

Duality for Modules and its Applications to the Theory of Rings with Minimum Condition

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Introduction

The purpose of this paper is to develop a theory of dualities for modules and to give some applications to the theory of rings with minimum condition for one-sided ideals. Dualities with which we are concerned are functorial dualities based on the notion of functors in the sense of Eilenberg and MacLane [5] and are not axiomatic ones such as discussed by MacLane [16] and Buchsbaum [2].

1. Let A and B be two rings with unit elements. Let \mathfrak{A} be a class of left (or right) A -modules and \mathfrak{B} a class of left (or right) B -modules. The category in which "objects" are modules in \mathfrak{A} (resp. \mathfrak{B}) and "maps" are all A -homomorphisms (resp. B -homomorphisms) will be denoted by the same letter \mathfrak{A} (resp. \mathfrak{B}). A function T which assigns to each module X in \mathfrak{A} a B -module $T(X)$ in \mathfrak{B} and to each A -homomorphism $f: X \rightarrow X'$ ($X, X' \in \mathfrak{A}$) a B -homomorphism $T(f): T(X) \rightarrow T(X')$, is called a covariant functor from the category \mathfrak{A} to the category \mathfrak{B} if the following conditions are satisfied:

- (0.1) If $f: X \rightarrow X$ is the identity, then $T(f)$ is the identity.
- (0.2) $T(f' \circ f) = T(f') \circ T(f)$ for $f: X \rightarrow X', f': X' \rightarrow X''$.
- (0.3) $T(f + f') = T(f) + T(f')$ for $f, f': X \rightarrow X'$.

In case T assigns to each A -homomorphism $f: X \rightarrow X'$ a B -homomorphism $T(f): T(X') \rightarrow T(X)$ and T satisfies (0.1), (0.3) and (0.2)' below:

- (0.2)' $T(f' \circ f) = T(f) \circ T(f')$ for $f: X \rightarrow X', f': X' \rightarrow X''$,

T is called a contravariant functor from \mathfrak{A} to \mathfrak{B} .

Let T_1 and T_2 be two functors from \mathfrak{A} to \mathfrak{B} , both covariant (resp. both contravariant). A natural transformation $\phi: T_1 \rightarrow T_2$ is defined to be a family of B -homomorphisms $\phi(X): T_1(X) \rightarrow T_2(X)$ such that the diagram

$$\begin{array}{ccc} T_1(X) & \xrightarrow{\phi(X)} & T_2(X) \\ \downarrow T_1(f) & & \downarrow T_2(f) \\ T_1(X') & \xrightarrow{\phi(X')} & T_2(X') \end{array}$$

is commutative (i.e. $T_2(f) \circ \phi(X) = \phi(X') \circ T_1(f)$) for all $f: X \rightarrow X'$ (resp. $f: X' \rightarrow X$). If each $\phi(X)$ is a B -isomorphism¹⁾ of $T_1(X)$ onto $T_2(X)$, then ϕ is called a

1) "Isomorphism" means "isomorphism onto".

natural equivalence. If there exists a natural equivalence $\theta: T_1 \rightarrow T_2$, T_1 and T_2 are said to be naturally equivalent. Eilenberg and MacLane [5] explained these notions by taking the duality relation between finite-dimensional real vector spaces as an example. In this paper we shall formulate dualities in terms of these notions.

2. Let \mathfrak{A} be a class of left A -modules and \mathfrak{B} a class of right B -modules such that \mathfrak{A} contains A as a left A -module and \mathfrak{B} contains B as a right B -module. A (functorial) *duality* between \mathfrak{A} and \mathfrak{B} is defined to be a pair (D_1, D_2) of a contravariant functor D_1 from \mathfrak{A} to \mathfrak{B} and a contravariant functor D_2 from \mathfrak{B} to \mathfrak{A} such that the covariant functors $D_2 D_1$ and $D_1 D_2$ are naturally equivalent to the identity functor; namely, if there exist a family of A -isomorphisms $\lambda_1(X): X \rightarrow D_2 D_1(X)$ and a family of B -isomorphisms $\lambda_2(Y): Y \rightarrow D_1 D_2(Y)$ such that the diagrams

$$(0.4) \quad \begin{array}{ccc} X & \xrightarrow{\lambda_1(X)} & D_2 D_1(X) & & Y & \xrightarrow{\lambda_2(Y)} & D_1 D_2(Y) \\ \downarrow f & & \downarrow D_2 D_1(f) & & \downarrow g & & \downarrow D_1 D_2(g) \\ X' & \xrightarrow{\lambda_1(X')} & D_2 D_1(X') & & Y' & \xrightarrow{\lambda_2(Y')} & D_1 D_2(Y') \end{array}$$

are commutative for any A -homomorphism $f: X \rightarrow X'$ and any B -homomorphism $g: Y \rightarrow Y'$ ($X, X' \in \mathfrak{A}$; $Y, Y' \in \mathfrak{B}$), then the pair (D_1, D_2) is called a duality. In this case, D_1 (resp. D_2) is called an *anti-isomorphism* from \mathfrak{A} onto \mathfrak{B} (resp. from \mathfrak{B} onto \mathfrak{A}).

3. A typical example of dualities is furnished by character modules which are algebraic analogues of character groups of locally compact commutative groups (cf. [33]). Let U be a two-sided A - B -module; by this we shall mean that U is a left A -module and a right B -module such that $a(ub) = (au)b$ for $a \in A$, $b \in B$, $u \in U$. For a left A -module X and a right B -module Y we put

$$(0.5) \quad \text{Char}_U X = \text{Hom}_A(X, U), \quad \text{Char}_U Y = \text{Hom}_B(Y, U).$$

Here $\text{Hom}_A(X, U)$ is a right B -module which consists of all A -homomorphisms of X into U with the usual compositions:

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x), \quad (\alpha b)(x) = (\alpha(x))b$$

where $\alpha, \beta \in \text{Hom}_A(X, U)$, $b \in B$, $x \in X$. $\text{Hom}_B(Y, U)$ is similarly defined. $\text{Char}_U X$ is called the *U-character module* of X ; the *U-character module* $\text{Char}_U Y$ of a right B -module Y is a left A -module. For any A -homomorphism $f: X \rightarrow X'$ we define

$$\text{Char}_U f: \text{Char}_U X' \rightarrow \text{Char}_U X$$

by putting

$$[\text{Char}_U f](\alpha) = \alpha \circ f \quad \text{for } \alpha \in \text{Char}_U X',$$

i.e. $\text{Char}_U f = \text{Hom}_A(f, 1)$ in the notations of Cartan and Eilenberg [5]. For a B -homomorphism $g: Y \rightarrow Y'$, $\text{Char}_U g: \text{Char}_U Y' \rightarrow \text{Char}_U Y$ is similarly defined.

For a left A -module X we define a natural homomorphism

$$(0.6) \quad \pi_U(X): X \rightarrow \text{Char}_U(\text{Char}_U X)$$

by

$$[\pi_U(X)(x)](\alpha) = \alpha(x) \quad \text{for } x \in X, \alpha \in \text{Char}_U X.$$

For a right B -module Y , a natural homomorphism $\pi_U(Y)$ is similarly defined.

Char_U is always a contravariant functor. If Char_U maps \mathfrak{A} into \mathfrak{B} and maps \mathfrak{B} into \mathfrak{A} , and $\pi_U(X), \pi_U(Y)$ are isomorphisms for every X in \mathfrak{A} and for every Y in \mathfrak{B} , Char_U defines a duality between \mathfrak{A} and \mathfrak{B} in our sense. In this case it is said that *the duality with respect to U -character modules holds between \mathfrak{A} and \mathfrak{B}* . Such a duality is an algebraic analogue of the Pontrjagin duality for locally compact commutative groups. It will be shown (Theorem 1.2 below) that if Char_U defines a duality between \mathfrak{A} and \mathfrak{B} , then $\pi_U(X)$ and $\pi_U(Y)$ are all isomorphisms for X in \mathfrak{A} and Y in \mathfrak{B} .

4. Another example of dualities is given by dual representation modules for finitely generated modules over an algebra of finite rank with respect to a field; a left A -module X is said to be a dual representation module of a right A -module Y if the representations of A determined by X and Y are equivalent (cf. Nesbitt and Thrall [22]). It has been proved quite recently by H. Tachikawa [27] that dual representation modules can be obtained also as U -character modules with a suitable two-sided A - A -module U .

5. Chapter I deals with dualities defined above as functors. In §1 we shall establish that any duality between \mathfrak{A} and \mathfrak{B} is equivalent to a duality which assigns to each module in \mathfrak{A} and \mathfrak{B} its U -character module with a suitable two-sided A - B -module U . Here two dualities (D_1, D_2) and (E_1, E_2) are said to be *equivalent* if the functors D_i and E_i are naturally equivalent for $i=1, 2$.

6. In case A is a commutative ring and \mathfrak{A} is a class of left A -modules containing A as a left A -module, a *duality* (or more precisely a self-duality) for \mathfrak{A} is defined to be a contravariant functor D from \mathfrak{A} to \mathfrak{A} such that D^2 is naturally equivalent to the identity functor; in this case there exists a family of A -homomorphisms $\lambda(X): X \rightarrow D^2(X)$ such that the diagram

$$(0.7) \quad \begin{array}{ccc} X & \xrightarrow{\lambda(X)} & D^2(X) \\ \downarrow f & & \downarrow D^2(f) \\ X' & \xrightarrow{\lambda(X')} & D^2(X') \end{array}$$

is commutative for any A -homomorphism $f: X \rightarrow X'$. In §5 we shall prove that any duality for \mathfrak{A} is equivalent to a duality defined by means of semi-linear (U, θ) -character modules where U is a left A -module and θ is a ring-automorphism of A with period ≤ 2 .

7. Semi-linear character modules are defined as follows. Let U be a left A -module with a *semi-linear isomorphism* ω ; that is, there exists a ring-automorphism θ of A such that $\omega(au) = \theta(a)\omega(u)$. We assume further that $\theta^2=1$. For

a left A -module X the set of all *semi-linear* (A, θ) -homomorphisms α of X into U (by this we mean that α is a mapping with the property $\alpha(ax) = \theta(a)\alpha(x)$) forms a left A -module under the definitions:

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x), \quad (a\alpha)(x) = a(\alpha(x))$$

where $x \in X$, $a \in A$. This module is called the *semi-linear* (U, θ) -character module of X and will be denoted by $\text{Char}_{U, \theta} X$. For any A -homomorphism $f: X \rightarrow X'$ we define an A -homomorphism

$$\text{Char}_{U, \theta} f: \text{Char}_{U, \theta} X' \rightarrow \text{Char}_{U, \theta} X$$

by $[\text{Char}_{U, \theta} f](\alpha) = \alpha \circ f: X \rightarrow U$ for $\alpha \in \text{Char}_{U, \theta} X'$. We define a natural A -homomorphism

$$\pi_{U, \theta}(X): X \rightarrow \text{Char}_{U, \theta}(\text{Char}_{U, \theta} X)$$

by putting $[\pi_{U, \theta}(X)(x)](\alpha) = \omega(\alpha(x))$ for $x \in X$, $\alpha \in \text{Char}_{U, \theta} X$. From the assumption that $\theta^2 = 1$, it follows that $\pi_{U, \theta}(X)$ is actually an A -homomorphism. As in the case of character modules, if $\text{Char}_{U, \theta}$ defines a duality for \mathfrak{A} then $\pi_{U, \theta}(X)$ is an A -isomorphism for X in \mathfrak{A} .

8. Let \mathfrak{A} be a class of left A -modules and \mathfrak{B} a class of left B -modules such that \mathfrak{A} contains A as a left A -module and \mathfrak{B} contains B as a left B -module. Let T_1 be a covariant functor from \mathfrak{A} to \mathfrak{B} and T_2 a covariant functor from \mathfrak{B} to \mathfrak{A} . If $T_2 T_1$ and $T_1 T_2$ are naturally equivalent to the identity functor, T_1 (resp. T_2) is called an *isomorphism* from \mathfrak{A} onto \mathfrak{B} (resp. from \mathfrak{B} to \mathfrak{A}). Under some conditions on \mathfrak{A} and \mathfrak{B} , it will be shown in § 3 that T_1 is naturally equivalent to each of covariant functors $U \otimes_A X$ and $\text{Hom}_A(V, X)$ where U is a two-sided B - A -module and V a two-sided A - B -module. It is to be noted that a composite of two anti-isomorphisms is an isomorphism.

9. Chapter II is devoted to a detailed investigation of dualities between \mathfrak{A} and \mathfrak{B} for the case where A and B are rings satisfying the minimum condition for left and right ideals, and \mathfrak{A} is the class ${}_A \mathfrak{M}$ of all finitely generated left A -modules and \mathfrak{B} is the class \mathfrak{M}_B of all finitely generated right B -modules. The necessary and sufficient condition in order that the duality with respect to U -character modules should hold between \mathfrak{A} and \mathfrak{B} , which is obtained in § 2, is simplified greatly in this case. This will be discussed in § 6; a half part of Theorem 6.3 was first proved by Tachikawa [27]. Our results obtained in §§ 6 and 10 may be considered as a generalization of the theory of quasi-Frobenius rings (§ 14).

10. In case A and B are algebras of finite rank over the same field, we shall restrict ourselves to dualities satisfying a certain condition depending on the ground field. We shall say that two algebras A and B over the same field are *similar* if there exists a duality between the categories ${}_A \mathfrak{M}$ and \mathfrak{M}_B , or what amounts to the same thing, if there exists an isomorphism between the categories ${}_A \mathfrak{M}$ and ${}_B \mathfrak{M}$. It will be shown in § 9 that A and B are similar if and only if their basic algebras A^0 and B^0 in the sense of Nesbitt and Scott [21] are isomorphic as algebras. Thus our notion of similarity is identical with that of similarity introduced by Osima [24], and is reduced to the classical notion of similarity for central simple algebras. In case A and B are similar, we can determine the

complete family of dualities between \mathfrak{M}_A and \mathfrak{M}_B with the aid of basic algebras. In determining the complete family of dualities for modules over a commutative ring with minimum condition, the work of Snapper [26] plays an important role. It will also be shown that a faithful module over a commutative ring with minimum condition which has a composition series is completely indecomposable in the sense of Snapper if and only if it is injective. The same fact remains valid for completely indecomposable modules in the sense of Feller [7].

11. In Chapter III we shall give some applications of our theory. In § 15 we shall discuss regular pairings introduced recently by Curtis [4]. His results will be generalized or refined. In § 16 we shall deal with the endomorphism rings of faithful modules over a quasi-Frobenius ring. We shall prove that for a quasi-Frobenius ring A the A -endomorphism rings of two faithful right A -modules U and V are isomorphic if and only if there exists semi-linear (A^0, θ) -isomorphism of Ue onto Ve with a ring-automorphism θ of A^0 where $A^0 = eAe$ is the basic ring of A and e is a sum of mutually orthogonal non-isomorphic primitive idempotents of A . For a quasi-Frobenius ring A , the A -endomorphism ring of a faithful right A -module U is not always quasi-Frobenius; we shall prove that it is quasi-Frobenius if and only if U is a projective right A -module. In case A is a quasi-Frobenius algebra, the A -endomorphism ring of a faithful right A -module is a QF-3 algebra in the sense of Thrall [29]. Thrall has introduced the notion of QF-3 algebras as a generalization of quasi-Frobenius algebras. In § 17 it will be shown how we can obtain QF-3 algebras.

12. In Appendix it will be shown that any duality for the category of all locally compact commutative groups is equivalent to Pontrjagin's duality defined by means of character groups. Here a duality D for this category is postulated to be such that D maps the topological space $\text{Hom}(X, X')$ with the compact-open topology continuously into $\text{Hom}(D(X'), D(X))$ for any locally compact commutative groups X and X' .

13. Throughout this paper, rings will be assumed to be associative rings with unit elements (=identity elements), and by a module with a ring as a left (or right) operator domain we shall mean always a module on which the unit element of the ring acts as the identity operator. The significances of the notations used in the Introduction will be retained throughout this paper.

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CHAPTER I. GENERAL THEORY OF DUALITIES

1. Dualities between two categories of modules

The purpose of this section is to prove the following theorem.

THEOREM 1.1. *Let A and B be rings whose unit elements will be denoted by 1 and $1'$ respectively. Let \mathfrak{A} be a class of left A -modules and \mathfrak{B} a class of right B -modules such that \mathfrak{A} contains A as a left A -module and \mathfrak{B} contains B as a right B -module. Then any duality $D=(D_1, D_2)$ between \mathfrak{A} and \mathfrak{B} is equivalent to a duality which assigns to each module in \mathfrak{A} and \mathfrak{B} its U -character module with a suitable two-sided A - B -module U ; more precisely, there exist a two-sided A - B -module U , a family of B -isomorphisms $\phi_1(X): D_1(X) \rightarrow \text{Char}_U X$ ($X \in \mathfrak{A}$) and a family of A -isomorphisms $\phi_2(Y): D_2(Y) \rightarrow \text{Char}_U Y$ ($Y \in \mathfrak{B}$) such that the diagrams*

$$\begin{array}{ccc} D_1(X') & \xrightarrow{D_1(f)} & D_1(X) & & D_2(Y') & \xrightarrow{D_2(g)} & D_2(Y) \\ \downarrow \phi_1(X') & & \downarrow \phi_1(X) & & \downarrow \phi_2(Y') & & \downarrow \phi_2(Y) \\ \text{Char}_U X' & \xrightarrow{\text{Char}_U f} & \text{Char}_U X & & \text{Char}_U Y' & \xrightarrow{\text{Char}_U g} & \text{Char}_U Y \end{array}$$

are commutative for $f: X \rightarrow X'$ ($X, X' \in \mathfrak{A}$) and $g: Y \rightarrow Y'$ ($Y, Y' \in \mathfrak{B}$). Moreover, the natural homomorphisms

$$\begin{aligned} \pi_U(X): X &\rightarrow \text{Char}_U(\text{Char}_U X), & X &\in \mathfrak{A} \\ \pi_U(Y): Y &\rightarrow \text{Char}_U(\text{Char}_U Y), & Y &\in \mathfrak{B} \end{aligned}$$

defined by

$$[\pi_U(X)(x)](\alpha) = \alpha(x), \quad [\pi_U(Y)(y)](\beta) = \beta(y)$$

where $x \in X, y \in Y, \alpha \in \text{Char}_U X, \beta \in \text{Char}_U Y$, are all isomorphisms.

PROOF. We shall divide our proof in several steps. For the sake of simplicity we shall denote D_1, D_2 by the same letter D and λ_1, λ_2 by the same letter λ . Further we put

$$(1) \quad \lambda^*(X) = D(\lambda(X)) \circ \lambda(D(X)), \quad \lambda^*(Y) = D(\lambda(Y)) \circ \lambda(D(Y))$$

where $X \in \mathfrak{A}, Y \in \mathfrak{B}$. $\lambda^*(X)$ and $\lambda^*(Y)$ are respectively a B -isomorphism and an

A -isomorphism.

I. i) Let us put

$$(2) \quad U = D(A) .$$

Then U is a right B -module. For any element a of A we define an A -homomorphism $\varphi_a: A \rightarrow A$ by putting $\varphi_a(x) = xa$ for $x \in X$. Since $D(\varphi_a)$ is a B -endomorphism of U , $D(\varphi_a)u$ is defined for any element u of U . We set

$$(3) \quad au = D(\varphi_a)u, \quad \text{for } u \in U, a \in A .$$

Then it is easy to see that U is a left A -module; e.g. we have $a'(au) = (a'a)u$ for $a, a' \in A, u \in U$ since $a'(au) = D(\varphi_{a'})(D(\varphi_a)u) = D(\varphi_a \circ \varphi_{a'})u = D(\varphi_{a'a})u = (a'a)u$. Moreover we have $a(ub) = (au)b$ for $a \in A, b \in B, u \in U$, since $D(\varphi_a)$ is a B -homomorphism and hence $D(\varphi_a)(ub) = (D(\varphi_a)u)b$. Thus U is a two-sided A - B -module.

ii) Let us put

$$(4) \quad V = D(B) .$$

Then V is a left A -module. For any element b of B we put

$$(5) \quad vb = D(\psi_b)v, \quad \text{for } v \in V,$$

where $\psi_b: B \rightarrow B$ is defined by $\psi_b(y) = by$ for $y \in B$. Similarly as in i) we can prove that V is a two-sided A - B -module.

II. i) Let X be any left A -module in \mathfrak{A} . We define a B -isomorphism $\Phi_0(X): D(X) \rightarrow \text{Char}_V X$ by

$$(6) \quad [\Phi_0(X)](y) = \lambda^*(B)^{-1} \circ D(\psi_y) \circ \lambda(X) \quad \text{for } y \in D(X),$$

where $\psi_y: B \rightarrow D(X)$ is defined by $\psi_y(b) = yb$ for $b \in B$. Furthermore, we shall consider a mapping $\Psi_0(X): \text{Char}_V X \rightarrow D(X)$ by putting

$$[\Psi_0(X)](\alpha) = (D(\alpha) \circ \lambda(B))(1') \quad \text{for } \alpha \in \text{Char}_V X .$$

We shall first prove that $\Phi_0(X) \circ \Psi_0(X) = 1$. If we put $y = \Psi_0(X)(\alpha)$, then $\psi_y = D(\alpha) \circ \lambda(B)$ and hence $D(\psi_y) \circ \lambda(X) = D(\lambda(B)) \circ D^2(\alpha) \circ \lambda(X)$. On the other hand, $D^2(\alpha) \circ \lambda(X) = [D_2 D_1(\alpha)] \circ \lambda(X) = \lambda(D(B)) \circ \alpha$ by the property of dualities (cf. (0.4) in Introduction) and hence we have $\Phi_0(X)(y) = \lambda^*(B)^{-1} \circ D(\lambda(B)) \circ \lambda(D(B)) \circ \alpha = \alpha$ by (1). Therefore $\Phi_0(X) \circ \Psi_0(X) = 1$.

We shall next prove that $\Psi_0(X)$ is a B -homomorphism. Let $\alpha \in \text{Char}_V X$. Then for any element x of X and for $b \in B$ we have $(\alpha b)(x) = (\alpha(x))b = D(\psi_b)(\alpha(x))$ by the definition of scalar multiplication for character modules and by the definition of multiplication of elements of V with elements of B . Therefore we have

$$\begin{aligned} [\Psi_0(X)](\alpha b) &= (D(\alpha b) \circ \lambda(B))(1') = (D(\alpha) \circ D^2(\psi_b) \circ \lambda(B))(1') \\ &= (D(\alpha) \circ \lambda(B) \circ \psi_b)(1') = (D(\alpha) \circ \lambda(B))(1' \cdot b) \\ &= [(D(\alpha) \circ \lambda(B))(1')]b = [\Psi_0(X)(\alpha)]b . \end{aligned}$$

Thus $\Psi_0(X)$ is a B -homomorphism since it is clear that $\Psi_0(X)(\alpha + \alpha') = \Psi_0(X)(\alpha) + \Psi_0(X)(\alpha')$.

Now $\Phi_0(X)$ is one-to-one since $D(\phi_y) = D(\phi_{y'})$ for $y, y' \in D(X)$ implies $D^2(\phi_y) = D^2(\phi_{y'})$ and hence $\phi_y = \phi_{y'}$, and consequently $y = y'$. Since $\Phi_0(X) \circ \Psi_0(X) = 1$, we see that $\Phi_0(X)$ is onto. Hence $\Psi_0(X)$ is also one-to-one and onto. Therefore $\Psi_0(X)$ is a B -isomorphism and hence $\Phi_0(X)$ is also a B -isomorphism. Thus we have

$$(7) \quad [\Phi_0(X)]^{-1}(\alpha) = (D(\alpha) \circ \lambda(B))(1'), \quad \text{for } \alpha \in \text{Char}_V X.$$

ii) We shall define an A -isomorphism $\Phi_0(Y): D(Y) \rightarrow \text{Char}_U Y$ for $Y \in \mathfrak{B}$ by

$$(8) \quad [\Phi_0(Y)](x) = (\lambda^*(A))^{-1} \circ D(\varphi_x) \circ \lambda(Y), \quad \text{for } x \in D(Y),$$

where $\varphi_x: A \rightarrow D(Y)$ is defined by $\varphi_x(a) = ax$ for $a \in A$. Similarly as in i) we can prove that $\Phi_0(Y)$ is actually an A -isomorphism and that

$$(9) \quad [\Phi_0(Y)]^{-1}(\beta) = (D(\beta) \circ \lambda(A))(1), \quad \text{for } \beta \in \text{Char}_U Y.$$

III. We shall prove that the diagrams

$$(10) \quad \begin{array}{ccc} D(X') & \xrightarrow{D(f)} & D(X) \\ \downarrow \Phi_0(X') & & \downarrow \Phi_0(X) \\ \text{Char}_V X' & \xrightarrow{\text{Char}_V f} & \text{Char}_V X \end{array} \quad \begin{array}{ccc} D(Y') & \xrightarrow{D(g)} & D(Y) \\ \downarrow \Phi_0(Y') & & \downarrow \Phi_0(Y) \\ \text{Char}_U Y' & \xrightarrow{\text{Char}_U g} & \text{Char}_U Y \end{array}$$

are commutative for any A -homomorphism $f: X \rightarrow X'$ ($X, X' \in \mathfrak{A}$) and for any B -homomorphism $g: Y \rightarrow Y'$ ($Y, Y' \in \mathfrak{B}$).

i) Let $y' \in D(X')$, $x \in X$ and put $y = D(f)(y')$. Then for $\phi_y: B \rightarrow D(X)$ defined by $\phi_y(b) = yb$, we have $\phi_y(b) = yb = (D(f)(y'))b = D(f)(y'b) = (D(f) \circ \phi_{y'})(b)$ and hence $D(\phi_y) \circ \lambda(X) = D(\phi_{y'}) \circ D^2(f) \circ \lambda(X) = D(\phi_{y'}) \circ \lambda(X') \circ f$. Therefore we have

$$\begin{aligned} [(\Phi_0(X) \circ D(f))(y')](x) &= [(\lambda^*(B))^{-1} \circ D(\phi_y) \circ \lambda(X)](x) \\ &= [(\lambda^*(B))^{-1} \circ D(\phi_{y'}) \circ \lambda(X') \circ f](x) = [(\text{Char}_V f) \circ \Phi_0(X')(y')](x). \end{aligned}$$

Thus the commutativity of the first diagram of (10) is proved.

ii) The second diagram of (10) is proved similarly as in i).

IV. We shall now prove that a mapping $\omega: U \rightarrow V$ defined by

$$(11) \quad \omega(u) = [(\lambda^*(B))^{-1} \circ D(\phi_u) \circ \lambda(A)](1), \quad \text{for } u \in U$$

is an A - B -isomorphism and that the inverse of ω is given by

$$(12) \quad \omega^{-1}(v) = [D(\varphi_v) \circ \lambda(B)](1'), \quad \text{for } v \in V,$$

where $\phi_u: B \rightarrow U$ and $\varphi_v: A \rightarrow V$ are defined by $\phi_u(b) = ub$, $\varphi_v(a) = av$, for $a \in A$, $b \in B$.

Let $\chi: \text{Char}_V A \rightarrow V$ be a mapping defined $\chi(\alpha) = \alpha(1)$ for $\alpha \in \text{Char}_V A$. Then χ is clearly a B -isomorphism. Since $\Phi_0(A)$ maps $D(A)$ ($=U$) B -isomorphically onto $\text{Char}_V A$ as is already proved in II and $\omega = \chi \circ \Phi_0(A)$, we see that ω is a B -isomorphism. Hence we have only to prove that ω is an A -homomorphism.

For an element a of A we have $\phi_{au}(b) = (au)b = a(ub) = [D(\varphi_a) \circ \phi_u](b)$ and hence

$D(\psi_{au}) \circ \lambda(A) = D(\psi_u) \circ D^2(\varphi_a) \circ \lambda(A) = D(\psi_u) \circ \lambda(A) \circ \varphi_a$ since $D^2(\varphi_a) \circ \lambda(A) = D_2 D_1(\varphi_a) \circ \lambda(A) = \lambda(A) \circ \varphi_a$ by the property of dualities (cf. (0.4) in Introduction). Therefore we have

$$\begin{aligned} \omega(au) &= [(\lambda^*(B))^{-1} \circ D(\psi_{au}) \circ \lambda(A)](1) = [(\lambda^*(B))^{-1} \circ D(\psi_u) \circ \lambda(A) \circ \varphi_a](1) \\ &= [(\lambda^*(B))^{-1} \circ D(\psi_u) \circ \lambda(A)](a \cdot 1) = a [(\lambda^*(B))^{-1} \circ D(\psi_u) \circ \lambda(A)](1) \\ &= a\omega(u). \end{aligned}$$

Thus ω is an A -homomorphism and hence ω is an A - B -isomorphism. It follows from (7) that ω^{-1} is given by the formula (12).

V. Let τ be any A - B -homomorphism of U into V . Then the mapping

$$\text{Hom}(1, \tau): \text{Char}_U M \rightarrow \text{Char}_V M$$

defined by $[\text{Hom}(1, \tau)](\gamma) = \tau \circ \gamma: M \rightarrow V$ for $\gamma \in \text{Char}_U M$, $M \in \mathfrak{A}$ or \mathfrak{B} , is clearly an A -homomorphism or a B -homomorphism according as $M \in \mathfrak{B}$ or $M \in \mathfrak{A}$.

Since $\omega: U \rightarrow V$ is an A - B -isomorphism as is proved in IV, $\text{Hom}(1, \omega^{-1})$ is a B -isomorphism if it is applied to a left A -module X . Let us put

$$\vartheta(X) = \text{Hom}(1, \omega^{-1}) \circ \vartheta_0(X), \quad \vartheta(Y) = \vartheta_0(Y), \quad \text{for } X \in \mathfrak{A}, Y \in \mathfrak{B}.$$

Then $\vartheta(X): D(X) \rightarrow \text{Char}_U X$ is a B -isomorphism and

$$(13) \quad \vartheta(X)(y) = \omega^{-1} \circ (\lambda^*(B))^{-1} \circ D(\psi_y) \circ \lambda(X), \quad \text{for } y \in D(X),$$

$$(14) \quad [\vartheta(X)]^{-1}(\alpha) = (D(\omega \circ \alpha) \circ \lambda(B))(1'), \quad \text{for } \alpha \in \text{Char}_U X.$$

The diagrams

$$(15) \quad \begin{array}{ccc} D(X') & \xrightarrow{D(f)} & D(X) \\ \downarrow \vartheta(X') & & \downarrow \vartheta(X) \\ \text{Char}_U X' & \xrightarrow{\text{Char}_U f} & \text{Char}_U X \end{array} \quad \begin{array}{ccc} D(Y') & \xrightarrow{D(g)} & D(Y) \\ \downarrow \vartheta(Y') & & \downarrow \vartheta(Y) \\ \text{Char}_U Y' & \xrightarrow{\text{Char}_U g} & \text{Char}_U Y \end{array}$$

are commutative for any A -homomorphism $f: X \rightarrow X'$ ($X, X' \in \mathfrak{A}$) and for any B -homomorphism $g: Y \rightarrow Y'$ ($Y, Y' \in \mathfrak{B}$); this is seen from the commutativity of the diagrams (10).

VI. i) Let us put

$$(16) \quad \begin{aligned} \mu(X) &= (\text{Char}_U(\vartheta(X) \circ \lambda^*(X))^{-1}) \circ \vartheta(D(X)) \circ \lambda(X) \\ X &\xrightarrow{\lambda(X)} D^2(X) \xrightarrow{\vartheta(D(X))} \text{Char}_U D(X) \xrightarrow{\text{Char}_U(\vartheta(X) \circ \lambda^*(X))^{-1}} \text{Char}_U(\text{Char}_U X) \end{aligned}$$

for any left A -module X in \mathfrak{A} . Then $\mu(X)$ is an A -isomorphism.

Let x be any element of X and put $x' = [\lambda(X)](x)$. Then we have, for any element a of A , $\varphi_{x'}(a) = ax' = a(\lambda(X)(x)) = \lambda(X)(ax) = (\lambda(X) \circ \varphi_x)(a)$ and hence $D(\varphi_{x'}) \circ \lambda(D(X)) = D(\varphi_x) \circ D(\lambda(X)) \circ \lambda(D(X)) = D(\varphi_x) \circ \lambda^*(X)$. Since $[\vartheta(D(X)) \circ \lambda(X)](x) = \vartheta(D(X))(x') = (\lambda^*(A))^{-1} \circ D(\varphi_{x'}) \circ \lambda(D(X))$, for any element α of $\text{Char}_U X$ we have

$$\begin{aligned} [\mu(X)(x)](\alpha) &= [\lambda^*(A)^{-1} \circ D(\varphi_x) \circ \lambda^*(X)]((\lambda^*(X))^{-1} \circ \vartheta(X)^{-1})(\alpha) \\ &= [\lambda^*(A)^{-1} \circ D(\varphi_x) \circ D(\omega \circ \alpha) \circ \lambda(B)](1') \\ &= [\lambda^*(A)^{-1} \circ D(\omega \circ \alpha \circ \varphi_x) \circ \lambda(B)](1'). \end{aligned}$$

On the other hand, we have, for any element a of A , $(\omega \circ \alpha \circ \varphi_x)(a) = \omega(\alpha(ax)) = \alpha(\omega(ax)) = \varphi_{\omega(ax)}(a)$. Hence from (12) we obtain $[\mu(X)(x)](\alpha) = [\lambda^*(A)^{-1} \circ D(\varphi_{\omega(ax)}) \circ \lambda(B)](1') = (\lambda^*(A)^{-1}(\omega^{-1}(\omega\alpha(x))))$, that is,

$$(17) \quad [\mu(X)(x)](\alpha) = (\lambda^*(A)^{-1}(\alpha(x))), \quad \text{for } x \in X, \alpha \in \text{Char}_U X.$$

The value of $[\mu(X)(x)](\alpha)$ can be calculated in another way. Indeed, since $\varphi_{\omega(ax)}(a) = a[\lambda^*(B)^{-1} \circ D(\psi_{\alpha(x)}) \circ \lambda(A)](1) = [\lambda^*(B)^{-1} \circ D(\psi_{\alpha(x)}) \circ \lambda(A)](a)$, we have

$$\begin{aligned} [\mu(X)(x)](\alpha) &= [\lambda^*(A)^{-1} \circ D(\lambda(A)) \circ D^2(\psi_{\alpha(x)}) \circ D(\lambda^*(B)^{-1}) \circ \lambda(B)](1') \\ &= [\lambda^*(A)^{-1} \circ D(\lambda(A)) \circ \lambda(D(A)) \circ \psi_{\alpha(x)} \circ \lambda(B)^{-1} \circ D(\lambda^*(B)^{-1}) \circ \lambda(B)](1') \\ &= (\alpha(x))b_0, \end{aligned}$$

where we put

$$(18) \quad b_0 = [\lambda(B)^{-1} \circ D(\lambda^*(B)^{-1}) \circ \lambda(B)](1').$$

Thus we get

$$(19) \quad [\mu(X)(x)](\alpha) = (\alpha(x))b_0 = (\lambda^*(A)^{-1}(\alpha(x))).$$

Applying (19) to the case where $X=A$, $\alpha = \varphi_u: A \rightarrow U$ (i.e. $\alpha(x) = \varphi_u(x) = xu$ and $x=1$), we have $\lambda^*(A)^{-1}u = ub_0$ for any element u of U . This shows that $\lambda^*(A)^{-1}$ is an A -homomorphism of U into itself. Since $\lambda^*(A)^{-1}$ is a B -isomorphism, $\lambda^*(A)^{-1}$ is an A - B -isomorphism. Therefore $\text{Hom}(1, \lambda^*(A)) \circ \mu(X)$ is an A -isomorphism.

Since $\pi_U(X) = \text{Hom}(1, \lambda^*(A)) \circ \mu(X)$ by (17), we see that $\pi_U(X)$ is itself an A -isomorphism.

ii) Let us put

$$(20) \quad \mu(Y) = (\text{Char}_U(\mathcal{O}(Y) \circ \lambda^*(Y)^{-1}) \circ \mathcal{O}(D(Y)) \circ \lambda(Y)) \\ Y \xrightarrow{\lambda(Y)} D^2(Y) \xrightarrow{\mathcal{O}(D(Y))} \text{Char}_U D(Y) \xrightarrow{\text{Char}_U(\mathcal{O}(Y) \circ \lambda^*(Y)^{-1})} \text{Char}_U(\text{Char}_U Y)$$

for any right B -module Y in \mathfrak{B} . Then $\mu(Y)$ is a B -isomorphism.

Let y be any element of Y and let us put $y' = \lambda(Y)(y)$. Then we have $\psi_{y'} = \lambda(Y) \circ \psi_y$. Since $[\mathcal{O}(D(Y)) \circ \lambda(Y)](y) = [\mathcal{O}(D(Y))](y') = \omega^{-1} \circ \lambda^*(B)^{-1} \circ D(\psi_{y'}) \circ \lambda(D(Y))$, for $\beta \in \text{Char}_U Y$ we have

$$\begin{aligned} [\mu(Y)(y)](\beta) &= [\omega^{-1} \circ \lambda^*(B)^{-1} \circ D(\psi_y) \circ D(\lambda(Y)) \circ \lambda(D(Y))](\lambda^*(Y)^{-1} \circ \mathcal{O}(Y)^{-1}(\beta)) \\ &= [\omega^{-1} \circ \lambda^*(B)^{-1} \circ D(\psi_y) \circ D(\beta) \circ \lambda(A)](1) \\ &= [\omega^{-1} \circ \lambda^*(B)^{-1} \circ D(\beta \circ \psi_y) \circ \lambda(A)](1). \end{aligned}$$

Since $\beta \circ \psi_y = \psi_{\beta(y)}$ and $[\lambda^*(B)^{-1} \circ D(\psi_{\beta(y)}) \circ \lambda(A)](1) = \omega(\beta(y))$, we obtain

$$(21) \quad [\mu(Y)(y)](\beta) = \beta(y) = [\pi_U(Y)(y)](\beta).$$

Therefore $\pi_U(Y)$ is a B -isomorphism.

Thus the functorial duality D is equivalent to a duality defined by means of U -character modules and the duality with respect to U -character modules holds. This completes our proof.

As an immediate consequence of the above proof we have

THEOREM 1.2. *Let U be a two-sided A - B -module. If Char_U determines a duality between \mathfrak{A} and \mathfrak{B} , then the natural homomorphisms $\pi_U(X)$, $\pi_U(Y)$ defined in Theorem 1.1 are all isomorphisms for $X \in \mathfrak{A}$, $Y \in \mathfrak{B}$.*

REMARK 1.3. Let us denote by $Z(A)$ and $Z(B)$ the centers of A and B respectively. Then for any element a of $Z(A)$ the correspondence $u \rightarrow au$ defines an A -homomorphism of U into itself. Since $\pi_U(B)$ is a B -isomorphism, there exists an element b of B such that

$$(22) \quad au = ub \quad \text{for any element } u \text{ of } U.$$

The element b belongs to the center of B . It is easy to see that (22) defines a ring-isomorphism

$$(23) \quad \zeta_U: Z(A) \rightarrow Z(B).$$

In particular, if A and B are commutative, then A and B are ring-isomorphic. The element b_0 defined in (18) belongs to $Z(B)$ since $u \rightarrow ub_0$ is a B -isomorphism.

2. Dualities defined by means of character modules

In this section we shall determine a necessary and sufficient condition that the duality with respect to character modules hold between two categories of modules. We shall begin with some lemmas. In the following, by $\text{Ann}(L; M)$ we denote the set of all elements of M which are annihilated by every element of L with respect to a multiplication under consideration.

LEMMA 2.1. *Let A be a ring. Let U be an injective left A -module such that for any left ideal I of A with $I \cong A$ there exists a non-zero A -submodule of U which is A -homomorphic to A/I . Then for any left A -module X and for any A -submodule X_0 of X we have*

$$X_0 = \text{Ann}(\text{Ann}(X_0; \text{Char}_U X); X).$$

PROOF. Let x_0 be any element of X such that $x_0 \notin X_0$. Let us put $X_1 = Ax_0 + X_0$. If we assign to each element a of A the coset $\{ax_0\}$ modulo X_0 , we have an A -homomorphism σ of A onto X_1/X_0 . If we denote by I the kernel of σ , there is an A -isomorphism σ_0 of A/I onto X_1/X_0 . Since $X_1 \cong X_0$, we have $I \cong A$. Hence by assumption there exists an A -homomorphism τ of A/I into U such that $\tau(A/I) \cong 0$. If we denote by i the natural A -homomorphism of X_1 onto X_1/X_0 , the A -homomorphism $\tau\sigma_0^{-1}i$ maps X_1 into U such that $\tau\sigma_0^{-1}i(x_0) \cong 0$. Since U is injective, $\tau\sigma_0^{-1}i$ is extended to an A -homomorphism α of X into U such that $\alpha(X_0) = 0$, $\alpha(x_0) \cong 0$. Hence $x_0 \notin \text{Ann}(\text{Ann}(X_0; \text{Char}_U X); X)$. This proves our lemma.

LEMMA 2.2. *Besides the hypothesis of Lemma 2.1, assume further that U is a two-sided A - B -module where B is another ring. If for a left A -module X the natural homomorphism $\pi_U(X): X \rightarrow \text{Char}_U(\text{Char}_U X)$ is an A -isomorphism, then $\pi_U(X_0): X_0 \rightarrow \text{Char}_U(\text{Char}_U X_0)$ is also an A -isomorphism for any left A -submodule X_0 of X .*

Proof is obvious from Lemma 2.1.

LEMMA 2.3. *Let A and B be two rings and U a two-sided A - B -module. Let X be a left A -module and X_0 a left A -submodule of X . If the natural homomorphism $\pi_U(X)$, $\pi_U(X/X_0)$ are A -isomorphisms, then*

$$X_0 = \text{Ann}(\text{Ann}(X_0; \text{Char}_U X); X).$$

PROOF. We put $X' = \text{Ann}(\text{Ann}(X_0; \text{Char}_U X); X)$. We have clearly $X_0 \subseteq X'$. Let x_0 be any element of X such that $x_0 \notin X_0$. Then the coset $\{x_0\}$ modulo X_0 is not zero in X/X_0 . Since $\pi_U(X/X_0)$ is an A -isomorphism, $\{x_0\}$ determines a non-zero B -homomorphism of $\text{Ann}(X_0; \text{Char}_U X)$ into U . Hence there exists $\alpha \in \text{Ann}(X_0; \text{Char}_U X)$ such that $\alpha(x_0) \neq 0$. Thus we have $x_0 \in X'$. Therefore $X_0 = X'$.

Throughout this section we shall assume that A and B are rings and U is a two-sided A - B -module. We denote by $\mathfrak{L}[A, U]$ the class of left A -modules which are obtained from the left A -modules A and U by taking finite direct sums, submodules and quotient modules; $\mathfrak{R}[B, U]$ denotes the class of right B -modules which are obtained from the right B -modules B and U by the same operations described above.

Now we shall prove the following theorem.

THEOREM 2.4. *The following conditions are equivalent.*

- I. $\pi_U(X)$, $\pi_U(Y)$ are respectively an A -isomorphism and a B -isomorphism for each module X in $\mathfrak{L}[A, U]$ and each module Y in $\mathfrak{R}[B, U]$.
- II. a) If a left A -module X is A -isomorphic to a quotient module of A or U , $\pi_U(X)$ is an A -isomorphism.
b) If a right B -module Y is B -isomorphic to a quotient module of B or U , $\pi_U(Y)$ is a B -isomorphism.
- III. a) $\pi_U(A)$ and $\pi_U(B)$ are respectively an A -isomorphism and a B -isomorphism; that is, A is isomorphic to the B -endomorphism ring of U and B is inverse-isomorphic to the A -endomorphism ring of U by the correspondences $a \rightarrow \varphi_a$, $b \rightarrow \psi_b$ where $\varphi_a(u) = au$, $\psi_b(u) = ub$.
b) U is injective as a left A -module, and for any left ideal I of A with $I \neq A$ there exists a non-zero A -submodule of U which is A -homomorphic to A/I .
c) U is injective as a right B -module, and for any right ideal J of B with $J \neq B$ there exists a non-zero B -submodule of U which is B -homomorphic to B/J .

PROOF. i) I \rightarrow II is obvious.

ii) III \rightarrow I. Assume III. Then by III a) $\pi_U(U)$ is an A -isomorphism or a B -isomorphism according as U is considered as a left A -module or as a right B -module. Let X be a left A -module such that $\pi_U(X)$ is an A -isomorphism. Put $Y = \text{Char}_U X$. Then $\pi_U(Y)$ is also a B -isomorphism. Let X_0 be any A -submodule of X , and put $Y_0 = \text{Ann}(X_0; Y)$. Then from III c) it follows by Lemma 2.2 that $\pi_U(Y_0)$ is a B -isomorphism. Since $X_0 = \text{Ann}(Y_0; X)$ and $\text{Char}_U Y_0 \approx X/X_0$, we see that $\pi_U(X/X_0)$ is also an A -isomorphism. Thus $\pi_U(X_0)$, $\pi_U(X/X_0)$ are A -isomorphisms for any A -submodule X_0 of X , and hence condition I holds.

iii) $II \rightarrow III$. Assume II. Then III a) holds clearly. Let I be any left ideal of A . In case it is necessary to emphasize that U is considered as a left A -module (resp. a right B -module), we write ${}_A U$ (resp. U_B). We put $W = \text{Ann}(I; U_B)$ where the pairing of A and U_B to U is considered. Since $\pi_U(A)$ and $\pi_U(A/I)$ are A -isomorphisms, by Lemma 2.3 we have $I = \text{Ann}(W; A)$. On the other hand, $\pi_U(U/W)$ is a B -isomorphism by II b), and $\text{Char}_U(U/W) \approx I$. Therefore any A -homomorphism of I into U can be extended to an A -homomorphism of A into U . This shows that U is injective.

Let I be any left ideal of A such that $I \not\approx A$. Then A/I is A -isomorphic to $\text{Char}_U(\text{Char}_U(A/I))$ and hence $\text{Char}_U(A/I) \not\approx 0$. We take a non-zero element α from $\text{Char}_U(A/I)$. Then we have $\alpha(A/I) \not\approx 0$. Thus U contains a non-zero A -submodule $\alpha(A/I)$ which is A -homomorphic to A/I . Therefore condition III b) holds. Since III c) is proved similarly, condition III holds.

THEOREM 2.5. *Suppose that the duality with respect to U -character modules holds between $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$. If a left A -module X in $\mathfrak{L}[A, U]$ is projective, then $\text{Char}_U X$ is an injective right B -module.*

PROOF. We put $Y = \text{Char}_U X$. Let J be any right ideal of B and consider any B -homomorphism g of J into Y . Let $j: J \rightarrow B$ be an injection. Since U is injective as a left A -module and a right B -module, the functor Char_U is exact (cf. [3]). Hence $\text{Char}_U j: \text{Char}_U B \rightarrow \text{Char}_U J$ is onto. Since $X \approx \text{Char}_U Y$ and X is projective, there exists an A -homomorphism $f: \text{Char}_U Y \rightarrow \text{Char}_U B$ such that $\text{Char}_U g = (\text{Char}_U j) \circ f$. Then if we define a B -homomorphism $g': B \rightarrow Y$ by putting $g' = \pi_U(Y)^{-1} \circ \text{Char}_U f \circ \pi_U(B)$ we have $g = g' \circ j$. This shows that Y is injective.

THEOREM 2.6. *Suppose that the duality with respect to U -character modules holds between $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$. Then the lattice of two-sided ideals of A is isomorphic to the lattice of two-sided ideals of B .*

PROOF. Let I be any two-sided ideal of A . Then $V = \text{Ann}(I; U)$ is an A - B -submodule of U since $a'(av) = (a'a)v$ for $a' \in I, a \in A, v \in V$. Conversely, if V' is a two-sided A - B -submodule of U , then $I' = \text{Ann}(V'; A)$ is a two-sided ideal of A . Hence the correspondence $I \rightarrow \text{Ann}(I; U) = V$ establishes an isomorphism between the lattice of two-sided ideals of A and the lattice of all A - B -submodules of U . Thus Theorem 2.6 is proved.

Let X be a left A -module and Y a right B -module. Suppose that to any element x of X and to any element y of Y an element u of a two-sided A - B -module U is assigned such that if we write $u = (x, y)$, then (x, y) is additive with respect to x and y , and $(ax, y) = a(x, y), (x, yb) = (x, y)b, a \in A, b \in B$. If $\text{Ann}(X; Y) = 0$ and $\text{Ann}(Y; X) = 0$, then X and Y are said to form an *orthogonal pair* to U . The following theorem is now easily proved by Theorem 2.4.

THEOREM 2.7. *Suppose that the duality with respect to U -character modules holds between $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$. If $X \in \mathfrak{L}[A, U]$ and $Y \in \mathfrak{R}[B, U]$ form an orthogonal pair to U , then each of X and Y is isomorphic to the U -character module of the other.*

The following theorem is essentially proved in [17].

THEOREM 2.8. *Let U be a two-sided A - B -module. Suppose that the U -character*

module of every simple left A -module and that of every simple right B -module are simple. If a left A -module X and a right B -module Y form an orthogonal pair to U and one of X and Y has a composition series, then the other has also a composition series, and for every A -submodule X_0 of X and every B -submodule Y_0 of Y , $\pi_U(X/X_0)$, $\pi_U(Y/Y_0)$ are isomorphisms and the annihilator relations $\text{Ann}(\text{Ann}(X_0; Y); X) = X_0$, $\text{Ann}(\text{Ann}(Y_0; X); Y) = Y_0$ hold.

3. Isomorphisms between two categories of modules

The following theorem is an analogue of Theorem 1.1.

THEOREM 3.1. *Let A and B be two rings. Let \mathfrak{A} be a class of left A -modules and \mathfrak{B} a class of left B -modules; we assume that \mathfrak{A} contains A as a left A -module and \mathfrak{B} contains B as a left B -module. Let T_1 be an isomorphism from the category \mathfrak{A} to the category \mathfrak{B} and T_2 an isomorphism from \mathfrak{B} to \mathfrak{A} such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor. Then the functors T_1 and T_2 are naturally equivalent to functors $\text{Hom}_A(V, \)$ and $\text{Hom}_B(U, \)$ respectively with a suitable two-sided A - B -module V and a suitable two-sided B - A -module U .*

Let λ_1 be a natural equivalence from T_2T_1 to the identity functor and λ_2 a natural equivalence from T_1T_2 to the identity functor; the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\lambda_1(X)} & T_2T_1(X) & & Y & \xrightarrow{\lambda_2(Y)} & T_1T_2(Y) \\ \downarrow f & & \downarrow T_2T_1(f) & & \downarrow g & & \downarrow T_1T_2(g) \\ X' & \xrightarrow{\lambda_1(X')} & T_2T_1(X') & & Y' & \xrightarrow{\lambda_2(Y')} & T_1T_2(Y') \end{array}$$

are commutative for any A -homomorphism $f: X \rightarrow X'$ and any B -homomorphism $g: Y \rightarrow Y'$, and $\lambda_1(X)$, $\lambda_2(Y)$ are isomorphisms ($X, X' \in \mathfrak{A}$; $Y, Y' \in \mathfrak{B}$).

I. Let us put

$$(24) \quad U = T_1(A).$$

Then U is a left B -module. For any element a of A we define an A -homomorphism $\varphi_a: A \rightarrow A$ by $\varphi_a(x) = xa$. We set

$$(25) \quad ua = T_1(\varphi_a)u, \quad u \in U, a \in A.$$

Then it is easy to see that U is a two-sided B - A -module; e.g. we have $(bu)a = T_1(\varphi_a)(bu) = b(T_1(\varphi_a)u) = b(ua)$ for $a \in A$, $b \in B$, $u \in U$.

Let us put

$$(26) \quad V = T_2(B).$$

If we put

$$(27) \quad vb = T_2(\psi_b)v, \quad v \in V, b \in B,$$

it is seen that V is a two-sided A - B -module, where $\psi_b: B \rightarrow B$ is defined by $\psi_b(y) = yb$.

II. i) For each module X in \mathfrak{A} we define a B -homomorphism $\phi_1(X): T_1(X) \rightarrow \text{Hom}_A(V, X)$ by

$$(28) \quad \Phi_1(X)(y) = \lambda_1(X)^{-1} \circ T_2(\psi_y) \circ \lambda_1^*(B)^{-1}, \quad y \in T_1(X),$$

where we put

$$(29) \quad \lambda_1^*(Y) = \lambda_1(T_2(Y))^{-1} \circ T_2(\lambda_2(Y)), \quad \text{for } Y \in \mathfrak{B}.$$

and $\psi_y: B \rightarrow T_1(X)$ is defined by $\psi_y(b) = by$. We shall prove that $\Phi_1(X)$ is a B -isomorphism. To prove this we shall consider a B -homomorphism $\Psi_1(X): \text{Hom}_A(V, X) \rightarrow T_1(X)$ defined by

$$\Psi_1(X)(\alpha) = (T_1(\alpha) \circ \lambda_2(B))(1'), \quad \text{for } \alpha \in \text{Hom}_A(V, X).$$

If we put $y = \Psi_1(X)(\alpha) = (T_1(\alpha) \circ \lambda_2(B))(1')$, then $\psi_y = T_1(\alpha) \circ \lambda_2(B)$ and hence $T_2(\psi_y) = T_2 T_1(\alpha) \circ T_2(\lambda_2(B))$. Therefore we have

$$\begin{aligned} \Phi_1(X)(y) &= \lambda_1(X)^{-1} \circ T_2 T_1(\alpha) \circ T_2(\lambda_2(B)) \circ \lambda_1^*(B)^{-1} \\ &= \alpha \circ \lambda_1(T_2(B))^{-1} \circ T_2(\lambda_2(B)) \circ \lambda_1^*(B)^{-1} = \alpha. \end{aligned}$$

This shows that $\Phi_1(X) \circ \Psi_1(X) = 1$.

Let $\alpha \in \text{Hom}_A(V, X)$ and $v \in V$. Then by the definition of multiplication of α with elements b of B on the left (cf. Cartan and Eilenberg [3, p. 22]), we have $(b\alpha)(v) = \alpha(vb)$. Since $\alpha(vb) = \alpha(T_2(\psi_b)v) = (\alpha \circ T_2(\psi_b))(v)$, we have $b\alpha = \alpha \circ T_2(\psi_b)$. Therefore we have

$$\begin{aligned} \Psi_1(X)(b\alpha) &= [T_1(b\alpha) \circ \lambda_2(B)](1') = [T_1(\alpha \circ T_2(\psi_b)) \circ \lambda_2(B)](1') \\ &= [T_1(\alpha) \circ T_1 T_2(\psi_b) \circ \lambda_2(B)](1') = [T_1(\alpha) \circ \lambda_2(B) \circ \psi_b](1') \\ &= (T_1(\alpha) \circ \lambda_2(B))(b \cdot 1') = b(T_1(\alpha) \circ \lambda_2(B))(1') = b[\Psi_1(X)(\alpha)]. \end{aligned}$$

This shows that $\Psi_1(X)$ is a B -homomorphism. From the fact that $\Phi_1(X) \circ \Psi_1(X) = 1$ it follows that $\Phi_1(X)$ is onto. On the other hand, $\Phi_1(X)$ is obviously one-to-one. Thus $\Phi_1(X)$ is a B -isomorphism and $\Psi_1(X) = \Phi_1(X)^{-1}$.

ii) For each module Y in \mathfrak{B} we can define an A -isomorphism $\Phi_2(Y): T_2(Y) \rightarrow \text{Hom}_B(U, Y)$ by

$$\Phi_2(Y)(x) = \lambda_2(Y)^{-1} \circ T_1(\varphi_x) \circ \lambda_2^*(A)^{-1}, \quad \text{for } x \in T_2(Y),$$

where

$$\lambda_2^*(X) = \lambda_2(T_1(X))^{-1} \circ T_1(\lambda_1(X)), \quad \text{for } X \in \mathfrak{A}$$

and $\varphi_x: A \rightarrow T_2(Y)$ is defined by $\varphi_x(a) = ax$. The inverse of $\Phi_2(Y)$ is given by

$$\Phi_2(Y)^{-1}(\beta) = [T_2(\beta) \circ \lambda_1(A)](1), \quad \text{for } \beta \in \text{Hom}_B(U, Y).$$

III. The diagrams

$$\begin{array}{ccc} T_1(X) & \xrightarrow{\Phi_1(X)} & \text{Hom}_A(V, X) & & T_2(Y) & \xrightarrow{\Phi_2(Y)} & \text{Hom}_B(U, Y) \\ \downarrow T_1(f) & & \downarrow \text{Hom}(1, f) & & \downarrow T_2(g) & & \downarrow \text{Hom}(1, g) \\ T_1(X') & \xrightarrow{\Phi_1(X')} & \text{Hom}_A(V, X') & & T_2(Y') & \xrightarrow{\Phi_2(Y')} & \text{Hom}_B(U, Y') \end{array}$$

are commutative for any A -homomorphism $f: X \rightarrow X'$ and any B -homomorphism

$g: Y \rightarrow Y'$ (for the notation $\text{Hom}(1, f)$ cf. [3, p. 20]). To prove this, let $y \in T_1(X)$ and put $y' = T_1(f)(y)$. Then $\psi_{y'} = T_1(f) \circ \psi_y$. Hence we have

$$\begin{aligned} [\Phi_1(X) \circ T_1(f)](y) &= \lambda_1(X')^{-1} \circ T_2(\psi_{y'}) \circ \lambda_1^*(B)^{-1} \\ &= \lambda_1(X')^{-1} \circ T_2 T_1(f) \circ T_2(\psi_y) \circ \lambda_1^*(B)^{-1} = f \circ \lambda_1(X)^{-1} \circ T_2(\psi_y) \circ \lambda_1^*(B)^{-1} \\ &= [\text{Hom}(1, f) \circ \Phi_1(X)](y). \end{aligned}$$

Thus the commutativity of the first diagram is proved. The commutativity of the second diagram is proved similarly.

Now our Theorem 3.1 follows readily from I to III.

IV. $\text{Hom}_A(V, A)$ is a left B -module. Since A is a right A -module $\text{Hom}_A(V, A)$ is a two-sided B - A -module. We shall prove that

$$\Phi_1(A): U \rightarrow \text{Hom}_A(V, A)$$

is a B - A -isomorphism. To prove this, let $a \in A$ and $u \in U$. Then $\Phi_1(A)(u) = \lambda_1(A)^{-1} \circ T_2(\psi_u) \circ \lambda_1^*(B)^{-1}$ where $\psi_u: B \rightarrow U$ is defined by $\psi_u(b) = bu$. Hence we have

$$\begin{aligned} \Phi_1(A)(ua) &= \lambda_1(A)^{-1} \circ T_2(\psi_{ua}) \circ \lambda_1^*(B)^{-1} = \lambda_1(A)^{-1} \circ T_2(T_1(\varphi_a) \circ \psi_u) \circ \lambda_1^*(B)^{-1} \\ &= \lambda_1(A)^{-1} \circ T_2 T_1(\varphi_a) \circ T_2(\psi_u) \circ \lambda_1^*(B)^{-1} = \varphi_a \circ \lambda_1(A)^{-1} \circ T_2(\psi_u) \circ \lambda_1^*(B)^{-1} \\ &= [\Phi_1(A)(u)]a. \end{aligned}$$

This shows that $\Phi_1(A)$ is a right A -isomorphism. Similarly we can prove that $\Phi_2(B): V \rightarrow \text{Hom}_B(U, B)$ is an A - B -isomorphism.

V. Let us consider an A -isomorphism

$$\mu(X): X \rightarrow \text{Hom}_B(U, \text{Hom}_A(V, X))$$

defined by

$$\mu(X) = \text{Hom}(1, (\Phi_1(X) \circ \lambda_2^*(X)^{-1})) \circ \Phi_2(T_1(X)) \circ \lambda_1(X).$$

Let $x \in X$ and $x' = \lambda_1(X)(x)$. Then

$$\begin{aligned} \Phi_2(T_1(X))(x') &= \lambda_2(T_1(X))^{-1} \circ T_1(\varphi_{x'}) \circ \lambda_2^*(A)^{-1} \\ &= \lambda_2(T_1(X))^{-1} \circ T_1(\lambda_1(X) \circ \varphi_x) \circ \lambda_2^*(A)^{-1} \\ &= \lambda_2(T_1(X))^{-1} \circ T_1(\lambda_1(X)) \circ T_1(\varphi_x) \circ \lambda_2^*(A)^{-1}. \end{aligned}$$

Hence for $u \in U$ we have

$$[\mu(X)(x)](u) = [\Phi_1(X) \circ T_1(\varphi_x) \circ \lambda_2^*(A)^{-1}](u).$$

On the other hand, if we put $y = [T_1(\varphi_x) \circ \lambda_2^*(A)^{-1}](u)$, we have $y \in T_1(X)$ and $\psi_y = T_1(\varphi_x) \circ \lambda_2^*(A)^{-1} \circ \psi_u$. Therefore we have

$$\begin{aligned} [\mu(X)(x)](u) &= \lambda_1(X)^{-1} \circ T_2(\psi_y) \circ \lambda_1^*(B)^{-1} \\ &= \lambda_1(X)^{-1} \circ T_2 T_1(\varphi_x) \circ T_2(\lambda_2^*(A)^{-1}) \circ T_2(\psi_u) \circ \lambda_1^*(B)^{-1} \\ &= \varphi_x \circ \lambda_1(A)^{-1} \circ T_2(\lambda_2^*(A)^{-1}) \circ \lambda_1(A) \circ \lambda_1(A)^{-1} \circ T_2(\psi_u) \circ \lambda_1^*(B)^{-1} \\ &= [\varphi_x \circ \{\lambda_1(A)^{-1} \circ T_2(\lambda_2^*(A)^{-1}) \circ \lambda_1(A)\}] \Phi_1(A)(u). \end{aligned}$$

Hence for $v \in V$ we have

$$(30) \quad [[\mu(X)(x)](u)](v) = [[\Phi_1(A)(u)](v)](a_0x), \quad \text{for } x \in X,$$

where

$$a_0 = [\lambda_1(A)^{-1} \circ T_2(\lambda_2^*(A)^{-1}) \circ \lambda_1(A)](1).$$

Since $\mu(X)(ax) = a[\mu(X)(x)]$, we have

$$(31) \quad [[\Phi_1(A)(u)](v)](aa_0x) = [[\Phi_1(A)(u)](v)](a_0ax), \quad a \in A, x \in X.$$

VI. We define a mapping

$$(32) \quad \omega: V \otimes_B U \rightarrow A$$

by putting $\omega(v \otimes u) = [\Phi_1(A)(u)](v)$. Then it is easy to see that ω is actually single valued. Since $\omega[a(v \otimes u)] = \omega(av \otimes u) = [\Phi_1(A)(u)](av) = a[[\Phi_1(A)(u)](v)] = a(\omega(v \otimes u))$ and $\omega[(v \otimes u)a] = \omega(v \otimes ua) = [\Phi_1(A)(ua)](v) = [[\Phi_1(A)(u)](a)](v) = [[\Phi_1(A)(u)](v)]a = [\omega(v \otimes u)]a$, ω is an A - A -homomorphism.

Therefore the image of ω is a two-sided ideal I of A . If every quotient module of the left A -module A belongs to \mathfrak{A} , then we have $I=A$. To prove this, assume that $I \not\approx A$, and put $X=A/I$. Then $\mu(X)$ must be an isomorphism. On the other hand, for any element x of X we have $Ix=0$ and hence $[[\mu(X)(x)](u)](v) = [\omega(v \otimes u)](a_0x) = 0$. This shows that $\mu(X)$ fails to be one-to-one. Therefore we conclude that ω is onto.

VII. The B -endomorphism ring of the left B -module B is isomorphic to the A -endomorphism ring of V by the correspondence $\alpha \rightarrow T_2(\alpha)$. Hence B is inverse-isomorphic to the A -endomorphism ring of V by the correspondence $b \rightarrow \phi_b$ where $\phi_b(v) = vb$, $v \in V$.

As is proved in VI, the image of ω coincides with A . Hence there exist a finite number of elements $v_i \in V$, $u_i \in U$, $i=1, \dots, n$ such that $\sum \omega(v_i \otimes u_i) = 1$. Therefore we have

$$\sum_{i=1}^n \alpha_i(v_i) = 1$$

where $\alpha_i \in \text{Hom}_A(V, A)$ are defined by the formula $\alpha_i(v) = \omega(v \otimes u_i)$. Hence by virtue of Lemma 3.3 below we see that V is a finitely generated, projective, right B -module and that A is isomorphic to the B -endomorphism ring of V by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(v) = av$, $v \in V$.

VIII. We assume further that every quotient module of the left B -module B belongs to \mathfrak{B} . Then we can prove similarly as in VII that U is a finitely generated, projective, right A -module and that B is isomorphic to the A -endomorphism ring of U by the correspondence $b \rightarrow \psi_b$ where $\psi_b(u) = bu$, $u \in U$.

From the relation $\sum \omega(v_i \otimes u_i) = 1$ we obtain

$$\sum_{i=1}^n \beta_i(u_i) = 1$$

where $\beta_i \in \text{Hom}_A(U, A)$ are defined by the formula $\beta_i(u) = \omega(v_i \otimes u)$. Hence by Lemma 3.3 below we see that U is a finitely generated, projective, left B -module

and that A is inverse-isomorphic to the B -endomorphism ring of U by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(u) = ua$, $u \in U$; the latter fact is proved also by an analogous method as in the first part of VII.

Thus we have proved the following theorem.

THEOREM 3.2. *Besides the assumption of Theorem 3.1 we assume further that every quotient module of the left A -module A belongs to \mathfrak{A} and that every quotient module of the left B -module B belongs to \mathfrak{B} . Then U (resp. V) is finitely generated and projective as a left B -module (resp. A -module) and as a right A -module (resp. B -module). The ring A is inverse-isomorphic (resp. isomorphic) to the B -endomorphism ring of U (resp. V) by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(u) = ua$ (resp. $\varphi_a(v) = av$). The ring B is isomorphic (resp. inverse-isomorphic) to the A -endomorphism ring of V (resp. U) by the correspondence $b \rightarrow \phi_b$ where $\phi_b(v) = vb$ (resp. $\phi_b(u) = bu$).*

LEMMA 3.3. *Let V be a two-sided A - B -module. Suppose that B is inverse-isomorphic to the A -endomorphism ring of V by the correspondence $b \rightarrow \phi_b$ where $\phi_b(v) = vb$. Then the following two conditions are equivalent.*

I. V is a finitely generated, projective, right B -module and A is isomorphic to the B -endomorphism ring of V by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(v) = av$, $v \in V$.

II. There exist a finite number of elements $v_i \in V$, $\alpha_i \in \text{Hom}_A(V, A)$, $i = 1, \dots, n$ such that

$$\sum_{i=1}^n \alpha_i(v_i) = 1.$$

PROOF. i) I \rightarrow II. Let $X = Bx_1 \oplus \dots \oplus Bx_n$ and $Y = y_1B \oplus \dots \oplus y_nB$ be free left and right B -modules. Let $\alpha = (\alpha_{ij})$ be a matrix of type (n, n) with coefficients in B . We set

$$\begin{aligned} (\sum b_i x_i) \alpha &= \sum_{i,j} b_i \alpha_{ij} x_j, & \alpha (\sum y_j c_j) &= \sum_{k,j} y_k \alpha_{kj} c_j \\ \omega'(\sum b_i x_i, \sum y_j c_j) &= \sum b_i c_i, \\ \omega(\sum y_j c_j, \sum b_i x_i) &= (c_i b_k)_{i,k} \in (B)_n, \end{aligned}$$

where $b_i, c_j \in B$ and $(c_i b_k)_{i,k}$ means a matrix with $c_i b_k$ as its (i, k) -coefficients ($i, k = 1, \dots, n$) and $(B)_n$ means the full matrix ring over B . Then we have

$$\begin{aligned} \omega'(x\alpha, y) &= \omega'(x, \alpha y), & \omega'(bx, yc) &= b\omega'(x, y)c, \\ x\omega(y, x') &= \omega'(x, y)x', & y\omega'(x, y') &= \omega(y, x)y', \\ \omega(\alpha y, x\beta) &= \alpha\omega(y, x)\beta, & \omega(yb, x) &= \omega(y, bx), \end{aligned}$$

where $x, x' \in X$; $y, y' \in Y$; $\alpha, \beta \in (B)_n$; $b, c \in B$. It is also clear that ω' and ω are additive with respect to each variable.

Let β be an idempotent in $(B)_n$. Then since $\sum \omega(y_i, x_i) = 1$, we have $\sum \omega(\beta y_i, x_i \beta) = \beta^2 = \beta$, and the B -endomorphism ring of βY is isomorphic to $\beta(B)_n \beta$. Hence if we put $A = \beta(B)_n \beta$ then βY is a left A -module and $\sum \alpha_i(\beta y_i) = \beta$ where $\alpha_i(y) = \omega(y, x_i \beta)$ for $y \in \beta Y$.

Any finitely generated, projective, right B -module is B -isomorphic to βY if we choose a suitable integer n and a suitable idempotent β . Hence the above consideration shows that I implies II.

ii) II \rightarrow I. Assume II. The mapping $v \rightarrow \alpha_i(v)v'$ is an A -endomorphism of V for each i and each element v' of V , and hence there exists a uniquely determined element $\beta_i(v')$ of B such that $v\beta_i(v') = \alpha_i(v)v'$. From the assumption that $\sum \alpha_i(v_i) = 1$ it follows that

$$v = \sum \alpha_i(v_i)v = \sum v_i\beta_i(v).$$

If we denote by β_i the mapping $v \rightarrow \beta_i(v)$, then β_i is a B -homomorphism of V into B . Therefore by a theorem of Cartan and Eilenberg [3, p. 132], V is a finitely generated, projective, right B -module.

Let β be any B -endomorphism of V . We set $\alpha_0 = \sum \alpha_i(\beta(v_i))$. Then we have $\alpha_0 v = \sum \alpha_i(\beta(v_i))v = \sum \beta(v_i)\beta_i(v) = \beta(\sum v_i\beta_i(v)) = \beta(v)$. This proves that II implies I.

The following theorem shows how we can construct an isomorphism between categories of modules.

THEOREM 3.4. *Let V be a two-sided A - B -module satisfying the two conditions:*

a) *V is projective and finitely generated as a left A -module and as a right B -module;*

b) *A is isomorphic to the B -endomorphism ring of V and B is inverse-isomorphic to the A -endomorphism ring of V by the correspondences $a \rightarrow \varphi_a$, $b \rightarrow \psi_b$ where $\varphi_a(v) = av$, $\psi_b(v) = vb$, $v \in V$.*

Then the natural homomorphisms

$$\rho(X): V \otimes_B \text{Hom}_A(V, X) \rightarrow X, \quad \nu(Y): Y \rightarrow \text{Hom}_A(V, V \otimes_B Y)$$

defined by the formulae $\rho(X)(v \otimes \alpha) = \alpha(v)$, $[\nu(Y)(y)](v) = v \otimes y$ where $v \in V$, $y \in Y$, $\alpha \in \text{Hom}_A(V, X)$ are respectively an A -isomorphism and a B -isomorphism for any left A -module X and any left B -module Y . The functors $T_1(X) = \text{Hom}_A(V, X)$ and $T_2(Y) = V \otimes_B Y$ are isomorphisms between the category of all left A -modules and that of all left B -modules.

PROOF. By [3, p. 120, Prop. 5], it follows that the mapping $\sigma(X): V \otimes_B \text{Hom}_A(V, X) \rightarrow \text{Hom}_A(\text{Hom}_B(V, V), X)$ defined by $[\sigma(X)(v \otimes \alpha)](\gamma) = \alpha(\gamma(v))$, $v \in V$, $\alpha \in \text{Hom}_A(V, X)$, $\gamma \in \text{Hom}_B(V, V)$, is an A -isomorphism. The mappings $\xi(X): \text{Hom}_A(\text{Hom}_B(V, V), X) \rightarrow \text{Hom}_A(A, X)$ and $\eta(X): \text{Hom}_A(A, X) \rightarrow X$ which are defined by $[\xi(X)(f)](a) = f(\varphi_a)$, $[\eta(X)](g) = g(1)$ where $\varphi_a(v) = av$, $a \in A$, $v \in V$, are clearly A -isomorphisms. Since $\rho(X) = \eta(X) \circ \xi(X) \circ \sigma(X)$, $\rho(X)$ is itself an A -isomorphism.

Similarly as in [3, p. 120] we can prove that the mapping $\tau: \text{Hom}_A(X, V) \otimes_B Y \rightarrow \text{Hom}_A(X, V \otimes_B Y)$ defined by $[\tau(\alpha \otimes y)](x) = \alpha(x) \otimes y$ is an isomorphism if X is a projective, finitely generated left A -module. Hence $\tau(Y): \text{Hom}_A(V, V) \otimes_B Y \rightarrow \text{Hom}_A(V, V \otimes_B Y)$ defined by $[\tau(Y)(\gamma \otimes y)](v) = \gamma(v) \otimes y$ is a B -isomorphism. Since $\text{Hom}_A(V, V) \approx B$ and $B \otimes_B Y \approx Y$, we can prove similarly as in the first part of the proof that $\nu(Y)$ is a B -isomorphism. This completes our proof.

We shall now prove the following theorem.

THEOREM 3.5. *Let \mathfrak{A} be a class of left A -modules containing A as a left A -module. Let T_1 and T_2 be isomorphisms from the category \mathfrak{A} onto itself such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor. Suppose that there exists an A -isomorphism $\sigma: A \rightarrow T_2(A)$. Then there exists a ring-automorphism θ of A such that*

$$(33) \quad \sigma^{-1} \circ T_2(\varphi_a) \circ \sigma = \varphi_{\theta(a)} ;$$

(33) is equivalent to

$$(34) \quad \sigma^{-1}(va) = [\sigma^{-1}(v)](\theta(a)) , \quad \text{for } v \in V, a \in A ,$$

where $V = T_2(A)$ and the multiplication of elements of V with elements of A on the right is defined by (25) with $B = A$. The functor T_1 is naturally equivalent to the identity functor if and only if θ is an inner automorphism.

PROOF. The first assertion is obvious. Suppose that θ is an inner automorphism; there exists an element a_0 of A such that $\theta(a) = a_0aa_0^{-1}$. Hence if we put $\sigma_0(v) = \sigma(v)a_0^{-1}$, we have $\sigma_0(va) = \sigma(va)a_0^{-1} = (\sigma(v))(a_0^{-1}aa_0)a_0^{-1} = \sigma_0(v)a$, and hence $\sigma_0: A \rightarrow V$ is an isomorphism as two-sided A - A -modules. This shows that T_1 is naturally equivalent to the identity functor.

Conversely, suppose that T_1 (and hence T_2) is naturally equivalent to the identity functor. Let θ be a natural equivalence from T_2 to 1. Then we have the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\sigma} & T_2(A) & \xrightarrow{\theta(A)} & A \\ \downarrow \varphi_{\theta(a)} & & \downarrow T_2(\varphi_a) & & \downarrow \varphi_a \\ A & \xrightarrow{\sigma} & T_2(A) & \xrightarrow{\theta(A)} & A \end{array}$$

If we put $\sigma_0 = \theta(A) \circ \sigma$, σ_0 is an A -isomorphism and hence there exists an element a_0 of A such that $\sigma_0(x) = \varphi_{a_0}(x)$ for $x \in A$. Thus we have $\varphi_{\theta(a)} = \varphi_{a_0^{-1}} \circ \varphi_a \circ \varphi_{a_0}$, that is, $\theta(a) = a_0aa_0^{-1}$.

REMARK 3.6. In Theorem 3.5 let $\Gamma(X): X \rightarrow \text{Hom}_A(V, X)$ be a mapping defined by $\Gamma(X)(x) = \varphi_x \circ \sigma^{-1}$ for $x \in X$. Then $\Gamma(X)$ is one-to-one and onto. For $v \in V$ we have

$$\begin{aligned} [\Gamma(X)(\theta(a)x)](v) &= [\varphi_{\theta(a)x} \circ \sigma^{-1}](v) = [\sigma^{-1}(v)](\theta(a)x) \\ &= [\sigma^{-1}(v)\theta(a)]x = [\sigma^{-1}(va)]x = (\varphi_x \circ \sigma^{-1})(va) \end{aligned}$$

and hence

$$(35) \quad \Gamma(X)(\theta(a)x) = a[\Gamma(X)(x)] .$$

From this we can conclude that for any given ring-automorphism θ of A there exists an isomorphism T_1 from \mathfrak{A} to \mathfrak{A} such that the automorphism θ defined by (33) for this T_1 is identical with the given θ .

4. Equivalence of dualities

Let A and B be two rings. Let \mathfrak{A} be a class of left A -modules and \mathfrak{B} a

class of right B -modules such that \mathfrak{A} contains A as a left A -module and \mathfrak{B} contains B as a right B -module. Let $D=(D_1, D_2)$ and $E=(E_1, E_2)$ be two dualities between \mathfrak{A} and \mathfrak{B} . Then by Theorem 1.1 D and E are respectively equivalent to dualities defined by means of U -character modules and V -character modules where U and V are two-sided A - B -modules constructed from D and E similarly as in the proof of Theorem 1.1. Then the following theorem holds.

THEOREM 4.1. *In order that two dualities D and E be equivalent it is necessary and sufficient that U and V be isomorphic as two-sided A - B -modules.*

The sufficiency of the condition is obvious. The necessity of the condition follows from Theorem 4.2 below.

THEOREM 4.2. *Let $D=(D_1, D_2)$ and $E=(E_1, E_2)$ be two dualities between \mathfrak{A} and \mathfrak{B} . Suppose that there exists a B -isomorphism $\tau: U \approx V$, that is, $\tau: D_1(A) \approx E_1(A)$. Then there exists a ring-automorphism θ of A such that*

$$(36) \quad \tau(\theta(a)u) = a(\tau(u)), \quad a \in A, u \in U;$$

(36) can also be written in the form

$$(37) \quad \tau^{-1} \circ E_1(\varphi_a) \circ \tau = D_1(\varphi_{\theta(a)}),$$

where $\varphi_a: A \rightarrow A$ is defined by $\varphi_a(x) = xa$. In this case, two dualities D and E are equivalent if and only if θ is an inner automorphism.

PROOF. Let us put $T_1 = E_2 D_1$, $T_2 = D_2 E_1$. Then T_1 and T_2 are isomorphisms from the category \mathfrak{A} into itself, and $T_2 T_1$ and $T_1 T_2$ are naturally equivalent to the identity functor. If we put $\sigma = D_2(\tau^{-1}) \circ \lambda_1(A)$, σ is an A -isomorphism from A onto $T_2(A)$. We now apply Theorem 3.5 to our case. Then from the commutativity of the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\lambda_1(A)} & D_2 D_1(A) & \xrightarrow{D_2(\tau^{-1})} & T_2(A) \\ \downarrow \varphi_{\theta(a)} & & \downarrow D_2 D_1(\varphi_{\theta(a)}) & & \downarrow T_2(\varphi_a) \\ A & \xrightarrow{\lambda_1(A)} & D_2 D_1(A) & \xrightarrow{D_2(\tau^{-1})} & T_2(A) \end{array}$$

where θ is an automorphism of A defined by the formula (33) in Theorem 3.5, it follows that $\tau^{-1} \circ E_1(\varphi_a) \circ \tau = D_1(\varphi_{\theta(a)})$. Conversely, (37) implies (33). Thus Theorem 4.2 is a direct consequence of Theorem 3.5, since D and E are equivalent if and only if T_1 is naturally equivalent to the identity functor.

It will be shown in § 14 that Nakayama's automorphism for a Frobenius algebra A is nothing but an automorphism connecting two dualities which are defined by means of A -character modules and by means of dual representation modules.

REMARK 4.3. In Remark 1.3 in § 1 we see that there exists a ring-isomorphism $\zeta_V: Z(A) \approx Z(B)$. Similarly we can define a ring-isomorphism $\zeta_V: Z(A) \approx Z(B)$ by means of V . Let a be any element of $Z(A)$; then for the B -isomorphism τ in Theorem 4.2 we have $a(\tau(u)) = \tau(\theta(a)u) = \tau[u(\zeta_V(\theta(a)))] = [\tau(u)]\zeta_V(\theta(a))$ and hence

$$(38) \quad \zeta_V(\theta(a)) = \zeta_V(a).$$

For the case of Frobenius algebras mentioned above, ζ_u and ζ_v may be considered as the identity and hence any element of the center of A remains invariant under Nakayama's automorphism.

THEOREM 4.4. *Let θ be an automorphism of the ring A . Let D be a duality between \mathfrak{A} and \mathfrak{B} . Then there exists a duality E between \mathfrak{A} and \mathfrak{B} such that there exists a B -isomorphism $\tau: D_1(A) \approx E_1(A)$ satisfying (37).*

PROOF. Let $U = D_1(A)$ be the two-sided A - B -module defined in the proof of Theorem 1.1. We define a new multiplication of elements of U with elements of A and B by

$$a * u = \theta(a)u, \quad u * b = ub, \quad a \in A, b \in B, u \in U.$$

Then we have a new two-sided A - B -module which will be denoted by V . Then Char_V defines a duality E satisfying the condition of the theorem. This is a direct description of E . The existence of E , however, is clear from Remark 3.6.

5. Dualities for modules over a commutative ring

In this section we shall discuss dualities for a class of A -modules with a commutative ring A as left operator domain.

THEOREM 5.1. *Let A be a commutative ring and \mathfrak{A} a class of left A -modules containing A as a left A -module. Then any duality D for \mathfrak{A} is equivalent to a duality which assigns to each module in \mathfrak{A} its semi-linear (U, θ) -character module where U is a left A -module and θ is a ring-automorphism of A with period ≤ 2 ; more precisely, there exist a left A -module U , a ring-automorphism θ of A with period ≤ 2 , a semi-linear (A, θ) -isomorphism ω of U onto itself, and a family of A -isomorphisms $\Phi(X): D(X) \rightarrow \text{Char}_{U, \theta} X$ ($X \in \mathfrak{A}$) such that the diagram*

$$\begin{array}{ccc} D(X') & \xrightarrow{D(f)} & D(X) \\ \downarrow \Phi(X') & & \downarrow \Phi(X) \\ \text{Char}_{U, \theta} X' & \xrightarrow{\text{Char}_{U, \theta} f} & \text{Char}_{U, \theta} X \end{array}$$

is commutative for any A -homomorphism $f: X \rightarrow X'$. Furthermore, the natural A -homomorphism

$$\pi_{U, \theta}(X): X \rightarrow \text{Char}_{U, \theta}(\text{Char}_{U, \theta} X)$$

defined by the formula $[\pi_{U, \theta}(X)(x)](\alpha) = \omega(\alpha(x))$, $x \in X$, $\alpha \in \text{Char}_{U, \theta} X$, is an A -isomorphism for each module X in \mathfrak{A} .

PROOF. For any left A -module X in \mathfrak{A} we denote by \tilde{X} a right A -module which is identical with X as an additive group and the elements of which are multiplied on the right with elements of A by the formula $xa = ax$, $a \in A$, $x \in X$. We put $\tilde{\mathfrak{A}} = \{\tilde{X} | X \in \mathfrak{A}\}$. Let us put next $E_1(X) = D(\tilde{X})$, $E_2(\tilde{X}) = D(X)$ for $X \in \mathfrak{A}$, and define $E_1(f): E_1(X') \rightarrow E_1(X)$, $E_2(\tilde{f}): E_2(\tilde{X}') \rightarrow E_2(\tilde{X})$ respectively by $E_1(f) = D(f)$, $E_2(\tilde{f}) = D(f)$ for $f = \tilde{f}: X \rightarrow X'$. Then E is a duality between \mathfrak{A} and $\tilde{\mathfrak{A}}$.

Therefore the proof of Theorem 1.1 can be applied to E . The results thus obtained may be stated in terms of left A -modules alone as follows (I to VI).

I. Let us put $U=D(A)$. Then U is a left A -module. For an element a of A we define an A -homomorphism $\varphi_a: A \rightarrow A$ by putting $\varphi_a(x)=xa, x \in A$. We introduce a new multiplication of the elements of U with elements of A :

$$(39) \quad a * u = D(\varphi_a)u, \quad a \in A, u \in U.$$

Then we have $a * (a'u) = a'(a * u)$ for $a, a' \in A, u \in U$. By U^* we denote the new left A -module thus obtained from U .

II. We can define an A -isomorphism $\omega: U^* \rightarrow U$ by the formula

$$(40) \quad \omega(u) = [\lambda^*(A)^{-1} \circ D(\varphi_u) \circ \lambda(A)](1), \quad \text{for } u \in U,$$

where $\varphi_u: A \rightarrow U$ is defined by $\varphi_u(x) = xu$ for $x \in A$ and $\lambda^*(A) = D(\lambda(A)) \circ \lambda(D(A))$. Then the inverse of ω is given by

$$(41) \quad \omega^{-1}(u) = [D(\varphi_u) \circ \lambda(A)](1), \quad \text{for } u \in U.$$

III. i) Let X be any module in \mathfrak{A} . For any A -homomorphism $\alpha: X \rightarrow U^*$ we define an A -homomorphism $a\alpha: X \rightarrow U^*$ by $(a\alpha)(x) = a(\alpha(x))$ for $a \in A, x \in X$. It is to be noted that the multiplication of $\alpha(x)$ with an element a of A is taken here in the original left A -module U . Then the set of all A -homomorphisms of X into U^* forms a left A -module which will be denoted by $C_1(X)$. We can define an A -isomorphism $\Phi_1(X): D(X) \rightarrow C_1(X)$ by putting

$$(42) \quad \Phi_1(X)(y) = \omega^{-1} \circ \lambda^*(A)^{-1} \circ D(\varphi_y) \circ \lambda(X), \quad \text{for } y \in D(X)$$

where $\varphi_y: A \rightarrow D(X)$ is defined by $\varphi_y(a) = ay$ for $a \in A$. The inverse of $\Phi_1(X)$ is written as follows: $[\Phi_1(X)]^{-1}(\alpha) = (D(\omega \circ \alpha) \circ \lambda(A))(1)$ for $\alpha \in C_1(X)$.

ii) Let $X \in \mathfrak{A}$. For any A -homomorphism $\beta: X \rightarrow U$ we define an A -homomorphism $a\beta: X \rightarrow U$ by $(a\beta)(x) = a * (\beta(x))$ for $a \in A, x \in X$. Then the set of all A -homomorphisms of X into U forms a left A -module which will be denoted by $C_2(X)$. We can define an A -isomorphism $\Phi_2(X): D(X) \rightarrow C_2(X)$ by the formula

$$(43) \quad \Phi_2(X)(y) = \lambda^*(A)^{-1} \circ D(\varphi_y) \circ \lambda(X), \quad \text{for } y \in D(X)$$

where φ_y has the same meaning as in i). The inverse of $\Phi_2(X)$ is given by $[\Phi_2(X)]^{-1}(\beta) = (D(\beta) \circ \lambda(A))(1)$, for $\beta \in C_2(X)$.

IV. Let $f: X \rightarrow X'$ be any A -homomorphism ($X, X' \in \mathfrak{A}$). For $\alpha' \in C_1(X'), \beta' \in C_2(X')$ we define A -homomorphisms $C_1(f): C_1(X') \rightarrow C_1(X), C_2(f): C_2(X') \rightarrow C_2(X)$ by putting $C_1(f)(\alpha') = \alpha' \circ f, C_2(f)(\beta') = \beta' \circ f$. Then the diagrams below are commutative.

$$\begin{array}{ccc} D(X') & \xrightarrow{D(f)} & D(X) \\ \downarrow \Phi_1(X') & & \downarrow \Phi_1(X) \\ C_1(X') & \xrightarrow{C_1(f)} & C_1(X) \end{array} \quad \begin{array}{ccc} D(X') & \xrightarrow{D(f)} & D(X) \\ \downarrow \Phi_2(X') & & \downarrow \Phi_2(X) \\ C_2(X') & \xrightarrow{C_2(f)} & C_2(X) \end{array}$$

V. Let us put

$$\mu(X) = C_2((\Phi_1(X) \circ D(\lambda(X)) \circ \lambda(D(X)))^{-1}) \circ \Phi_2(D(X)) \circ \lambda(X) .$$

$$X \xrightarrow{\lambda(X)} D^2(X) \xrightarrow{\Phi_2(D(X))} C_2(D(X)) \xrightarrow{C_2((\Phi_1(X) \circ D(\lambda(X)) \circ \lambda(D(X)))^{-1})} C_2(C_1(X))$$

Then we have

$$(44) \quad [\mu(X)(x)](\alpha) = a_0(\alpha(x)) \quad \text{for } x \in X, \alpha \in C_1(X)$$

where $a_0 \in A$ and a_0^{-1} exists. $\mu(X)$ is an A -isomorphism.

VI. We have $\lambda^*(A)^{-1}u = a_0u$ for every element u of U . On the other hand, for any element a of A there exists an element $\theta(a)$ of A such that

$$(45) \quad a * u = \theta(a)u, \quad \text{for } u \in U,$$

as is shown in Remark 1.3. $\theta(a)$ is determined uniquely by a . It is obvious that θ is a ring-automorphism of A .

Now we shall proceed to the final step of the proof of Theorem 5.1. Since $\omega: U^* \rightarrow U$ is an A -isomorphism, by (45) we have

$$(46) \quad \omega(\theta(a)u) = a(\omega(u)), \quad \text{for } a \in A, u \in U.$$

From (40) and (41) it follows that $\omega(u) = a_0(\omega^{-1}(u))$ for $u \in U$. Hence by (46) we have $a_0(\omega^{-1}(\theta(a)u)) = (aa_0)(\omega^{-1}(u))$ and so $a_0\theta^2(a)\omega^{-1}(u) = aa_0\omega^{-1}(u)$, and consequently

$$(47) \quad \theta^2(a) = a, \quad \text{for } a \in A.$$

Let us put $\nu(X) = C_1((\Phi_1(X) \circ \lambda^*(X))^{-1}) \circ \Phi_1(D(X)) \circ \lambda(X)$, where $\lambda^*(X) = D(\lambda(X)) \circ \lambda(D(X))$. Then we have, for $x \in X, \alpha \in C_1(X)$,

$$[\nu(X)(x)](\alpha) = [(\Phi_1(D(X)) \circ \lambda(X))(x)](\lambda^*(X)^{-1} \circ \Phi_1(X)^{-1})(\alpha) \\ = \omega^{-1}(\mu(X)(x))(\alpha) = \omega^{-1}(a_0\alpha(x)) = a_0^{-1}\theta(a_0)\omega(\alpha(x)).$$

Since $[\pi_{U,\theta}(X)(x)](\alpha) = \omega(\alpha(x))$ we have

$$[\nu(X)(x)](\alpha) = \theta(a_0)a_0^{-1}[\pi_{U,\theta}(X)(x)](\alpha)$$

where $x \in X, \alpha \in C_1(X)$. Since $\nu(X)$ is an A -isomorphism, $\pi_{U,\theta}(X)$ is itself an A -isomorphism.

In terms of notations used in Introduction we have $\text{Char}_{U,\theta} X = C_1(X)$, $\text{Char}_{U,\theta} f = C_1(f)$ where $f: X \rightarrow X'$ is an A -homomorphism. Thus our theorem is completely proved.

THEOREM 5.2. *Let A be a commutative ring and U a left A -module with a semi-linear (A, θ) -isomorphism ω of U onto itself where θ is a ring-automorphism of A with period ≤ 2 . Let $\mathfrak{A}[A, U]$ be a class of left A -modules which are obtained from the left A -modules A and U by taking finite direct sums, submodules, and quotient modules. Let $\pi_{U,\theta}(X): X \rightarrow \text{Char}_{U,\theta}(\text{Char}_{U,\theta} X)$ be the natural A -homomorphism defined in Theorem 5.1 (and in Introduction). Then the following conditions are equivalent.*

- I. $\pi_{U,\theta}(X)$ is an A -isomorphism for each module X in $\mathfrak{A}[A, U]$.
- II. $\pi_{U,\theta}(X)$ is an A -isomorphism for any left A -module X which is A -isomorphic to a quotient module of A or U .

III. a) $\pi_U(A)$ is an A -isomorphism; that is, A is isomorphic to the A -endomorphism ring of U by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(u) = au$, $u \in U$.

b) U is injective and for any ideal I of A with $I \approx A$ there exists a non-zero A -submodule of U which is A -homomorphic to A/I .

PROOF. We shall make U into a two-sided A - A -module by putting $a * u = \theta(a)u$, $u * a = au$ for $a \in A$, $u \in U$. This module will be denoted by V . By a device described in the proof of Theorem 5.1 and notations used there it is seen that condition I is equivalent to the condition that V -character modules define a duality between the categories \mathfrak{A} and $\tilde{\mathfrak{A}}$ where $\mathfrak{A} = \mathfrak{L}[A, U]$. Hence the implication $\text{II} \rightarrow \text{III}$ is proved by Theorem 2.4. Suppose that III holds. Then V is injective as a right A -module and for any right ideal I of A with $I \approx A$ there exists a non-zero right A -submodule of V which is A -homomorphic to A/I . From this it follows further that V is injective as a left A -module and for any left ideal I of A with $I \approx A$ there exists a non-zero left A -submodule of V which is A -homomorphic to A/I . Therefore V -character modules define a duality between \mathfrak{A} and $\tilde{\mathfrak{A}}$ by virtue of Theorem 2.4, and consequently condition I holds as is noted at the beginning of the proof. Since $\text{I} \rightarrow \text{II}$ is valid clearly, the proof is completed.

As an immediate consequence of Theorem 4.2 we obtain

THEOREM 5.3. *Let A be a commutative ring and U a left A -module satisfying condition III of Theorem 5.2. Let θ, θ' be two ring-automorphisms of A with period ≤ 2 such that there exist semi-linear isomorphisms ω and ω' : $\omega(au) = \theta(a)\omega(u)$, $\omega'(au) = \theta'(a)\omega'(u)$, $a \in A$, $u \in U$. Then two dualities defined by means of semi-linear (U, θ) -character modules and semi-linear (U, θ') -character modules are equivalent if and only if $\theta = \theta'$.*

REMARK 5.4. Let A be a complete discrete valuation ring and \mathfrak{A} a class of left A -modules such that $A \in \mathfrak{A}$ and a left A -module X belongs to \mathfrak{A} with every quotient module of X . Then any duality for \mathfrak{A} must be equivalent to a duality defined by means of semi-linear (U, θ) -character modules where U satisfies condition III of Theorem 5.2 and θ is a ring-automorphism of A with period ≤ 2 . Since A contains no zero-divisor, U must be indecomposable and hence isomorphic to K/A where K is the quotient field of A (cf. [14, p. 53]). Since K/A is divisible and A is a Dedekind ring, K/A is injective (cf. [3, p. 134]) and actually satisfies condition III of Theorem 5.2. Since K is the quotient field of A , any ring-automorphism θ of A with period ≤ 2 is extended to a ring-automorphism of K which will be denoted by the same letter θ . If we put $\omega(\{u\}) = \{\theta(u)\}$ for $\{u\} \in K/A$, ω is a semi-linear (A, θ) -isomorphism of U onto itself where $U = K/A$. Hence U and θ define actually a duality for the category $\mathfrak{L}[A, U]^2$. In this case the family of all the equivalence classes of dualities for \mathfrak{A} is in a one-to-one correspondence with the set of all ring-automorphisms of A with period ≤ 2 . The

2) It has been proved by Kaplansky [13] and Schöneborn [25] (cf. also Leptin [15]) that the duality with respect to K/A -character modules holds for some classes of topological modules.

same situation prevails for the case where A satisfies the minimum condition for ideals (cf. §12 below).

CHAPTER II. DUALITIES FOR MODULES OVER RINGS WITH MINIMUM CONDITION

6. Dualities defined by means of character modules

For the sake of convenience we shall first state the following theorem (for the proof cf. Morita, Kawada and Tachikawa [18]). Throughout Chapter II we shall denote by $N(R)$ the radical of a ring R with minimum condition.

THEOREM 6.1. *Let A be a ring satisfying the minimum condition for left ideals. Let U and V be injective left A -modules, and let us denote by $S(W)$ the semi-simple part of a left A -module W : $S(W) = \text{Ann}(N(A); W) = \{w | w \in W, N(A)w = 0\}$. Then the following propositions hold.*

1°. U and V are A -isomorphic if and only if $S(U)$ and $S(V)$ are A -isomorphic.

2°. U is indecomposable if and only if $S(U)$ is simple.

3°. U is a direct sum of indecomposable injective left A -modules.

4°. For any semi-simple left A -module X there exists an injective left A -module W such that $S(W)$ is A -isomorphic to X .

We shall first prove the following lemma.

LEMMA 6.2. *Let A and B be two rings and U a two-sided A - B -module. Suppose that U -character modules define a duality between $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$ where $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$ have the same meanings as in §2. If A satisfies the minimum condition for left ideals and every indecomposable injective left A -module is finitely generated³⁾, then B satisfies the minimum condition for right ideals and U is a finitely generated left A -module. Furthermore, if A is an algebra of finite rank over a commutative field K , then B is an algebra of finite rank over a field which is isomorphic to K .*

PROOF. Let A be a ring satisfying the minimum condition for left ideals. Then by the hypothesis of the theorem the ring B is inverse-isomorphic to the A -endomorphism ring of U . Let us put $N_0 = \{b | [S(U)]b = 0, b \in B\}$ where $S(U)$ is the semi-simple part of U . Then N_0 is nilpotent and B/N_0 is inverse-isomorphic to the A -endomorphism ring of $S(U)$, and $S(U) = \text{Ann}(N_0, U)$ as is proved in [18]. Let us denote by \bar{A} and \bar{B} respectively the residue class rings $A/N(A)$ and B/N_0 . Then $S(U)$ is a faithful left \bar{A} -module, and for any A -submodule V of $S(U)$ and for any right ideal J of \bar{B} we have $V = \text{Ann}(\text{Ann}(V; \bar{B}); S(U))$, $J = \text{Ann}(\text{Ann}(J; S(U)); \bar{B})$. For an A -submodule V_0 of $S(U)$ there exists another

3) As examples of such rings we can mention quasi-Frobenius rings, commutative rings (cf. §12 below) and algebras of finite rank over a commutative field. The problem, whether any ring with minimum condition for left and right ideals enjoys such a property remains open. It has been recently shown by Tachikawa that a ring with minimum condition for left ideals does not always possess such a property.

A -submodule V_1 of $S(U)$ such that $S(U) = V_0 \oplus V_1$. Hence there exists an element \bar{b}_0 of \bar{B} which induces the projection of $S(U)$ onto V_0 , and we have $J_0 = \text{Ann}(V_0; \bar{B}) = \{\bar{b} \mid V_0 \bar{b} = 0, \bar{b} \in \bar{B}\} = \{\bar{b} \mid S(U) \bar{b}_0 \bar{b} = 0, \bar{b} \in \bar{B}\} = \{\bar{b} \mid \bar{b}_0 \bar{b} = 0, \bar{b} \in \bar{B}\}$.

Thus any right ideal of \bar{B} is an annulet. We decompose $S(U)$ into a direct sum of left A -submodules $V_i, i=1, 2, \dots, k$ such that any two simple left A -modules contained in $S(U)$ are A -isomorphic if and only if they are contained in the same summand. Then $\bar{B} \approx B_1 \oplus \dots \oplus B_k$ where B_i is inverse-isomorphic to the A -endomorphism ring of V_i . Each ring B_i is a primitive ring such that every right ideal is an annulet. Hence by a theorem of Wolfson [32] we see that B_i is isomorphic to the ring of all matrices of finite order over a division ring. Therefore $S(U)$ is a finitely generated left A -module. Hence the ring B is semi-primary and N_0 is the radical $N(B)$ of B .

Assume that each indecomposable injective left A -module is finitely generated. Then by Theorem 6.1 U is also a finitely generated left A -module. Since $B \approx \text{Char}_U U, B$ satisfies the minimum condition for right ideals.

Now we shall consider the case where A is an algebra of finite rank over a commutative field K . Then the center of A contains K and hence the center of B contains a field K' which is isomorphic to K by the isomorphism ζ_U defined in Remark 1.3. The left A -module U is a vector space of finite rank over K and hence the right B -module U is a vector space of finite rank over K' . Since B satisfies the minimum condition for right ideals as is proved above and every simple right B -module is B -isomorphic to a B -submodule of U (and hence of $S(U)$), by Theorem 6.1 B is B -isomorphic to a B -submodule of $U_{B^{(n)}}$ with some integer n where $U_{B^{(n)}}$ is a direct sum of n copies of the right B -module U . Hence B is a vector space of finite rank over K' . This completes the proof of Lemma 6.2.

Now we shall prove the following theorem.⁴⁾

THEOREM 6.3. *Let A be a ring satisfying the minimum condition for left ideals. Let B be another ring and U a two-sided A - B -module. Then the following conditions are equivalent.*

- I. a) $\pi_U(X)$ and $\pi_U(Y)$ are isomorphisms for each module X in $\mathfrak{L}[A, U]$ and for each module Y in $\mathfrak{R}[B, U]$ where $\mathfrak{L}[A, U]$ and $\mathfrak{R}[B, U]$ are the classes of modules defined in § 2.
 b) Every indecomposable injective left A -module is finitely generated.
- II. a) B satisfies the minimum condition for right ideals.
 b) $\pi_U(X)$ and $\pi_U(Y)$ are isomorphisms for every finitely generated left A -module X and for every finitely generated right B -module Y .
- III. a) U is faithful as a left A -module and as a right B -module.
 b) For every simple left A -module X and for every simple right B -module Y , there exist the isomorphisms $X \approx \text{Char}_U(\text{Char}_U X)$ and $Y \approx \text{Char}_U(\text{Char}_U Y)$.
 c) B satisfies the minimum condition for right ideals.

4) The implications VI \rightarrow IV, IV \rightarrow II were proved by Tachikawa [27].

- IV. a) U is faithful as a left A -module and as a right B -module.
 b) The U -character modules of every simple left A -module and of every simple right B -module are simple.
 c) U is finitely generated as a left A -module.
- V. a) For any left ideal I of A and for any right B -submodule W of U the annihilator relations $I = \text{Ann}(\text{Ann}(I; U); A)$ and $W = \text{Ann}(\text{Ann}(W; A); U)$ hold.
 b) For any left A -submodule V of U and for any right ideal J of B the annihilator relations $V = \text{Ann}(\text{Ann}(V; B); U)$ and $J = \text{Ann}(\text{Ann}(J; U); B)$ hold.
 c) U is finitely generated as a left A -module.
- VI. a) $\pi_U(B)$ is a B -isomorphism (i.e. B is inverse-isomorphic to the endomorphism ring of U by the correspondence $b \rightarrow \phi_b$ where $\phi_b(u) = ub$, $u \in U$).
 b) U is injective as a left A -module and every simple left A -module is A -isomorphic to an A -submodule of U .
 c) U is finitely generated as a left A -module.
- VII. U -character modules define a duality between the category of all finitely generated left A -modules and the category of all finitely generated right B -modules.

PROOF. i) I \rightarrow II is a direct consequence of Lemma 6.2.

ii) II \rightarrow III is obvious.

iii) III \rightarrow IV. Assume III. Let X be a simple left A -module. For $x \in X$, $\alpha \in \text{Char}_U X$ we put $X_0 = \text{Ann}(\text{Char}_U X; X)$. Then $X_0 = 0$, since otherwise we would have $X_0 = X$ and hence $\text{Char}_U X = 0$ which contradicts the assumption that $X \approx \text{Char}_U(\text{Char}_U X)$. Therefore $\pi_U(X)$ is an A -isomorphism. Let Y_0 be a B -submodule of $\text{Char}_U X$ such that $(\text{Char}_U X)/Y_0$ is simple; such a Y_0 exists by virtue of III c). We put $X_1 = \text{Ann}(Y_0; X)$. Then we have either $X_1 = X$ or $X_1 = 0$. If $X_1 = 0$, then $\text{Char}_U((\text{Char}_U X)/Y_0) \approx X_1 = 0$. Hence $X_1 = X = \text{Ann}(Y_0; X)$. This shows that $Y_0 = 0$. Thus $\text{Char}_U X$ is simple. Similarly the U -character module of every simple right B -module is simple. By III a) the left A -module U and the right B -module B form an orthogonal pair to U . Hence by Theorem 2.8 the left A -module U has a composition series. Thus IV holds.

iv) IV \rightarrow V is a direct consequence of Theorem 2.8.

v) V \rightarrow IV. Assume V. Let I be a maximal left ideal of A . Then from V a) it follows that $\text{Ann}(I; U)$ is a minimal right B -submodule of U . Since $\tau: \text{Char}_U A \approx U$ where $\tau(\alpha) = \alpha(1)$, $\alpha \in \text{Char}_U A$, we have $\text{Char}_U(A/I) \approx \text{Ann}(I; U)$. Hence the U -character module of every simple left A -module is simple. Similarly the U -character module of every simple right B -module is simple. From V a) it follows that $0 = \text{Ann}(\text{Ann}(0; U); A) = \text{Ann}(U; A)$. This shows that U is faithful as a left A -module. Likewise U is faithful as a right B -module.

vi) I \rightarrow VI is an immediate consequence of Theorem 2.4 combined with Lemma 6.2.

vii) VI \rightarrow IV. Assume VI. Then there exist mutually orthogonal primitive

idempotents $e_{k,i}$, $k=1, \dots, m$; $i=1, \dots, r(k)$ such that $1 = \sum_{k,i} e_{k,i}$ and that $Ae_{k,i} \approx Ae_{l,j}$ if and only if $k=l$. According to Theorem 6.1 the left A -module U is decomposed into a direct sum of indecomposable A -submodules $U_{k,j}$, $k=1, \dots, m$; $j=1, \dots, s(k)$ such that

$$S(U_{k,j}) \approx Ae_k/N(A)e_k, \quad k=1, \dots, m$$

where $e_k = e_{k,1}$ and $N(A)$ is the radical of A . Let $e'_{k,j}$ be the idempotent of B which induces the projection of U onto $U_{k,j}$. Then we have $B = \sum e'_{k,j}B$ where the sum means a direct sum, and $e'_{k,j}$ are mutually orthogonal primitive idempotents. As is shown in the proof of Lemma 6.2, $\text{Ann}(S(U); B)$ is the radical $N(B)$ of B . We put $e_k' = e'_{k,1}$. Since $\text{Ann}(S(U)e_k'; B) = \{b|e_k'b \in N(B)\} = (1-e_k')B + e_k'N(B)$ and U is injective as a left A -module, we have

$$(48) \quad \text{Char}_U(S(U)e_k') \approx B/\text{Ann}(S(U)e_k'; B) \approx e_k'B/e_k'N(B).$$

Since $S(U)e_k' = S(Ue_k') \approx Ae_k/N(A)e_k$, we obtain $\text{Char}_U(Ae_k/N(A)e_k) \approx e_k'B/e_k'N(B)$.

Now let X be any finitely generated left A -module. Then X is A -isomorphic to an A -submodule of ${}_A U^{(n)}$ for some positive integer n where ${}_A U^{(n)}$ means a direct sum of n copies of the left A -module U . Hence $\pi_U(X)$ is an A -isomorphism by virtue of Lemma 2.2 and VI a). Thus $\pi_U(A)$ and $\pi_U(Ae_k/N(A)e_k)$ are A -isomorphisms. Therefore from (48) we have

$$(49) \quad \text{Char}_U(e_k'B/e_k'N(B)) \approx Ae_k/N(A)e_k.$$

Since any simple right B -module is B -isomorphic to $e_k'B/e_k'N(B)$ with some k , the U -character module of every simple right B -module is simple. This proves VI \rightarrow IV.

viii) IV \rightarrow I. Assume IV. Then by IV a) the left A -module U and the right B -module B form an orthogonal pair to U . From Theorem 2.8 and IV c) it follows that B satisfies the minimum condition for right ideals. The left A -module A and the right B -module U form an orthogonal pair to U . Hence by Theorem 2.8 U is also a finitely generated right B -module. Let X be any finitely generated left A -module. Then X is isomorphic to a quotient module of $A^{(n)}$ for some n where $A^{(n)}$ means a direct sum of n copies of the left A -module A . Applying Theorem 2.8 to the orthogonal pair $(A^{(n)}, U_B^{(n)})$ we see that $\pi_U(X)$ is an A -isomorphism. Similarly $\pi_U(Y)$ is a B -isomorphism for every finitely generated right B -module Y . Since U is finitely generated as a left A -module (resp. a right B -module), $\mathfrak{L}[A, U]$ (resp. $\mathfrak{R}[A, U]$) is identical with the class of all finitely generated left A -modules (resp. right B -modules). Thus I a) holds. From I a) and Theorem 2.4 we obtain VI b); that is, U is an injective left A -module and every simple left A -module is A -isomorphic to an A -submodule of U . Hence by Theorem 6.1 every indecomposable injective left A -module is A -isomorphic to an A -submodule of U and is finitely generated. Thus I b) holds.

ix) II \rightarrow VII. If II holds, then U is finitely generated as a left A -module and as a right B -module, and hence VII holds.

x) VII \rightarrow II. Let Y be any finitely generated right B -module. We set $X = \text{Char}_U Y$. Then X is finitely generated as a left A -module. Let Y_0 be any right

B -submodule of Y . Then the right B -module Y/Y_0 is finitely generated and hence $\pi_{\sigma}(Y/Y_0)$ is a B -isomorphism. Hence we have $Y_0 = \text{Ann}(\text{Ann}(Y_0; X); Y)$ by Lemma 2.3. Since $\text{Char}_{\sigma}(X/\text{Ann}(Y_0; X)) \approx \text{Ann}(\text{Ann}(Y_0; X); Y) = Y_0$, we see that Y_0 is finitely generated. Thus any B -submodule of Y is finitely generated and hence Y satisfies the maximum condition for right B -submodules. Similarly X satisfies the maximum condition for left A -submodules and hence Y satisfies the minimum condition for right B -submodules in view of the annihilator relation $Y_0 = \text{Ann}(\text{Ann}(Y_0; X); Y)$. Thus II a) holds and hence II holds.

REMARK. The above proof of the implication VII \rightarrow II shows that if the condition VII holds for arbitrary rings A and B (with or without the minimum condition) then A and B satisfy necessarily the minimum condition for left and right ideals respectively.

7. Isomorphisms between categories of modules

Let A be a ring satisfying the minimum condition for left ideals. Then there exist mutually orthogonal primitive idempotents $e_{k,i}$, $k=1, \dots, m$; $i=1, \dots, r(k)$ such that $1 = \sum_{k,i} e_{k,i}$ and $Ae_{k,i} \approx Ae_{l,j}$ if and only if $k=l$. There exist m systems of matrix units $c_{k,ij}$, $k=1, \dots, m$; $i, j=1, \dots, r(k)$ such that $c_{k,ii} = e_{k,i}$, $c_{k,ij}c_{r,i\ell} = \delta_{kr}\delta_{js}c_{k,il}$ where δ_{kr} and δ_{js} mean Kronecker's δ . We set

$$(50) \quad A^0 = eAe, \quad e = \sum_{k=1}^m e_k; \quad e_k = e_{k,1}, \quad k=1, \dots, m.$$

After Osima [24] we shall call A^0 the basic ring of A ; as is shown in [24] the basic ring of A is determined by A uniquely up to an inner automorphism.

Any element a of A can be written uniquely in the form

$$(51) \quad a = \sum_{k,i,l,j} c_{k,il} b_{kl,ij} c_{l,1j}$$

with elements $b_{kl,ij}$ from $e_k A e_l$ ($b_{kl,ij} = c_{k,1i} a c_{l,1j}$).

LEMMA 7.1. *If we regard Ae as a two-sided A - A^0 -module, then Ae is projective as a left A -module and as a right A^0 -module, and any A^0 -endomorphism of Ae is obtained by the left multiplication of an element of A and any A -endomorphism of Ae is obtained by the right multiplication of an element of A^0 . The similar proposition holds for the two-sided A^0 - A -module eA .*

PROOF. For each k and i with $1 \leq i \leq r(k)$, let us define a mapping $\alpha_{ki}: Ae \rightarrow A$ by

$$\alpha_{ki}(x) = xc_{k,1i}, \quad \text{for } x \in Ae.$$

Then α_{ki} is clearly an A -homomorphism and we have

$$\sum_{k=1}^m \sum_{i=1}^{r(k)} \alpha_{ki}(c_{k,1i}) = 1.$$

Thus Lemma 7.1 follows readily from Lemma 3.3.

LEMMA 7.2. *Let us set*

$$T_1(X) = eA \otimes_A X, \quad T_2(Y) = \text{Hom}_{A^0}(eA, Y)$$

$$R_1(X) = \text{Hom}_A(Ae, X), \quad R_2(Y) = Ae \otimes_{A^0} Y,$$

where X is a left A -module and Y a left A^0 -module. Then T_1 and T_2 (resp. R_1 and R_2) are isomorphisms between the category of all left A -modules and the category of all left A^0 -modules such that T_2T_1 and T_1T_2 (resp. R_2R_1 and R_1R_2) are naturally equivalent to the identity functor. T_i and R_i are naturally equivalent for $i=1, 2$.

PROOF. The first part is a direct consequence of Theorem 3.4 in view of Lemma 7.1. The mapping $ea \rightarrow \varphi_{ea}$ gives an A^0 - A -isomorphism of eA onto $\text{Hom}_A(Ae, A)$ where $\varphi_{ea}(x) = xea$ for $x \in Ae$. Hence it follows from the proof of Theorem 3.1 that R_i and T_i are naturally equivalent for $i=1, 2$.

THEOREM 7.3. Let A be a ring satisfying the minimum condition for left ideals, and let B be another ring. Let V be a two-sided A - B -module and set

$$T_1(X) = \text{Hom}_A(V, X), \quad T_2(Y) = V \otimes_B Y,$$

where X is a left A -module and Y a left B -module. Then the following conditions are equivalent.

- I. T_1 and T_2 are isomorphisms between the category of all finitely generated left A -modules and the category of all finitely generated left B -modules such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor.
- II. a) B is inverse-isomorphic to the A -endomorphism ring of V by the correspondence $b \rightarrow \phi_b$ where $\phi_b(v) = vb, v \in V$.
 b) V is projective and finitely generated as a left A -module.
 c) Every simple left A -module is A -isomorphic to a quotient module of V .
- III. The natural homomorphisms $\rho(X): V \otimes_B \text{Hom}_A(V, X) \rightarrow X, \nu(Y): Y \rightarrow \text{Hom}_A(V, V \otimes_B Y)$ defined by the formula $\rho(X)(v \otimes \alpha) = \alpha(v), [\nu(Y)(y)](v) = v \otimes y$ where $v \in V, y \in Y, \alpha \in \text{Hom}_A(V, X)$, are isomorphisms for any left A -module X and for any left B -module Y .

In case these conditions hold B satisfies the minimum condition for left ideals.

PROOF. i) I \rightarrow II. Assume I. Then we see the validity of II a) and II b) in virtue of Theorem 3.2. II c) follows readily from the fact that $\text{Hom}_A(V, X) \cong 0$ for $X \cong 0$.

ii) II \rightarrow III. Assume II. Then V can be decomposed into a direct sum of indecomposable left A -modules $V_{kj}, k=1, \dots, m; j=1, \dots, s(k)$ such that $V_{kj} \cong Ae_k$ for $j=1, \dots, s(k)$. Let $e'_{k,j}$ be the idempotent of B inducing the projection of V onto V_{kj} , and set $e' = \sum_{k=1}^m e'_{k,1}$. Then $e'Be'$ is inverse-isomorphic to the A -endomorphism ring of Ve' . On the other hand, $Ve' \cong Ae$ as left A -modules and hence the basic ring $B^0 = e'Be'$ of B is ring-isomorphic to $A^0 = eAe$.

Since $Ve' \cong Ae$ and $V = V(1-e') \oplus Ve'$, it follows from the proof of Lemma 7.1 that there exist a finite number of elements $\alpha_{ki} \in \text{Hom}_A(V, A), v_{ki} \in V, k=1, \dots, m; i=1, \dots, r(k)$ such that

$$\sum_{k=1}^m \sum_{i=1}^{r(k)} \alpha_{ki}(v_{ki}) = 1.$$

Hence by virtue of Lemma 3.3 and Theorem 3.4 the condition III holds.

iii) III \rightarrow I follows from Theorems 3.2 and 3.4.

COROLLARY 7.4. *Suppose that V satisfies the condition II of Theorem 7.3. Then the correspondences*

$$X \rightarrow R_1(X) = \{b \mid Vb \subseteq X, b \in B\}, \quad J \rightarrow R_2(J) = VJ$$

between the class of all left A -submodules X of V and the class of all left ideals J of B are one-to-one and are inverses of each other.

PROOF. Since V is projective, there exists an A -isomorphism: $V \otimes_B J \approx VJ$. On the other hand, it is obvious that $\text{Hom}_A(V, X)$ is B -isomorphic to $R_1(X)$ for $X \subseteq V$. Since $J \subseteq R_1(R_2(J))$ and $R_2(R_1(X)) \subseteq X$ and A, B satisfy the minimum condition for left ideals, we have $J = R_1(R_2(J))$ and $X = R_2(R_1(X))$ by Theorem 7.3.

In case A is semi-simple, any faithful, finitely generated, left A -module V is projective and satisfies the condition II of Theorem 7.3 where B is defined to be a ring inverse-isomorphic to the A -endomorphism ring of V . Hence Corollary 7.4 is applicable to the case, and we obtain a theorem of Fitting [8]. Thus Fitting's correspondences are, so to speak, restrictions of isomorphisms between the categories of all left A -modules and of all left B -modules. The correspondences given by Weyl [31, Chap. III] in connection with the representation theory of full linear groups coincide with Fitting's.

The following theorem shows that the notion of basic rings holds an important position in our theory.

THEOREM 7.5. *Let A and B be two rings satisfying the minimum condition for left ideals. Then there exists an isomorphism between the categories of all (finitely generated) left A -modules and of all (finitely generated) left B -modules if and only if the basic rings A^0 and B^0 of A and B are isomorphic.*

PROOF. The "only if" part is actually proved in the proof of Theorem 7.3. The "if" part follows readily from Lemma 7.2.

8. The complete family of dualities between categories of modules

Let A and B be two rings satisfying the minimum condition for left and right ideals. Let $D = (D_1, D_2)$ be any duality between the categories ${}_A\mathfrak{M}$ and \mathfrak{M}_B . Here ${}_A\mathfrak{M}$ (resp. \mathfrak{M}_B) means the category of all finitely generated left A -modules (resp. right B -modules). Then by Theorem 1.1 we may assume without loss of generality that

$$D_1(X) = \text{Char}_U X, \quad D_2(Y) = \text{Char}_U Y, \quad \text{for } X \in {}_A\mathfrak{M}, Y \in \mathfrak{M}_B$$

where U is a two-sided A - B -module. By Theorem 6.3 U satisfies each of the condition from I to VII of Theorem 6.3. By using the notations in §7 we set

$$A^0 = eAe, \quad B^0 = e'Be', \quad \text{where } e = \sum_k e_{k,1}, \quad e' = \sum_k e'_{k,1}.$$

Then A^0 and B^0 are basic rings of A and B respectively. Let us put

$$\begin{aligned}
 D_1^0(X) &= \text{Char}_{eUe'} X', & D_2^0(Y) &= \text{Char}_{eUe'} Y', & \text{for } X' \in {}_A^0\mathfrak{M}, Y' \in \mathfrak{M}_{B^0} \\
 E_1(X) &= \text{Char}_{Ue'} X, & E_2(Y') &= \text{Char}_{Ue'} Y', & \text{for } X \in {}_A\mathfrak{M}, Y' \in \mathfrak{M}_{B^0} \\
 F_1(X') &= \text{Char}_{eU} X', & F_2(Y) &= \text{Char}_{eU} Y, & \text{for } X' \in {}_A^0\mathfrak{M}, Y \in \mathfrak{M}_B \\
 P_1(X) &= eA \otimes_A X, & P_2(X') &= Ae \otimes_{A^0} X', & \text{for } X \in {}_A\mathfrak{M}, X' \in {}_A^0\mathfrak{M} \\
 Q_1(Y) &= Y \otimes_B Be', & Q_2(Y') &= Y' \otimes_{B^0} e'B, & \text{for } Y \in \mathfrak{M}_B, Y' \in \mathfrak{M}_{B^0}.
 \end{aligned}$$

Then by Theorem 6.3 (D_1^0, D_2^0) , (E_1, E_2) , (F_1, F_2) are dualities between the respective categories, while P_i and Q_i are isomorphisms between the respective categories by Lemma 7.2.

We shall write $T_1 \simeq T_2$ if two contravariant (or covariant) functors T_1 and T_2 are naturally equivalent. Then we can easily prove the propositions that

$$\begin{aligned}
 (52) \quad & R_1 \simeq R_2 \text{ and } T_1 \simeq T_2 \text{ imply } T_1 R_1 \simeq T_2 R_2, \\
 (53) \quad & T_i R_i \simeq 1, R_i T_i \simeq 1, i=1, 2 \text{ and } R_1 \simeq R_2 \text{ imply } T_1 \simeq T_2.
 \end{aligned}$$

THEOREM 8.1. *The following diagram is commutative:*

$$\begin{array}{ccc}
 & D_1 & \\
 {}_A\mathfrak{M} & \xleftrightarrow{\quad} & \mathfrak{M}_B \\
 P_2 \uparrow & P_1 \begin{array}{c} D_2 \\ D_1^0 \end{array} Q_1 \downarrow & Q_2 \\
 {}_A^0\mathfrak{M} & \xleftrightarrow{\quad} & \mathfrak{M}_{B^0} \\
 & D_2^0 &
 \end{array}$$

where “commutative” means that the composites of functors indicated by arrows are naturally equivalent if they are defined over the same category and have values in the same category. In particular,

$$(54) \quad D_1 \simeq Q_2 D_1^0 P_1, \quad D_2 \simeq P_2 D_2^0 Q_1.$$

We shall first note that the following lemma holds.

LEMMA 8.2. *There exists a natural equivalence Φ from P_1 to $F_2 D_1$; that is, a family of A^0 -isomorphisms*

$$\Phi(X) : eA \otimes_A X \rightarrow \text{Char}_{eU} (\text{Char}_U X)$$

defined by the formula $[\Phi(X)(ex)](\alpha) = \alpha(ex)$ for $x \in X$, $\alpha \in \text{Char}_U X$ gives a natural equivalence.

Lemma 8.2 is proved easily (directly or by appealing to a result in [3, p. 120]) and is omitted here. If we apply Lemma 8.2 to the case with U replaced by Ue' , we see that $P_1 \simeq D_2^0 E_1$. Likewise $Q_1 \simeq E_1 D_2$, $Q_1 \simeq D_1^0 F_2$. Hence the diagram

$$\begin{array}{ccc}
 {}_A\mathfrak{M} & \xleftrightarrow{\quad D_1 \quad} & \mathfrak{M}_B \\
 P_1 \downarrow & \begin{array}{c} E_1 \quad D_2 \\ F_2 \quad D_1^0 \end{array} & \downarrow Q_1 \\
 {}_A^0\mathfrak{M} & \xleftrightarrow{\quad D_2^0 \quad} & \mathfrak{M}_{B^0}
 \end{array}$$

is commutative. In view of the formulae (52) and (53) we obtain Theorem 8.1.

THEOREM 8.3. *The family of all the equivalence classes of dualities between ${}_A\mathfrak{M}$ and \mathfrak{M}_B is in a one-to-one correspondence with the family of all the equivalence classes of dualities between ${}_{A^0}\mathfrak{M}$ and \mathfrak{M}_{B^0} , and hence with the factor group of the group of all automorphisms of the ring A^0 modulo the subgroup of all inner automorphisms. Here we assume, of course, that there exists at least a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B (or ${}_{A^0}\mathfrak{M}$ and \mathfrak{M}_{B^0}).*

PROOF. The first part is a direct consequence of Theorem 8.1. Let $D=(D_1, D_2)$ and $E=(E_1, E_2)$ be any two dualities between ${}_{A^0}\mathfrak{M}$ and \mathfrak{M}_{B^0} . Since B^0 is a basic ring, D_1 and E_1 satisfy the assumption of Theorem 4.2. Hence the second part follows readily from Theorem 4.2.

THEOREM 8.4. *Let A be a quasi-Frobenius ring. Then there exists a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B if and only if the basic rings of A and B are isomorphic.*

PROOF. In this case A -character modules define a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A (cf. § 14 below). Hence there exists a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B if and only if there exists an isomorphism between the categories \mathfrak{M}_A and \mathfrak{M}_B . Therefore the theorem follows at once from Theorem 7.5.

9. Similarity of algebras

Let K be a commutative field. Throughout this chapter, by an algebra over K we mean one which is of finite rank over K ; thus K can be identified with a subring of the center of any algebra over K (in this paper a ring is assumed to have a unit element). Let A and B be two algebras over K . Then by a duality between the categories ${}_A\mathfrak{M}$ and \mathfrak{M}_B we shall mean a duality $D=(D_1, D_2)$ such that the condition (K) below is satisfied:

(K) If $f: X \rightarrow X'$, $g: Y \rightarrow Y'$ ($X, X' \in {}_A\mathfrak{M}$, $Y, Y' \in \mathfrak{M}_B$) are defined by $f(x) = \kappa x$, $g(y) = y\kappa$ with a fixed element κ in K , then $D_1(f)(y) = y\kappa$, $D_2(g)(x) = \kappa x$.

By an isomorphism from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$ we shall mean an isomorphism satisfying a similar condition as (K).

Let A be an algebra over K . We set

$$J_1(X) = \text{Hom}_K(X, K), \quad J_2(Y) = \text{Hom}_K(Y, K) \quad \text{for } X \in {}_A\mathfrak{M}, Y \in \mathfrak{M}_A$$

where X and Y are considered respectively as a right K -module and as a left K -module in forming $\text{Hom}_K(X, K)$ and $\text{Hom}_K(Y, K)$. $J_1(X)$ and $J_2(Y)$ are considered as a right A -module and a left A -module respectively by the formulae

$$(\alpha a)(x) = \alpha(ax), \quad (a\beta)(y) = \beta(ya)$$

where $x \in X$, $y \in Y$, $a \in A$, $\alpha \in J_1(X)$, $\beta \in J_2(Y)$. A left A -homomorphism $f: X \rightarrow X'$ can be considered as a right K -homomorphism. We define $J_1(f): J_1(X') \rightarrow J_1(X)$ by $J_1(f) = \text{Hom}_K(f, 1)$; $J_1(f)$ is a right A -homomorphism. Similarly, for a right A -homomorphism $g: Y \rightarrow Y'$ ($Y, Y' \in \mathfrak{M}_A$) $J_2(g)$ is defined to be $\text{Hom}_K(g, 1)$; $J_2(g)$ is a left A -homomorphism. Now we define a natural homomorphisms

$$\nu(X): X \rightarrow J_2 J_1(X), \quad \nu(Y): Y \rightarrow J_1 J_2(Y)$$

by putting $[\nu(X)(x)](\alpha) = \alpha(x)$, $[\nu(Y)(y)](\beta) = \beta(y)$ where $x \in X$, $y \in Y$, $\alpha \in J_1(X)$, $\beta \in J_2(Y)$. Then $\nu(X)$ and $\nu(Y)$ are respectively a left and a right A -isomorphisms. Hence $J = (J_1, J_2)$ is a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A . Since the representations of A in K defined by taking X and $J_1(X)$ as representation modules are equivalent, $J_1(X)$ (resp. $J_2(Y)$) is called the *dual representation module* of X (resp. Y).

Thus there exists always a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A . Hence for any two algebras A, B over the same field K there exists a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B if and only if there exists an isomorphism from ${}_A\mathfrak{M}$ onto ${}_B\mathfrak{M}$. Therefore from Theorem 7.5 we obtain at once the following theorem.

THEOREM 9.1. *Let A and B be two algebras over a commutative field K . Then the following conditions are equivalent.*

- 1° *There exists a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B .*
- 2° *There exists an isomorphism from ${}_A\mathfrak{M}$ onto ${}_B\mathfrak{M}$.*
- 3° *The basic algebras of A and B are isomorphic as algebras over K .*

THEOREM 9.2. *Let A be an algebra over K and A^0 the basic algebra of A . Let J be the duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A which is defined by means of dual representation modules. In the commutative diagram*

$$\begin{array}{ccc}
 & J_1 & \\
 & \longleftarrow & \longrightarrow \\
 {}_A\mathfrak{M} & & \mathfrak{M}_A \\
 P_2 \uparrow & P_1 & J_2 \quad Q_1 \downarrow \\
 & \downarrow & J_1^0 \quad \uparrow Q_2 \\
 {}_{A^0}\mathfrak{M} & & \mathfrak{M}_{A^0} \\
 & \longleftarrow & \longrightarrow \\
 & J_2^0 &
 \end{array}$$

given in Theorem 8.1 with $B = A$, the duality J^0 between ${}_{A^0}\mathfrak{M}$ and \mathfrak{M}_{A^0} is equivalent to the duality defined by means of dual representation modules.

PROOF. Let $J^* = (J_1^*, J_2^*)$ be the duality between ${}_{A^0}\mathfrak{M}$ and \mathfrak{M}_{A^0} defined by means of dual representation modules. In view of Theorem 8.1, if we prove that $J_1^*P_1 \simeq Q_1J_1$, we can conclude that $J_1^* \simeq Q_1J_1P_2$ and our Theorem 9.2 follows immediately.

Let X be any finitely generated left A -module. Since there exists a natural isomorphism from $P_1(X)$ to eX , we have only to prove that there exists a natural equivalence Φ consisting of

$$\Phi(X): \text{Hom}_K(eX, K) \rightarrow [\text{Hom}_K(X, K)]e$$

where $e = \sum_k e_{k,1}$ is defined by (50). Since any right K -homomorphism f of eX into K can be extended to a right K -homomorphism φ of X into K and $(\varphi e)(x) = \varphi(ex) = f(ex)$ holds for any extension φ of f and for any x in X , the existence of Φ is obvious. Thus Theorem 9.2 is proved.

Two algebras A and B over the same field K are said to be *similar* if there exists an isomorphism between the categories ${}_A\mathfrak{M}$ and ${}_B\mathfrak{M}$. In view of Theorem 9.1 our notion of similarity is identical with "similarity" in the sense of Osima [24], and for central simple algebras it is equivalent to the classical notion of similarity.

THEOREM 9.3. *Suppose that the basic rings of two rings A and B satisfying the minimum condition for left and right ideals are isomorphic. Then if A has one of the properties listed below, B has also the same property:*

- (1) *quasi-Frobenius, (2) generalized uni-serial, (3) uni-serial,*
- (4) *primary-decomposable, (5) primary, (6) weakly symmetric,*
- (7) *almost symmetric, (8) semi-simple, (9) simple.*

PROOF. Let $T_1: {}_A\mathfrak{M} \rightarrow {}_B\mathfrak{M}$ and $T_2: {}_B\mathfrak{M} \rightarrow {}_A\mathfrak{M}$ be isomorphisms between the categories such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor. If A is quasi-Frobenius, then, since any projective A -module is injective, $T_2(B)$ is injective and hence B is injective as a left B -module, and consequently B is quasi-Frobenius by Theorem 14.1 below. Similarly we can prove by using isomorphisms T_1 and T_2 that if A is generalized uni-serial, so also is B . As for the properties (6) and (7) the theorem is proved by Osima [23]. For the remaining properties the theorem is known or easily proved.

THEOREM 9.4. *Suppose that two algebras A and B over the same commutative field K are similar. If A is symmetric, so also is B .*

PROOF. It is sufficient to prove that an algebra A over K is symmetric if and only if its basic algebra A^0 is symmetric. This proposition is proved by Nesbitt and Scott [21] for the case where K is algebraically closed and by Osima [23] for the case where K is arbitrary. Here, as an application of our theory, we shall give a simple proof to this proposition. As will be shown later (§ 14), A is symmetric if and only if the duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A defined by means of A -character modules is equivalent to the duality defined by means of dual representation modules. Hence the above proposition follows immediately from Theorems 8.1 and 9.2.

10. Tensor products of dualities for modules over algebras

Let A and B be two algebras over a commutative field K . In case A and B are similar (cf. § 9) we shall write $A \sim B$; in particular, if a two-sided A - B -module U defines a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B we write $A \sim B [U]$. The purpose of this section is to prove the following theorem.

THEOREM 10.1. *Let A, B, P and Q be algebras over a commutative field K . If $A \sim B [U]$ and $P \sim Q [V]$, then*

- i) $A \oplus P \sim B \oplus Q [U \oplus V]$,
- ii) $A \otimes_K P \sim B \otimes_K Q [U \otimes_K V]$.

Here P and Q may be algebras (not necessarily of finite rank) over K which satisfy the minimum condition for left and right ideals, while A and B must be of finite rank over K .

For the sake of simplicity we shall assume in the following that P and Q are of finite rank over K and that \otimes means the tensor product over K . We shall begin with some lemmas.

LEMMA 10.2. *If $A \sim B [U]$ and $P \sim Q [V]$, then $A \oplus P \sim B \oplus Q [U \oplus V]$, where $U \oplus V$ is regarded as a two-sided $A \oplus P$ - $B \oplus Q$ -module by setting $av = vb = 0$,*

$pu=ug=0$ for $a \in A, b \in B, p \in P, q \in Q, u \in U, v \in V$.

Proof is obvious.

LEMMA 10.3. *If U is a faithful, finitely generated, left A -module, and V a faithful, finitely generated, left P -module, then $U \otimes V$ is a faithful left $A \otimes P$ -module.*

PROOF. Let $\{a_i | i=1, \dots, r\}$ and $\{p_j | j=1, \dots, s\}$ be K -bases of A and P respectively. Under the representations of A and P obtained by taking U and V as representation modules, let $a_i \rightarrow L_U(a_i), i=1, \dots, r$ and $p_j \rightarrow L_V(p_j), j=1, \dots, s$ where $L_U(a_i)$ are m by m matrices with coefficients in K and $L_V(p_j)$ n by n matrices with coefficients in K . Then $\{L_U(a_i) | i=1, \dots, r\}$ and $\{L_V(p_j) | j=1, \dots, s\}$ are linearly independent systems with respect to K since U and V are faithful. Hence $\{L_U(a_i) \otimes L_V(p_j) | i=1, \dots, r; j=1, \dots, s\}$ is also independent where \otimes means the Kronecker product of matrices. Therefore $U \otimes V$ is faithful as a left $A \otimes P$ -module.

LEMMA 10.4. *If A, B, P and Q are semi-simple algebras over K and $A \sim B [U], P \sim Q [V]$, then $A \otimes P \sim B \otimes Q [U \otimes V]$.*

PROOF. i) We shall first assume that A, B, P, Q are simple algebras. Then there exist division algebras R and T such that A, B and P, Q are full matrix rings over R and T respectively: $A=(R)_m, B=(R)_n; P=(T)_r, Q=(T)_s$.

Let G be any Frobenius algebra over K . Then we have clearly $G \sim G [G]$. Let $G^{(k)}$ be a direct sum of k copies of the right G -module G ; then $G^{(k)}$ can be considered as a left $(G)_k$ -module, and hence $G^{(k)}$ is a two-sided $(G)_k$ - G -module and $(G)_k \sim G [G^{(k)}]$. If we denote by ${}^{(j)}(G^{(k)})$ a direct sum of j copies of the left $(G)_k$ -module $G^{(k)}$, then ${}^{(j)}(G^{(k)})$ can be considered as a two-sided $(G)_k$ - $(G)_j$ -module and $(G)_k \sim (G)_j [{}^{(j)}(G^{(k)})]$. Applying this result to the case $G=R, G=T$ and $G=R \otimes T$ (it is well-known (and is easily proved) that $R \otimes T$ is a Frobenius algebra) we obtain

$$A \sim B [{}^{(n)}(R^{(m)})], \quad P \sim Q [{}^{(s)}(T^{(r)})], \\ A \otimes P \sim B \otimes Q [{}^{(ns)}((R \otimes T)^{(mr})].$$

Let us put $U^* = {}^{(n)}(R^{(m)}), V^* = {}^{(s)}(T^{(r)})$; then we have $U^* \otimes V^* \approx {}^{(ns)}((R \otimes T)^{(mr})$ as two-sided $A \otimes P$ - $B \otimes Q$ -module. Therefore we have

$$A \sim B [U^*], \quad P \sim Q [V^*], \quad A \otimes P \sim B \otimes Q [U^* \otimes V^*].$$

Since B and Q are simple algebras, there exist a B -isomorphism $\sigma: U \approx U^*$ and a Q -isomorphism $\tau: V \approx V^*$. Hence by Theorem 4.2 there exist a ring-automorphism θ of A and a ring-automorphism φ of P such that $\sigma(\theta(a)u) = a\sigma(u), \tau(\varphi(p)v) = p\tau(v)$ where $a \in A, u \in U, p \in P, v \in V$. By our convention concerning dualities for modules over algebras we have $\kappa u = u\kappa, \kappa v = v\kappa$ for $u \in U, v \in V, \kappa \in K$. Hence $\theta(\kappa) = \kappa, \varphi(\kappa) = \kappa$ for $\kappa \in K$.

Let $\{a_i | i=1, \dots, k\}$ and $\{p_j | j=1, \dots, l\}$ be K -bases of A and P respectively. Then we can define an automorphism ψ of $A \otimes P$ by putting

$$\psi(\sum \kappa_{ij}(a_i \otimes p_j)) = \sum \kappa_{ij}(\theta(a_i) \otimes \varphi(p_j)).$$

For this ψ we have

$$(\sigma \otimes \tau)(\psi(c)w) = c[(\sigma \otimes \tau)(w)], \quad \text{for } c \in A \otimes P, w \in U \otimes V,$$

where $\sigma \otimes \tau: U \otimes V \rightarrow U^* \otimes V^*$ is a $B \otimes Q$ -isomorphism defined by $(\sigma \otimes \tau)(u \otimes v) = \sigma(u) \otimes \tau(v)$. Since $A \otimes P \sim B \otimes Q [U^* \otimes V^*]$, by Theorem 4.5 we see that $A \otimes P \sim B \otimes Q [U \otimes V]$.

ii) Let $A = A_1 \oplus \cdots \oplus A_r$ and $P = P_1 \oplus \cdots \oplus P_s$ be direct-sum decompositions of A and P into simple subalgebras respectively. As is easily seen from the part vii) of the proof of Theorem 6.3, we obtain the following direct-sum decompositions:

$$\begin{aligned} B &= B_1 \oplus \cdots \oplus B_r, & U &= U_1 \oplus \cdots \oplus U_r, & A_i &\sim B_i [U_i], \\ Q &= Q_1 \oplus \cdots \oplus Q_s, & V &= V_1 \oplus \cdots \oplus V_s, & P_j &\sim Q_j [V_j]. \end{aligned}$$

According to i) we have $A_i \otimes P_j \sim B_i \otimes Q_j [U_i \otimes V_j]$. Hence by Lemma 10.2 we have

$$\sum (A_i \otimes P_j) \sim \sum (B_i \otimes Q_j) [\sum (U_i \otimes V_j)]$$

where \sum means a direct sum. Therefore $A \otimes P \sim B \otimes Q [U \otimes V]$. This proves Lemma 10.4.

We shall now proceed to the proof of Theorem 10.1.

PROOF OF THEOREM 10.1. Suppose that $A \sim B [U]$ and $P \sim Q [V]$. As before the radical of an algebra R will be denoted by $N(R)$. We shall first prove

$$(55) \quad \text{Ann}(N(A) \otimes P + A \otimes N(P); U \otimes V) = \text{Ann}(N(A); U) \otimes \text{Ann}(N(P); V).$$

To prove this, let $\{u_i | i=1, \dots, m\}$ and $\{v_j | j=1, \dots, n\}$ be K -bases of U and V respectively such that $\text{Ann}(N(A); U) = \sum_{i=1}^r K u_i$, $\text{Ann}(N(P); V) = \sum_{j=1}^s K v_j$. Let $w = \sum \kappa_{ij}(u_i \otimes v_j)$ ($\kappa_{ij} \in K$) be any element of $\text{Ann}((N(A) \otimes P + A \otimes N(P)); U \otimes V)$. Then $(x \otimes 1)w = 0$ for any element x of $N(A)$. Hence we have $0 = \sum \kappa_{ij}(x u_i \otimes v_j) = \sum_j (\sum_i \kappa_{ij} x u_i) \otimes v_j$. Therefore $\sum \kappa_{ij} x u_i = 0$, and hence we have $\kappa_{ij} = 0$ for $i > r$. Similarly we have $\kappa_{ij} = 0$ for $j > s$. Thus (55) is proved.

On the other hand, we can prove the following relation by making use of K -bases of A and P similarly as above:

$$(56) \quad (A \otimes P)/(N(A) \otimes P + A \otimes N(P)) \approx A/N(A) \otimes P/N(P).$$

As before we set $\bar{A} = A/N(A)$, $\bar{P} = P/N(P)$, $S(U) = \text{Ann}(N(A); U) = \text{Ann}(N(B); U)$, $S(V) = \text{Ann}(N(P); V) = \text{Ann}(N(Q); V)$.

Let X be any simple left $A \otimes P$ -module and let $\alpha: X \rightarrow U \otimes V$ be any $A \otimes P$ -homomorphism. Then, since $N(A) \otimes P + A \otimes N(P) \subseteq N(A \otimes P)$, by (56) X may be considered as a left $\bar{A} \otimes \bar{P}$ -module. Then $\alpha: X \rightarrow U \otimes V$ is also considered as an $\bar{A} \otimes \bar{P}$ -homomorphism of X into $S(U) \otimes S(V)$ since $[N(A) \otimes P + A \otimes N(P)]\alpha(X) = 0$ (cf. (55)). On the other hand, since $A \sim B [U]$ and $P \sim Q [V]$, we have $\bar{A} \sim \bar{B} [S(U)]$ and $\bar{P} \sim \bar{Q} [S(V)]$ as is shown in the proof of Lemma 6.2. Hence

we have $\bar{A} \otimes \bar{P} \sim \bar{B} \otimes \bar{Q} [S(U) \otimes S(V)]$ by Lemma 10.4. Therefore $\text{Char}_{S(U) \otimes S(V)} X$ is a simple right $\bar{B} \otimes \bar{Q}$ -module and consequently $\text{Char}_{U \otimes V} X$ is a simple right $B \otimes Q$ -module.

Thus the $U \otimes V$ -character module of every simple left $A \otimes P$ -module is simple. Similarly the $U \otimes V$ -character module of every simple right $B \otimes Q$ -module is simple. According to Lemma 10.3, $U \otimes V$ is faithful as a left $A \otimes P$ -module and as a right $B \otimes Q$ -module. Therefore by Theorem 6.3 we see that $A \otimes P \sim B \otimes Q [U \otimes V]$. This proves the part ii) of Theorem 10.1 in case P and Q are of finite rank over K . In case P and Q are not necessarily of finite rank over K , by an easy modification we see the validity of ii) of the theorem in this general case. Thus Theorem 10.1 is completely proved.

As an immediate corollary we obtain Osima's result [24]:

COROLLARY 10.2. *If $A \sim B$, $P \sim Q$ and L is a field which contains K as a subfield, then $A \otimes P \sim B \otimes Q$ and $A_L \sim B_L$.*

11. Completely indecomposable modules

Recently E. H. Feller [7] has extended the notion of completely indecomposable modules, which is introduced by E. Snapper [26] for modules with a commutative ring as operator domain, to the case of modules with a non-commutative ring as operator domain. In this section it will be shown that the notion of completely indecomposable modules is closely related with the notion of injective modules.

Let A be a ring (commutative or non-commutative). A left A -module X is said to be *completely indecomposable* if the following conditions a) and b) are satisfied:

a) X satisfies the minimum and maximum conditions for left A -submodules and every left A -submodule of X is indecomposable.

b) There exists another ring B such that (i) X is a two-sided A - B -module, (ii) X satisfies the minimum and maximum conditions for right B -submodules and (iii) every right B -submodule of X is indecomposable.

Feller's definition is given by taking A as right operator domain of X . In case A is commutative, the condition b) is automatically satisfied with $B=A$ if the condition a) is satisfied; this is the case treated originally by Snapper [26].

Feller [7] has proved the following theorem as a generalization of the main theorem of Snapper [26] under an additional assumption that any left A -submodule of X (resp. X') is a right B -submodule (resp. B' -submodule) and any right B -submodule of X (resp. right B' -submodule of X') is a left A -submodule of X (resp. X'). Our first remark is to show that this additional assumption is redundant.

THEOREM 11.1. *Let X and X' be left A -modules which are completely indecomposable. Let m and m' be the annihilator ideals of X and X' in A respectively: $m = \{a | a \in A, ax = 0 \text{ for } x \in X\}$, $m' = \{a | a \in A, ax' = 0 \text{ for } x' \in X'\}$. Then X and X' are A -isomorphic if and only if $m = m'$.*

To prove this theorem (under an additional assumption stated above), Feller proves the following theorem (which is also a generalization of Snapper's theorem [26]): If a left A -module X is completely indecomposable, then, for any left ideal I of A containing the annihilator ideal \mathfrak{m} , any left A -submodule X_0 of X , any right ideal J of B containing \mathfrak{n} and any right B -submodule Y_0 of X , the annihilator relations

$$\begin{aligned} \text{Ann}(\text{Ann}(I; X); A) &= I, & \text{Ann}(\text{Ann}(Y_0; A); X) &= Y_0, \\ \text{Ann}(\text{Ann}(J; X); B) &= J, & \text{Ann}(\text{Ann}(X_0; B); X) &= X_0. \end{aligned}$$

hold where $\mathfrak{n} = \{b \mid b \in B, xb = 0 \text{ for } x \in X\}$. In this case A/\mathfrak{m} (resp. B/\mathfrak{n}) satisfies the minimum and maximum conditions for left (resp. right) ideals. We now apply Theorem 6.3 to this case. Then we see that X is injective as a left A/\mathfrak{m} -module and as a right B/\mathfrak{n} -module. Our second remark is to prove the following theorem.

THEOREM 11.2. *Let X be a left A -module and \mathfrak{m} the annihilator ideal of X in A . Suppose that X satisfies the minimum and maximum conditions for left A -submodules. Then X is completely indecomposable as a left A -module if and only if i) A/\mathfrak{m} is a completely primary ring satisfying the minimum condition for left ideals and ii) X is an indecomposable, injective, left A/\mathfrak{m} -module which has a composition series.*

PROOF. The "only if" part is already proved above. The "if" part follows readily from Theorem 6.3. Indeed, if we let B be a ring which is inverse-isomorphic to the A/\mathfrak{m} -endomorphism ring of X and regard B as a right operator domain of X , then by Theorem 6.3 X is injective as a right B -module and, since A/\mathfrak{m} is completely primary, X is indecomposable as a right B -module.

PROOF OF THEOREM 11.1. Theorem 11.1 is now a direct consequence of Theorem 6.1 in view of Theorem 11.2. Indeed, if $\mathfrak{m} = \mathfrak{m}'$, then the semi-simple parts of X and X' are simple A/\mathfrak{m} -modules and hence A/\mathfrak{m} -isomorphic (because simple left A/\mathfrak{m} -modules are all A/\mathfrak{m} -isomorphic since A/\mathfrak{m} is a primary ring) and consequently X and X' are A/\mathfrak{m} -isomorphic by Theorems 11.2 and 6.1. The "only if" part is obvious. Thus Theorem 11.1 is proved.

The following corollary, which is due to Feller [7], is an immediate consequence of Theorem 6.3 in view of Theorem 11.2.

COROLLARY 11.3. *If X is a completely indecomposable left A -module, then B/\mathfrak{n} is inverse-isomorphic to the A -endomorphism ring of X and A/\mathfrak{m} is isomorphic to the B -endomorphism ring of X .*

12. Dualities for modules over a commutative ring with minimum condition

Throughout this section A is assumed to be a commutative ring with the minimum condition for ideals. We shall begin with a lemma.

LEMMA 12.1. *Every indecomposable, injective, left A -module is finitely generated.*

PROOF. Since A is commutative, A is primary-decomposable. Hence it is sufficient to treat the case where A is primary (and hence completely primary). In this case, by Snapper [26, Corollary 9.4] there exists a faithful left A -module Q which is completely indecomposable. By Theorem 11.2 Q is an injective left A -module. Since Q is completely indecomposable, Q has a composition series and hence is finitely generated as a left A -module. Since A is completely primary, every indecomposable injective left A -module must be A -isomorphic to Q by virtue of Theorem 6.1. Thus Lemma 12.1 is proved.

Now we shall prove the following theorem.

THEOREM 12.2. *Let A be a commutative ring satisfying the minimum condition for ideals. Then there exists a one-to-one correspondence between the family of all the equivalence classes of dualities for the category of all finitely generated left A -modules and the set of all ring-automorphism of A with period ≤ 2 .*

PROOF. Let U be an injective left A -module such that $s(k)=1, k=1, \dots, m$ (in this case $r(k)=1$ since A is commutative) in the notations of § 6 (in particular, cf. part vii) of the proof of Theorem 6.3). Then the A -endomorphism ring of U is isomorphic to A by virtue of Lemma 12.1 and Corollary 11.3. Hence any left A -module V such that V -character modules define a duality for ${}_A\mathfrak{M}$ must be A -isomorphic to U . Here ${}_A\mathfrak{M}$ means the category of all finitely generated left A -modules as before.

Let θ be a ring-automorphism of A with period ≤ 2 . If we define a new multiplication of elements of U with elements of A by putting $a * u = \theta(a)u$ for $u \in U, a \in A$, then the new left A -module U^* defines likewise a duality for ${}_A\mathfrak{M}$. Hence U^* is A -isomorphic to U as is observed above. Any A -isomorphism ω of U onto U^* satisfies the relation: $\omega(au) = \theta(a)\omega(u)$ for $u \in U, a \in A$ when ω is considered as a mapping from U to U . This relation shows that ω is a semi-linear (A, θ) -isomorphism of U onto itself.

Therefore any ring-automorphism θ of A with period ≤ 2 defines a duality for ${}_A\mathfrak{M}$ which assigns to each finitely generated left A -module X the semi-linear (U, θ) -character module; this is seen by Theorem 5.2. Conversely, Theorem 5.1 shows that any duality for ${}_A\mathfrak{M}$ is equivalent to a duality defined by means of semi-linear (U, θ) -character modules. According to Theorem 5.3, two dualities defined by means of semi-linear (U, θ) -character modules and semi-linear (U, θ') -character modules are equivalent if and only if $\theta = \theta'$. Thus our Theorem is completely proved.

13. Some remarks concerning dualities and isomorphisms

The following theorem has been used tacitly (cf. the proof of Theorem 9.3). Here we shall state it explicitly for the sake of completeness.

THEOREM 13.1. *Let A and B be two rings satisfying the minimum condition for left and right ideals. Let $D = (D_1, D_2)$ (resp. T_1 and T_2) be a duality (resp. isomorphisms) between the categories ${}_A\mathfrak{M}$ and \mathfrak{M}_B (resp. ${}_A\mathfrak{M}$ and ${}_B\mathfrak{M}$) such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor. If $X \in {}_A\mathfrak{M}$ and*

$Y = D_1(X)$ (resp. $Y = T_1(X)$), then Y is projective or injective according as X is injective or projective (resp. projective or injective).

PROOF. Suppose that X is injective and $Y = D_1(X)$. Then there exists a finitely generated, free, right B -module Y_0 such that Y_0/Y_1 is B -isomorphic to Y with some $Y_1 \subseteq Y_0$. We set $\text{Char}_U Y_0 = X_0$, $\text{Ann}(Y_1; X_0) = X_1$ where we assume that D is defined by U -character modules with a suitable two-sided A - B -module U . Then $X \approx X_1$. Since X is injective, X_1 is a direct summand of X_0 and hence Y_1 is B -isomorphic to a direct summand of Y_0 . Thus Y is projective. By Theorems 1.1 and 2.5 we see that if X is projective then $D_1(X)$ is injective. As for T_1 and T_2 the theorem is obvious.

As is observed in § 9, in case A is an algebra of finite rank over a commutative field the dual representation modules define a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_A . For this case Theorem 13.1 is proved by Nagao and Nakayama [19].

In this chapter we have discussed dualities and isomorphisms exclusively for the categories of all finitely generated left or right modules. If we restrict ourselves to narrower categories, the situation is different. For example, we have the following theorems in which A and B are assumed to be arbitrary rings and U is a two-sided A - B -module.

THEOREM 13.2. *In order that U -character modules define a duality between the categories $\mathfrak{L}^*[A, U]$ and $\mathfrak{R}^*[B, U]$ it is necessary and sufficient that $\pi_U(A)$ and $\pi_U(B)$ be isomorphisms, where we denote by $\mathfrak{L}^*[A, U]$ the class of all left A -modules which are finite direct sums of left A -modules each A -isomorphic to a direct summand of either A or U and $\mathfrak{R}^*[B, U]$ is the class of right B -modules defined similarly.*

THEOREM 13.3. *Suppose that B is inverse-isomorphic to the A -endomorphism ring of U by the correspondence $b \rightarrow \phi_b$ where $\phi_b(u) = ub$. Then the functors $T_1(X) = \text{Hom}_A(U, X)$, $T_2(Y) = U \otimes_B Y$ are isomorphisms between the category of all left A -modules which are finite direct sums of left A -modules each A -isomorphic to a direct summand of U and the category of all left B -modules which are finite direct sums of left B -modules each B -isomorphic to a direct summand of B , and T_2T_1 and T_1T_2 are naturally equivalent to the identity functor.*

These theorems are easy to prove, and the proof is omitted. The correspondence given by Curtis [4] may be viewed as one given in Theorem 13.3.

CHAPTER III. APPLICATIONS

14. Quasi-Frobenius rings

As an application we obtain the following well-known theorem (cf. Nakayama [20], Ikeda [11], Ikeda and Nakayama [12], Eilenberg and Nakayama [6], Morita and Tachikawa [17]).

THEOREM 14.1. *Let A be a ring satisfying the minimum condition for left and right ideals. Then the following statements are equivalent.*

I. *The A -character modules define a duality between the category ${}_A\mathfrak{R}$ of all*

finitely generated left A -modules and the category \mathfrak{M}_A of all finitely generated right A -modules (i.e. $\pi_A(X)$ and $\pi_A(Y)$ are isomorphisms for $X \in {}_A\mathfrak{M}$, $Y \in \mathfrak{M}_A$).

II. For every simple left A -module X and for every simple right A -module Y , we have

$$X \approx \text{Char}_A(\text{Char}_A X), \quad Y \approx \text{Char}_A(\text{Char}_A Y).$$

III. The A -character module of every simple left A -module and that of every simple right A -module are simple.

IV. For every left ideal I of A and for every right ideal J of A the annihilator relations

$$I = l(r(I)), \quad J = r(l(J))$$

hold where $r(S) = \{x | ax = 0 \text{ for every } a \in S\}$, $l(S) = \{x | xa = 0 \text{ for every } a \in S\}$ for a subset S of A .

V. A is injective as a left A -module.

VI. A is a quasi-Frobenius ring.

PROOF. From Theorem 6.3 it follows readily that I, II, III and IV are equivalent. Likewise $I \rightarrow V$ holds by Theorem 6.3. Now assume V. Then for any primitive idempotents e_k, e_j of A , Ae_k is A -isomorphic to Ae_j if and only if their semi-simple parts (which are simple left ideals in this case) are A -isomorphic, by virtue of Theorem 6.1. Thus every simple left A -module is A -isomorphic to an A -submodule of A . Hence by Theorem 6.3 we see again that $V \rightarrow I$ holds. Since $V \rightarrow I$ is valid, V implies that A is injective as a right A -module. Hence the above proof for $V \rightarrow I$ shows at the same time that $V \rightarrow VI$ holds. Finally assume VI. Then we have $r(N) = l(N)$ as is shown in Nakayama [20, p. 9] where N is the radical of A . If e_k is a primitive idempotent of A , the A -character module of Ae_k/Ne_k is A -isomorphic to $e_k r(N) = e_k l(N)$, and $e_k l(N)$ is simple by the definition of quasi-Frobenius rings. Thus $VI \rightarrow III$ is proved (this proof is the same as given in Morita and Tachikawa [17]).

THEOREM 14.2. If A and P are quasi-Frobenius rings and they are algebras over a field K one of which is of finite rank over K , then $A \otimes_K P$ is also quasi-Frobenius.

PROOF. This theorem is due to Nakayama [20]. If we put $B = A$, $U = A$, $Q = P$, $V = P$ in Theorem 10.1, we obtain at once $A \otimes P \sim A \otimes P [A \otimes P]$, which shows by Theorem 14.1 that $A \otimes P$ is quasi-Frobenius.

Now let A be a quasi-Frobenius ring and U an injective left A -module such that every simple left A -module is A -isomorphic to an A -submodule of U . Let a ring B be inverse-isomorphic to the A -endomorphism ring of U by the correspondence $b \rightarrow \psi_b$; then U is considered as a two-sided A - B -module by the formula $ub = \psi_b(u)$. Then B is also quasi-Frobenius by Theorems 8.4 and 9.3. Since A is quasi-Frobenius, U is a projective left A -module and every simple left A -module is A -isomorphic to a quotient module of U . Let us set

$$(57) \quad D_1(X) = \text{Char}_U X, \quad D_2(Y) = \text{Char}_U Y, \quad \text{for } X \in {}_A\mathfrak{M}, Y \in \mathfrak{M}_B.$$

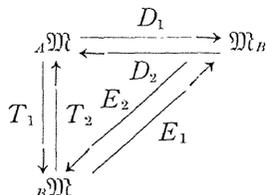
$$(58) \quad T_1(X) = \text{Hom}_A(U, X), \quad T_2(Y) = U \otimes_B Y, \quad \text{for } X \in {}_A\mathfrak{M}, Y \in {}_B\mathfrak{M}.$$

Then by Theorem 6.3 $D=(D_1, D_2)$ is a duality between ${}_A\mathfrak{M}$ and \mathfrak{M}_B , and by Theorem 7.3 T_1 and T_2 are isomorphisms between ${}_A\mathfrak{M}$ and ${}_B\mathfrak{M}$ such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor. Since B is quasi-Frobenius, the functors E_1 and E_2 defined by

$$(59) \quad E_1(Y)=\text{Char}_B Y, \quad E_2(Y')=\text{Char}_B Y', \quad \text{for } Y \in {}_B\mathfrak{M}, Y' \in \mathfrak{M}_B$$

form a duality between ${}_B\mathfrak{M}$ and \mathfrak{M}_B . Since there exists a natural isomorphism $\text{Hom}_A(U \otimes_B Y, U) \approx \text{Hom}_B(Y, \text{Hom}_A(U, U))$ where $Y \in {}_B\mathfrak{M}$, and $\text{Hom}_A(U, U)$ is isomorphic to B as a two-sided B - B -module, D_1T_2 is naturally equivalent to E_1 . Thus we have proved the following theorem.

THEOREM 14.3. *The diagram below is commutative:*



where D_i, E_i, T_i are defined by (57), (58), (59).

Let A and B be two algebras of finite rank over a commutative field K . Let U and V be two-sided A - B -modules satisfying the conditions of Theorem 6.3; U -character modules and V -character modules define respectively dualities between ${}_A\mathfrak{M}$ and \mathfrak{M}_B . By the convention stated in §8 we assume that $\kappa u = u\kappa, \kappa v = v\kappa$ for $\kappa \in K, u \in U, v \in V$. Let (u_1, \dots, u_m) and (v_1, \dots, v_n) be K -bases of U and V respectively, and let

$$a(u_1, \dots, u_m) = (u_1, \dots, u_m)L_U(a), \quad a(v_1, \dots, v_n) = (v_1, \dots, v_n)L_V(a),$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} b = R_U(b) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} b = R_V(b) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Suppose that $m=n$ and the representations $b \rightarrow R_U(b), b \rightarrow R_V(b)$ of B are equivalent. Then there exists a non-singular matrix P such that $R_U(b) = PR_V(b)P^{-1}, b \in B$. If we define a K -isomorphism τ of U onto V by putting $(\tau(u_1), \dots, \tau(u_m)) = (v_1, \dots, v_n)P'$, then $\tau(ub) = \tau(u)b$ for $b \in B, u \in U$. Hence by Theorem 4.2 there exists an automorphism θ of A such that $\tau(\theta(a)u) = a\tau(u), u \in U$. Since $a(\tau(u_1), \dots, \tau(u_m)) = (v_1, \dots, v_n)L_V(a)P'$, we have $L_U(\theta(a)) = (P')^{-1}L_V(a)P'$.

In case A is a Frobenius algebra, $B=A, U=A$ and V is a two-sided A - A -module such that the V -character module of any finitely generated left or right A -module X is A -isomorphic to the dual representation module, we have $L_U(a) = S(a), R_V(a) = R(a), L_V(a) = R(a), R_V(a) = S(a)$ where $a \rightarrow S(a)$ and $a \rightarrow R(a)$ mean the left and the right regular representations of A respectively. Thus θ coincides with Nakayama's automorphism (cf. Nakayama [20, p. 3]).

According to Nakayama [20], a Frobenius algebra is symmetric if and only if

Nakayama's automorphism is inner. Therefore we obtain the following theorem by Theorem 4.2.

THEOREM 14.4. *A quasi-Frobenius algebra is a symmetric algebra if and only if the two dualities between ${}_A\mathfrak{M}$ and \mathfrak{M}_A , one defined by means of A -character modules and the other by means of dual representation modules, are equivalent.*

15. Regular pairings

In a recent paper [4] C. W. Curtis has introduced the notion of regular pairings; his research has its origin in Weyl's paper ([30], cf. also [3], chap. III) on the centralizer of a finite group of collineations. In this section we shall discuss regular pairings from our standpoint.

Let A be a ring, and let U and V be a right and a left A -modules respectively such that they are paired to A by a function $\omega(v, u)$:

$$\omega(av, u) = a\omega(v, u), \quad \omega(v, ua) = \omega(v, u)a \quad \text{for } u \in U, v \in V, a \in A,$$

and ω is additive with respect to each variable. In case V and U form an orthogonal pair to A with respect to ω and the additive group A_0 generated by $\omega(v, u)$ for all $v \in V, u \in U$ contains an idempotent e_0 such that $a = ae_0 = e_0a$ for all $a \in A_0$, Curtis [4] calls the system (V, U, ω) a regular pairing. In this case we have

$$\omega(v, u(1-e_0)) = \omega(v, u)(1-e_0) = \omega(v, u)e_0(1-e_0) = 0 \quad \text{for } v \in V, u \in U$$

and hence $u(1-e_0) = 0$ for all $u \in U$; similarly $(1-e_0)v = 0$ for all $v \in V$. Since A is shown to be a direct sum of two-sided ideals $Ae_0 (= A_0)$ and $A(1-e_0)$, we may assume without loss of generality that $A_0 = A$; this assumption shall be made throughout this section. Then V and U are faithful as A -modules. Indeed, there exist a finite number of elements $v_i \in V, u_i \in U, i = 1, \dots, n$ such that

$$(60) \quad \sum_{i=1}^n \omega(v_i, u_i) = 1.$$

Hence for $a \neq 0, a \in A$ we have $av_i \neq 0, u_ja \neq 0$ for some i, j since $a = \sum \omega(av_i, u_i) = \sum \omega(v_i, u_ia)$.

Let B be the ring of all A -endomorphisms α of U such that there exists an A -endomorphism α^* of V satisfying $\omega(\alpha^*(v), u) = \omega(v, \alpha(u))$ for all $u \in U, v \in V$. We shall consider B as a left operator domain of U ; U is a two-sided B - A -module. Curtis calls B the centralizer of U relative to A . The correspondence $u \rightarrow u'\omega(v, u)$ defines an A -endomorphism of U for $v \in V, u' \in U$. As is proved by Curtis, this endomorphism belongs to B ; we shall write $\omega'(u', v)$ for this endomorphism. Thus

$$(61) \quad \omega'(u', v)u = u'\omega(v, u), \quad u, u' \in U, v \in V.$$

Then we have clearly $\omega'(bu, v) = b\omega'(u, v)$ for $u \in U, v \in V, b \in B$, and ω' is additive with respect to each variable. Hence for any element u of U we have

$$(62) \quad u = \sum \omega'(u, v_i)u_i$$

and the correspondence $u \rightarrow \omega'(u, v_i)$ is a B -homomorphism of U into B . Let B_0

be any subring of B which contains 1 and $\omega'(u, v)$ for all $u \in U, v \in V$. Then, since the implication II \rightarrow I in Lemma 3.3 remains valid if B there is replaced by any subring of B which contains 1 and $\beta_i(v)$ for all $v \in V, i=1, \dots, n$, we have the first part of the following theorem.

THEOREM 15.1. *Let (V, U, ω) be a regular pairing. Then U is a finitely generated, projective, left B_0 -module and A is inverse-isomorphic to the B_0 -endomorphism ring of U . V is B_0 -isomorphic to the B_0 -character module of U .*

To prove the second part, let $\beta: U \rightarrow B_0$ be any B_0 -homomorphism. If we set $v_0 = \sum v_i \beta(u_i)$, we have $\beta(u) = \beta(u \sum \omega(v_i, u_i)) = \beta(\sum \omega'(u, v_i) u_i) = \sum \omega'(u, v_i) \beta(u_i) = \omega'(u, \sum v_i \beta(u_i)) = \omega'(u, v_0)$. Hence $V \approx \text{Char}_{v_0} U$ and the second part of Theorem 15.1 is proved.

For $v \in V, b \in B$ we define vb by the formula $\omega(vb, u) = \omega(v, bu)$ for all $u \in U$. Then V becomes a two-sided A - B -module and we have

$$(63) \quad v\omega'(u, v') = \omega(v, u)v', \quad u \in U, v, v' \in V,$$

since

$$\omega(v\omega'(u, v'), u') = \omega(v, \omega'(u, v')u') = \omega(v, u\omega(v', u')) = \omega(v, u)\omega(v', u') = \omega(\omega(v, u)v', u')$$

for all $u' \in U$. If $\omega'(u, v') = 0$ for all $v' \in V$, then for all $v \in V$ we get $\omega(v, u)v' = 0$ and hence $\omega(v, u) = 0$, and consequently $u = 0$. Similarly, if $\omega'(u', v) = 0$ for all $u' \in U$, then we have $v = 0$. Thus U and V form an orthogonal pair to B with respect to ω' .

Following Curtis [4], let us denote by $\omega(V, Y)$ the left ideal of A generated by $\omega(v, y)$ for all $v \in V, y \in Y$ where Y is any left B_0 -submodule of U . Then we have

$$(64) \quad \omega(V, Y) = \{a \mid Ua \subseteq Y, a \in A\},$$

because if $Ua \subseteq Y$ then $a = \sum \omega(v_i, u_i a) \in \omega(V, Y)$, and conversely $u\omega(v, y) = \omega'(u, v)y \in Y$ for $y \in Y$.

LEMMA 15.2. *Let I and I' be left ideals of A . Then we have*

- 1) $\omega(V, UI) = I$;
- 2) UI and UI' are B_0 -isomorphic if I and I' are A -isomorphic.

PROOF. Let a be any element of I . Then we have $a = (\sum \omega(v_i, u_i))a = \sum \omega(v_i, u_i a)$. Since $u_i a \in UI$, this relation shows that $I \subseteq \omega(V, UI)$. The converse relation $\omega(V, UI) \subseteq I$ being obvious, we obtain 1). Now suppose that there exists an A -isomorphism $\theta: I \approx I'$. If $\sum x_i a_i = 0$ for $x_i \in U, a_i \in I, i=1, \dots, m$, then we have, for any element v of $V, \omega(v, \sum x_i \theta(a_i)) = \sum \omega(v, x_i) \theta(a_i) = \theta(\sum \omega(v, x_i) a_i) = 0$ and hence $\sum x_i \theta(a_i) = 0$. Hence, if we set

$$\varphi(\sum x_j a_j) = \sum x_j \theta(a_j),$$

where $x_j \in U, a_j \in I, j=1, \dots, n, \varphi$ defines a single valued mapping from UI to UI' . Since θ is an A -isomorphism, φ is one-to-one and onto. Thus φ is a B_0 -isomorphism of UI onto UI' .

THEOREM 15.3. *Let (V, U, ω) be a regular pairing. Suppose that B_0 satisfies the minimum condition for left and right ideals and U is finitely generated as a*

right A -module. Then A is a quasi-Frobenius ring if and only if U and V are finitely generated and injective as B_0 -modules. Here B_0 is a subring of B which contains 1 and $\omega(u, v)$ for all $u \in U, v \in V$.

PROOF. Suppose that A is a quasi-Frobenius ring. Then, since V and U form an orthogonal pair to A with respect to ω , V and U are A -isomorphic to the A -character modules of each other by Theorems 2.8 and 14.1. By Theorem 15.1, V is a finitely generated, projective, right B_0 -module. Hence by a theorem of Cartan and Eilenberg [3, p. 107] U is an injective left B_0 -module. Similarly, V is injective as a right B_0 -module. Thus the "only if" part is proved.

To prove the "if" part, suppose that U and V are injective as B_0 -modules. Since U has a composition series as a left B_0 -module, it follows from the relation 1) of Lemma 15.2 that A satisfies the minimum condition for left ideals. Let $\{e_i | i=1, \dots, m\}$ be a maximal set of primitive idempotents of A such that Ae_i and Ae_j are not A -isomorphic for $i \neq j$ (cf. §7). Let L_i be any minimal left subideal of Ae_i . Then L_i is the unique minimal left subideal of Ae_i . To prove this, suppose that there exists another minimal subideal L'_i of Ae_i such that $L'_i \cong L_i$. Then we have $UL_i \cap UL'_i = 0$; because if there exists an element x contained in $UL_i \cap UL'_i$, then for any element v of V we have $\omega(v, x) \in L_i \cap L'_i = 0$ by Lemma 15.2 and hence $x=0$ by the orthogonality of ω . Since UL_i and UL'_i are contained in the indecomposable injective left B_0 -module Ue_i , this is a contradiction. Thus L_i is the unique minimal left subideal of Ae_i .

We shall next prove that $L_i \cong L_j$ implies $Ae_i \cong Ae_j$. Suppose that $L_i \cong L_j$. Then by Lemma 15.2 UL_i and UL_j are B_0 -isomorphic, and hence the semi-simple parts of UL_i and UL_j are B_0 -isomorphic. Since the semi-simple part of Ue_i (resp. Ue_j) coincides with that of UL_i (resp. UL_j), we see by Theorem 6.1 that Ue_i and Ue_j are B_0 -isomorphic. Therefore Ae_i and Ae_j are A -isomorphic.

The above considerations are applied equally well to V and the right ideals of A . Hence A is a quasi-Frobenius ring. This completes the proof of Theorem 15.3.

THEOREM 15.4. *Let (V, U, ω) be a regular pairing. Suppose that A satisfies the minimum condition for left ideals. Then the set $N_0 = \{b | bU \subseteq UN(A), b \in B_0\}$ is a nilpotent two-sided ideal of B_0 and $N_0U = UN(A)$. If e_i and e_j are primitive idempotents of A , then $Ue_i/UN(A)e_i$ and $Ue_j/UN(A)e_j$ are B_0 -isomorphic if and only if Ue_i and Ue_j are B_0 -isomorphic.*

PROOF. Since it is obvious that $N_0U \subseteq UN(A)$, it is sufficient to prove that $UN(A) \subseteq N_0U$. Let $u \in U, r \in N(A)$. Then $\omega'(u, rv) \in N_0$ for any $v \in V$ since $\omega'(u, rv)u' = u(r\omega(v, u'))$ for $u' \in U$. Hence $ur = \sum ur\omega(v_i, u_i) = \sum \omega'(u, rv_i)u_i \in N_0U$. Thus the first part is proved. The second part follows immediately from the fact that U is a projective left B_0 -module and N_0 is nilpotent (cf. [18]).

Now we shall prove the following theorem.

THEOREM 15.5. *Let (V, U, ω) be a regular pairing. Then the following three conditions are equivalent.*

- I. (U, V, ω') is a regular pairing.
- II. a) U is a finitely generated, projective, right A -module.

b) $\Phi: V \rightarrow \text{Hom}_A(U, A)$ is an A -isomorphism.

III. a) The functors $T_1(X) = U \otimes_A X$, $T_2(Y) = \text{Hom}_B(U, Y)$ are isomorphisms between the category of all left A -modules X and the category of all left B -modules Y such that T_2T_1 and T_1T_2 are naturally equivalent to the identity functor.

b) $\Phi: V \rightarrow \text{Hom}_A(U, A)$ is an A -isomorphism.

Here $\Phi: V \rightarrow \text{Hom}_A(U, A)$ is the A -homomorphism defined by $[\Phi(v)](u) = \omega(v, u)$; Φ is a monomorphism.

PROOF. i) I \rightarrow II. Assume I. Then the additive group generated by $\omega'(u, v)$ for all $u \in U$, $v \in V$ coincides with B since U is faithful as a B -module. As is proved by Curtis [4] and in Theorem 15.1, A is inverse-isomorphic to the B -endomorphism ring of U by the correspondence $a \rightarrow \varphi_a$ where $\varphi_a(u) = ua$. Since it follows from (63) that $\omega'(ua, v') = \omega'(u, av')$ for $u \in U$, $v' \in V$, A is the centralizer of U relative to B . Applying Theorem 15.1 to the regular pairing (U, V, ω') we see that U is finitely generated and projective as a right A -module. Similarly as in the proof of Theorem 15.1 we can prove the validity of II b).

ii) II \rightarrow III. Assume II. Let C be the A -endomorphism ring of U . For $v \in V$, $c \in C$, we have $[\Phi(v)c](u) = \Phi(v)(cu) = \omega(v, cu)$ for $u \in U$. If we put $vc = \Phi^{-1}(\Phi(v)c)$, then we have $\omega(vc, u) = \omega(v, cu)$. Hence we have $B = C$. On the other hand, in virtue of Theorem 15.1 U is a finitely generated, projective, left B -module. Therefore the validity of III a) follows from Theorem 3.4.

iii) III \rightarrow II is a direct consequence of Theorem 3.2.

iv) II \rightarrow I. Assume II. Since U is a finitely generated, projective, right A -module, it follows that there exist a finite number of elements $u_j^* \in U$, $\varphi_j \in \text{Hom}_A(U, A)$, $j = 1, \dots, m$ such that $u = \sum u_j^* \varphi_j(u)$ holds for any element u of U . Since Φ is an A -isomorphism, if we set $v_j^* = \Phi^{-1}(\varphi_j)$, then we have $u = \sum u_j^* \omega(v_j^*, u)$. By (61) we get $u = \sum \omega'(u_j^*, v_j^*)u$. Since U is faithful as a left B -module, this relation shows that $\sum \omega'(u_j^*, v_j^*) = 1$. Thus (U, V, ω') is a regular pairing.

COROLLARY 15.6. Let (V, U, ω) be a regular pairing such that (U, V, ω') is also a regular pairing. If A is a quasi-Frobenius ring, so also is B .

PROOF. As is proved in ii) of the proof of Theorem 15.5 B coincides with the A -endomorphism ring of U . Hence by virtue of Theorem 7.5 the basic rings of A and B are isomorphic. Therefore by Theorem 9.3, B is quasi-Frobenius.

The results obtained by Curtis [4] are related with ours as follows. In case A is quasi-Frobenius, the condition II b) of Theorem 15.5 is always satisfied since in this case U and V form an orthogonal pair to A and II b) holds by Theorems 2.8 and 14.1. Thus Proposition 5 of [4] follows immediately from our Theorem 15.5. Our Corollary 15.6 is also a generalization of Theorem 6 and its Corollary in [4]. By our Theorems 8.4 and 9.3 we see that Theorem 7 of [4] is true without the assumption that the ring \mathfrak{A} is commutative. We have stated Theorem 15.4 as a supplement of Curtis's Theorem 4 in [4].

Finally we shall prove the following theorem.

THEOREM 15.7. *Let U be a finitely generated, faithful, projective, left B -module and V the B -character module of U . Suppose that V is faithful as a right B -module, and let A be the B -endomorphism ring of V . Then V is a left A -module and U can be considered as a right A -module, and (V, U, ω) forms a regular pairing if we choose ω suitably.*

PROOF. We set $\omega'(u, v) = v(u)$ for $u \in U, v \in V$. The mapping $v \rightarrow v'\omega'(u, v)$ defines a B -homomorphism and hence there exists a unique element $\omega(v', u)$ of A such that $\omega(v', u)v = v'\omega'(u, v)$. Since $\omega'(u'\omega(v', u), v) = \omega'(u', \omega(v' u)v) = \omega'(u', v'\omega'(u, v)) = \omega'(u', v')\omega'(u, v) = \omega'(\omega'(u', v')u, v)$, we have $u'\omega(v', u) = \omega'(u', v')u$ for $u, u' \in U, v \in V$. By the proof of Lemma 3.3 there exist a finite number of elements $u_i \in U, v_i \in V, i = 1, \dots, m$ such that $\sum \omega(v_i, u_i) = 1$. Suppose that $\omega(v, u_0) = 0$ for all $v \in V$. Then we have $\omega(v, u_0)v' = 0$ for all $v, v' \in V$, and hence $v\omega'(u_0, v') = 0$ for all $v, v' \in V$. Therefore $\omega'(u_0, v') = 0$ for all $v' \in V$ since V is faithful as a right B -module. Therefore we conclude that $u_0 = 0$. Similarly we can prove that if $\omega(v_0, u) = 0$ for all $u \in U$ then $v_0 = 0$. Since it is easily seen that $a\omega(v, u) = \omega(av, u), \omega(v, u)a = \omega(v, ua)$ and that ω is additive with respect to each variable, (V, U, ω) is a regular pairing.

16. The endomorphism rings of faithful modules over a quasi-Frobenius ring

Let A be a ring and U a right A -module. Let B be the A -endomorphism ring of U ; we shall consider B as a left operator domain of U .

THEOREM 16.1. *Suppose that U is a direct sum of two right A -modules U' and U'' such that U' is finitely generated and projective as a right A -module and as a left C -module and that any C -endomorphism of U' is obtained by the right multiplication of an element of A where C is the A -endomorphism ring of U' . Then any B -endomorphism of U is obtained by the right multiplication of an element of A and U is a finitely generated, projective, left B -module.*

PROOF. From the assumption it follows in virtue of Lemma 3.3 that there exist a finite number of elements $u'_i \in U', \alpha'_i \in \text{Hom}_A(U', A), i = 1, \dots, n$ such that $\sum \alpha'_i(u'_i) = 1$. If we define $\alpha_i \in \text{Hom}_A(U, A)$ by setting $\alpha_i(u') = \alpha'_i(u')$ for $u' \in U', \alpha_i(u'') = 0$ for $u'' \in U''$, then we have $\sum \alpha_i(u'_i) = 1$. Hence Theorem 16.1 follows immediately from Lemma 3.3.

THEOREM 16.2. *Let U be a direct sum of two right A -modules U' and U'' such that the conditions below are satisfied where C is the A -endomorphism ring of U' and we regard U' as a two-sided C - A -module:*

- 1) U' is injective as a right A -module and as a left C -module, and any C -endomorphism of U' is obtained by the right multiplication of an element of A .
- 2) For any right ideal I of A with $I \cong A$ there exists a non-zero A -submodule of U' which is A -homomorphic to $A|I$, and for any left ideal J of C with $J \cong C$ there exists a non-zero C -submodule of U' which is C -homomorphic to $C|J$.

Then any B -endomorphism of U is obtained by the right multiplication of an element of A . Furthermore, if U'' is obtained from A and U' by taking finite

direct sums, submodules and quotient modules, then U is an injective left B -module.

PROOF. Let α be any B -endomorphism of U . If we denote by e' the idempotent of B which induces the projection of U onto U' , we have $\alpha(e'u) = e'(\alpha u)$ for any element u of U . Hence α induces a B' -endomorphism of U' where $B' = e'Be'$. By assumption we may assume that $C = B'$. Hence by assumption we can find an element a of A such that $\alpha(e'u) = (e'u)a$ for any element u of U . Let $\alpha'(u) = \alpha(u) - ua$. Then α' is also a B -endomorphism of U and $\alpha'(e'u) = 0$ for all $u \in U$. We shall prove that $\alpha' = 0$. For this purpose, suppose that there exists an element u_0 of U such that $\alpha'(u_0) = u_1 \neq 0$. Then by Theorem 2.4 we can find an element β of $\text{Char}_{U'}(u_1A)$ such that $\beta(u_1) \neq 0$. Since U' is injective, β is extended to an A -homomorphism β^* of U into U' . Hence there exists an element b_0 of B such that $b_0u = \beta^*(u)$ for $u \in U$. Thus we have $\beta(u_1) = b_0u_1 = b_0(\alpha'(u_0)) = \alpha'(b_0u_0) = 0$ since $b_0u_0 \in U'$. This is a contradiction. Therefore the first part is proved.

Next we shall prove the second part. Let V be the U' -character module of U . Then V is a left C -module and is decomposed into a direct sum of two left C -modules V' and V'' such that V' is C -isomorphic to the left C -module C . Moreover, B is considered as a right operator domain of V and any C -endomorphism of V is obtained by the right multiplication of an element of B ; this is seen by Theorem 2.4. Hence by Theorem 16.1 V is a projective right B -module. Since U is considered as the U' -character module of the left C -module V , by a theorem of Cartan and Eilenberg [3, p. 107], U is injective as a left B -module.

In case A satisfies the minimum condition for right ideals, Theorems 6.3 and 7.3 enable us to state the above theorems in the following forms.

THEOREM 16.3. *Suppose that U is decomposed into a direct sum of two right A -modules U' and U'' such that U' is a finitely generated, projective right A -module and every simple right A -module is A -isomorphic to a quotient module of U' . Then any B -endomorphism of U is obtained by the right multiplication of an element of A and U is a finitely generated, projective, left B -module.*

THEOREM 16.4. *Suppose that U is a direct sum of two finitely generated right A -modules U' and U'' such that U' is injective as a right A -module and such that every simple right A -module is A -isomorphic to an A -submodule of U' . Then any B -endomorphism of U is obtained by the right multiplication of an element of A and U is an injective left B -module.*

For the case $U' = A$ the first part of Theorem 16.1 is proved by Nesbitt and Thrall [22], and the first part of Theorem 16.3 is proved by Nesbitt and Thrall [22] and by Osima [24]. The first part of Theorem 16.4 is proved also by Tachikawa independently.

The following theorem is a generalization of a theorem of Nesbitt and Thrall [22] and is due to G. Azumaya; it is proved similarly as in Nesbitt and Thrall [22] in view of the fact that projective modules over a quasi-Frobenius ring are injective (cf. Theorem 14.1).

THEOREM 16.5. *Let A be a quasi-Frobenius ring. Then any faithful right A -module contains a direct summand which is A -isomorphic to Ae where $e = \sum_k e_{k,1}$ is defined by (50), § 7.*

Now we are in a position to establish the following theorem.

THEOREM 16.6. *Let A be a quasi-Frobenius ring. Let U be a finitely generated, faithful, right A -module, and let B be the A -endomorphism ring of U ; we shall consider B as a left operator domain of U . Then U is finitely generated, projective and injective as a left B -module. The ring B is a quasi-Frobenius ring if and only if U is projective as a right A -module, and if this is the case the basic rings of A and B are isomorphic.*

PROOF. The first part follows readily from Theorems 16.3, 16.4 and 16.5. Suppose that U is projective as a right A -module. Then by Theorem 16.5 U satisfies the conditions of Theorem 6.3 (or Theorem 7.3) and hence by Theorems 8.4 and 9.3 the basic rings of A and B are isomorphic and B is quasi-Frobenius. Conversely, if B is quasi-Frobenius, then U is projective as a right A -module as is proved by the first part. Thus our theorem is completely proved.

As a corollary to this theorem we obtain the following theorem in virtue of Theorem 9.3.

THEOREM 16.7. *Theorem 16.6 remains true if we replace “a quasi-Frobenius ring” by “a uni-serial ring,” “a weakly symmetric ring,” “an almost symmetric ring” or “a symmetric algebra.”*

The following theorem gives a characterization of the endomorphism ring of a faithful module over a quasi-Frobenius ring.

THEOREM 16.8. *Let B be a ring satisfying the minimum condition for left and right ideals. Then B is isomorphic to the A -endomorphism ring of a finitely generated, faithful, right A -module with a suitable quasi-Frobenius ring A if and only if there exists an idempotent e_0 of B such that Be_0 and e_0B are faithful injective (left and right respectively) B -modules and such that any e_0Be_0 -endomorphism of Be_0 (resp. e_0B) is obtained by the left (resp. right) multiplication of an element of B .*

PROOF. Let A be a quasi-Frobenius ring and U a finitely generated, faithful, right A -module. If we denote by V the A -character module of U , then the A -endomorphism ring B of U is inverse-isomorphic to the A -endomorphism ring of V . In this case (V, U, ω) forms a regular pairing where $\omega(v, u) = v(u)$ for $v \in V, u \in U$. Hence by Theorem 15.3 V is considered as the B -character module of U , and U, V are faithful, projective, injective B -modules. Thus the “only if” part is proved. Conversely, suppose that Be_0 and e_0B are faithful, injective (left and right respectively) ideals. In virtue of Theorems 15.7 and 15.3 it is seen that the B -endomorphism ring e_0Be_0 of e_0B is a quasi-Frobenius ring. Hence the “if” part is proved.

The following theorem shows the extent to which a faithful module over a quasi-Frobenius ring can be recaptured from its ring of operator-endomorphisms.

THEOREM 16.9. *Let A be a quasi-Frobenius ring such that its basic ring*

coincides with A itself. Let U and V be two finitely generated, faithful, right A -modules. Then, for any ring-isomorphism φ from the A -endomorphism ring B of U onto the A -endomorphism ring C of V , there exists a semi-linear (A, θ) -isomorphism ω from U onto V (i.e. $\omega(ua) = \omega(u)\theta(a)$ for $u \in U, a \in A$, and θ is a ring-automorphism of A) such that $\omega(bu) = \varphi(b)\omega(u)$ for $u \in U, b \in B$.

PROOF. According to Theorems 16.3, 16.4 and 16.5, the left B -module U is finitely generated, faithful, projective and injective. We set

$$b * v = \varphi(b)v \quad \text{for } v \in V, b \in B.$$

Then V is a left B -module and moreover V is finitely generated, faithful, projective and injective as a left B -module. Since the B -endomorphism rings of U and V are inverse-isomorphic to A and the basic ring of A coincides with A itself, the left B -modules U and V are B -isomorphic to Be_1 and Be_2 where each of e_1 and e_2 is a finite sum of mutually orthogonal primitive idempotents such that any two of them generate non-isomorphic primitive left ideals of B . Here it is to be noted that the ring B is semi-primary in the sense that its radical is nilpotent and the residue class ring modulo its radical satisfies the minimum condition for left and right ideals. Since $Be_i \subseteq B$ for $i=1, 2$ and U, V are faithful as B -modules, the semi-simple parts of the left B -modules U and V are B -isomorphic. Since U and V are injective as left B -modules and Theorem 6.1 (except statement 3°) holds for semi-primary rings in the sense mentioned above, we see that U and V are B -isomorphic. Let $\omega: U \rightarrow V$ be a B -isomorphism from U onto V . Then we have

$$\omega(bu) = b * \omega(u) = \varphi(b)\omega(u) \quad \text{for } u \in U, b \in B.$$

On the other hand, for an element a of A the mapping $v \rightarrow \omega(\omega^{-1}(v)a)$ defines a C -endomorphism of V and hence there exists an element $\theta(a)$ of A such that $\omega(\omega^{-1}(v)a) = v\theta(a)$. Thus we have $\omega(ua) = \omega(u)\theta(a)$. Now it is easy to see that θ is a ring-automorphism of A . This completes our proof.

K. Asano [1] has proved that for two finitely generated faithful right modules U and V over a commutative uni-serial ring there exists a semi-linear isomorphism of U onto V if their operator-endomorphism rings are isomorphic. Even in this special case our theorem states much more than Asano's theorem.

Let U be a faithful right A -module. Then for any element a of the center $Z(A)$ of the ring A , the mapping $u \rightarrow ua$ defines an A -endomorphism φ_a of U . The correspondence $a \rightarrow \varphi_a$ is easily shown to give an isomorphism from $Z(A)$ onto the center of the A -endomorphism ring B of U ; we shall identify these two centers and regard $Z(A)$ as the center of B .

THEOREM 16.10. *Let A be a quasi-Frobenius ring such that its basic ring coincides with A itself and every ring-automorphism of A leaving its center $Z(A)$ elementwise fixed is inner. Let U and V be two finitely generated, faithful, right A -modules. Then, for any ring-isomorphism φ from the A -endomorphism ring B of U onto the A -endomorphism ring C of V such that φ is the identity on $Z(A)$, there exists an A -isomorphism ω from U onto V such that $\omega(bu) = \varphi(b)\omega(u)$*

for $u \in U, b \in B$.

PROOF. By Theorem 16.9 there exists a semi-linear (A, θ) -isomorphism ω from U onto V such that $\omega(bu) = \varphi(b)\omega(u)$ for $u \in U, b \in B$. Let a be any element of $Z(A)$. Then we have $\omega(u)\theta(a) = \omega(ua) = \omega(au) = \varphi(a)\omega(u) = a\omega(u) = \omega(u)a$, and hence $\theta(a) = a$. By the assumption of the theorem there exists an element a_0 of A such that $\theta(a) = a_0aa_0^{-1}$. If we set $\omega'(u) = \omega(u)a_0$, then we have $\omega'(ua) = \omega'(u)a, \omega'(bu) = \varphi(b)\omega'(u)$ for $u \in U, a \in A, b \in B$. Thus ω' satisfies the conditions of our theorem.

COROLLARY 16.11. *Let A be the same as in Theorem 16.10. Let B be the A -endomorphism ring of a finitely generated, faithful, right A -module U . Then any ring-automorphism φ of B leaving its center elementwise fixed is inner.*

PROOF. There exists an A -isomorphism ω of U onto itself such that $\omega(bu) = \varphi(b)\omega(u)$ for $u \in U, b \in B$. Since ω is an A -isomorphism of U onto itself, there exists an element b_0 of B such that $\omega(u) = b_0u$ for every $u \in U$ and b_0^{-1} exists. Hence we have $b_0bu = \varphi(b)b_0u$ and consequently $\varphi(b) = b_0bb_0^{-1}$ for $b \in B$.

As is easily shown, Theorem 16.9 fails to be true if the condition that the basic ring of A coincides with A itself is dropped. By Lemma 7.2 we can deduce the following theorem from Theorem 16.9.

THEOREM 16.12. *Let A be a quasi-Frobenius ring and $A^0 = eAe$ the basic ring of A where e is an idempotent defined by (50). Let U and V be finitely generated, faithful, right A -modules. Then the A -endomorphism rings of U and V are isomorphic if and only if there exists a semi-linear (A^0, θ) -isomorphism of Ue onto Ve with a ring-automorphism θ of A^0 .*

17. QF-3 algebras

An algebra is called a QF-3 algebra if it has a unique minimal faithful representation. This notion was introduced by Thrall [29] as a generalization of quasi-Frobenius algebras. In this section we shall show how to obtain QF-3 algebras. We shall first state the following lemma due to Tachikawa [28].

LEMMA 17.1. *An algebra A is a QF-3 algebra if and only if there exists a faithful, projective, injective, left A -module.*

According to Theorem 16.6, if A is a quasi-Frobenius subalgebra of the full matrix ring $(K)_n$ over a commutative field K , then the commutator algebra of A in $(K)_n$ is a QF-3 algebra.

THEOREM 17.2. *Let A be an algebra (of finite rank) over a commutative field K . Let U be a finitely generated, faithful, right A -module such that every indecomposable projective right A -module as well as every indecomposable injective right A -module is A -isomorphic to a direct summand of U . Then U is a faithful, projective, injective, left B -module and B is a QF-3 algebra where B is the A -endomorphism ring of U and is considered as a left operator domain of U .*

This theorem is a direct consequence of Theorems 16.3 and 16.4.

THEOREM 17.3. *Let B be a QF-3 algebra and let e_1, e_2 be two idempotents such that Be_1 and e_2B are dual representation modules of each other and they are*

faithful as B -modules. If B_0 is a subalgebra of B which contains 1 , Be_1 and e_2B , then B_0 is also a QF-3 algebra.

PROOF. By the assumption B_0 contains e_1 and e_2 , and we have $B_0e_1=Be_1$, $e_2B_0=e_2B$. Since Be_1 and e_2B are dual representation modules of each other, B_0e_1 and e_2B_0 are also dual representation modules of each other as B_0 -modules and they are faithful as B_0 -modules. Hence B_0 is a QF-3 algebra by Lemma 17.1.

LEMMA 17.4. Let B be a QF-3 algebra and let $\{e_1, \dots, e_n\}$ be a maximal set of mutually orthogonal primitive idempotents of B such that Be_i is not B -isomorphic to Be_j for $i \neq j$. Suppose that Be is a faithful injective left B -module where $e=e_1+\dots+e_m$ ($m \leq n$) and that the dual representation module of Be_i is B -isomorphic to $e_{\pi(i)}B$ for $i=1, \dots, m$. Let U be a faithful, projective, injective, left B -module and let A be the B -endomorphism ring of U (which is considered as a right operator domain of U).

0) Every indecomposable projective or injective right A -module is A -isomorphic to a direct summand of the right A -module U .

1) If π is a permutation of $(1, \dots, m)$, then A is a quasi-Frobenius algebra.

2) If $\{1, \dots, m\} \setminus \{\pi(1), \dots, \pi(m)\} = \{1, \dots, n\}$,⁵⁾ then every indecomposable direct summand of the right A -module U is either projective or injective.

3) If B is a QF-2 algebra in the sense of Thrall [29], then every indecomposable direct summand of the right A -module U has a unique maximal A -submodule and a unique minimal A -submodule (in particular, A is a QF-2 algebra).

In case any A -endomorphism of U is obtained by the left multiplication of an element of B , the converses of 1), 2) and 3) are true.

PROOF. The left B -module Be is B -isomorphic to a direct summand of the left B -module U and eBe is isomorphic to the basic algebra of A . By Lemma 7.2 and Theorem 13.1 we see that it is sufficient to treat the case where $U=Be$ and $A=eBe$.

We set $A=eBe$. Then the dual representation module of the left A -module eBe_i ($i \leq m$) is A -isomorphic to $e_{\pi(i)}Be$ and hence $e_{\pi(i)}Be$ is an indecomposable injective right A -module, while e_iBe is an indecomposable projective right A -module ($i \leq m$). Therefore 0) and 2) hold. If B is a QF-2 algebra, then e_jB is B -isomorphic to a B -submodule of $e_{\pi(i)}B$ with some $i \leq m$ and hence e_jBe is A -isomorphic to an A -submodule of the right A -module $e_{\pi(i)}Be$, and consequently e_jBe has a unique minimal A -submodule. By consideration of dual representation modules, we see that 3) holds. 1) is a direct consequence of Theorem 16.8.

Suppose that any A -endomorphism of Be is obtained by the left multiplication of an element of B . Since e_j is a primitive idempotent, e_jBe is indecomposable as a right A -module. $\{e_iBe | i=1, \dots, m\}$ and $\{e_{\pi(i)}Be | i=1, \dots, m\}$ are respectively the totality of non-isomorphic indecomposable projective right A -modules and the totality of non-isomorphic indecomposable injective right A -modules. Therefore, according as e_jBe is projective or injective, we have $j \leq m$ or $j=\pi(i)$ with some $i \leq m$. Thus the converse of 2) holds.

5) This condition is satisfied if B is a QF-1 algebra (cf. Thrall [29]).

Since $e'B$ is faithful as a right B -module where $e' = e_{\pi(1)} + \dots + e_{\pi(m)}$, any simple right ideal of B is B -isomorphic to the semi-simple part of $e_{\pi(i)}B$ with some $i \leq m$, and the latter is B -isomorphic to e_iB/e_iN where N means the radical of B . Therefore for any simple right ideal I of B we have always $Ie \cong 0$. Hence, if e_jB has two distinct simple right ideals I_1 and I_2 , then e_jBe has two distinct simple right A -modules I_1e and I_2e . Therefore if e_jBe has a unique minimal right A -submodule, then e_jB has also a unique minimal right B -submodule. By consideration of dual representation modules, we see that the converse of 3) holds.

Since the converse of 1) is a direct consequence of Theorem 16.8, our theorem is completely proved.

THEOREM 17.5. *Let B be a QF-3 algebra over a commutative field K and e an idempotent of B such that Be is faithful and injective as a left B -module. Then Be is a right A -module where we set $A = eBe$, and every indecomposable projective or injective right A -module is A -isomorphic to a direct summand of the right A -module Be . If we denote by B^* the A -endomorphism ring of Be , then B^* is a QF-3 algebra and B may be considered as a subring of B^* with the property described in Theorem 17.3. More precisely, let $b \rightarrow L(b) = (\lambda_{ij}(b))$ be the representation of B in K with degree n determined by the representation module Be ; then B is isomorphic to a subring $L(B) = \{L(b) | b \in B\}$ of $(K)_n$ and if we denote by $L(B^*)$ the commutator algebra of the commutator algebra of $L(B)$ in $(K)_n$, $L(B)$ is a subring of $L(B^*)$ with the property described in Theorem 17.3.*

PROOF. The first part is a direct consequence of Lemma 17.4 and Theorem 17.2. Let $\{u_1, \dots, u_n\}$ be a K -basis of Be . Then we have

$$\begin{aligned}
 bu_i &= \sum_j u_j \lambda_{ji}(b), \quad \text{for } b \in B, \\
 u_i a &= \sum_j u_j \mu_{ji}(a), \quad \text{for } a \in A = eBe.
 \end{aligned}$$

Since $u_i u_j = \sum_r u_r \lambda_{rj}(u_i) = u_i e u_j e = \sum_r u_r \mu_{ri}(e u_j e)$, we have $\lambda_{rj}(u_i) = \mu_{ri}(e u_j e)$. An element C in $(K)_n$ belongs to $L(B^*)$ if and only if $CM(e u_j e) = M(e u_j e)C$ for all $j = 1, \dots, n$ where $M(e u_j e) = (\mu_{rs}(e u_j e))$. Hence we have, for $C = (c_{ij}) \in L(B^*)$, $\sum_i c_{ir} \lambda_{rk}(u_j) = \sum_i \lambda_{ik}(u_s) c_{sj}$. Thus we have

$$(65) \quad CL(u_j) = \sum L(u_i) c_{ij}, \quad \text{for } C \in L(B^*),$$

and $L(B^*)$ is defined to be the set of all elements C of $(K)_n$ satisfying the relation (65) for all $j = 1, \dots, n$.

If $e'B$ is the dual representation module of Be , we can find a K -basis $\{v_1, \dots, v_n\}$ of $e'B$ such that

$$v_i b = \sum \lambda_{ij}(b) v_j, \quad \text{for } b \in B.$$

Then $L(B^*)$ is obtained as the set of all elements $C = (c_{ij})$ of $(K)_n$ satisfying

$$(66) \quad L(v_r)C = \sum c_{rs} L(v_s), \quad \text{for } r = 1, \dots, n.$$

By the relations (65) and (66) we have

$$L(B^*)L(e)=L(B)L(e), \quad L(e')L(B^*)=L(e')L(B).$$

Thus the theorem is completely proved.

According to Theorem 17.5, it is seen that any QF-3 algebra is obtained by the method indicated in Theorems 17.2 and 17.3.

LEMMA 17.6. *Let A and B be two algebras over a commutative field K . Suppose that U and V are finitely generated left A - and B -modules respectively. If U and V are projective (resp. injective), then $U \otimes_K V$ is projective (resp. injective) as a left $A \otimes_K B$ -module.*

PROOF. If U and V are projective, then $U \otimes V$ is clearly projective as a left $A \otimes B$ -module. Suppose that U and V are injective. Then their dual representation modules $J(U)$ and $J(V)$ are projective and hence $J(U) \otimes J(V)$ is projective as a right $A \otimes B$ -module. Since the dual representation module of $U \otimes V$ is $A \otimes B$ -isomorphic to $J(U) \otimes J(V)$, we see that $U \otimes V$ is injective as a left $A \otimes B$ -module. Thus Lemma 17.6 is proved.

THEOREM 17.7. *If A and B are two QF-3 algebras over a commutative field K , then their tensor product $A \otimes_K B$ over K is also a QF-3 algebra.*

PROOF. Let U (resp. V) be a finitely generated, faithful, projective, injective, left A -module (resp. B -module). Then $U \otimes V$ is faithful, projective and injective as a left $A \otimes B$ -module by virtue of Lemmas 10.3 and 17.6. Hence $A \otimes B$ is a QF-3 algebra by Lemma 17.1.

Finally we shall prove the following theorem⁶⁾; it may be of some interest in view of the known fact that an algebra A is uni-serial if and only if the residue class algebra A/I is a Frobenius algebra for every two-sided ideal I of A (cf. Nakayama [20]).

THEOREM 17.8. *An algebra A is generalized uni-serial if and only if the residue class algebra A/I is a QF-3 algebra for every two-sided ideal I of A .*

PROOF. Suppose that A is generalized uni-serial. If I is a two-sided ideal of A , then A/I is generalized uni-serial and hence it is a QF-3 algebra. This proves the "only if" part.

Conversely, suppose that the residue class algebra A/I is a QF-3 algebra for every two-sided ideal I of A . Let L be any indecomposable left A -module. If we denote by I the set of all elements x of A such that $xL=0$. Then I is a two-sided ideal of A and L is a faithful left A/I -module. Since A/I is a QF-3 algebra, we see that L is A/I -isomorphic to the unique minimal faithful left A/I -module. Hence L is A -homomorphic to a primitive left ideal of A . The same situation prevails for indecomposable right A -modules. Therefore, by a theorem of Nakayama⁷⁾, we conclude that A is generalized uni-serial. Thus our theorem is proved.

6) Added in proof.

7) T. Nakayama, Note on uni-serial and generalized uni-serial rings, Proc. Acad. Tokyo, **16**, 285-289 (1941).

APPENDIX

18. The uniqueness of duality for locally compact commutative groups

Let \mathfrak{G}_{LC} be the category consisting of all locally compact commutative groups and of all continuous homomorphisms. A contravariant functor D from \mathfrak{G}_{LC} to itself is said to be a *duality* for \mathfrak{G}_{LC} if the following two conditions are satisfied:

- 1) D^2 is naturally equivalent to the identity functor.
- 2) D maps the topological space $\text{Hom}(X, X')$ continuously into $\text{Hom}(D(X'), D(X))$ for any X, X' in \mathfrak{G}_{LC} .

Here (and throughout this section) $\text{Hom}(X, X')$ means the topological space which consists of all continuous homomorphisms of X into X' and which has the compact-open topology (i.e. the family of all $[C, G]$ forms a sub-basis where $[C, G] = \{\alpha \mid \alpha(C) \subseteq G, \alpha \in \text{Hom}(X, X')\}$ and C ranges over all compact sets of X and G ranges over all open sets of X').

Let P be the additive group of real numbers reduced modulo 1 and set

$$\text{Char } X = \text{Hom}(X, P).$$

For a continuous homomorphism $f: X \rightarrow X'$ we define $\text{Char } f: \text{Char } X' \rightarrow \text{Char } X$ by

$$[\text{Char } f](\alpha') = \alpha' \circ f \quad \text{for } \alpha' \in \text{Char } X'.$$

Then $\text{Char } f$ is a continuous homomorphism. The natural homomorphism

$$\pi(X): X \rightarrow \text{Char}(\text{Char } X)$$

defined by $[\pi(X)(x)](\alpha) = \alpha(x)$ for $\alpha \in \text{Char } X$ is continuous and moreover is a topological isomorphism. This is the Pontrjagin duality. Furthermore, Char induces a topological isomorphism of $\text{Hom}(X, X')$ onto $\text{Hom}(\text{Char } X', \text{Char } X)$ by the correspondence $f \rightarrow \text{Char } f$. This fact has been already observed by Eilenberg and MacLane [5, p. 256]. Thus the Pontrjagin duality is actually a duality for \mathfrak{G}_{LC} in our sense. The purpose of this section is to establish the following theorem.

THEOREM 18.1. *There exists essentially a unique duality for \mathfrak{G}_{LC} ; that is, any duality D for \mathfrak{G}_{LC} is equivalent to the Pontrjagin duality.*

PROOF. Let D be a duality for \mathfrak{G}_{LC} . Then there exists a natural equivalence λ from the identity functor to D^2 ; for each X in \mathfrak{G}_{LC} a topological isomorphism $\lambda(X): X \rightarrow D^2(X)$ is defined and the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow \lambda(X) & & \downarrow \lambda(X') \\ D^2(X) & \xrightarrow{D^2(f)} & D^2(X') \end{array}$$

is commutative for any continuous homomorphism $f: X \rightarrow X'$.

Let us denote by A the additive group of all integers with the discrete topology. For any locally compact commutative group X we define a continuous homomorphism $\varphi_x: A \rightarrow X$ by the formula $\varphi_x(a) = ax$ for $a \in A$. Then the mapping

$$(67) \quad \Gamma: X \rightarrow \text{Hom}(A, X)$$

defined by $\Gamma(x) = \varphi_x$ is a continuous homomorphism. Because for any compact set K of A and an open set G of X such that $\varphi_{x_0} \in [K, G]$ where x_0 is an element of X , we can determine a finite number of open sets $L_i, i=1, \dots, m$ of A and a finite number of open neighbourhoods $W_i(x_0)$ of x_0 such that

$$K \subseteq \bigcup_{i=1}^m L_i; \quad L_i W_i(x_0) \subseteq G, \quad i=1, \dots, m,$$

and hence we have

$$\varphi_x \in [K, G] \quad \text{for } x \in \bigcap_{i=1}^m W_i(x_0).$$

Thus Γ is continuous.

We now apply the proof of Theorem 5.1 to the present case. Let us set

$$U = D(A).$$

Since for any positive integer n we have $D(\varphi_n) = D(\varphi_1) + \dots + D(\varphi_1), D(\varphi_1) = 1$ where $\varphi_n: A \rightarrow A$ is defined by $\varphi_n(a) = na$, the relation $D(\varphi_n)u = nu$ holds for any $u \in U$ and for any positive (and hence negative) integer n . Hence the group U^* defined by (39) coincides with U . Since Γ is continuous, the mapping $\omega: U \rightarrow U$ defined by

$$\omega(u) = [D(\lambda(A)) \circ \lambda(D(A))]^{-1} \circ D(\varphi_u) \circ \lambda(A)(1)$$

is continuous (cf. (40)). Since $\omega^{-1}(u) = [D(\varphi_u) \circ \lambda(A)](1), \omega^{-1}$ is also continuous. Hence ω is a topological isomorphism. The mapping $\Phi_1(X) \rightarrow \text{Hom}(X, U)$ defined by (42), i.e.

$$[\Phi_1(X)](y) = \omega^{-1} \circ [D(\lambda(A)) \circ \lambda(D(A))]^{-1} \circ D(\varphi_y) \circ \lambda(X)$$

where $y \in D(X)$ and $\varphi_y(a) = ay$ for $a \in A$, is continuous. Since $[\Phi_1(X)]^{-1}(\alpha) = [D(\omega \circ \alpha) \circ \lambda(A)](1)$ for $\alpha \in \text{Hom}(X, U), \Phi_1(X)^{-1}$ is also continuous. Thus $\Phi_1(X)$ is a topological isomorphism. The ring-automorphism θ defined by (45) is the identity.

Therefore if we set, for any X in \mathfrak{G}_{LC} ,

$$\text{Char}_U X = \text{Hom}(X, U)$$

and define $\text{Char}_U f: \text{Char}_U X' \rightarrow \text{Char}_U X$ by the formula $[\text{Char}_U f](\alpha') = \alpha' \circ f$ where $f: X \rightarrow X'$ is a continuous homomorphism and $\alpha' \in \text{Char}_U X'$, then the functors D and Char_U are naturally equivalent and the natural homomorphism

$$\pi_U(X): X \rightarrow \text{Char}_U(\text{Char}_U X)$$

defined by $[\pi_U(X)(x)](\alpha) = \alpha(x)$ for $x \in X, \alpha \in \text{Char}_U X$, is a topological isomorphism; the latter fact is seen from (44).

Now we apply a theorem of Pontrjagin [33, p. 164], which is easily shown to be valid for a general locally compact commutative group Q , to the present case. Then we see that U must be topologically isomorphic to the additive group P of real numbers reduced modulo 1. Thus our theorem is proved.

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Added in proof. Several months after this paper had been submitted for publication, I received from Prof. G. Azumaya a copy of the manuscript of his forthcoming paper: A duality theory for injective modules, submitted to Amer. J. Math. I have found that his paper has some points in common with ours (for example, Theorems 6, 8, 10 and Proposition 10 in his paper are included in our paper). I wish to express my thanks to him for his kindness.

As for QF-1 algebras see my paper: On algebras for which every faithful representation is its own second commutator, forthcoming in Math. Zeitschr.