix} \delta_{\mu\nu} & & & \\ & -\delta_{ij} & & \\ & & 0 & i \\ & & -i & 0 \end{pmatrix} \end{array} \qquad \text{with } \ \mu,\nu \in \mathrm{N} \ , \ i,j \in \mathrm{D} \ . \tag{4.4}$$

Since the norm of the boundary state $|B_0\rangle$ diverges, we need to regularize it. In order to do so, we introduce

$$|B_0\rangle^T = e^{-\frac{T}{|\alpha|}(L_0+\tilde{L}_0-2)}|B_0\rangle \tag{4.5}$$

for $T > 0$, and consider $|B_0\rangle^\epsilon$ with $0 < \epsilon \ll 1$ as a regularized version of $|B_0\rangle$ (See also Fig. 4.1.). Notice that the operator $e^{-\frac{T}{|\alpha|}(L_0+\tilde{L}_0-2)}$ commutes with the BRST operator Q_{B},

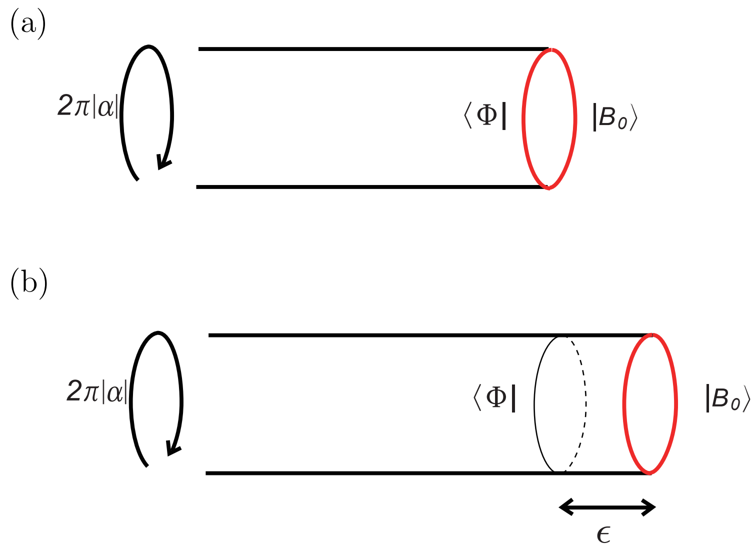


Figure 4.1: (a) The inner product $\langle\Phi\,|B_0\rangle$. (b) The inner product $\langle\Phi\,|B_0\rangle^\epsilon$.

which can be seen from eq.(2.59).

The boundary state $|B_0\rangle$ and $|B_0\rangle^\epsilon$ is not invariant under the BRST transformation. For the BRST invariance, we need the additional factor $\frac{1}{\alpha}$:

$$Q_{\mathrm{B}}\left(|B_0\rangle\frac{1}{\alpha}\right) = Q_{\mathrm{B}}\left(|B_0\rangle^\epsilon\frac{1}{\alpha}\right) = 0 \ . \tag{4.6}$$

4.2 Overlap of Three-String Vertex with One Boundary State

In this section, we will evaluate the overlaps of the three-string vertices with one boundary state:

$$\int d'3\langle V_3^0(1,2,3)|B_0\rangle_3^\epsilon \ , \tag{4.7}$$

and

$$\int d'3 \langle V_3(1,2,3)|B_0\rangle_3^{\epsilon} \tag{4.8}$$

for $|\alpha_3| > |\alpha_1|, |\alpha_2|$ (See Fig. 4.2). Here the integration measure $d'r$ is defined as

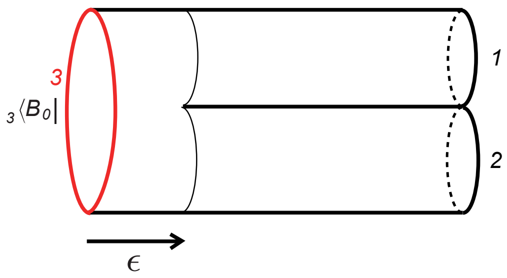


Figure 4.2: Overlap of the three-string vertex with one boundary state.

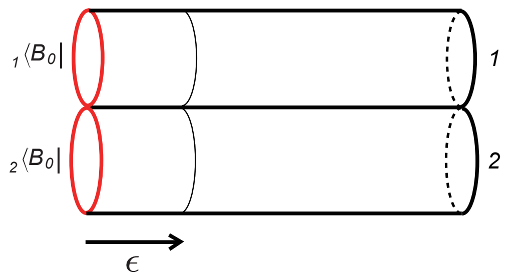


Figure 4.3: Product of two boundary states.

$$d'r \equiv \frac{d^{26}p_r}{(2\pi)^{26}} i d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \ , \tag{4.9}$$

and the three-string vertices $\langle V_3^0(1,2,3)|$ and $\langle V_3(1,2,3)|$ are defined in eqs.(2.47) and (2.56), respectively. Naively, one may expect that eqs.(4.7) and (4.8) are proportional to ${}_1^{\epsilon}\langle B_0|\,{}_2^{\epsilon}\langle B_0|$ (See Fig. 4.3.). In fact, as we will show, these are

$$\begin{aligned}
&\int d'3 \, \langle V_3^0(1,2,3)|B_0\rangle_3^{\epsilon} \\
&\quad \sim \mathrm{sgn}(\alpha_3) 2\delta\left(\alpha_1+\alpha_2+\alpha_3\right) \frac{1}{(16\pi)^{\frac{p+1}{2}}} \frac{4}{\epsilon^3(-\ln\epsilon)^{\frac{p+1}{2}}} \, {}_1^{\epsilon}\langle B_0|\,{}_2^{\epsilon}\langle B_0| \ , \qquad (4.10) \\
&\int d'3 \, \langle V_3(1,2,3)|B_0\rangle_3^{\epsilon} \\
&\quad \sim 2\delta\left(\alpha_1+\alpha_2+\alpha_3\right) \frac{1}{(16\pi)^{\frac{p+1}{2}}} \frac{4}{\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}} \, {}_1^{\epsilon}\langle B_0|\,{}_2^{\epsilon}\langle B_0| \left(\frac{i}{\alpha_1}\pi_0^{(1)} + \frac{i}{\alpha_2}\pi_0^{(2)}\right) \ , \quad (4.11)
\end{aligned}$$

in the leading order. These equations will be used in the next chapter.

In order to obtain eqs.(4.10) and (4.11), we calculate the string vertices $\langle V_2^0(1,2);T|$ and $\langle V_2(1,2);T|$ defined as follows:

$$\begin{aligned}
\langle V_2^0(1,2);T| &\equiv \int d'3 \, \langle V_3^0(1,2,3)|B_0\rangle_3^T = \langle v_2^0(1,2);T|\mathcal{P}_{12} \ , \\
\langle V_2(1,2);T| &\equiv \int d'3 \, \langle V_3(1,2,3)|B_0\rangle_3^T = \langle v_2^0(1,2);T|C(\rho_I,\bar{\rho}_I)\mathcal{P}_{12} \ , \\
\langle v_2^0(1,2);T| &= \int d'3 \, \langle v_3^0(1,2,3)|B_0\rangle_3^T \ ,
\end{aligned} \tag{4.12}$$

and then we take $T = \epsilon \ll 1$. Here we present the calculations for the case $\alpha_1, \alpha_2 < 0$.[2]

[2]The case $\alpha_1, \alpha_2 > 0$ can be treated in the same way.

4.2.1 Mandelstam mapping

The vertex $\langle v_2^0(1,2);T|$ introduced in eq.(4.12) takes the form

$$\langle v_2^0(1,2);T| = 2\delta(\alpha_1+\alpha_2+\alpha_3)(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p_1+p_2)\mathcal{K}_2(1,2;T)\langle v_{2,\mathrm{LPP}}^0(1,2);T| \ , \quad (4.13)$$

where $\langle v_{2,\mathrm{LPP}}^0(1,2);T|$ is the LPP vertex, and the factor $\mathcal{K}_2(1,2;T)$ depends only on the zero-modes and the moduli. We can obtain these objects by using the prescription explained in appendix A. In order to do so, we need conformal mappings which map two unit disks $|w_r| \leq 1$ corresponding to the string r ($r=1,2$) (Fig. 4.4 (a)) to a Riemann surface. Since the worldsheet in Fig. 4.2 has one hole and two punctures at infinity corresponding to the two external strings and its topology is a disk with two punctures, it is useful to use the complex upper half z-plane (Fig. 4.4 (c)) with two punctures as a Riemann surface.

Let us introduce a complex coordinate ρ on the worldsheet so that the string diagram in Fig. 4.2 can be identified with the region depicted in Fig. 4.4 (b) on the ρ-plane. Each portion of the ρ-plane corresponding to the r-th external string ($r=1,2$) is identified with the unit disk $|w_r| \leq 1$ of string r by the relation

$$\begin{aligned} &\rho = \alpha_r\zeta_r + T + i\beta_r \ , \quad \beta_r = -\alpha_2\pi - \alpha_r\sigma_I^{(r)} \ , \\ &\zeta_r (= \tau_r + i\sigma_r) = \ln w_r \ , \qquad \tau_r \leq 0 \ , \quad -\pi \leq \sigma_r \leq \pi \ . \end{aligned} \quad (4.14)$$

Here $\rho_I = T - i\pi\alpha_2$ is the interaction point on the ρ-plane and $\sigma_I^{(r)}$ is the value of the σ_r coordinate where the r-th string interacts. We set $\sigma_I^{(1)} = \pi$ and $\sigma_I^{(2)} = -\pi$. Therefore we have

$$\beta_1 = -(\alpha_1+\alpha_2)\pi \ , \quad \beta_2 = 0 \ . \quad (4.15)$$

The ρ-plane and the z-plane are related by the Mandelstam mapping

$$\rho(z) = \alpha_1 \ln\frac{z-Z_1}{z-\bar{Z}_1} + \alpha_2 \ln\frac{z-Z_2}{z-\bar{Z}_2} \ , \quad (4.16)$$

where the point $z = Z_r$ ($r=1,2$) is the puncture corresponding to the origin of the unit disk $|w_r| \leq 1$ of string r. We fix the $SL(2,\mathbb{R})$ gauge symmetry on the upper half plane by setting $Z_1 = iy$ and $Z_2 = i$, where y is a real parameter with $0 < y < 1$. The interaction point z_I on the z-plane is determined by $\frac{d\rho}{dz}(z_I) = 0$. This yields

$$z_I = i\sqrt{\frac{(\alpha_1+\alpha_2 y)y}{\alpha_1 y + \alpha_2}} \ . \quad (4.17)$$

Here we have used $\alpha_1, \alpha_2 < 0$, $0 < y < 1$ and $\operatorname{Im} z_I > 0$. Eq.(4.17) leads to

$$T = \operatorname{Re}\rho(z_I) = \alpha_1 \ln\left|\frac{z_I - iy}{z_I + iy}\right| + \alpha_2 \ln\left|\frac{z_I - i}{z_I + i}\right| \ . \quad (4.18)$$

From this relation, we find that in the small T limit, $T = \epsilon \ll 1$ ($y \ll 1$), we have

$$\begin{aligned} \epsilon &\simeq 4\sqrt{\alpha_1\alpha_2}\, y^{1/2} + \frac{2}{3}\frac{\alpha_1^2+\alpha_2^2}{\sqrt{\alpha_1\alpha_2}} y^{3/2} + \mathcal{O}(y^{5/2}) \ , \\ y &\simeq \frac{1}{16\alpha_1\alpha_2}\epsilon^2\left(1 - \frac{1}{48}\left(\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \ . \end{aligned} \quad (4.19)$$

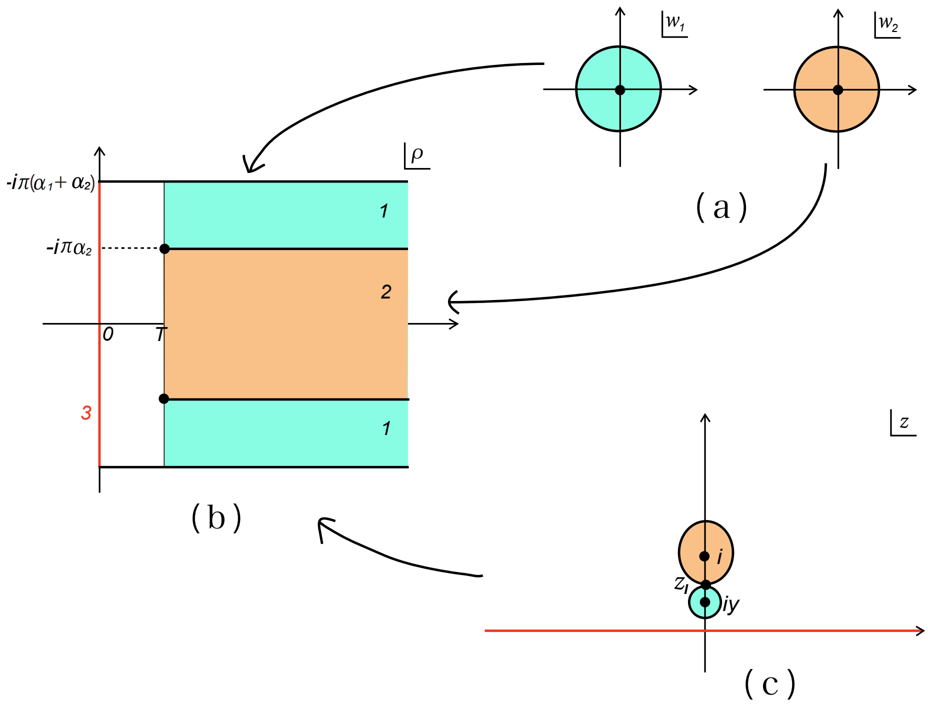


Figure 4.4: (a) The unit disks corresponding to each string. (b) The ρ-plane corresponding to the string diagram depicted in Fig. 4.2. (c) The complex upper half z-plane. These coordinates are related through eqs.(4.14) and (4.16).

For later use, we consider the limit $T \to \infty$ as well. In this limit, $y \sim 1$. In fact,

$$T \simeq \hat{\tau}_0 - (\alpha_1 + \alpha_2)\ln 2 + (\alpha_1 + \alpha_2)\ln(1-y) + \mathcal{O}(1-y)\ , \tag{4.20}$$

where

$$\hat{\tau}_0 = \alpha_1 \ln|\alpha_1| + \alpha_2 \ln|\alpha_2| - (\alpha_1+\alpha_2)\ln|\alpha_1+\alpha_2|\ . \tag{4.21}$$

4.2.2 LPP vertex and Neumann coefficients

First, we construct the LPP vertex $\langle v^0_{2,\mathrm{LPP}}(1,2);T|$. To do so, we have to know the two-point functions of the worldsheet variables $X^N = (X^\mu, X^i, C, \bar{C})$ on the z-plane. The real axis of the z-plane corresponds to the worldsheet boundary attached to $|B_0\rangle_3$. Because of the boundary conditions (4.1) satisfied by the worldsheet variables X^Non the boundary state $|B_0\rangle$, the two-point functions of $X^N(z,\bar{z})$ on the z-plane become

$$G^{NM}_{\mathrm{UHP}}(z,\bar{z};z',\bar{z}') = \langle X^N(z,\bar{z})X^M(z',\bar{z}')\rangle_{\mathrm{UHP}} = -\eta^{NM}\ln|z-z'|^2 - D^{NM}\ln|z-\bar{z}'|^2\ , \tag{4.22}$$

where D^{NM} is the tensor introduced in eq.(4.4).

By using the prescription explained in appendix A, we can obtain the LPP vertex $\langle v^0_{2,\mathrm{LPP}}(1,2);T|$ as

$$\langle v^0_{2,\mathrm{LPP}}(1,2);T|$$
$$= {}_{12}\langle 0|\exp\Bigg[\sum_{n,m=0}^{\infty}\sum_{r,s=1,2}\Bigg\{\frac{1}{2}\left(\bar{N}^{(2)rs}_{nm}\,\alpha^{N(r)}_n\,\alpha^{M(s)}_m + \bar{N}^{(2)\tilde{r}\tilde{s}}_{nm}\,\tilde{\alpha}^{N(r)}_n\tilde{\alpha}^{M(s)}_m\right)\eta_{NM}$$

$$+\frac{1}{2}\left(\bar{N}^{(2)r\tilde{s}}_{nm}\,\alpha^{N(r)}_{n}\tilde{\alpha}^{M(s)}_{m}+\bar{N}^{(2)\tilde{r}s}_{nm}\,\tilde{\alpha}^{N(r)}_{n}\alpha^{M(s)}_{m}\right)D_{NM}\Big\}\Big], \quad (4.23)$$

where the Neumann coefficients $\bar{N}^{(2)rs}_{nm}$, $\bar{N}^{(2)\tilde{r}\tilde{s}}_{nm}$ and $\bar{N}^{(2)r\tilde{s}}_{nm}$ are given as

$$\begin{aligned}
\bar{N}^{(2)rs}_{nm} &= \left(\bar{N}^{(2)\tilde{r}\tilde{s}}_{nm}\right)^{*} = \frac{1}{nm}\oint_{Z_r}\frac{dz_r}{2\pi i}\oint_{Z_s}\frac{dz'_s}{2\pi i}\frac{1}{(z_r-z'_s)^2}\Big(w_r(z_r)\Big)^{-n}\Big(w'_s(z'_s)\Big)^{-m} ,\\
\bar{N}^{(2)r\tilde{s}}_{nm} &= \left(\bar{N}^{(2)\tilde{r}s}_{nm}\right)^{*} = \frac{1}{nm}\oint_{Z_r}\frac{dz_r}{2\pi i}\oint_{\bar{Z}_s}\frac{d\bar{z}'_s}{2\pi i}\frac{1}{(z_r-\bar{z}'_s)^2}\Big(w_r(z_r)\Big)^{-n}\Big(\bar{w}'_s(\bar{z}'_s)\Big)^{-m} ,\\
\bar{N}^{(2)rs}_{n0} &= \left(\bar{N}^{(2)\tilde{r}\tilde{s}}_{n0}\right)^{*} = \frac{1}{n}\oint_{Z_r}\frac{dz_r}{2\pi i}\frac{1}{z_r-Z_s}\Big(w_r(z_r)\Big)^{-n} ,\\
\bar{N}^{(2)r\tilde{s}}_{n0} &= \left(\bar{N}^{(2)\tilde{r}s}_{n0}\right)^{*} = \frac{1}{n}\oint_{Z_r}\frac{dz_r}{2\pi i}\frac{1}{z_r-\bar{Z}_s}\Big(w_r(z_r)\Big)^{-n} ,\\
\bar{N}^{(2)rs}_{00} &= \left(\bar{N}^{(2)\tilde{r}\tilde{s}}_{00}\right)^{*} = \ln(Z_r-Z_s) \quad (r\neq s) ,\\
\bar{N}^{(2)r\tilde{s}}_{00} &= \left(\bar{N}^{(2)\tilde{r}s}_{00}\right)^{*} = \ln(Z_r-\bar{Z}_s) \quad (r\neq s) ,\\
\bar{N}^{(2)rr}_{00} &= \left(\bar{N}^{(2)\tilde{r}\tilde{r}}_{00}\right)^{*} = \ln(Z_r-\bar{Z}_r)-\sum_{s\neq r}\frac{\alpha_s}{\alpha_r}\left\{\ln(Z_r-Z_s)-\ln(Z_r-\bar{Z}_s)\right\}+\frac{T+i\beta_r}{\alpha_r} ,\\
\bar{N}^{(2)r\tilde{r}}_{00} &= \left(\bar{N}^{(2)\tilde{r}r}_{00}\right)^{*} = \ln(Z_r-\bar{Z}_r)
\end{aligned} \quad (4.24)$$

for $n,m\geq 1$. Here we have used the convention for the orientation of the $\bar{z}$ integration such that $\oint_0 \frac{d\bar{z}}{2\pi i}\frac{1}{\bar{z}}=1$. From the conformal mappings (4.14) and (4.16), we can easily find that $w_r(z)$ appeared in the definition of the Neumann coefficients are

$$\begin{aligned}
w_1(z) &= -\frac{z-iy}{z+iy}\left(\frac{1+iz}{1-iz}\right)^{\frac{\alpha_2}{\alpha_1}}e^{-\frac{T}{\alpha_1}} ,\\
w_2(z) &= \frac{z-i}{z+i}\left(\frac{z-iy}{z+iy}\right)^{\frac{\alpha_1}{\alpha_2}}e^{-\frac{T}{\alpha_2}} .
\end{aligned} \quad (4.25)$$

The behaviors of the Neumann coefficients in the limit $T=\epsilon\to 0$ are derived in appendix C.1. By using the results, we can obtain

$$(2\pi)^{p+1}\delta^{p+1}_{\mathrm{N}}(p_1+p_2)\langle v^{0}_{2,\mathrm{LPP}}(1,2);\epsilon| \sim \frac{1}{(16\pi)^{\frac{p+1}{2}}(-\ln\epsilon)^{\frac{p+1}{2}}}\,{}^{\epsilon}_{1}\langle B_0|\,{}^{\epsilon}_{2}\langle B_0| , \quad (4.26)$$

in the leading order. Here we have used the following relation:

$$\begin{aligned}
&\exp\left[\frac{1}{2}\sum_{r,s=1,2}\left(\bar{N}^{(2)rs}_{00}+\bar{N}^{(2)\tilde{r}\tilde{s}}_{00}\right)p^N_r p^M_s\eta_{NM}+\frac{1}{2}\sum_{r,s=1,2}\left(\bar{N}^{(2)r\tilde{s}}_{00}+\bar{N}^{(2)\tilde{r}s}_{00}\right)p^N_r p^M_s D_{NM}\right]\\
\sim\ &\exp\Big[2\ln\epsilon\ p^N_1 p^M_1(\eta_{NM}+D_{NM})-\ln(8\alpha_1\alpha_2)p^N_1 p^M_1(\eta_{NM}+D_{NM})\\
&\qquad +\ln 2\ p^N_2 p^M_2(\eta_{NM}+D_{NM})-\sum_{r=1,2}\frac{\epsilon}{|\alpha_r|}p^N_r p^M_r\eta_{NM}+\mathcal{O}(\epsilon^2)\Big]\\
\sim\ &\frac{1}{(16\pi)^{\frac{p+1}{2}}(-\ln\epsilon)^{\frac{p+1}{2}}}(2\pi)^{p+1}\delta^{p+1}_{\mathrm{N}}(p_1)4^{\sum_{\mu\in\mathrm{N}}p^{\mu}_2 p_{2,\mu}}\,e^{-\sum_{r=1,2}\frac{\epsilon}{|\alpha_r|}p^N_r p^M_r\eta_{NM}} .
\end{aligned} \quad (4.27)$$

4.2.3 $\mathcal{K}_2(1,2;T)$

Next, we evaluate $\mathcal{K}_2(1,2;T)$. By using the prescription explained in appendix A, we have[3]:

$$\mathcal{K}_2(1,2;T) \propto \frac{|c_\infty|^2}{|c_I||\alpha_1|^2|\alpha_2|^2}\left|\frac{\partial w_1}{\partial z}(Z_1)\right|^2\left|\frac{\partial w_2}{\partial z}(Z_2)\right|^2 , \tag{4.28}$$

where

$$c_\infty \equiv \lim_{z\to\infty}\left(z^2\frac{\partial\rho}{\partial z}\right) = 2i(\alpha_1 y+\alpha_2) , \qquad c_I \equiv \left.\frac{\partial^2\rho}{\partial z^2}\right|_{z=z_I} = \frac{c_\infty^3 z_I}{2\alpha_1\alpha_2 y(1-y^2)^2} \tag{4.29}$$

Therefore we can see that $\mathcal{K}_2(1,2;T)$ is expressed as

$$\mathcal{K}_2(1,2;T) = \mathcal{K}_0\frac{1}{\alpha_1\alpha_2}\sqrt{\frac{(\alpha_1 y+\alpha_2)y}{\alpha_1+\alpha_2 y}\frac{(1-y^2)^2}{(\alpha_1 y+\alpha_2)16y^2}}\; e^{-2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)T-2\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1}{\alpha_2}\right)\ln\frac{1+y}{1-y}} , \tag{4.30}$$

where $\mathcal{K}_0$ is a constant independent of α_1, α_2 and T. $\mathcal{K}_0$ can be determined by comparing the left and right hand sides of the equation

$$\begin{aligned}
&\int d'1d'2d'3\,\langle V_3^0(1,2,3)|B_0\rangle_3^T \\
&\qquad\qquad \times|0\rangle_{12}(2\pi)^{26}\delta^{26}(p_1)i\bar{\pi}_0^{(1)}\pi_0^{(1)}(2\pi)^{25-p}\delta_{\mathrm{D}}^{25-p}(p_2)i\bar{\pi}_0^{(2)}\pi_0^{(2)} \\
&= \int d'1d'2\;(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p_1+p_2)\;2\delta(\alpha_1+\alpha_2+\alpha_3)\mathcal{K}_2(1,2;T)\,\langle v_{2,\mathrm{LPP}}^0(1,2);T| \\
&\qquad\qquad \times|0\rangle_{12}(2\pi)^{26}\delta^{26}(p_1)i\bar{\pi}_0^{(1)}\pi_0^{(1)}(2\pi)^{25-p}\delta_{\mathrm{D}}^{25-p}(p_2)i\bar{\pi}_0^{(2)}\pi_0^{(2)} ,
\end{aligned} \tag{4.31}$$

in the $T\to\infty$ $(y\to 1)$ limit. Here $\delta_{\mathrm{D}}^{25-p}(p)$ denotes the delta function of the momentum in the Dirichlet directions: $\delta_{\mathrm{D}}^{25-p}(p) = \prod_{i\in\mathrm{D}}\delta(p_i)$. One can readily evaluate the left hand side of the above equation because the non-zero oscillation modes do not contribute in this limit:

$$\begin{aligned}
\text{LHS of eq.(4.31)} \;&\sim\; 2\delta(\alpha_1+\alpha_2+\alpha_3)\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3}e^{2\frac{T}{\alpha_3}} \\
&\sim\; 2\delta(\alpha_1+\alpha_2+\alpha_3)\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3}\frac{4}{(1-y)^2}e^{\frac{2}{\alpha_3}\hat{\tau}_0} .
\end{aligned} \tag{4.32}$$

Here we have used eq.(4.20). On the other hand, the right hand side of eq.(4.31) behaves as

$$\begin{aligned}
\text{RHS of eq.(4.31)} \;&=\; 2\delta(\alpha_1+\alpha_2+\alpha_3)\mathcal{K}_2(1,2;T) \\
&\sim\; 2\delta(\alpha_1+\alpha_2+\alpha_3)(-\mathcal{K}_0)\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3}\frac{4}{(1-y)^2}e^{\frac{2}{\alpha_3}\hat{\tau}_0} .
\end{aligned} \tag{4.33}$$

By comparing them, we find $\mathcal{K}_0 = -1$.

In the limit $T=\epsilon\to 0$ $(y\to 0)$, $\mathcal{K}_2(1,2;\epsilon)$ behaves as

$$\begin{aligned}
\mathcal{K}_2(1,2;\epsilon) \;&=\; \frac{1}{16(\alpha_1\alpha_2 y)^{\frac{3}{2}}}e^{-2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\epsilon}\left(1-\frac{9}{2}\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1}{\alpha_2}\right)y+\mathcal{O}(y^2)\right) \\
&=\; \frac{4}{\epsilon^3}e^{-2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\epsilon}\left(1-\frac{1}{4}\left(\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}\right)\epsilon^2+\mathcal{O}(\epsilon^4)\right) .
\end{aligned} \tag{4.34}$$

Combining this and eq.(4.26), we can prove eq.(4.10).

[3]The contribution from $z\sim\infty$ to eq.(4.28) is half of that to eq.(A.20), because now we are dealing with the upper half z-plane with semicircles cut out.

4.2.4 $C(\rho_I, \bar{\rho}_I)$

Finally, we consider the effect of inserting $C(\rho_I, \bar{\rho}_I)$. This can be described as follows:

$$\begin{aligned} \langle v_{2,\mathrm{LPP}}(1,2);T| &\equiv \langle v^0_{2,\mathrm{LPP}}(1,2);T|C(\rho_I,\bar{\rho}_I) \\ &= \langle v^0_{2,\mathrm{LPP}}(1,2);T|\left[\sum_{n=0}^{\infty}\sum_{r=1,2}\left(-iM_{\mathrm{UHP}}{}^r_n\alpha_n^{C(r)} - iM_{\mathrm{UHP}}{}^{\tilde{r}}_n\tilde{\alpha}_n^{C(r)}\right)\right] \end{aligned} \tag{4.35}$$

The coefficients $M_{\mathrm{UHP}}{}^r_n$ and $M_{\mathrm{UHP}}{}^{\tilde{r}}_n$ can be determined by the LPP prescription, i.e. we require that

$$\begin{aligned} &\int d'1d'2\langle v_{2,\mathrm{LPP}}(1,2);T|\bar{C}^{(r)}(w_r,\bar{w}_r)|0\rangle_{12}\prod_{r=1,2}(2\pi)^{26}\delta^{26}(p_r)i\bar{\pi}_0^{(r)}\pi_0^{(r)} \\ &= G^{C\bar{C}}_{\mathrm{UHP}}(z_I,\bar{z}_I;z_r,\bar{z}_r) = i\Big[\ln(z_I - z_r) + \ln(\bar{z}_I - \bar{z}_r) - \ln(z_I - \bar{z}_r) - \ln(\bar{z}_I - z_r)\Big]. \end{aligned} \tag{4.36}$$

This yields

$$\begin{aligned} M_{\mathrm{UHP}}{}^r_0 + M_{\mathrm{UHP}}{}^{\tilde{r}}_0 &= \ln(z_I - Z_r) + \ln(\bar{z}_I - \bar{Z}_r) - \ln(z_I - \bar{Z}_r) - \ln(\bar{z}_I - Z_r)\ , \\ M_{\mathrm{UHP}}{}^r_n = (M_{\mathrm{UHP}}{}^{\tilde{r}}_n)^* &= -\frac{i}{n}\oint_{Z_r}\frac{dz_r}{2\pi i}\left(\frac{i}{z_r - z_I} - \frac{i}{z_r - \bar{z}_I}\right)\Big(w_r(z_r)\Big)^{-n} \end{aligned} \tag{4.37}$$

for $n \geq 1$. The values of these coefficients at $T = \epsilon \ll 1$ are summarized in appendix C.1. These are of order ϵ. Since eq.(4.26) is proportional to a product of boundary states, in evaluating $\langle V_2(1,2);\epsilon| = \langle v_2^0(1,2);\epsilon|C\left(\rho_I,\bar{\rho}_I\right)\mathcal{P}_{12}$, only the term proportional to $\pi_0^{(r)}$ from $C(\rho_I, \bar{\rho}_I)$ survives the level matching condition. Combining this fact and the results obtained in the above, we find eq.(4.11).

4.3 Overlap of Three-String Vertex with Two Boundary States

In this section, we investigate the overlap of three-string vertices with two boundary states:

$$\int d'1d'2\langle V_3^0(1,2,3)|B_0\rangle_1^\epsilon|B_0\rangle_2^\epsilon\ , \tag{4.38}$$

and

$$\int d'1d'2\langle V_3(1,2,3)|B_0\rangle_1^\epsilon|B_0\rangle_2^\epsilon \tag{4.39}$$

for $|\alpha_3| > |\alpha_1|, |\alpha_2|$ (See Fig. 4.5.). Naively, one may expect that these are proportional to ${}_3^\epsilon\langle B_0|$ (See Fig. 4.6.). In fact, these are

$$\begin{aligned} &\int d'1d'2\langle V_3^0(1,2,3)|B_0\rangle_1^\epsilon|B_0\rangle_2^\epsilon \\ &\qquad\sim \mathrm{sgn}(\alpha_3)2\delta\left(\alpha_1+\alpha_2+\alpha_3\right)\frac{(4\pi^3)^{\frac{p+1}{2}}}{(2\pi)^{25}}\frac{4}{\epsilon^3(-\ln\epsilon)^{\frac{p+1}{2}}}{}_3^\epsilon\langle B_0|\ , \end{aligned} \tag{4.40}$$

$$\begin{aligned} &\int d'1d'2\langle V_3(1,2,3)|B_0\rangle_1^\epsilon|B_0\rangle_2^\epsilon \\ &\qquad\sim -2\delta\left(\alpha_1+\alpha_2+\alpha_3\right)\frac{(4\pi^3)^{\frac{p+1}{2}}}{(2\pi)^{25}}\frac{4}{\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}}{}_3^\epsilon\langle B_0|\frac{2i}{\alpha_3}\pi_0^{(3)}\ , \end{aligned} \tag{4.41}$$

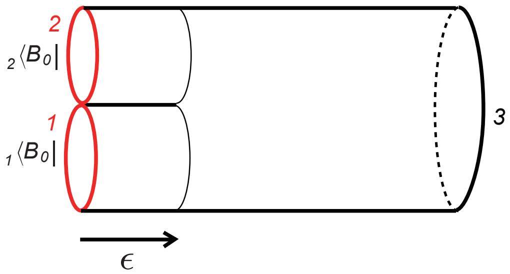


Figure 4.5: Overlap of the three-string vertex with two boundary state.

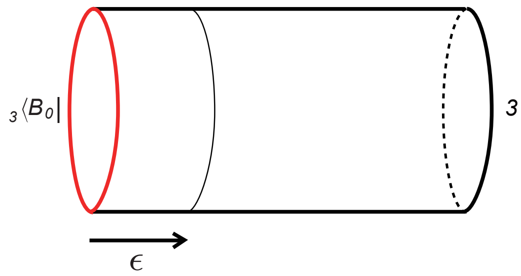


Figure 4.6: A boundary state.

in the leading order. These are the idempotency equations [7] satisfied by the boundary states in the OSp invariant string field theory. We will use these equations in the next chapter to construct D-brane states.

The calculation proceeds in the same way as the one above. We begin by evaluating

$$\begin{aligned}\langle V_1^0(3);T| &\equiv \int d'1d'2\langle V_3^0(1,2,3)|B_0\rangle_1^T|B_0\rangle_2^T = \langle v_1^0(3);T|\mathcal{P}_3\ ,\\ \langle V_1(3);T| &\equiv \int d'1d'2\langle V_3(1,2,3)|B_0\rangle_1^T|B_0\rangle_2^T = \langle v_1^0(3);T|C(\rho_I,\bar\rho_I)\mathcal{P}_3\ ,\\ &\langle v_1^0(3);T| = \int d'1d'2\langle v_3^0(1,2,3)|B_0\rangle_1^T|B_0\rangle_2^T\ . \end{aligned} \tag{4.42}$$

and then take $T=\epsilon\ll 1$. Here we present the calculations for the case $\alpha_1,\alpha_2>0$.[4]

4.3.1 Mandelstam mapping

The vertex $\langle v_1^0(3);T|$ can be expressed as

$$\langle v_1^0(3);T| = 2\delta(\alpha_1+\alpha_2+\alpha_3)(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p_3)\mathcal{K}_1(3;T)\langle v_{1,\mathrm{LPP}}^0(3);T|\ , \tag{4.43}$$

where $\langle v_{1,\mathrm{LPP}}^0(3);T|$ is the LPP vertex, and the remaining factor $\mathcal{K}_1(3;T)$ is independent of the non-zero oscillation modes.

The complex ρ-plane indicating the string diagram Fig.4.5 is described by Fig. 4.7 (b). The region of the ρ-plane corresponding to the external string, string 3, is identified with the unit disk $|w_3|\le 1$ (Fig. 4.7 (a)) of this string through the relation

$$\begin{aligned}&\rho=\alpha_3\zeta_3+T+i\beta_3\ ,\quad \beta_3=\alpha_1\pi-\alpha_3\sigma_I^{(3)}\ ,\\ &\zeta_3(=\tau_3+i\sigma_3)=\ln w_3\ ,\qquad \tau_3\le 0\ ,\quad -\pi\le\sigma_3\le\pi\ .\end{aligned} \tag{4.44}$$

Here $\rho_I=T+i\pi\alpha_1$ (and $\bar\rho_I$) is the interaction point on the ρ-plane and $\sigma_I^{(3)}$ denotes the value of the σ_3 coordinate of the interaction point of string 3. We set $\sigma_I^{(3)}=\pi\alpha_1/\alpha_3$ so that

$$\beta_3=0\ . \tag{4.45}$$

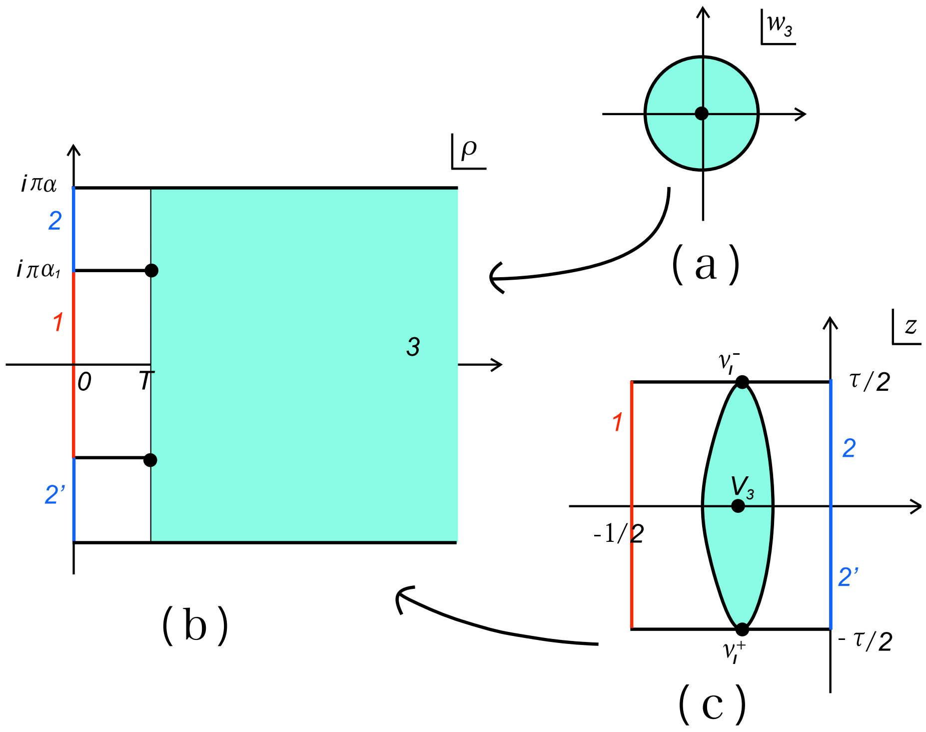


Figure 4.7: (a) The unit disk corresponding to string 3. (b) The ρ-plane corresponding to the string diagram depicted by Fig.4.5. (c) The ν-plane. These coordinates are related through eqs.(4.44) and (4.46).

The topology of the string diagram Fig. 4.5 is an annulus with a puncture corresponding to string 3. Therefore the ρ-plane can be mapped to a rectangle with a puncture on the complex ν-plane (Fig. 4.7 (c)). We take this rectangle to be the region defined by $-\frac{1}{2} \leq \mathrm{Re}\,\nu \leq 0$ and $i\frac{\tau}{2} \leq \mathrm{Im}\,\nu \leq -i\frac{\tau}{2}$. Here τ ($\tau \in i\mathbb{R}$) is the moduli parameter and the identification $\nu \cong \nu + \tau$ should be made. These two surfaces are related by the Mandelstam mapping[5]

$$\rho(\nu) = \alpha \ln \frac{\vartheta_1(\nu + V_3|\tau)}{\vartheta_1(\nu - V_3|\tau)} \ , \tag{4.46}$$

where $\alpha = \alpha_1 + \alpha_2 = -\alpha_3 > 0$, $V_3 = -\frac{\alpha_1}{2\alpha}$ and $\vartheta_i(\nu|\tau)$ ($i = 1, \ldots, 4$) are the theta functions. The point $\nu = V_3$ is the puncture corresponding to the origin $w_3 = 0$ of the unit disk $|w_3| \leq 1$ of string 3. We may parametrize the interaction points ν_I^- and ν_I^+ on the ν-plane corresponding to ρ_I and $\bar{\rho}_I$ on the ρ-plane as $\nu_I^{\pm} = -y \mp \frac{\tau}{2}$ with $y \in \mathbb{R}$, $0 \leq y \leq \frac{1}{2}$. These are determined by $\frac{d\rho}{d\nu}(\nu_I^{\pm}) = 0$. This yields

$$g_4\left(y - V_3 \,|\, \tau\right) - g_4\left(y + V_3 \,|\, \tau\right) = 0 \ , \tag{4.47}$$

where

$$g_4(\nu|\tau) \equiv \frac{\partial_\nu \vartheta_4(\nu|\tau)}{\vartheta_4(\nu|\tau)} = 4\pi \sum_{n=1}^{\infty} \sin(2\pi n\nu) \frac{q^{\frac{n}{2}}}{1 - q^n} \ , \qquad q \equiv e^{2\pi i\tau} \ . \tag{4.48}$$

[4]One can treat the case $\alpha_1, \alpha_2 < 0$ in the same way and show that these equations also hold in this case.

[5]The Mandelstam mapping (4.46) is essentially the same as the one in [34]. The rectangle on the ν-plane introduced here is the dual annulus of the rectangle on the u-plane considered in [34]. These are related by $\nu = \frac{u}{\tilde{\tau}}$, where $\tilde{\tau} = -\frac{1}{\tau}$. See also [35][36][37].

The relation $\mathrm{Re}\,\rho(\nu_I^{\pm}) = T$ leads to

$$T = \alpha \ln \frac{\vartheta_4\left(y - V_3 \,|\, \tau\right)}{\vartheta_4\left(y + V_3 \,|\, \tau\right)} \ . \tag{4.49}$$

It follows from eqs.(4.47) and (4.49) that in the small T limit, $T = \epsilon \ll 1$, the parameters τ and y behave as follows [34][36]:

$$\begin{aligned} q^{\frac{1}{2}} \equiv e^{i\pi\tau} &\simeq \frac{-\epsilon}{4\alpha \sin(2\pi V_3)} \left(1 - \frac{\epsilon^2}{8\alpha^2 \sin^2(2\pi V_3)} \left(1 - \frac{1}{3}\sin^2(2\pi V_3)\right) + \mathcal{O}(\epsilon^4)\right) \ , \\ y &\simeq \frac{1}{4} - \frac{1}{\pi}\cos(2\pi V_3)\, q^{\frac{1}{2}} + \frac{2}{\pi}\cos(2\pi V_3)\left(1 - \frac{1}{3}\cos^2(2\pi V_3)\right) q^{\frac{3}{2}} + \mathcal{O}(q^{\frac{5}{2}}) \ . \end{aligned} \tag{4.50}$$

Therefore we find that in this limit the moduli parameter $-i\tau$ becomes infinity.

For later use, we consider the behavior of τ and y in the $T \to \infty$ limit as well. In this limit, the moduli parameter τ tends to 0. In fact, by using the relations derived from eqs.(4.47) and (4.49):

$$\begin{aligned} & g_2\left(\frac{y - V_3}{\tau}\middle| \frac{-1}{\tau}\right) - g_2\left(\frac{y + V_3}{\tau}\middle| \frac{-1}{\tau}\right) = -4\pi i V_3 \ , \\ & T = \alpha \ln \frac{\vartheta_2\left(\frac{y-V_3}{\tau}\middle| \frac{-1}{\tau}\right)}{\vartheta_2\left(\frac{y+V_3}{\tau}\middle| \frac{-1}{\tau}\right)} + \frac{4\pi i \alpha y V_3}{\tau} \ , \\ & g_2(\nu|\tau) \equiv \frac{\partial_\nu \vartheta_2(\nu|\tau)}{\vartheta_2(\nu|\tau)} = -\pi\tan(\pi\nu) + 4\pi \sum_{n=1}^{\infty} \sin(2\pi n\nu) \frac{(-1)^n e^{2\pi i n\tau}}{1 - e^{2\pi i n\tau}} \ , \end{aligned} \tag{4.51}$$

we have

$$T \sim \frac{\alpha_1\alpha_2\pi}{\alpha}\frac{i}{\tau} + \hat{\tau}_0 \ , \qquad y \sim \frac{\alpha_1}{2\alpha} + \frac{i}{2\pi}\tau \ln\frac{\alpha_1}{\alpha_2} \ . \tag{4.52}$$

4.3.2 LPP vertex and Neumann coefficients

The part $-\pi\alpha_1 \leq \mathrm{Im}\,\rho \leq \pi\alpha_1$ of the boundary $\mathrm{Re}\,\rho = 0$ of the ρ-plane where the ρ-plane is attached to $|B_0\rangle_1$ corresponds to the side $\mathrm{Re}\,\nu = -\frac{1}{2}$ of the rectangle on the ν-plane. The remaining part of the boundary of the ρ-plane where the ρ-plane is attached to $|B_0\rangle_2$ corresponds to the the other side $\mathrm{Re}\,\nu = 0$ of the rectangle on the ν-plane. Therefore, on the ν-plane the worldsheet variables $X^N(\nu,\bar{\nu})$ satisfy the Neumann and the Dirichlet boundary conditions according to eq.(4.1) on the two sides, $\mathrm{Re}\,\nu = -\frac{1}{2}$ and $\mathrm{Re}\,\nu = 0$, of the rectangle and the periodic boundary condition $X^N(\nu+\tau, \bar{\nu}-\tau) = X^N(\nu,\bar{\nu})$ along the imaginary axis. It follows that the two-point functions of $X^N(\nu,\bar{\nu})$ on the ν-plane become

$$\begin{aligned} G^{NM}_{\mathrm{rectan.}}(\nu,\bar{\nu};\nu',\bar{\nu}') &= \langle X^N(\nu,\bar{\nu}) X^M(\nu',\bar{\nu}')\rangle_{\mathrm{rectan.}} \\ &= -\eta^{NM}\ln\vartheta_1\left(\nu-\nu' \,|\, \tau\right) - \eta^{NM}\ln\vartheta_1\left(\bar{\nu}-\bar{\nu}' \,|\, \tau\right) \\ &\quad - D^{NM}\ln\vartheta_1\left(\nu+\bar{\nu}' \,|\, \tau\right) - D^{NM}\ln\vartheta_1\left(\bar{\nu}+\nu' \,|\, \tau\right) + f^{NM}(\nu,\bar{\nu};\nu',\bar{\nu}') \ , \end{aligned} \tag{4.53}$$

where $f^{NM}(\nu,\bar{\nu};\nu',\bar{\nu}')$ are the terms necessary for the periodicity of the two-point functions $G^{NM}_{\mathrm{rectan.}}(\nu,\bar{\nu};\nu',\bar{\nu}')$ along the imaginary axis of the ν-plane, defined as

$$f^{\mu\lambda}(\nu,\bar{\nu};\nu',\bar{\nu}') = -\eta^{\mu\lambda}\frac{\pi i}{\tau}\left(\nu-\nu'-\bar{\nu}+\bar{\nu}'\right)^2 \tag{4.54}$$

for $\mu, \lambda \in \mathrm{N}$, and 0 otherwise. In eq.(4.53), we have used the relation

$$\overline{\vartheta_1(\nu|\tau)} = \vartheta_1(\bar{\nu}|\tau) \tag{4.55}$$

for $\tau \in i\mathbb{R}$.

Because the two-point functions on the ν-plane is eq.(4.53), we find that the LPP vertex $\langle v^0_{1,\mathrm{LPP}}(3); T|$ is of the form

$$\begin{aligned}
&(2\pi)^{p+1}\delta^{p+1}_{\mathrm{N}}(p_3)\langle v^0_{1,\mathrm{LPP}}(3); T| \\
&\quad = (2\pi)^{p+1}\delta^{p+1}_{\mathrm{N}}(p_3)\, {}_3\langle 0| \times \\
&\qquad \times \exp\Bigg[\frac{1}{2}\sum_{n,m=0}^{\infty}\sum_{N,M}\Big\{\big(\bar{N}^{hh}_{nm}\,\alpha^{N(3)}_n\,\alpha^{M(3)}_m + \bar{N}^{aa}_{nm}\,\tilde{\alpha}^{N(3)}_n\tilde{\alpha}^{M(3)}_m\big)\,\eta_{NM} \\
&\qquad\qquad + \big(\bar{N}^{ha}_{nm}\,\alpha^{N(3)}_n\tilde{\alpha}^{M(3)}_m + \bar{N}^{ah}_{nm}\,\tilde{\alpha}^{N(3)}_n\alpha^{M(3)}_m\big)\,D_{NM}\Big\} \\
&\qquad + \frac{1}{2}\sum_{n,m=1}^{\infty}\sum_{\mu,\nu\in\mathrm{N}}\Big\{\bar{F}^{hh}_{nm}\,\alpha^{\mu(3)}_n\alpha^{\nu(3)}_m + \bar{F}^{aa}_{nm}\,\tilde{\alpha}^{\mu(3)}_n\tilde{\alpha}^{\nu(3)}_m \\
&\qquad\qquad + \bar{F}^{ha}_{nm}\,\alpha^{\mu(3)}_n\tilde{\alpha}^{\nu(3)}_m + \bar{F}^{ah}_{nm}\,\tilde{\alpha}^{\mu(3)}_n\alpha^{\nu(3)}_m\Big\}\eta_{\mu\nu}\Bigg] ,
\end{aligned} \tag{4.56}$$

with

$$\begin{aligned}
\bar{N}^{hh}_{nm} &= \big(\bar{N}^{aa}_{nm}\big)^* = \frac{-1}{nm}\oint_{V_3}\frac{d\nu}{2\pi i}\oint_{V_3}\frac{d\nu'}{2\pi i}\Big(w_3(\nu)\Big)^{-n}\Big(w_3(\nu')\Big)^{-m}\partial_\nu g_1(\nu-\nu'|\tau) , \\
\bar{N}^{ha}_{nm} &= \big(\bar{N}^{ah}_{nm}\big)^* = \frac{1}{nm}\oint_{V_3}\frac{d\nu}{2\pi i}\oint_{V_3}\frac{d\bar{\nu}'}{2\pi i}\Big(w_3(\nu)\Big)^{-n}\Big(\bar{w}'_3(\bar{\nu}')\Big)^{-m}\partial_\nu g_1(\nu+\bar{\nu}'|\tau) , \\
\bar{N}^{hh}_{n0} &= \big(\bar{N}^{aa}_{n0}\big)^* = \frac{1}{n}\oint_{V_3}\frac{d\nu}{2\pi i}\Big(w_3(\nu)\Big)^{-n} g_1(\nu - V_3|\tau) , \\
\bar{N}^{ha}_{n0} &= \big(\bar{N}^{ah}_{n0}\big)^* = \frac{1}{n}\oint_{V_3}\frac{d\nu}{2\pi i}\Big(w_3(\nu)\Big)^{-n} g_1(\nu + V_3|\tau) , \\
\bar{N}^{hh}_{00} &= \big(\bar{N}^{aa}_{00}\big)^* = \ln\vartheta_1(2V_3|\tau) + \frac{T}{\alpha_3} \\
\bar{N}^{ha}_{00} &= \big(\bar{N}^{ah}_{00}\big) = \ln\vartheta_1(2V_3|\tau) \\
\bar{F}^{hh}_{nm} &= \big(\bar{F}^{aa}_{nm}\big)^* = -\frac{1}{nm}\frac{2\pi i}{\tau}\oint_{V_3}\frac{d\nu}{2\pi i}\oint_{V_3}\frac{d\nu'}{2\pi i}\Big(w_3(\nu)\Big)^{-n}\Big(w_3(\nu')\Big)^{-m} , \\
\bar{F}^{ha}_{nm} &= \big(\bar{F}^{ah}_{nm}\big)^* = \frac{1}{nm}\frac{2\pi i}{\tau}\oint_{V_3}\frac{d\nu}{2\pi i}\oint_{V_3}\frac{d\bar{\nu}'}{2\pi i}\Big(w_3(\nu)\Big)^{-n}\Big(\bar{w}'_3(\bar{\nu}')\Big)^{-m} .
\end{aligned} \tag{4.57}$$

Here $\bar{F}_{nm}$ are the contributions from the source f and $g_1(\nu|\tau)$ is defined as

$$\begin{aligned}
g_1(\nu|\tau) &\equiv \frac{\partial_\nu\vartheta_1(\nu|\tau)}{\vartheta_1(\nu|\tau)} = \pi\cot(\pi\nu) + 4\pi\sum_{n=1}^{\infty}\sin(2\pi n\nu)\frac{q^n}{1-q^n} , \\
\partial_\nu g_1(\nu|\tau) &= -\pi^2\frac{1}{\sin^2(\pi\nu)} + 8\pi^2\sum_{n=1}^{\infty} n\cos(2\pi n\nu)\frac{q^n}{1-q^n} .
\end{aligned} \tag{4.58}$$

From the mappings eqs.(4.44) and (4.46), we find

$$w_3(\nu) = e^{\frac{T}{\alpha}} \frac{\vartheta_1(\nu - V_3|\tau)}{\vartheta_1(\nu + V_3|\tau)} \ . \tag{4.59}$$

The behaviors of the Neumann coefficients in the limit $T = \epsilon \to 0$ are summarized in appendix C.2. Again it is intuitively obvious and straightforward to show that

$$(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p_3)\langle v^0_{1,\mathrm{LPP}}(3);\epsilon| \sim {}_3^{\epsilon}\langle B_0| \ , \tag{4.60}$$

in the leading order.

4.3.3 $\mathcal{K}_1(3;T)$

Next we calculate the prefactor $\mathcal{K}_1(3;T)$. This can be determined through the method explained in appendix A again. We excise small semi-circles around the interaction points $\nu = \nu_I^{\pm}$ and a small circle around the puncture $\nu = V_3$ on the ν-plane. Performing the same calculation, we obtain the contributions from these holes[6],

$$\frac{1}{|c_I||\alpha|^2}\left|\frac{\partial w_3}{\partial \nu}(V_3)\right|^2 \ , \tag{4.61}$$

where c_I is defined by

$$\begin{aligned} c_I &\equiv \frac{\partial^2 \rho}{\partial \nu^2}(\nu_I^-) \\ &= \alpha\Big[g_4'(y - V_3|\tau) - g_4'(y + V_3|\tau)\Big] \\ &= \frac{\alpha}{\tau^2}\left[g_2'\left(\frac{y - V_3}{\tau}\middle|\frac{-1}{\tau}\right) - g_2'\left(\frac{y + V_3}{\tau}\middle|\frac{-1}{\tau}\right)\right] \ , \end{aligned} \tag{4.62}$$

where $g_i'(\nu|\tau)$ denotes the $\partial_\nu g_i(\nu|\tau)$. This time, we should include the moduli dependence of the partition function on the ν-plane, and we find

$$\mathcal{K}_1(3;T) \propto \frac{1}{|c_I||\alpha|^2}\left|\frac{\partial w_3}{\partial \nu}(V_3)\right|^2 |\tau|^{-\frac{p+1}{2}} \eta(\tau)^{-24} \ , \tag{4.63}$$

where $\eta(\tau)$ is the Dedekind eta function. Thus we obtain

$$\mathcal{K}_1(3;T) = \mathcal{K}_0' \frac{(2\pi)^2 e^{\frac{2T}{\alpha}}}{(-i\tau)^{\frac{p+1}{2}} \eta(\tau)^{18} \alpha^2 c_I \, \vartheta_1(2V_3|\tau)^2} \ , \tag{4.64}$$

where $\mathcal{K}_0'$ is a numerical factor which cannot be determined by this method. The factor $\mathcal{K}_0'$ can be fixed by comparing the behaviors in the $T \to \infty$ limit of the left and right hand sides of the following equation:

$$\begin{aligned} &\int d'1 d'2 d'3 \langle v_3^0(1,2,3)|B_0\rangle_1^T |B_0\rangle_2^T |0\rangle_3 (2\pi)^{25-p}\delta_{\mathrm{D}}^{25-p}(p_3) i\bar{\pi}_0^{(3)}\pi_0^{(3)} \\ &\quad = \int d'3\ 2\delta(\alpha_1 + \alpha_2 + \alpha_3)(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p_3)\mathcal{K}_1(3;T) \\ &\qquad\qquad \times \langle v^0_{1,\mathrm{LPP}}(3);T|0\rangle_3 (2\pi)^{25-p}\delta_{\mathrm{D}}^{25-p}(p_3) i\bar{\pi}_0^{(3)}\pi_0^{(3)} \ . \end{aligned} \tag{4.65}$$

[6] Since the ν-plane is finite, we do not have the factor corresponding to c_∞.

The left hand side of this equation behaves as

$$\begin{aligned}
&\sim \quad 2\delta(\alpha_1+\alpha_2+\alpha_3)\frac{|\mu(1,2,3)|^2}{\alpha_1\alpha_2\alpha_3}e^{2\frac{T}{\alpha_1}+2\frac{T}{\alpha_2}} \\
&\qquad \times \int\prod_{i\in\mathrm{D}}\left(\frac{dp_i}{2\pi}\right) id\bar{\pi}_0 d\pi_0 \,\exp\left[-\left(\frac{T}{\alpha_1}+\frac{T}{\alpha_2}\right)\left(\sum_{i\in\mathrm{D}}p_ip_i+2i\pi_0\bar{\pi}_0\right)\right] \\
&\sim \quad 2\delta(\alpha_1+\alpha_2+\alpha_3)\frac{1}{\alpha_1\alpha_2\alpha_3}e^{-2\frac{\hat{\tau}_0}{\alpha_3}}e^{\frac{2\pi i}{\tau}}\frac{1}{2\pi}\left(\frac{-i\tau}{4\pi^2}\right)^{\frac{23-p}{2}} \quad , \qquad (4.66)
\end{aligned}$$

where we have used eq.(4.52). On the other hand, the right hand side of eq.(4.65) becomes to

$$2\delta(\alpha_1+\alpha_2+\alpha_3)\mathcal{K}_1(3;T) \quad \sim \quad 2\delta(\alpha_1+\alpha_2+\alpha_3)\mathcal{K}'_0\frac{1}{\alpha_1\alpha_2\alpha_3}e^{-2\frac{\hat{\tau}_0}{\alpha_3}}e^{\frac{2\pi i}{\tau}}(-i\tau)^{\frac{23-p}{2}} \ . \quad (4.67)$$

By comparing these equations, we find that

$$\mathcal{K}'_0 = \frac{(2\pi)^{p+1}}{(2\pi)^{25}} \ . \qquad (4.68)$$

In the case $T=\epsilon \ll 1$, the normalization factor behaves as

$$\begin{aligned}
\mathcal{K}_1(3;\epsilon) \quad &= \quad \frac{(2\pi)^{p+1}}{(2\pi)^{25}}\frac{4e^{-\frac{2\epsilon}{\alpha_3}}}{(-i\tau)^{\frac{p+1}{2}}(4\alpha\sin(2\pi V_3))^3}(1+18q+\mathcal{O}(q^2)) \\
&\sim \quad -\frac{(4\pi^3)^{\frac{p+1}{2}}}{(2\pi)^{25}}\frac{4}{(-\ln\epsilon)^{\frac{p+1}{2}}\epsilon^3} \ . \qquad (4.69)
\end{aligned}$$

From eqs.(4.60) and (4.69), we find eq.(4.40).

4.3.4 $C(\rho_I,\bar{\rho}_I)$

Finally, let us consider the effect of the insertion of the ghost field C at the interaction point. In the same way as eq.(4.35), this can be described by

$$\begin{aligned}
\langle v_{1,\mathrm{LPP}}(3);T| \quad &\equiv \quad \langle v^0_{1,\mathrm{LPP}}(3);T|C(\rho_I,\bar{\rho}_I) \\
&= \quad \langle v^0_{1,\mathrm{LPP}}(3);T|\sum_{n=0}^{\infty}\left(-iM_{\mathrm{rectan}.n}{}^{h}\,\alpha_n^{C(3)}-iM_{\mathrm{rectan}.n}{}^{a}\,\tilde{\alpha}_n^{C(3)}\right) \ . \qquad (4.70)
\end{aligned}$$

The coefficients $M_{\mathrm{rectan}.n}{}^{h}$ and $M_{\mathrm{rectan}.n}{}^{a}$ can be determined through the LPP prescription by requiring that

$$\begin{aligned}
&\int d'3\langle v_{1,\mathrm{LPP}}(3);T|\bar{C}^{(3)}(w_3,\bar{w}_3)|0\rangle_3(2\pi)^{26}\delta^{26}(p_3)i\bar{\pi}_0^{(3)}\pi_0^{(3)} \\
&\quad = G^{C\bar{C}}_{\mathrm{rectan.}}(\nu_I^-,\bar{\nu}_I^-;\nu_3,\bar{\nu}_3) \\
&\quad = i\Big[\ln\vartheta_1(\nu_I^--\nu_3|\tau)+\ln\vartheta_1(\bar{\nu}_I^--\bar{\nu}_3|\tau)-\ln\vartheta_1(\nu_I^-+\bar{\nu}_3|\tau)-\ln\vartheta_1(\bar{\nu}_I^-+\nu_3|\tau)\Big] \ .(4.71)
\end{aligned}$$

It follows that the coefficients of the non-zero modes are

$$M_{\text{rectan}.n}^{\ \ h} = (M_{\text{rectan}.n}^{\ \ a})^* = -\frac{1}{n}\oint_{\nu=V_3}\frac{d\nu}{2\pi i}\Big(w_3(\nu)\Big)^{-n}\Big[g_1(\nu_I^- - \nu|\tau) + g_1(\bar{\nu}_I^- + \nu|\tau)\Big] \tag{4.72}$$

for $n \geq 1$ and that of the zero-mode $\alpha_0^{C(3)} = \tilde{\alpha}_0^{C(3)} = -\pi_0^{(3)}$ is

$$\begin{aligned} & M_{\text{rectan}.0}^{\ \ h} + M_{\text{rectan}.0}^{\ \ a} \\ & \quad = \ln\vartheta_1(\nu_I^- - V_3|\tau) + \ln\vartheta_1(\bar{\nu}_I^- - V_3|\tau) - \ln\vartheta_1(\nu_I^- + V_3|\tau) - \ln\vartheta_1(\bar{\nu}_I^- + V_3|\tau) \\ & \quad = 2\ln\frac{\vartheta_4\ (y+V_3\,|\,\tau)}{\vartheta_4\ (y-V_3\,|\,\tau)} = -\frac{2T}{\alpha}\ . \end{aligned} \tag{4.73}$$

The values of these coefficients at the $T = \epsilon \ll 1$ are calculated in appendix C.2.

Since the vertex $\langle v^0_{1,\text{LPP}}(3); \epsilon|$ is proportional to a boundary state in the leading order, only the $\pi_0^{(3)}$ from $C(\rho_I, \bar{\rho}_I)$ survives the level matching projection. Thus we have eq.(4.41).

Chapter 5

D-brane States

In the previous chapter, we defined the boundary states and studied their overlaps with three-string vertices. Now, using these results, let us construct BRST invariant second quantized states in which D-branes are excited. In chapter 3, we constructed the observables corresponding to on-shell asymptotic states. They are BRST invariant up to the nonlinear term. Since the D-branes are off-shell objects, we should consider the BRST transformation including the non-linear term. What we would like to do is to construct the BRST invariant states including the non-linear term by using the boundary states. In this chapter, we will construct such states, D-brane states. We will calculate the Disk amplitudes and show that they coincide with those in usual string theory.

5.1 D-brane States

5.1.1 States with one D-brane

D-branes can be regarded as the source of the boundary states. In the presence of the D-branes, worldsheets may have boundaries. We should represent the D-branes as an object which can generate boundaries on the worldsheet.

Using the regularized boundary state $|B_0\rangle^\epsilon$, let us construct a state in the following form in the Hilbert space of the OSp invariant string field theory,[1]

$$|D\rangle\!\rangle \equiv \lambda \int d\zeta\, \bar{\mathcal{O}}_D(\zeta)|0\rangle\!\rangle \ , \tag{5.1}$$

where

$$\bar{\mathcal{O}}_D(\zeta) = \exp\left[a\int_{-\infty}^{0} \frac{dr}{\alpha_r}\, e^{\zeta\alpha_r}\, {}_r^{\epsilon}\langle B_0|\Phi\rangle_r + F(\zeta)\right] \ . \tag{5.2}$$

Here λ and a are constants, $F(\zeta)$ is a function of ζ and the limits of the zero-mode integration dr in the exponent of $\bar{\mathcal{O}}_D(\zeta)$ denotes the integration region of the string length α_r. Since the integration is over $-\infty < \alpha_r < 0$, only the creation operators contribute to $\bar{\mathcal{O}}_D(\zeta)$. Assuming that the integration over α_r is convergent with $\mathrm{Re}\,\zeta > 0$ sufficiently large, we define $\bar{\mathcal{O}}_D(\zeta)$ by analytic continuation.

[1] The form of the operator eq.(5.2) is very similar to that of a creation operator of the D-branes [21] in a field theoretical description of the noncritical strings. We need to add a function $F(\zeta)$ in order to make the state $|D\rangle\!\rangle$ be BRST invariant.

Expanding the exponential in terms of the string field, it is easy to see that the state $|D\rangle\rangle$ has the effect of generating boundaries in the worldsheet, with a weight which depends on a and $F(\zeta)$. Let us impose the condition that the state $|D\rangle\rangle$ is BRST invariant in the leading order of ϵ. As we will see, we can determine a and $F(\zeta)$ from this condition.

BRST transformation

In order to evaluate $\delta_{\rm B}|D\rangle\rangle$, we should calculate the BRST transformation of the operator in the exponent of $\bar{\mathcal{O}}_D(\zeta)$:

$$\begin{aligned}\delta_{\rm B}\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|\Phi\rangle_r &= \int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|Q_{\rm B}^{(r)}|\Phi\rangle_r \\ &\quad + g\int_{0}^{\infty}\frac{d3}{\alpha_3}\,e^{-\zeta\alpha_3}\int d1d2\,\langle V_3(1,2,3)|\Phi\rangle_1|\Phi\rangle_2|B_0\rangle_3^{\epsilon}\ .\end{aligned} \tag{5.3}$$

By using eq.(4.6), one can recast the first term on the right hand side of eq.(5.3) into

$$\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|Q_{\rm B}^{(r)}|\Phi\rangle_r = \zeta\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|i\pi_0^{(r)}|\bar{\psi}\rangle_r\ . \tag{5.4}$$

Let us here introduce shorthand notations

$$\begin{aligned}\bar{\phi}(\zeta) &\equiv \int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|\Phi\rangle_r\ ,\\ \bar{\chi}(\zeta) &\equiv \int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|i\pi_0^{(r)}|\bar{\psi}\rangle_r\ ,\end{aligned} \tag{5.5}$$

in terms of which eq.(5.2) can be expressed as

$$\bar{\mathcal{O}}_D(\zeta) = \exp\left(a\bar{\phi}(\zeta)+F(\zeta)\right)\ , \tag{5.6}$$

and eq.(5.4) can be written as

$$\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}_r^{\epsilon}\langle B_0|Q_{\rm B}^{(r)}|\Phi\rangle_r = \zeta\bar{\chi}(\zeta)\ . \tag{5.7}$$

Notice that $\bar{\phi}$ and $\bar{\chi}$ are made only from the creation modes and commute with each other.

For the second term on the right hand side of eq.(5.3), we decompose $|\Phi\rangle$ into the creation and annihilation parts as $|\Phi\rangle = |\psi\rangle + |\bar{\psi}\rangle$, and obtain

$$\begin{aligned}&g\int_{0}^{\infty}\frac{d3}{\alpha_3}\,e^{-\zeta\alpha_3}\int d1d2\,\langle V_3(1,2,3)|\Phi\rangle_1|\Phi\rangle_2|B_0\rangle_3^{\epsilon}\\ &\quad = g\int_{0}^{\infty}\frac{d3}{\alpha_3}\,e^{-\zeta\alpha_3}\left[\int_{-\infty}^{0}d1\int_{0}^{\infty}d2\,\langle V_3(1,2,3)|\bar{\psi}\rangle_1|\psi\rangle_2|B_0\rangle_3^{\epsilon}\right.\\ &\qquad\qquad + \int_{0}^{\infty}d1\int_{-\infty}^{0}d2\,\langle V_3(1,2,3)|\psi\rangle_1|\bar{\psi}\rangle_2|B_0\rangle_3^{\epsilon}\\ &\qquad\qquad \left. + \int_{-\infty}^{0}d1\int_{-\infty}^{0}d2\,\langle V_3(1,2,3)|\bar{\psi}\rangle_1|\bar{\psi}\rangle_2|B_0\rangle_3^{\epsilon}\right]\ .\end{aligned} \tag{5.8}$$

It follows from the relation $\langle V_3(1,2,3)| = \langle V_3(2,1,3)|$ that the first and the second terms on the right hand side of this equation are equal to each other.

In this form, it is straightforward to calculate the BRST transformation of $|D\rangle\rangle$. Using the commutation relation (2.62), we have

$$\begin{aligned}
\delta_{\mathrm{B}}|D\rangle\rangle &= \lambda \int d\zeta \, \exp\left(a\bar{\phi}(\zeta) + F(\zeta)\right) \\
&\quad \times \left[a\zeta\bar{\chi}(\zeta) + ga^2 \int_{-\infty}^{0} d1 \int_0^\infty \frac{d2}{\alpha_2} \int_0^\infty \frac{d3}{\alpha_3} \, e^{\zeta\alpha_1} \langle V_3(1,2,3)|\bar{\psi}\rangle_1 |B_0\rangle_2^\epsilon |B_0\rangle_3^\epsilon \right. \\
&\quad \left. + ga \int_{-\infty}^{0} d1 \int_{-\infty}^{0} d2 \int_0^\infty \frac{d3}{\alpha_3} \, e^{\zeta(\alpha_1+\alpha_2)} \langle V_3(1,2,3)|\bar{\psi}\rangle_1 |\bar{\psi}\rangle_2 |B_0\rangle_3^\epsilon \right] |0\rangle\rangle \\
&= \lambda \int d\zeta \left[a\zeta\bar{\chi}(\zeta) + ga^2 C_1 \partial_\zeta \bar{\chi}(\zeta) + 2gaC_2 \bar{\chi}(\zeta)\partial_\zeta\bar{\phi}(\zeta) \right] e^{a\bar{\phi}(\zeta)+F(\zeta)} \, |0\rangle\rangle \, , \qquad (5.9)
\end{aligned}$$

where

$$C_2 \equiv \frac{1}{(16\pi)^{\frac{p+1}{2}}} \frac{4}{\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}} \, , \qquad C_1 \equiv \frac{(4\pi^3)^{\frac{p+1}{2}}}{(2\pi)^{25}} \frac{4}{\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}} \, . \tag{5.10}$$

In the second equality of eq.(5.9), we have used the idempotency relations (4.11), (4.41) and following identity

$$\int_0^\infty dl_1 \int_0^\infty dl_2 e^{-\zeta_1 l_1 - \zeta_2 l_2} f(l_1 + l_2) = -\frac{\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2)}{\zeta_1 - \zeta_2} \, , \tag{5.11}$$

where

$$\tilde{f}(\zeta) \equiv \int_0^\infty dl e^{-\zeta l} f(l) \, . \tag{5.12}$$

Now, in order to make $|D\rangle\rangle$ BRST invariant, we choose $F(\zeta)$ to be of the form

$$F(\zeta) = b\zeta^2 \, . \tag{5.13}$$

Then the right hand side of eq.(5.9) becomes

$$\lambda \int d\zeta \, \partial_\zeta \left[\frac{a}{2b} \bar{\chi}(\zeta) \, \exp\left(a\bar{\phi}(\zeta) + b\zeta^2\right) \right] |0\rangle\rangle \, , \tag{5.14}$$

provided the constants a, b satisfy

$$\frac{a}{2b} = ga^2 C_1 \, , \quad \frac{a^2}{2b} = 2gaC_2 \, . \tag{5.15}$$

These equations have the solutions $(a,b) = \pm(A,B)$, where

$$A = \frac{(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}} \, , \quad B = \frac{(2\pi)^{13}\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}}{16\left(\frac{\pi}{2}\right)^{\frac{p+1}{2}}\sqrt{\pi}g} \, . \tag{5.16}$$

Therefore, by choosing (a,b) as $\pm(A,B)$ and taking the integration contour for ζ appropriately, we can obtain a state BRST invariant in the leading order of ϵ. Let us define

$$|D_\pm\rangle\rangle \equiv \lambda_\pm \int d\zeta \, \bar{\mathcal{O}}_{D_\pm}(\zeta)|0\rangle\rangle \, , \tag{5.17}$$

with

$$\bar{\mathcal{O}}_{D_\pm}(\zeta) = \exp\left[\pm\frac{(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,e^{\zeta\alpha_r}\,{}^{\epsilon}_{r}\langle B_0|\bar{\psi}\rangle_r \pm \frac{(2\pi)^{13}\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}}{16\left(\frac{\pi}{2}\right)^{\frac{p+1}{2}}\sqrt{\pi}g}\zeta^2\right] . \tag{5.18}$$

These states are considered as states in which one D-brane or one ghost D-brane is excited. We will show that $|D_\pm\rangle\rangle$ generate the worldsheets with boundaries with the right weight and disk amplitudes are reproduced. In this thesis, we take $g > 0$. In this convention, as we will see in the next section, $|D_+\rangle\rangle$ corresponds to the D-brane and $|D_-\rangle\rangle$ corresponds to the ghost D-brane.

In perturbation theory, $|D_\pm\rangle\rangle$ can be recast into a more tractable form as follows. In the integrand (5.18) of the integration (5.17), the factor

$$\exp\left[\pm\frac{(2\pi)^{13}\epsilon^2\left(-\ln\epsilon\right)^{\frac{p+1}{2}}}{16\left(\frac{\pi}{2}\right)^{\frac{p+1}{2}}\sqrt{\pi}g}\zeta^2\right] \tag{5.19}$$

becomes the most dominant perturbatively. Therefore, we carry out the saddle point approximation to obtain

$$|D_\pm\rangle\rangle \simeq \lambda'_\pm \exp\left[\pm\frac{(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int_{-\infty}^{0}\frac{dr}{\alpha_r}\,{}^{\epsilon}_{r}\langle B_0|\bar{\psi}\rangle_r\right]|0\rangle\rangle , \tag{5.20}$$

where $\lambda'_\pm$ is given as

$$\lambda'_\pm \equiv \sqrt{\mp\frac{16\left(\frac{\pi}{2}\right)^{\frac{p+1}{2}}\pi^{\frac{3}{2}}g}{(2\pi)^{13}\epsilon^2(-\ln\epsilon)^{\frac{p+1}{2}}}}\,\lambda_\pm . \tag{5.21}$$

Notice that for $|D_+\rangle\rangle$ the exponent of the Gaussian factor (5.19) has the wrong sign, which makes the factor in front of λ_+ in eq.(5.21) pure imaginary. This is a sign of instability.

One comment is in order. The variable ζ and the factor eq.(5.19) may be interpreted as a constant mode of open string tachyon and its potential. Indeed ζ appears in the form of $\exp(-\zeta|\alpha|)|B_0\rangle$ in D-brane states. Since $|\alpha|$ is the length of the boundary, ζ can be considered as a constant tachyon background.

5.1.2 States with N D-branes

We can construct BRST invariant states with N D-branes in a similar way. Let us consider a state in the following form

$$|D_{N+}\rangle\rangle \equiv \lambda_{N+}\int\prod_{i=1}^{N}d\zeta_i\,\bar{\mathcal{O}}_{D_{N+}}(\zeta_1,\cdots,\zeta_N)|0\rangle\rangle , \tag{5.22}$$

where

$$\bar{\mathcal{O}}_{D_{N+}}(\zeta_1,\cdots,\zeta_N) = \exp\left[\sum_{i=1}^{N}\left(A\bar{\phi}(\zeta_i)+B\zeta_i^2\right)+F_N(\zeta_1,\cdots,\zeta_N)\right] . \tag{5.23}$$

Here the coefficients A and B are given in eq.(5.16), and the function $F_N(\zeta_1,\cdots,\zeta_N)$ is to be determined.

It is now straightforward to evaluate the BRST variation of this state:

$$
\begin{aligned}
\delta_{\mathrm{B}}|D_{N+}\rangle\!\rangle \;&=\; \lambda_{N+}\int\prod_{i=1}^{N}d\zeta_i\exp\left[\sum_{i=1}^{N}\left(A\bar{\phi}(\zeta_i)+B\zeta_i^2\right)+F_N(\zeta_1,\cdots,\zeta_N)\right]\\
&\quad\times\left[\sum_{i=1}^{N}\Big(A\zeta_i\bar{\chi}(\zeta_i)+gA^2C_1\partial_{\zeta_i}\bar{\chi}(\zeta_i)+2gAC_2\bar{\chi}(\zeta_i)\partial_{\zeta_i}\bar{\phi}(\zeta_i)\Big)\right.\\
&\qquad\left.+gA^2C_1\sum_{i\neq j}\frac{\bar{\chi}(\zeta_i)-\bar{\chi}(\zeta_j)}{\zeta_i-\zeta_j}\right]|0\rangle\!\rangle\ . \qquad (5.24)
\end{aligned}
$$

Using eq.(5.15), one can easily deduce that the right hand side of eq.(5.24) can be recast into the form

$$
\lambda_{N+}\int\prod_{i=1}^{N}d\zeta_i\sum_{j=1}^{N}\partial_{\zeta_j}\left[\frac{A}{2B}\bar{\chi}(\zeta_j)\exp\left\{\sum_{i=1}^{N}\left(A\bar{\phi}(\zeta_i)+B\zeta_i^2\right)+F_N(\zeta_1,\cdots,\zeta_N)\right\}\right]|0\rangle\!\rangle\ , \quad (5.25)
$$

provided $F_N(\zeta_1,\cdots,\zeta_N)$ satisfies

$$
\partial_{\zeta_i}F_N(\zeta_1,\ldots,\zeta_N)=\sum_{j\neq i}\frac{2}{\zeta_i-\zeta_j}\ . \qquad (5.26)
$$

Thus we get

$$
F_N(\zeta_1,\cdots,\zeta_N)=2\sum_{i>j}\ln(\zeta_i-\zeta_j)\ , \qquad (5.27)
$$

and

$$
|D_{N+}\rangle\!\rangle=\lambda_{N+}\int\prod_{i=1}^{N}d\zeta_i\,\triangle_N^2\,(\zeta_1,\cdots,\zeta_N)\exp\left[\sum_{i=1}^{N}\left(A\bar{\phi}(\zeta_i)+B\zeta_i^2\right)\right]|0\rangle\!\rangle\ . \qquad (5.28)
$$

Here $\triangle_N$ is the Vandermonde determinant.

Notice that the integration measure

$$
\prod_{i=1}^{N}d\zeta_i\,\triangle_N^2\,(\zeta_1,\cdots,\zeta_N) \qquad (5.29)
$$

coincides with that of the matrix models. This is natural if we regard ζ as the constant mode of tachyon on the D-brane. When there exist N D-branes, the tachyon field becomes a matrix and we should consider ζ_i as its eigenvalues. Therefore we here encounter a matrix model of constant tachyons.

$|D_{N+}\rangle\!\rangle$ can be considered as a state with N D-branes. We can also construct a state with N D-branes and M ghost D-branes as

$$
\begin{aligned}
|D_{N+,M-}\rangle\!\rangle \;&\equiv\; \lambda_{N+,M-}\int\prod_{i=1}^{N}d\zeta_i\prod_{\bar{\imath}=1}^{M}d\zeta_{\bar{\imath}}\prod_{i>j}(\zeta_i-\zeta_j)^2\prod_{\bar{\imath}>\bar{\jmath}}(\zeta_{\bar{\imath}}-\zeta_{\bar{\jmath}})^2\prod_{i,\bar{\jmath}}(\zeta_i-\zeta_{\bar{\jmath}})^{-2}\\
&\quad\times\exp\left[A\left(\sum_{i=1}^{N}\bar{\phi}(\zeta_i)-\sum_{\bar{\imath}=1}^{M}\bar{\phi}(\zeta_{\bar{\imath}})\right)+B\left(\sum_{i=1}^{N}\zeta_i^2-\sum_{\bar{\imath}=1}^{M}\zeta_{\bar{\imath}}^2\right)\right]|0\rangle\!\rangle \qquad (5.30)
\end{aligned}
$$

This time the integration measure is that of the supermatrix model.

Before closing this section, one comment is in order. It is possible to express the state $|D_{N+,M-}\rangle\rangle$ as

$$|D_{N+,M-}\rangle\rangle \propto \left(\int d\zeta\, \mathcal{V}_{D_+}(\zeta)\right)^N \left(\int d\zeta'\, \mathcal{V}_{D_-}(\zeta')\right)^M |0\rangle\rangle\ . \tag{5.31}$$

Here $\mathcal{V}_{D_\pm}(\zeta)$ are of the form

$$\mathcal{V}_{D_\pm}(\zeta) = \bar{\mathcal{O}}_{D_\pm}(\zeta)\mathcal{O}_{D_\pm}(\zeta)\ , \tag{5.32}$$

where $\bar{\mathcal{O}}_{D_\pm}(\zeta)$ are the operators given in eq.(5.18) and $\mathcal{O}_{D_\pm}(\zeta)$ are defined as

$$\mathcal{O}_{D_\pm}(\zeta) = \exp\left[\pm\int_0^\infty dr\, \frac{e^{\zeta\alpha_r}}{\alpha_r}\, {}_r\langle v|\psi\rangle_r\right]\ , \tag{5.33}$$

with $|v\rangle$ satisfying

$$\int d'r\, {}_r\langle v|B_0\rangle_r^\epsilon = -\frac{4}{A}\ . \tag{5.34}$$

$\mathcal{V}_{D_\pm}(\zeta)$ look like vertex operators and may be considered as creation operators of D-branes and ghost D-branes. $|v\rangle$ can be any state as long as it satisfies eq.(5.34). For example, $|v\rangle$ can be taken to be proportional to $|B_0\rangle^\epsilon$. However, it seems difficult to make $\int d\zeta \mathcal{V}_{D_\pm}(\zeta)$ itself be invariant under the BRST transformation.

5.2 Disk Amplitudes

Now that we have BRST invariant states made from the boundary states, we would like to calculate the scattering amplitudes involving these states and show that the amplitudes involving D-branes are reproduced.

We consider the amplitudes in the presence of one (ghost) D-brane, as an example. Since $|D_\pm\rangle\rangle$ is a BRST invariant state, we may be able to calculate these amplitudes by starting from the correlation function

$$\langle\langle 0|\mathrm{T}\mathcal{O}_1(t_1)\cdots\mathcal{O}_N(t_N)|D_\pm\rangle\rangle\ . \tag{5.35}$$

Indeed, from $|D_\pm\rangle\rangle$ we get insertions of the boundary states and the worldsheets with boundaries are generated. However, because the formulation of the theory is similar to the light-cone gauge string field theory, we cannot generate boundaries on the worldsheets without any external line insertions by considering

$$\langle\langle 0|D_\pm\rangle\rangle\ . \tag{5.36}$$

Such vacuum amplitudes are constants. Therefore they can be considered to be included in the definition of the unknown constant $\lambda_\pm$.

In order to normalize the correlation function (5.35), we divide it by the vacuum amplitude as in the usual field theory, and consider

$$\left\langle\!\!\left\langle \mathcal{O}_1(t_1)\cdots\mathcal{O}_N(t_N)\right\rangle\!\!\right\rangle_{D_\pm} = \frac{\langle\langle 0|\mathrm{T}\mathcal{O}_1(t_1)\cdots\mathcal{O}_N(t_N)|D_\pm\rangle\rangle}{\langle\langle 0|D_\pm\rangle\rangle}\ . \tag{5.37}$$

Therefore, starting from this normalized correlation function, we can calculate the amplitudes in the usual way.

In this section, we will calculate the disk amplitudes. Using these disk amplitudes, we will determine which of the states $|D_\pm\rangle\rangle$ corresponds to the D-brane.

5.2.1 Disk amplitudes

Let us calculate the disk amplitudes. We evaluate the disk amplitude with two external closed string tachyons in the presence of one (ghost) D-brane, as an example. The calculation goes in the same way as that for the three-point amplitudes evaluated in subsection 3.2.1. We show that our results coincide with those for a (ghost) D-brane in string theory.

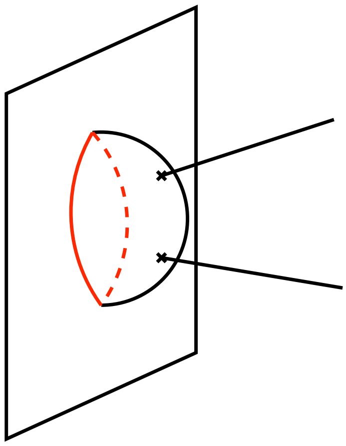
To obtain the S-matrix elements, we calculate the correlation functions for two closed string tachyons in the presence of the D-brane:

$$\left\langle\!\!\left\langle \mathcal{O}_1^T(t_1,k_1)\mathcal{O}_2^T(t_2,k_2)\right\rangle\!\!\right\rangle_{D_\pm} = \frac{\langle\!\langle 0|\mathcal{O}_1^T(t_1,k_1)\mathcal{O}_2^T(t_2,k_2)|D_\pm\rangle\!\rangle}{\langle\!\langle 0|D_\pm\rangle\!\rangle}\ . \tag{5.38}$$

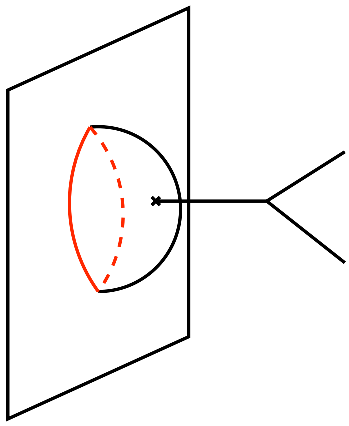
Here $\mathcal{O}_r^T$ is the observable corresponding to the tachyon state, and $t_1 > t_2$. The lowest order contributions to this correlation function give the propagator and tadpole for the tachyon. The $\mathcal{O}(g)$ term is what we should look at. This can be given as

$$\begin{aligned}
&G_{TTD_\pm}(k_1,k_2)\\
&=\frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\left[\int_{t_3}^{t_2} dT \prod_{s=1,2}\left(-\int_{-\infty}^{0}\frac{ds}{\alpha_s}\right)\left(\int_0^\infty \frac{d3}{\alpha_3}\right)\right.\\
&\qquad\qquad\left.+\int_{t_2}^{t_1} dT\left(-\int_{-\infty}^{0}\frac{d1}{\alpha_1}\right)\prod_{s=2,3}\left(\int_0^\infty\frac{ds}{\alpha_s}\right)\right]\langle V_3^0(1,2,3)|\\
&\quad\times\prod_{r=1}^{2}\left(e^{-i\frac{|T-t_r|}{|\alpha_r|}\left(p_r^2+2i\pi_0^{(r)}\bar{\pi}_0^{(r)}-2\right)}|0\rangle_r(2\pi)^{26}\delta^{26}(p_r-k_r)\right)e^{-i\frac{T-t_3}{\alpha_3}\left(L_0^{(3)}+\tilde{L}_0^{(3)}-2\right)}|B_0\rangle_3^\epsilon\ ,
\end{aligned}\tag{5.39}$$

where t_3 $(< t_1, t_2)$ is the proper time of the D-brane state. In what follows, we will show that this correctly provides the contribution of the disk attached to the (ghost) D-brane corresponding to $|D_\pm\rangle\!\rangle$. The worldsheet diagram of this process is depicted in Fig. 5.1(a).



(a)



(b)

Figure 5.1: (a) The worldsheet diagram of the two-tachyon disk amplitudes. (b) The worldsheet diagram that contributes to the pole of intermediate closed string states.

Eq.(5.39) is quite similar to eq.(3.38) and can be calculated in the same way. Looking for the singular behavior at $k_2^2-2=0$, we can get

$$
\begin{aligned}
&G_{TTD_\pm}(k_1,k_2)\\
\sim\ & \frac{1}{k_2^2-2}\frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int_{t_3}^{t_1}dT\int_0^\infty \frac{d3}{{\alpha_3}^2}e^{-i\frac{t_1-T}{\alpha_3}\left(k_1^2+2i\pi_0^{(3)}\bar{\pi}_0^{(3)}-2\right)}\\
&\quad\times(2\pi)^{26}\delta^{26}(p_3+k_1)\,{}_3\Big({}_X\langle 0|\Big)\,\mathbin{\circ\!\!\circ}\, e^{ik_{2,\mu}X^{\mu(3)}}(0)\,\mathbin{\circ\!\!\circ}\, e^{-i\frac{T-t_3}{\alpha_3}\left(L_0^{X(3)}+\tilde{L}_0^{X(3)}+2i\pi_0^{(3)}\bar{\pi}_0^{(3)}-2\right)}\Big(|B_0\rangle_X\Big)_3\\
=\ & \frac{1}{k_2^2-2}\frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\,i\int_0^\infty dT'\int_0^\infty dT'' e^{-iT'\left(k_1^2-2\right)}\\
&\quad\times\int\frac{d^{26}p}{(2\pi)^{26}}(2\pi)^{26}\delta^{26}(p+k_1)\,{}_X\langle 0|\,\mathbin{\circ\!\!\circ}\, e^{ik_{2,\mu}X^\mu}(0)\,\mathbin{\circ\!\!\circ}\, e^{-iT''\left(L_0^X+\tilde{L}_0^X-2\right)}|B_0\rangle_X\\
=\ & \frac{1}{k_1^2-2}\frac{1}{k_2^2-2}\frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\\
&\quad\times\int\frac{d^{26}p}{(2\pi)^{26}}(2\pi)^{26}\delta^{26}(p+k_1)\,{}_X\langle 0|\,\mathbin{\circ\!\!\circ}\, e^{ik_{2,\mu}X^\mu}(0)\,\mathbin{\circ\!\!\circ}\,\frac{-i}{L_0^X+\tilde{L}_0^X-2}|B_0\rangle_X\ ,
\end{aligned}
\tag{5.40}
$$

where L_0^X and $\tilde{L}_0^X$ are the zero-modes of the Virasoro generators defined in eq.(3.58) and $|B_0\rangle_X$ is the boundary state in the X^μ sector:

$$
|B_0\rangle_X \;=\; \exp\left[-\sum_{\mu,\nu\in\mathrm{N,D}}\sum_{n=1}^\infty\frac{1}{n}\alpha_{-n}^\mu\tilde{\alpha}_{-n}^\nu D_{\mu\nu}\right]|0\rangle_X(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p)\ . \tag{5.41}
$$

Carrying out the Wick rotation, we find that the S-matrix element for this process is

$$
S_{TTD_\pm}=\frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int\frac{d^{26}p}{(2\pi)^{26}}(2\pi)^{26}\delta^{26}(p+k_1)\,{}_X\langle 0|\,\mathbin{\circ\!\!\circ}\, e^{ik_{2,\mu}X^\mu}(0)\,\mathbin{\circ\!\!\circ}\,\frac{1}{L_0^X+\tilde{L}_0^X-2}|B_0\rangle_X\ , \tag{5.42}
$$

where the momenta $k_{r,\mu}$ $(r=1,2)$ are subject to the on-shell condition for the tachyon: $k_r^2=2$. It is clear that the amplitude is proportional to the usual disk amplitude.

It is straightforward to generalize the above calculations for other closed string states, just by replacing the state and the vertex operator. Also it is quite obvious that we can reproduce the disk amplitudes with more than two external lines. In order to consider the situation in which there are more than one D-branes, we should replace $|D_\pm\rangle\!\rangle$ by $|D_{N+,M-}\rangle\!\rangle$. The leading order contribution in perturbation theory is from $\zeta_i=\zeta_{\bar{\imath}}=0$ in eq.(5.30) and we obtain the S-matrix element as S_{TTD_+} in eq.(5.42) multiplied by $N-M$.

5.2.2 D-brane and ghost D-brane states

Let us check if the disk amplitude (5.42) has the correct normalization. At the on-shell pole of an intermediate closed string state $|\mathrm{primary};k\rangle_X$, it is factorized as

$$
\begin{aligned}
S_{TTD_\pm}\sim\int\frac{d^{26}k}{(2\pi)^{26}}&\left[4ig\int\frac{d^{26}p'}{(2\pi)^{26}}(2\pi)^{26}\delta^{26}(p'+k_1)\,{}_X\langle 0|\,\mathbin{\circ\!\!\circ}\, e^{ik_{2,\mu}X^\mu}(0)\,\mathbin{\circ\!\!\circ}\,|\mathrm{primary};k\rangle_X\right]\\
&\times\frac{-i}{k^2+M^2}\times\left[\frac{\pm i(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int\frac{d^{26}p}{(2\pi)^{26}}\,{}_X\langle\mathrm{primary};-k|B_0\rangle_X\right]\ ,
\end{aligned}
\tag{5.43}
$$

where M denotes the mass of the state. Since the D-brane can be considered as a source of closed string states, the low energy effective action should have source terms at $x^i = 0$ ($i \in \mathrm{D}$) due to the presence of (ghost) D-branes. From eq.(5.43), we can read off the source terms as

$$S'_{\pm} = \pm \frac{(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}} \int d^{26}x \prod_{i\in \mathrm{D}} \delta(x^i) \left[T(x) - 2 \sum_{\mu,\nu\in \mathrm{N}} h_{\mu\nu}(x)\eta^{\mu\nu} + \cdots \right] , \tag{5.44}$$

where the ellipsis denotes the contribution from the states other than the tachyon $T(x)$ and the graviton $h_{\mu\nu}(x)$. This can be compared with the DBI action for a flat Dp-brane located at $x^i = 0$ ($i \in \mathrm{D}$):

$$S_p = -\tau_p \int d^{26}x \prod_{i\in \mathrm{D}} \delta(x^i) \sqrt{- \det_{\mu,\nu\in \mathrm{N}} G_{\mu\nu}(x)} , \tag{5.45}$$

where τ_p is the Dp-brane tension in bosonic string theory defined as [38][39][2]

$$\tau_p = \frac{\sqrt{\pi}}{16\kappa}(8\pi^2)^{\frac{11-p}{2}} . \tag{5.46}$$

Using eq.(3.53) we can expand S_p in terms of $h_{\mu\nu}(x)$, and obtain the source term for $h_{\mu\nu}(x)$ which coincides with that in S'_+ in eq.(5.44). Therefore the disk amplitude S_{TTD_+} coincides with that for a D-brane and S_{TTD_-} coincides with that for a ghost D-brane.

Hence we should identify $|D_+\rangle\rangle$ with the state with one D-brane and $|D_-\rangle\rangle$ with the state with one ghost D-brane. This identification is quite consistent. D-branes in bosonic string theory are unstable due to the lack of the RR-charge and the brane corresponding to the state $|D_+\rangle\rangle$ is also unstable, as was mentioned below eq.(5.21).

[2] In this thesis, we use the units in which $\alpha' = 2$.

Chapter 6

Discussion

In this thesis, after constructing the observables for on-shell asymptotic states, we construct D-brane states in the OSp invariant closed string field theory. We also check that the disk amplitudes calculated from these states coincide with the usual string theory results.

In constructing D-brane states, we impose the BRST invariance only in the leading order of ϵ. Since the BRST variation in eq.(5.14) is of order $\epsilon^{-2}(-\ln\epsilon)^{-\frac{p+1}{2}}$, higher order corrections do not go to 0 in the limit $\epsilon \to 0$. For $p \neq -1$, the correction terms are of order $\epsilon^{-2}(-\ln\epsilon)^{-\frac{p+1}{2}-n}$ $(n > 0)$ and for $p = -1$, the next leading term is of order ϵ^0. It might be possible to prove that by modifying the exponent of $\bar{\mathcal{O}}_{D\pm}(\zeta)$ as

$$\exp\left[\pm A\bar{\phi}(\zeta) \pm B\zeta^2 + (\text{terms higher order in } \epsilon)\right] , \tag{6.1}$$

it becomes BRST invariant. As is clear from the calculation of the disk amplitudes, the higher order terms do not contribute to the amplitudes in the limit $\epsilon \to 0$. Of course, we need to examine the form of the BRST transformation to show that this actually happens. We do not try doing so, because here we are dealing with bosonic strings and we are destined to have insurmountable divergences any way. Hopefully, we may be able to show the BRST invariance more completely in the superstring case.

The calculation of the disk amplitudes arrives at the oscillator calculation in first quantized string theory. The amplitudes with more boundaries, things are not so simple. For example, let us consider the annulus amplitudes which are calculated from the worldsheet with two boundaries. Repeating the technic used in calculation of the four-point amplitudes, we finally reach a string diagram such as Fig. 6.1. We need to another method to continue the calculations. It should be checked that the results agree with the corresponding amplitude in usual string theory including the normalizations.

In the presence of D-branes, the worldsheet may have boundaries. This implies that the presence of the open strings around the D-brane vacuum. Indeed the disk two tachyon amplitude eq.(5.42) includes open strings as intermediate states. This can be seen from rewriting the disk two tachyon amplitudes as

$$\begin{aligned} S_{TTD_\pm} &= \frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int_0^\infty dT \int \frac{d^{26}p}{(2\pi)^{26}}(2\pi)^{26}\delta^{26}(p+k_1) \\ &\qquad \times {}_X\langle 0| \circ\, e^{ik_{2,\mu}X^\mu}(0) \circ\, e^{-T\left(L_0^X+\tilde{L}_0^X-2\right)}|B_0\rangle_X \\ &= \frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}}\int_0^\infty dT \end{aligned}$$

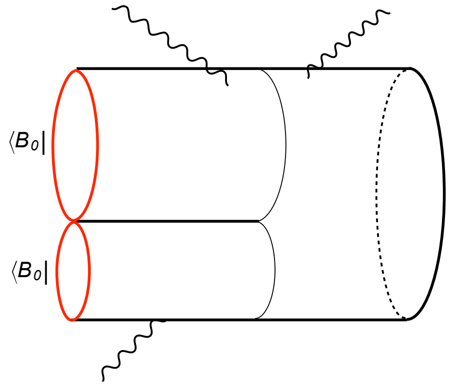


Figure 6.1: A string diagram that contributes to annulus amplitudes.

$$
\begin{aligned}
&\times \int \frac{d^{26}p}{(2\pi)^{26}} (2\pi)^{26}\delta^{26}(p+k_1) e^{-k_2\cdot\frac{\partial}{\partial p}} e^{-T\left(p^2-2\right)} (2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(p) \\
&\times \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} e^{-2nT} \sum_{\mu,\nu\in \mathrm{N,D}} k_2^\mu k_2^\nu D_{\mu\nu} \right) \\
= \frac{\pm 4ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}} &(2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(k_1+k_2) \int_0^\infty dT e^{-T\left(\sum_{i\in\mathrm{D}}(k_1+k_2)^i(k_1+k_2)^i-2\right)} \\
&\times \left(1-e^{-2T}\right)^{\sum_{\mu,\nu\in\mathrm{N,D}} k_2^\mu k_2^\nu D_{\mu\nu}} \quad . \qquad (6.2)
\end{aligned}
$$

By using the Mandelstam variables s and t defined as (See also Fig. 6.2 (a) and (b).)

$$
s = -\sum_{\mu=0}^{p} k_{1\mu}k_1{}^{\mu} = -\sum_{\mu=0}^{p} k_{2\mu}k_2{}^{\mu} \ , \quad t = -\sum_{\mu=0}^{25}(k_1+k_2)_\mu(k_1+k_2)^\mu \ , \qquad (6.3)
$$

we have

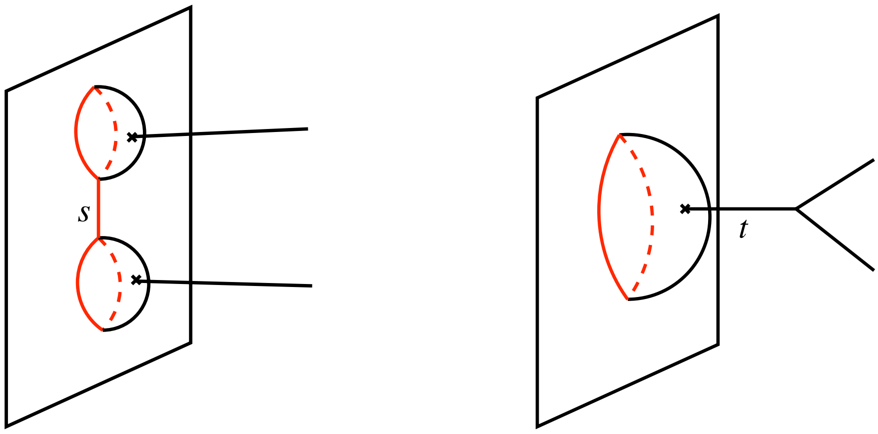
$$
S_{TTD_\pm} = \frac{\pm 8ig(2\pi)^{13}}{(8\pi^2)^{\frac{p+1}{2}}\sqrt{\pi}} (2\pi)^{p+1}\delta_{\mathrm{N}}^{p+1}(k_1+k_2) \int_0^1 dx (1-x)^{-2s-2} x^{-\frac{1}{2}t-2} \ , \qquad (6.4)
$$

where

$$
x = e^{-2T} \ . \qquad (6.5)
$$

From eq.(6.4), we can see that this amplitude has the poles at $t = -2-2n$ $(n = 1, 2, 3, \cdots)$ and $s = -\frac{1}{2}+\frac{1}{2}n$ $(n = 1, 2, 3, \cdots)$. The first series has considered in eq.(5.43) and they can be regarded as a resonance corresponding to propagation of closed strings over long spacetime distance. The amplitude also has the s-channel poles and these correspond to the exchange of open strings.

Unitarity requires that the open strings must also appear as external lines. They may be introduced by deforming the boundary state by the marginal operators corresponding to the open string vertex operators. It is an intriguing problem to examine if the higher order open string amplitudes are reproduced correctly. Another problem is to calculate the open string amplitudes without closed string insertions.



(a) (b)

Figure 6.2: Worldsheet diagrams of the two-tachyon disk amplitudes.

The variable ζ appeared in the definition of the D-brane states can be regarded as constant tachyon. In the presence of N D-branes, ζ should become $N \times N$ matrix. Since we are considered only the case that N D-branes are coincide, we cannot see the off-diagonal elements. To study more about the tachyon and its off-diagonal components will be interesting problem.

Of course, our calculations should be generalized to the superstring case. In order to do so, we have to construct the OSp invariant closed string field theory for superstrings.

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Appendix A

String Vertices from Conformal Field Theory

The N-string vertex $\langle V_N^0(1,2,\ldots,N)|$ is a fundamental object which specify the interaction of N strings. It lives in the tensor product of N string Hilbert spaces. The vertex $\langle V_N^0(1,2,\ldots,N)|$ has the following structure:

$$\langle V_N^0(1,2,\ldots,N)| = \delta_N(1,2,\ldots,N)\,\mathcal{K}_N\,\langle v_{N,\mathrm{LPP}}^0(1,2,\ldots,N)|\prod_{r=1}^{N}\mathcal{P}_r\ , \tag{A.1}$$

where $\delta_N(1,2,\ldots,N)$ denotes the delta function for total momentum conservation and $\mathcal{P}_r$ denotes the level matching projection for string r. The factor $\mathcal{K}_N$ is independent of the non-zero oscillation modes and the vertex $\langle v_{N,\mathrm{LPP}}^0(1,2,\ldots,N)|$ is determined by the prescription of LeClair, Peskin and Preitschopf (LPP) [40]. We refer to the latter as the LPP vertex. In this appendix, we explain how to obtain the LPP vertices and the normalization factors $\mathcal{K}_N$.

A.1 LPP vertices

To define the LPP vertex $\langle v_{N,\mathrm{LPP}}^0(1,2,\ldots,N)|$, we need to know the followings:

1. N conformal mappings $h_r(w_r)$ $(r=1,\ldots N)$ from N unit disks with coordinates w_r to a Riemann surface Σ. These mappings specify the interaction of N strings.
2. The correlation functions on the Riemann surface Σ.

If we know these, the LPP vertex $\langle v_{N,\mathrm{LPP}}^0(1,2,\ldots,N)|$ is defined to satisfy

$$\int d'1d'2\cdots d'N\,\langle v_{N,\mathrm{LPP}}^0(1,2,\ldots,N)|\prod_{r=1}^{N}\left(\prod_{i_r}\Big(\mathcal{O}_{i_r}\left(w_{r,i_r},\bar{w}_{r,i_r}\right)\Big)|0\rangle_r(2\pi)^{26}\delta^{26}(p_r)i\bar{\pi}_0^{(r)}\pi_0^{(r)}\right)$$
$$= \left\langle\prod_{r=1}^{N}\prod_{i_r}h_r\Big[\mathcal{O}_{i_r}\left(w_{r,i_r},\bar{w}_{r,i_r}\right)\Big]\right\rangle_\Sigma\ , \tag{A.2}$$

where $d'r$ denotes $d'r = \frac{d^{26}p_r}{(2\pi)^{26}} i d\bar{\pi}_0^{(r)} d\pi_0^{(r)}$ [1] and $\mathcal{O}_{i_r}(w_{r,i_r}, \bar{w}_{r,i_r})$ denotes an operator on the unit disk w_r and $h_r\Big[\mathcal{O}_{i_r}(w_{r,i_r}, \bar{w}_{r,i_r})\Big]$ is this operator applied the conformal transformation h_r. The right hand side of eq.(A.2) is a correlation function of conformal field theory (CFT) on Σ.

Since we are interested in the case that the string can be describe as free CFT, the LPP vertex have the structure[2]

$$\begin{aligned}
\langle v^0_{N,\mathrm{LPP}}| &= {}_{12\ldots N}\langle 0|e^{E_N} \ , \\
{}_{12\cdots N}\langle 0| &= {}_1\langle 0|{}_2\langle 0|\cdots{}_N\langle 0| \\
E_N &= \frac{1}{2}\sum_{n,m\geq 0}\sum_{r,s=1}^{N}\sum_{N,M}\Big[\bar{N}^{rs}_{nm,NM}\alpha^{N(r)}_n\alpha^{M(s)}_m + \bar{N}^{\tilde{r}\tilde{s}}_{nm,NM}\tilde{\alpha}^{N(r)}_n\tilde{\alpha}^{M(s)}_m \\
&\qquad + \bar{N}^{r\tilde{s}}_{nm,NM}\alpha^{N(r)}_n\tilde{\alpha}^{M(s)}_m + \bar{N}^{\tilde{r}s}_{nm,NM}\tilde{\alpha}^{N(r)}_n\alpha^{M(s)}_m\Big] \ , \qquad \text{(A.4)}
\end{aligned}$$

for Wick's theorem to hold. Here $\bar{N}^{rs}_{nm,NM}$, $\bar{N}^{\tilde{r}\tilde{s}}_{nm,NM}$, $\bar{N}^{r\tilde{s}}_{nm,NM}$, $\bar{N}^{\tilde{r}s}_{nm,NM}$ are called the Neumann coefficients and determined by requiring that the following equation should hold

$$\begin{aligned}
&G^{NM}_{\Sigma}(z_r,\bar{z}_r;z_s,\bar{z}_s) \qquad \text{(A.5)}\\
&= \int d'1\,d'2\cdots d'N\langle v^0_{N,\mathrm{LPP}}|X^{N(r)}(w_r,\bar{w}_r)X^{M(s)}(w_s,\bar{w}_s)\prod_{r'=1}^{N}|0\rangle_{r'}(2\pi)^{26}\delta^{26}(p_{r'})i\bar{\pi}_0^{(r')}\pi_0^{(r')} \ ,
\end{aligned}$$

where z_r and z'_s are the coordinates on Σ corresponding to the points w_r and w'_s on the unit disk of strings r and s, respectively. We obtain

$$\begin{aligned}
\bar{N}^{rs,NM}_{nm} &= \left(\bar{N}^{\tilde{r}\tilde{s},MN}_{nm}\right)^* \\
&= \frac{-1}{nm}\oint_{Z_r}\frac{dz_r}{2\pi i}\oint_{Z_s}\frac{dz'_s}{2\pi i}\Big(w_r(z_r)\Big)^{-n}\Big(w'_s(z'_s)\Big)^{-m}\partial_{z_r}\partial_{z'_s}G^{NM}_{\Sigma}(z_r,\bar{z}_r;z'_s,\bar{z}'_s) \ , \\
\bar{N}^{r\tilde{s},NM}_{nm} &= \left(\bar{N}^{\tilde{r}s,MN}_{nm}\right)^* \\
&= \frac{-1}{nm}\oint_{Z_r}\frac{dz_r}{2\pi i}\oint_{\bar{Z}_s}\frac{d\bar{z}'_s}{2\pi i}\Big(w_r(z_r)\Big)^{-n}\Big(\bar{w}'_s(\bar{z}'_s)\Big)^{-m}\partial_{z_r}\partial_{\bar{z}'_s}G^{NM}_{\Sigma}(z_r,\bar{z}_r;z'_s,\bar{z}'_s) \ , \\
\bar{N}^{rs,NM}_{n0} + \bar{N}^{r\tilde{s},NM}_{n0} &= \left(\bar{N}^{\tilde{r}\tilde{s},MN}_{n0} + \bar{N}^{\tilde{r}s,MN}_{n0}\right)^* \\
&= -\frac{1}{n}\oint_{Z_r}\frac{dz_r}{2\pi i}\Big(w_r(z_r)\Big)^{-n}\partial_{z_r}G^{NM}_{\Sigma}(z_r,\bar{z}_r;Z_s,\bar{Z}_s) \ , \\
\bar{N}^{rs,NM}_{00} + \bar{N}^{r\tilde{s},NM}_{00} &+ \bar{N}^{\tilde{r}s,NM}_{00} + \bar{N}^{\tilde{r}\tilde{s},NM}_{00} \\
&= -\lim_{z_r\to Z_r}\Big(G^{NM}_{\Sigma}(z_r,\bar{z}_r;Z_s,\bar{Z}_s) + \eta^{NM}\delta_{r,s}\ln w_r(z_r) + \eta^{NM}\delta_{r,s}\ln\bar{w}_r(\bar{z}_r)\Big) \ , \qquad \text{(A.6)}
\end{aligned}$$

where $w_r(z_r)$ denotes the inverse mapping of $h_r(w_r)$ and $Z_r = h_r(0)$ denote the z-coordinate corresponding to the origin of the unit disk w_r. Here we have used the convention for the orientation of the $\bar{z}$ integration such that $\oint_0 \frac{d\bar{z}}{2\pi i}\frac{1}{\bar{z}} = 1$.

[1]For the light-cone gauge string field theory, we should replace

$$\frac{d^{26}p_r}{(2\pi)^{26}} i d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \to \frac{d^{24}p_r}{(2\pi)^{24}} \ , \quad \text{and} \quad (2\pi)^{26}\delta^{26}(p_r)i\bar{\pi}_0^{(r)}\pi_0^{(r)} \to (2\pi)^{24}\delta^{24}(p_r) \ . \qquad \text{(A.3)}$$

[2]For the light-cone gauge string field theory, we consider the superscript M runs over $M = 1, 2, \cdots 24$.

Three-string LPP vertex $\langle v^0_{3,\mathrm{LPP}}(1,2,3)|$

Let us derive the three-string LPP vertex $\langle v^0_{3,\mathrm{LPP}}(1,2,3)|$ for joining and splitting type interaction as example. This interaction is specified by the pants-shaped worldsheet Fig. A.1. Let us introduce a complex coordinate ρ on it (Fig. A.2 (b)). Each portion of the ρ-plane

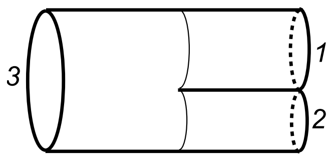

Figure A.1: The string diagram which specify the three-string interaction.

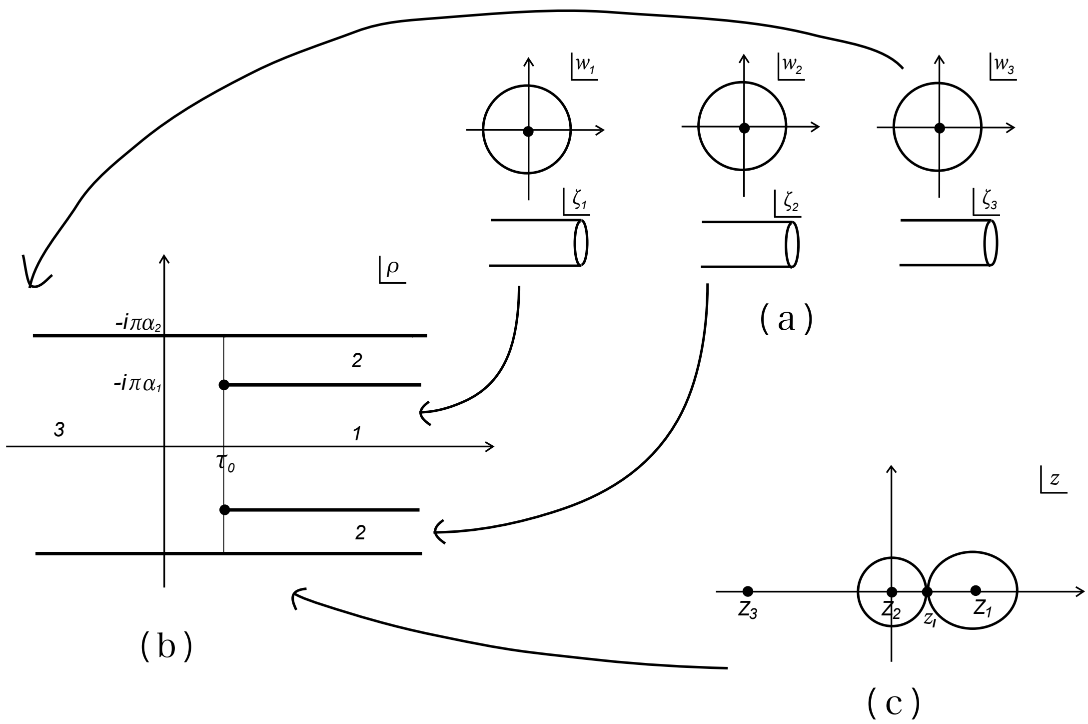

Figure A.2: (a) The unit disks corresponding to each string. (b) The ρ-plane corresponding to the string diagram depicted in Fig. A.1. (c) The complex z-plane. These coordinates are related through eqs.(A.7) and (A.8) .

corresponds to the unit disk $|w_r| \le 1$ of string r ($r = 1,2,3$) (See Fig. A.2 (a)) by the relation

$$\rho = \alpha_r \zeta_r + \tau_0 + i\beta_r \ , \quad \beta_r = \pi \sum_{s=1}^{r-1} \alpha_s \ ,$$
$$\zeta_r = \tau_r + i\sigma_r = \ln w_r \ , \qquad \tau_r \le 0 \ , \quad -\pi \le \sigma_r \le \pi \ . \tag{A.7}$$

The Mandelstam mapping from the complex z-plane to the ρ-plane is given as

$$\rho(z) = \sum_{r=1}^{3} \alpha_r \ln(z - Z_r) \ , \tag{A.8}$$

where Z_r are arbitrary complex numbers. The interaction point z_I is a solution of

$$\frac{\partial \rho}{\partial z}(z_I) = 0 \ , \qquad \tau_0 = \mathrm{Re}\left[\rho(z_I)\right] \ . \tag{A.9}$$

From eqs.(A.7) and (A.8), we find the conformal mappings from the unit disks w_r to the complex z-plane. As is well known, two point functions on the z-plane are given as

$$G^{NM}(z, \bar{z}; z', \bar{z}') = -\eta^{NM} \ln |z - z'|^2 \ . \tag{A.10}$$

Thus the three-string LPP vertex is

$$\langle v^0_{3,\mathrm{LPP}}(1,2,3)| = {}_{123}\langle 0| \exp\left(\frac{1}{2} \sum_{n,m \geq 0} \sum_{r,s} (\bar{N}^{rs}_{nm} \alpha^{(r)N}_n \alpha^{(s)M}_m + \bar{N}^{\tilde{r}\tilde{s}}_{nm} \tilde{\alpha}^{(r)N}_n \tilde{\alpha}^{(s)M}_m) \eta_{NM} \right) \ , \tag{A.11}$$

where $\bar{N}^{rs}_{nm}$ denote the Neumann coefficients associated with the joining-splitting type of three-string interaction[8][9][10]:

$$\begin{aligned}
\bar{N}^{rs}_{nm} &= \frac{1}{nm} \oint_{Z_r} \frac{dz}{2\pi i} \oint_{Z_s} \frac{dz'}{2\pi i} \frac{1}{(z - z')^2} \Big(w_r(z) \Big)^{-n} \Big(w_s(z') \Big)^{-m} \ , \\
\bar{N}^{rs}_{n0} &= \bar{N}^{sr}_{0n} = \frac{1}{n} \oint_{Z_r} \frac{dz}{2\pi i} \frac{1}{z - Z_s} \Big(w_r(z) \Big)^{-n} \ , \\
\bar{N}^{rs}_{00} &= \begin{cases} \ln(Z_r - Z_s) \ , & r \neq s \\ -\sum_{i \neq r} \frac{\alpha_i}{\alpha_r} \ln(Z_r - Z_i) + \frac{1}{\alpha_r}(\tau_0 + i\beta_r) \ , & r = s \end{cases} \ ,
\end{aligned} \tag{A.12}$$

for $n > 0$.

A.2 Moduli dependence of normalization factor

In this thesis, we consider the light-cone gauge string theory and the OSp invariant string theory. These theories are the CFT with total central charge $c = 24$ and not 0. Therefore the Generalized Gluing and Resmoothing Theorem [41] does not hold in this case and thus $\mathcal{K}_N \neq 1$. Since the overall constant is a matter of the conventions, we are interested in the moduli dependence of $\mathcal{K}_N$. This can be determined through the method explained in [42]. In this section, we calculate the factor $\mathcal{K}_3(1,2,3)$ for tree-level three closed string interaction explicitly. One can easily generalize for the other case.

The three-string vertex $\langle V^0_3(1,2,3)|$ is defined assuming that the ρ-plane is endowed with the flat metric. In order to avoid the singularities, we cut out small circles of radii r_I and r_∞ around the interaction point ρ_I and the point $\rho_\infty = \rho(z = \infty)$ in the ρ-plane (Shown in Fig. A.3 (a)). We also cut the points corresponding to incoming and outgoing strings at $\tau_r = -\infty$ by terminating each string at $\tau_r = T_r$ $(r = 1,2,3)$. These correspond to cutting circles out of the z-plane centered on $z = z_I, Z_r$ $(r = 1,2,3)$ with radii $\varepsilon_I, \varepsilon_r$, and excise the region $|z| > 1/\varepsilon_\infty$, where $\varepsilon_I, \varepsilon_r, \varepsilon_\infty$ are small (See also Fig. A.3 (b).). The

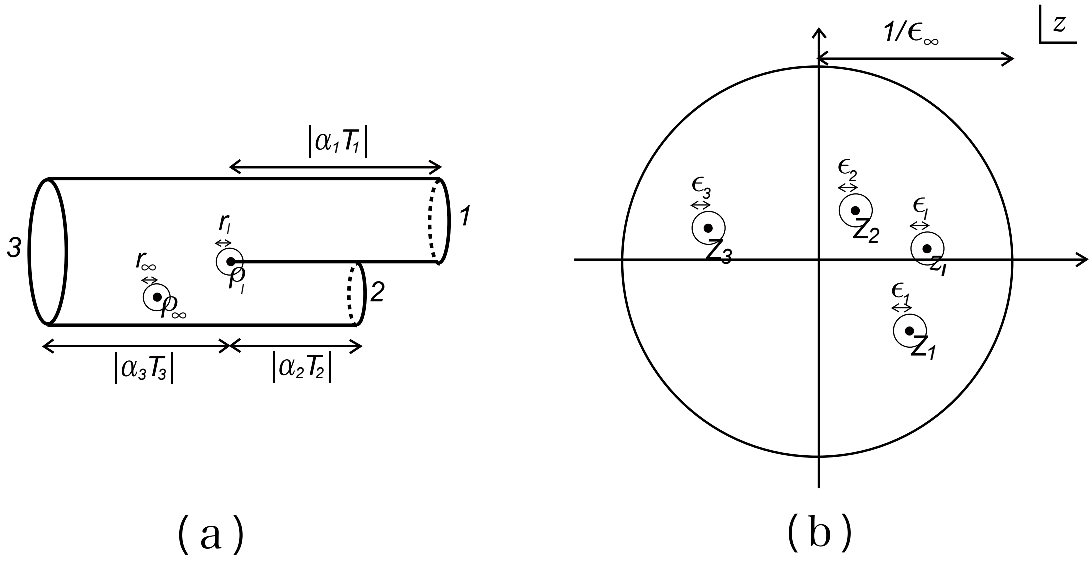

Figure A.3: (a) The regularized ρ-plane. (b) The regularized z-plane.

oscillator independent part $\mathcal{K}_3(1,2,3)$ is the partition function of the CFT on the regularized ρ-plane (Fig. A.3 (a)) with the flat metric.

The ρ-plane is equipped with the flat metric:

$$ds^2 = d\rho d\bar{\rho} = e^{\phi} dz d\bar{z} \ , \quad \phi = \ln \left| \frac{\partial \rho}{\partial z} \right|^2 \ . \tag{A.13}$$

We can obtain $\mathcal{K}_3(1,2,3)$ to calculate the Liouville action for the Liouville field ϕ in eq.(A.13), because the determinant of the Laplacian is expressed as

$$\ln \det{}' \Delta|_{\phi} - \ln \det{}' \Delta|_{\phi=0} \sim -\frac{1}{48\pi} \left[\int d^2\sigma \partial_a \phi \partial^a \phi + 4 \int_{\partial \mathcal{M}} ds\, \hat{k} \phi \right] \ , \tag{A.14}$$

where s denotes the variable parameterizing the boundary of the worldsheet $\mathcal{M}$; $\hat{k}$ denotes the geodesic curvature of the boundary defined as

$$\hat{k} = n_b t^a \hat{\nabla}_a t^b \ , \tag{A.15}$$

where t^a is the unit vector tangential to the boundary while $n^a = -\frac{\epsilon^{ab}}{\sqrt{\hat{h}}} t_b$ is normal, and $\hat{\nabla}_a$ denotes the covariant derivative associated with the metric $ds^2 = dz d\bar{z}$.

The dependence of $\ln \det{}' \Delta|_{\phi=0}$ on ε_p was calculated in Appendix 11.A of [43] [3]:

$$\ln \det{}' \Delta|_{\phi=0} \sim -\frac{1}{3} \sum_p \ln \varepsilon_p \ , \tag{A.16}$$

[3] Eq.(A.16) is twice eq.(11.A.26) in [43] because we are dealing with the closed string case.

where $\sum_p$ denotes the sum over all the values I, r and ∞. By exploring the Mandelstam mapping (4.16) near the cuts, we find that ε_p depend on α_r as follows:

$$
\begin{aligned}
\ln \varepsilon_r &\sim \lim_{z \to Z_r} \ln |z - Z_r| \sim \lim_{z \to Z_r} \left(\ln |w_r(z)| - \ln \left| \frac{w_r(z)}{z - Z_r} \right| \right) \\
&\sim T_r - \ln \left| \frac{\partial w_r}{\partial z}(Z_r) \right| \quad (r = 1, 2, 3) \ , \\
\ln \varepsilon_I &\sim \lim_{z \to z_I} \ln |z - z_I| \sim \frac{1}{2} \lim_{z \to z_I} \left(\ln \left| \frac{2(\rho(z) - \rho(z_I))}{c_I} \right| \right) \\
&\sim \frac{1}{2} (\ln 2 r_I - \ln |c_I|) \ , \\
\ln \varepsilon_\infty &\sim \lim_{z \to \infty} \ln \left| \frac{1}{z} \right| \sim \lim_{z \to \infty} \left(\ln \left| \frac{\rho(\infty) - \rho(z)}{c_\infty} \right| \right) \\
&\sim \ln r_\infty - \ln |c_\infty| \ ,
\end{aligned} \tag{A.17}
$$

where

$$
\begin{aligned}
c_\infty &\equiv - \lim_{z \to \infty} \left(\frac{\partial \rho(z)}{\partial\, (1/z)} \right) = \lim_{z \to \infty} \left(z^2 \frac{\partial \rho(z)}{\partial z} \right) = \sum_{r=1}^{3} \alpha_r Z_r \ , \\
c_I &\equiv \left. \frac{\partial^2 \rho}{\partial z^2} \right|_{z = z_I} = \frac{c_\infty^4}{\alpha_1 \alpha_2 \alpha_3 \prod_{r > s} (Z_r - Z_s)^2} \ .
\end{aligned} \tag{A.18}
$$

Collecting these results, we obtain

$$
\begin{aligned}
\ln \det{}' \Delta &\sim \frac{1}{6} \sum_{r = 1,2,3} (\ln |\alpha_r| - \ln \varepsilon_r) + \frac{1}{2} (\ln |c_I| + \ln \varepsilon_I) - \left(- \frac{1}{3} \sum_p \ln \varepsilon_p \right) \\
&\sim - \frac{1}{6} \sum_{r=1}^{3} \ln \left| \frac{\partial w_r}{\partial z}(Z_r) \right| + \frac{1}{6} \sum_{r=1}^{2} \ln |\alpha_r| + \frac{1}{12} \ln |c_I| - \frac{1}{3} \ln |c_\infty| \ ,
\end{aligned} \tag{A.19}
$$

and

$$
\mathcal{K}_3(1,2,3) \propto (\det{}' \Delta)^{-12} \propto \frac{|c_\infty|^4}{|c_I|} \prod_{r=1}^{3} \left(\frac{1}{|\alpha_r|^2} \left| \frac{\partial w_r}{\partial z}(Z_r) \right|^2 \right) \ . \tag{A.20}
$$

Therefore we can define $\mathcal{K}_3(1,2,3)$ as

$$
\mathcal{K}_3(1,2,3) = \frac{|\mu(1,2,3)|^2}{\alpha_1 \alpha_2 \alpha_3} \ , \tag{A.21}
$$

where

$$
\mu(1,2,3) = \exp \left(- \hat{\tau}_0 \sum_{r=1}^{3} \frac{1}{\alpha_r} \right) \ , \quad \hat{\tau}_0 = \sum_{r=1}^{3} \alpha_r \ln |\alpha_r| \ . \tag{A.22}
$$

Appendix B

Lorentz Invariance of the Light-Cone Gauge String Field Theory

In subsections 2.1.2 and 2.2.2, in order to show the Lorentz invariance of the light-cone gauge string field theory and the BRST invariance of the OSp invariant string field theory, we have used the relations derived from the followings:

$$\langle V_3^0(1,2,3)|\sum_{r=1}^{3} M^{N-(r)} = \langle V_3^0(1,2,3);X^N|\sum_{r=1}^{3} \frac{L_0^{(r)}+\tilde{L}_0^{(r)}-2}{2\alpha_r} \ , \tag{B.1}$$

$$\langle V_3^0(1,2,3);X^M|\sum_{r=1}^{3} M^{N-(r)} - (-1)^{|N||M|}\langle V_3^0(1,2,3);X^N|\sum_{r=1}^{3} M^{M-(r)} = 0 \ , \tag{B.2}$$

$$\begin{aligned}\int d5d6\Big[&\langle V_3^0(1,2,5)|\langle V_3^0(3,4,6);X^N| - \langle V_3^0(1,2,5);X^N|\langle V_3^0(3,4,6)| \\ &+\langle V_3^0(2,3,5)|\langle V_3^0(1,4,6);X^N| - \langle V_3^0(2,3,5);X^N|\langle V_3^0(1,4,6)| \\ &+\langle V_3^0(3,1,5)|\langle V_3^0(2,4,6);X^N| - \langle V_3^0(3,1,5);X^N|\langle V_3^0(2,4,6)|\Big]|R(5,6)\rangle = 0 \ ,\end{aligned} \tag{B.3}$$

$$\begin{aligned}&\int d5d6\langle V_3^0(1,2,5);X^M|\langle V_3^0(3,4,6);X^N|R(5,6)\rangle \\ &\quad+\int d5d6\langle V_3^0(2,3,5);X^M|\langle V_3^0(1,4,6);X^N|R(5,6)\rangle \\ &\quad+\int d5d6\langle V_3^0(3,1,5);X^M|\langle V_3^0(2,4,6);X^N|R(5,6)\rangle - (-)^{|M||N|}\Big(X^M \leftrightarrow X^N\Big) = 0 \ ,\end{aligned} \tag{B.4}$$

where

$$\begin{aligned}M^{N-(r)} = &\frac{x_r^N}{2\alpha_r}(L_0+\tilde{L}_0-2) + ip_r^N\frac{\partial}{\partial\alpha_r} \\ &-\frac{i}{\alpha_r}\sum_{n=1}^{\infty}\left(\frac{\alpha_{-n}^{N(r)}L_n^{(r)}-\tilde{\alpha}_{-n}^{N(r)}\tilde{L}_n^{(r)}}{n}+\frac{L_{-n}^{(r)}\alpha_n^{N(r)}-\tilde{L}_{-n}^{(r)}\tilde{\alpha}_n^{N(r)}}{n}\right) \ ,\end{aligned} \tag{B.5}$$

$$\langle V_3^0(1,2,3);X^M| = \langle v_3^0(1,2,3)|X^M(\rho_I,\bar{\rho}_I)\mathcal{P}_{123} \ . \tag{B.6}$$

Here the three-string vertex $\langle v_3^0(1,2,3)|$ is defined in eq.(2.47) and ρ_I is the interaction point of the three strings. In this appendix, we show these relations. We present the calculations in terms of the OSp invariant string field theory. One can think that the subscript "LC" is omitted in the light-cone gauge string field theory.

B.1 $\mathcal{O}(g)$

First we derive the $\mathcal{O}(g)$ relations eqs.(B.1) and (B.2). As was performed in [29] (See also [28][44].), we use conformal field theory (CFT) technique. Let us split X^N into holomorphic and antiholomorphic parts

$$\begin{aligned} X_L^N(w_r) &= x_L^{N(r)} - i\alpha_0^{N(r)} \ln w_r + i\sum_{n\neq 0} \frac{1}{n}\alpha_n^{N(r)} w_r^{-n} \ , \\ \tilde{X}_R^N(\bar{w}_r) &= \tilde{x}_R^{N(r)} - i\tilde{\alpha}_0^{N(r)} \ln \bar{w}_r + i\sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^{N(r)} \bar{w}_r^{-n} \ , \end{aligned} \tag{B.7}$$

where w_r denotes the coordinate of the unit disk for r-th string. We have splitted the zero mode x_r^N into

$$x_r^N = x_L^{N(r)} + \tilde{x}_R^{N(r)} \ , \qquad \left[x_L^{N(r)}, \alpha_0^{M(s)}\right\} = \left[\tilde{x}_R^{N(r)}, \tilde{\alpha}_0^{M(s)}\right\} = i\eta^{NM}\delta_{rs} \ . \tag{B.8}$$

The energy momentum tensor of this CFT can be defined as

$$\begin{aligned} T(w_r) &\equiv -\frac{1}{2} : \partial X_L^N(w_r) \partial X_L^M(w_r) : \eta_{NM} \ , \\ \tilde{T}(\bar{w}_r) &\equiv -\frac{1}{2} : \bar{\partial} \tilde{X}_R^N(\bar{w}_r) \bar{\partial} \tilde{X}_R^M(\bar{w}_r) : \eta_{NM} \ , \end{aligned} \tag{B.9}$$

where the symbol : : denotes the conformal normal ordering defined such as

$$\begin{aligned} : X_L^N(w_r) X_L^M(w'_r) : &\equiv X_L^N(w_r) X_L^M(w'_r) + \eta^{NM} \ln(w_r - w'_r) \ , \\ : \tilde{X}_R^N(\bar{w}_r) \tilde{X}_R^M(\bar{w}'_r) : &\equiv \tilde{X}_R^N(\bar{w}_r) \tilde{X}_R^M(\bar{w}'_r) + \eta^{NM} \ln(\bar{w}_r - \bar{w}'_r) \ , \\ : X_L^N(w_r) \tilde{X}_R^M(\bar{w}'_r) : &\equiv X_L^N(w_r) \tilde{X}_R^M(\bar{w}'_r) \ . \end{aligned} \tag{B.10}$$

By using the terms of CFT, we can rewrite the Virasoro generators $L_0, \tilde{L}_0$ and the Lorentz generator M^{N-} as[1]

$$\begin{aligned} L_0^{(r)} - 1 &= \oint_0 \frac{dw_r}{2\pi i} w_r \Big(T(w_r) - \frac{1}{w_r^2} \Big) = \oint_{C_r} \frac{d\zeta_r}{2\pi i} T(\zeta_r) \ , \\ \tilde{L}_0^{(r)} - 1 &= \oint_0 \frac{d\bar{w}_r}{2\pi i} \bar{w}_r \Big(\tilde{T}(\bar{w}_r) - \frac{1}{\bar{w}_r^2} \Big) = \oint_{C_r} \frac{d\bar{\zeta}_r}{2\pi i} T(\bar{\zeta}_r) \ , \\ M^{N-(r)} &= M_L^{N-(r)} + \tilde{M}_R^{N-(r)} + \frac{i}{2} \left(\alpha_0^{N(r)} + \tilde{\alpha}_0^{N(r)} \right) \frac{\partial}{\partial \alpha_r} \ , \\ M_L^{N-(r)} &= \frac{1}{\alpha_r} \oint_0 \frac{dw_r}{2\pi i} w_r : \left(X_L^N(w_r) + i\alpha_0^{N(r)} \ln w_r \right) \left(T(w_r) - \frac{1}{w_r^2} \right) : \end{aligned} \tag{B.11}$$

[1] Eq.(B.12) is correct only under the level matching projection.

$$= \frac{1}{\alpha_r}\oint_{C_r}\frac{d\zeta_r}{2\pi i} : \left(X_L^N(\zeta_r) + i\alpha_0^{N(r)}\zeta_r\right) T(\zeta_r) : + \frac{i}{2\alpha_r}\alpha_0^{N(r)} ,$$

$$\tilde{M}_R^{N-(r)} = \frac{1}{\alpha_r}\oint_0 \frac{d\bar{w}_r}{2\pi i}\bar{w}_r : \left(\tilde{X}_R^N(\bar{w}_r) + i\tilde{\alpha}_0^{N(r)}\ln\bar{w}_r\right)\left(\tilde{T}(\bar{w}_r) - \frac{1}{\bar{w}_r^2}\right) :$$

$$= \frac{1}{\alpha_r}\oint_{C_r}\frac{d\bar{\zeta}_r}{2\pi i} : \left(\tilde{X}_R^N(\bar{\zeta}_r) + i\tilde{\alpha}_0^{N(r)}\bar{\zeta}_r\right)\tilde{T}(\bar{\zeta}_r) : + \frac{i}{2\alpha_r}\tilde{\alpha}_0^{N(r)} , \qquad \text{(B.12)}$$

where $\zeta_r = \ln w_r$ denotes the cylinder coordinate. Here we have used the relations

$$(\partial_{w_r}f)^2 T(f) = T(w_r) - 2\{f, w_r\} ,$$

$$(\partial_{w_r}f)^2 : X_L^N(f)T(f) : = : X_L^N(w_r)T(w_r) : -2\{f, w_r\}X_L^N(w_r) - \frac{\partial_{w_r}^2 f}{2\partial_{w_r}f}\partial X_L^N(w_r) ,$$

$$\{f, w_r\} \equiv \frac{\partial_{w_r}^3 f}{\partial_{w_r}f} - \frac{3}{2}\left(\frac{\partial_{w_r}^2 f}{\partial_{w_r}f}\right)^2 . \qquad \text{(B.13)}$$

with the conformal mapping $f(w_r) = \ln w_r$ and the same relations for antiholomorphic sector. The contours C_r $(r = 1, 2, 3)$ are appeared in Fig. B.1.

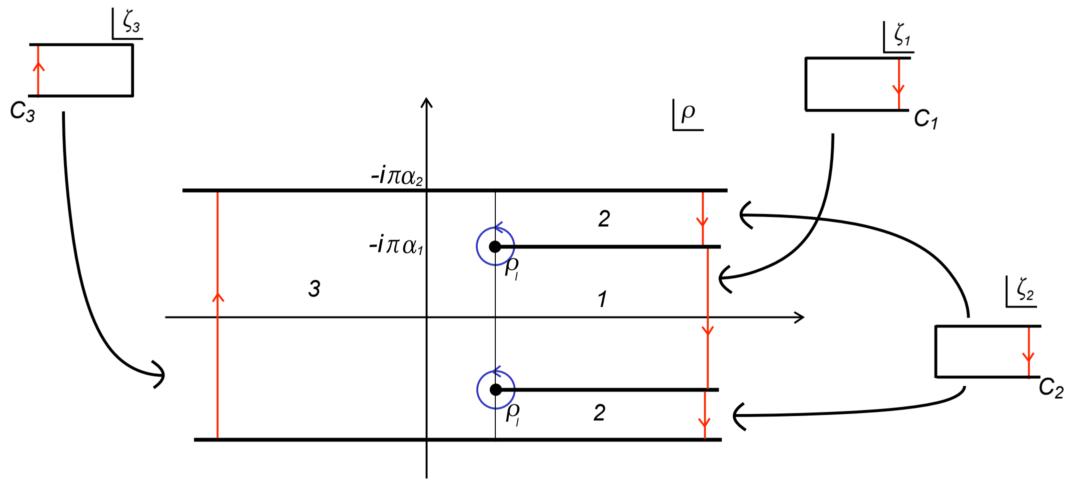

Figure B.1: The contours C_r $(r = 1, 2, 3)$.

Now let us calculate the right hand side of eq.(B.1) in terms of the CFT:

$$\langle V_3^0(1,2,3); X^N| \sum_{r=1}^{3}\frac{L_0^{(r)} + \tilde{L}_0^{(r)} - 2}{2\alpha_r}$$

$$= \langle v_3^0(1,2,3)|X^N(\rho_I, \bar{\rho}_I)\sum_{r=1}^{3}\frac{1}{2\alpha_r}\left(\oint_{C_r}\frac{d\zeta_r}{2\pi i}T^{(r)}(\zeta_r) + \oint_{C_r}\frac{d\bar{\zeta}_r}{2\pi i}\tilde{T}(\bar{\zeta}_r)\right)\mathcal{P}_{123}$$

$$= -\frac{1}{2}\langle v_3^0(1,2,3)|\left(X_L^N(\rho_I) + \tilde{X}_R^N(\bar{\rho}_I)\right)\left(\oint_{\rho_I}\frac{d\rho}{2\pi i}T(\rho) + \oint_{\bar{\rho}_I}\frac{d\bar{\rho}}{2\pi i}\tilde{T}(\bar{\rho})\right)\mathcal{P}_{123}$$

$$= -\frac{1}{2}\langle v_3^0(1,2,3)|\left(X_L^N(\omega_I) + \tilde{X}_R^N(\bar{\omega}_I)\right)$$

$$
\begin{aligned}
&\times\left(\oint_{\omega_I}\frac{d\omega}{2\pi i}\frac{1}{\omega}\left(T(\omega)+\frac{3}{\omega^2}\right)+\oint_{\bar{\omega}_I}\frac{d\bar{\omega}}{2\pi i}\frac{1}{\bar{\omega}}\left(\tilde{T}(\bar{\omega})+\frac{3}{\bar{\omega}^2}\right)\right)\mathcal{P}_{123}\\
=\;& -\frac{1}{2}\langle v_3^0(1,2,3)|\Bigg(:\left(X_L^N(\omega_I)+\tilde{X}_R^N(\bar{\omega}_I)\right)\left(T(\omega_I)+\tilde{T}(\bar{\omega}_I)\right):\\
&\qquad +\partial^2 X_L^N(\omega_I)+\bar{\partial}^2\tilde{X}_R^N(\bar{\omega}_I)\Bigg)\mathcal{P}_{123}\ ,
\end{aligned}
\tag{B.14}
$$

where ω is defined as

$$
\rho-\rho_I=\frac{1}{2}\omega^2\ ,\qquad \omega_I=0\ ,
\tag{B.15}
$$

and we have used eq.(B.13).

Since $L_0^{(r)}-\tilde{L}_0^{(r)}$ is equal to zero under the projection operator $\mathcal{P}_r$, we find the following relations

$$
\begin{aligned}
0 \;=\;& \langle v_3^0(1,2,3)|\left(X_L^N(\rho_I)-\tilde{X}_R^N(\bar{\rho}_I)\right)\sum_{r=1}^{3}\frac{L_0^{(r)}-\tilde{L}_0^{(r)}}{2\alpha_r}\mathcal{P}_{123}\\
=\;& \langle v_3^0(1,2,3)|\left(X_L^N(\rho_I)-\tilde{X}_R^N(\bar{\rho}_I)\right)\sum_{r=1}^{3}\frac{1}{2\alpha_r}\left(\oint_{C_r}\frac{d\zeta_r}{2\pi i}T^{(r)}(\zeta_r)-\oint_{C_r}\frac{d\bar{\zeta}_r}{2\pi i}\tilde{T}(\bar{\zeta}_r)\right)\mathcal{P}_{123}\\
=\;& -\frac{1}{2}\langle v_3^0(1,2,3)|\Bigg(:\left(X_L^N(\omega_I)-\tilde{X}_R^N(\bar{\omega}_I)\right)\left(T(\omega_I)-\tilde{T}(\bar{\omega}_I)\right):\\
&\qquad +\partial^2 X_L^N(\omega_I)+\bar{\partial}^2\tilde{X}_R^N(\bar{\omega}_I)\Bigg)\mathcal{P}_{123}\ ,
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
0 \;=\;& \langle v_3^0(1,2,3)|:X_L^N(\rho_I)\tilde{X}_R^M(\bar{\rho}_I):\sum_{r=1}^{3}\frac{L_0^{(r)}-\tilde{L}_0^{(r)}}{2\alpha_r}\mathcal{P}_{123}\\
=\;& \langle v_3^0(1,2,3)|:X_L^N(\rho_I)\tilde{X}_R^M(\bar{\rho}_I):\sum_{r=1}^{3}\frac{1}{2\alpha_r}\left(\oint_{C_r}\frac{d\zeta_r}{2\pi i}T^{(r)}(\zeta_r)-\oint_{C_r}\frac{d\bar{\zeta}_r}{2\pi i}\tilde{T}(\bar{\zeta}_r)\right)\mathcal{P}_{123}\\
=\;& -\frac{1}{2}\langle v_3^0(1,2,3)|\Bigg(:X_L^N(\omega_I)\tilde{X}_R^M(\bar{\omega}_I)\left(T(\omega_I)-\tilde{T}(\bar{\omega}_I)\right):\\
&\qquad +\partial^2 X_L^N(\omega_I)\tilde{X}_R^M(\bar{\omega}_I)-X_L^N(\omega_I)\bar{\partial}^2\tilde{X}_R^M(\bar{\omega}_I)\Bigg)\mathcal{P}_{123}\ .
\end{aligned}
\tag{B.17}
$$

Combining eqs.(B.14) and (B.16), one can obtain

$$
\begin{aligned}
&\langle V_3^0(1,2,3);X^N|\sum_{r=1}^{3}\frac{L_0^{(r)}+\tilde{L}_0^{(r)}-2}{2\alpha_r}\\
=\;& -\langle v_3^0(1,2,3)|\Bigg(:X_L^N(\omega_I)T(\omega_I):+:\tilde{X}_R^N(\bar{\omega}_I)\tilde{T}(\bar{\omega}_I):\\
&\qquad +\partial^2 X_L^N(\omega_I)+\bar{\partial}^2\tilde{X}_R^N(\bar{\omega}_I)\Bigg)\mathcal{P}_{123}\ .
\end{aligned}
\tag{B.18}
$$

We should also calculate

$$
\langle V_3^0(1,2,3);A|\sum_{r=1}^{3}M^{N-(r)}
\tag{B.19}
$$

for $A = 1, X^M$. Here we define $\langle V_3^0(1,2,3); 1| = \langle V_3^0(1,2,3)|$. In order to do so, we need the α_r derivative of the three-string vertex. This can be derived as follows. Let us remember that

$$
\begin{aligned}
&\int d'1 d'2 d'3_{123}\langle 0|e^{E(1,2,3)} A(\rho_I, \bar{\rho}_I) \\
&\qquad\qquad \times \prod_{r=1}^{3}\left(\left(\frac{\partial w_r}{\partial z}\right)^{h_r}\left(\frac{\partial \bar{w}_r}{\partial \bar{z}}\right)^{\bar{h}_r} \mathcal{O}_r(w_r = 0)\,|0\rangle_r\,(2\pi)^{26}\delta^{26}(p_r) i\bar{\pi}_0^{(r)}\pi_0^{(r)}\right) \\
&\qquad = \left\langle A(z_I, \bar{z}_I) \prod_{r=1}^{3} \mathcal{O}_r(Z_r, \bar{Z}_r)\right\rangle ,
\end{aligned}
\tag{B.20}
$$

where $(h_r, \bar{h}_r)$ $(r = 1, 2, 3)$ are the conformal weights of the operators $\mathcal{O}_r$ and the right hand side denotes the correlation function of these operators on the z-plane. Performing the infinitesimal transformation

$$
\alpha_r \to \alpha_r + \delta\alpha_r , \qquad w_r \to w_r + \delta w_r , \qquad \text{with } Z_r \text{ fixed} ,
\tag{B.21}
$$

we can obtain

$$
\begin{aligned}
&\delta\Big(\alpha_1\alpha_2\alpha_3\langle v_3^0(1,2,3)|A(z_I, \bar{z}_I)\mathcal{P}_{123}\Big) \\
= {}& \alpha_1\alpha_2\alpha_3\langle v_3^0(1,2,3)|\Big(\delta z_I \partial A(z_I) + \delta\bar{z}_I\bar{\partial}A(\bar{z}_I)\Big)\mathcal{P}_{123} \\
&-\alpha_1\alpha_2\alpha_3\langle v_3^0(1,2,3)|A(z_I, \bar{z}_I)\sum_{s=1}^{3}\left(\oint_0 \frac{dw_s}{2\pi i}\delta w_s T(w_s) + \oint_0 \frac{d\bar{w}_s}{2\pi i}\delta\bar{w}_s\tilde{T}(\bar{w}_s)\right)\mathcal{P}_{123} \\
&+\delta(1,2,3)_{123}\langle 0|e^{E(1,2,3)}A(z_I, \bar{z}_I)\mathcal{P}_{123}\delta\Big(|\mu(1,2,3)|^2\Big) .
\end{aligned}
\tag{B.22}
$$

By using the fact

$$
\begin{aligned}
\oint_0 \frac{dw_s}{2\pi i}\delta w_s T(w_s) &= \oint_{C_s}\frac{d\zeta_s}{2\pi i}\delta\zeta_s\Big(T(\zeta_s) + 1\Big) , \\
\sum_{s=1}^{3}\oint_{C_s}\frac{d\zeta_s}{2\pi i}\delta\zeta_s &= \sum_{s=1}^{3}\oint_{Z_s}\frac{dz}{2\pi i}\frac{\partial\zeta_s}{\partial z}\delta\zeta_s = \sum_{s=1}^{3}\delta\zeta_s|_{z=Z_s} \\
&= \delta\Big(\ln|\mu(1,2,3)|\Big) + \text{pure imaginary} ,
\end{aligned}
\tag{B.23}
$$

we can derive the α_r derivative of the three-string vertex:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha_r}\Big(\alpha_1\alpha_2\alpha_3\langle V_3^0(1,2,3); A|\Big) &= \alpha_1\alpha_2\alpha_3\langle v_3^0(1,2,3)|\left(\frac{\partial z_I}{\partial\alpha_r}\partial A(z_I) + \frac{\partial\bar{z}_I}{\partial\alpha_r}\bar{\partial}A(\bar{z}_I)\right)\mathcal{P}_{123} \\
&\quad -\alpha_1\alpha_2\alpha_3\langle v_3^0(1,2,3)|A(z_I, \bar{z}_I)\sum_{s=1}^{3}\left(\oint_{C_s}\frac{d\zeta_s}{2\pi i}\frac{\partial\zeta_s}{\partial\alpha_r}T(\zeta_s) + \oint_{C_s}\frac{d\bar{\zeta}_s}{2\pi i}\frac{\partial\bar{\zeta}_s}{\partial\alpha_r}\tilde{T}(\bar{\zeta}_s)\right)\mathcal{P}_{123} .
\end{aligned}
\tag{B.24}
$$

Combining the relations derived from the definition of ζ_r:

$$\begin{aligned} \zeta_r &= \frac{1}{\alpha_r}\Big(\rho(z)-\rho(z_I)\Big) + z\text{-independent imaginary terms }, \\ \frac{\partial\zeta_r}{\partial\alpha_s} &= -\frac{\delta_{r,s}}{\alpha_r}\zeta_r + \frac{1}{\alpha_r}\Big(\ln(z-Z_s)-\ln(z_I-Z_s)\Big) + z\text{-independent imaginary terms }, \end{aligned} \tag{B.25}$$

and eqs.(B.12) and (B.24), we obtain

$$\begin{aligned} &\langle V_3^0(1,2,3);A|\sum_{r=1}^{3} M^{N-(r)} \\ &= \langle v_3^0(1,2,3)|A(\rho_I,\bar\rho_I)\sum_{r=1}^{3}\left[\frac{1}{\alpha_r}\oint_{C_r}\frac{d\zeta_r}{2\pi i}:X_L^N(\zeta_r)T(\zeta_r):+\sim\right]\mathcal{P}_{123} \\ &\quad +i\,\langle v_3^0(1,2,3)|A(\rho_I,\bar\rho_I) \\ &\qquad \times\sum_{r=1}^{3}\left[\frac{1}{\alpha_r}\oint_{C_r}\frac{d\zeta_r}{2\pi i}\sum_{s=1}^{3}\alpha_0^{N(s)}\Big(\ln(z-Z_s)-\ln(z_I-Z_s)\Big)T(\zeta_r)+\frac{\alpha_0^{N(r)}}{2\alpha_r}\right. \\ &\qquad\qquad \left.+\ \sim\ \right]\mathcal{P}_{123} \\ &\quad -i\sum_{r=1}^{3}\alpha_0^{N(r)}\langle v_3^0(1,2,3)|\left(\frac{\partial z_I}{\partial\alpha_r}\partial A(z_I)+\frac{\partial\bar z_I}{\partial\alpha_r}\bar\partial A(\bar z_I)\right)\mathcal{P}_{123}\ , \end{aligned} \tag{B.26}$$

where the symbol $\sim$ denotes the contributions from antiholomorphic sector. The first term of the right hand side of this equation is

$$\begin{aligned} &-\langle v_3^0(1,2,3)|A(\rho_I,\bar\rho_I)\left(\oint_{\rho_I}\frac{d\rho}{2\pi i}:X_L^N(\rho)T(\rho):+\sim\right)\mathcal{P}_{123} \\ &= -\langle v_3^0(1,2,3)|A(\omega_I,\bar\omega_I) \\ &\qquad\times\left(\oint_{\omega_I}\frac{d\omega}{2\pi i}\frac{1}{\omega}\left[:X_L^N(\omega)T(\omega):+\frac{3}{\omega^2}X_L^N(\omega)-\frac{1}{2\omega}\partial X_L^N(\omega)\right]+\sim\right)\mathcal{P}_{123} \end{aligned} \tag{B.27}$$

If we set $A=1$, this serves eq.(B.18). If we set $A=X^M$, this contribution is cancel in eq.(B.2). In doing so, we use eq.(B.17).

The remaining task is to show that the second and the third contributions in the right hand side of eq.(B.26) are cancel each other. As was performed in [29][44], let us expand $\rho(z)$ around the interaction point:

$$\rho(z)=\rho_I-\frac{a_I}{2}\varepsilon^2-\frac{b_I}{3}\varepsilon^3+\mathcal{O}\left(\varepsilon^4\right)\ , \tag{B.28}$$

where $\varepsilon=z-z_I$. By using the expansion coefficients $a_I, b_I, \ldots$, we find

$$\begin{aligned} &\frac{1}{\rho'}\Big(\ln(z-Z_s)-\ln(z_I-Z_s)\Big) \\ &\qquad\sim\ -\frac{1}{a_I(z_I-Z_s)}+\left(\frac{b_I}{a_I^2(z_I-Z_s)}+\frac{1}{2a_I(z_I-Z_s)^2}\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\ , \\ &\{\rho,z\}\ \sim\ -\frac{3}{2\varepsilon^2}-\frac{b_I}{a_I\varepsilon}+\mathcal{O}(1)\ , \end{aligned} \tag{B.29}$$

and we can rewrite the second term in the right hand side of eq.(B.26) as

$$
\begin{aligned}
& -i\langle v_3^0(1,2,3)|A(\rho_I,\bar{\rho}_I)\Bigg[\oint_{\rho_I}\frac{d\rho}{2\pi i}\sum_{s=1}^{3}p_s^N\Big(\ln(z-Z_s)-\ln(z_I-Z_s)\Big)T(\rho) \\
& \qquad\qquad\qquad + \ \sim \ -\sum_{r=1}^{3}\frac{p_r^N}{\alpha_r}\Bigg] \\
= \ & -i\langle v_3^0(1,2,3)|A(z_I,\bar{z}_I)\Bigg[\oint_{z_I}\frac{dz}{2\pi i}\frac{1}{\rho'}\sum_{s=1}^{3}p_s^N\Big(\ln(z-Z_s)-\ln(z_I-Z_s)\Big)(T(z)-2\{\rho,z\}) \\
& \qquad\qquad\qquad + \ \sim \ -\sum_{r=1}^{3}\frac{p_r^N}{\alpha_r}\Bigg] \\
= \ & -i\langle v_3^0(1,2,3)|A(\rho_I,\bar{\rho}_I)\left(\frac{3}{2a_I}\sum_{s=1}^{3}\frac{p_s^N}{(z_I-Z_s)^2}+\frac{b_I}{a_I^2}\sum_{s=1}^{3}\frac{p_s^N}{z_I-Z_s}+ \ \sim \ -\sum_{r=1}^{3}\frac{p_r^N}{\alpha_r}\right) \\
& \qquad -i\langle v_3^0(1,2,3)|\left[\partial A(z_I)\left(-\frac{1}{a_I}\sum_{s=1}^{3}\frac{p_s^N}{z_I-Z_s}\right)+ \ \sim \ \right] .
\end{aligned} \tag{B.30}
$$

From the following formulae, one can show that this cancel the third term of the right hand side of eq.(B.26):

$$
\frac{\partial z_I}{\partial\alpha_s}=\frac{1}{a_I}\frac{1}{z_I-Z_s} , \qquad \frac{3}{2a_I}\sum_{s=1}^{3}\frac{p_s^N}{(z_I-Z_s)^2}+\frac{b_I}{a_I^2}\sum_{s=1}^{3}\frac{p_s^N}{z_I-Z_s}=\frac{1}{2}\sum_{r=1}^{3}\frac{p_r^N}{\alpha_r} , \tag{B.31}
$$

up to the conservation of p_r and α_r.

B.2 $\mathcal{O}(g^2)$

Next we show the $\mathcal{O}(g^2)$ relations (B.3) and (B.4). These relations can be shown from more general relation:

$$
\begin{aligned}
& \int d5d6\langle V_3^0(1,2,5);A|\langle V_3^0(3,4,6);B|R(5,6)\rangle \\
& \quad +\int d5d6\langle V_3^0(2,3,5);A|\langle V_3^0(1,4,6);B|R(5,6)\rangle \\
& \quad +\int d5d6\langle V_3^0(3,1,5);A|\langle V_3^0(2,4,6);B|R(5,6)\rangle-(-1)^{|A||B|}(A\leftrightarrow B)=0 ,
\end{aligned} \tag{B.32}
$$

for $A,B\in\{1,X^N\}$. Indeed we can obtain eq.(B.3) from the case $A=1,B=X^N$ and eq.(B.4) from the case $A=X^N,B=X^M$. Eq.(B.32) are called Jacobi identity and was shown in [28][29] (See also [14].). Here we will explain their calculations briefly. It is convenient to define the vertex $\langle\Delta(1,2;3,4);\theta|$ as

$$
\langle\Delta(1,2;3,4);\theta|\equiv\int d5d6\langle v_3^0(1,2,5)|\langle v_3^0(3,4,6)|e^{i(L_0^{(5)}-\tilde{L}_0^{(5)})\theta}|R(5,6)\rangle , \tag{B.33}
$$

where $e^{i(L_0^{(5)}-\tilde{L}_0^{(5)})\theta}$ is the σ-translation operator for the intermediate string. By using this notation, one can rewrite eq.(B.32) as

$$\int d\theta_P \langle \Delta(1,2;3,4);\theta_P| A(\rho_I^{125}) B(\rho_I^{346}) + \int d\theta_Q \langle \Delta(2,3;1,4);\theta_Q| A(\rho_I^{235}) B(\rho_I^{146}) \\ + \int d\theta_R \langle \Delta(3,1;2,4);\theta_R| A(\rho_I^{315}) B(\rho_I^{246}) - (-1)^{|A||B|}(A \leftrightarrow B) = 0 \quad \text{(B.34)}$$

for $A, B = \{1, X^N\}$. Here θ integrations come from the projection operators for intermediate strings and ρ_I^{ijk} denotes the ρ coordinate of the interaction point of three strings i, j, k.

The clue of the proof is the fact that the vertex $\langle \Delta(1,2;3,4);\theta|$ depends only on the form of the string diagram and independent from how it's constructed from the three-string vertex. This fact was shown in [14] from the explicit calculation. By using this fact, let us show eq.(B.34). There are some cases of sign combinations of α_r. It is sufficient to consider the following two cases: (i) $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_4 < 0$, (ii) $\alpha_1, \alpha_2 > 0$ and $\alpha_3, \alpha_4 < 0$. We explain how cancel the left hand side of eq.(B.34) diagrammatically.

Firstly we consider the case (i). In this case, the string diagrams corresponding to the left hand side of eq.(B.34) are depicted in Figs. B.2, B.3 and B.4. It is easy to see that the contributions from Figs. B.2-(b), B.3-(b) and B.4-(c) cancel the contributions from Figs. B.3-(c), B.4-(b) and B.2-(c), respectively.

Next, we see the case (ii). Now, the string diagrams corresponding to the left hand side of eq.(B.34) are described in Figs. B.5, B.6 and B.7. One can find that the contributions from Figs. B.5-(b), -(d) and -(c) cancel the contributions from Figs. B.6-(b), B.7-(b) and B.5-(d), respectively.

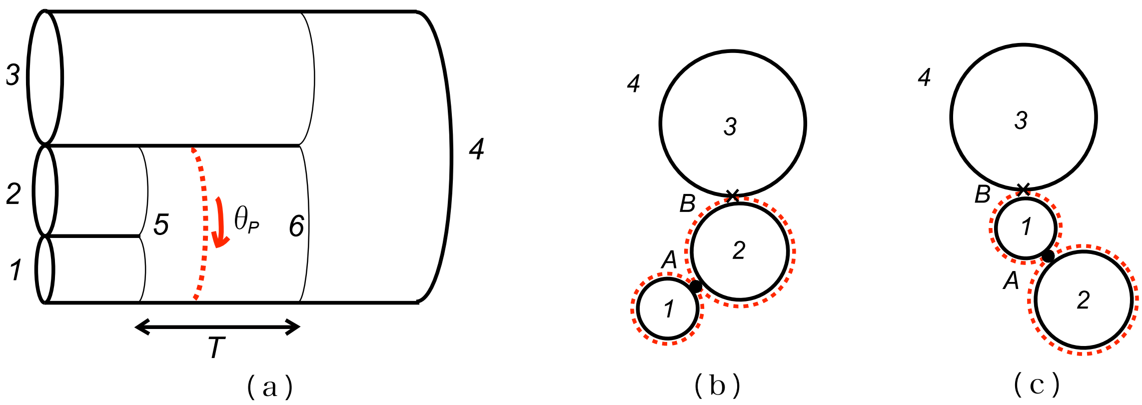

Figure B.2: The string diagrams corresponding to the first term in eq.(B.34) for the case (i). The limit $T \to 0$ should be taken.

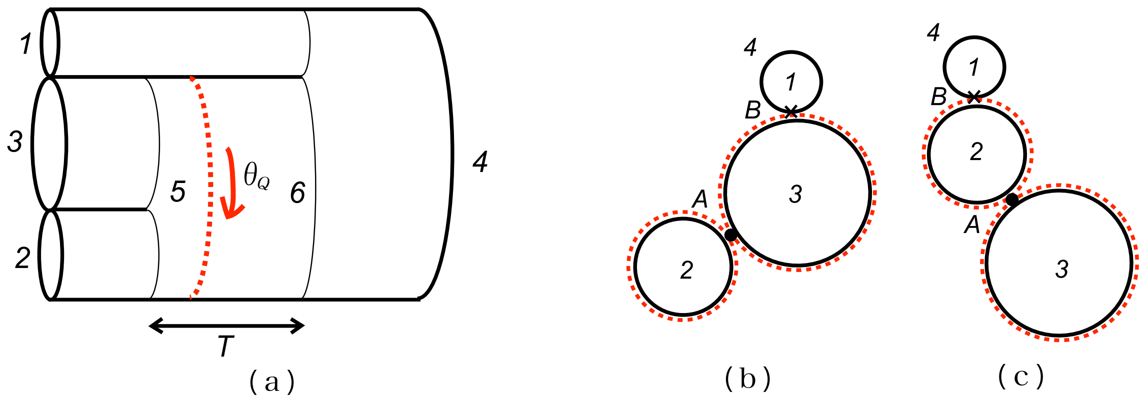

Figure B.3: The string diagrams corresponding to the second term in eq.(B.34) for the case (i). The limit $T \to 0$ should be taken.

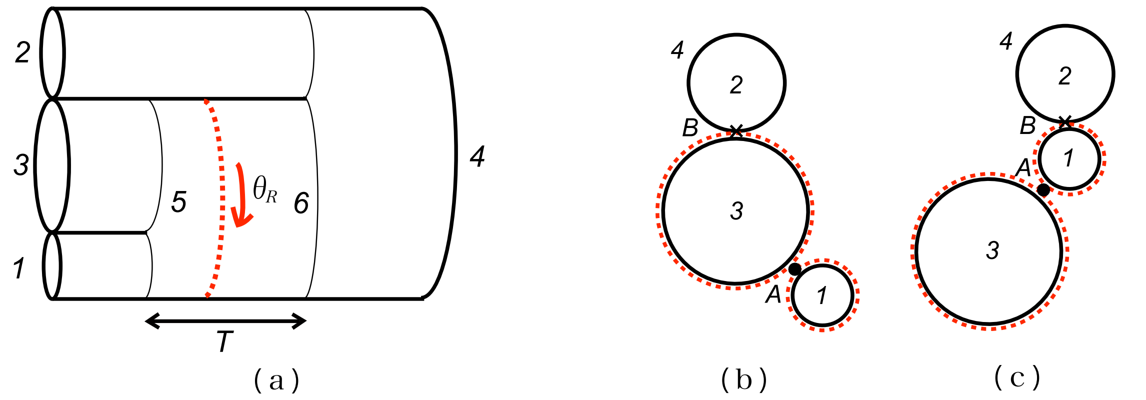

Figure B.4: The string diagrams corresponding to the third term in eq.(B.34) for the case (i). The limit $T \to 0$ should be taken.

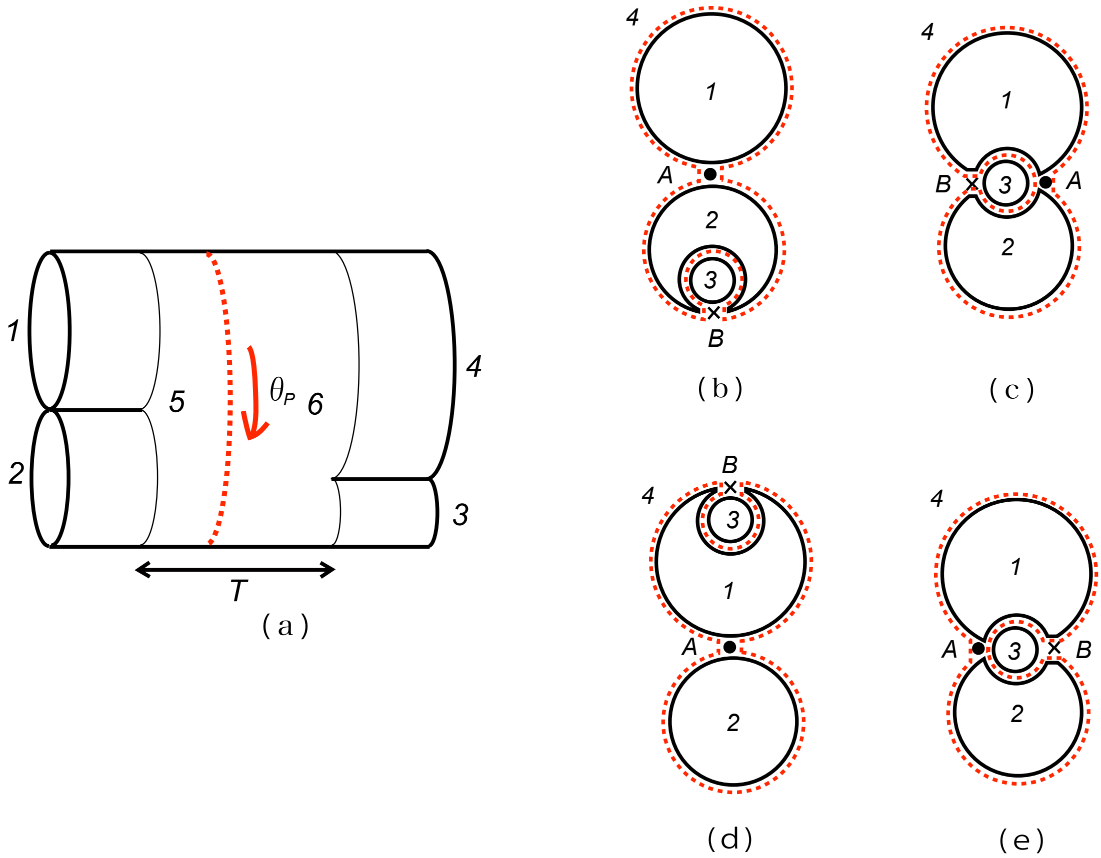

Figure B.5: The string diagrams corresponding to the first term in eq.(B.34) for the case (ii). The limit $T \to 0$ should be taken.

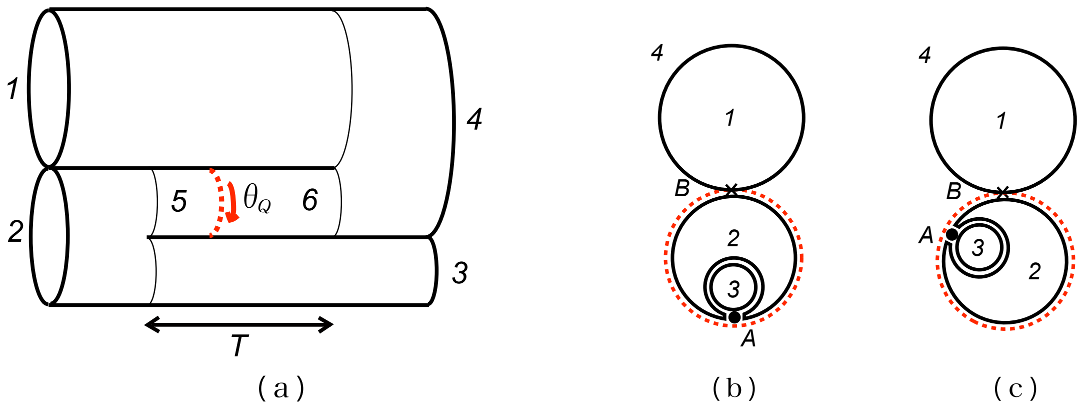

Figure B.6: The string diagrams corresponding to the second term in eq.(B.34) for the case (ii). The limit $T \to 0$ should be taken.

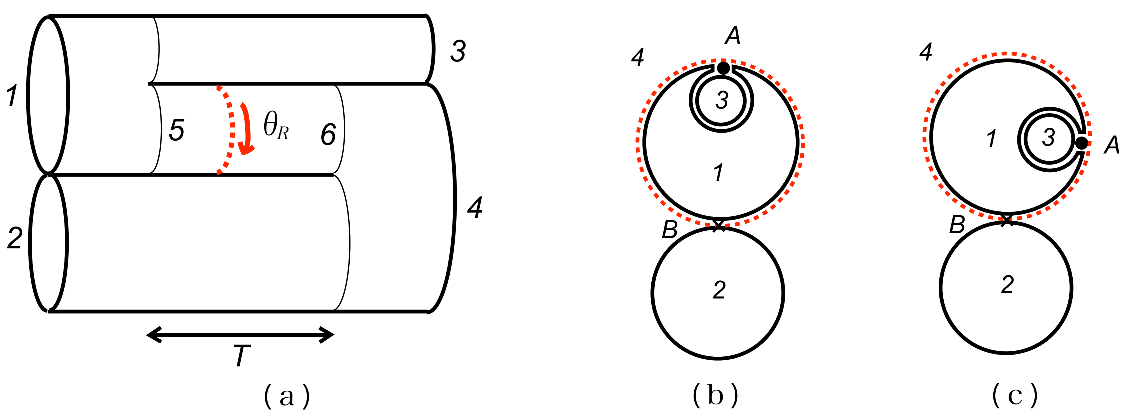

Figure B.7: The string diagrams corresponding to the third term in eq.(B.34) for the case (ii). The limit $T \to 0$ should be taken.

Appendix C

Neumann Coefficients at $T = \epsilon \ll 1$

In this appendix, we calculate the Neumann coefficients defined in eqs.(4.24), (4.37), (4.57) and (4.72) at $T = \epsilon \ll 1$. Although we consider the leading order in this thesis, we will partly calculate subleading contributions, since they have non-zero corrections to idempotency relations even in the limit $\epsilon \to 0$.

C.1 Neumann Coefficients Appeared in Section 4.2

$\bar{N}^{(2)ij}_{\ 00}$

Firstly, we consider the Neumann coefficients $\bar{N}^{(2)ij}_{\ 00}$ $(i, j \in \{1, 2, \tilde{1}, \tilde{2}\})$. From the definitions, we can obtain

$$
\begin{aligned}
\mathrm{Re}\left[\bar{N}^{(2)11}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)\tilde{1}\tilde{1}}_{\ 00}\right] = \ln(2y) + \frac{\epsilon}{\alpha_1} + 2\frac{\alpha_2}{\alpha_1}y + \mathcal{O}(y^2) \\
&= 2\ln\epsilon - \ln(8\alpha_1\alpha_2) + \frac{\epsilon}{\alpha_1} + \frac{1}{8\alpha_1^2}\epsilon^2 - \frac{1}{48}\left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\ , \\
\mathrm{Re}\left[\bar{N}^{(2)22}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)\tilde{2}\tilde{2}}_{\ 00}\right] = \ln 2 + \frac{\epsilon}{\alpha_2} + 2\frac{\alpha_1}{\alpha_2}y + \mathcal{O}(y^2) \\
&= \ln 2 + \frac{\epsilon}{\alpha_2} + \frac{1}{8\alpha_2^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\ , \\
\mathrm{Re}\left[\bar{N}^{(2)12}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)21}_{\ 00}\right] = \mathrm{Re}\left[\bar{N}^{(2)\tilde{1}\tilde{2}}_{\ 00}\right] = \mathrm{Re}\left[\bar{N}^{(2)\tilde{2}\tilde{1}}_{\ 00}\right] = -y + \mathcal{O}(y^2) \\
&= -\frac{1}{16\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\ , \\
\mathrm{Re}\left[\bar{N}^{(2)1\tilde{2}}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)\tilde{1}2}_{\ 00}\right] = \mathrm{Re}\left[\bar{N}^{(2)2\tilde{1}}_{\ 00}\right] = \mathrm{Re}\left[\bar{N}^{(2)\tilde{2}1}_{\ 00}\right] = y + \mathcal{O}(y^2) \\
&= \frac{1}{16\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\ , \\
\mathrm{Re}\left[\bar{N}^{(2)1\tilde{1}}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)\tilde{1}1}_{\ 00}\right] = \ln(2y) \\
&= 2\ln\epsilon - \ln(8\alpha_1\alpha_2) - \frac{1}{48}\left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\ , \\
\mathrm{Re}\left[\bar{N}^{(2)2\tilde{2}}_{\ 00}\right] &= \mathrm{Re}\left[\bar{N}^{(2)\tilde{2}2}_{\ 00}\right] = \ln 2\ .
\end{aligned}
\tag{C.1}
$$

$\bar{N}^{(2)ij}_{n0}$

Secondly, we calculate the Neumann coefficients $\bar{N}^{(2)ij}_{n0}$ for $n > 0$. By using eq.(4.25) in the definitions of these coefficients, we get

$$
\begin{aligned}
\bar{N}^{(2)11}_{n0} &= \left(\bar{N}^{(2)\tilde{1}\tilde{1}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n\frac{1}{n} + (-1)^n(4n+2)\frac{\alpha_2}{\alpha_1}y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n\frac{1}{n} + (-1)^n\frac{2n+1}{8\alpha_1^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)1\tilde{1}}_{n0} &= \left(\bar{N}^{(2)\tilde{1}1}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n\frac{1}{n} + (-1)^n(4n-2)\frac{\alpha_2}{\alpha_1}y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n\frac{1}{n} + (-1)^n\frac{2n-1}{8\alpha_1^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)22}_{n0} &= \left(\bar{N}^{(2)\tilde{2}\tilde{2}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n} + (-1)^n 2\frac{\alpha_1}{\alpha_2}y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n} + (-1)^n\frac{1}{8\alpha_2^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)2\tilde{2}}_{n0} &= \left(\bar{N}^{(2)\tilde{2}2}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n} - (-1)^n 2\frac{\alpha_1}{\alpha_2}y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n} - (-1)^n\frac{1}{8\alpha_2^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)12}_{n0} &= \left(\bar{N}^{(2)\tilde{1}\tilde{2}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_1}}\left(-(-1)^n 2y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_1}}\left(-(-1)^n\frac{1}{8\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)\tilde{1}2}_{n0} &= \left(\bar{N}^{(2)1\tilde{2}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n 2y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_1}}\left((-1)^n\frac{1}{8\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)21}_{n0} &= \left(\bar{N}^{(2)\tilde{2}\tilde{1}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n}(1-(-1)^n) - 2(-1)^n\left(2n\frac{\alpha_1}{\alpha_2}+1\right)y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n}(1-(-1)^n) - (-1)^n\left(2n\frac{\alpha_1}{\alpha_2}+1\right)\frac{1}{8\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
\bar{N}^{(2)\tilde{2}1}_{n0} &= \left(\bar{N}^{(2)2\tilde{1}}_{n0}\right)^* = e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n}(1-(-1)^n) - 2(-1)^n\left(2n\frac{\alpha_1}{\alpha_2}-1\right)y + \mathcal{O}(y^2)\right) \\
&= e^{n\frac{\epsilon}{\alpha_2}}\left(\frac{1}{n}(1-(-1)^n) - (-1)^n\left(2n\frac{\alpha_1}{\alpha_2}-1\right)\frac{1}{8\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) . \qquad \text{(C.2)}
\end{aligned}
$$

$\bar{N}^{(2)ij}_{nm}$

By using the definition of the conformal mappings (4.14) and (4.16), we can easily obtain

$$
\sum_k \alpha_k \frac{1}{z_i - Z_k}\frac{1}{z'_j - Z_k} = \frac{-1}{z_i - z'_j}\sum_k \alpha_k\left(\frac{1}{z_i - Z_k} - \frac{1}{z'_j - Z_k}\right)
$$

$$
\begin{aligned}
&= \frac{-1}{z_i - z'_j}\left(\frac{\partial}{\partial z_i}\rho(z_i) - \frac{\partial}{\partial z'_j}\rho(z'_j)\right) \\
&= \frac{-1}{z_i - z'_j}\left(\alpha_i \frac{1}{w_i(z_i)}\frac{\partial}{\partial z_i}w_i(z_i) - \alpha_j \frac{1}{w'_j(z'_j)}\frac{\partial}{\partial z'_j}w'_j(z'_j)\right) \qquad (C.3)
\end{aligned}
$$

for $i, j \in (1, 2, \tilde{1}, \tilde{2})$. Here the summation runs over $(1, 2, \tilde{1}, \tilde{2})$ and $\alpha_{\tilde{r}} = -\alpha_r, Z_{\tilde{r}} = \bar{Z}_r, z_{\tilde{r}} = \bar{z}_r, w_{\tilde{r}} = \bar{w}_r$ for $r = 1, 2$. From this relation and the definition of the Neumann coefficients, we have the following formula:

$$
\bar{N}^{(2)ij}_{nm} = \left(\frac{\alpha_i}{n} + \frac{\alpha_j}{m}\right)^{-1} \sum_k \alpha_k \bar{N}^{(2)ik}_{n0} \bar{N}^{(2)jk}_{m0} \; . \qquad (C.4)
$$

For the Neumann coefficients $\bar{N}^{(2)ij}_{nm}$ for $n, m > 0$ with $\frac{\alpha_i}{n} + \frac{\alpha_j}{m} \neq 0$, we can use the formula (C.4) and we find

$$
\begin{aligned}
\bar{N}^{(2)11}_{nm} &= \left(\bar{N}^{(2)\tilde{1}\tilde{1}}_{nm}\right)^* = e^{\frac{n+m}{\alpha_1}\epsilon}\left((-1)^{n+m} 4\frac{\alpha_2}{\alpha_1}y + \mathcal{O}(y^2)\right) \\
&= e^{\frac{n+m}{\alpha_1}\epsilon}\left((-1)^{n+m}\frac{1}{4\alpha_1^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; , \\
\bar{N}^{(2)22}_{nm} &= \left(\bar{N}^{(2)\tilde{2}\tilde{2}}_{nm}\right)^* = e^{\frac{n+m}{\alpha_2}\epsilon}\left((-1)^{n+m} 4\frac{\alpha_1}{\alpha_2}y + \mathcal{O}(y^2)\right) \\
&= e^{\frac{n+m}{\alpha_2}\epsilon}\left((-1)^{n+m}\frac{1}{4\alpha_2^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; , \\
\bar{N}^{(2)12}_{nm} &= \left(\bar{N}^{(2)\tilde{1}\tilde{2}}_{n0}\right)^* = e^{\left(\frac{n}{\alpha_1}+\frac{m}{\alpha_2}\right)\epsilon}\left(-(-1)^{n+m} 4y + \mathcal{O}(y^2)\right) \\
&= e^{\left(\frac{n}{\alpha_1}+\frac{m}{\alpha_2}\right)\epsilon}\left(-(-1)^{n+m}\frac{1}{4\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; , \\
\bar{N}^{(2)1\tilde{2}}_{nm} &= \left(\bar{N}^{(2)\tilde{1}2}_{n0}\right)^* = e^{\left(\frac{n}{\alpha_1}+\frac{m}{\alpha_2}\right)\epsilon}\left((-1)^{n+m} 4y + \mathcal{O}(y^2)\right) \\
&= e^{\left(\frac{n}{\alpha_1}+\frac{m}{\alpha_2}\right)\epsilon}\left((-1)^{n+m}\frac{1}{4\alpha_1\alpha_2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; . \qquad (C.5)
\end{aligned}
$$

For the Neumann coefficients $\bar{N}^{(2)r\tilde{r}}_{nn}$ for $n > 0$, we cannot use the formula (C.4) and need to calculate from the definitions (4.24). We can obtain

$$
\begin{aligned}
\bar{N}^{(2)1\tilde{1}}_{nm} &= -\frac{1}{n}e^{\frac{2n}{\alpha_1}\epsilon}\delta_{nm} + e^{\frac{n+m}{\alpha_1}\epsilon}\left(-(-1)^{n+m} 4\frac{\alpha_2}{\alpha_1}y + \mathcal{O}(y^2)\right) \\
&= e^{\frac{n+m}{\alpha_1}\epsilon}\left(-\frac{1}{n}\delta_{nm} - (-1)^{n+m}\frac{1}{4\alpha_1^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; , \\
\bar{N}^{(2)2\tilde{2}}_{nm} &= -\frac{1}{n}e^{\frac{2n}{\alpha_2}\epsilon}\delta_{nm} + e^{\frac{n+m}{\alpha_2}\epsilon}\left(-(-1)^{n+m} 4\frac{\alpha_1}{\alpha_2}y + \mathcal{O}(y^2)\right) \\
&= e^{\frac{n+m}{\alpha_2}\epsilon}\left(-\frac{1}{n}\delta_{nm} - (-1)^{n+m}\frac{1}{4\alpha_2^2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) \; . \qquad (C.6)
\end{aligned}
$$

${M_{\mathrm{UHP}}}_n^{\ i}$

Finally, we consider the coefficients ${M_{\mathrm{UHP}}}_n^{\ i}$ $\left(i \in \{1, 2, \tilde{1}, \tilde{2}\}\right)$ defined in eq.(4.37). In the limit $\epsilon \to 0$ $(y \to 0)$, the interaction point z_I behaves as

$$z_I = i y^{\frac{1}{2}} \sqrt{\frac{\alpha_1}{\alpha_2}} \left(1 + \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{\alpha_2}\right) y + \mathcal{O}(y^2)\right) . \tag{C.7}$$

From the definitions and this relation, we can obtain

$$\begin{aligned}
{M_{\mathrm{UHP}}}_0^{\ 1} + {M_{\mathrm{UHP}}}_0^{\ \tilde{1}} &= -4\sqrt{y}\sqrt{\frac{\alpha_2}{\alpha_1}} \left(1 + \frac{1}{2}\frac{\alpha_1}{\alpha_2} y - \frac{1}{6}\frac{\alpha_2}{\alpha_1} y + \mathcal{O}(y^2)\right) \\
&= \frac{\epsilon}{\alpha_1}\left(1 - \frac{1}{48}\left(\frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
{M_{\mathrm{UHP}}}_0^{\ 2} + {M_{\mathrm{UHP}}}_0^{\ \tilde{2}} &= -4\sqrt{y}\sqrt{\frac{\alpha_1}{\alpha_2}} \left(1 + \frac{1}{2}\frac{\alpha_2}{\alpha_1} y - \frac{1}{6}\frac{\alpha_1}{\alpha_2} y + \mathcal{O}(y^2)\right) \\
&= \frac{\epsilon}{\alpha_2}\left(1 - \frac{1}{48}\left(\frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) ,
\end{aligned}$$

$$\begin{aligned}
{M_{\mathrm{UHP}}}_n^{\ 1} &= ({M_{\mathrm{UHP}}}_n^{\ \tilde{1}})^* = -(-1)^n 4\sqrt{y}\sqrt{\frac{\alpha_2}{\alpha_1}} e^{\frac{n}{\alpha_1}\epsilon} \left(1 + \frac{8}{3} n^2 \frac{\alpha_2}{\alpha_1} y + \frac{1}{2}\frac{\alpha_1}{\alpha_2} y - \frac{1}{6}\frac{\alpha_2}{\alpha_1} y + \mathcal{O}(y^2)\right) \\
&= (-1)^n \frac{\epsilon}{\alpha_1} e^{\frac{n}{\alpha_1}\epsilon} \left(1 + \frac{1}{6} n^2 \frac{1}{\alpha_1^2}\epsilon^2 - \frac{1}{48}\left(\frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) , \\
{M_{\mathrm{UHP}}}_n^{\ 2} &= ({M_{\mathrm{UHP}}}_n^{\ \tilde{2}})^* = -(-1)^n 4\sqrt{y}\sqrt{\frac{\alpha_1}{\alpha_2}} e^{\frac{n}{\alpha_2}\epsilon} \left(1 + \frac{8}{3} n^2 \frac{\alpha_1}{\alpha_2} y + \frac{1}{2}\frac{\alpha_2}{\alpha_1} y - \frac{1}{6}\frac{\alpha_1}{\alpha_2} y + \mathcal{O}(y^2)\right) \\
&= (-1)^n \frac{\epsilon}{\alpha_2} e^{\frac{n}{\alpha_2}\epsilon} \left(1 + \frac{1}{6} n^2 \frac{1}{\alpha_2^2}\epsilon^2 - \frac{1}{48}\left(\frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\right) .
\end{aligned} \tag{C.8}$$

C.2 Neumann Coefficients Appeared in Section 4.3

$\bar{N}^{ij}_{nm}$, $\bar{F}^{ij}_{nm}$ $\quad (i, j \in \{h, a\})$

To evaluate the Neumann coefficients (4.57) in $T = \epsilon \to 0$ limit, it is useful to change the integration variables from $\nu, \bar{\nu}$ to $\mathcal{Z} \equiv e^{2\pi i(\nu - V_3)}, \bar{\mathcal{Z}} \equiv e^{-2\pi i(\bar{\nu} - V_3)}$. By using these variables, we find following relations:

$$\begin{aligned}
w_3(\nu) &= e^{-\frac{\epsilon}{\alpha_3}} \frac{\mathcal{Z} - 1}{\mathcal{Z} e^{2\pi i V_3} - e^{-2\pi i V_3}} \left[1 - \left(\mathcal{Z}(1 - e^{4\pi i V_3}) + \mathcal{Z}^{-1}(1 - e^{-4\pi i V_3})\right) q + \mathcal{O}(q^2)\right] , \\
g_1(\nu - V_3|\tau) &= i\pi \left(\frac{\mathcal{Z} + 1}{\mathcal{Z} - 1} - 2\left(\mathcal{Z} - \mathcal{Z}^{-1}\right) q + \mathcal{O}(q^2)\right) , \\
g_1(\nu + V_3|\tau) &= i\pi \left(\frac{\mathcal{Z} + e^{-4\pi i V_3}}{\mathcal{Z} - e^{-4\pi i V_3}} - 2\left(\mathcal{Z} e^{4\pi i V_3} - \mathcal{Z}^{-1} e^{-4\pi i V_3}\right) q + \mathcal{O}(q^2)\right) , \\
\partial_\nu g_1(\nu - \nu'|\tau) &= 4\pi^2 \mathcal{Z}\mathcal{Z}' \left(\frac{1}{(\mathcal{Z} - \mathcal{Z}')^2} + \left(\frac{1}{\mathcal{Z}'^2} + \frac{1}{\mathcal{Z}^2}\right) q + \mathcal{O}(q^2)\right) , \\
\partial_\nu g_1(\nu + \bar{\nu}'|\tau) &= 4\pi^2 \mathcal{Z}\bar{\mathcal{Z}}' \left(\frac{e^{-4\pi i V_3}}{(\mathcal{Z} - \bar{\mathcal{Z}}' e^{-4\pi i V_3})^2} + \left(\frac{1}{\bar{\mathcal{Z}}'^2} e^{4\pi i V_3} + \frac{1}{\mathcal{Z}^2} e^{-4\pi i V_3}\right) q + \mathcal{O}(q^2)\right) .
\end{aligned} \tag{C.9}$$

From these relations and the definition of the Neumann coefficients (4.57), we get

$$
\begin{aligned}
\bar{N}^{hh}_{nm} = \left(\bar{N}^{aa}_{nm}\right)^* &= -16 e^{\frac{n+m}{\alpha_3}\epsilon} q \sin^2(2\pi V_3)\sin(2\pi n V_3)\sin(2\pi m V_3) + \mathcal{O}(q^2) \\
&= -\frac{\epsilon^2}{\alpha_3^2} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3) + \mathcal{O}(\epsilon^4) \ , \\
\bar{N}^{ha}_{nm} = \left(\bar{N}^{ah}_{nm}\right)^* &= -\frac{1}{n} e^{\frac{2n}{\alpha_3}\epsilon} \delta_{n,m} \\
&\quad -16 e^{\frac{n+m}{\alpha_3}\epsilon} q \sin^2(2\pi V_3)\sin(2\pi n V_3)\sin(2\pi m V_3) + \mathcal{O}(q^2) \\
&= -\frac{1}{n} e^{\frac{2n}{\alpha_3}\epsilon} \delta_{n,m} - \frac{\epsilon^2}{\alpha_3^2} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3) + \mathcal{O}(\epsilon^4) \ , \\
\bar{N}^{hh}_{n0} = \left(\bar{N}^{aa}_{n0}\right)^* &= \frac{1}{n} e^{\frac{n}{\alpha_3}\epsilon} \cos(2\pi n V_3) + 4nq e^{\frac{n}{\alpha_3}\epsilon} \sin^2(2\pi V_3)\cos(2\pi n V_3) \\
&\qquad + 8q e^{\frac{n}{\alpha_3}\epsilon} \sin(2\pi V_3)\cos(2\pi V_3)\sin(2\pi n V_3) + \mathcal{O}(q^2) \\
&= \frac{1}{n} e^{\frac{n}{\alpha_3}\epsilon} \cos(2\pi n V_3) \\
&\qquad \times \left(1 + \frac{n^2\epsilon^2}{4\alpha_3^2} + \frac{n\epsilon^2}{2\alpha_3^2}\cot(2\pi V_3)\tan(2\pi n V_3) + \mathcal{O}(\epsilon^4)\right) \ , \\
\bar{N}^{ha}_{n0} = \left(\bar{N}^{ah}_{n0}\right)^* &= \frac{1}{n} e^{\frac{n}{\alpha_3}\epsilon} \cos(2\pi n V_3) + 4nq e^{\frac{n}{\alpha_3}\epsilon} \sin^2(2\pi V_3)\cos(2\pi n V_3) \\
&\qquad + 8q e^{\frac{n}{\alpha_3}\epsilon} \sin(2\pi V_3)\cos(2\pi V_3)\sin(2\pi n V_3) + \mathcal{O}(q^2) \\
&= \frac{1}{n} e^{\frac{n}{\alpha_3}\epsilon} \cos(2\pi n V_3) \\
&\qquad \times \left(1 + \frac{n^2\epsilon^2}{4\alpha_3^2} + \frac{n\epsilon^2}{2\alpha_3^2}\cot(2\pi V_3)\tan(2\pi n V_3) + \mathcal{O}(\epsilon^4)\right) \ , \\
\bar{F}^{hh}_{nm} = \left(\bar{F}^{aa}_{nm}\right)^* &= -\frac{1}{nm}\frac{4}{(-\ln q)} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3) \\
&\qquad \times \left(1 + 4(n^2+m^2) q \sin^2(2\pi V_3) + \mathcal{O}(q^2)\right) \\
&= -\frac{1}{nm}\frac{2}{(-\ln \epsilon)} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3)\left(1 + \mathcal{O}\left(\frac{1}{-\ln\epsilon}\right)\right) \ , \\
\bar{F}^{ha}_{nm} = \left(\bar{F}^{ah}_{nm}\right)^* &= \frac{1}{nm}\frac{4}{(-\ln q)} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3) \\
&\qquad \times \left(1 + 4(n^2+m^2) q \sin^2(2\pi V_3) + \mathcal{O}(q^2)\right) \\
&= \frac{1}{nm}\frac{2}{(-\ln \epsilon)} e^{\frac{n+m}{\alpha_3}\epsilon} \sin(2\pi n V_3)\sin(2\pi m V_3)\left(1 + \mathcal{O}\left(\frac{1}{-\ln\epsilon}\right)\right) . \qquad (C.10)
\end{aligned}
$$

${M_{\mathrm{rectan.}n}}^{i} \quad (i,j \in \{h,a\})$

By using the variable $\mathcal{Z}$, we can rewrite the integrand of eq.(4.72) as

$$
\begin{aligned}
g_1(\nu_I^- - \nu|\tau) + g_1(\bar{\nu}_I^- + \nu|\tau) &= g_4(-y-\nu|\tau) + g_4(-y+\nu|\tau) \\
&= -4\pi q^{\frac{1}{2}}\left(\mathcal{Z}e^{2\pi i V_3} + \mathcal{Z}^{-1}e^{-2\pi i V_3}\right) + \mathcal{O}(q^{\frac{3}{2}}) \ . \qquad (C.11)
\end{aligned}
$$

Combining this relation and the definition (4.72), we can obtain

$$
\begin{aligned}
{M_{\mathrm{rectan}.n}}^{h} = ({M_{\mathrm{rectan}.n}}^{a})^{*} &= 8e^{\frac{n}{\alpha_3}\epsilon}\sin(2\pi V_3)\cos(2\pi n V_3)q^{\frac{1}{2}}\Big(1+\mathcal{O}(q)\Big) \\
&= \frac{2\epsilon}{\alpha_3}e^{\frac{n}{\alpha_3}\epsilon}\cos(2\pi n V_3)\Big(1+\mathcal{O}(\epsilon^2)\Big)\ , \\
{M_{\mathrm{rectan}.0}}^{h} + {M_{\mathrm{rectan}.0}}^{a} &= \frac{2\epsilon}{\alpha_3}\ .
\end{aligned}
\tag{C.12}
$$

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