Localization in categories of modules. IV

By

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Let A be a ring with an identity and ${}_A\mathfrak{M}$ the category of all unitary left A-modules.

A full subcategory $\mathfrak{B}$ of ${}_A\mathfrak{M}$ is called a reflective subcategory if there is a covariant additive functor $S: {}_A\mathfrak{M} \to \mathfrak{B}$ which is a left adjoint of the inclusion functor $T: \mathfrak{B} \to {}_A\mathfrak{M}$; in this case S is called a reflector from ${}_A\mathfrak{M}$ to $\mathfrak{B}$. For a left A-module V let us denote by $\mathfrak{D}(V)$ the full subcategory of ${}_A\mathfrak{M}$ consisting of all left A-modules of V-dominant dimension $\geqq 2$.

In a previous paper [6] we have proved that a full subcategory $\mathfrak{B}$ of ${}_A\mathfrak{M}$ which is closed under isomorphic images is a reflective subcategory which is itself a Grothendieck category iff (=if and only if) there is a left A-module V of type FI with $\mathfrak{B}=\mathfrak{D}(V)$.

Let V be a left A-module of type FI and S a reflector from ${}_A\mathfrak{M}$ to $\mathfrak{D}(V)$. Then the module $S({}_AA)$ carries a ring structure so that the natural A-homomorphism ${}_AA \to S({}_AA)$ turns out to be a ring homomorphism, and in this way we have all quotient rings by letting V range over all injective modules.

In view of these results, in the present paper we shall give a new construction of a reflector S from ${}_A\mathfrak{M}$ to $\mathfrak{D}(V)$ for the case of V being either of type FI or injective.

Let V be a left A-module and let $B=\mathrm{End}({}_AV)$; we consider B as a right operator domain of V. Thus V is an A-B-bimodule. Let us put

(1) $$D(X)={}_A[\mathrm{Hom}_B(\mathrm{Hom}_A(X, {}_AV_B), {}_AV_B)] \qquad \text{for} \quad X\in {}_A\mathfrak{M};$$

the map $\pi(X): X \to D(X)$ defined by

(2) $$[\pi(X)(x)](f)=f(x) \qquad \text{for} \quad x\in X, f\in \mathrm{Hom}_A(X, V)$$

is a natural homomorphism. Let us put further

(3) $$\tilde{D}(X)=\cap\{\mathrm{Ker}\, f \mid f\in \mathrm{Hom}_A(\mathrm{D}(X), V) \quad \text{with} \quad f\pi(X)=0\}.$$

If we denote by $\xi(X)$ the inclusion map from $\tilde{D}(X)$ to $D(X)$, then $\pi(X)$ is factored as

(4) $$\pi(X)=\xi(X)\cdot\tilde{\pi}(X) \quad \text{with} \quad \tilde{\pi}(X): X \to \tilde{D}(X).$$

Now, let us assume that V is either of type FI or injective, and let $X\in {}_A\mathfrak{M}$.

Then it will be established in § 2 that $\tilde{D}$ is a reflector S from ${}_A\mathfrak{M}$ to $\mathfrak{D}(V)$.

As for the problem: "Under what condition does $\tilde{D}(X)$ coincide with $D(X)$?" we can give a number of necessary and sufficient conditions in § 3; the condition that $D^i(X)$ be V-reflexive (that is, $\pi(D^i(X))$ be an isomorphism) for some $i\geqq 1$ (or all $i\geqq 1$) is one of these conditions. However, a sufficient condition for $\tilde{D}(X)=D(X)$, which is sometimes more useful, is that $D(X)$ be isomorphic to an A-submodule of V^n for some n. where V^n is a direct sum of n copies of V. In particular, it will be shown that in case $X\subset V^p, X$ is V-reflexive iff $X\in\mathfrak{D}(V)$.

In § 4 we shall discuss conditions for the V-reflexivity of a module X without imposing any such restriction on V that is described in §§ 2 and 3. It will be shown that these conditions lead naturally to the conditions for the double centralizer prperty of a module V.

Finally, in § 5 we shall give a characterization of $\mathfrak{D}(V)$ with V injective among full reflective Grothendieck subcategories of ${}_A\mathfrak{M}$.

Our results above were obtained at the end of 1971 and reported at the Symposium on Ring Theory, Matsumoto, Japan, August 28-31, 1972. Our results for the case of V being injective and those in Lambek [9] overlap, although the formulation and the methods of proof are different. (As for the features of the situation for the case of V being a module of type FI, one should refer to Remark preceding Example 2.8) Almost the same result as Theorem 4.1 was established also by Onodera [10].

§ 1. Generalities on D

Throughout this paper V will denote a left A-module, and $B=\mathrm{End}\,({}_AV)$. With respect to V, we shall use the same notations as in the introduction.

Let $f: X\to Y$ be an A-homomorphism. Let us put

$$(5)\qquad D(f)=\mathrm{Hom}\,(\mathrm{Hom}\,(f, 1_V).1_V):\; D(X)\to D(Y).$$

We shall first prove

$$[D(f)\circ\xi(X)](y)\in\tilde{D}(Y)\qquad \text{for}\quad y\in\tilde{D}(X).$$

Let $h\in\mathrm{Hom}_A(D(Y), V)$ such that $h\circ\pi(Y)=0$. Then we have

$$h\circ D(f)\circ\pi(X)=h\circ\pi(Y)\circ f=0,$$

and hence $h\circ D(f)\circ\xi(X)=0$. This shows that $[D(f)\circ\xi(X)](y)\in\tilde{D}(Y)$. Let us denote by $\tilde{D}(f)$ the map $D(f)\circ\xi(X)$ with $\tilde{D}(Y)$ as its range. Then

$$(6)\qquad D(f)\circ\xi(X)=\xi(Y)\circ\tilde{D}(f).$$

Thus, we have

LEMMA 1.1. *$\tilde{D}$ is a covariant additive functor and $\tilde{\pi}(X): X\to\tilde{D}(X)$ is a natural homomorphism.*

PROOF. Since $\xi(Y)\circ\tilde{D}(f)\circ\tilde{\pi}(X)=D(f)\circ\pi(X)=\pi(Y)\circ f=\xi(Y)\circ\tilde{\pi}(Y)\circ f$ and $\xi(Y)$ is a monomorphism, we have $\tilde{D}(f)\tilde{\pi}(X)=\tilde{\pi}(Y)f$.

PROPOSITION 1.2. *If* $Y\in\mathfrak{M}_B$, *then* ${}_A[\mathrm{Hom}_B[Y_B, {}_AV_B)]\in\mathfrak{D}(V)$.

PROOF. Let

$$0\leftarrow Y\leftarrow Y_1\leftarrow Y_2\leftarrow\cdots$$

be a free resolution of Y_B. Then the sequence

$$0\rightarrow \mathrm{Hom}_B(Y, {}_AV_B)\rightarrow \mathrm{Hom}_B(Y_1, {}_AV_B)\rightarrow \mathrm{Hom}_B(Y_2, {}_AV_B)$$

is exact and $\mathrm{Hom}_B(Y_i, {}_AV_B)$ are direct products of copies of ${}_AV$.

COROLLARY 1.3. $D(X)\in\mathfrak{D}(V)$ *for* $X\in{}_A\mathfrak{M}$.

On the other hand, by the definition of $\tilde{D}(X)$ we have

PROPOSITION 1.4. *V-dom. dim* $D(X)/\tilde{D}(X)\geqq 1$.

Now for any positive integer n let us put

$$V^n=\sum_{i=1}^{n}\oplus V.$$

Then we have easily

PROPOSITION 1.5. $\pi(V^n): V^n\cong D(V^n)$ *and* $\tilde{D}(V^n)=D(V^n)$. *If* X *is a direct product of copies of* V, *then* $\pi(X)$ *is a monomorphism.*

Next, for $f\in\mathrm{Hom}_A(X, V^n)$ let us put

$$\lambda_n(X)(f)=\pi(V^n)^{-1}\circ D(f):\quad D(X)\rightarrow V^n,$$
$$\tilde{\lambda}_n(X)(f)=\pi(V^n)^{-1}\circ\xi(V^n)\circ\tilde{D}(f):\quad \tilde{D}(X)\rightarrow V^n.$$

Then we have

$$f=\lambda_n(X)(f)\circ\pi(X)=\tilde{\lambda}_n(X)(f)\circ\tilde{\pi}(X),$$

$$\tilde{\lambda}_n(X)(f)=\lambda_n(X)(f)\circ\xi(X).$$

PROPOSITION 1.6. $\lambda_n(X):\mathrm{Hom}(X, V^n)\rightarrow\mathrm{Hom}(D(X), V^n)$ *is a monomorphism.*

PROPOSITION 1.7. $\lambda_1(X)(h\circ f)=h\circ\lambda_n(X)(f)$ for

$$f\in\mathrm{Hon}_A(X, V^n)\quad\text{and}\quad h\in\mathrm{Hom}_A(V^n, V).$$

PROOF.

$$\begin{aligned}\lambda_1(X)(h\circ f)&=\pi(V)^{-1}\circ D(h)\circ D(f)\\&=\pi(V)^{-1}\circ D(h)\circ\pi(V^n)\circ\pi(V^n)^{-1}\circ D(f)\\&=h\circ\lambda_n(X)(f).\end{aligned}$$

In what follows, instead of $\lambda_1(X)$ we write also $\lambda(X)$.

PROPOSITION 1.8. *For* $f\in\mathrm{Hom}_A(X, V^n)$, *let us put*

$$f(x)=(f_1(x), \cdots, f_n(x)), \quad x\in X.$$

Then for $\alpha\in D(X)$ *we have*

$$[\lambda_n(X)(f)](\alpha)=(\alpha(f_1), \cdots, \alpha(f_n)).$$

PROOF. For $f\in\mathrm{Hom}_A(X, V)$ it is easy to see that $[\lambda(X)(f)](\alpha)=\alpha(f)$, and the general case is obtained from this.

§2. The functor $\tilde{D}$ as a reflector

Let $B=\mathrm{End}({}_AV)$, $C=\mathrm{End}(V_B)$; $\phi: A\to C$ is a canonical ring homomorphism. Henceforth let us make the following assumption.

ASSUMPTION (*): There exists a subring R of C such that
(a) $\phi(A)\subset R\subset C$,
(b) ${}_RV_B\cong\mathrm{Hom}_A({}_AR_R, {}_AV_B)$, and
(c) ${}_RV$ is injective.

This assumption is satisfied if V is either of type FI or injective; we have only to put $R=C$ or $R=\phi(A)$.

Let us put

$$K(X)=\mathrm{Hom}_A({}_AR_R, X), \qquad X\in\mathfrak{D}({}_AV),$$

$$L(Y)={}_AR_R\otimes Y, \qquad Y\in\mathfrak{D}({}_RV).$$

Then by [3, Lemma 3.1] K carries $\mathfrak{D}({}_AV)$ into $\mathfrak{D}({}_RV)$, L carries $\mathfrak{D}({}_RV)$ into $\mathfrak{D}({}_AV)$, and LK (resp. KL) is naturally equivalent to the identity functor on $\mathfrak{D}({}_AV)$ (resp. $\mathfrak{D}({}_RV)$). Hence we have

LEMMA 2.1. *If* $X, X'\in\mathfrak{D}({}_AV)$ *and* $f\in\mathrm{Hom}_A(X, X')$, *then* X *and* X' *are left* R-*modules contained in* $\mathfrak{D}({}_RV)$ *and* f *is an* R-*homomorphism.*

PROPOSITION 2.2. ${}_RV$-dom. $\dim D(X)/\tilde{D}(X)\geqq 1$ *for* $X\in{}_A\mathfrak{M}$.

PROOF. By Corollary 1.3 $D(X)\in\mathfrak{D}(V)$. Since $V\in\mathfrak{D}(V)$, by Lemma 2.1 every A-homomorphism $f: D(X)\to V$ is an R-homomorphism. Hence by definition $\tilde{D}(X)$ is an R-module. This proves Proposition 2.2 by Proposition 1.4.

LEMMA 2.3. $\tilde{D}(X)\in\mathfrak{D}(V)$ *for* $X\in{}_A\mathfrak{M}$.

PROOF. ${}_RV$-dom. $\dim D(X)\geqq 2$ by Corollary 1.3 and Lemma 2.1, and ${}_RV$ is injective. Hence we have $\tilde{D}(X)\in\mathfrak{D}_R(V)$ by Proposition 2.2.

LEMMA 2.4. $\tilde{\lambda}(X): \mathrm{Hom}_A(X, V)\cong\mathrm{Hom}_A(\tilde{D}(X), V)$ *and*

$$\mathrm{Hom}\,(\tilde{\pi}(X), 1): \mathrm{Hom}_A\,(\hat{D}(X), V) \cong \mathrm{Hom}_A\,(X, V)$$

for $X \in {}_A\mathfrak{M}$.

PROOF. Let $g \in \mathrm{Hom}_A\,(D(X), V)$. Then by Lemma 2.1 g is an R-homomorphism. Since $\xi(X): \tilde{D}(X) \to D(X)$ is an R-monomorphism and ${}_RV$ is injective, there is $h \in \mathrm{Hom}_R\,(D(X), V)$ such that $g = h \circ \xi(X)$. If we put $f = g \circ \tilde{\pi}(X)$, then $f = h \circ \pi(X)$. On the oter hand, $f = \lambda(X)(f) \circ \pi(X)$. Hence $(h - \lambda(X)(f)) \circ \pi(X) = 0$. Thus by definition we have $(h - \lambda(X)(f)) \circ \xi(X) = 0$, that is, $g = h \circ \xi(X) = \lambda(X)(f) \circ \xi(X)(f) = \tilde{\lambda}(X)(f)$, which shows that $\tilde{\lambda}(X)$ is onto. Since $\tilde{\lambda}(X)$ is a monomorphism by Proposition 1.6, $\tilde{\lambda}(X)$ is an isomorphism.

LEMMA 2.5. *For* $X \in {}_A\mathfrak{M}$ *and* $Y \in \mathfrak{D}(V)$, *we have a natural isomorphism*

$$\mathrm{Hom}\,(\tilde{\pi}(X), 1_Y): \mathrm{Hom}_A\,(\hat{D}(X), Y) \cong \mathrm{Hom}_A\,(X, Y).$$

PROOF. Since $Y \in \mathfrak{D}(V)$, there is an exact sequence in ${}_A\mathfrak{M}$

$$0 \to Y \to Y_1 \to Y_2$$

such that each Y_i is a direct product of copies of ${}_AV$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_A\,(\tilde{D}(X), Y) & \longrightarrow & \mathrm{Hom}_A\,(\tilde{D}(X), Y_1) & \longrightarrow & \mathrm{Hom}_A\,(\tilde{D}(X), Y_2) \\
& {\scriptstyle \mathrm{Hom}\,(\tilde{\pi}(X), 1_Y)}\downarrow & & {\scriptstyle \mathrm{Hom}\,(\tilde{\pi}(X), 1_{Y_1})}\downarrow & & {\scriptstyle \mathrm{Hom}\,(\tilde{\pi}(X), 1_{Y_2})}\downarrow \\
0 \longrightarrow & \mathrm{Hom}_A\,(X, Y) & \longrightarrow & \mathrm{Hom}_A\,(X, Y_1) & \longrightarrow & \mathrm{Hom}_A\,(X, Y_2)
\end{array}$$

in which each row is exact. It follows from Lemma 2.4 that the two vertical homomorphisms on the right hand side are isomorphisms. Hence by the Five Lemma we get the desired isomorphisms.

We are now in a position to establish the following theorem.

THEOREM 2.6. *Let* V *be a left* A*-module satisfying the standing assumption*(*). *Then* $\mathfrak{D}(V)$ *is a reflective subcategory of* ${}_A\mathfrak{M}$ with $\tilde{D}$ *as its reflector and* $\mathfrak{D}(V)$ *is a Grothendieck category.*

PROOF. The first assertion is a restatement of Lemma 2.5, while the second follows from the fact that ${}_RV$ is injective, $\mathfrak{D}({}_AV) \cong \mathfrak{D}({}_RV)$ and that $\mathfrak{D}({}_RV)$ is a Grothendieck category.

COROLLARY 2.7. $\tilde{\pi}(X): X \to \tilde{D}(X)$ *is an isomorphism iff* $X \in \mathfrak{D}(V)$.

PROOF. The "only if" part is clear by Lemma 2.3. To prove the "if" part, suppose that $X \in \mathfrak{D}(V)$. Then by Lemma 2.5 there exists $\psi \in \mathrm{Hom}_A\,(\tilde{D}(X), X)$ such that $1_X = \psi \circ \tilde{\pi}(X)$. Thus we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\pi}(X)} & \tilde{D}(X) \\ \Big\downarrow 1 & \swarrow \psi & \Big\downarrow \tilde{D}(1)=1 \\ X & \xrightarrow{\tilde{\pi}(X)} & \tilde{D}(X) \end{array}$$

In fact, $1_{\tilde{D}(X)}\circ\tilde{\pi}(X)=\tilde{\pi}(X)\circ 1_X=(\tilde{\pi}(X)\circ\psi)\circ\tilde{\pi}(X)$, and hence by Lemma 2.5 we have $1_{\tilde{D}(X)}=\tilde{\pi}(X)\circ\psi$.

REMARK. Let ${}_AV$ be injective or of type FI. Then by [3] and [6] there exists a reflector S from ${}_A\mathfrak{M}$ to $\mathfrak{D}(V)$. The kernel of the natural homomorphism $\Phi(X)$: $X\to S(X)$ is coincident with Ker $\pi(X)$. Since S is a reflector, by Corollary 1.3 there is a unique A-homomorphism $\eta(X): S(X)\to D(X)$ such that $\pi(X)=\eta(X)\circ\Phi(X)$. Hence $\operatorname{Ker}\eta(X)\cap\operatorname{Im}\Phi(X)=0$. In case V is injective, $S(X)$ is an essential extension of $\operatorname{Im}\Phi(X)$ and hence we can conclude that $\operatorname{Ker}\eta(X)=0$, from which Theorem 2.6 follows readily. However, $S(X)$ is not an essential extension of $\operatorname{Im}\Phi(X)$ in general in case V is not injective, as will be seen from Example 2.8 below.

EXAMPLE 2.8. Let A be a K-subalgebra of $(K)_3$ with a K-basis $\{c_{11}, c_{22}, c_{33}, c_{21}, c_{31}\}$, where K is a commutative field and c_{ik} are matrix units. Let us put $V=Ac_{11}$ and $X=Ac_{21}+Ac_{31}$. Then V is of type FI and $B=\operatorname{End}({}_AV)=K$. Let $f_i\in\operatorname{Hom}_A(X, V)$, $i=1,2$, be maps defined by

$$\begin{cases} f_1(c_{21})=c_{21} \\ f_1(c_{31})=0, \end{cases} \qquad \begin{cases} f_2(c_{21})=0 \\ f_2(c_{31})=c_{31}. \end{cases}$$

Then $\operatorname{Hom}_A(X, V)=f_1K+f_2K$. Now, for $(v_1, v_2)\in V\oplus V$, let us define $\psi(v_1, v_2)\in D(X)$ by

$$\psi(v_1, v_2)(f_1)=v_1, \qquad \psi(v_1, v_2)(f_2)=v_2.$$

Then it is easy to see that $\psi: V\oplus V\to D(X)$ is an isomorphism and that

$$\pi(X)(c_{21})=\psi(c_{21}, 0), \qquad \pi(X)(c_{31})=\psi(0, c_{31}).$$

Thus, we have $X\subset V$, $\tilde{D}(X)=D(X)\cong V\oplus V$, and $D(X)$ is not an essential extension of $\operatorname{Im}\pi(X)$.

§ 3. Conditions for $\tilde{D}(X)=D(X)$

Let V be a left A-module satisfying the standing assumption (*) in § 2.

PROPOSITION 3.1. *A left A-module X is V-reflexive, that is,* $\pi(X): X\cong D(X)$ *if* $X\in\mathfrak{D}(V)$ *and* $\tilde{D}(X)=D(X)$.

PROOF. This proposition follows readily from Corollary 2.7.

THEOREM 3.2. *For $X \in {}_A\mathfrak{M}$ the following conditions are equivalent.*

(a) $\tilde{D}(X) = D(X)$.

(b) $\mathrm{Hom}(\pi(X), 1): \mathrm{Hom}_A(D(X), V) \cong \mathrm{Hom}_A(X, V)$.

(c) $\pi(\tilde{D}(X)): \tilde{D}(X) \cong D(\tilde{D}(X))$.

(d) $\pi(D(X)): D(X) \cong D(D(X))$.

(e) $\pi(D^i(X)): D^i(X) \cong D^{i+1}(X)$ *for some* $i \geqq 1$.

(f) $\pi(D^i(X)): D^i(X) \cong D^{i+1}(X)$ *for all* $i \geqq 1$.

PROOF. Since $\tilde{D}(X)$ and $D(X)$ are R-modules and ${}_RV$ is injective, we have an exact sequence

$$0 \to \mathrm{Hom}_R(D(X)/\tilde{D}(X), V) \to \mathrm{Hom}_R(D(X), V) \xrightarrow{\mathrm{Hom}(\xi(X),1)} \mathrm{Hom}(\tilde{D}(X), V) \to 0.$$

By Proposition 2.2 $\tilde{D}(X) = D(X)$ iff $\mathrm{Hom}_R(D(X)/\tilde{D}(X), V) = 0$. On the other hand, if (b) holds, then by Lemma 2.4 $\mathrm{Hom}(\xi(X), 1_V)$ is an isomorphism since $\pi(X) = \xi(X) \circ \tilde{\pi}(X)$. This proves (a) $\Leftrightarrow$ (b).

Next, let us consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\pi}(X)} & \tilde{D}(X) \\ {\scriptstyle \pi(X)}\downarrow & \swarrow{\scriptstyle \xi(X)} & \downarrow{\scriptstyle \pi(\tilde{D}(X))} \\ D(X) & \xrightarrow{D(\tilde{\pi}(X))} & D(\tilde{D}(X)) \end{array} .$$

Since $D(\tilde{\pi}(X)) \circ \xi(X) \circ \tilde{\pi}(X) = D(\tilde{\pi}(X)) \circ \pi(X) = \pi(\tilde{D}(X)) \circ \tilde{\pi}(X)$, we have $D(\tilde{\pi}(X)) \circ \xi(X) = \pi(\tilde{D}(X))$ by Lemma 2.5. On the other hand, by Lemma 2.4 $D(\tilde{\pi}(X))$ is an isomorphism. Hence $\xi(X)$ is an isomorphism iff $\pi(\tilde{D}(X))$ is an isomorphism. This proves (a) $\Leftrightarrow$ (c).

If (a) holds, then we have (d) by (c). To prove (d) $\Rightarrow$ (c), let us consider a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \tilde{D}(X) & \longrightarrow & D(X) & \longrightarrow & D(X)/\tilde{D}(X) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \pi(\tilde{D}(X))} & & \downarrow{\scriptstyle \pi(D(X))} & & \downarrow{\scriptstyle \pi(D(X)/\tilde{D}(X))} & \\ 0 \longrightarrow & D(\tilde{D}(X)) & \longrightarrow & D(D(X)) & \longrightarrow & D(D(X)/\tilde{D}(X)), & \end{array}$$

in which the upper row is exact. Here we note that the lower row is also exact since ${}_RV$ is injective. By Proposition 2.2 $\pi(D(X)/\tilde{D}(X))$ is a monomorphism. Hence, if $\pi(D(X))$ is an isomorphism, then $\pi(\tilde{D}(X))$ is an epimorphism by Five Lemma and so an isomorphism. This shows that (d) $\Rightarrow$ (c).

Finally, assume that $\pi(D^{i+1}(X)): D^{i+1}(X) \cong D^{i+2}(X)$ for some $i \geqq 1$. Let us put $Y = D^i(X)$. Then, by the equivalence (a) $\Leftrightarrow$ (d) which has been just proved above, we have $\tilde{D}(Y)=D(Y)$. Since $Y\in\mathfrak{D}(V)$, this shows that $\pi(Y): Y\cong D(Y)$. Thus we have $\pi(D^i(X)): D^i(X)\cong D^{i+1}(X)$. By repeating this argument, we see that (d) holds. This proves (e) $\Rightarrow$ (d).

Since (d) $\Rightarrow$ (f) and (f) $\Rightarrow$ (e) are obvious, this completes our proof of Theorem 3.2.

THEOREM 3.3. *Let X be an A-submodule of $V^n=\sum_{i=1}^{n}\oplus V$. Then $\pi(X): X\cong D(X)$ iff $X\in\mathfrak{D}(V)$.*

PROOF. The "only if" part is obvious from Corollary 1.3. To prove the "if" part, suppose that $X\in\mathfrak{D}(V)$. Then, there is an exact sequence in ${}_R\mathfrak{M}$

$$0 \longrightarrow X \longrightarrow V^n \longrightarrow W$$

such that W is a direct product of copies of V. By using the injectivity of ${}_RV$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & X & \longrightarrow & V^n & \longrightarrow & W \\ & \downarrow{\scriptstyle \pi(X)} & & \downarrow{\scriptstyle \pi(V^n)} & & \downarrow{\scriptstyle \pi(W)} \\ 0 \longrightarrow & D(X) & \longrightarrow & D(V^n) & \longrightarrow & D(W) \end{array}$$

in which each row is exact. Since $\pi(V^n)$ is an isomorphism and $\pi(W)$ is a monomorphism, $\pi(X)$ is an isomorphism.

THEOREM 3.4. *If $D(X)$ is isomorphic to an A-submodule of V^n for some n, then $\tilde{D}(X)=D(X)$.*

PROOF. In view of Theorem 3.3, Theorem 3.4 is a direct consequence of Theorem 3.2.

PROPOSITION 3.5. *The following conditions are equivalent for $X\in{}_A\mathfrak{M}$; in particular, (b) and (c) are equivalent without the assumption (*).*

(a) *There exists a monomorphism $g: D(X)\to V^n$ for some positive integer n.*

(b) *$[\mathrm{Hom}_A(X, {}_AV)]_B$ is finitely generated as as a right B-module.*

(c) *There exists an A-homomorphism $\phi; X\to V^n$ for some n such that for any $f\in\mathrm{Hom}_A(X, V)$ there is $h\in\mathrm{Hom}_A(V^n, V)$ with $f=h\circ\phi$.*

PROOF. (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) are obvious. Suppose that (b) holds. Then there are a finite number of elements $f_i\in\mathrm{Hom}_A(X, V), i=1, \cdots, n,$ such that

$$[\mathrm{Hom}_A(X, V)]_B=\sum_{i=1}^{n} f_iB.$$

Let us define an A-homomorphism $\phi: X\to V^n$ by putting $\phi(X)=(f_1(x), \cdots, f_n(x))$ for $x\in X$. Then for any $f\in\mathrm{Hom}_A(X, V)$ there are $b_i\in B, i=1, \cdots, n,$ such that $f=\sum_{i=1}^{n} f_ib_i$. If we put $h(v_1, \cdots, v_n)=\sum_{i=1}^{n} v_ib_i$, then $h\in\mathrm{Hom}_A(V^n, V)$ and $f=h\circ\phi$. Thus, we have

(b) $\Rightarrow$ (c). Here we note that assumption (*) is not used in this proof. Hence (b) is equivalent to (c) without the assumption (*).

Finally, assume (a). Let us put $\psi = g \circ \pi(X): X \to V^n$. With the notation in §1 we have $f = \lambda(X)(f) \circ \pi(X)$ for $f \in \mathrm{Hom}_A(X, V)$. Since $\lambda(X)(f): D(X) \to V$ is an R-homomorphism by Lemma 2.1 and ${}_R V$ is injective, there is $h \in \mathrm{Hom}_R(V^n, V)$ with $\lambda(X)(f) = h \circ g$. Thus $f = h \circ \psi$. This proves (c).

COROLLARY 3.6. *In case ${}_A V$ is injective, each of the conditions in Proposition 3.5 is equivalent to (d) below.*

(d) *X is finitely cogenerated by V in the sense of* [3, §2].

PROOF. (a) $\Rightarrow$ (d). The map $\psi = g \circ \pi(X): X \to V^n$ shows the validity of (d).

(d) $\Rightarrow$ (c). Assume (d). Then there exists $\psi \in \mathrm{Hom}_A(X, V^n)$ for some n such that $\mathrm{Ker}\,\psi = \mathrm{Ker}\,\pi(X)$. Then for any $f \in \mathrm{Hom}_A(X, V)$ there exists $h \in \mathrm{Hom}_A(V^n, V)$ such that $f = h \circ \psi$, since V is injective. This proves (c).

REMARK. From the above proof we may say that in case V is injective $X \subsetneq V^n$ implies $D(X) \subsetneq V^n$. (Here by "$\subsetneq$" we mean "is isomorphic to a submodule of".) This implication, however, fails to be true if V is not injective, as we have shown already in Example 2.8. The problem whether $X \subsetneq V^n$ implies $D(X) \subsetneq V^m$ for some $m \geqq n$ when V is of type FI remains open.

§4. Conditions for the V-reflexivity of a module X

The following theorem is a direct consequence of Proposition 3.1 and Theorem 3.2 if V satisfies the assumption (*) in §2 which is not assumed in this section.

THEOREM 4.1. *Let V be a left A-module. Then a left A-module X is V-reflexive, that is, $\pi(X): X \cong D(X)$ iff $X \in \mathfrak{D}(V)$ and* $\mathrm{Hom}(\pi(X), 1): \mathrm{Hom}_A(D(X), V) \cong \mathrm{Hom}_A(X, V)$.

PROOF. The "only if" part is obvious from Corollary 1.3. To prove the "if" part, suppose that $X \in \mathfrak{D}(V)$ and $\mathrm{Hom}(\pi(X), 1_V)$ is an isomorphism. Then there is an exact sequence

$$0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2$$

such that each X_i is a direct product of copies of V. Now, let us consider the following commutative diagram in which each row is exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(D(X), X) & \longrightarrow & \mathrm{Hom}(D(X), X_1) & \longrightarrow & \mathrm{Hom}(D(X), X_2) \\
& \Big\downarrow {\scriptstyle \mathrm{Hom}(\pi(X), 1_X)} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\pi(X), 1_{X_1})} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\pi(X), 1_{X_2})} \\
0 \longrightarrow & \mathrm{Hom}(X, X) & \longrightarrow & \mathrm{Hom}(X, X_1) & \longrightarrow & \mathrm{Hom}(X, X_2).
\end{array}$$

From the assumption that $\mathrm{Hom}(\pi(X), 1_V)$ is an isomorphism it follows that each $\mathrm{Hom}(\pi(X), 1_{X_i}): \mathrm{Hom}(D(X), X_i) \to \mathrm{Hom}(X, X_i)$ is an isomorphism. Hence $\mathrm{Hom}(\pi(X),$

$1_X)$: Hom $(D(X), X) \cong$ Hom (X, X).

Therefore, there exists $f \in$ Hom $(D(X), X)$ such that $1_X = f \circ \pi(X)$. Thus, we have

$$D(X) = \text{Im } \pi(X) \oplus \text{Ker} f,$$

and hence from an exact sequence

$$0 \longrightarrow X \xrightarrow{\pi(X)} D(X) \longrightarrow \text{Ker} f \longrightarrow 0$$

we get another exact sequence

$$0 \longrightarrow \text{Hom}(\text{Ker} f, V) \longrightarrow \text{Hom}(D(X), V) \xrightarrow{\text{Hom}(\pi(X), 1_V)} \text{Hom}(X, V).$$

Since Hom $(\pi(X), 1_V)$ is an isomorphism, we have Hom $(\text{Ker} f, V) = 0$.

On the other hand, $D(X) \in \mathfrak{D}(V)$ by Corollary 1.3, and hence V-dom. dim Ker $f \geqq 1$, since Ker $f \subset D(X)$. Therefore, we have Ker $f = 0$. Thus, f, and hence $\pi(X)$, is an isomorphism. This completes our proof of Theorm 4.1.

The following theorem provides us with another condition for V-reflexivity, which is valid in the restricted case but is easier to handle.

THEOREM 4.2. *Suppose that condition* (c) *in Proposition* 3.5 *holds for a left A-module X. Then $\pi(X)$ is surjective iff $V^n/\phi(X) \subseteq \Pi V$, where $\phi: X \rightarrow V^n$ is the map described in condition* (c).

PROOF. Let us put

$$\phi(x) = (f_1(x), \cdots, f_n(x)) \in V^n$$

where $f_i \in \text{Hom}_A(X, V)$, $i = 1, \cdots, n$. Then any $f \in$ Hom (X, V) can be written as $f = \sum_{i=1}^{n} f_i b_i$ with $b_i \in B, i = 1, \cdots, n$. Next, let us define a map $\psi: D(X) \rightarrow V^n$ by

$$\psi(\xi) = (\xi(f_1), \cdots, \xi(f_n)) \quad \text{for} \quad \xi \in D(X).$$

Then we have $\phi = \psi \circ \pi(X)$ and

$$\psi(D(X)) = \left\{ (v_1, \cdots, v_n) \middle| \sum_{i=1}^{n} v_i b_i = 0 \text{ for all } b_i \in B, i = 1, n, \text{ such that } \sum_{i=1}^{n} f_i b_i = 0 \right\}.$$

Indeed, if $(v_1, \cdots, v_n)$ belongs to the set on the right hand side of the above equality, the map ξ : Hom$_A(X, V) \rightarrow V$ defined by

$$\xi\left(\sum_{i=1}^{n} f_i b_i\right) = \sum_{i=1}^{n} v_i b_i$$

is one-valued and hence $\xi \in D(X)$, from which we have

$$\xi(f_i) = v_i \quad \text{for} \quad i = 1, \cdots, n.$$

Since the other inclusion is obvious, we have the above equality.

Then we have

$$\begin{aligned}\pi(X)\ \text{surjective} &\Longleftrightarrow \phi(D(X))=\phi(X)\\ &\Longleftrightarrow \cap\{\operatorname{Ker} g \mid g\in \operatorname{Hom}_A(V^n/\phi(X), V)\}=\phi\\ &\Longleftrightarrow V^n/\phi(X)\subsetneq // V.\end{aligned}$$

Thus, the theorem is proved.

As before, let $B=\operatorname{End}({}_AV)$ and $C=\operatorname{End}(V_B)$. Then $D({}_AA)={}_AC$. Let $\phi: V\to \operatorname{Hom}_A({}_AC, {}_AV)$ be a map defined by $\phi(v)(c)=cv$ for $v\in V, c\in C$. Then we have

$$\operatorname{Hom}(\pi({}_AA), 1_V)\circ\phi: V\cong \operatorname{Hom}_A({}_AA, {}_AV).$$

Therefore, in view of [3, Lemma 2.1], we have the equivalence of (a), (b) and (c) below:

(a) $\phi: V\cong \operatorname{Hom}_A({}_AC, {}_AV)$,

(b) ${}_CV_B\cong \operatorname{Hom}_A({}_AC_C, {}_AV_B)$,

(c) $\operatorname{Hom}(\pi({}_AA), 1_V): \operatorname{Hom}_A({}_AC, {}_AV)\cong \operatorname{Hom}_A({}_AA, {}_AV)$.

Thus, as a direct consequence of Theorem 4.1 we get the following corollary, which was obtained by Morita [5] and by Y. Suzuki [8] independently.

COROLLARY 4.3. *A faithful left A-module V has the double centralizer property iff* ${}_AA\in\mathfrak{D}(V)$ *and* ${}_CV_B\cong \operatorname{Hom}_A({}_AC_C, {}_AV_B)$.

On the other hand, Theorem 4.2 leads us to the following corollary, which was obtained by Morita [7] and by T. Kato and Y. Suzuki independently.

COROLLARY 4.4. *Let V be a faithful left A-module such that* V_B *is finitely generated. Then there is an A-monomorphism* $\phi: A\to V^n$ *for some integer* $n\geqq 1$ *such that for any* $f\in \operatorname{Hom}_A({}_AA, V)$ *there is* $g\in \operatorname{Hom}_A(V^n, V)$ *with* $f=g\circ\phi$, *and V has the double centralizer property iff* $V^n/\phi(A)\subsetneq // V$.

§ 5. A characterization of $\mathfrak{D}(V)$ with V injective.

In a previous paper [4] we have given a characterization of $\mathfrak{D}(V)$ with V injective. Here we shall establish another characterization.

THEOREM 5.1. *Let* $\mathfrak{B}$ *be a full reflective subcategory of* ${}_A\mathfrak{M}$ *closed under isomorphic images. Then there is an injective module V with* $\mathfrak{B}=\mathfrak{D}(V)$ *iff* $\mathfrak{B}$ *is a Grothendieck category and* $X\in\mathfrak{B}$ *implies* $E(x)\in\mathfrak{B}$, *where* $E(X)$ *is the injective envelope of X.*

PROOF. The "only if" part is obvious. To prove the "if" part, suppose that $\mathfrak{B}$ is a Grothendieck category and that $X\in\mathfrak{B}$ implies $E(X)\in\mathfrak{B}$. Then by [6, Theorem 1.11] there is a left A-module W of type FI such that $\mathfrak{B}=\mathfrak{D}(W)$. Let C be the double centralizer of W and let us put

$$\begin{aligned} K(X)&=\operatorname{Hom}_A({}_AC_C, X), && X\in {}_A\mathfrak{M},\\ L(Z)&={}_AC_C\otimes Z, && Z\in {}_C\mathfrak{M}.\end{aligned}$$

Then by [3, Theorem 3.3] K (resp. L) carries $\mathfrak{D}({}_AW)$ (resp. $\mathfrak{D}({}_CW)$) into $\mathfrak{D}({}_CW)$

(resp. $\mathfrak{D}({}_A W)$) and there is an A-homomorphism $\Gamma(X): LK(X) \to X$ which is natural in $X \in {}_A\mathfrak{M}$ and is an isomorphism for $X \in \mathfrak{D}({}_A W)$.

Suppose that $X = X_1 \oplus X_2$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & LK(X_1) & \longrightarrow & LK(X) & \longrightarrow & LK(X_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Gamma(X_1)} & & \downarrow{\scriptstyle \Gamma(X)} & & \downarrow{\scriptstyle \Gamma(X_2)} & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & X_2 & \longrightarrow & 0 \end{array}$$

in which each row is exact. If $\Gamma(X)$ is an isomorphism, then $\Gamma(X_1)$ is a monomorphism and $\Gamma(X_2)$ is an epimorphism. By exchanging X_1 with X_2, we see that if $\Gamma(X)$ is an isomorphism, so is $\Gamma(X_i)$ for $i = 1, 2$. This shows that $\mathfrak{B}$ satisfies condition (b) in [4, Theorem 1.1], since $\mathfrak{D}({}_C W)$ is proved to satisfy this condition by this theorem.

From the proof of [4, Theorem 1.1] it follows that there is a finitely cogenerating injective left A-module V such that

$$V\text{-dom.}\dim X \geqq 1 \iff W\text{-dom.}\dim X \geqq 1;$$

we have only to apply the arguments there to the present case by putting $\mathfrak{B}' = \{X \in {}_A\mathfrak{M} \mid E(X) \in \mathfrak{D}(W)\}$. Since $E(V) \in \mathfrak{D}(W)$, we have $V \in \mathfrak{D}(W)$. Then ${}_C V = K(V) \in \mathfrak{D}({}_C W)$ and ${}_C V$ is injective. Since W-dom. dim $V \geqq 1$ and V-dom. dim $W \geqq 1$, we have ${}_C W$-dom. dim ${}_C V \geqq 1$ and ${}_C V$-dom. dim ${}_C W \geqq 1$. Hence $\mathfrak{D}({}_C V) = \mathfrak{D}({}_C W)$.

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