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The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (ver.003)

by

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Alice's Adventures in Wonderland*

The Integrated Theory of Selling and Buying Problems Based on Concepts of Symmetry and Analogy

— ver003 —

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Keywords

selling problem, buying problem, optimal stopping problem, search, symmetry, analogy, quitting penalty price, market restriction, recognizing time, staring time, initiating time, null time zone, deadline-engulfing

Abstract

A trading problem can be classified into the following four types: a selling problem and a buying problem, each of which can be categorized as a problem with a reservation price mechanism (where the counter trader proposes the trading price) and a problem with a posted price mechanism (where the leading trader proposes the trading price). Let us refer to this group of four problems as the *quadruple-asset-trading-problems*. The main objective of this paper is twofold: to construct a general theory that integrates the quadruple-asset-trading-problems and to analyze these problems by using the theory. To achieve this objectives, several novel concepts are introduced, say symmetry, analogy, initiating time, quitting penalty price, market restriction, etc. These concepts lead us to a new horizon that has not been previously explored by any researchers. The most notable findings resulting from the analyses of these models are twofold: first, there is a significant breakdown of symmetry between the selling problem and the buying problem; second, the existence of *null-time-zone*, a time period during which any decision-making activity is entirely senseless. Particularly, the latter discovery challenges us to re-examine the entire discussions that have been conducted so far regarding conventional trading problems as decision-making processes. Moreover interestingly, when this time zone encompasses all points in time on the planning horizon except the deadline, it follows that all decision-making activities scheduled throughout the entire planning horizon are *engulfed* in the deadline, which is reminiscent of all matter, even light, falling into a black hole. Lastly, we present an extensive range of models for asset trading problems that have not yet been proposed, concluding this study by emphasizing that the treatment of these problems is nearly impossible without the integrated theory.

♡

It was a spring afternoon in March, 1966, and the distant song of a bird filled the air. I (Ikuta) was in the office of my academic supervisor Dr. (Eng.) Shizuo Senju. Sunbeams streamed through leaves, casting a gentle sway on window glasses. The professor silently rose from the chair and drew a picture of one apple on the blackboard. He turned to me and said "*Would you take this apple? If you do, you can eat it and that will be the end of it. However, if you choose not to, this apple will disappear, and another one may appear — either greater or smaller than the one that vanished. In considering this situation, how would you decide whether or not to take this apple ?*". After a few moments of contemplation, the professor softly continued "*Many decision problems in corporate management have a similar structure ⋯. This is the subject of your master's thesis !*". With that, he left the room. Even now, the sound of the chalk sliding on the blackboard echoes in the depths of my ears. With those words, he exited the room, leaving behind the lingering resonance of chalk sliding on the blackboard, a sound forever etched in the recesses of my memory.

Version History

- ver000 (2004/07) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1084 [26,Ikuta], — The original paper
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*Readers will be bewildered by the indication in Alice 1(p.15), 2(p.44), 3(p.46), and 4(p.46).

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Contents

Abbreviations

$\mathbb{R}$-mechanism	$\mathbb{R}$eservation price mechanism
$\mathbb{P}$-mechanism	$\mathbb{P}$osted price mechanism
opt-$\mathbb{R}$-price	optimal reservation price
opt-$\mathbb{P}$-price	optimal posted price
$\mathbb{R}$-model	model with $\mathbb{R}$-mechanism
$\mathbb{P}$-model	model with $\mathbb{P}$-mechanism
$\mathbb{R}$-problem	problem with $\mathbb{R}$-mechanism
$\mathbb{P}$-problem	problem with $\mathbb{P}$-mechanism
ATP	Asset Trading Problem
ASP	Asset Selling Problem
ABP	Asset Buying Problem
ATM	Asset Trading Model
ASM	Asset Selling Model
ABM	Asset Buying Model
ATP[$\mathbb{R}$]	Asset Trading Problem with $\mathbb{R}$-mechanism
ATP[$\mathbb{P}$]	Asset Trading Problem with $\mathbb{P}$-mechanism
ASP[$\mathbb{R}$]	Asset Selling Problem with $\mathbb{R}$-mechanism
ASP[$\mathbb{P}$]	Asset Selling Problem with $\mathbb{P}$-mechanism
ABP[$\mathbb{R}$]	Asset Buying Problem with $\mathbb{R}$-mechanism
ABP[$\mathbb{P}$]	Asset Buying Problem with $\mathbb{P}$-mechanism
Tom	Lemma on Total market ($\mathscr{F}$)
Pom	Lemma on Positive market ($\mathscr{F}^+$)
Mim	Lemma on Mixed market ($\mathscr{F}^\pm$)
Nem	Lemma on Negative market ($\mathscr{F}^-$)
A	Assertion
$\mathscr{A}$	$\mathscr{A}$ssertion system
M:x[$\mathbb{X}$][X]	Model of asset selling problem ($x = 1, 2, 3$, $\mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
$\tilde{\text{M}}$:x[$\mathbb{X}$][X]	$\tilde{\text{M}}$odel of asset buying problem ($x = 1, 2, 3$, $\mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
$\mathcal{Q}$(Models)	quadruple-asset-trading-models
SOE	System of Optimality Equations [soé]
OIT	Optimal-Initiating-Time [òuít]
dOITs	degenerate OIT to the starting time
dOITd	degenerate OIT to the deadline
ndOIT	nondegenerate OIT
odr	optimal decision rule h
Null-TZ	Null-Time-Zone [nʌ́ltí:zí:]
sE-case	search-Enforced-case
sA-case	search-Allowed-case
tE-model	tackle-Enforced-model
tA-model	tackle-Allowed-model
sE-model	search-Enforced-model
sA-model	search-Allowed-model
iiE-Case	immediate-initiation-Enforced-case
iiA-Case	immediate-initiation-Allowed-case
iiE-model	immediate-initiation-Enforced-model
iiA-model	immediate-initiation-Allowed-model
tE-case	tackle-Enforced-case
tA-case	tackle-Allowed-case
C⇝S	abbreviation of C/S-switch
S⇝C	abbreviation of S/C-switch
C/S-switch	switch from "Conduct-search" to "Skip-search"
S/C-switch	switch from "Skip-search" to "Conduct-search"
[F.S.]	future subject
[F.S]	reference of [F.S.]

Symbols

$\mapsto$	reduction
$\twoheadrightarrow$	running-back
$\looparrowright$	migration
$\neq$	"not equal"
$\doteqdot$	"not always equal"
A/B	A and B
$A\backslash B$	A or B
$A\backslash\!\backslash B$	Either of A and B

Part 1

Introduction

This part provides all the necessary information that will be required for the construction of the integrated theory in Part 2 (p.51).

Chapter 1

Preface

1.1 Overview

First, let us focus on the fact that an economic behaviour is fundamentally constituted by various types of transactions. Thus far, different types of models for trading assets (house, car, a lot of land, $\cdots$), commodities (wheat, copper, gasoline, $\cdots$), and goods (fruit, fish, clothes, $\cdots$) have been proposed and examined. These trading problems can be classified into the following four types: an asset selling problem† and an asset buying problem,‡ each of which can be categorized as a problem with a reservation price mechanism (where the counter trader proposes the trading price ($\mathbb{R}$-mechanism)) and a problem with a posted price mechanism (where the leading trader proposes the trading price ($\mathbb{P}$-mechanism)). Let us refer to this group of four problems as the *quadruple-asset-trading-problems* (see Section 1.4.5(p.7)). In addition, taking into account the two events, "whether or not to conduct the search for counter trader (see A5(p.12))" and "presence or absence of quitting penalty price (see A7(p.12))", we can make up the six types of quadruple-asset-trading-problems in all, the whole of which is called the "*structured-unit-of-models.*" (see Section 3.3(p.18)). The main objective of this paper is twofold: to construct a general theory that integrates the quadruple-asset-trading-problems and to analyze these problems by using the theory. To achieve this objectives, several novel concepts are introduced, say symmetry, analogy, initiating time, quitting penalty price, market restriction, etc. These concepts lead us to a new horizon that has not been previously explored by any researchers. The most notable findings resulting from the analyses of these models are twofold: first, there is a significant breakdown of symmetry between the selling problem and the buying problem; second, the existence of *null-time-zone* (see Section 7.2.4.5(p.46)), a time period during which any decision-making activity is entirely senseless. Particularly, the latter discovery challenges us to re-examine the entire discussions that have been conducted so far regarding conventional trading problems as decision-making processes. Moreover interestingly, when this time zone encompasses all points in time on the planning horizon except the deadline, it follows that all decision-making activities scheduled throughout the entire planning horizon are *engulfed* in the deadline (see Section 7.2.4.6(p.46)), which is reminiscent of all matter, even light, falling into a *black hole*. Lastly, we present an extensive range of models for asset trading problems that have not yet been proposed (see Section 30.1(p.287) and F1(p.287)), concluding this study by emphasizing that the treatment of these problems is nearly impossible without the integrated theory.

1.2 Two Motives

While considering the four problems in the *quadruple-asset-trading-problems*, two questions as shown below naturally come to appear. The exploration of these questions has formed the two main motivations that have driven the present paper.

Motive 1 **Is a buying problem always symmetrical to a selling problem ?**

Long before the inception of this study, we held a naive perspective on the selling and buying problems: "Could a buying problem always be symmetrical to a selling problem ?" In other words, if we understand the nature of a seller's problem, could we immediately grasp the nature of its corresponding buyer's problem by merely altering the signs of variables, parameters, constants, etc. defined in the seller's problem? Our ultimate response to this viewpoint in this study is a resolute "no !"

Motive 2 **Does a general theory integrating quadruple-asset-trading-problems exist ?**

Before beginning to write this paper, we extensively reviewed numerous papers related to the buying and selling problems and naturally developed a *preliminary expectation* that there could potentially be a "*common denominator*" underlying all discussions presented therein. This intuition guided us to the insight (realization) that this common denominator is closely connected to a function known as the T-function defined by (5.1.1(p.25)). Urged by this insight, we soon developed a *faint anticipation* that a general theory integrating the quadruple-asset-trading-problems might exist. As we delved deeper into our exploration, *a ray of hope* emerged that constructing such a theory might

†[32,1962], [33,1963], [2,1977], [39,1983], [38,1983], [41,1990], [6,1991], [34,1993], [45,1993], [37,1995], [29,1995], [46,1995], [3,1995], [48,1997], [8,1997], [11,1998], [19,1999], [1,1999], [12,2001], [36,2002], [10,2002], [14,2004], [18,2005], [15,2005]

‡[8,1998], [10,2002]

indeed be possible. This hope was buoyed by introducing the concepts of symmetry (see Chap. 12(p.69)) and analogy (see Chap. 13(p.89)), and fortunately our attempt over more than fifty years led to the successful construction of this theory (see Chap. 16(p.115)).

1.3 Philosophical Background of This paper

Before proceeding with our discussions, below let us outline our philosophical background that underpins the entire writing of this paper.

1.3.1 Outset

When I (Ikuta) was a high-school student (1958), during a physics lesson, the teacher placed one cotton ball and one iron ball in a glass tube of one-meter length, setting it upright. Not surprisingly, the iron ball fell with a thud, and the cotton ball fell slowly as if chasing the iron ball. Afterward, the air in the tube was evacuated with a turn of the motor switch, and the tube was again set upright. This time, both balls fell alongside. Why? A surprise passed through my mind. The teacher then drew a picture on the blackboard and explained the rationality of this phenomenon; it was my first introduction to the power of real experiments and thought experiments in physics. After an interval, he mentioned that Galileo conducted an experiment of a free fall in the Tower of Pisa and harked back that it took several thousand years to recognize the shift from the earth-centered theory to the sun-centered theory (the Copernican revolution). Shortly afterwards, the teacher tossed a sponge ball from the platform toward us (students) and explained that the trajectory of an object tossed over forms a parabola expressed by the quadratic curve. Without air, a speed at which an object thrown horizontally will loop back around the earth, drawing a circular orbit, is approximately 7.9 kilometers per second, and the speed at which it flies out of the orbit is about 11.2 kilometers per second. After graduating from high-school, I enrolled in the engineering department of Keio University, where I learned high-level physics. In the spring afternoon of March, 1966, I visited the laboratory office of my academic supervisor, Dr. Professor Shizuo Senju (see the episode on the title page of this paper). In the process of this personal history, I gradually came to recognize not only natural phenomena but also human behaviors physically. This is the fundamental outset that has influenced the entirety of my investigative life.

1.3.2 Decision Theory as Physics

Basically, every human being's behavior is influenced by their underlying philosophical background. Therefore, naturally, the authors (Ikuta & Kang, both holding Dr.Engineering) consistently approached their research with a deep-rooted focus on the *physical* perspective. Since physics is a research discipline free from preconceived premises, assumptions, hypotheses, or preconceptions, it requires researchers to actively engage both ears and eyes in observing the research object, calmly listening to every sound from its depths and carefully observing every light emerging within. While the authors are open to integrating concepts, knowledge, and techniques from business administration, economics, and mathematics as necessary, their core viewpoint is that decision processes are inherently connected to human-driven *physical* phenomena. Accordingly, for us who are both natural scientists, it follows that the decision theory discussed in this paper is *a decision theory as physics.* If we were not researchers in the field of natural science, this paper would not saw the light of day at all.

1.4 Structure of Asset Trading Problems

The section clarifies the structure of asset trading problems which are dealt with in this paper.

1.4.1 Definitions of Terms

Before moving on, let us establish definitions for some key terms that will be used in our upcoming discussion.

- For the subject matter of transaction, whether properties, commodities, or goods, we refer to it as the *asset* in a general term.
- For the decision-making problem related to the trading of asset, we refer to it as the *asset trading problem,* `ATP` for short, consisting of *asset selling problem* and *asset buying problem*, simply `ASP` and `ABP` respectively.
- For the parts involved in a trading, we use the terms "*leading-trader*" and "*counter-trader*" to distinguish between the part leading the trading and its counterpart. Accordingly, in `ASP` (`ABP`), the seller (buyer) is a *leading-trader* and the buyer (seller) is an *counter-trader.*

1.4.2 Asset Trading Problem (ATP)

Below, let us conceptualize the asset trading problem as a drama involving a *leading-trader* and an *counter-trader* on unfolding two scenes below:

- `Scene` $\mathbb{R}$ in which
  - first a counter-trader appears and posts his trading price,
  - then a leading-trader appears and answers whether or not to accept it based on his reservation price.†
- `Scene` $\mathbb{P}$ in which
  - first a leading-trader appears and posts his trading price,

◦ then a counter-trader appears and answers whether or not to accept it based on his reservation price.

Let us refer to the trading in `Scene` $\mathbb{R}$ (`Scene` $\mathbb{P}$) as the *asset trading problem with the reservation price mechanism* (*posted price mechanism*), simply `ATP` with $\mathbb{R}$-mechanism[¶] (`ATP` with $\mathbb{P}$-mechanism[‖]), further abbreviated as

$$\texttt{ATP}[\mathbb{R}]\ (\texttt{ATP}[\mathbb{P}]).$$

The above asset *trading* problem (`ATP`) can be translated into the asset *selling* problem (`ASP`) and the asset *buying* problem (`ABP`) which are presented in the two sections that follows.

1.4.3 Asset Selling Problem (ASP)

In the asset selling problem (`ASP`), a leading-trader is a seller and its counter-trader is a buyer, hence the drama of the asset selling problem can be rewritten as below:

`Scene` $\mathbb{R}$ in which

- ◦ first a buyer (counter-trader) appears and posts his buying price,
- ◦ then a seller (leading-trader) appears and answers whether or not to accept it based on his reservation price.

`Scene` $\mathbb{P}$ in which

- ◦ first a seller (leading-trader) appears and posts his selling price,
- ◦ then a buyer (counter-trader) appears and answers whether or not to accept it based on his reservation price.

Let us refer to the asset selling problem in `Scene` $\mathbb{R}$ (`Scene` $\mathbb{P}$) as the *asset selling problem with reservation price mechanism* (*posted price mechanism*), simply `ASP` with $\mathbb{R}$-mechanism ($\mathbb{P}$-mechanism), further abbreviated as

$$\texttt{ASP}[\mathbb{R}]\ (\texttt{ASP}[\mathbb{P}]).$$

The following two examples convey a flavor of the above asset selling problem, which mirror the "mental conflict" (see Section 7.3(p.47)) of a seller (leading-trader) in the above drama.

□ *Example* **1.4.1 (`Scene` $\mathbb{R}$)** Suppose you (seller, leading-trader) have to sell your car by a specified deadline due to a compelling reason, such as being required to suddenly return to your mother country by order of the head office when you are stationed in a foreign country. Suppose a potential buyer (counter-trader) has just appeared. In this situation, if the buyer offers a high buying price, you (leading-trader) would likely sell the car. However, if the offered price is very low, you might hesitate. In either case, you are faced with a decision that involves the following risks. Selling the car carries the risk of missing out a higher-paying buyer that may appear in the future. On the other hand, not selling the car carries the risk that a higher-paying buyer may not appear before the deadline, or even worse, no buyers may appear at all, leading to the necessity of selling the car at a very low price (a giveaway price) or incurring costs to dispose of it. Considering these risks, you must decide whether or not to sell your car to each successive buyer. This perspective implies that, as the deadline approaches, it is necessary to gradually lower *the minimum permissible selling price* (*reservation price*). The above *expectation* reflects a mental conflict that you must more and more become "selling spree" as the deadline approaches. ▯

The above example is what has been defined and investigated under the name "optimal stopping problem". To the best of the authors' knowledge, the earliest papers related to the problem can be traced back to 1960's
[44,1961][32,1962][9,1971][35,1973].

□ *Example* **1.4.2 (`Scene` $\mathbb{P}$)** In the same example as mentioned above, let us suppose that you (leading-trader) set a selling price for your car to buyers who appear successively in front of you. In the situation, if you set your price too low, a buyer will buy the car, conversely, if your price is excessively high, the buyer will leave (walk away). This indicates that a low posted price carries the risk of missing an opportunity that a potential buyer willing to pay a higher price appears in the future. On the other hand, setting a high posted price carries the risk of no buyer who buys for such a price appearing before the deadline; if so, then you are compelled to sell your car at a significantly reduced price (a rock-bottom price) or dispose of it at a cost. Considering these risks, you must decide whether or not to sell your car to each successive buyer. Similarly to in *Example* 1.4.1(p.5), this perspective implies that, as the deadline approaches, it is necessary to gradually lower *the selling price to propose* (*posted price*). The above *expectation* reflects a mental conflict that you must more and more become "selling spree" as the deadline approaches. ▯

[†] A threshold based on which it is judged whether or not to accept it.
[¶] [3,1995],[5,2001]
[‖] [4,1998],[5,2001],[20,1994],[45,1993],[46,1995]

1.4.4 Asset Buying Problem (ABP)

In the asset buying problem (ABP), a leading-trader is a buyer and its counter-trader is a seller, hence the drama of the asset buying problem can be rewritten as below:

Scene $\mathbb{R}$ in which

- first a seller (counter-trader) appears and posts his selling price,
- then a buyer (leading-trader) appears and answers whether or not to accept it based on his reservation price.

Scene $\mathbb{P}$ in which

- first a buyer (leading-trader) appears and posts his buying price,
- then a seller (counter-trader) appears and answers whether or not to accept it based on his reservation price.

Let us refer to the asset buying problem in Scene $\mathbb{R}$ (Scene $\mathbb{P}$) as the *asset buying problem with reservation price mechanism* (*posted price mechasnism*), simply ABP with $\mathbb{R}$-mechanism (ABP with $\mathbb{P}$-mechanism), further abbreviated as

$$\text{ABP}[\mathbb{R}] \ (\text{ABP}[\mathbb{P}]).$$

One may say that since the above two problems seem to be *mere inverses* of the asset selling problem aforementioned, they are redundant and unnecessary. However, it will be known later on that fine differences between the asset selling problem (ASP) and the asset buying problem (ABP) produces a significant difference between both. The following two examples convey a flavor of the models of the above asset buying problem, which mirror the "mental conflict" (see Section 7.3(p.47)) of a buyer (leading-trader) in the above drama.

□ *Example* **1.4.3 (Scene $\mathbb{R}$)** Suppose you (buyer, leading-trader) have to buy a car by a specified date (deadline), and then you find a potential seller. In this situation, if the price offered by the seller is low enough, you will buy the car from the seller. However, if it is very high, you will hesitate. Buying the car carries the risk of missing an opportunity that you can find a potential seller offering a lower price in the future. On the other hand, not buying a car carries the risk that a lower-offering seller may not appears before the deadline. Considering these risks, you must decide whether or not to buy a car from each successive seller. This perspective implies that, as the deadline approaches, it is necessary to gradually raise *the maximum permissible buying price* (*reservation price*). The above *expectation* reflects a mental conflict that you must more and more become "buying spree" as the deadline approaches. ■

□ *Example* **1.4.4 (Scene $\mathbb{P}$)** Suppose that you (leading-trader) propose your buying price to a potential seller. Then, if your proposed price is high enough, the seller will sell the car, conversely, if it is very low, the seller will reject the offer. Buying the car carries the risk that a seller offering a lower price may appear in the future. On the other hand, not buying a car carries the risk that a lower-offering seller may not appear before the deadline. Considering these risks, you must determine your buying price to propose. Similarly to in *Example* 1.4.3(p.6), this perspective implies that, as the deadline approaches, it is necessary to gradually raise *the buying price to propose* (*proposed price*). The above *expectation* reflects a mental conflict that you must more and more become "buying spree" as the deadline approaches. ■

1.4.5 Quadruple-Asset-Trading-Problems

Let us refer to the set of the four asset trading problems ASP[ℝ], ABP[ℝ], ASP[ℙ], and ABP[ℙ] defined above as the *quadruple-asset-trading-problems*, represented as

$$\mathtt{qATP} = \{\mathtt{ASP}[\mathbb{R}], \mathtt{ABP}[\mathbb{R}], \mathtt{ASP}[\mathbb{P}], \mathtt{ABP}[\mathbb{P}]\}. \tag{1.4.1}$$

The interconnectedness among these problems are somewhat akin to a drama played across the *looking glass*, depicted as in Figure 1.4.1(p.7) below.

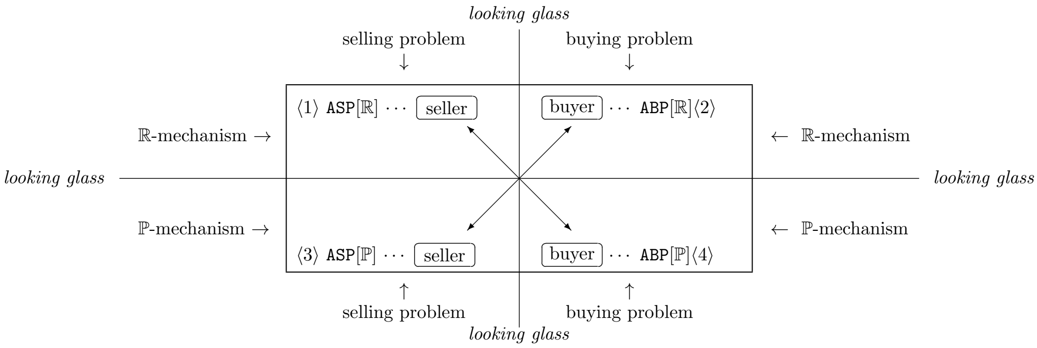

Figure 1.4.1: Interconnectedness among the quadruple-asset trading-problems

The aslant arrows ✕ in the above figure symbolizes a drama which revolves between a leading-trader in ASP and a leading-trader in ABP, i.e.,

- ↘ The leading-trader seller in ⟨1⟩ ASP[ℝ] faces, across the *looking glass*, the leading-trader buyer in ⟨4⟩ ABP[ℙ],†
- ↖ The leading-trader buyer in ⟨4⟩ ABP[ℙ] faces, across the *looking glass*, the leading-trader seller in ⟨1⟩ ASP[ℝ],‡
- ↙ The leading-trader buyer in ⟨2⟩ ABP[ℝ] faces, across the *looking glass*, the leading-trader seller in ⟨3⟩ ASP[ℙ],
- ↗ The leading-trader seller in ⟨3⟩ ASP[ℙ] faces, across the *looking glass*, the leading-trader buyer in ⟨2⟩ ABP[ℝ].

1.4.6 Symmetry and Analogy

The concepts of symmetry and analogy† play pivotal role in the construction of the integrated theory as stated in Motive 2(p.3). We delve into these concepts further by illustrating their significance in Figure 1.4.2(p.7) below.

(i) A symmetry is observed between ⟨1⟩ ASP[ℝ] and ⟨2⟩ ABP[ℝ],
(ii) A symmetry is observed between ⟨3⟩ ASP[ℙ] and ⟨4⟩ ABP[ℙ],
(iii) An analogy is observed between ⟨1⟩ ASP[ℝ] and ⟨3⟩ ASP[ℙ],
(iv) An analogy is observed between ⟨2⟩ ABP[ℝ] and ⟨4⟩ ABP[ℙ].

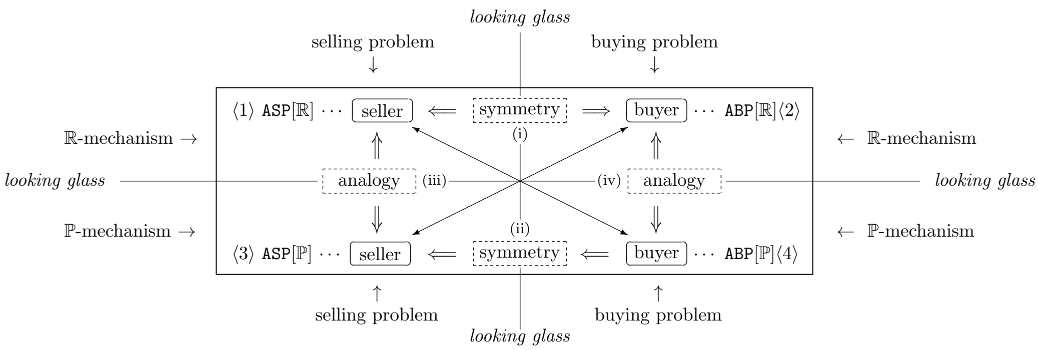

Figure 1.4.2: Symmetry and analogy among the quadruple-asset-trading-problems

Roughly speaking, the two concepts implies the following. The symmetry relation in (i) and (ii) means that for each of $\mathbb{X} = \mathbb{R}, \mathbb{P}$, a given *transformation* of some variables in an assertion on ASP[𝕏] yields its corresponding assertion on ABP[𝕏] and vice versa, and the analogy relation in (iii) and (iv) means that a given *replacement* of some variables in an assertion on ATP[ℝ] yields its corresponding assertion on ATP[ℙ] and vice versa.

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† The leading-trader buyer in ⟨4⟩ ABP[ℙ] is an counter-trader from the standpoint of the leading-trader seller in ⟨1⟩ ASP[ℝ].
‡ The leading-trader seller in ⟨1⟩ ABP[ℝ] is an counter-trader from the standpoint of the leading-trader buyer in ⟨4⟩ ASP[ℙ].
† The strict definitions and deep implications of symmetry and analogy will be given in Chaps. 12(p.69), 13(p.89), 14(p.101), and 15(p.111).

1.5 Highlights of This Paper

Before proceeding with our discussions, let us outline the key points of this paper.

H1. Five points in time

The following five points in time (see Section 7.1(p.43)) are essential requisites that inevitably arise from the philosophical background of "*decision theory as physics*" (see Section 1.3.2(p.4)). Below are summaries of implications that they have:

a. Recognizing time t_r

Since a decision is, after all, what is made by a human-being, it eventually follows that a behaviour of "decision" first materializes only when being recognized in the bottom of heart of a person; let us refer to the time point of this recognition as the *recognizing time* t_r. Now, when a decision-making problem is recognized, the first question to answer is whether or not it is enforced to tackle with the decision problem.

i. tE-case: Let us refer to the former case as the *tackle-Enforced-case*. In this case, even if it is known to yield no profit when tackling the problem, a decision-maker must accept the red ink.

ii. tA-case: Let us refer to the latter case as the *tackle-Allowed* (*not enforced*) *case*. In this case, a decision-maker has the option "whether to tackle the problem or not". Therefore, when it is known that no profit yields even if tackling the problem, it suffices not to tackle it.

b. Starting time τ

Whether in tE-case or when it is determined to tackle the problem in tA-case, after a period of preparation, it arrives at the time when the decision-maker can *start* to initiate the attack of the decision-making problem. Let us refer to the time point as the *starting time* τ.

c. Initiating time t_i

Before moving further on ahead, let us suppose the following two cases related to "whether or not it is *enforced* to *immediately initiate* the attack of the problem at the starting time τ":

i. iiE-Case: The case in which it is *enforced* to immediately initiate the attack, called the *immediate-initiation-enforced-case*.

ii. iiA-Case: The case in which it is *allowed* (*not enforced*) to immediately initiate the attack, called the *immediate-initiation-allowed-case*. In this case, it is possible to postpone its initiation; in other words, we have the options "initiation at the starting time τ", "initiation at the time $\tau - 1$", $\cdots$, "initiation at the deadline (time 0)". Then, if it is determined to initiate the attack of the decision-making problem at time t ($\tau \geq t \geq 0$), let us refer to this time point as the *initiating time* t_i. Here it is naturally questioned how to determine the *optimal initiating time*, denoted by t^*_τ ($\tau \geq t^*_\tau \geq 0$) (see Section 7.2.4.1(p.44)).

d. Stopping time t_s

When the attack of the decision-making problem initiates at the optimal initiating time t^*_τ and the asset's sale (in ASP) or the asset's purchase (in ABP) occurs thereafter, the process stops at that time. We refer to this point in time as the *stopping time* t_s .

e. Deadline 0

In this paper, from a practical viewpoint, we stress that a decision process with an *infinite* planning horizon is a product of mathematical imagination beyond the real world; in fact, considering a planning horizon spanning over 135 hundred millions years is nonsensical and futile. Therefore, in this paper, we will focus on only models with *finite* planning horizons. Then, let us refer to the terminal (final) point in time of the decision process as *deadline*. However, we can have the two reasons for which it becomes still meaningful to discuss the model with the *infinite* planning horizon. One is that it can become an approximation for the models with an enough long (finite) planning horizon, the other is that results mathematically derived from it can provide an important information for the analyses of models with the *finite* planning horizon.

f. The flow of the five points in time

All physical phenomena are not alien to a time concept; in other words, there do not exist physical phenomena alien to the time concept. The above five points in time are concepts that were yielded by our physical recognition. The flow of these points in time can be depicted as below.

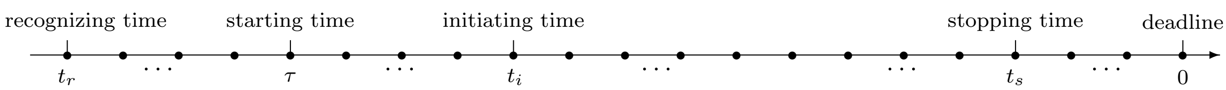

Figure 1.5.1: Five points in time

H2. Deadline and Decision-Making Behaviour

A decision process with a *finite* planning horizon is akin to a conveyor-belt machinery which willy-nilly moves on to a deadline with the passage of time, usually leading to undesirable results, say a sale for a giveaway price in *Example* 1.4.1(p.5), a bankruptcy in the business management, and a ruination of state in the political decision. This event which is brought forth by the deadline becomes stronger as it gets nearer to the deadline and conversely weaker as it get away from the deadline. The mental conflict of a seller (decision-maker) stated in *Examples* 1.4.1(p.5) graphically reflects this situation in the sense that the reservation price of a seller becomes smaller as the distance from the deadline get shorter. The above phenomenon also implies that a decision-making behaviour at any point in time is, in varying degrees, touched off by the existence of deadline. For this reason, the existence of deadline should be said to be an imperative requirement of decision process in the real world. In other words, the decision process with *infinite* planning horizon (without deadline) is what can be considered only at an abstract level (see A11(p.13)), implying that the existence of such decision process should be said to be *a creature of fantasy* from the realistic viewpoint.

H3. Null-time-zone and Deadline-engulfing

Before delving into the implication of the two terms in the title, let us recall here the definitions of the starting time τ (see H1b(p.8)) and the initiating time t_i (see H1c(p.8)). Then, the case of $\tau > t_\tau^*$ indicates that no action is taken at every point in time $t \in \{\tau, \tau-1, \cdots, t_\tau^*\}$. In this case, we will refer to this period of time as the *null-time-zone* (see Section 7.2.4.5(p.46)). Next, consider an interesting case in which the *optimal initiating time t_τ^* coincides with the deadline, i.e., $t_\tau^* = 0$*. This situation ultimately implies that any actions undertaken prior to the deadline are rendered meaningless, suggesting "Don't do anything until the deadline." Using a metaphorical comparison, it is akin to "All actions that must be undertaken before the deadline being engulfed by the deadline", much like all forms of matter, including light, being absorbed into a black hole. Taking this into consideration, we refer to this phenomenon as *deadline-engulfing* (see Section 7.2.4.6(p.46)). Then, when we regard a decision process with the *infinite* planning horizon as the limiting process of the finite planning horizon process, the existence of "deadline-engulfing" implies that the decision process with the *finite* planning horizon fades away in time toward the infinite future. What are presented above can be said to be one of the most remarkable discoveries in this paper, compelling us to undertake a comprehensive re-examination of the entire theory of decision processes that have been explored so far without taken into account the phenomenon of "deadline-engulfing".

H4. Symmetry

The notion of the adjective "symmetrical" used in the description of Motive 1(p.3) was initially sparked by a vague inspiration. This notion is shaped in the process in which transforming some of variables and constants related to the asset selling problem with $\mathbb{R}$-mechanism ($\texttt{ASP}[\mathbb{R}]$) produces its corresponding asset buying problem with $\mathbb{R}$-mechanism $\texttt{ABP}[\mathbb{R}]$ (see Chap. 12(p.69)).

H5. Analogy

At the earlier stage of this study we could not absolutely imagine that there will exist a relationship between the asset selling problem with $\mathbb{R}$-mechanism ($\texttt{ASP}[\mathbb{R}]$) and the asset selling problem with $\mathbb{P}$-mechanism ($\texttt{ASP}[\mathbb{P}]$). However, in the process of delving into discussions, we observed certain similarity between the two problems. This insight led us, before long, to a procedure, called the *analogy replacement operation*, replacing the two parameters a and μ[†] included within $\texttt{ASP}[\mathbb{R}]$ by $a^{\star}$[‡] and a respectively yields $\texttt{ASP}[\mathbb{P}]$ (see Chap. 13(p.89)).

H6. Integrated Theory

One of the most important results obtained in this paper is the successful construction of the theory integrating selling and buying problems based on concepts of symmetry and analogy. The two concepts were derived through a highly complicated discussion in Chaps. 12(p.69), 13(p.89), 14(p.101), and 15(p.111). The full spectrum of this theory can be schematized by Figure 16.2.1(p.115).

H7. Collapse of symmetry

The symmetry and analogy in H6(p.9) is discussed under the premise that the price ξ is defined on the interval $(-\infty, \infty)$, which allows for the possibility of negative values. However, in a typical the real-world, prices ξ are always positive, i.e., $\xi \in (0, \infty)$. Consequently, if $(-\infty, \infty)$ is constrained to $(0, \infty)$, then a natural question arises: "Is the symmetry inherited ?" (see Motive 1(p.3)). Contrary to this expectation, it will be observed later that it is not inherited.

H8. Underlying functions

The introduction of the underlying functions T, L, K, and $\mathcal{L}$ (see Chap. 5(p.25)) stands as a significant highlights in this paper. While T-function has been widely recognized thus far in fields of statistics, operational research, and economics (see [13,Deg1970]), the remaining underlying functions L, K, and $\mathcal{L}$ are all what are first defined in the present paper. It will be known later on that the properties of these functions (see Chap. 10(p.55)) play a central role in the analyses of all the models dealt with in the present paper. Without properties of these functions, not only could we challenge systematic analysis of these models, but also the successful construction of the integrated theory would have been nearly impossible.

[†] The lower bound a and the expectation μ of the distribution function of ξ (see A9(p.13))

[‡] See (5.1.26(p.26))

H9. Structured-unit-of models

This paper addresses two types of models, no-recall model and recall model (see Section 3.2(p.17)). For each model we define 24 distinct models. In this paper we refer to the whole of these 24 models as the *structured-unit-of-model* (see Section 3.3(p.18)). Now, these 24 models are not what were *capriciously* defined but what were *inevitably* established based on the principles of search enforced/allowed-case (see (A5(p.12))) and quitting penalty price ρ (see A7(p.12)). In this paper, through treating the entirety of these 24 models as a cohesive unit, we endeavored to comprehensively analyze all of them. Although so many models of asset trading problems have been posed so far,† all of them have been one-by-one and independently treated thus far without touching upon any relationships each other. Against this, in the present paper, we aim to clarify the *interconnectedness* among all models included in the structured-unit-of-model.

† [32,1962], [33,1963], [2,1977], [39,1983], [38,1983], [41,1990], [6,1991], [34,1993], [45,1993], [37,1995], [29,1995], [46,1995], [3,1995], [48,1997], [8,1997], [11,1998], [19,1999], [1,1999], [12,2001], [36,2002], [10,2002], [14,2004], [18,2005], [15,2005] ‡

Chapter 2

Preliminaries

2.1 Simplification of Models

In addressing a given real-world problem, two distinct approaches emerge. One is the construction of a model that faithfully represents its research object to the greatest extent possible. The other involves building the simplest model conceivable where further simplification risks the loss of its existence itself. Here, we label research based on the former as *experimental study* and the latter as *theoretical study*. While there is no inherent superiority between these two approaches, our overall stance in the present paper aligns with the latter. The methodology classification into these two categories acts as a *dividing ridge*, causing a study to bifurcate in counter directions. The first drop of water from the former follows the east wall, and the first drop of water from the latter follows the west wall. Eventually, both converge in a lake with a common bottom, and shortly thereafter, a flower blooms. This amalgamation of results from both methodologies leads us to a genuine understanding of the reality in question.

2.2 Assumptions

In order to realize the simplification of models that was presented above, let us configure the following assumptions:

A1 Points in time

The asset trading process occurs intermittently at points in time equally spaced along a finite length of the time axis as depicted in Figure 2.2.1(p.11) below. We shall backward label each point in time from the final point in time, denoted as time 0 (deadline), as 0, 1, and so forth. Accordingly, when the *present* point in time is designated as time t, the two adjacent points in time, $t+1$ and $t-1$, are the *previous* and *next* points in time respectively.

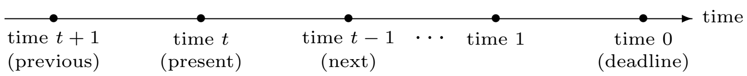

Figure 2.2.1: Points in time

A2 Absolutely necessary condition

In `ASP` (`ABP`), the leading-trader acting as a seller (buyer) must sell (buy), by all means, the trading asset to a buyer (from a seller) by the deadline. To rephrase, the seller (buyer) is not allowed to quit the selling (buying) process without completing the sale (purchase) of the asset.

A3 Stop of process

The process *stops* when the leading-trader accepts a price proposed by an counter-trader in `ATP`$[\mathbb{R}]$ and when an counter-trader accepts a price proposed by the leading-trader in `ATP`$[\mathbb{P}]$.

A4 Search cost

A cost $s \geq 0$ (*search cost*) must be paid to search for counter-traders, which includes expenses for advertising, communication, transfer, and so on.

A5 Search-Enforced-Model and search-Allowed-Model

The existence of the search cost s inevitably leads us to a question whether conducting the search activity always become profitable or not. Here we consider the following two cases in the paper.

a. search-Enforced-model (sE-model): This refers to the case in which, once the process has initiated, conducting the search is mandatory at every subsequent point in time. In this case, the above question loses its meaning. Then, as illustrated in Figure 2.2.2(p.12) below, a decision-maker must continue to conduct the search until the process stops.

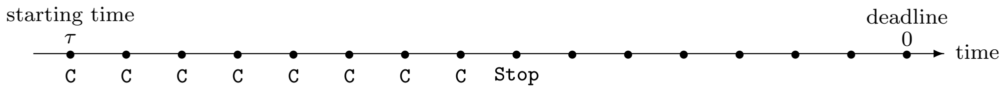

Figure 2.2.2: Flow of search-Conducts in the search-Enforced-model

b. search-Allowed-model (sA-model): This refers to the case in which, after the process has initiated, it is *permissible* to skip the search at every subsequent point in time. In other words, a leading-trader has the option to conduct the search or to skip at every point in time as long as the process does not stop. In this case, it becomes a necessary subject of study how to answer the above question. Then, we can consider different types of flows for search-Conduct and search-Skip, as illustrated in Figure 2.2.3(p.12) below, where "⇝" represents the transition from search-Skip to search-Conduct or from search-Conduct to search-Skip.

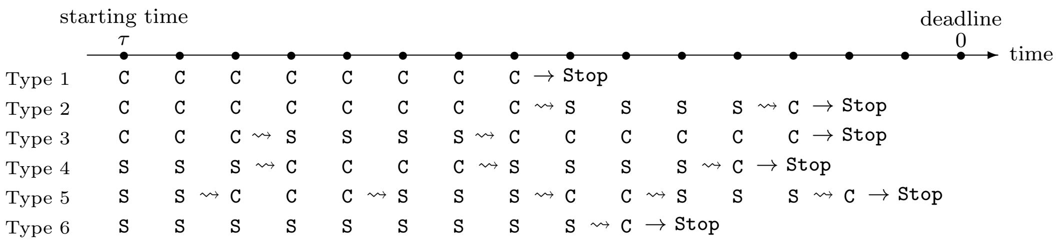

Figure 2.2.3: Different flows of search-Conduct and search-Skip

Definition 2.2.1 By C⇝S (S⇝C) let us denote the switch from search-Conduct to search-Skip (search-Skip to search-Conduct). □

A6 Opposite-trader's appearance probability λ

In this paper, it is assumed that when the search is conducted at a certain point in time, a counter-trader appears at the next point in time with a known probability λ $(0 < \lambda \leq 1)$.

A7 Quitting penalty price

Suppose that a counter-trader appearing probability λ is less than 1, i.e., $0 < \lambda < 1$. Then it is possible that no counter-trader appears in the subsequent points in time even if conducting the search. This situation can lead to the risk that a leading-trader potentially has to quit the process at the final point in time point (deadline) without executing the trade for the asset, which contradicts the requirement of A2. When facing with such a circumstance, the leading-trader will take the following actions at the deadline:

- In ASP, the seller (leading-trader) will attempt to find ways to sell the asset by proposing a giveaway price ρ to any available buyer (counter-trader).
- In ABP, the buyer (leading-trader) will strive to acquire the asset by presenting a notably high-price ρ to any available seller (counter-trader).

Let us refer to such a price ρ as the *terminal quitting penalty price* ρ, implying that, at the deadline, the leading-trader can quit the process in exchange for the ρ. Additionally, we can consider the case that such a ρ is available also at every point in time including the deadline. Then let us refer to it as the *intervening quitting penalty price*. In the explanation above, the ρ is implicitly assumed to be positive $\rho \in (0, \infty)$; however, to generalize discussions that follows, we define it to be $\xi \in (-\infty, \infty)$.

A8 Range of price

Whether a price ξ proposed by an appearing counter-trader or the reservation price ξ of an appearing counter-trader, it should be defined on $(0, \infty)$ in the normal market of the real-world (see Section 17.2(p.117)). However, in this paper, to successfully construct the integrated theory in Part 2 (p.51) we dare to define it on $(-\infty, \infty)$.

A9 Distribution function

In ATP[$\mathbb{R}$] (ATP[$\mathbb{P}$]) we assume that the prices proposed by successively appearing counter-trader, $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\cdots$ (the reservation prices of successively appearing counter-trader, $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\cdots$) are independent identically distributed random variables having a *continuous* distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with a finite expectation μ where

$$\begin{aligned} F(\xi) &= 0 \quad \cdots (1) \quad \xi \le a, \\ 0 < F(\xi) &< 1 \quad \cdots (2) \quad a < \xi < b, \\ F(\xi) &= 1 \quad \cdots (3) \quad b \le \xi, \end{aligned} \tag{2.2.1}$$

for given constants a and b such that

$$-\infty < a < \mu < b < \infty. \tag{2.2.2}$$

Furthermore, for its probability density function $f(\xi)$ let us assume

$$\begin{aligned} f(\xi) &= 0 \quad \cdots (1) \quad \xi < a, \\ 0 < f(\xi) &< 1 \quad \cdots (2) \quad a \le \xi \le b, \\ f(\xi) &= 0 \quad \cdots (3) \quad b < \xi. \end{aligned} \tag{2.2.3}$$

Here assume that there exits $\underline{f}$ such that

$$\underline{f} = \inf_{a \le \xi \le b} f(\xi) d\xi > 0. \tag{2.2.4}$$

Let us represent the set consisting of all possible distribution functions with (2.2.2(p.13)) by $\mathscr{F}$, i.e.,

$$\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}, \tag{2.2.5}$$

called the *total distribution function space*, simply the *total*-DF-*space* .

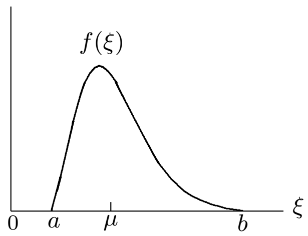

Figure 2.2.4: Probability density function $f(\xi)$

A10 Recallability of once rejected counter-trader

Whether model with $\mathbb{R}$-mechanism or model with $\mathbb{P}$-mechanism, if a once-rejected counter-trader can be *recalled* later and accepted at the discretion of the leading-trader, then it is referred to as the *recall-model* or *model-with-recall* (see Section 3.2.2(p.18)). Conversely, if such recallability is not allowed, then it is referred to as the *no-recall-model*, *model-with-no-recall*, or *model-without-recall*.

A11 Finiteness of planning horizon

In the present paper we consider only models with the *finite* planning horizon (see H1e(p.8)). Our basic standpoint over the whole of this paper lies in a *grim reality* that a process with the *infinite* planning horizon is a *mere product of fantasy* created by mathematics, which does not exist in the real world at all; in fact, it is an inanity to consider a model with the planning horizon of more than 135 hundred millions years. However, we can have the two reasons for which it becomes still meaningful to discuss the model with the *infinite* planning horizon. One is that it can become an approximation for the process with an enough long (finite) planning horizon, the other is that results obtained from it can provide a meaningful information for the analyses of models with the *finite* planning horizon.

2.3 Discount factor

This section presents the actual and theoretical implication of the discount factor which will be used in describing the systems of optimality equations for any decision processes (see Chap. 6(p.29))

2.3.1 Our Basic Stance

In whether mathematics or natural science, what to do first of all before proceeding with discussions is to clearly define the concepts and terms employed there. Going back this stance, in this section we try to provide rigorous definitions for terms *profit* and *cost* which seem to be quite commonplace at a glance in the fields of business science and economic science.

2.3.2 Definitions

To start with, we provide the following four definitions.

(a) *Fund F*: We refer to the total amount of available money on hand as the *fund F*, which can be always and freely invested.

(b) *Interest rate r*: We denote the interest rate per period by $r \geq 0$, implying that the today's fund of one unit increases to the $1+r$ units tomorrow. Here let us define $\beta = (1+r)^{-1}$ $(1 \geq \beta > 0)$, called the *discount factor*. Then $1+r = \beta^{-1}$.

(c) *Profit P*: Let us refer to the *incremented amount* of the fund F yielded by a managerial and/or economic activity as the *profit P*, i.e.,

$$\text{profit } P = \text{incremented fund } F^i \cdots (1^\bullet),$$

(d) *Cost C*: The definition of cost is rather complicated.

- Suppose that an amount of fund F has been paid away for a reason. Here we refer to the amount of fund paid away as the *expense E* and to the amount of fund *lost* by the *expense E* as the *decremented fund F^d*, i.e.,

$$\text{expense } E = \text{decremented fund } F^d \cdots (2^\bullet).$$

  Here anew register that the profit P is defined by the incremented fund F^i and that the expense E is defined by the decremented fund F^d.

- Although the *decremented fund F^d* is what was *once* paid away as an *expense E*, supposedly let us assume here that the expense E were not paid away. Then, it backs to the fund F, hence it is squirreled away as a *savings* on hand, so let us refer to this savings as the *conditional savings S^c* in the sense that it remains on hand under the *condition* of not having been paid away, i.e.,

$$\text{decremented fund } F^d = \text{conditional savings } S^c \cdots (3^\bullet).$$

- Now, since this *conditional savings S^c* is *eventually* reduced to what must be paid away *in the end*, we refer to this *conditional savings S^c* as the *conditional expense E^c*, i.e.,

$$\text{conditional savings } S^c = \text{conditional expense } E^c \cdots (4^\bullet).$$

- In fact, this *conditional expense E^c* is what is conventionally referred to as the *cost C*, i.e.,

$$\text{conditional expence } E^c = \text{cost } C \cdots (5^\bullet).$$

- From $(2^\bullet)$ – $(5^\bullet)$ we have the following translation flow.

$$\text{expence } E = \text{decremented fund } F^d = \text{conditional savings } S^c = \text{conditional expence } E^c = \text{cost } C \cdots (6^\bullet).$$

Now, the above interpretation may seem to be somewhat periphrastic; however, when trying to introduce the interest rate r to the evaluation of cost on the time axis, we will see that the above flow in $(6^\bullet)$ become *decisively essential* (see Section 2.3.5(p.15)) .

What should be especially noted here is that while the *profit P* is defined via *only* F^i (see $(1^\bullet)$), the *cost C* is defined via E^c, S^c, F^d, and E (see $(6^\bullet)$).

2.3.3 Discount Factor for Fund

Suppose you have the fund F today. Then, since it can be invested at a given interest rate r, the today's fund F increases to $(1+r)^n F = \beta^{-n} F$ after n days, i.e., $F \to \beta^{-n} F$. Multiplying this relation by β^n leads to $\beta^n F \to F$, implying that if we have the fund $\beta^n F$ today, it increase to F after n days. Accordingly, denoting the fund of $n = 0, 1, \cdots$ days later by F_n,[†] we have

$$\beta^n F_n \to F_n \cdots (1^\circ),$$

meaning that if you have the fund $\beta^n F_n$ today, it increase to F_n after n days. This implies that the fund F_n of n days later can be evaluated as the fund $\beta^n F_n$ of today; in other words, as an economical value, the fund F_n of n days later is equivalent to the fund $\beta^n F_n$ of today, vice versa. In this sense, $\beta^n F_n$ is usually called the *present* (*today*) *value* of the fund F_n of n days later.

2.3.4 Discount Factor for Profit

Since $P = F^i$ due to $(1^\bullet)$, defining P and F^i of n days later by P_n and F^i_n respectively, we have $P_n = F^i_n$. Hence, since $\beta^n P_n \to P_n$ due to (1°), the present (today) value of P_n is given by $\beta^n P_n$. Thus, it follows that the total present value of profit for the whole actions with profits $P_0, P_1, P_2, \cdots$ is given by not $P_0 + P_1 + P_2 + \cdots$ but $P_0 + \beta P_1 + \beta^2 P_2 + \cdots$.[§] This is a reason why the discount factor β is introduced in the description of the system of optimality equations for the "selling problem" with the *profit maximization*.

[†]Hence F_0 represent the fund of 0 day later, i.e., today with $n = 0$.

[§]Let $V_0 \stackrel{\text{def}}{=} P_0 + \beta P_1 + \beta^2 P_2 + \cdots$ and $V_1 \stackrel{\text{def}}{=} P_1 + \beta P_2 + \beta^2 P_3 + \cdots$. Then we have $V_0 = P_0 + \beta V_1$.

2.3.5 Discount Factor for Cost

♡ Alice 1 *Here Alice wandered round with the following question. "In the asset buying problem, since a buying price is what have been already paid away, it does not remain on hand, hence it cannot invest !. But, but —, if so, the concept of the discount factor cannot be applied to the asset buying problem which is a cost minimizing problem !. Then, what will happen ?. Then, Dr. Rabbit clad in the waistcoat-pocket suddenly appeared in front of her and told "Well, it's, it's puzzled $\cdots$.". And, after looking dead at her for a while, taking a watch out of its waistcoat-pocket and then murmuring "Oh dear! Oh dear !, I shall be too late for the faculty meeting", he disappeared down the hole.* □

Below is our answer to the Alice's question. Let us denote the expense E of n days later by E_n. Then, let us represent the decremented fund F^d corresponding to the E_n by F^d_n (see ($2^\bullet$)), next the conditional savings S^c corresponding to the F^d_n by S^c_n (see ($3^\bullet$)), furthermore the conditional expense E^c corresponding to the S^c_n by E^c_n (see ($4^\bullet$)), and finally the cost C corresponding to the E^c_n by C_n (see ($5^\bullet$)). Accordingly, from ($6^\bullet$) we have

$$E_n = F^d_n = S^c_n = E^c_n = C_n.$$

Thus, since $F^d_n = C_n$, we have $\beta^n C_n \to C_n$ from (1°), hence the present (today) value of C_n is given by $\beta^n C_n$. Consequently, it follows that the total cost for the whole behavior consisting of actions with costs $C_0, C_1, C_2, \cdots$ is given by not $C_0 + C_1 + C_2 + \cdots$ but $C_0 + \beta C_1 + \beta^2 C_2 + \cdots$. This is a reason why the discount factor β is introduced in the description of the system of optimality equations for the "buying problem" with the *cost minimization*.

2.3.6 Essential Point

Presumably, this paper will be the first to define the concepts of *profit* and *cost* through the third concept of *fund*. To be honest, we have always found certain inconsistencies in conventional approaches to *profit* and *cost* where clear definitions are often lacking. Despite the extensive discussions about the discount factor for *profit*, it is surprising that the discount factor for *cost* has been addressed so infrequently. We believe this oversight stems from a misguided assumption that the buying problem is of little importance, as it is merely considered the inverse of the selling problem. This assumption implies that the buying problem can be fully explained by simply reversing the signs of the variables, parameters, constants, etc., defined in the selling problem. However, we emphasize here that this paper demonstrates that the two problems are not inversely related at all.

Chapter 3

Classification of Models

3.1 Model Classification Factors

The paper categorizes models based on the following four factors:

(A) The first factor is whether selling model or buying model, represented as:
- Selling model → M.
- Buying model → $\tilde{\mathsf{M}}$.

(B) The second factor is the presence or absence of the quitting penalty price ρ (see A7(p.12)), classified as:
- Model 1 in which the quitting penalty price ρ is not available.
- Model 2 in which the only *terminal* quitting penalty price ρ is available.
- Model 3 in which both *terminal* quitting penalty price ρ and *intervening* quitting penalty ρ are available.

(C) The third factor is whether $\mathbb{R}$-mechanism or $\mathbb{P}$-mechanism (see Section 1.4(p.4)), denoted as:
- $\mathbb{R}$-mechanism-model ($\mathbb{R}$-`model`) → $[\mathbb{R}]$.
- $\mathbb{P}$-mechanism-model ($\mathbb{P}$-`model`) → $[\mathbb{P}]$.

(D) The last factor is whether search-Enforced-model or search-Allowed-model (see A5(p.12)), symbolized as:
- search-Enforced-model (sE-model) → [E].
- search-Allowed-model (sA-model) → [A].

3.2 Tables of Models

3.2.1 No-Recall-Model

Let us designate the no-recall-model by

$$\mathsf{M}{:}x[\mathbb{X}][\mathtt{X}] \quad (\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{X}]) \quad x = 1, 2, 3, \quad \mathbb{X} = \mathbb{R}, \mathbb{P}, \quad \mathtt{X} = \mathtt{E}, \mathtt{A}^{\ddagger}$$

Then let us define the set

$$\mathcal{Q}\langle \mathsf{M}:x[\mathtt{X}]\rangle \overset{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}], \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]\}, \quad x = 1, 2, 3, \;\; \mathtt{X} = \mathtt{E}, \mathtt{A},$$

called the *quadruple-asset-trading-models-with-no-recall*, consisting of the 24 models in the table below:

Table 3.2.1: Twenty Four No-recall-Models

	ASP[ℝ]	ABP[ℝ]	ASP[ℙ]	ABP[ℙ]
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{E}]\}$ = {	M:1[ℝ][E],	M̃:1[ℝ][E],	M:1[ℙ][E],	M̃:1[ℙ][E] }
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{A}]\}$ = {	M:1[ℝ][A],	M̃:1[ℝ][A],	M:1[ℙ][A],	M̃:1[ℙ][A] }
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{E}]\}$ = {	M:2[ℝ][E],	M̃:2[ℝ][E],	M:2[ℙ][E],	M̃:2[ℙ][E] }
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\}$ = {	M:2[ℝ][A],	M̃:2[ℝ][A],	M:2[ℙ][A],	M̃:2[ℙ][A] }
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{E}]\}$ = {	M:3[ℝ][E],	M̃:3[ℝ][E],	M:3[ℙ][E],	M̃:3[ℙ][E] }
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\}$ = {	M:3[ℝ][A],	M̃:3[ℝ][A],	M:3[ℙ][A],	M̃:3[ℙ][A] }

‡Throughout the paper, the model of the asset *buying* problem (ABP) is represented by the symbol upon which the tilde "˜" is capped like $\tilde{\mathsf{M}}$.

3.2.2 Recall-Model

Let us designate the recall-model by

$$\text{rM:}x[\mathbb{X}][\texttt{X}] \quad (\text{r}\tilde{\text{M}}\text{:}x[\mathbb{X}][\texttt{X}]) \quad x = 1, 2, 3, \quad \mathbb{X} = \mathbb{R}, \mathbb{P}, \quad \texttt{X} = \texttt{E}, \texttt{A}.$$

Then let us define the set

$$\mathcal{Q}\langle \text{rM}: x[\texttt{X}]\rangle \stackrel{\text{def}}{=} \{\text{rM:}x[\mathbb{R}][\texttt{X}], \text{r}\tilde{\text{M}}\text{:}x[\mathbb{R}][\texttt{X}], \text{rM:}x[\mathbb{P}][\texttt{X}], \text{r}\tilde{\text{M}}\text{:}x[\mathbb{P}][\texttt{X}]\}, \quad x = 1, 2, 3, \quad \texttt{X} = \texttt{E}, \texttt{A},$$

called the *quadruple-asset-trading-models-with-recall*, consisting of the 24 models in the table below:

Table 3.2.2: Twenty Four Recall-Models

	ASP[ℝ]	ABP[ℝ]	ASP[ℙ]	ABP[ℙ]
$\mathcal{Q}\{\text{rM:1}[\texttt{E}]\}$ = {	rM:1[ℝ][E],	r$\tilde{\text{M}}$:1[ℝ][E],	rM:1[ℙ][E],	r$\tilde{\text{M}}$:1[ℙ][E] }
$\mathcal{Q}\{\text{rM:1}[\texttt{A}]\}$ = {	rM:1[ℝ][A],	r$\tilde{\text{M}}$:1[ℝ][A],	rM:1[ℙ][A],	r$\tilde{\text{M}}$:1[ℙ][A] }
$\mathcal{Q}\{\text{rM:2}[\texttt{E}]\}$ = {	rM:2[ℝ][E],	r$\tilde{\text{M}}$:2[ℝ][E],	rM:2[ℙ][E],	r$\tilde{\text{M}}$:2[ℙ][E] }
$\mathcal{Q}\{\text{rM:2}[\texttt{A}]\}$ = {	rM:2[ℝ][A],	r$\tilde{\text{M}}$:2[ℝ][A],	rM:2[ℙ][A],	r$\tilde{\text{M}}$:2[ℙ][A] }
$\mathcal{Q}\{\text{rM:3}[\texttt{E}]\}$ = {	rM:3[ℝ][E],	r$\tilde{\text{M}}$:3[ℝ][E],	rM:3[ℙ][E],	r$\tilde{\text{M}}$:3[ℙ][E] }
$\mathcal{Q}\{\text{rM:3}[\texttt{A}]\}$ = {	rM:3[ℝ][A],	r$\tilde{\text{M}}$:3[ℝ][A],	rM:3[ℙ][A],	r$\tilde{\text{M}}$:3[ℙ][A] }

3.3 Structured-Unit-of-Models

Let us refer to the set of 24 models defined in each of Tables 3.2.1(p.17) and 3.2.2(p.18) as the *structured-unit-of-models*. Here note that all models within each structured-unit-of-model are not ones *blindly* defined but ones *systematically* and *inevitably* defined according to the four factors in Section 3.1(p.17). The big difference from all other studies that have been conventionally made by many researchers, including the authors in the past, lies in clarifying the *overall interconnectedness* among these models. In addition, let us refer to the whole of problems corresponding to models included in a given structured-unit-of-models as the *structured-unit-of-problems*.

3.4 Decisions

What a leading-trader should determine in each of models defined in Tables 3.2.1(p.17) and 3.2.2(p.18) are as follows:

⟨1⟩ Whether or not to accept the price proposed by a counter-trader (only for ℝ-model) (see Section 7.2.1(p.43)),

⟨2⟩ What price to post (only for ℙ-model) (see Section 7.2.2(p.44)),

⟨3⟩ Whether or not to conduct the search (only for sÅ-model) (see Section 7.2.3(p.44)),

⟨4⟩ When to initiate the process (for all models) (see Section 7.2.4(p.44)).

3.5 Trading Problem with Negative Trading Price

In A8(p.12) we defined a price ξ on $(-\infty, \infty)$. However, this seemingly unrealistic assumption can be justified for the following reason. First let us note here that "sell" means "deliver" and "buy" means "receive"; more precisely speaking:

- In a selling problem, a seller (leading-trader) *delivers* the asset to a buyer (counter-trader), who *receives* it from the seller.
- In a buying problem, a buyer (leading-trader) *receives* the asset from a seller (counter-trader), who *delivers* it to the buyer.

The above two scenarios can be schematized as below.

	leading-trader ↓				counter-trader ↓
selling problem:	seller	(delivering-side)	↔	(recieving-side)	buyer
buying problem:	buyer	(recieving-side)	↔	(delivering-side)	seller

In other words, "selling problem" and "buying problem" can be said to be "delivering problem" and "receiving problem" respectively. Now let us consider here a transaction in which the asset traded there is a worthless debris such as surplus soil, concrete blocks and so on which are disposed of when a building is broken up. In this case, a receiving-side (buyer), in whether selling problem or buying problem, rightly requires some amount of money as a disposal cost, nevertheless being a buyer. Seeing the problem from the standpoint of the seller (delivering-side), the seller gives some amount of money to the buyer (receiving-side), nevertheless being a seller. This interpretation implies that the trading problem stated above can be regarded as "a trading problem with a *negative* trading price" whether selling problem or buying problem. To discuss the trading problem more generally taking into account the above reason, expanding the range of the trading price to $(-\infty, \infty)$ can be said to be reasonable from a practical viewpoint. See Section A 7.5(p.317) for a further economic implication.

3.6 Simplified Notation of Models

In the paper we will sometimes use the following simplified notational convention for the no-recall-model.

- By $\mathsf{M}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$ let us denote $\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]$ and $\mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]$.
- By $\tilde{\mathsf{M}}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$ let us denote $\tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]$ and $\tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]$.
- By $\mathsf{M}/\tilde{\mathsf{M}}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$ let us denote $\mathsf{M}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$ and $\tilde{\mathsf{M}}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$.
- By $\mathsf{M}{:}1/2/3[\mathbb{X}][\mathtt{X}]$ let us denote $\mathsf{M}{:}1[\mathbb{X}][\mathtt{X}]$, $\mathsf{M}{:}2[\mathbb{X}][\mathtt{X}]$, and $\mathsf{M}{:}3[\mathbb{X}][\mathtt{X}]$.
- By $\mathsf{M}{:}x[\mathbb{X}][\mathtt{E}/\mathtt{A}]$ let us denote $\mathsf{M}{:}x[\mathbb{X}][\mathtt{E}]$ and $\mathsf{M}{:}x[\mathbb{X}][\mathtt{A}]$.
- By $\tilde{\mathsf{M}}{:}1/2/3[\mathbb{X}][\mathtt{X}]$ let us denote $\tilde{\mathsf{M}}{:}1[\mathbb{X}][\mathtt{X}]$, $\tilde{\mathsf{M}}{:}2[\mathbb{X}][\mathtt{X}]$, and $\tilde{\mathsf{M}}{:}3[\mathbb{X}][\mathtt{X}]$.
- By $\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}/\mathtt{A}]$ let us denote $\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}]$ and $\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{A}]$.

Also for the recall-model we define the same symbols, say $\mathrm{r}\mathsf{M}/\tilde{\mathsf{M}}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$, $\mathrm{r}\tilde{\mathsf{M}}{:}x[\mathbb{R}/\mathbb{P}][\mathtt{X}]$, $\cdots$

Chapter 4

Definitions of Models

4.1 No-Recall-Model

4.1.1 Model 1

4.1.1.1 Search-Enforced-Model: $\mathcal{Q}\langle \mathsf{M}\!:\!1[\mathsf{E}]\rangle = \{\mathsf{M}\!:\!1[\mathbb{R}][\mathsf{E}],\ \mathsf{M}\!:\!1[\mathbb{P}][\mathsf{E}],\ \tilde{\mathsf{M}}\!:\!1[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}\!:\!1[\mathbb{P}][\mathsf{E}]\}$

4.1.1.1.1 $\mathsf{M}\!:\!1[\mathbb{R}][\mathsf{E}]$ and $\mathsf{M}\!:\!1[\mathbb{P}][\mathsf{E}]$

The two are the most basic models of the asset selling problem [7,Ber1995,p.158-162][47,You1998], which are defined by the following assumptions:

*A*1. Once the process initiates, at every point in time after that it is enforced to conduct the search for buyers (see (A5a(p.12))), hence the search cost $s \geq 0$ is paid at every point in time (see A4(p.11)).

*A*2. After the search has been conducted at a point in time $t > 0$, a buyer certainly appears at time $t-1$ (next point in time), i.e., the buyer appearing probability $\lambda = 1$ (see A6(p.12)).

*A*3. The prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ proposed by successively appearing buyers in $\mathsf{M}\!:\!1[\mathbb{R}][\mathsf{E}]$ and the reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ of successively appearing buyers in $\mathsf{M}\!:\!1[\mathbb{P}][\mathsf{E}]$ are both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$ (see A9(p.13)).†

*A*4. Both terminal quitting penalty price ρ and intervening quitting penalty price ρ are not available (see A7(p.12)).

*A*5. The selling process stops at the point in time when the asset is sold to an appearing buyer (see A3(p.11)).

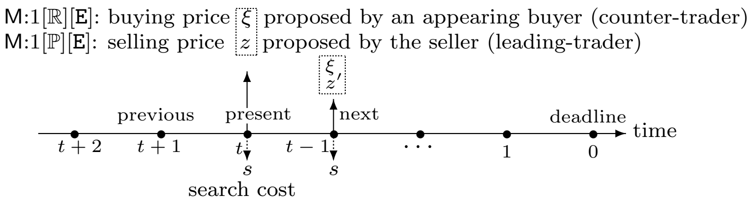

Figure 4.1.1: $\mathsf{M}\!:\!1[\mathbb{R}][\mathsf{E}]$ and $\mathsf{M}\!:\!1[\mathbb{P}][\mathsf{E}]$

The objective is to maximize the total expected present discounted *profit*, i.e., the expected present discounted value of the price for which the asset is sold, *minus* the total expected present discounted value of the search costs which will be paid until the process stops with selling the asset.

Remark 4.1.1

(a) The starting time τ must be greater than or equal to 1, i.e., $\tau \geq 1$ for the following reason. If $\tau = 0$, there exists no buyer at time 0, hence the process must stop without selling the asset, which contradicts A2(p.11).

(b) Suppose the process has proceeded up to time 1. Then, since the search is conducted at that time due to *A*1(p.21), a buyer certainly appears at time 0 (deadline) due to *A*2(p.21).

1. In $\mathsf{M}\!:\!1[\mathbb{R}][\mathsf{E}]$, due to A2(p.11) the seller must sell the asset to the buyer however small the price proposed by the buyer may be.
2. In $\mathsf{M}\!:\!1[\mathbb{P}][\mathsf{E}]$, the seller must propose the price a to the buyer where a is the lower bound of the distribution function F for the reservation price $\boldsymbol{\xi}$ of the buyer (see Figure 2.2.4(p.13)). Then, the buyer certainly buys the asset. □

† $\boldsymbol{\xi}$ and ξ represent a random variable and a realized value respectively.

4.1.1.1.2 $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$

The two are both the models of the asset *buying* problem, defined by the following assumptions:

*A*1. Once the process initiates, at every point in time after that it is enforced to conduct the search for sellers, hence the search cost $s \geq 0$ is paid at every point in time.

*A*2. After the search has been conducted at a point in time $t > 0$, a seller certainly appears at time $t-1$ (next point in time), i.e., the seller appearing probability $\lambda = 1$.

*A*3. The prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ proposed by successively appearing sellers in $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$ and the reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ of successively appearing sellers in $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$ are both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$.†

*A*4. Both terminal quitting penalty price ρ and intervening quitting penalty price ρ are not available.

*A*5. The buying process stops at the point in time when the asset is bought by an appearing seller.

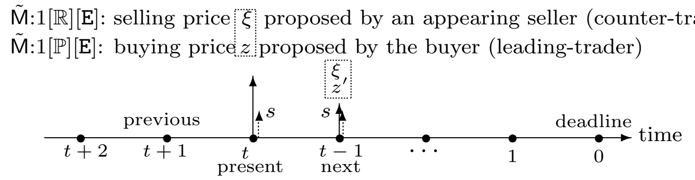

Figure 4.1.2: $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$

The objective is to minimize the total expected present discounted *cost*, i.e., the expected present discounted value of the price for which the asset is bought, *plus* the total expected present discounted value of the search costs which will be paid until the process stops with buying the asset.

Remark 4.1.2 Here it should be noted that although in 4.1.2(p.22), ξ, z, and s are all in an upward direction, in 4.1.1(p.21), only s is in a downward direction. □

4.1.1.2 Search-Allowed-Model 1: $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}],\ \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

4.1.1.2.1 $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ and $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$

The two are the same as $\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]$ and $\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]$ in Section 4.1.1.1.1(p.21) only except that *A*1(p.21) is changed into as follows:

*A*1. At every point in time $t > 0$, it is allowed to skip the search (see (A5b(p.12))); in other words, the seller has an option whether to conduct the search or to skip.

Remark 4.1.3

(a) The starting time τ must be greater than 0, i.e., $\tau > 0$ for the same reason as in Remark 4.1.1(p.21) (a).

(b) Suppose the process has proceeded up to time $t = 1$. Then, if the search is skipped at that time, no buyer appears at time $t = 0$, hence the seller is faced with the situation of having to quit the process without selling the asset, which contradicts A2(p.11). Accordingly, also in this case *the search must be necessarily conducted at time* $t = 1$; as a result, a buyer certainly appears at time 0 due to the assumption *A*2. □

4.1.1.2.2 $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$

The two are the same as $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$ in Section 4.1.1.1.2(p.22) only except that after the process has initiated, it is allowed to skip the search.

4.1.2 Model 2

4.1.2.1 Search-Enforced-Model 2: $\mathcal{Q}\langle\mathsf{M}{:}2[\mathtt{E}]\rangle = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}],\ \mathsf{M}{:}2[\mathbb{P}][\mathtt{E}],\ \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}],\ \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\}$

The quadruple models indicated in the above brace are the same as in Section 4.1.1.1.1(p.21) only except that the assumptions *A*2(p.21) and *A*4(p.21) are changed into as follows:

*A*2. After the search has been conducted at time $t > 0$, a buyer appears at the next point in time with a probability $\lambda \leq 1$.

*A*4. The terminal quitting penalty price ρ is available.

Remark 4.1.4 In these models it is possible to stop the process by accepting the terminal quitting penalty price ρ at time 0 (deadline), hence the starting time $\tau = 0$ is permitted since the leading-trader can quit the process with accepting the ρ at time 0 even if no counter-trader exists at time 0. Accordingly, in these models it follows that the starting time τ is greater than or equal to 0, i.e., $\tau \geq 0$. □

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† $\boldsymbol{\xi}$ and ξ represent a random variable and a realized variable respectively.

4.1.2.2 Search-Allowed-Model 2: $\mathcal{Q}\langle \mathsf{M}:2[\mathtt{A}]\rangle = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}],\ \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$

The quadruple models indicated in the above brace are the same as in Section 4.1.2.1(p.22) only except that *A*1(p.21) is changed as follows:

*A*1. After the process has initiated, it is allowed to skip the search at every point in time $t > 0$.

4.1.3 Model 3

4.1.3.1 Search-Enforced-Model 3: $\mathcal{Q}\langle \mathsf{M}:3[\mathtt{E}]\rangle = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}],\ \mathsf{M}{:}3[\mathbb{P}][\mathtt{E}],\ \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}],\ \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]\}$

The quadruple models are the same as in Section 4.1.2.1(p.22) only except that the assumption *A*4(p.22) is changed as follows:

*A*4. In addition to the terminal quitting penalty price ρ, the intervening quitting penalty price ρ is also available.

4.1.3.2 Search-Allowed-Model 3: $\mathcal{Q}\langle \mathsf{M}:3[\mathtt{A}]\rangle = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}],\ \mathsf{M}{:}3[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

The quadruple models are the same as those in Section 4.1.2.2(p.23) only except that after the process has initiated, it is allowed to skip the search.

4.2 Recall-Model

See Chap. 23(p.237) .

4.3 Spaces

Let us refer to $\lambda \in (0,1]$, $\beta \in (0,1]$, $s \in [0,\infty)$, and $\rho \in (-\infty,\infty)$ as the *parameter* of models, all of which are independent of the distribution function F. Then, let $\boldsymbol{p} = (\lambda,\beta,s)$ for Model 1 and $\boldsymbol{p} = (\lambda,\beta,s,\rho)$ for Models 2 and 3, which are called the *parameter vector*. We represent the set of all possible $\boldsymbol{p}$'s by

$$\mathscr{P} = \{\boldsymbol{p} \mid \lambda = 1,\ 0 < \beta \le 1,\ 0 \le s\} \qquad \text{for Model 1,} \tag{4.3.1}$$

$$\mathscr{P} = \{\boldsymbol{p} \mid 0 < \lambda \le 1,\ 0 < \beta \le 1,\ 0 \le s,\ -\infty < \rho < \infty\} \qquad \text{for Models 2,3,} \tag{4.3.2}$$

called the *total parameter space*, simply total-P-space. Then, let us refer to the direct product (Cartesian product) of the total-P-space $\mathscr{P}$ and total-DF-space $\mathscr{F}$ (see (2.2.5(p.13))), i.e.,

$$\mathscr{P} \times \mathscr{F} = \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}, F \in \mathscr{F}\} \tag{4.3.3}$$

as the total-P/DF-space.

Chapter 5

Underlying Functions

This chapter defines some functions called the *underlying function*, which will be used to derive the system of optimality equations of the 24 model in Table 3.2.1(p.17) .

5.1 Definition

5.1.1 T, L, K, and $\mathcal{L}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$T(x) = \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}] \quad (5.1.1)$$
$$= \int_{-\infty}^{\infty} \max\{\xi - x, 0\} f(\xi) d\xi,^{\dagger\ddagger} \quad (5.1.2)$$

and then define

$$L(x) = \lambda\beta T(x) - s, \quad (5.1.3)$$
$$K(x) = \lambda\beta T(x) - (1-\beta)x - s,^{\S} \quad (5.1.4)$$
$$\mathcal{L}(s) = L(\lambda\beta\mu - s), \quad (5.1.5)$$
$$\kappa = \lambda\beta T(0) - s \quad (5.1.6)$$
$$= L(0) = K(0) = \lambda\beta T(0) - s \quad (5.1.7)$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{R}$ and to κ as the *κ-value* of Type $\mathbb{R}$. The formula below will be sometimes used in the rest of the paper.

$$K(x) + (1-\beta)x = L(x), \quad (5.1.8)$$
$$K(x) + x = L(x) + \beta x, \quad (5.1.9)$$
$$\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x\}] + (1-\lambda)\beta x - s = K(x) + x \quad (5.1.10)$$

5.1.2 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$\tilde{T}(x) = \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}] \quad (5.1.11)$$
$$= \int_{-\infty}^{\infty} \min\{\xi - x, 0\} f(\xi) d\xi, \quad (5.1.12)$$

and then define

$$\tilde{L}(x) = \lambda\beta\tilde{T}(x) + s, \quad (5.1.13)$$
$$\tilde{K}(x) = \lambda\beta\tilde{T}(x) - (1-\beta)x + s, \quad (5.1.14)$$
$$\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s), \quad (5.1.15)$$
$$\tilde{\kappa} = \lambda\beta\tilde{T}(0) + s \quad (5.1.16)$$
$$= \tilde{L}(0) = \tilde{K}(0). \quad (5.1.17)$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{R}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{R}$.

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† See [13,DeGroot70].
‡ See Figure A 7.3(p.315) (I) ,
§ See Figure A 7.3(p.315) (II) ,

5.1.3 T, L, K, and $\mathcal{L}$ of Type $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$p(z) = \Pr\{z \leq \boldsymbol{\xi}\}, \tag{5.1.18}$$
$$T(x) = \max_z p(z)(z-x)^\dagger \tag{5.1.19}$$

and then define

$$L(x) = \lambda\beta T(x) - s, \tag{5.1.20}$$
$$K(x) = \lambda\beta T(x) - (1-\beta)x - s, \tag{5.1.21}$$
$$\mathcal{L}(s) = L(\lambda\beta a - s), \tag{5.1.22}$$
$$\kappa = \lambda\beta T(0) - s \tag{5.1.23}$$
$$= L(0) = K(0) \tag{5.1.24}$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{P}$ and to κ as the *κ-value* of Type $\mathbb{P}$. Let us denote z maximizing $p(z)(z-x)$ by $z(x)$ if it exists, i.e.,

$$T(x) = p(z(x))(z(x)-x). \tag{5.1.25}$$

Definition 5.1.1 If there exists multiple $z(x)$, let us define the *smallest* of them as $z(x)$. □

Furthermore, for convenience of later discussions, let us define

$$a^\star = \inf\{x \mid T(x) + x > a\} = \inf\{x \mid T(x) > a - x\}, \tag{5.1.26}$$
$$x^\star = \inf\{x \mid z(x) > a\}. \tag{5.1.27}$$

Noting that (5.1.18(p.26)) can be rewritten as $p(z) = 1 - \Pr\{\boldsymbol{\xi} < z\} = 1 - \Pr\{\boldsymbol{\xi} \leq z\}$ due to the assumption of F being continuous (see A9(p.13)), we have $p(z) = 1 - F(z)$. Accordingly, it can be immediately seen that

$$p(z) \begin{cases} = 1, & z \leq a \quad \cdots (1) \quad \text{due to (2.2.1 (1) (p.13))}, \\ < 1, & a < z \quad \cdots (2) \quad \text{due to (2.2.1 (2,3) (p.13))}, \end{cases} \tag{5.1.28}$$

$$p(z) \begin{cases} > 0, & z < b \quad \cdots (1), \quad \text{due to (2.2.1 (1,2) (p.13))}, \\ = 0, & b \leq z \quad \cdots (2), \quad \text{due to (2.2.1(p.13))3}. \end{cases} \tag{5.1.29}$$

In general $p(z)(z-x)$ can be depicted as below.

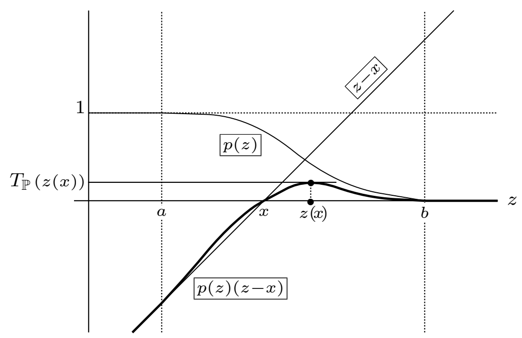

Figure 5.1.1: Graph of $p(z)(z-x)$

When F is the uniform distribution function on $[a, b]$, we have

$$a^\star = 2a - b \quad \text{(see (A 7.6 (1) (p.316)))}. \tag{5.1.30}$$

5.1.4 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of Type $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}, \tag{5.1.31}$$
$$\tilde{T}(x) = \min_z \tilde{p}(z)(z-x), \tag{5.1.32}$$

---

†See Figure A 7.4(p.315) .

and then define

$$\tilde{L}(x) = \lambda\beta\tilde{T}(x) + s, \tag{5.1.33}$$

$$\tilde{K}(x) = \lambda\beta\tilde{T}(x) - (1-\beta)x + s, \tag{5.1.34}$$

$$\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta b + s), \tag{5.1.35}$$

$$\tilde{\kappa} = \lambda\beta\tilde{T}(0) + s \tag{5.1.36}$$

$$= \tilde{L}(0) = \tilde{K}(0). \tag{5.1.37}$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{P}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{P}$. Let us denote z minimizing $\tilde{p}(z)(z-x)$ by $z(x)$ if it exists, i.e.,

$$\tilde{T}(x) = \tilde{p}(z(x))(z(x) - x). \tag{5.1.38}$$

Definition 5.1.2 If there exists multiple $z(x)$, let us define the *largest* of them as $z(x)$. □

Furthermore, for convenience of later discussions, let us define

$$b^\star = \sup\{x \mid \tilde{T}(x) + x < b\} = \sup\{x \mid \tilde{T}(x) < b - x\}, \tag{5.1.39}$$

$$\tilde{x}^\star = \sup\{x \mid z(x) < b\}. \tag{5.1.40}$$

Noting that (5.1.31(p.26)) can be rewritten as $\tilde{p}(z) = F(z)$, we can immediately see that

$$\tilde{p}(z) \begin{cases} = 0, & z \le a \quad \cdots (1) \quad \text{due to } (2.2.1\,(1)\,(\text{p.13})), \\ > 0, & a < z \quad \cdots (2) \quad \text{due to } (2.2.1\,(2.3)\,(\text{p.13})), \end{cases} \tag{5.1.41}$$

$$\tilde{p}(z) \begin{cases} < 1, & z < b \quad \cdots (1) \quad \text{due to } (2.2.1\,(1,2)\,(\text{p.13})), \\ = 1, & b \le z \quad \cdots (2) \quad \text{due to } (2.2.1\,(3)\,(\text{p.13})). \end{cases} \tag{5.1.42}$$

5.2 Solutions

The solutions defined below are commonly used in the analyses of all models in the whole paper.

(a) Let us define the solutions of the equations $L(x) = 0$, $K(x) = 0$, and $\mathcal{L}(s) = 0$ (whether Type $\mathbb{R}$ or Type $\mathbb{P}$) by x_L, x_K, and $s_{\mathcal{L}}$ respectively if they exist, i.e.,

$$L(x_L) = 0 \cdots (1), \qquad K(x_K) = 0 \cdots (2), \qquad \mathcal{L}(s_{\mathcal{L}}) = 0 \cdots (1). \tag{5.2.1}$$

If multiple solutions exist for each of the above three equations, we employ the *smallest* as its solution.

(b) Let us define the solutions of the equations $\tilde{L}(x) = 0$, $\tilde{K}(x) = 0$, and $\tilde{\mathcal{L}}(s) = 0$ (whether $\tilde{\text{T}}$ype $\mathbb{R}$ or $\tilde{\text{T}}$ype $\mathbb{P}$) by $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$ respectively if they exist.

$$\tilde{L}(x_{\tilde{L}}) = 0 \cdots (1), \qquad \tilde{K}(x_{\tilde{K}}) = 0 \cdots (2), \qquad \tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0 \cdots (1). \tag{5.2.2}$$

If multiple solutions exist for each of the above three equations, we employ the *largest* as its solution.

5.3 Primitive Underlying Functions and Derivative Underlying Functions

Sometimes let us refer to each of T- and $\tilde{T}$-functions as the *primitive underlying function* and to each of L-, K-, $\mathcal{L}$-, $\tilde{L}$-, $\tilde{L}$-, and $\tilde{\mathcal{L}}$-functions as the *derivative underlying function*, which are defined by use of primitive underlying functions T and $\tilde{T}$.

5.4 Identical Representation and Explicit Representation

In the rest of the paper, when we need to distinguish

$$T,\ L,\ K,\ \mathcal{L},\ \kappa,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ \tilde{\kappa},\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}} \tag{5.4.1}$$

between Type $\mathbb{R}$ and Type $\mathbb{P}$, let us denote them by

$$T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}, x_{L_{\mathbb{R}}}, x_{K_{\mathbb{R}}}, s_{\mathcal{L}_{\mathbb{R}}}, \tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}, x_{\tilde{L}_{\mathbb{R}}}, x_{\tilde{K}_{\mathbb{R}}}, s_{\tilde{\mathcal{L}}_{\mathbb{R}}}, \tag{5.4.2}$$

$$T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}, x_{L_{\mathbb{P}}}, x_{K_{\mathbb{P}}}, s_{\mathcal{L}_{\mathbb{P}}}, \tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}, x_{\tilde{L}_{\mathbb{P}}}, x_{\tilde{K}_{\mathbb{P}}}, s_{\tilde{\mathcal{L}}_{\mathbb{P}}}. \tag{5.4.3}$$

Let us refer to (5.4.1) as the *identical representation* and to (5.4.2) and (5.4.3) as the *explicit representation.*

5.5 Characteristic Vector and Characteristic Element

Let us here define the two vectors, $\boldsymbol{C}_{\mathbb{R}}$ consisting of (5.1.3(p.25))-(5.1.6(p.25)) and $\tilde{\boldsymbol{C}}_{\mathbb{R}}$ consisting of (5.1.13(p.25))-(5.1.16(p.25)), i.e,

$$\boldsymbol{C}_{\mathbb{R}} = (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{R}} = (\tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}).$$

Likewise, let us define the two vectors, $\boldsymbol{C}_{\mathbb{P}}$ consisting of (5.1.20(p.26))-(5.1.23(p.26)) and $\tilde{\boldsymbol{C}}_{\mathbb{P}}$ consisting of (5.1.33(p.27))-(5.1.36(p.27)), i.e.,

$$\boldsymbol{C}_{\mathbb{P}} = (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{P}} = (\tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}).$$

Furthermore, adding T- and $\tilde{T}$-functions to the above vectors, let us define

$$\boldsymbol{C}^T_{\mathbb{R}} = (T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \quad \tilde{\boldsymbol{C}}^T_{\mathbb{R}} = (\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}),$$
$$\boldsymbol{C}^T_{\mathbb{P}} = (T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \quad \tilde{\boldsymbol{C}}^T_{\mathbb{P}} = (\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}).$$

Let us call each of the vectors defined above the *characteristic vector* and its element the *characteristic element*. In the identical representation, the above vectors are all represented by $\boldsymbol{C}$, $\tilde{\boldsymbol{C}}$, $\boldsymbol{C}^T$, and $\tilde{\boldsymbol{C}}^T$ respectively.

Chapter 6

Systems of Optimality Equations

In this chapter we derive the system of optimality equations (SOE) for each of the 24 models in Table 3.2.1(p.17) (see Chap. 24(p.239) for models in Table 3.2.2(p.18)).

6.1 Preliminary

Definition 6.1.1 Throughout the paper let us represent the action

"Conduct the search at time t" ("Skip the search at time t")

as $\texttt{Conduct}_t$ ($\texttt{Skip}_t$) for short. Then, when this action is *simply* optimal, *indifferently* optimal, or *strictly* optimal, let us represent it as respectively

$$\texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}),\quad \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}),\quad \text{or}\quad \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}).\ \square$$

Remark 6.1.1 (relationship between SOE and assertion) In general, a model M of a decision process, whether in this paper or not, has the system of optimality equations, denoted by SOE{M}, which should be said to be a mirror exhaustively reflecting the entire aspect of the model M. In other words, SOE{M} involves the exhaustive information of the model M as if a gene has the exhaustive information of a life. This implies that any assertion which is characterized by the sequence $\{V_t\}$ generated from SOE{M} can be regarded as an assertion on the model M; conversely, an assertion which is not characterized by the sequence $\{V_t\}$ cannot be said to be an assertion on the M. □

Below let us represent "buyer (seller) proposing a price w" by "buyer (seller) w" for short.

6.2 No-Recall-Model

6.2.1 Search-Allowed-Model

6.2.1.1 Model 1

Let us note here that $\lambda = 1$ is assumed in this model.

6.2.1.1.1 M:1[ℝ][A]

By $v_t(w)$ ($t \geq 0$) and V_t ($t > 0$) let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively. Then, we have

$$v_0(w) = w, \tag{6.2.1}$$

$$v_t(w) = \max\{w, V_t\}, \quad t > 0, \tag{6.2.2}$$

where V_t is the maximum of the total expected present discounted profit from rejecting the proposed price w. Then, we have

$$V_1 = \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] - s = \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s \quad \text{(see Remark 4.1.3(p.22) (b))}, \tag{6.2.3}$$

$$V_t = \max\{\texttt{C} : \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s,\ \texttt{S} : \beta V_{t-1}\}, \quad t > 1, \tag{6.2.4}$$

where C and S represent the actions of Conducting the search and Skipping the search respectively. Then, since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = T(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.1(p.25))), hence (6.2.4(p.29)) can be written as

$$\begin{aligned} V_t &= \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4(p.25)) with } \lambda = 1) \qquad (6.2.5) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.8(p.25)))}. \qquad (6.2.6) \end{aligned}$$

□ SOE{M:1[ℝ][A]} is given by the set of (6.2.1(p.29)) – (6.2.4(p.29)). However, since the sequence $\{V_t\}$ is generated from the two expressions (6.2.3(p.29)) and (6.2.5(p.29)), due to Remark 6.1.1(p.29) it can be reduced to only the two in Table 6.4.1(p.41) (I). □

Now, let us here define

$$\mathbb{S}_t = \beta(\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 1. \tag{6.2.7}$$

Then, (6.2.4(p.29)) can be rewritten as

$$\begin{aligned} V_t &= \max\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \beta V_{t-1} - s, 0\} + \beta V_{t-1} \\ &= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.8}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.9}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.10}$$

$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.11}$$

$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.12}$$

Then, from (6.2.2(p.29)) we can rewrite (6.2.7(p.30)) as

$$\mathbb{S}_t = \beta(\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s.$$

Accordingly, from (5.1.1(p.25)) and (5.1.3(p.25)) with $\lambda = 1$ we have

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.13}$$

$$= L(V_{t-1}), \quad t > 1. \tag{6.2.14}$$

6.2.1.1.2 $\tilde{\mathsf{M}}$:1[ℝ][A]

By $v_t(w)$ ($t \geq 0$) and V_t ($t > 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller w and with no seller respectively. Then, we have

$$v_0(w) = w, \tag{6.2.15}$$

$$v_t(w) = \min\{w, V_t\}, \quad t > 0, \tag{6.2.16}$$

where V_t is the minimum of the total expected present discounted cost from rejecting the proposed price w. Then, we have

$$V_1 = \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] + s = \beta\,\mathbf{E}[\boldsymbol{\xi}] + s = \beta\mu + s, \tag{6.2.17}$$

$$V_t = \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s,\ \beta V_{t-1}\}, \quad t > 1. \tag{6.2.18}$$

Then, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.11(p.25))), hence (6.2.18(p.30)) can be written as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.25)) with } \lambda = 1) \end{aligned} \tag{6.2.19}$$

$$\begin{aligned} &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.14(p.25)) and (5.1.13(p.25)) with } \lambda = 1). \end{aligned} \tag{6.2.20}$$

□ SOE{$\tilde{\mathsf{M}}$:1[ℝ][A]} can be reduced to (6.2.17(p.30)) and (6.2.19(p.30)), listed in Table 6.4.1(p.41) (II). ■

Remark 6.2.1 Note here that the same notations $v_t(w)$ and V_t are used for both M:1[ℝ][A] and $\tilde{\mathsf{M}}$:1[ℝ][A]. For explanatory convenience, later on we sometimes represent the $v_t(w)$ and V_t for $\tilde{\mathsf{M}}$:1[ℝ][A] by $\tilde{v}_t(w)$ and $\tilde{V}_t$ respectively. Then (6.2.15(p.30))-(6.2.18(p.30)) are written as respectively

$$\tilde{v}_0(w) = w,$$

$$\tilde{v}_t(w) = \min\{w, \tilde{V}_t\},$$

$$\tilde{V}_1 = \beta\mu + s,$$

$$\tilde{V}_t = \min\{\beta\,\mathbf{E}[\tilde{v}_{t-1}(\boldsymbol{\xi})] + s, \beta\tilde{V}_{t-1}\}. \quad ■$$

Now, let us here define

$$\tilde{\mathbb{S}}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 1. \tag{6.2.21}$$

Then, (6.2.18(p.30)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.22}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.23}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.24}$$

$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.25}$$

$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.26}$$

Then, from (6.2.16(p.30)) we can rewrite (6.2.21(p.31)) as

$$\tilde{\mathbb{S}}_t = \beta(\,\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s.$$

Accordingly, from (5.1.11(p.25)) and (5.1.13(p.25)) with $\lambda = 1$ we have

$$\tilde{\mathbb{S}}_t \;=\; \beta\tilde{T}(V_{t-1}) + s \tag{6.2.27}$$

$$\;=\; \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.28}$$

6.2.1.1.3 M:1[ℙ][A]

By v_t $(t \ge 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \ge 0$ by z_t. In this model, since the search must be necessarily conducted at time 1 (see Remark 4.1.3(p.22) (b)), there exists a buyer at time 0. Suppose the process has proceeded up to time 0. Then, since the seller must necessarily sell the asset at that time, he must propose the price $a^\dagger$ to a buyer appearing at that time (see Remark 4.1.1(p.21) (b2)), thus we have

$$z_0 = a. \tag{6.2.29}$$

Hence, the profit that the seller obtains at time 0 becomes a, i.e.,

$$v_0 = a. \tag{6.2.30}$$

Now, since the search is conducted at time 1 (see Remark 4.1.3(p.22) (b)), we have

$$V_1 = \beta v_0 - s = \beta a - s. \tag{6.2.31}$$

In addition, we have

$$V_t \;=\; \max\{\beta v_{t-1} - s,\, \beta V_{t-1}\}, \quad t > 1. \tag{6.2.32}$$

If the seller proposes a price z, the probability of a buyer buying the asset is given by $p(z) = \Pr\{z \le \boldsymbol{\xi}\}$ (see (5.1.18(p.26))), hence we have

$$v_t \;=\; \max_z\{p(z)z + (1 - p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.33}$$

due to (5.1.19(p.26)), implying that the optimal price z_t which the seller should propose is given by

$$z_t = z(V_t), \quad t > 0, \quad \text{(see (5.1.25(p.26)))}. \tag{6.2.34}$$

Now, since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$, we can rearrange (6.2.32(p.31)) as follows

$$\begin{aligned} V_t &= \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.26)) with } \lambda = 1) \end{aligned} \tag{6.2.35}$$

$$\begin{aligned} &= \max\{K(V_{t-1}) + (1 - \beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \quad \text{(see (5.1.21(p.26)) and (5.1.20(p.26)) with } \lambda = 1) \end{aligned} \tag{6.2.36}$$

□ SOE{M:1[ℙ][A]} is given by (6.2.31(p.31)) and (6.2.35(p.31)), listed in Table 6.4.1(p.41) (III). ■

Now, let us here define

$$\mathbb{S}_t \;=\; \beta(v_{t-1} - V_{t-1}) - s, \quad t > 1. \tag{6.2.37}$$

---

†The lower bound of the distribution function for the reservation price (maximum permissible buying price) of the buyer.

Then, (6.2.32(p.31)) can be rewritten as

$$\begin{aligned} V_t &= \max\{\beta v_{t-1} - \beta V_{t-1} - s, 0\} + \beta V_{t-1} \\ &= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.38}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.39}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.40}$$

$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.41}$$

$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.42}$$

Then, from (6.2.33(p.31)) with $t-1$ we have $v_{t-1} = T(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = T(V_{t-1})$, thus, noting (5.1.20(p.26)), we can rewrite (6.2.37(p.31)) as below

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.43}$$

$$= L(V_{t-1}), \quad t > 1. \tag{6.2.44}$$

6.2.1.1.4 $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][$\mathtt{A}$]

By v_t $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. In this model, since the search must be necessarily conducted at time 1, there exists a seller at time 0. Suppose the process has proceeded up to time 0. Then, since the buyer must necessarily buy the asset at that time, he must propose the price b[†] to a seller appearing at that time, thus we have

$$z_0 = b. \tag{6.2.45}$$

Hence, the cost that the buyer pays at time 0 becomes b, i.e.,

$$v_0 = b. \tag{6.2.46}$$

Now, since the search is conducted at time 1, we have

$$V_1 = \beta v_0 + s = \beta b + s. \tag{6.2.47}$$

In addition, we have

$$V_t = \min\{\beta v_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 1. \tag{6.2.48}$$

If the buyer proposes a price z, the probability of a seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.26))), hence we have

$$v_t = \min_z\{\tilde{p}(z)z + (1-\tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.49}$$

due to (5.1.32(p.26)), implying that the optimal price z_t which the buyer should propose is given by

$$z_t = z(V_t), \quad t > 0, \quad (\text{see } (5.1.38(\text{p.27}))). \tag{6.2.50}$$

Now, since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$, we can rearrange (6.2.48(p.32)) as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.34(\text{p.27})) \text{ with } \lambda = 1) \end{aligned} \tag{6.2.51}$$

$$\begin{aligned} &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad (\text{see } (5.1.34(\text{p.27})) \text{ and } (5.1.33(\text{p.27})) \text{ with } \lambda = 1) \end{aligned} \tag{6.2.52}$$

□ SOE{$\tilde{\mathsf{M}}$:1[$\mathbb{P}$][$\mathtt{A}$]} is given by (6.2.47(p.32)) and (6.2.51(p.32)), listed in Table 6.4.1(p.41) (IV). ∎

Now, let us here define

$$\tilde{\mathbb{S}}_t = \beta(v_{t-1} - V_{t-1}) + s, \quad t > 1. \tag{6.2.53}$$

Then, (6.2.48(p.32)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\beta v_{t-1} - \beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.54}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.55}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

[†]The upper bound of the distribution function for the reservation price (minimum permissible selling price) of the seller

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.56}$$
$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.57}$$
$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.58}$$

Then, from (6.2.49(p.32)) with $t-1$ we have $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = \tilde{T}(V_{t-1})$, thus, noting (5.1.33(p.27)), we can rewrite (6.2.53(p.32)) as below

$$\mathbb{S}_t \;=\; \beta\tilde{T}(V_{t-1}) + s \tag{6.2.59}$$
$$\;=\; \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.60}$$

6.2.1.2 Model 2

6.2.1.2.1 M:2[ℝ][A]

By $v_t(w)$ $(t \ge 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively. Then we have

$$v_0(w) \;=\; \max\{w, \rho\}, \tag{6.2.61}$$
$$v_t(w) \;=\; \max\{w, V_t\}, \quad t > 0, \tag{6.2.62}$$

where

$$V_t \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.63}$$

Let us here define

$$V_0 = \rho. \tag{6.2.64}$$

Then (6.2.62(p.33)) holds for $t \ge 0$ instead of $t > 0$, i.e.,

$$v_t(w) \;=\; \max\{w, V_t\}, \quad t \ge 0. \tag{6.2.65}$$

Since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.1(p.25))), from (6.2.63(p.33)) we have

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.4(\text{p.25}))) \end{aligned} \tag{6.2.66}$$
$$\begin{aligned} &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad (\text{see } (5.1.8(\text{p.25}))). \end{aligned} \tag{6.2.67}$$

□ SOE{M:2[ℝ][A]} is given by (6.2.64(p.33)) and (6.2.66(p.33)), listed in Table 6.4.3(p.41) (I). ■

Let us here define

$$\mathbb{S}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \tag{6.2.68}$$

Then, (6.2.63(p.33)) can be rewritten as

$$V_t \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \lambda\beta V_{t-1} - s, 0\} + \beta V_{t-1}$$
$$\;=\; \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{6.2.69}$$

implying that

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{6.2.70}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.71}$$
$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.72}$$
$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.73}$$

Then, from (6.2.68(p.33)) we can rewrite (6.2.62(p.33)) as

$$\mathbb{S}_t = \beta(\,\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s.$$

Accordingly, from (5.1.1(p.25)) and (5.1.3(p.25)) we have

$$\mathbb{S}_t \;=\; \beta T(V_{t-1}) - s \tag{6.2.74}$$
$$\;=\; L(V_{t-1}), \quad t > 0. \tag{6.2.75}$$

6.2.1.2.2 $\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller w and with no seller respectively. Then, we have

$$v_0(w) \;=\; \min\{w, \rho\}, \tag{6.2.76}$$
$$v_t(w) \;=\; \min\{w, V_t\}, \quad t > 0, \tag{6.2.77}$$

where

$$V_t \;=\; \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\; \beta V_{t-1}\}, \quad t > 0. \tag{6.2.78}$$

Let us here define

$$V_0 = \rho. \tag{6.2.79}$$

Then (6.2.77(p.34)) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(w) \;=\; \min\{w, V_t\}, \quad t \geq 0. \tag{6.2.80}$$

Since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.11(p.25))), from (6.2.78(p.34)) we have

$$\begin{aligned} V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.25)))} \qquad (6.2.81) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0. \quad \text{(see (5.1.14(p.25)) and (5.1.13(p.25)))} \qquad (6.2.82) \end{aligned}$$

□ SOE{$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]} is given by (6.2.79(p.34)) and (6.2.81(p.34)), listed in Table 6.4.3(p.41) (II). ∎

Let us here define

$$\tilde{\mathbb{S}}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{6.2.83}$$

Then, (6.2.78(p.34)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \lambda\beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.84}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.85}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\tilde{\mathbb{S}}_t \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.86}$$
$$\tilde{\mathbb{S}}_t = (=)\; 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.87}$$
$$\tilde{\mathbb{S}}_t < (>)\; 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.88}$$

Then, from (6.2.77(p.34)) we can rewrite (6.2.83(p.34)) as

$$\tilde{\mathbb{S}}_t = \beta(\,\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s.$$

Accordingly, from (5.1.11(p.25)) and (5.1.13(p.25)) we have

$$\tilde{\mathbb{S}}_t \;=\; \beta\tilde{T}(V_{t-1}) + s \tag{6.2.89}$$
$$\;=\; \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.90}$$

6.2.1.2.3 M:2[$\mathbb{P}$][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, the seller must determine whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. Let the ρ is accepted. Then the profit which the seller can obtain is ρ. On the other hand, let the asset be sold to the buyer. Then since the seller must necessarily sell the asset to the buyer due to A2(p.11), the price a[†] must be proposed to the buyer; in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \tag{6.2.91}$$

hence the profit which the seller can obtain at that time is a. Accordingly, it follows that the profit that the seller can obtain at time 0 is given by

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[†]The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer.

$$v_0 = \max\{\rho, a\}. \tag{6.2.92}$$

Suppose there exists a buyer at a time $t > 0$. Then, since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \leq \boldsymbol{\xi}\}$ (see (5.1.18(p.26))). Hence we have

$$v_t = \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.93}$$

due to (5.1.19(p.26)), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.94}$$

due to (5.1.25(p.26)). Finally V_t can be expressed as follows.

$$V_0 = \rho, \tag{6.2.95}$$

$$V_t = \max\{\lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0. \tag{6.2.96}$$

For $t = 1$ we have

$$\begin{aligned} V_1 &= \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &= \max\{\lambda\beta \max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &= \max\{\lambda\beta \max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \end{aligned} \tag{6.2.97}$$

Since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.93(p.35)), we can rearrange (6.2.96(p.35)) as follows.

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.26)))} \qquad (6.2.98) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.21(p.26)) and (5.1.20(p.26)))}. \qquad (6.2.99) \end{aligned}$$

□ SOE{M:1[ℙ][A]} is given by (6.2.95(p.35)), (6.2.97(p.35)), and (6.2.98(p.35)), listed in Table 6.4.3(p.41) (III). ■

Now let us here define

$$\mathbb{S}_t = \lambda\beta(v_{t-1} - V_{t-1}) - s, \quad t > 0. \tag{6.2.100}$$

Then (6.2.96(p.35)) can be rewritten as

$$\begin{aligned} V_t &= \max\{\lambda\beta v_{t-1} - \lambda\beta V_{t-1} - s, 0\} - \beta V_{t-1} \\ &= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.101}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.102}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.103}$$

$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.104}$$

$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.105}$$

Then, from (6.2.93(p.35)) with $t-1$ we have $v_{t-1} = T(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = T(V_{t-1})$, thus, noting (5.1.20(p.26)), we can rewrite (6.2.100(p.35)) as below

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.106}$$

$$= L(V_{t-1}), \quad t > 0. \tag{6.2.107}$$

6.2.1.2.4 $\tilde{\mathsf{M}}$:2[ℙ][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, the buyer must determine whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. Let the ρ is accepted. Then the cost which the buyer pays is ρ. On the other hand, let an asset be bought from the seller. Them since the buyer must necessarily buy the asset from the seller due to A2(p.11), the price $b^\dagger$ must be proposed to the seller; in other words, the optimal price to propose at time $t = 0$ is given by

† The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

$$z_0 = b, \tag{6.2.108}$$

hence the cost which the buyer pays at that time is b. Accordingly, the cost that the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\}. \tag{6.2.109}$$

Suppose there exists a seller at a time $t > 0$. Then, since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes a price z, the probability of the seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \le z\}$ (see (5.1.31(p.26))). Hence we have

$$v_t = \min_z\{\tilde{p}(z)z + (1 - p(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.110}$$

due to (5.1.32(p.26)), implying that the optimal buying price z_t which the buyer should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.111}$$

due to (5.1.38(p.27)). Finally V_t can be expressed as follows.

$$V_0 = \rho, \tag{6.2.112}$$
$$V_t = \min\{\lambda\beta v_{t-1} + (1 - \lambda)\beta V_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.113}$$

For $t = 1$ we have

$$\begin{aligned} V_1 &= \min\{\lambda\beta v_0 + (1 - \lambda)\beta V_0 + s, \beta V_0\} \\ &= \min\{\lambda\beta \min\{\rho, b\} + (1 - \lambda)\beta\rho + s, \beta\rho\} \\ &= \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}. \end{aligned} \tag{6.2.114}$$

Since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.110(p.36)), we can rearrange (6.2.113(p.36)) as follows.

$$\begin{aligned} V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1 - \lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.27)))} \qquad (6.2.115) \\ &= \min\{\tilde{K}(V_{t-1}) + (1 - \beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad \text{(see (5.1.34(p.27)) and (5.1.33(p.27)))} \qquad (6.2.116) \end{aligned}$$

□ $\texttt{SOE}\{\tilde{\texttt{M}}{:}2[\mathbb{P}][\texttt{A}]\}$ can be reduced to (6.2.112(p.36)), (6.2.114(p.36)), and (6.2.115(p.36)), listed in Table 6.4.3(p.41) (IV). ■

Now, let us here define

$$\tilde{\mathbb{S}}_t = \lambda\beta(v_{t-1} - V_{t-1}) + s, \quad t > 0. \tag{6.2.117}$$

Then, (6.2.113(p.36)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\lambda\beta v_{t-1} - \lambda\beta V_{t-1} + s, 0\} - \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.118}$$

implying that

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.119}$$

which can be rewritten as, due to Def. 6.1.1(p.29),

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.120}$$
$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.121}$$
$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.122}$$

Then, from (6.2.110(p.36)) with $t - 1$ we have $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = \tilde{T}(V_{t-1})$, thus, noting (5.1.33(p.27)), we can rewrite (6.2.117(p.36)) as below

$$\mathbb{S}_t = \beta\tilde{T}(V_{t-1}) + s \quad t > 0. \tag{6.2.123}$$
$$= \tilde{L}(V_{t-1}), \quad t > 0. \tag{6.2.124}$$

6.2.1.3 Model 3

6.2.1.3.1 M:3[ℝ][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively, expressed as

$$\begin{aligned} v_0(w) &= \max\{w, \rho\}, && (6.2.125)\\ v_t(w) &= \max\{w, \rho, U_t\}, \quad t > 0, && (6.2.126)\\ V_0 &= \rho, && (6.2.127)\\ V_t &= \max\{\rho, U_t\}, \qquad t > 0, && (6.2.128)\end{aligned}$$

where U_t is the maximum of the total expected present discounted *profit* from rejecting both the price w and intervening quitting penalty ρ in (6.2.126(p.37)) and from rejecting the intervening quitting penalty ρ in (6.2.128(p.37)). Then, U_t can be expressed as

$$U_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.129}$$

For convenience, let us here define $U_0 = \rho$, hence from (6.2.127(p.37)) we have

$$V_0 = U_0 = \rho. \tag{6.2.130}$$

Then, it follows that both (6.2.126(p.37)) and (6.2.128(p.37)) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$\begin{aligned} v_t(w) &= \max\{w, \rho, U_t\}, \quad t \geq 0, && (6.2.131)\\ V_t &= \max\{\rho, U_t\}, \qquad t \geq 0, && (6.2.132)\end{aligned}$$

thus (6.2.131(p.37)) can be expressed as

$$v_t(w) = \max\{w, V_t\}, \quad t \geq 0. \tag{6.2.133}$$

Accordingly, since $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] = \mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.1(p.25)), we can rewrite (6.2.129(p.37)) as

$$\begin{aligned} U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}\\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\}\\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.4(\text{p.25}))) && (6.2.134)\\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}\\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad (\text{see } (5.1.8(\text{p.25}))). && (6.2.135)\end{aligned}$$

□ SOE{M:3[ℝ][A]} can be reduced to (6.2.130(p.37)), (6.2.132(p.37)), and (6.2.134(p.37)), listed in Table 6.4.5(p.41) (I). ∎

6.2.1.3.2 M̃:3[ℝ][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time $t \geq 0$ with a seller w and with no seller respectively, expressed as

$$\begin{aligned} v_0(w) &= \min\{w, \rho\}, && (6.2.136)\\ v_t(w) &= \min\{w, \rho, U_t\}, \quad t > 0, && (6.2.137)\\ V_0 &= \rho, && (6.2.138)\\ V_t &= \min\{\rho, U_t\}, \qquad t > 0, && (6.2.139)\end{aligned}$$

where U_t is the minimum of the total expected present discounted *cost* from rejecting both the price w and intervening quitting penalty ρ in (6.2.137(p.37)) and from rejecting the intervening quitting penalty ρ in (6.2.139(p.37)). Then, U_t can be expressed as

$$U_t = \min\{\mathtt{C} : \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\ \mathtt{S} : \beta V_{t-1}\}, \quad t > 0. \tag{6.2.140}$$

For convenience, let us here define $U_0 = \rho$, hence from (6.2.138(p.37)) we have

$$V_0 = U_0 = \rho. \tag{6.2.141}$$

Then, it follows that both (6.2.137(p.37)) and (6.2.139(p.37)) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$\begin{aligned} v_t(w) &= \min\{w, \rho, U_t\}, \quad t \geq 0, && (6.2.142)\\ V_t &= \min\{\rho, U_t\}, \qquad t \geq 0, && (6.2.143)\end{aligned}$$

thus (6.2.137(p.37)) can be expressed as

$$v_t(w) = \min\{w, V_t\}, \quad t \geq 0. \tag{6.2.144}$$

Accordingly, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.11(p.25)), we can rewrite (6.2.140(p.37)) as follows.

$$\begin{aligned} U_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.25)))} \quad (6.2.145) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.14(p.25)) and (5.1.13(p.25)))}. \quad (6.2.146) \end{aligned}$$

□ SOE{$\tilde{\mathsf{M}}$:3[ℝ][A]} can be reduced to (6.2.141(p.37)), (6.2.143(p.37)), and (6.2.145(p.38)), listed in Table 6.4.5(p.41) (II). ∎

6.2.1.3.3 M:3[ℙ][A]

By v_t $(t \ge 0)$ and V_t $(t \ge 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \ge 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, the seller must determine whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. Let the ρ be accepted. Then the profit which the seller can obtain is ρ. On the other hand, let the asset be sold to the buyer. Then, since the seller must sell the asset to the buyer due to A2(p.11), the price $a^\dagger$ must be proposed to the buyer, in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \quad (6.2.147)$$

hence the profit which the seller obtains at that time is a. Accordingly, the profit that the seller obtains at time 0 becomes

$$v_0 = \max\{\rho, a\}. \quad (6.2.148)$$

Next we have

$$v_t = \max\{\rho, H_t\}, \quad t > 0, \quad (6.2.149)$$

$$V_0 = \rho, \quad (6.2.150)$$

$$V_t = \max\{\rho, U_t\}, \quad t > 0, \quad (6.2.151)$$

where H_t and U_t are defined as follows. Firstly H_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since a buyer exists due to the above definition of v_t and since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \le \boldsymbol{\xi}\}$ (see (5.1.18(p.26))). Hence we have

$$H_t = \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0 \quad (6.2.152)$$

due to (5.1.19(p.26)), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \quad (6.2.153)$$

due to (5.1.25(p.26)). Finally U_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since no buyer exists due to the above definition of V_t, it can be expressed as follows.

$$U_t = \max\{\,\mathtt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s,\ \mathtt{S}\colon \beta V_{t-1}\}, \quad t > 0. \quad (6.2.154)$$

For $t = 1$ we have

$$\begin{aligned} U_1 &= \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &= \max\{\lambda\beta\max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &= \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \quad (6.2.155) \end{aligned}$$

Now, from (6.2.152(p.38)) we have $H_t - V_t = T(V_t)$ for $t > 0$, hence from (6.2.149(p.38)) we have $v_t - V_t = \max\{\rho - V_t, H_t - V_t\} = \max\{\rho - V_t, T(V_t)\} \cdots (1)$ for $t > 0$. Since $V_t \ge \rho$ for $t > 0$ from (6.2.151(p.38)), we have $\rho - V_t \le 0$ for $t > 0$. In addition, since $p(b) = 0$ due to (5.1.29 (2) (p.26)), from (5.1.19(p.26)) we have $T(V_t) \ge p(b)(b - V_t) = 0$. Therefore, since $\rho - V_t \le 0 \le T(V_t)$, from (1) we have $v_t - V_t = T(V_t)$ for $t > 0$, i.e., $v_t = T(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.154(p.38)) with $t > 1^\ddagger$ can be rearranged as

$$\begin{aligned} U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.26)))} \quad (6.2.156) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.21(p.26)) and (5.1.20(p.26)))}. \quad (6.2.157) \end{aligned}$$

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† The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer

‡ Instead of $t > 0$.

For convenience, let $U_0 = \rho$. Then, due to (6.2.150(p.38)) we have

$$V_0 = U_0 = \rho, \tag{6.2.158}$$

hence it follows that (6.2.151(p.38)) holds true for $t \geq 0$ instead of $t > 0$, i.e.,

$$V_t = \max\{\rho, U_t\}, \quad t \geq 0. \tag{6.2.159}$$

□ SOE{M:3[ℙ][A]} is given by (6.2.158(p.39)), (6.2.159(p.39)), (6.2.155(p.38)), and (6.2.156(p.38)), listed in Table 6.4.5(p.41) (III). ∎

6.2.1.3.4 $\tilde{\text{M}}$:3[ℙ][A]

By v_t ($t \geq 0$) and V_t ($t \geq 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, the buyer must determine whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. Let the ρ be accepted. Then, the cost which the buyer pays at time 0 is ρ. On the other hand, let the asset be bought for the buyer. Then, since the buyer must buy the asset from the seller due to A2(p.11), the price b[†] must be is proposed to the seller; in other words, the optimal price to propose is given by

$$z_0 = b, \tag{6.2.160}$$

hence the cost which the buyer pays at that time is b. Accordingly, the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\}. \tag{6.2.161}$$

Next we have

$$v_t = \min\{\rho, H_t\}, \quad t > 0. \tag{6.2.162}$$

$$V_0 = \rho, \tag{6.2.163}$$

$$V_t = \min\{\rho, U_t\}, \quad t > 0, \tag{6.2.164}$$

where H_t and U_t are defined as follows. Firstly H_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since a seller exists due to the above definition of v_t and since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes the price z to an appearing seller, the probability of the seller selling the asset for the price z is $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.26))). Hence we have

$$H_t = \min_z\{\tilde{p}(z)z + (1 - \tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.165}$$

due to (5.1.32(p.26)), implying that the optimal buying price which the buyer should pay is given by

$$z_t = z(V_t), \qquad t \geq 0, \tag{6.2.166}$$

due to (5.1.38(p.27)). Finally U_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since no seller exists due to the above definition of V_t, it can be expressed as follows.

$$U_t = \min\{\texttt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} + s,\ \texttt{S}\colon \beta V_{t-1}\}, \quad t > 0. \tag{6.2.167}$$

For $t = 1$ we have

$$\begin{aligned} U_1 &= \min\{\lambda\beta v_0 + (1-\lambda)\beta V_0 + s, \beta V_0\} \\ &= \min\{\lambda\beta \min\{\rho, b\} + (1-\lambda)\beta\rho + s, \beta\rho\} \\ &= \min\{\lambda\beta \min\{0, b-\rho\} + \beta\rho + s, \beta\rho\}. \end{aligned} \tag{6.2.168}$$

Now, from (6.2.165(p.39)) we have $H_t - V_t = \tilde{T}(V_t)$ for $t > 0$, hence from (6.2.162(p.39)) we have $v_t - V_t = \min\{\rho - V_t, H_t - V_t\} = \min\{\rho - V_t, \tilde{T}(V_t)\} \cdots (2)$ for $t > 0$. Since $V_t \leq \rho$ for $t > 0$ from (6.2.164(p.39)), we have $\rho - V_t \geq 0$ for $t > 0$. In addition, since $\tilde{p}(a) = 0$ due to (5.1.41 (1) (p.27)), from (5.1.32(p.26)) we have $\tilde{T}(V_t) \leq \tilde{p}(a)(a - V_t) = 0$. Therefore, since $\rho - V_t \geq 0 \geq \tilde{T}(V_t)$, from (2) we have $v_t - V_t = \tilde{T}(V_t)$ for $t > 0$, i.e., $v_t = \tilde{T}(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.167(p.39)) with $t > 1$ can be rearranged as

$$\begin{aligned} U_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.27)))} \qquad (6.2.169) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}, \quad t > 1 \\ &= \max\{\tilde{L}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.27)) and (5.1.33(p.27)))} \qquad (6.2.170) \end{aligned}$$

[†]The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

For convenience, let $U_0 = \rho$. Then, due to (6.2.163(p.39)) we have

$$V_0 = U_0 = \rho, \tag{6.2.171}$$

hence it follows that (6.2.164(p.39)) holds true for $t \geq 0$ instead of $t > 0$, i.e.,

$$V_t = \min\{\rho, U_t\}, \quad t \geq 0. \tag{6.2.172}$$

□ SOE{$\tilde{\mathsf{M}}$:3[$\mathbb{R}$][A]} is given by (6.2.171(p.40)), (6.2.172(p.40)), (6.2.168(p.39)), and (6.2.169(p.39)), listed in Table 6.4.5(p.41) (IV). ▯

6.2.2 Search-Enforced-Model

In sE-model (M:x[$\mathbb{X}$][E] and $\tilde{\mathsf{M}}$:x[$\mathbb{X}$][E] with $x = 1, 2, 3$ and $\mathbb{X} = \mathbb{R}, \mathbb{P}$) a leading-trader needs to take no decision activity regarding whether or not to conduct the search. This implies that eliminating the terms related to this decision from the systems of optimality equations in sA-model (SOE{M:x[$\mathbb{X}$][A]} and SOE{$\tilde{\mathsf{M}}$:x[$\mathbb{X}$][A]}) produces the systems of optimality equations in sE-model (SOE{M:x[$\mathbb{X}$][E]} and SOE{$\tilde{\mathsf{M}}$:x[$\mathbb{X}$][E]}). Noting this, from Tables 6.4.1(p.41), 6.4.3(p.41), and 6.4.5(p.41) we can immediately obtain the systems of optimality equations for `E-model`, which are given by Tables 6.4.2(p.41), 6.4.4(p.41), and 6.4.6(p.41).

6.2.3 Assertion and Assertion System of Model

In general, let us refer to a description on whether or not a given statement is true as the *assertion*, denoted by A, and as a set consisting of some assertions as the *assertion system*, denoted by $\mathscr{A}$. In addition, let us denote an assertion and an assertion system for a given Model by respectively A{Model} and $\mathscr{A}${Model}.

6.3 Recall-Model

See Chap. 24(p.239).

6.4 Summary of the System of Optimality Equations (SOE)

Model 1

Table 6.4.1: **Search-Allowed-Model** 1

(I) SOE{M:1[ℝ][A]}

$$V_1 = \beta\mu - s, \tag{6.4.1}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.2}$$

(II) SOE{M̃:1[ℝ][A]}

$$V_1 = \beta\mu + s, \tag{6.4.3}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.4}$$

(III) SOE{M:1[ℙ][A]}

$$V_1 = \beta a - s, \tag{6.4.5}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.6}$$

(IV) SOE{M̃:1[ℙ][A]}

$$V_1 = \beta b + s, \tag{6.4.7}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.8}$$

Table 6.4.2: **Search-Enforced-Model** 1

(I) SOE{M:1[ℝ][E]}

$$V_1 = \beta\mu - s, \tag{6.4.9}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.4.10}$$

(II) SOE{M̃:1[ℝ][E]}

$$V_1 = \beta\mu + s, \tag{6.4.11}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.4.12}$$

(III) SOE{M:1[ℙ][E]}

$$V_1 = \beta a - s, \tag{6.4.13}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.4.14}$$

(IV) SOE{M̃:1[ℙ][E]}

$$V_1 = \beta b + s, \tag{6.4.15}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.4.16}$$

Model 2

Table 6.4.3: **Search-Allowed-Model** 2

(I) SOE{M:2[ℝ][A]}

$$V_0 = \rho, \tag{6.4.17}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.4.18}$$

(II) SOE{M̃:2[ℝ][A]}

$$V_0 = \rho, \tag{6.4.19}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.4.20}$$

(III) SOE{M:2[ℙ][A]}

$$V_0 = \rho, \tag{6.4.21}$$
$$V_1 = \max\{\lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.4.22}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.23}$$

(IV) SOE{M̃:2[ℙ][A]}

$$V_0 = \rho, \tag{6.4.24}$$
$$V_1 = \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.4.25}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.26}$$

Table 6.4.4: **Search-Enforced-Model** 2

(I) SOE{M:2[ℝ][E]}

$$V_0 = \rho, \tag{6.4.27}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.4.28}$$

(II) SOE{M̃:2[ℝ][E]}

$$V_0 = \rho, \tag{6.4.29}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.4.30}$$

(III) SOE{M:2[ℙ][E]}

$$V_0 = \rho, \tag{6.4.31}$$
$$V_1 = \lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \tag{6.4.32}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.4.33}$$

(IV) SOE{M̃:2[ℙ][E]}

$$V_0 = \rho, \tag{6.4.34}$$
$$V_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \tag{6.4.35}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.4.36}$$

Model 3

Table 6.4.5: **Search-Allowed-Model** 3

(I) SOE{M:3[ℝ][A]}

$$V_0 = U_0 = \rho, \tag{6.4.37}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.4.38}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.4.39}$$

(II) SOE{M̃:3[ℝ][A]}

$$V_0 = U_0 = \rho, \tag{6.4.40}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.4.41}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.4.42}$$

(III) SOE{M:3[ℙ][A]}

$$V_0 = U_0 = \rho, \tag{6.4.43}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.4.44}$$
$$U_1 = \max\{\lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.4.45}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.46}$$

(IV) SOE{M̃:3[ℙ][A]}

$$V_0 = U_0 = \rho, \tag{6.4.47}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.4.48}$$
$$U_1 = \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.4.49}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.4.50}$$

Table 6.4.6: **Search-Enforced-Model** 3

(I) SOE{M:3[ℝ][E]}

$$V_0 = U_0 = \rho, \tag{6.4.51}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.4.52}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.4.53}$$

(II) SOE{M̃:3[ℝ][E]}

$$V_0 = U_0 = \rho, \tag{6.4.54}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.4.55}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.4.56}$$

(III) SOE{M:3[ℙ][E]}

$$V_0 = U_0 = \rho, \tag{6.4.57}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.4.58}$$
$$U_1 = \lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \tag{6.4.59}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.4.60}$$

(IV) SOE{M̃:3[ℙ][E]}

$$V_0 = U_0 = \rho, \tag{6.4.61}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.4.62}$$
$$U_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \tag{6.4.63}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.4.64}$$

Chapter 7

Optimal Decision Rules

This chapter clarifies the structure of the optimal decision rules for the 24 no-recall-models in Table 3.2.1(p.17).

7.1 Points in Time

To start with, let us note here that the optimal decision rules are closely related to the following six points in time (see H1(p.8)).

- *Recognizing time* t_r (see H1a(p.8)),
- *Starting time* τ (see H1b(p.8)),
- *Initiating time* t_i (see H1c(p.8)),
- *Stopping time* t_s (see H1d(p.8)),
- *Deadline* $t_d = 0$ (see H1e(p.8)),
- *Quasi-deadline* t_{qd}, the smallest of all possible initiating times.
  - For Model 1, the initiating time t_i must be greater than or equal to 1 (i.e., $t_i \geq 1$) for the following reason. If $t_i = 0$, there exists no buyer at time 0, hence the process must stop without selling the asset at that time, which contradicts A2(p.11). Accordingly, the initiating time t_i must be 1 by definition.
  - For Models 2/3, suppose the initiating time t_i is equal to the deadline 0 (i.e., $t_i = 0$). Then, although there exists no buyer at that time, the process can stop by accepting the terminal quitting penalty price ρ, hence the smallest initiating time t_i is 0 by definition.

  Thus it follows that we have

$$t_{qd} = 1 \text{ for Model 1} \tag{7.1.1}$$

$$t_{qd} = 0 \text{ for Model 2/3} \tag{7.1.2}$$

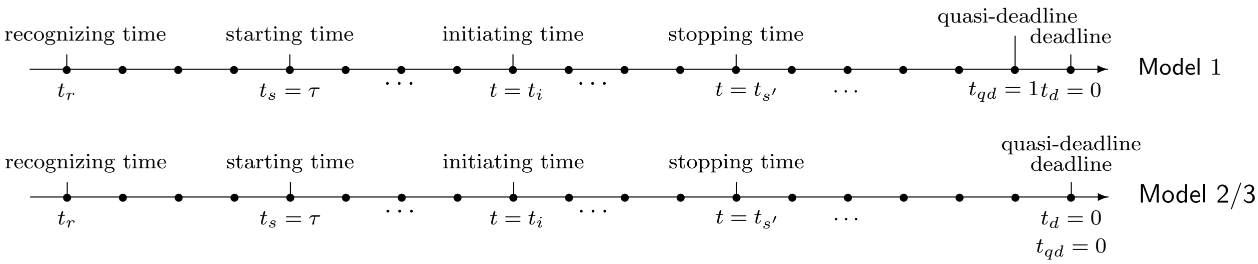

Figure 7.1.1: Six points in time related to the optimal decision rules

7.2 Four Types of Decisions

Below let us provide the strict definitions for the four types of decisions prescribed in Section 3.4(p.18).

7.2.1 Whether or Not to Accept the Proposed Price

This is the decision only for $\mathbb{R}$-`model`. Below let us represent

$$\text{Accept a price } w \text{ at time } t \;\rightarrow\; \texttt{Accept}_t\langle w\rangle, \tag{7.2.1}$$

$$\text{Reject a price } w \text{ at time } t \;\rightarrow\; \texttt{Reject}_t\langle w\rangle. \tag{7.2.2}$$

First, in the selling model, suppose that a buyer appearing at a time t has proposed a buying price w. Then, from (6.2.2(p.29)) and (6.2.62(p.33)) we have

$$w \geq (\leq)\, V_t \Rightarrow \texttt{Accept}_t\langle w\rangle \;\; (\texttt{Reject}_t\langle w\rangle). \tag{7.2.3}$$

Similarly, in the buying model, suppose that a seller appearing at a time t has proposed a selling price w. Then, from (6.2.16(p.30)) and (6.2.77(p.34)) we have

$$w \leq (\geq)\, V_t \Rightarrow \texttt{Accept}_t\langle w\rangle \;\; (\texttt{Reject}_t\langle w\rangle).$$

Then, we refer to the V_t as the `optimal-reservation-price`, opt-$\mathbb{R}$-price for short.

7.2.2 What Price to Propose

This is the decision only for $\mathbb{P}$-`model`. In the selling model, the optimal selling price which a seller (leading-trader) should propose at a time t is given by

$$z_t = z(V_t) \quad \text{(see (6.2.34(p.31)) and (6.2.94(p.35)))}.$$

Similarly, in the buying model, the optimal buying price which a buyer (leading-trader) should propose at a time t is given by

$$z_t = z(V_t) \quad \text{(see (6.2.50(p.32)) and (6.2.111(p.36)))}.$$

Then, we refer to the z_t as the `optimal-posted-price`, opt-$\mathbb{P}$-price for short.

7.2.3 Whether or not to Conduct the Search

This is the decision only for sA-model (see (A5b(p.12)). Then, its decision rule is given by (6.2.9(p.30)), (6.2.23(p.31)), (6.2.39(p.32)), (6.2.55(p.32)), (6.2.70(p.33)), (6.2.85(p.34)), (6.2.102(p.35)), and (6.2.119(p.36)).

Remark 7.2.1 (`Conduct⇝Skip` (`C⇝S`) (see Figure 2.2.3(p.12))) Figure 7.2.1(p.44) (I) below sketches the case (Model 1) that the search-Conduct starts at the optimal initiating time t_τ^* and continue up to the quasi-deadline $t_{qd} = 1$ so long as the process does not stop; it will be known later on that this case occurs everywhere in the paper. Contrary to this, Figure 7.2.1(p.44) (II) schematizes the case (Model 2) that the search-Conduct starts at the optimal initiating time t_τ^*, continues for a while, and switches to the search-Skip at a certain point in time $t' > t_{qd} = 1$; this is a very rare case that occurs only in `Tom`'s 20.1.4(p.160) (b3iii), 20.1.12(p.169) (b3iii), and 20.1.15(p.170) (b3iii). Let us represent the case as `Conduct⇝Skip`, simply `C⇝S` (Def. 2.2.1(p.12)). □

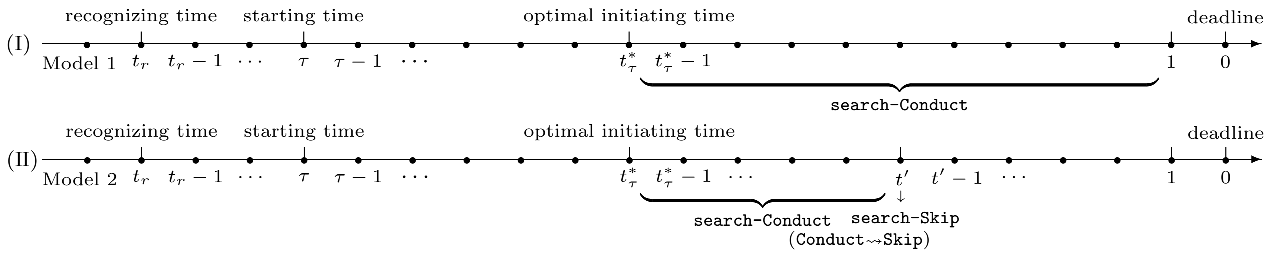

Figure 7.2.1: `Conduct⇝Skip` (`C⇝S`)

♡ Alice **2 (jumble of intuition and theory)** *Herein, Alice was hit by the following question. Suppose that* $\mathbb{S}_t < 0$ *at a time* t (see (6.2.12(p.30))), *i.e., the search-skip becomes strictly optimal at that time. Then, since* $\max\{\mathbb{S}_t, 0\} = 0$, *we have* $V_t = \beta V_{t-1}$ *from* (6.2.8(p.30))*, implying that initiating the process at time t becomes indifferent to initiating the process at time t* − 1*; nevertheless, the search skip is strictly optimal! After having mumbled, letting out a strange noise "Is this a little bit funny ?", she gave a shout "Such a laughable affair !". Then, Dr. Rabbit again appeared and pedantically told to Alice "The above two results are both ones based on a theory of mathematics, but your confusion is caused by an intuition; there does not exist any logical relationship between the two! Well··· your confusion is what is caused by a jumble of intuition and theory!!", and he again disappeared down the hole as murmuring "Oh dear! Oh dear! I shall be too late for the faculty meeting!".* ▮

7.2.4 When to Initiate the Process (Optimal Initiating Time)

This is the decision only for iiA-`Case` (see H1cii(p.8)).

7.2.4.1 Definition

The definition below is only for a selling model with $t_{qd} = 1$ (Model 1 ($t_{qd} = 0$ for Model 2)). Suppose that the process has started at the starting time τ and that the seller (leading-trader) has determined to *initiate* the process at a given time t after that ($\tau \geq t \geq t_{qd}$), i.e., $\tau - t$ periods hence. Then, the total expected present discounted profit at the *starting time* τ is given by

$$I_\tau^t \overset{\text{def}}{=} \beta^{\tau - t} V_t, \quad \tau \geq t \geq t_{qd}. \tag{7.2.4}$$

See (6.2.3(p.29)) and (6.2.4(p.29)) for the definition of V_t. Then, by t_τ^* let us denote t maximizing I_τ^t on $\tau \geq t \geq t_{qd}$, i.e.,

$$I_\tau^{t_\tau^*} = \max_{\tau \geq t \geq t_{qd}} I_\tau^t \quad \text{or equivalently} \quad I_\tau^{t_\tau^*} \geq I_\tau^t, \quad \tau \geq t \geq t_{qd}. \tag{7.2.5}$$

Let us call the t^*_τ the *optimal initiating time*, denoted by $\texttt{OIT}_\tau\langle t^*_\tau\rangle_\vartriangle$. If

$$I_\tau^{t^*_\tau} > I_\tau^t \ \text{ for } \ t \neq t^*_\tau, \tag{7.2.6}$$

then it is called the <u>strictly</u> *optimal initiating time*, denoted by $\texttt{OIT}_\tau\langle t^*_\tau\rangle_\blacktriangle$.

Remark 7.2.2 (strictness of optimality ($\blacktriangle$)) Suppose that the initiating time t^*_τ is <u>strictly</u> optimal in a sense of (7.2.6(p.45)). Then, since $I_\tau^{t^*_\tau} > I_\tau^{t^*_\tau-1}$, we have $\beta^{\tau-t^*_\tau}V_{t^*_\tau} > \beta^{\tau-t^*_\tau+1}V_{t^*_\tau-1}$, hence $V_{t^*_\tau} > \beta V_{t^*_\tau-1}$. Accordingly, since $V_{t^*_\tau} = \max\{\mathbb{S}_{t^*_\tau}, 0\} + \beta V_{t^*_\tau-1}$ from (6.2.8(p.30)) with $t = t^*_\tau$, we have $\max\{\mathbb{S}_{t^*_\tau}, 0\} > 0$, hence $\mathbb{S}_{t^*_\tau} > 0$, implying that it becomes <u>*strictly*</u> optimal to conduct the search due to (6.2.12(p.30)); in other words, it is not allowed to skip the search. □

Throughout the paper, let us employ the following preference rule.

Preference Rule **7.2.1** *Let $I^t_\tau = I^{t-1}_\tau$ for a given t. Then, the seller (leading-trader) prefers $t-1$ to t as the initiating time, implying that "Postpone the initiation of the process so long as it is not unprofitable to do so."* □

7.2.4.2 β-adjusted sequence $V_{\beta[\tau]}$

First, let us denote the sequence consisting of V_τ, $V_{\tau-1}$, $V_{\tau-2}$, $\cdots$, $V_{t_{qd}}$ by

$$V_{[\tau]} \stackrel{\text{def}}{=} \{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{t_{qd}}\}, \tag{7.2.7}$$

called the *original sequence* and let

$$t^{*\prime}_\tau = \arg\max V_{[\tau]} = \arg\max\{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{t_{qd}}\}. \tag{7.2.8}$$

Next, let us denote the sequence

$$V_{\beta[\tau]} \stackrel{\text{def}}{=} \{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^{\tau-t_{qd}} V_{t_{qd}}\} = \{I^\tau_\tau, I^{\tau-1}_\tau, I^{\tau-2}_\tau, \cdots, I^{t_{qd}}_\tau\}, \tag{7.2.9}$$

called the *β-adjusted sequence* of $V_{[\tau]}$. By definition, the optimal initiating time t^*_τ is given by t attaining the maximum of elements within *β-adjusted sequence* $V_{\beta[\tau]}$, i.e.,

$$t^*_\tau = \arg\max V_{\beta[\tau]} = \arg\max\{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^{\tau-t_{qd}} V_{t_{qd}}\}. \tag{7.2.10}$$

Note here that the monotonicity of the original sequence $V_{[\tau]}$ is not always inherited to the β-adjusted sequence $V_{\beta[\tau]}$, i.e., $t^*_\tau \neq t^{*\prime}_\tau$ (see Section A 5.2.2(p.310)).

7.2.4.3 Three Possibilities

Below let us define the three types of the optimal initiating time (OIT).

1. **Degeneration to the starting time τ**

   Let $t^*_\tau = \tau$, i.e., it is optimal to initiate the process at the starting time τ, denoted by Ⓢ. Then, the optimal initiating time t^*_τ is said to *degenerate* to the *starting time* τ, represented by $\boxed{\text{Ⓢ}\ \texttt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ (Ⓢ$_\vartriangle$ for short). If the optimal initiating time t^*_τ is *strict* (see (7.2.6(p.45))), it is called the *strictly degenerate* OIT, represented by $\boxed{\text{Ⓢ}\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ (Ⓢ$_\blacktriangle$ for short).

2. **Non-degeneration ($\tau > t^*_\tau > t_{qd}$)**

   Let $\tau > t^*_\tau > t_{qd}$, i.e., the optimal initiating time is between the starting time τ and the quasi-deadline t_{qd}, denoted by ◎. Then, the optimal initiating time t^*_τ is said to be the *non-degenerate* OIT, represented by $\boxed{\text{◎}\ \texttt{ndOIT}_\tau\langle t^*_\tau\rangle}_\vartriangle$ (◎$_\vartriangle$ for short). If

   $$I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^{t^*_\tau}_\tau \geq I^{t_{qd}}_\tau \tag{7.2.11}$$

   as a special case, then it is said to be the *indifferently* non-degenerate OIT (see Preference Rule 7.2.1), represented by $\boxed{\text{◎}\ \texttt{ndOIT}_\tau\langle t^*_\tau\rangle}_\parallel$ (◎$_\parallel$ for short). If $I^{t^*_\tau}_\tau > I^t_\tau$ for all $t \neq t^*_\tau$, then it is said to be the *strictly* non-degenerate OIT, represented by $\boxed{\text{◎}\ \texttt{ndOIT}_\tau\langle t^*_\tau\rangle}_\blacktriangle$ (◎$_\blacktriangle$ for short).

3. **Degeneration to the deadline t_{qd}**

   Let $t^*_\tau = t_{qd} = 1\,(0)$ for Model 1 (Model 2/3), i.e., the optimal initiating time is the quasi-deadline, denoted by ⓓ. Then, the optimal initiating time t^*_τ is said to *degenerate* to the *quasi-deadline* t_{qd}, represented by $\boxed{\bullet\ \texttt{dOITd}_\tau\langle t_{qd}\rangle}_\vartriangle$ (ⓓ$_\vartriangle$ for short). If its optimality is *strict*, then it is called the *strictly degenerate* OIT, represented by $\boxed{\bullet\ \texttt{dOITd}_\tau\langle t_{qd}\rangle}_\blacktriangle$ (ⓓ$_\blacktriangle$ for short). If

   $$I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^{t_{qd}}_\tau, \cdots (1)$$

   then the degeneration is said to be *indifferent*, represented by $\boxed{\bullet\ \texttt{dOITd}_\tau\langle t_{qd}\rangle}_\parallel$ (ⓓ$_\parallel$ for short).

Remark 7.2.3 When (7.2.11(p.45)) is possible, as an optimal initiating time we can define Ⓢ$_\parallel$ if Preference Rule 7.2.1(p.45) is ignored. However, this definition is not permitted since the preference rule is applied throughout the paper. □

7.2.4.4 First Search Conducing Time

There might exist a person who thinks that the optimal initiating time can be given also by the first search conducing time. Here, for example, consider M:2[ℝ][A] ($t_{qd} = 0$) with the starting time $\tau = 6$ where

$$\texttt{Skip}_{6\vartriangle},\quad \texttt{Skip}_{5\vartriangle},\quad \texttt{Skip}_{4\vartriangle},\quad \texttt{Conduct}_{3\blacktriangle},\quad \texttt{Conduct}_{2\vartriangle},\quad \texttt{Conduct}_{1\vartriangle},$$

meaning that the *first search conducting time* is $t_\tau^{**} \overset{\text{def}}{=} 3 \cdots$ (2). Then, since

$$\mathbb{S}_6 \le 0,\quad \mathbb{S}_5 \le 0,\quad \mathbb{S}_4 \le 0,\quad \mathbb{S}_3 > 0,\quad \mathbb{S}_2 \ge 0,\quad \mathbb{S}_1 \ge 0$$

from (6.2.71(p.33)) and (6.2.73(p.33)), we have

$$\max\{\mathbb{S}_6, 0\} = 0,\quad \max\{\mathbb{S}_5, 0\} = 0,\quad \max\{\mathbb{S}_4, 0\} = 0,\quad \max\{\mathbb{S}_3, 0\} > 0,\quad \max\{\mathbb{S}_2, 0\} \ge 0,\quad \max\{\mathbb{S}_1, 0\} \ge 0.$$

Thus, from (6.2.69(p.33)) we have

$$V_6 = \beta V_5,\quad V_5 = \beta V_4,\quad V_4 = \beta V_3,\quad V_3 > \beta V_2,\quad V_2 \ge \beta V_1,\quad V_1 \ge \beta V_0,$$

so

$$V_6 = \beta V_5 = \beta^2 V_4 = \beta^3 V_3 > \beta^4 V_2 \ge \beta^5 V_1 \ge \beta^6 V_0 \quad \text{or equivalently} \quad I_6^6 = I_6^5 = I_6^4 = I_6^3 > I_6^2 \ge I_6^1 \ge I_6^0$$

due to (7.2.4(p.44)), hence we have the *optimal initiating time* $t_\tau^* = 3 \cdots$ (3) by definition.

♡ Alice **3 (first search conducting time)** *When the story has come up to here, after a moment's reverse, Alice happened to conceive of an idea; "Since $t_\tau^{**} = t_\tau^* = 3$ due to (2) and (3), as an optimal initiating time we can employ the first search conducting time $t_\tau^{**} = 3$ instead of $t_\tau^* = 3$!". Then, Dr. Rabbit appeared and told to her "Surely you are not incorrect, Miss Alice !. But, but— the profit attained by initiating the process at the first search conducting time t_τ^{**} is the same as the profit attained by initiating the process at the optimal initiating time t_τ^*; in other words, since the former profit does not become greater than the latter profit, we have no reason why t_τ^{**} must be used instead of t_τ^*; accordingly, it suffices to employ t_τ^* !! Miss Alice !!! ". And then, taking a watch out of the waistcoat-pocket and murmuring "Oh dear! Oh dear! I shall be too late for the faculty meeting", he again disappeared down the hole.* ▮

7.2.4.5 Null-Time-Zone

In this section let us raise a *perplexing* question caused by the optimal initiating time t_τ^*. Here, let $\tau > t_\tau^*$, i.e., the optimal initiating time t_τ^* is not the starting time τ (see Figure 7.2.2(p.46) below), implying that no decision-making action is taken at every point in time $t = \tau, \tau - 1, \cdots, t_\tau^* + 1$. Let us refer to each of $\tau, \tau - 1, \cdots, t_\tau^* + 1$ as the *null point in time* and the whole of these time points as the *null-time-zone*, denoted as `Null-TZ`.

$$\texttt{Null-TZ} \overset{\text{def}}{=} \langle \tau, \tau - 1, \cdots, t_\tau^* + 1 \rangle.$$

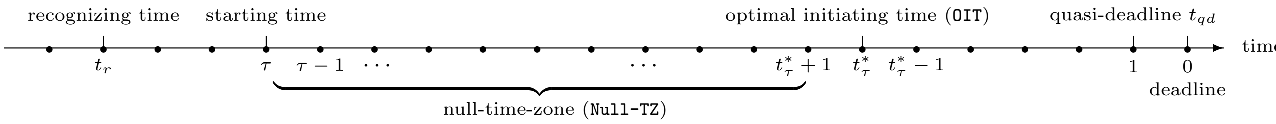

Figure 7.2.2: Null-time-zone in Model 1 with $t_{qd} = 1$ (Null-TZ)

The above event implies that, without noticing the existence of `Null-TZ`, so far we *unwittingly* or *unconsciously* might have continued to fall into the senselessness of engaging in *unnecessary decision-making activities* over these points in time.

7.2.4.6 Deadline-Engulfing

♡ Alice **4 (black hole)** *Hereupon, Alice supposed "If the optimal initiating time t_τ^* degenerates to the deadline (time 0), then what will ever happen ?", and screamed out "If so, it follows that don't conduct any decision-making activity up to the deadline !; If that happens, the whole of decision-making activities which are scheduled at the starting time τ come to nought as if being engulfed in the deadline !". Alice was heavily nonplused and cried "It ⋯, it is the same as that black hole into which all physical matters, even light, are squeezed into ! If so, ⋯, a decision process with an infinite planning horizon vanishes away in time toward an infinite future ! Oh dear!! Oh dear !!! ⋯ " She hunkered down, and then buried her head in her hands. Then, Dr. Rabbit again appeared and told to her a little bit ungraciously "This is an undeniable conclusion that is theoretically derived !."* ▮

□ *Example* **7.2.1** In fact, consider `Tom` 20.2.4(p.196) (d2i) with the condition of "$\beta < 1$, $s > 0$, $\rho > x_K$, and $\kappa \le 0$", which has $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$ (ⓓ▲). Thus, it follows that the model vanishes away in time toward an infinite future under this condition. □

□ *Example* **7.2.2** What should be noteworthy is here that `Pom` 20.2.1(p.198) (b) with the condition of "$a > 0$, $\beta = 1$, $s = 0$, and $\rho \ge b$" has $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ (ⓓ‖), implying that there can exist an instance of vanishing away in time toward an infinite future *even under the most simple condition of $\beta = 1$ and $s = 0$.* □

In this paper, let us refer to "engulfed in the deadline" as "*deadline-engulfing*", represented by ⓓ-engulfing. This situation can be depicted as the two figures below.

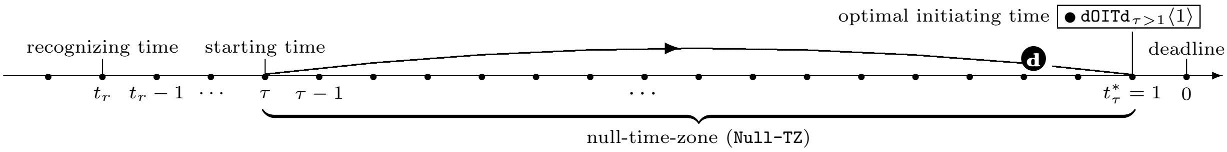

Figure 7.2.3: Deadline-engulfing (ⓓ) for Model 1

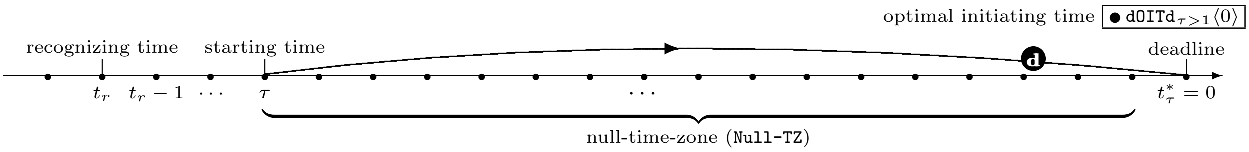

Figure 7.2.4: Deadline-engulfing (ⓓ) for Model 2/3

Later on we will demonstrate that the ⓓ-engulfing is not a rare event but a phenomenon which is very often possible; amazingly, it can occur even in the simplest case "$\beta = 1$ and $s = 0$" (see Pom's 20.2.1(p.198), 20.2.5(p.200), 20.2.9(p.210), and 20.2.17(p.217)). Taking this fact into consideration, we will inevitably be led to a serious re-examination of the whole discussion that have been made so far for all decision processes, including Markovian decision processes [21,Howard,1960] (see Section A 5(p.310)).

7.3 Mental Conflicts

Below let us represent the collective term of

opt-$\mathbb{R}$-price (V_t) (`optimal-reservation-price` (see Section 7.2.1(p.43)))
opt-$\mathbb{P}$-price (z_t) (`optimal-posted-price` (see Section 7.2.2(p.44)))

as opt-$\mathbb{R}/\mathbb{P}$-price (V_t/z_t). One of our main concerns on the opt-$\mathbb{R}/\mathbb{P}$-price (V_t/z_t) is its monotonicity.

7.3.1 Normality

Suppose that the monotonicity over the entire planning horizon is

- nondecreasing in t (see Figure 7.3.1(p.47) (I)) or
- nonincreasing in t (see Figure 7.3.1(p.47) (II)).

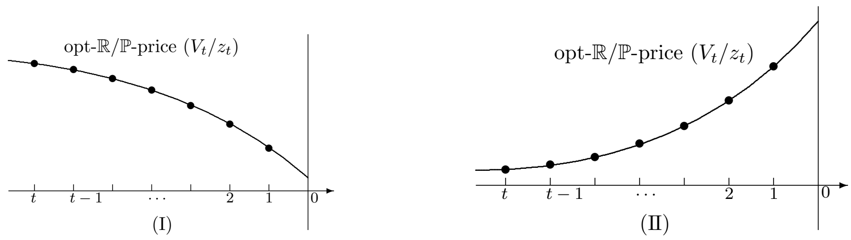

Figure 7.3.1: Normal Mental Conflict

Remark 7.3.1 (normal mental conflict) The monotonicity of the opt-$\mathbb{R}/\mathbb{P}$-price reflects the mental conflict of decision-maker that was presented within the expectation of *Examples* 1.4.1(p.5) - 1.4.4(p.6). This mental conflict can be restated as follows. As the deadline approaches,

- a seller becomes "selling spree" in the selling problem.
- a buyer becomes "buying spree" in the buying problem.

Let us refer to this as the *normal mental conflict*.. □

7.3.2 Abnormality

Suppose that the monotonicity over the entire planning horizon shifts

- from “nondecreasing” to “nonincreasing” in t (see Figure 7.3.2(p.48) (I)) or
- from “nonincreasing” to “nondecreasing ”in t (see Figure 7.3.2(p.48) (I)).

Remark 7.3.2 (abnormal mental conflict) The above monotonicity of the opt-$\mathbb{R}/\mathbb{P}$-price reflects the mental conflict stated below. As the deadline approaches

- A seller shift from “selling spree” to “buying spree”in the selling problem.
- A buyer shift from “buying spree” to “selling spree” in the buying problem.

Let us refer to this as the *abnormal mental conflict.*. □

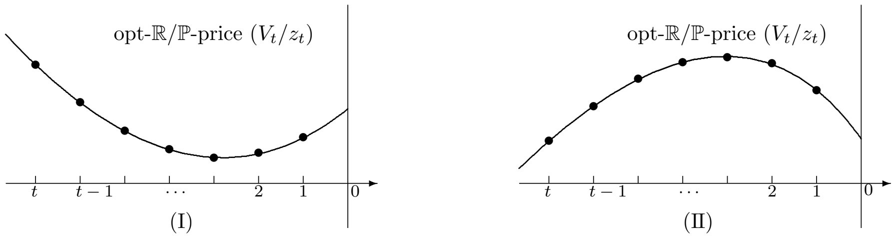

Figure 7.3.2: Abnormal Mental Conflict

Chapter 8

Conclusions of Part 1 (Introduction)

The whole discussions over Chaps. 1(p.3) - 7(p.43) are summarized as below.

$\overline{\overline{\text{C}}}$1. Two motives

Behavior of human-beings, whether a little action or a significant one, often starts with subtle motives. In an early stage of this study, the authors observed similarities between selling problem and buying problem as well as resemblances between methodologies used to analyze the two problems. This observation led us, before long, to the motives with the following questions (see Section 1.2(p.3)): (1) *Is a buying problem symmetrical to a selling problem ?* and (2) *Does a general theory integrating quadruple-asset-trading-problems exist ?* This study, spanning over near half a century, was inspired by the desire to answer the two questions. Our final conclusions are “No” for (1) and “Yes” for (2).

$\overline{\overline{\text{C}}}$2. Philosophical background

Refer to Section 1.3(p.4) for the philosophical background of “how and why we came to perceive *a decision theory as physics*”, which fundamentally informs the entire writing of this paper. Generally, a physical viewpoint stems from a mental process involving unfiltered observation of a subject, free from any preconceived premises, assumptions, hypotheses, biases, and so on. It is crucial to recognize the difficulty of this task, even for modern individuals who consider themselves enough scientifically aware. In fact, Prior to Galileo’s era (pre-1600s), no one would have questioned the belief that the heaven revolved around the Earth (Ptolemaic system). Similarly, in the absence of modern knowledge, individuals, including the authors, would adhere to this theory without question. It is essential to acknowledge that the transition to the sun-centered theory (Copernican system) took thousands of years. History demonstrates that the natural science has successfully undergone this rigorous examination. Scientists must be open to the existence of “as-yet-unrecognized knowledge” and embrace the acknowledgment of ignorance. Those familiar with physics will quickly grasp the essence of Albert Einstein “*As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.*” However, for those without this experience, the understanding of this context may require significant time or might never fully materialize.

$\overline{\overline{\text{C}}}$3. Time concept

Guided by the above philosophical background, we came to regard *human beings* as real entities that scientists study as their research objects. Now, there are no physical existence devoid of the time concept; accordingly, we inevitably and/or unconsciously began to recognize the existence of the five points in time: *recognizing time*, *starting time*, *initiating time*, *stopping time*, and *deadline* (see H1(p.8) and Section 7.1(p.43)), which dominate the whole description of this paper.

$\overline{\overline{\text{C}}}$4. Optimal initiating time

Especially noteworthy one among the above five points in time is the *initiating time*, which leads us to the *optimal initiating time* (`OIT`) (see Section 7.2.4.1(p.44)). This yields three kinds of points in time: ⓢ (starting time), ◎ (non-degenerate time), and 🅓 (deadline) (see Section 7.2.4.3(p.45)).

$\overline{\overline{\text{C}}}$5. Null-time-zone and deadline-engulfing

It is striking that the two optimal initiating times (`OIT`) ◎ and 🅓 inevitably gives rise to the events of *null-time-zone* and *deadline-engulfing* (see H3(p.9) and Sections 7.2.4.5(p.46) and 7.2.4.6(p.46)), which can be said to be novel findings in the sense that they have not been previously recognized by any researchers, including the authors in the past. What is furthermore remarkable is that the existence of ◎ and 🅓 are not rare but rather frequent (see 22.2% and 33.4% in Table 22.1.1(p.234)). Here it should be emphasized that ◎$_{\blacktriangle}$ and 🅓$_{\blacktriangle}$ (strictly optimal) occur although at the very small rates of 2.6% and 3.2% respectively (see Table 22.1.1(p.234)). Lastly, note that the existence of the above two events suggests the need for a comprehensive re-examination of all results derived in the conventional investigations of decision processes without incorporating the concept of the optimal initiating time.

$\overline{\overline{\mathsf{C}}}$6. Structured-unit-of-models

Before delving into the core of the study, we endeavored to clarify the general structure of asset trading problems (see Section 1.4(p.4)), which gave rise to the concepts of the quadruple-asset-trading-problems (see Section 1.4.5(p.7)) and the structured-unit-of-models (see Section 3.3(p.18)). One of the key points in this paper is not to analyze respective models included in the structured-unit-of-models *discretely* and *individualistically* but to clarify the *interconnectedness* among these models *systematically* and *comprehensively* by using the integrated theory in Part 2(p.51).

$\overline{\overline{\mathsf{C}}}$7. Assumptions

In Section 2.2(p.11) we presented the eleven assumptions, A1(p.11) - A11(p.13), which become necessary for providing strict definitions of all models related to asset trading problems discussed in this paper. Presumably, the three of them, A5(p.12), A7(p.12), and A11(p.13), are all what are first introduced in this paper. The first one, A5(p.12) (search-Enforced-Model and search-Allowed-Model), is from the realistic requirement, the second one, A7(p.12) (quitting penalty price), from the inevitable assumption due to $\lambda < 1$, and the last one (finiteness of planning horizon), from the physical recognition that there does not exist a problem with the infinite planning horizon in the real world.

$\overline{\overline{\mathsf{C}}}$8. **Discount factor for cost**

Refer to [40, Ross] for a description concerning an economic implications of the discount factor β for *profit*. However, surprisingly, to the best of the authors' knowledge, we have not encountered references providing a persuasive explanation for the implications of the discount factor β for *cost*. We provided a clear interpretation for this issue in Section 2.3(p.13).

$\overline{\overline{\mathsf{C}}}$9. Underlying functions.

The systems of optimality equations (see Chap. 6(p.29)) for all models (see Table 3.2.1(p.17)) are expressed by using functions T, L, K, and $\mathcal{L}$, referred to as the *underlying function* (see Chap. 5(p.25)). The function T has been often defined and used thus far in the fields of mathematical statistics, operational research, and economics (see [13,Deg1970]); however, the introduction of remaining functions L, K, and $\mathcal{L}$ (see (5.1.3(p.25)) - (5.1.5(p.25))) is presumably first in this paper. Moreover, the different properties of these functions are consistently utilized in the analyses of these models. All properties of these underlying functions (see Lemmas 10.1.1(p.55) - 10.3.1(p.59)) were derived through the repeated arrangement and rearrangement, as if solving a jigsaw puzzle, of many results that were obtained, over more than ten years, for various models. It was demonstrated in [25,Iku1996] that various results for wide-ranging types of decision problems that have been posed and examined in many references thus far can be derived by using these functions.

$\overline{\overline{\mathsf{C}}}$10. Mental conflict As illustrated in *Examples* 1.4.1(p.5)-1.4.4(p.6), the *normal* mental conflict experienced by a leading trader (see Remark 7.3.1(p.47)) can be intuitively understood. On the other hand, the *abnormal* mental conflict (see Remark 7.3.2(p.48)) is hard to immediately grasp, which is possible in fact as presented in $\overline{\overline{\mathsf{C}}}$1b2(p.233).

Part 2

Integrated Theory

In this part we attempt to construct the integrated theory.

Chapter 9

Flow of the Construction of Integrated Theory

9.1 Bird's-Eye View of Integrated Theory

Figure 9.1.1(p.53) below provides a bird's-eye view of the flow of discussions which constructs the integrated theory.

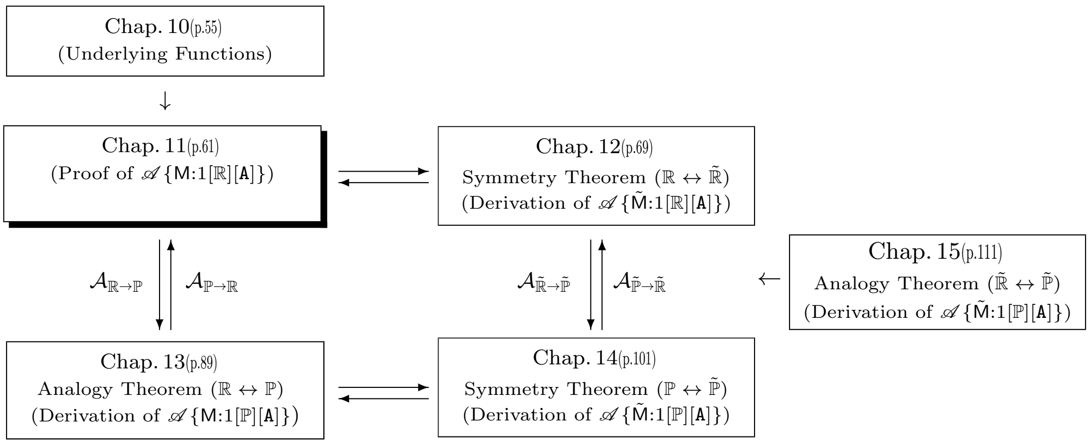

Figure 9.1.1: The flow of the construction of the integrated theory

The above figure presents the following:

- In Chap. 10(p.55), lemmas and corollaries for underlying functions are proven.
- In Chap. 11(p.61), $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is proven by using the results in Chap. 10(p.55).
- In Chap. 12(p.69), the symmetry theorem ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
- In Chap. 13(p.89), the analogy theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$) is proven, by which $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
- In Chap. 14(p.101), the symmetry theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$.
- In Chap. 15(p.111), the analogy theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$) is proven, which gives the relationship between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$.

9.2 Connection with Both Directions

In the flow of Figure 9.1.1(p.53) we should note the following:

- It is only $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}}$ that is directly proven.
- The remaining three $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}}$, $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}}$, and $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}}$ are derived by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}}$.
- The above four boxes are connected with both directions ($\leftrightarrow \updownarrow$). This interrelationship implies that any given box can be derived from any other box by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$, $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$, $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$, which are defined in Chaps. 12(p.69)-15(p.111).

Chapter 10

Properties of Underlying Functions

This chapter examines the properties of underlying functions $T_{\mathbb{R}}$, $L_{\mathbb{R}}$, $K_{\mathbb{R}}$, and $\mathcal{L}_{\mathbb{R}}$ and the $\kappa_{\mathbb{R}}$-value (see (5.1.1(p.25))-(5.1.6(p.25))), which are used to clarify the properties of the optimal decision rules for M:1[ℝ][A] (see Chap. 11(p.61)).

Definition 10.0.1 ($A\{X_{\mathbb{R}}\}$and $\mathscr{A}\{X_{\mathbb{R}}\}$) Let us denote an assertion on $X_{\mathbb{R}} = T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}$ by $A\{X_{\mathbb{R}}\}$ and an assertion system consisting of some assertions $A\{X_{\mathbb{R}}\}$'s by $\mathscr{A}\{X_{\mathbb{R}}\}$. □

10.1 Primitive Underlying Function $T_{\mathbb{R}}$

To begin with, let us prove the following lemma for the assertion system $\mathscr{A}\{T_{\mathbb{R}}\}$.

Lemma 10.1.1 ($\mathscr{A}\{T_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b]$.
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $T(x) + x$ *is strictly increasing on* $[a, \infty)$.
(f) $T(x) = \mu - x$ *on* $(-\infty, a]$ *and* $T(x) > \mu - x$ *on* (a, ∞).
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty)$.
(h) $T(x) \geq \max\{0, \mu - x\}$ *on* $(-\infty, \infty)$.
(i) $T(0) = \mu$ *if* $a > 0$ *and* $T(0) = 0$ *if* $b < 0$.
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x < y$ *and* $a < y$, *then* $T(x) + x < T(y) + y$.
(m) $\lambda\beta T(\lambda\beta\mu - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(n) $a < \mu$.[†] □

● *Proof* First, for any x and y let us prove the following two inequalities:

$$-(x-y)(1-F(y)) \leq T(x) - T(y) \leq -(x-y)(1-F(x)) \cdots \textbf{(1)}, \tag{10.1.1}$$

$$(x-y)F(y) \leq T(x) + x - T(y) - y \leq (x-y)F(x) \cdots \textbf{(2)}. \tag{10.1.2}$$

Then, let $T(x,y) \stackrel{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)]$ for any x and y.[‡] Since $1 \geq I(\boldsymbol{\xi} > y) \geq 0$ and since $\max\{\boldsymbol{\xi} - x, 0\} \geq 0$ and $\max\{\boldsymbol{\xi} - x, 0\} \geq \boldsymbol{\xi} - x$, we have

$$\max\{\boldsymbol{\xi} - x, 0\} \geq \max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > y) \geq (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y),$$

hence from (5.1.1(p.25)) we get $T(x) \geq \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] = T(x,y)$. Accordingly, for any x and y we have

$$T(x) - T(y) \geq T(x,y) - T(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} > y)] = -(x-y)\,\mathbf{E}[I(\boldsymbol{\xi} > y)].$$

Since $I(\boldsymbol{\xi} \leq y) + I(\boldsymbol{\xi} > y) = 1$, we have

$$T(x) - T(y) \geq -(x-y)(\mathbf{E}[1 - I(\boldsymbol{\xi} \leq y)]) = -(x-y)(1 - \mathbf{E}[I(\boldsymbol{\xi} \leq y)]).$$

Then, since

$$\mathbf{E}[I(\boldsymbol{\xi} \leq y)] = \textstyle\int_{-\infty}^{\infty} I(\xi \leq y)f(\xi)d\xi = \int_{-\infty}^{y} 1 \times f(\xi)d\xi = \int_{-\infty}^{y} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} \leq y\} = F(y),$$

we have $T(x) - T(y) \geq -(x-y)(1-F(y))$, hence the far left inequality of (1) holds. Multiplying both sides of the inequality by -1 leads to $-T(x) + T(y) \leq (x-y)(1-F(y))$ or equivalently $T(y) - T(x) \leq -(y-x)(1-F(y))$. Then, interchanging the

[†]The self-evident assertion is intentionally added here in order to keep the consistency with Lemma 13.2.1(p.93)(n).
[‡]If a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

notations x and y yields $T(x)-T(y) \le -(x-y)(1-F(x))$, hence the far right inequality of (1) holds. (2) is immediate from adding $x-y$ to (1). Let us note here that $T(x)$ defined by (5.1.1(p.25)) can be rewritten as

$$T(x) = \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(a \le \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} < a)] \cdots (3), \tag{10.1.3}$$

$$= \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(b < \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b)] \cdots (4). \tag{10.1.4}$$

(a,b) Immediate from (5.1.1(p.25)) and from the fact that $\max\{\boldsymbol{\xi}-x,0\}$ is continuous and nonincreasing in $x \in (-\infty,\infty)$ for any given $\boldsymbol{\xi}$.

(c) Let $y < x < b$, hence $x-y > 0$. Then, since $F(x) < 1$ due to (2.2.1 (1,2) (p.13)), we have $-(x-y)(1-F(x)) < 0$, hence $T(x)-T(y) < 0$ due to (1), so $T(x) < T(y)$, i.e., $T(x)$ is *strictly* decreasing on $x < b \cdots$ (5). Let us assume $T(x) = T(b)$ on $x < b$. Then, for any sufficiently small $\varepsilon > 0$ such that $b-x > 2\varepsilon$ we have $b > b-\varepsilon > x+\varepsilon > x$, hence $T(b) = T(x) > T(b-\varepsilon) \ge T(b)$ due to the *strict* decreasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, it must be that $T(x) \ne T(b)$ on $x < b$, so $T(x) > T(b)$ or $T(x) < T(b)$ on $x < b$; however, the latter is impossible due to (b), hence it follows that $T(x) > T(b)$ on $x < b$. From this fact and (5) it inevitably follows that $T(x)$ is strictly decreasing on $x \le b$, i.e., $T(x)$ is strictly decreasing on $(-\infty, b]$.

(d) Evident from the fact that $T(x)+x = \mathbf{E}[\max\{\boldsymbol{\xi}, x\}]$ from (5.1.1(p.25)) and $\max\{\boldsymbol{\xi}, x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $a < y < x$, hence $F(y) > 0$ due to (2.2.1 (2,3) (p.13)). Then, since $(x-y)F(y) > 0$, we have $0 < T(x)+x-T(y)+y$ from (2), hence $T(y)+y < T(x)+x$, i.e., $T(x)+x$ is *strictly* increasing on $a < x \cdots$ (6). Let us assume $T(a)+a = T(x)+x$ on $a < x$. Then, for any sufficiently small $\varepsilon > 0$ such that $x-a > \varepsilon$ we have $a < a+\varepsilon < x$, hence $T(a)+a = T(x)+x > T(a+\varepsilon)+a+\varepsilon \ge T(a)+a$ due to the *strict* increasingness shown above and the nondecreasing in (d), which is a contradiction. Thus, it must be that $T(x)+x \ne T(a)+a$ on $a < x$, so we have $T(x)+x > T(a)+a$ or $T(x)+x < T(a)+a$ on $a < x$; however, the latter is impossible due to (d), hence it follows that $T(x)+x > T(a)+a$ on $a < x$. From this fact and (6) it inevitably follows that $T(x)+$ is strictly increasing on $a \le x$, i.e., $T(x)+x$ is strictly increasing on $[a,\infty)$.

(f) Let $x \le a$. If $a \le \boldsymbol{\xi}$, then $x \le \boldsymbol{\xi}$, hence $\max\{\boldsymbol{\xi}-x,0\} = \boldsymbol{\xi}-x$ and if $\boldsymbol{\xi} < a$, then $f(\boldsymbol{\xi}) = 0 \cdots$ (7) due to (2.2.3 (1) (p.13)). Thus, from (3) we have $T(x) = \mathbf{E}[(\boldsymbol{\xi}-x)I(a \le \boldsymbol{\xi})]+0$. Then, since $\mathbf{E}[(\boldsymbol{\xi}-x)I(\boldsymbol{\xi} < a)] = \int_{\infty}^{a}(\xi-x)f(\xi)d\xi = 0$ due to (7), we have

$$T(x) = \mathbf{E}[(\boldsymbol{\xi}-x)I(a \le \boldsymbol{\xi})] + \mathbf{E}[(\boldsymbol{\xi}-x)I(\boldsymbol{\xi} < a)] = \mathbf{E}[(\boldsymbol{\xi}-x)(I(a \le \boldsymbol{\xi}) + I(\boldsymbol{\xi} < a)] = \mathbf{E}[\boldsymbol{\xi}-x] = \mu - x,$$

hence the former half is true. Then, since $T(a) = \mu - a$ or equivalently $T(a)+a = \mu$, if $a < x$, from (e) we have $T(x)+x > T(a)+a = \mu$, hence $T(x) > \mu - x$, thus the latter half is true.

(g) Let $b \le x$. If $b < \boldsymbol{\xi}$, then $f(\boldsymbol{\xi}) = 0$ due to (2.2.3 (3) (p.13)), hence $\mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(b < \boldsymbol{\xi})] = \int_b^\infty \max\{\xi-x,0\}f(\xi)d\xi = 0$ and if $\boldsymbol{\xi} \le b$, then $\boldsymbol{\xi} \le x$, hence $\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b) = 0$, so $\mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b)] = 0$. Accordingly, from (4) we have $T(x) = 0 \cdots$ (8), so the latter half is true. Let $x < b$. Then, since $T(x) > T(b)$ from (c) and $T(b) = 0$ from (8), we have $T(x) > 0$, hence the former half is true.

(h) Since $T(x) \ge \mu - x$ on $(-\infty,\infty)$ from (f) and $T(x) \ge 0$ on $(-\infty,\infty)$ from (g), it follows that $T(x) \ge \max\{0, \mu-x\}$ on $(-\infty,\infty)$.

(i) From (5.1.1(p.25)) and (2.2.3 (1,3) (p.13)) we have $T(0) = \mathbf{E}[\max\{\boldsymbol{\xi},0\}] = \mathbf{E}[\max\{\boldsymbol{\xi},0\}I(a \le \boldsymbol{\xi} \le b)]$. Hence, if $a > 0$, then $T(0) = \mathbf{E}[\boldsymbol{\xi}I(a \le \boldsymbol{\xi} \le b)] = \mathbf{E}[\boldsymbol{\xi}] = \mu$ and if $b < 0$, then $T(0) = \mathbf{E}[0I(a \le \boldsymbol{\xi} \le b)] = 0$.

(j) If $\beta = 1$, then $\beta T(x)+x = T(x)+x$, hence the assertion is true from (d).

(k) Since $\beta T(x)+x = \beta(T(x)+x)+(1-\beta)x$, if $\beta < 1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x < y$ and $a < y$. If $x \le a$, then $T(x)+x \le T(a)+a < T(y)+y$ due to (d,e), and if $a < x$, then $a \le x < y$, hence $K(x)+x < K(y)+y$ due to (e). Thus, whether $x \le a$ or $a < x$, we have $T(x)+x < T(y)+y$

(m) From (5.1.1(p.25)) we have

$$\begin{aligned} \lambda\beta T(\lambda\beta\mu - s) - s &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}-\lambda\beta\mu+s,0\}]-s \\ &= \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu+\lambda\beta s,0\}]-s \\ &= \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu-(1-\lambda\beta)s,-s\}], \end{aligned}$$

which is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.

(n) Evident. ∎

10.2 Derivative Underlying Functions

First let us define

$$\delta = 1-(1-\lambda)\beta. \tag{10.2.1}$$

Then, since $0 < \beta \le 1$ and $1 \ge \lambda > 0$, we have

$$\delta \ge 1-(1-\lambda)\times 1 = \lambda > 0 \cdots (1), \qquad \delta \le 1-(1-\lambda)\times 0 = 1 \cdots (2). \tag{10.2.2}$$

Now, from (5.1.3(p.25)) and (5.1.4(p.25)) and from Lemma 10.1.1(p.55) (f) we obtain

$$L(x)\begin{cases} = \lambda\beta\mu - s - \lambda\beta x \text{ on } (-\infty, a] & \cdots(1),\\ > \lambda\beta\mu - s - \lambda\beta x \text{ on } (a, \infty) & \cdots(2),\end{cases} \tag{10.2.3}$$

$$K(x)\begin{cases} = \lambda\beta\mu - s - \delta x \quad\text{on}\quad (-\infty, a] & \cdots(1),\\ > \lambda\beta\mu - s - \delta x \quad\text{on}\quad (a, \infty) & \cdots(2).\end{cases} \tag{10.2.4}$$

In addition, from (5.1.4(p.25)) and Lemma 10.1.1(p.55) (g) we have

$$K(x)\begin{cases} > -(1-\beta)x - s \text{ on } \quad (-\infty, b) & \cdots(1),\\ = -(1-\beta)x - s \text{ on } \quad [b, \infty) & \cdots(2),\end{cases} \tag{10.2.5}$$

from which we obtain

$$K(x) + x \geq \beta x - s \quad\text{on}\quad (-\infty, \infty). \tag{10.2.6}$$

Then, from (10.2.4 (1) (p.57)) and (10.2.5 (2) (p.57)) we get

$$K(x) + x = \begin{cases} \lambda\beta\mu - s + (1-\lambda)\beta x \text{ on } \quad (-\infty, a] & \cdots(1),\\ \beta x - s \qquad\qquad\qquad\quad \text{on } \quad [b, \infty) & \cdots(2).\end{cases} \tag{10.2.7}$$

From (5.1.8(p.25)) we have $K(x) = L(x) - (1-\beta)x$ and $L(x) = K(x) + (1-\beta)x$. Accordingly, if x_L and x_K exist, then we get

$$K(x_L) = -(1-\beta)\,x_L \cdots(1), \qquad L(x_K) = (1-\beta)\,x_K \cdots(2). \tag{10.2.8}$$

Lemma 10.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) *$L(x)$ is continuous.*
(b) *$L(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *Let $s = 0$. Then $x_L = b$ where $x_L > (\leq)\, x \Leftrightarrow L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.*
(e) *Let $s > 0$.*

1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\, x \Leftrightarrow L(x) > (= (<))\, 0$.*
2. *$(\lambda\beta\mu - s)/\lambda\beta \leq (>)\, a \Leftrightarrow x_L = (>)\, (\lambda\beta\mu - s)/\lambda\beta$.* □

• *Proof* (a-c) Immediate from (5.1.3(p.25)) and Lemma 10.1.1(p.55) (a-c).

(d) Let $s = 0$. Then, since $L(x) = \lambda\beta T(x)$, from Lemma 10.1.1(p.55) (g) we have $L(x) > 0$ for $b > x$ and $L(x) = 0$ for $b \leq x$, hence $x_L = b$ by the definition of x_L (see Section 5.2(p.27) (a)), thus $x_L > (\leq)\, x \Rightarrow L(x) > (=)\, 0$. The inverse is true by contraposition. In addition, since $L(x) = 0 \Rightarrow L(x) \leq 0$, we have $L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.

(e) Let $s > 0$.

(e1) From (10.2.3 (1) (p.57)) and from $\lambda > 0$ and $\beta > 0$ we have $L(x) > 0$ for a sufficiently small $x < 0$ such that $x \leq a$. In addition, we have $L(b) = \lambda\beta T(b) - s = -s < 0$ due to Lemma 10.1.1(p.55) (g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L < b$ is immediate from $L(b) < 0$. The latter half is evident.

(e2) If $(\lambda\beta\mu - s)/\lambda\beta \leq (>)\, a$, from (10.2.3(p.57)) we have

$$L((\lambda\beta\mu - s)/\lambda\beta) = (>)\, \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s)/\lambda\beta = 0,$$

hence $x_L = (>)\, (\lambda\beta\mu - s)/\lambda\beta$ from (e1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

Corollary 10.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) $x_L > (\leq)\, x \Leftrightarrow L(x) > (\leq)\, 0$.
(b) $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. □

• *Proof* (a) "⇒" is immediate from Lemma 10.2.1(p.57) (d,e1). "⇐" is evident by contraposition.

(b) Since $x_L > (\leq)\, x \Rightarrow L(x) > (\leq)\, 0$ due to (a) and since $L(x) > (\leq)\, 0 \Rightarrow L(x) \geq (\leq)\, 0$, we have $x_L > (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. In addition, if $x_L = x$, then $L(x) = L(x_L) = 0$ or equivalently $x_L = x \Rightarrow L(x) = 0$, hence $x_L = x \Rightarrow L(x) \geq 0$. Accordingly, it follows that $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. ∎

Lemma 10.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(f) *$K(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(g) *$K(x)+x$ is strictly increasing on $[a,\infty)$.*
(h) *If $x<y$ and $a<y$, then $K(x)+x<K(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_K=b$ where $x_K>(\leq)\,x \Leftrightarrow K(x)>(=)\,0 \Rightarrow K(x)>(\leq)\,0$.*
(j) *Let $\beta<1$ or $s>0$.*

1. *There uniquely exists x_K where $x_K>(=(<))\,x \Leftrightarrow K(x)>(=(<))\,0$.*
2. *$(\lambda\beta\mu-s)/\delta\leq(>)\,a \Leftrightarrow x_K=(>)\,(\lambda\beta\mu-s)/\delta$.*
3. *Let $\kappa>(=(<))\,0$. Then $x_K>(=(<))\,0$.* □

• *Proof* (a-c) Immediate from (5.1.4(p.25)) and Lemma 10.1.1(p.55) (a-c).

(d) Immediate from (5.1.4(p.25)) and Lemma 10.1.1(p.55) (b).

(e) From (5.1.4(p.25)) we have

$$K(x)+x=\lambda\beta T(x)+\beta x-s=\lambda\beta(T(x)+x)+(1-\lambda)\beta x-s\cdots\textbf{(1)},$$

hence the assertion holds from Lemma 10.1.1(p.55) (d).

(f) Obvious from (1) and Lemma 10.1.1(p.55) (d).

(g) Clearly from (1) and Lemma 10.1.1(p.55) (e).

(h) Let $x<y$ and $a<y$. If $x\leq a$, then $K(x)+x\leq K(a)+a<K(y)+y$ due to (e,g), and if $a<x$, then $a<x<y$, hence $K(x)+x<K(y)+y$ due to (g). Thus, whether $x\leq a$ or $a<x$, we have $K(x)+x<K(y)+y$

(i) Let $\beta=1$ and $s=0$. Then, since $K(x)=\lambda T(x)$ due to (5.1.4(p.25)), from Lemma 10.1.1(p.55) (g) we have $K(x)>0$ for $x<b$ and $K(x)=0$ for $b\leq x$, hence $x_K=b$ by the definition of x_K (see Section 5.2(p.27) (a)). Thus $x_K>(\leq)\,x\Rightarrow K(x)>(=)\,0$. The inverse holds by contraposition. In addition, since $K(x)=0\Rightarrow K(x)\leq 0$, we have $K(x)>(=)\,0\Rightarrow K(x)>(\leq)\,0$.

(j) Let $\beta<1$ or $s>0$.

(j1) This proof consists of the following six steps:

- First note (10.2.5 (2) (p.57)). If $\beta<1$, then $K(x)<0$ for any sufficiently large $x>0$ with $x\geq b$ and if $s>0$, then, whether $\beta<1$ or $\beta=1$, we have $K(x)<0$ for any sufficiently large $x>0$ with $x\geq b$. Hence, whether $\beta<1$ or $s>0$, we have $K(x)<0$ for any sufficiently large $x>0$ with $x\geq b$.
- Next note (10.2.4 (1) (p.57)). Then, since $\delta>0$ from (10.2.2 (1) (p.56)), whether $\beta<1$ or $s>0$ we have $K(x)>0$ for any sufficiently small $x<0$ with $x\leq a$.
- Hence, whether $\beta<1$ or $s>0$, it follows that there exists the solution x_K.

◦ Let $\beta<1$. Then, the solution x_K is unique from (d).

◦ Let $s>0$. If $\beta<1$, the solution x_K is unique for the reason just above. If $\beta=1$, we have $K(b)=-s<0$ from (10.2.5 (2) (p.57)), hence $x_K<b$ due to (c), so $K(x)$ is strictly decreasing on the neighbourhood of $x=x_K$ due to (c), hence the solution x_K is unique. Therefore, whether $\beta<1$ or $\beta=1$, it follows that the solution x_K is unique.

◦ Accordingly, whether $\beta<1$ or $s>0$, it follows that the solution x_K is unique.

From all the above, whether $\beta<1$ or $s>0$, it follows that the solution x_K uniquely exists and hence that the latter half becomes true.

(j2) Let $(\lambda\beta\mu-s)/\delta\leq(>)\,a$. Then, from (10.2.4 (1(2)) (p.57)) we have

$$K((\lambda\beta\mu-s)/\delta)=(>)\,\lambda\beta\mu-s-\delta(\lambda\beta\mu-s)/\delta=0,$$

hence $x_K=(>)\,(\lambda\beta\mu-s)/\delta$ due to (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition.

(j3) If $\kappa>(=(<))\,0$, then $K(0)>(=(<))\,0$ from (5.1.7(p.25)), hence $x_K>(=(<))\,0$ from (j1). ∎

Corollary 10.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) $x_K>(\leq)\,x\Leftrightarrow K(x)>(\leq)\,0$.
(b) $x_K\geq(\leq)\,x\Rightarrow K(x)\geq(\leq)\,0$. □

• *Proof* (a) "⇒" is immediate from Lemma 10.2.2(p.57) (i,j1). "⇐" is evident by contraposition.

(b) Since $x_K>(\leq)\,x\Rightarrow K(x)>(\leq)\,0$ due to (a) and since $K(x)>(\leq)\,0\Rightarrow K(x)\geq(\leq)\,0$, we have $x_K>(\leq)\,x\Rightarrow K(x)\geq(\leq)\,0$. In addition, if $x_K=x$, then $K(x)=K(x_K)=0$ or equivalently $x_K=x\Rightarrow K(x)=0$, hence $x_K=x\Rightarrow K(x)\geq 0$. Accordingly, it follows that $x_K\geq(\leq)\,x\Rightarrow K(x)\geq(\leq)\,0$. ∎

Lemma 10.2.3 ($\mathscr{A}\{L_{\mathbb{R}}/K_{\mathbb{R}}\}$)

(a) *Let $\beta=1$ and $s=0$. Then $x_L=x_K=b$.*
(b) *Let $\beta=1$ and $s>0$. Then $x_L=x_K$.*
(c) *Let $\beta<1$ and $s=0$. Then $b>(=(<))\,0\Leftrightarrow x_L>(=(<))\,x_K\Rightarrow x_K>(=(=))\,0$.*
(d) *Let $\beta<1$ and $s>0$. Then $\kappa>(=(<))\,0\Leftrightarrow x_L>(=(<))\,x_K\Rightarrow x_K>(=(<))\,0$.* □

● *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma 10.2.1(p.57) (d) and $x_K = b$ from Lemma 10.2.2(p.57) (i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (10.2.8 (1) (p.57)), hence $x_K = x_L$ from Lemma 10.2.2(p.57) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots$ **(1)** from Lemma 10.2.1(p.57) (d).

- ∘ If $b > 0$, then $x_L > 0$, hence $K(x_L) < 0$ from (10.2.8 (1) (p.57)), so $x_L > x_K$ from Lemma 10.2.2(p.57) (j1). If $b = (<)\ 0$, then $x_L = (<)\ 0$, hence $K(x_L) = (>)\ 0$ from (10.2.8 (1) (p.57)), so $x_L = (<)\ x_K$ from Lemma 10.2.2(p.57) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.
- ∘ If $b > 0$, from (5.1.7(p.25)) we have $K(0) = \lambda\beta T(0) > 0$ due to Lemma 10.1.1(p.55) (g), hence $x_K > 0 \cdots$ **(2)** from Lemma 10.2.2(p.57) (j1). If $b = (<)\ 0$, from (5.1.7(p.25)) we have $K(0) = \lambda\beta T(0) = 0$ due to Lemma 10.1.1(p.55) (g), hence $x_K = (<)\ 0$ from Lemma 10.2.2(p.57) (j1). Accordingly, we have the *second relation* "⇒".

(d) Let $\beta < 1$ and $s > 0$. Now, since $\kappa = K(0)$ from (5.1.7(p.25)), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0 \cdots$ **(3)** from Lemma 10.2.2(p.57) (j1). Accordingly $L(x_K) > (= (<))\ 0$ from (10.2.8 (2) (p.57)), hence $x_L > (= (<))\ x_K$ from Lemma 10.2.1(p.57) (e1). Thus, "⇒" in the *first relation* "⇔" holds and its inverse "⇐" is immediate by contraposition. Finally, the *first relation* "⇒" is immediate from (3). ∎

Lemma 10.2.4 ($\mathcal{L}_{\mathbb{R}}$)

(a) *$\mathcal{L}(s)$ is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.*

(b) *Let $\lambda\beta\mu \geq b$.*

1. *$x_L \leq \lambda\beta\mu - s$.*
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta\mu - s$.*

(c) *Let $\lambda\beta\mu < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\leq)\ s$, then $x_L > (\leq)\ \lambda\beta\mu - s$.* □

● *Proof* (a) From (5.1.5(p.25)) and (5.1.3(p.25)) we have

$$\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta T(\lambda\beta\mu - s) - s \cdots \textbf{(1)},$$

hence the assertion holds from Lemma 10.1.1(p.55) (m).

(b) Let $\lambda\beta\mu \geq b$. Then, from (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) = 0 \cdots$ **(2)** due to Lemma 10.1.1(p.55) (g).

(b1) Since $s \geq 0$, from (a) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (2) or equivalently $L(\lambda\beta\mu - s) \leq 0$ due to (1), hence $x_L \leq \lambda\beta\mu - s$ from Corollary 10.2.1(p.57) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0 \cdots$ **(3)** due to (2) or equivalently $L(\lambda\beta\mu - s) < 0$ due to (1), hence $x_L < \lambda\beta\mu - s$ from Lemma 10.2.1(p.57) (e1).

(c) Let $\lambda\beta\mu < b$. From (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) > 0 \cdots$ **(4)** due to Lemma 10.1.1(p.55) (g). Note (10.2.3 (1) (p.57)). Then, for any sufficiently large $s > 0$ such that $\lambda\beta\mu - s \leq a$ and $\lambda\beta\mu - s < 0$ we have

$$\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s) = (1 - \lambda\beta)(\lambda\beta\mu - s) \leq 0.$$

Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}}$ of $\mathcal{L}(s) = 0$ where $s_{\mathcal{L}} > 0$ due to (4). Then, since $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta\mu - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta\mu - s) \leq 0$ for $s \geq s_{\mathcal{L}}$, from Corollary 10.2.1(p.57) (a) we get $x_L > \lambda\beta\mu - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta\mu - s$ for $s \geq s_{\mathcal{L}}$. ∎

10.3 $\kappa_{\mathbb{R}}$-value

Lemma 10.3.1 ($\mathscr{A}\{\kappa_{\mathbb{R}}\}$)

(a) *$\kappa = \lambda\beta\mu - s$ if $a > 0$ and $\kappa = -s$ if $b < 0$.*

(b) *Let $\beta < 1$ or $s > 0$, Then $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$.* □

● *Proof* (a) Immediate from (5.1.6(p.25)) and Lemma 10.1.1(p.55) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.7(p.25)), hence $x_K > (= (<))\ 0$ from Lemma 10.2.2(p.57) (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

Chapter 11

Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

11.1 Preliminary

From (6.2.8(p.30)) and (6.2.14(p.30)) we have

$$\begin{aligned} V_t - \beta V_{t-1} &= \max\{\mathbb{S}_t, 0\} \\ &= \max\{L(V_{t-1}), 0\}, \quad t > 1. \end{aligned} \tag{11.1.1}$$

Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence from (5.1.9(p.25)) we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{11.1.2}$$

2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$, hence

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{11.1.3}$$

Now, from (6.4.2(p.41)) with $t = 2$ we have

$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{11.1.4}$$

Finally, from (6.2.14(p.30)) and (6.2.12(p.30)) we have

$$\mathbb{S}_t = L(V_{t-1}) > (<)\, 0 \Rightarrow \ \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle}), \quad t > 1.. \tag{11.1.5}$$

11.2 Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Definition 11.2.1 (assertion and assertion system) By $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ let us represent an *assertion* included in each of `Tom`'s 11.2.1(p.61) and 11.2.2(p.62) below and by $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ the *assertion system* consisting of all assertions included in each `Tom`. □

Definition 11.2.2 (primitive `Tom` (▣) and derivative `Tom` (⧈)) Let us refer to a `Tom` the assertions included in which are *directly proven* as the *primitive* `Tom` (▣) and to a `Tom` the assertions included in which are *derived by transforming* assertions included in a primitive `Tom` (▣) as the *derivative* `Tom` (⧈). □

Below, note that $\lambda = 1$ is assume in the model (See Section 4.1.1.2.1(p.22) for the meaning of symbol ▣ which is used below).

□ **`Tom` 11.2.1** (▣ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ } \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.4(p.25)) we have $K(x) = T(x) \geq 0 \cdots (1)$ for any x due to Lemma 10.1.1(p.55) (g), hence from (6.4.2(p.41)) and (1) we have

$$V_t = \max\{T(V_{t-1}) + V_{t-1}, V_{t-1}\} = \max\{T(V_{t-1}), 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1} \cdots (2), \quad t > 1.$$

(a) Since $V_2 = T(V_1) + V_1$, we have $V_2 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.1.1(p.55) (d) we have $V_t \leq T(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Since $V_1 = \mu$ from (6.4.1(p.41)), we have $V_1 < b$. Suppose $V_{t-1} < b$. Then, from (2) we have $V_t < T(b) + b = b$ due to Lemma 10.1.1(p.55) (l,g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.57) (d); accordingly, $L(V_{t-1}) > 0 \cdots (3)$ for $\tau \geq t > 1$. Thus, from (11.1.1(p.61)) we obtain $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly, since $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$, we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\text{ⓢ } \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$, hence we have $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (11.1.5(p.61)). ∎

Let us define

$$\mathtt{S}_1\boxed{\circledS\blacktriangle \mid \circledcirc\,\|} = \left\{\begin{array}{ll} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1) \quad \boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}, \\ (2) \quad \boxed{\circledcirc\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\|} \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}. \end{array}\right\}$$

□ **Tom 11.2.2** ($\blacksquare\mathscr{A}\{\mathrm{M{:}1}[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.

(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
   ii. *Let $\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
   i. *Let $b > 0\,(\!(\kappa > 0)\!)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $b = 0\,(\!(\kappa = 0)\!)$.*
      1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
      2. *Let $\beta\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $b < 0\,(\!(\kappa < 0)\!)$.*
      1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
      2. *Let $\beta\mu - s > a$ and $s < s_{\mathcal{L}}$. Then* $\mathtt{S}_1$(p.62)$\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ *is true.* □

● *Proof* Let $\beta < 1$ or $s > 0$. In this model, note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.22) (b)) and that $\delta = 1 \cdots (1)$ (see (10.2.1(p.56))) due to the assumption $\lambda = 1 \cdots (2)$.

(a) Since $x_K \geq \beta\mu - s = V_1$ due to Lemma 10.2.2(p.57) (j2) and (6.4.1(p.41)), we have $K(V_1) \geq 0$ due to Lemma 10.2.2(p.57) (j1), hence $V_2 - V_1 \geq 0$ from (11.1.4(p.61)), i.e., $V_1 \leq V_2$. Suppose $V_{t-1} \leq V_t$. Then, from (6.4.2(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Consider a sufficiently large $M > 0$ with $\beta\mu - s \leq M$ and $b \leq M$, hence $V_1 \leq M$ from (6.4.1(p.41)). Suppose $V_{t-1} \leq M$. Then, from (6.4.2(p.41)), Lemma 10.2.2(p.57) (e), and (10.2.7 (2) (p.57)) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\} \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \leq M$ for $t > 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.4.2(p.41)) we have $V = \max\{K(V) + V, \beta V\}$, hence $0 = \max\{K(V), -(1-\beta)\beta V\}$. Thus, since $K(V) \leq 0$, we have $V \geq x_K$ from Lemma 10.2.2(p.57) (j1).

(b) Let $\beta\mu \geq b$. Then $x_L \leq \beta\mu - s = V_1$ from Lemma 10.2.4(p.59) (b1) with $\lambda = 1$, hence $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Hence, from (11.1.3(p.61)) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ (see Preference Rule 7.2.1(p.45)).

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots (3)$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then, from (3) , (1) , (2) we have $(\lambda\beta\mu - s)/\delta = \mu - s \cdots (4)$. In addition, since $x_L = x_K \cdots (5)$ from Lemma 10.2.3(p.58) (b), we have $K(x_L) = K(x_K) = 0 \cdots (6)$.

(c1i) Let $\mu - s \leq a$. Then $x_L = x_K = \mu - s = V_1$ from (5) , Lemma 10.2.2(p.57) (j2), (4) , and (6.4.1(p.41)). Accordingly, since $x_L \leq V_{t-1}$ for $t > 1$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 1$ due to Lemma 10.2.1(p.57) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.

(c1ii) Let $\mu - s > a$. Then $x_L = x_K > \mu - s = V_1 > a$ from (5) and Lemma 10.2.2(p.57) (j2), hence $a < V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} < x_L$, hence $L(V_{t-1}) > 0$ from Lemma 10.2.1(p.57) (e1). Then, from (11.1.2(p.61)), Lemma 10.2.2(p.57) (g), and (5) we have $V_t < K(x_L) + x_L = K(x_K) + x_L = x_L$. Accordingly, by induction $V_{t-1} < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a). Thus, for the same reason as in the proof of Tom 11.2.1(p.61) (b) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(c2i) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots (7)$ from Lemma 10.2.3(p.58) (c (d)). Now, since $x_K \geq \beta\mu - s$ due to Lemma 10.2.2(p.57) (j2), (1) , and (2) , we have $x_K \geq V_1$ from (6.4.1(p.41)). Suppose $x_K \geq V_{t-1}$. Then, from (6.4.2(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to (7) . Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (7) , thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a). Hence, for the same reason as in the proof of Tom 11.2.1(p.61) (b) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2ii) Let $b = 0\,(\!(\kappa = 0)\!)$. Then $x_L = x_K \cdots (8)$ from Lemma 10.2.3(p.58) (c (d)).

(c2ii1) Let $\beta\mu - s \leq a$. Then, $x_K = \beta\mu - s = V_1$ from Lemma 10.2.2(p.57) (j2). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (8) , so $L(V_{t-1}) = L(x_L) = 0$. Then, from (11.1.2(p.61)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (8) . Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (11.1.3(p.61)), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$ from (6.4.1(p.41)), we have $V_{t-1} > a$ for $t > 1$ due to (a). In addition, we have $x_K > \beta\mu - s = V_1$ from Lemma 10.2.2(p.57) (j2). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (8). Then, since $L(V_{t-1}) > 0$ due to Corollary 10.2.1(p.57) (a), from (11.1.2(p.61)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, so $x_L > V_{t-1}$ for $t > 1$ due to (8). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a), for the same reason as in the proof of (c1ii) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2iii) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots$ (9) from Lemma 10.2.3(p.58) (c (d)).

(c2iii1) Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. First let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s = V_1$ from Lemma 10.2.2(p.57) (j2), we have $x_L < V_1$ from (9), hence $x_L \leq V_1$. Next, let $s_{\mathcal{L}} \leq s$. Then, since $x_L \leq \beta\mu - s$ due to Lemma 10.2.4(p.59) (c), we have $x_L \leq V_1$. Accordingly, whether $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, thus $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta\mu - s > a \cdots$ (10) and $s < s_{\mathcal{L}}$. Then, from (9) and Lemma 10.2.4(p.59) (c) we have $x_K > x_L > \beta\mu - s = V_1 \cdots$ (11), hence $K(V_1) > 0 \cdots$ (12) from Lemma 10.2.2(p.57) (j1). In addition, since $V_1 > a$ due to (10), we have $V_{t-1} > a$ for $t > 0$ from (a). Now, from (11.1.4(p.61)) and (12) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.57) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (11). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \leq x_L$ due to (a). Then, from (9) and from $V \geq x_K$ due to (a) we have the contradiction of $V \geq x_K > x_L \geq V$. Hence, it is impossible that $V_{t-1} < x_L$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau+1} < V_{t^\bullet_\tau+2} < \cdots,$$

from which

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t-1}, \quad t > t^\bullet_\tau. \tag{11.2.1}$$

Therefore, from Corollary 10.2.1(p.57) (a) we have

$$L(V_{t-1}) > 0 \cdots (13), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \leq 0 \cdots (14), \quad t > t^\bullet_\tau.$$

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots$ (15) for $\tau \geq t > 1$ from (13), for the same reason as in the proof of (c1ii) we have $\boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau\geq\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. Hence $\mathtt{S}_1(1)$ is true.

2. Let $\tau > t^\bullet_\tau$. First, let $\tau \geq t > t^\bullet_\tau$. Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (14), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (11.1.3(p.61)), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (16).$$

Next let $t^\bullet_\tau \geq t > 1$. Then, from (13) and (11.1.1(p.61)) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet_\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \beta^2 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (17).$$

From (16) and (17) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1,$$

hence we obtain $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\|}$ due to Preference Rule 7.2.1(p.45). In addition, we have $\mathtt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (13) and (11.1.5(p.61)). Hence $\mathtt{S}_1(2)$ is true. ∎

Definition 11.2.3 (model-migration) If "$\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$" holds in M:1[ℝ][A], then the search is conducted over $\tau \geq t > 1$, implying that the model M:1[ℝ][A] is substantively reduced to the model in which the search is enforced over $\tau \geq t > 1$, i.e. , M:1[ℝ][E]. We refer to this event as " M:1[ℝ][A] *migrates* over to M:1[ℝ][E] ", represented as

$$\text{M:1}[\mathbb{R}][\mathtt{A}] \looparrowright \text{M:1}[\mathbb{R}][\mathtt{E}]. \quad \square$$

11.3 Structure of Assertion System $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

In this section we clarify the structure of the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Def. 11.2.1(p.61)). It will be known later on that its structure will play an essential role in the discussions in Step 6 (p.78).

11.3.1 Breakdown and Aggregation

Before proceeding with our discussions, let us define the following two perspectives (see Figure 11.3.1(p.64) below ($k = 3$)).

(I) The breakdown of a given set $\mathscr{X}$ into k mutually disjoint subsets $\mathscr{X}_1$, $\mathscr{X}_2$, $\cdots$, and $\mathscr{X}_k$ ($k > 0$), i.e.,

$$\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2 \cup \cdots \cup \mathscr{X}_k \text{ where } \mathscr{X}_i \cap \mathscr{X}_j = \emptyset \text{ for any } i \neq j.$$

This is called the *breakdown scenario*, represented as $\mathscr{X} \Rightarrow \{\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_k\}$.

(II) The aggregation of k mutually disjoint subsets $\mathscr{X}'_1$, $\mathscr{X}'_2$, $\cdots$, and $\mathscr{X}'_k$ ($k > 0$) of a given set $\mathscr{X}$, i.e.,

$$\mathscr{X}' \stackrel{\text{def}}{=} \mathscr{X}'_1 \cup \mathscr{X}'_2 \cup \cdots \cup \mathscr{X}'_k \subseteq \mathscr{X} \text{ where } \mathscr{X}'_i \cap \mathscr{X}'_j = \emptyset \text{ for any } i \neq j.$$

This is called the *aggregation scenario*, represented as $\{\mathscr{X}'_1, \mathscr{X}'_2, \cdots, \mathscr{X}'_k\} \Rightarrow \mathscr{X}'$. □

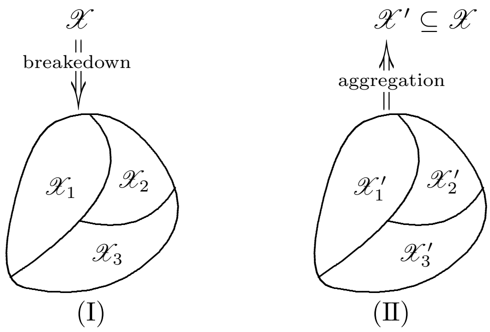

Figure 11.3.1: Breakdown and aggregation

11.3.2 Structure of Assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

11.3.2.1 Condition Space $\mathscr{C}\langle A\rangle$

In general, any given assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ consists of a *statement* S and a *condition expression* CE, schematized as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}=\{\mathsf{S} \text{ holds if } \mathsf{CE} \text{ is satisfied}\}. \tag{11.3.1}$$

□ *Example* **11.3.1** The assertion given by Tom 11.2.2(p.62) (b) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}=\{\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{||} \text{ holds if } \beta\mu \geq b \text{ is satisfied}\}$$

where $\mathsf{S} = \{\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{||}\}$ and $\mathsf{CE} = \{\beta\mu \geq b\}$. □

In general, for a given parameter space $\mathscr{P}_A \subseteq \mathscr{P}$ (see (4.3.1(p.23)) and (4.3.2(p.23))) and for a given distribution function space $\mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}$ (see (2.2.5(p.13))) related to a given $\boldsymbol{p} \in \mathscr{P}_A$, the condition expression CE is given as a conditional on a parameter vector $\boldsymbol{p}$ and a distribution function F where

$$\boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P},$$
$$F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}.$$

Then (11.3.1(p.64)) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds for } \boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{11.3.2}$$

□ *Example* **11.3.2** For the assertion A given by Tom 11.2.2(p.62) (c1i) we have

$$\mathscr{P}_A = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta = 1 \cap s > 0\},^\dagger$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{F \mid \beta\mu < b \cap \mu - s \leq a\}. \ \square$$

---

†When $\beta = 1$, we have $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$".

□ *Example* **11.3.3** For the assertion A given by Tom 11.2.2(p.62) (c2iii2) we have

$$\mathscr{P}_A = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta < 1 \cap\ s = 0 \,(\!(s > 0)\!)\,\},$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{F \mid \beta\mu < b \cap\ b < 0 \,(\!(\kappa < 0)\!)\, \cap \beta\mu - s > a \cap s <\ s_{\mathcal{L}}\ \}. \quad \square$$

Here let us define

$$\mathscr{C}\langle A\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P}, F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}\}, \quad (11.3.3)$$

called the *condition-space* of a given assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. Then, (11.3.2(p.64)) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds on } \mathscr{C}\langle A\rangle\ \}. \quad (11.3.4)$$

Throughout the rest of the paper, let us *alternatively* express the whole of (11.3.4(p.65)) as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A\rangle \quad (11.3.5)$$

for short

11.3.2.2 Structure of Tom

Definition 11.3.1

(a) We sometimes represent Tom 11.2.1(p.61) and Tom 11.2.2(p.62) by "Tom" for short, removing "11.2.1" and "11.2.2".

(b) For multiple Tom's we sometimes use terms $\mathtt{Tom}_1$, $\mathtt{Tom}_2$, $\cdots$. For example, $\mathtt{Tom}_1$ = Tom 11.2.1(p.61) and $\mathtt{Tom}_2$ = Tom 11.2.2(p.62).

(c) In order to stress that an assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is included in a given Tom, i.e., $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \in \mathtt{Tom}$, let us represent it as $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and an assertion system consisting of all $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ as $\mathscr{A}_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. □

Then (11.3.2(p.64)) - (11.3.5(p.65)) can be rewritten as respectively

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \quad (11.3.6)$$

$$\mathscr{C}\langle A_{\mathtt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P}, F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \quad (11.3.7)$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\ \}, \quad (11.3.8)$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle. \quad (11.3.9)$$

Closely looking into the structure of Tom's 11.2.1(p.61) and 11.2.2(p.62), in general we see that a given Tom consists of a *basic premise* $\mathsf{BP}_{\mathtt{Tom}}$ and some *assertions* $A^1_{\mathtt{Tom}}$, $A^2_{\mathtt{Tom}}$, $\cdots$, i.e.,

$$\mathtt{Tom} = \{\text{Let BP be true. Then assertions } A^1_{\mathtt{Tom}},\ A^2_{\mathtt{Tom}},\ \cdots \text{ hold.}\}$$

or equivalently

$$\mathtt{Tom} = \{\text{Assertions } A^1_{\mathtt{Tom}},\ A^2_{\mathtt{Tom}},\ \cdots \text{ hold if } \mathsf{BP}_{\mathtt{Tom}} \text{ be true.}\} \quad (11.3.10)$$

in which the basic premise BP is given as a conditional on a parameter vector $\boldsymbol{p}$ and a distribution function F where, for given subsets $\mathscr{P}_{\mathtt{Tom}} \subseteq \mathscr{P}$ and $\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}$,

$$\boldsymbol{p} \in \mathscr{P}_{\mathtt{Tom}} \subseteq \mathscr{P},$$
$$F \in \mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F} \quad (11.3.11)$$

Then the basic premise $\mathsf{BP}_{\mathtt{Tom}}$ can be written as

$$\mathsf{BP}_{\mathtt{Tom}} = \{\text{a condition on } \boldsymbol{p} \in \mathscr{P}_{\mathtt{Tom}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \quad (11.3.12)$$

□ *Example* **11.3.4** For $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Section 11.2(p.61) we have

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta = 1 \cap s = 0\}$ for Tom 11.2.1(p.61)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda = 1 \cap (\beta < 1 \cup s > 0)\}$ for Tom 11.2.2(p.62)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \mathscr{F}$ for Tom 11.2.1(p.61)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \mathscr{F}$ for Tom 11.2.2(p.62)

For $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ in Section 20.1.3(p.156) we have

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap \beta = 1 \cap s = 0 \cap -\infty < \rho < \infty\}$ for Tom 20.1.1(p.156)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\}$ for Tom 20.1.2(p.156)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\}$ for Tom 20.1.3(p.159)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\}$ for Tom 20.1.4(p.160)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F \mid -\infty < a < \mu < b < \infty\} = \mathscr{F}$ for Tom 20.1.1(p.156)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho < x_K\}$ for Tom 20.1.2(p.156)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho = x_K\}$ for Tom 20.1.3(p.159)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho > x_K\}$ for Tom 20.1.4(p.160) □

11.3.2.3 Condition Space $\mathscr{C}\langle\texttt{Tom}\rangle$

For a given Tom let us define

$$\mathscr{C}\langle\texttt{Tom}\rangle \stackrel{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{\texttt{Tom}} \subseteq \mathscr{P}, F \in \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{11.3.13}$$

called the *condition space* of Tom. Then (11.3.12(p.65)) can be rewritten as

$$\mathsf{BP}_{\texttt{Tom}} = \{\text{a condition on } \mathscr{C}\langle\texttt{Tom}\rangle\}, \tag{11.3.14}$$

hence (11.3.10(p.65)) can be rewritten as

$$\texttt{Tom} = \{\text{Assertions } A^1_{\texttt{Tom}},\ A^2_{\texttt{Tom}}.\ \cdots \text{ hold on } \mathsf{BP}_{\texttt{Tom}}\}, \tag{11.3.15}$$

alternatively as

$$\texttt{Tom} = \{\text{Assertions } A^1_{\texttt{Tom}},\ A^2_{\texttt{Tom}},\ \cdots \text{ hold on } \mathscr{C}\langle\texttt{Tom}\rangle\}. \tag{11.3.16}$$

For explanatory convenience, we will sometimes express "$A^j_{\texttt{Tom}}$ is included in Tom" as "$A^j_{\texttt{Tom}} \in \texttt{Tom}$" or sometimes as "$A_{\texttt{Tom}} \in \texttt{Tom}$" removing the superscript ^{j}.

11.3.3 Construction of Assertion System $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

breakdown scenario

↓

11.3.3.1 Completeness of Tom on $\mathscr{C}\langle\texttt{Tom}\rangle$

(11.3.16(p.66)) means that assertions $A^1_{\texttt{Tom}}$, $A^2_{\texttt{Tom}}$, $\cdots$ included in Tom are all over *all possible parameters* $(\boldsymbol{p}, F) \in \mathscr{C}\langle\texttt{Tom}\rangle$. In this paper we refer to this fact as the *completeness* of Tom on $\mathscr{C}\langle\texttt{Tom}\rangle$. Here note that this completeness is not *what should be proven* but a *necessary condition* to be satisfied, implying that Tom must be constructed so as for the completeness to be attained.

11.3.3.2 Breakdown of $\mathscr{C}\langle\texttt{Tom}\rangle$

The completeness of Tom is *what is given as a necessary condition* as stated just above. This requirement can be attained by the breakdown of the condition space $\mathscr{C}\langle\texttt{Tom}\rangle$ to the condition spaces $\mathscr{C}\langle A^1_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^2_{\texttt{Tom}}\rangle$, $\cdots$, i.e.,

$$\mathscr{C}\langle\texttt{Tom}\rangle = \cup_{j=1,2,\cdots}\mathscr{C}\langle A^j_{\texttt{Tom}}\rangle = \cup_{A_{\texttt{Tom}}\in\texttt{Tom}}\mathscr{C}\langle A_{\texttt{Tom}}\rangle, \tag{11.3.17}$$

depicted as in Figure 11.3.2(p.66) $(k = 3)$ below.

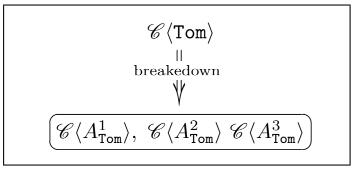

Figure 11.3.2: Breakedown of $\mathscr{C}\langle\texttt{Tom}\rangle$ to $\mathscr{C}\langle A^1_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^2_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^3_{\texttt{Tom}}\rangle$ $(k = 3)$

11.3.3.3 Construction of $\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Consider the list of (11.3.9(p.65)) over Tom, i.e., $A^1_{\texttt{Tom}}, A^2_{\texttt{Tom}}, \cdots \in \texttt{Tom}$, or equivalently

$$\text{“}A^1_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A^1_{\texttt{Tom}}\rangle\text{”},$$
$$\text{“}A^2_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A^2_{\texttt{Tom}}\rangle\text{”},$$
$$\vdots$$

Then, gathering the above list with noting (11.3.17(p.66)), we get

$$\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\texttt{Tom}\rangle \tag{11.3.18}$$

where

$$\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \stackrel{\text{def}}{=} \{A^1_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, A^2_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \cdots\}. \tag{11.3.19}$$

11.3.3.4 Condition Space $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$

For explanatory convenience, let us represent $\mathtt{Tom}$ 11.2.1(p.61) and $\mathtt{Tom}$ 11.2.2(p.62) by $\mathtt{Tom}_1$ and $\mathtt{Tom}_2$ respectively; in general, let $\mathtt{Tom}_1, \mathtt{Tom}_2, \cdots$. Then, let us define

$$\mathcal{T}\mathtt{om} \stackrel{\text{def}}{=} \{\mathtt{Tom}_1, \mathtt{Tom}_2, , \cdots\} = \{\mathtt{Tom}\}.$$

□ *Example* **11.3.5** For example we have

$$\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1 = \mathtt{Tom}\ 11.2.1(\text{p.61}), \mathtt{Tom}_2 = \mathtt{Tom}\ 11.2.2(\text{p.62})\},$$
$$\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1 = \mathtt{Tom}\ 20.1.1(\text{p.156}), \mathtt{Tom}_2 = \mathtt{Tom}\ 20.1.2(\text{p.156}), \mathtt{Tom}_3 = \mathtt{Tom}\ 20.1.3(\text{p.159}), \mathtt{Tom}_4 = \mathtt{Tom}\ 20.1.4(\text{p.160})\}. \quad ■$$

Here let us define

$$\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle \stackrel{\text{def}}{=} \cup_{i=1,2,\cdots}\mathscr{C}\langle \mathtt{Tom}_i\rangle = \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}}\mathscr{C}\langle \mathtt{Tom}\rangle, \tag{11.3.20}$$

called the *condition space* of $\mathcal{T}\mathtt{om}$, schematized as in Figure 11.3.3(p.67) below.

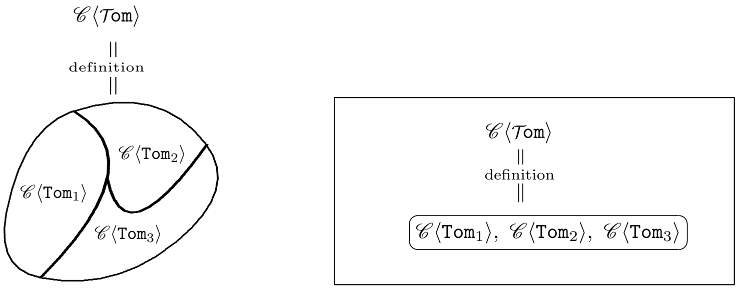

Figure 11.3.3: Condition space $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$

For convenience of discussions that follows, as one corresponding to (11.3.16(p.66)), let us define, for $i = 1, 2, \cdots$,

$$\mathtt{Tom}_i = \{\text{Assertions } A^1_{\mathtt{Tom}_i}, A^2_{\mathtt{Tom}_i}, \cdots \text{ hold on } \mathscr{C}\langle \mathtt{Tom}_i\rangle\}. \tag{11.3.21}$$

11.3.3.5 Construction of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Using (11.3.17(p.66)), we can express (11.3.20(p.67)) as below

$$\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle = \cup_{i=1,2,\cdots} \cup_{j=1,2,\cdots} \mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle \tag{11.3.22}$$
$$= \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \cup_{j=1,2,\cdots} \mathscr{C}\langle A^j_{\mathtt{Tom}}\rangle \tag{11.3.23}$$
$$= \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}} \mathscr{C}\langle A_{\mathtt{Tom}}\rangle \tag{11.3.24}$$

This relation implies the *breakdown* of $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $\mathscr{C}\langle A^j_{\mathtt{Tom}}\rangle$, and $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$.

□ *Example* **11.3.6** As an example let us consider $\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, \mathtt{Tom}_3\}$ where $\mathtt{Tom}_1 = \{A^1_{\mathtt{Tom}_1}, A^2_{\mathtt{Tom}_1}, A^3_{\mathtt{Tom}_1}\}$, $\mathtt{Tom}_2 = \{A^1_{\mathtt{Tom}_2}, A^2_{\mathtt{Tom}_2}, A^3_{\mathtt{Tom}_2}\}$, and $\mathtt{Tom}_3 = \{A^1_{\mathtt{Tom}_3}, A^2_{\mathtt{Tom}_3}, A^3_{\mathtt{Tom}_3}\}$. ■

Then, fetching Figure 11.3.2(p.66) in Figure 11.3.3(p.67), we see that (11.3.22(p.67)) can be depicted as Figure 11.3.4(p.67) below, demonstrating the *breakdown* of $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$.

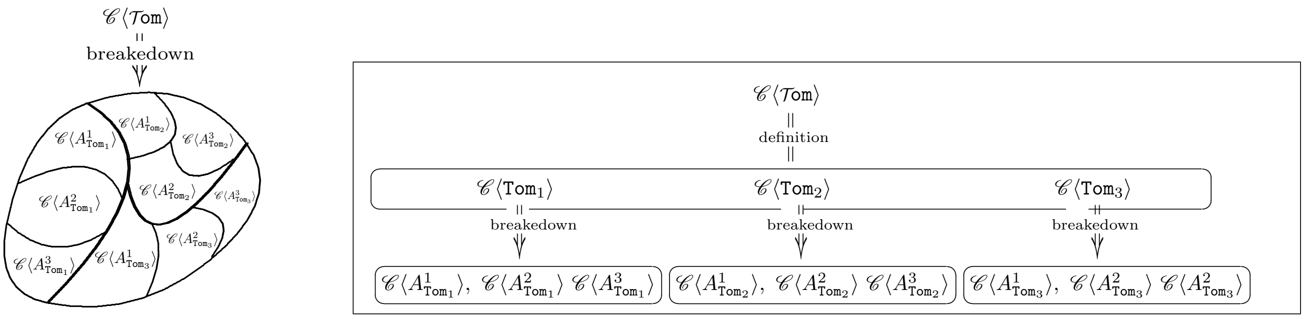

Figure 11.3.4: Breakdown of $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i, j = 1, 2, 3$

Figure 11.3.4(p.67) above implies that first

$$\text{“}\boxed{\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle} \text{ is broken down to } \boxed{\mathscr{C}\langle \mathtt{Tom}_i\rangle},\ i = 1, 2, 3\text{”},$$

and then

“ each $\boxed{\mathscr{C}\langle \mathtt{Tom}_i\rangle}$, $i = 1,2,3$ is broken down to $\boxed{\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle}$, $i,j = 1,2,3$. ”

The above two successive breakdown procedures eventually yields

“ $\boxed{\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle}$ is broken down to $\boxed{\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle}$ for $i,j = 1,2,3$”,

more generally

“ $\boxed{\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle}$ is broken down to $\boxed{\mathscr{C}\langle A_{\mathtt{Tom}}\rangle}$ with $\mathtt{Tom} \in \mathcal{T}\mathtt{om}$ ”

Here, consider the list of (11.3.18(p.66)) over $\mathtt{Tom}_1, \mathtt{Tom}_2, , \cdots \in \mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, \cdots\}$, i.e.,

$$\text{“}\mathscr{A}_{\mathtt{Tom}_1}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle \mathtt{Tom}_1\rangle \text{ ”.}$$

$$\text{“}\mathscr{A}_{\mathtt{Tom}_2}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle \mathtt{Tom}_2\rangle \text{ ”.}$$

$$\vdots$$

Then, gathering the above list with noting (11.3.22(p.67)), we obtain

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle \tag{11.3.25}$$

where

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \stackrel{\text{def}}{=} \{\mathscr{A}_{\mathtt{Tom}_1}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \mathscr{A}_{\mathtt{Tom}_2}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \cdots\}.$$

11.3.3.6 Completeness of $\mathcal{T}\mathtt{om}$ on $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$

Closely looking at the contents of Tom's 11.2.1(p.61) and 11.2.2(p.62), we see that the whole of assertions presented there is over all possible parameters $\boldsymbol{p}$ and distribution functions F; in other words, over the total-P/DF-space $\mathscr{P} \times \mathscr{F}$ (see (4.3.3(p.23))). This means that the condition space $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ is constructed so as to become equal to $\mathscr{P} \times \mathscr{F}$, i.e.,

$$\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}. \tag{11.3.26}$$

This implies that the whole of assertions $A^j_{\mathtt{Tom}_i}$, $i,j = 1,2,\cdots$ is all over $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$. Let us refer to this as the *completeness of* $\mathcal{T}\mathtt{om}$ on $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$.

Remark 11.3.1 (a priori requirement) What should be especially noted here is that this is not *what should be proven* but *what should be satisfied as a priori requirement.* □

The above perspective can be depicted as in Figure 11.3.4(p.67) as below.

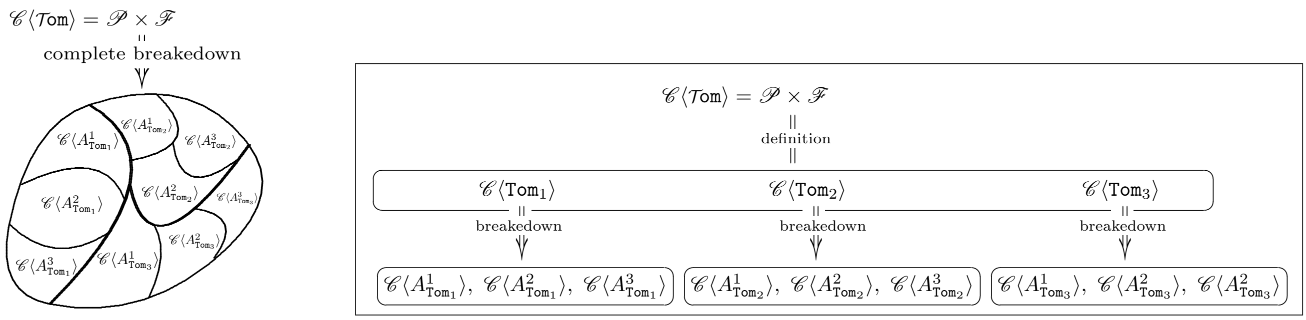

Figure 11.3.5: The completeness of $\mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ to $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i,j = 1,2,3$

↑

breakdown scenario

Chapter 12

Symmetry Theorem ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$)

12.1 Two Kinds of Equality

12.1.1 Correspondence Equality

For $\boldsymbol{\xi}$, a, μ, b, $T(x)$, $\cdots$, which are all dependent on a given distribution function $F \in \mathscr{F}$ (see (2.2.5(p.13))), let us define $\hat{\boldsymbol{\xi}} = -\boldsymbol{\xi}$, $\hat{a} = -a$, $\hat{\mu} = -\mu$, $\hat{b} = -b$, $\hat{T}(x) = -T(x)$, $\cdots$ respectively, called the *reverse operation* $\mathcal{R}$. Then, for any given distribution function $F \in \mathscr{F}$, i.e.,

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} \subseteq \mathscr{F}, \tag{12.1.1}$$

let us define the distribution function of $\hat{\boldsymbol{\xi}}$ by $\check{F}$, i.e.,

$$\check{F}(\xi) \stackrel{\text{def}}{=} \Pr\{\hat{\boldsymbol{\xi}} \le \xi\}, \tag{12.1.2}$$

where its probability density function is represented by $\check{f}$ and the set of all possible $\check{F}$ is denoted by $\check{\mathscr{F}}$, i.e.,

$$\check{\mathscr{F}} \stackrel{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}\}. \tag{12.1.3}$$

Now, since $\check{\check{F}}(\xi) = \Pr\{\hat{\hat{\boldsymbol{\xi}}} \le \xi\}$ for any ξ due to the definition (12.1.2(p.69)) and since

$$\hat{\hat{\boldsymbol{\xi}}} = \widehat{-\boldsymbol{\xi}} = -(-\boldsymbol{\xi}) = \boldsymbol{\xi}, \tag{12.1.4}$$

we have $\check{\check{F}}(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = F(\xi)$ for any ξ due to (12.1.1(p.69)), i.e.,

$$\check{\check{F}} \equiv F. \tag{12.1.5}$$

For any subset $\mathscr{F}' \subseteq \mathscr{F}$ let us define

$$\check{\mathscr{F}}' \stackrel{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}'\}. \tag{12.1.6}$$

Then we have

$$\check{\check{\mathscr{F}}}' = \{\check{\check{F}} \mid \check{F} \in \check{\mathscr{F}}'\} = \{F \mid \check{F} \in \check{\mathscr{F}}'\} \tag{12.1.7}$$

due to (12.1.5(p.69)). If $F \in \mathscr{F}'$, then $\check{F} \in \check{\mathscr{F}}'$ from (12.1.6(p.69)), hence $F \in \check{\check{\mathscr{F}}}'$ due to (12.1.7(p.69)); accordingly, we have $\mathscr{F}' \subseteq \check{\check{\mathscr{F}}}' \cdots (*)$. If $F \in \check{\check{\mathscr{F}}}'$, then $\check{F} \in \check{\mathscr{F}}'$ due to (12.1.7(p.69)), hence $F \in \mathscr{F}'$ from (12.1.6(p.69)); therefore, we have $\check{\check{\mathscr{F}}}' \subseteq \mathscr{F}'$. From this and $(*)$ it follows that

$$\check{\check{\mathscr{F}}}' = \mathscr{F}'. \tag{12.1.8}$$

By $\check{a}$, $\check{\mu}$, and $\check{b}$ let us denote the lower bound, expectation, and upper bound of $\check{F} \in \check{\mathscr{F}}$ corresponding to any given $F \in \mathscr{F}$ with the lower bound a, expectation μ, and upper bound b. Then, from Figure 12.1.1(p.70) just below we clearly have, for any ξ,

$$f(\xi) = \check{f}(\hat{\xi}), \tag{12.1.9}$$

called the *correspondence equality*, where

$$\hat{a} = \check{b}, \quad \hat{\mu} = \check{\mu}, \quad \hat{b} = \check{a}. \tag{12.1.10}$$

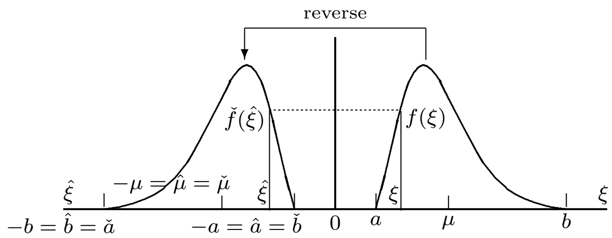

Figure 12.1.1: Relationship between probability density functions f and $\check{f}$

12.1.2 Identity Equality

Lemma 12.1.1

(a) *$\mathscr{F}$ and $\check{\mathscr{F}}$ are one-to-one correspondent where $\mathscr{F} = \check{\mathscr{F}}$.*
(b) *For any $\check{F} \in \check{\mathscr{F}}$ there exists a $F \in \mathscr{F}$ which is identical to the $\check{F}$, i.e., $F \equiv \check{F}$.*[†]
(c) *For any $F \in \mathscr{F}$ there exists a $\check{F} \in \check{\mathscr{F}}$ which is identical to the F, i.e., $\check{F} \equiv F$.*

● *Proof* If $F \in \mathscr{F}$, then $\check{F} \in \check{\mathscr{F}}$ from (12.1.3(p.69)), hence $F \in \mathscr{F} \Rightarrow \check{F} \in \check{\mathscr{F}} \cdots$ **(1)**. Conversely, if $\check{F} \in \check{\mathscr{F}}$, then F from which $\check{F} \in \check{\mathscr{F}}$ is defined is clearly an element of $\mathscr{F}$ due to (12.1.3(p.69)), i.e., $F \in \mathscr{F}$, hence $\check{F} \in \check{\mathscr{F}} \Rightarrow F \in \mathscr{F} \cdots$ **(2)**.

(a) First, for any $F \in \mathscr{F}$ and for the $\check{F} \in \check{\mathscr{F}}$ corresponding to the F we have

$$\begin{aligned}\check{F}(\xi) &= \Pr\{\hat{\boldsymbol{\xi}} \le \xi\} = \Pr\{-\hat{\hat{\boldsymbol{\xi}}} \le -\hat{\xi}\} = \Pr\{\hat{\hat{\boldsymbol{\xi}}} \ge \hat{\xi}\} = \Pr\{\boldsymbol{\xi} \ge \hat{\xi}\} \quad \text{(due to (12.1.4(p.69)))} \\ &= 1 - \Pr\{\boldsymbol{\xi} < \hat{\xi}\} = 1 - \Pr\{\boldsymbol{\xi} \le \hat{\xi}\}^{\ddagger} = 1 - F(\hat{\xi}) \cdots \mathbf{(3)}.\end{aligned}$$

Suppose any $F \in \mathscr{F}$ yields the two different $\check{F}_1 \in \check{\mathscr{F}}$ and $\check{F}_2 \in \check{\mathscr{F}}$, meaning that there exists at least one ξ' such that $\check{F}_1(\xi') \neq \check{F}_2(\xi')$. Then, since $\check{F}_1(\xi') = 1 - F(\hat{\xi}')$ and $\check{F}_2(\xi') = 1 - F(\hat{\xi}')$ due to (3), we have the contradiction of $\check{F}_1(\xi') = \check{F}_2(\xi')$, hence the $F \in \mathscr{F}$ must correspond to a *unique* $\check{F} \in \check{\mathscr{F}}$.

Next, for any $\check{F} \in \check{\mathscr{F}}$ and for $F \in \mathscr{F}$ from which $\check{F} \in \check{\mathscr{F}}$ is defined we have

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = \Pr\{-\hat{\boldsymbol{\xi}} \le -\hat{\xi}\} = \Pr\{\hat{\boldsymbol{\xi}} \ge \hat{\xi}\} = 1 - \Pr\{\hat{\boldsymbol{\xi}} < \hat{\xi}\} = 1 - \Pr\{\hat{\boldsymbol{\xi}} \le \hat{\xi}\}^{\ddagger} = 1 - \check{F}(\hat{\xi}) \cdots \mathbf{(4)}.$$

Suppose any $\check{F} \in \check{\mathscr{F}}$ is yielded from the two different $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}$, meaning that there exists at least one ξ' such that $F_1(\xi') \neq F_2(\xi')$. Then, since $F_1(\xi') = 1 - \check{F}(\hat{\xi}')$ and $F_2(\xi') = 1 - \check{F}(\hat{\xi}')$ due to (4), we have the contradiction of $F_1(\xi') = F_2(\xi')$, hence the $\check{F} \in \check{\mathscr{F}}$ must correspond to a unique $F \in \mathscr{F}$. Thus, the former half of the assertion is true.

The latter half can be proven as follows. First, consider any $F \in \check{\mathscr{F}}$. Then, since $F \in \mathscr{F}$ by definition, we have $\check{\mathscr{F}} \subseteq \mathscr{F} \cdots$ **(5)**. Next, consider any $F \in \mathscr{F}$. Then, since $\check{F} \in \check{\mathscr{F}}$ due to (1), we have $\check{F} \in \mathscr{F}$ due to (5). Hence $\check{\check{F}} \in \check{\mathscr{F}}$ due to (1(p.70)), so $F \in \check{\mathscr{F}}$ due to (12.1.5(p.69)), thus we have $\mathscr{F} \subseteq \check{\mathscr{F}}$. From this and (5) we have $\check{\mathscr{F}} = \mathscr{F} \cdots$ **(6)**.

(b) Consider any $\check{F} \in \check{\mathscr{F}}$, hence $\check{F} \in \mathscr{F} \cdots$ **(7)** due to (6). Suppose every $F \in \mathscr{F}$ is not identical to the $\check{F}$, i.e., $F \not\equiv \check{F}$, implying that the $\check{F}$ lies outside $\mathscr{F}$,[§] hence cannot become an element of $\mathscr{F}$, i.e., $\check{F} \notin \mathscr{F}$, which contradicts (7). Hence, it follows that there must exist at least one F such that $F \equiv \check{F}$, thus the assertion holds.

(c) Consider any $F \in \mathscr{F}$, hence $F \in \check{\mathscr{F}} \cdots$ **(8)** due to (6). Suppose every $\check{F} \in \check{\mathscr{F}}$ is not identical to the F, i.e., $\check{F} \not\equiv F$, implying that the F lies outside $\check{\mathscr{F}}$[‖], hence cannot become an element of $\check{\mathscr{F}}$, i.e., $F \notin \check{\mathscr{F}}$, which contradicts (8). Hence, it follows that there must exist at least one $\check{F}$ such that $\check{F} \equiv F$, thus the assertion holds. ∎

Lemma 12.1.1(p.70) (b,c) implies that there always exist F and $\check{F}$ such that $F \equiv \check{F}$ holds; in other words, there always exist f and $\check{f}$ such that $f \equiv \check{f}$ or equivalently

$$f(\xi) \equiv \check{f}(\xi), \tag{12.1.11}$$

called the *identity equality*.

[†] This means $F(x) = \check{F}(x)$ for all $x \in (-\infty, \infty)$.
[‡] Due to the assumption of F being continuous (see A9(p.13))
[§] Note that $\mathscr{F}$ is a set consisting of al possible F's by definition.
[‖] Note that $\check{\mathscr{F}}$ is a set consisting of al possible $\check{F}$'s by definition.

12.2 Definitions of Underlying Functions

The functions defined in the successive two sections are all the variations of ones that were defined in Sections 5.1.1(p.25) and 5.1.2(p.25) .

12.2.1 $\check{T}$, $\check{L}$, $\check{K}$, $\check{\mathcal{L}}$, and $\check{\kappa}$ of Type $\mathbb{R}$

Let us define the underlying functions of Type $\mathbb{R}$ (see Section 5.1.1(p.25)) *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ as follows.

$$\check{T}(x) \; = \; \check{\mathbf{E}}[\max\{\boldsymbol{\xi} - x, 0\}] = \textstyle\int_{-\infty}^{\infty} \max\{\xi - x, 0\}\check{f}(\xi)d\xi, \tag{12.2.1}$$

$$\check{L}(x) \; = \; \lambda\beta\check{T}(x) - s, \tag{12.2.2}$$

$$\check{K}(x) \; = \; \lambda\beta\check{T}(x) - (1-\beta)x - s, \tag{12.2.3}$$

$$\check{\mathcal{L}}(s) \; = \; \check{L}(\lambda\beta\check{\mu} - s). \tag{12.2.4}$$

Let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ respectively if they exist. If each of the equations has the multiple solutions, let us employ the *smallest* one (see (a) of Section 5.2(p.27)). Let us define

$$\check{\kappa} \; = \; \lambda\beta\check{T}(0) - s. \tag{12.2.5}$$

By $\check{\mathsf{M}}$:1[$\mathbb{R}$][A] let us define M:1[$\mathbb{R}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for M:1[$\mathbb{R}$][A] we can express SOE{$\check{\mathsf{M}}$:1[$\mathbb{R}$][A]} as (see Table 6.4.1(p.41) (I))

$$\texttt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} - s,\ V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}.$$

12.2.2 $\check{\tilde{T}}$, $\check{\tilde{L}}$, $\check{\tilde{K}}$, $\check{\tilde{\mathcal{L}}}$, and $\check{\tilde{\kappa}}$ of $\tilde{\text{T}}$ype $\mathbb{R}$

Let us define the underlying functions of $\tilde{\text{T}}$ype $\mathbb{R}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ as follows.

$$\check{\tilde{T}}(x) \; = \; \check{\mathbf{E}}[\min\{\boldsymbol{\xi} - x, 0\}] = \textstyle\int_{-\infty}^{\infty} \min\{\xi - x, 0\}\check{f}(\xi)d\xi, \tag{12.2.6}$$

$$\check{\tilde{L}}(x) \; = \; \lambda\beta\check{\tilde{T}}(x) + s, \tag{12.2.7}$$

$$\check{\tilde{K}}(x) \; = \; \lambda\beta\check{\tilde{T}}(x) - (1-\beta)x + s, \tag{12.2.8}$$

$$\check{\tilde{\mathcal{L}}}(s) \; = \; \check{\tilde{L}}(\lambda\beta\check{\mu} + s). \tag{12.2.9}$$

Let the solutions of $\check{\tilde{L}}(x) = 0$, $\check{\tilde{K}}(x) = 0$, and $\check{\tilde{\mathcal{L}}}(s) = 0$ be denoted by $x_{\check{\tilde{L}}}$, $x_{\check{\tilde{K}}}$, and $s_{\check{\tilde{\mathcal{L}}}}$ respectively if they exist. If each of the equations has the multiple solutions, let us employ the *largest* one (see (b) of Section 5.2(p.27)). Let us define

$$\check{\tilde{\kappa}} \; = \; \lambda\beta\check{\tilde{T}}(0) + s. \tag{12.2.10}$$

By $\check{\tilde{\mathsf{M}}}$:1[$\mathbb{R}$][A] let us define $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] we can express SOE{$\check{\tilde{\mathsf{M}}}$:1[$\mathbb{R}$][A]} as (see Table 6.4.1(p.41) (II))

$$\texttt{SOE}\{\check{\tilde{\mathsf{M}}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} + s,\ V_t = \min\{\check{\tilde{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}.$$

12.2.3 List of the Underline Functions of Type $\mathbb{R}$ and $\tilde{\text{T}}$ype $\mathbb{R}$

So far we have defined the four kinds of underlying functions, which may cause confusions. To give a clearer picture of these functions, we shall coordinate them as in Table 12.2.1(p.71) .

Table 12.2.1: List of the underlying functions of Type $\mathbb{R}$ and $\tilde{\text{T}}$ype $\mathbb{R}$

Type $\mathbb{R}$	$\tilde{\text{T}}$ype $\mathbb{R}$
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \int_a^b \max\{\xi - x, 0\}f(\xi)d\xi$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta\mu - s)$ See Section 5.1.1(p.25)	$\check{T}(x) = \int_a^b \max\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{\mu} - s)$ See Section 12.2.1(p.71)
$\tilde{T}(x) = \int_a^b \min\{\xi - x, 0\}f(\xi)d\xi$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta\mu + s)$ See Section 5.1.2(p.25)	$\check{\tilde{T}}(x) = \int_a^b \min\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\check{\tilde{L}}(x) = \beta\check{\tilde{T}}(x) + s$ $\check{\tilde{K}}(x) = \beta\check{\tilde{T}}(x) - (1-\beta)x + s$ $\check{\tilde{\mathcal{L}}}(x) = \check{\tilde{L}}(\beta\check{\mu} + s)$ See Section 12.2.2(p.71)

12.3 Two Kinds of Replacements

12.3.1 Correspondence Replacement

Lemma 12.3.1 ($\mathcal{C}_{\mathbb{R}}$) *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}, \quad \hat{\mu} = \check{\mu}, \quad \hat{b} = \check{a}$.
(c) $\hat{T}(x) = \check{\check{T}}(\hat{x})$.
(d) $\hat{L}(x) = \check{\check{L}}(\hat{x})$.
(e) $\hat{K}(x) = \check{\check{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \check{\check{\mathcal{L}}}(s)$.
(g) $\hat{x}_L = x_{\check{\check{L}}}$.
(h) $\hat{x}_K = x_{\check{\check{K}}}$.
(i) $s_{\mathcal{L}} = s_{\check{\check{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \check{\check{\kappa}}$. □

● *Proof* (a) The same as (12.1.9(p.70)).

(b) The same as (12.1.10(p.70)).

(c) The function $T(x)$ for any F (see (5.1.2(p.25))) can be rewritten as

$$\begin{aligned} T(x) &= \int_{-\infty}^{\infty} \max\{-\hat{\xi} + \hat{x}, 0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi} - \hat{x}, 0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi} - \hat{x}, 0\} \check{f}(\hat{\xi}) d\xi \quad \text{due to (a).} \end{aligned}$$

Let $\eta \overset{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then, we have

$$\begin{aligned} T(x) &= \int_{\infty}^{-\infty} \min\{\eta - \hat{x}, 0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\eta - \hat{x}, 0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\xi - \hat{x}, 0\} \check{f}(\xi) d\xi \quad \text{(without loss of generality}^{\dagger}) \\ &= -\check{\check{T}}(\hat{x}) \quad \text{(see (12.2.6(p.71))),} \end{aligned}$$

hence $\hat{T}(x) = \check{\check{T}}(\hat{x})$.

(d) From (5.1.3(p.25)) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\check{T}}(\hat{x}) - s = -\check{\check{L}}(\hat{x})$ from (12.2.7(p.71)), hence $\hat{L}(x) = \check{\check{L}}(\hat{x})$.

(e) From (5.1.4(p.25)) and (c) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\check{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\check{K}}(\hat{x})$ from (12.2.8(p.71)), hence $\hat{K}(x) = \check{\check{K}}(\hat{x})$.

(f) From (5.1.5(p.25)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta\mu - s)$, hence from (d) we obtain $\mathcal{L}(s) = -\check{\check{L}}(\widehat{\lambda\beta\mu - s}) = -\check{\check{L}}(-\lambda\beta\mu + s) = -\check{\check{L}}(\lambda\beta\hat{\mu} + s) = -\check{\check{L}}(\lambda\beta\check{\mu} + s)$ due to (b). Accordingly, from (12.2.9(p.71)) we obtain $\mathcal{L}(s) = -\check{\check{\mathcal{L}}}(s)$, hence $\hat{\mathcal{L}}(s) = \check{\check{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $\hat{L}(x_L) = 0$, which can be rewritten as $\check{\check{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\check{L}}(x) = 0$ has the solution $x_{\check{\check{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\hat{K}(x_K) = 0$, which can be rewritten as $\check{\check{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\check{K}}(x) = 0$ has the solution $x_{\check{\check{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, which can be rewritten as $\check{\check{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\check{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\check{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.6(p.25)) we have $\kappa = -\lambda\beta\hat{T}(0) - s$, which can be rewritten as $\kappa = -\lambda\beta\check{\check{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\check{T}}(0) - s = -\check{\check{\kappa}}$ from (12.2.10(p.71)), thus $\hat{\kappa} = \check{\check{\kappa}}$. ∎

Definition 12.3.1 (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.1(p.72) by its right-hand the *correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$. □

Lemma 12.3.2 ($\tilde{\mathcal{C}}_{\mathbb{R}}$) *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{b} = \check{a}, \quad \hat{\mu} = \check{\mu}, \quad \hat{a} = \check{b}$.
(c) $\hat{\hat{T}}(x) = \check{T}(\hat{x})$.

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† The mere replacement of the symbol η by ξ.

(d) $\hat{\check{L}}(x) = \check{L}(\hat{x})$.
(e) $\hat{\check{K}}(x) = \check{K}(\hat{x})$.
(f) $\hat{\check{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.
(g) $\hat{x}_{\tilde{L}} = x_{\check{L}}$.
(h) $\hat{x}_{\tilde{K}} = x_{\check{K}}$.
(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.
(j) $\hat{\check{\kappa}} = \check{\kappa}$. □

● *Proof* (a) The same as (12.1.9(p.70)).

(b) The same as (12.1.10(p.70)).

(c) The function $\tilde{T}(x)$ for any F (see (5.1.12(p.25))) can be rewritten as

$$\begin{aligned}\tilde{T}(x) &= \int_{-\infty}^{\infty} \min\{-\hat{\xi}+\hat{x},0\}f(\xi)d\xi\\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\}f(\xi)d\xi\\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\}\check{f}(\hat{\xi})d\xi \quad \text{(due to (a(p.72))).}\end{aligned}$$

Let $\eta = \hat{\xi} = -\xi$. Then, since $d\eta = -d\xi$, we have

$$\begin{aligned}\tilde{T}(x) &= \int_{\infty}^{-\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta\\ &= -\int_{-\infty}^{\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta\\ &= -\int_{-\infty}^{\infty} \max\{\xi-\hat{x},0\}\check{f}(\xi)d\xi \quad \text{(without loss of generality}^{\dagger}\text{)}\\ &= -\check{T}(\hat{x}) \quad \text{(see (12.2.1(p.71))),}\end{aligned}$$

hence $\hat{\check{T}}(x) = \check{T}(\hat{x})$.

(d) From (5.1.13(p.25)) and (c) we have $\tilde{L}(x) = -\lambda\beta\hat{\check{T}}(x) + s = -\lambda\beta\check{T}(\hat{x}) + s = -\check{L}(\hat{x})$ from (12.2.2(p.71)), hence $\hat{\check{L}}(x) = \check{L}(\hat{x})$.

(e) From (5.1.14(p.25)) and (c) we have $\tilde{K}(x) = -\lambda\beta\hat{\check{T}}(x) + (1-\beta)\hat{x} + s = -\lambda\beta\check{T}(\hat{x}) + (1-\beta)\hat{x} + s = -\check{K}(\hat{x})$ from (12.2.3(p.71)), hence $\hat{\check{K}}(x) = \check{K}(\hat{x})$.

(f) From (5.1.15(p.25)) and (d) we have $\tilde{\mathcal{L}}(s) = -\hat{\check{L}}(\lambda\beta\mu+s) = -\check{L}(\widehat{\lambda\beta\mu+s}) = -\check{L}(-\lambda\beta\mu-s) = -\check{L}(\lambda\beta\hat{\mu}-s) = -\check{L}(\lambda\beta\check{\mu}-s)$ due to (b), hence from (12.2.4(p.71)) we obtain $\tilde{\mathcal{L}}(s) = -\check{\mathcal{L}}(s)$, hence $\hat{\check{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $\hat{\check{L}}(x_{\tilde{L}}) = 0$, which can be rewritten as $\check{L}(\hat{x}_{\tilde{L}}) = 0$ from (d), implying that $\check{L}(x) = 0$ has the solution $x_{\check{L}} = \hat{x}_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $\hat{\check{K}}(x_{\tilde{K}}) = 0$, which can be rewritten as $\check{K}(\hat{x}_{\tilde{K}}) = 0$ from (e), implying that $\check{K}(x) = 0$ has the solution $x_{\check{K}} = \hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $\hat{\check{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, which can be rewritten as $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), implying that $\check{\mathcal{L}}(s) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.16(p.25)) we have $\tilde{\kappa} = -\lambda\beta\hat{\check{T}}(0) + s$, which can be rewritten as $\tilde{\kappa} = -\lambda\beta\check{T}(\hat{0}) + s$ from (c), hence $\tilde{\kappa} = -\lambda\beta\check{T}(0) + s = -\check{\kappa}$ from (12.2.5(p.71)), thus $\hat{\check{\kappa}} = \check{\kappa}$. ∎

Definition 12.3.2 (correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.2(p.72) by its right-hand the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{R}}$. □

Definition 12.3.3 (reversible element and non-reversible element) It should be noted that the left-hand of each of the equalities in Lemmas 12.3.1(p.72) (i) and 12.3.2(p.72) (i) have not the hat symbol " ˆ ". In other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reverse. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reversible element* and to each of all the other elements as the *reversible element*. □

12.3.2 Identity Replacement

Lemma 12.3.3 ($\mathcal{I}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right-hand side is for $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots$ [1*].*[†]

(a) $\check{F}(\xi) = F(\xi) \cdots$ [2*] *and* $\check{f}(\xi) = f(\xi) \cdots$ [3*] *for any* ξ.
(b) $\check{a} = a$, $\check{\mu} = \mu$, $\check{b} = b$.
(c) $\check{\tilde{T}}(x) = \tilde{T}(x)$.
(d) $\check{\tilde{L}}(x) = \tilde{L}(x)$.
(e) $\check{\tilde{K}}(x) = \tilde{K}(x)$.

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† The mere replacement of the symbol η by ξ.
† See Lemma 12.1.1(p.70) (b,c).

(f) $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$.
(g) $x_{\check{\tilde{L}}} = x_{\tilde{L}}$.
(h) $x_{\check{\tilde{K}}} = x_{\tilde{K}}$.
(i) $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$.
(j) $\check{\tilde{\kappa}} = \tilde{\kappa}$. □

• *Proof* (a) Clear from [1*].

(b) Obvious from (a).

(c) Evident from (12.2.6(p.71)), (5.1.12(p.25)), and [3*].

(d) From (12.2.7(p.71)) and (c) we have $\check{\tilde{L}}(x) = \lambda\beta\tilde{T}(x) + s$, hence $\check{\tilde{L}}(x) = \tilde{L}(x)$ from (5.1.13(p.25)).

(e) From (12.2.8(p.71)) and (c) we have $\check{\tilde{K}}(x) = \lambda\beta\tilde{T}(x) - (1-\beta)x + s$, hence $\check{\tilde{K}}(x) = \tilde{K}(x)$ from (5.1.14(p.25)).

(f) From (12.2.9(p.71)) and (d) we have $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta\check{\mu} + s)$, hence $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta\mu + s)$ from (b), so $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$ (5.1.15(p.25)).

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $\check{\tilde{L}}(x_{\tilde{L}}) = 0$ from (d), hence $\check{\tilde{L}}(x) = 0$ has the solution $x_{\check{\tilde{L}}} = x_{\tilde{L}}$.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $\check{\tilde{K}}(x_{\tilde{K}}) = 0$ from (e), hence $\check{\tilde{K}}(x) = 0$ has the solution $x_{\check{\tilde{K}}} = x_{\tilde{K}}$.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $\check{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), hence $\check{\tilde{\mathcal{L}}}(x) = 0$ has the solution $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (12.2.10(p.71)) and (c) with $x = 0$ we have (5.1.16(p.25)). ∎

Definition 12.3.4 (identity replacement operation $\mathcal{I}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand side of each equality in Lemma 12.3.3(p.73) by its right-hand side the *identity replacement operation* $\mathcal{I}_{\mathbb{R}}$. □

Lemma 12.3.4 ($\tilde{\mathcal{I}}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right-hand side is for $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots [1^*]$.*[†]

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ.
(b) $\check{a} = a,\ \check{\mu} = \mu,\ \check{b} = b$.
(c) $\check{T}(x) = T(x)$.
(d) $\check{L}(x) = L(x)$.
(e) $\check{K}(x) = K(x)$.
(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$.
(g) $x_{\check{L}} = x_L$.
(h) $x_{\check{K}} = x_K$.
(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$.
(j) $\check{\kappa} = \kappa$. □

• *Proof* (a) Clear from [1*].

(b) Obvious from (a).

(c) Evident from (12.2.1(p.71)), (5.1.2(p.25)), and [3*].

(d) From (12.2.2(p.71)) and (c) we have $\check{L}(x) = \lambda\beta T(x) - s$, hence $\check{L}(x) = L(x)$ from (5.1.3(p.25)).

(e) From (12.2.3(p.71)) and (c) we have $\check{K}(x) = \lambda\beta T(x) - (1-\beta)x - s$, hence $\check{K}(x) = K(x)$ from (5.1.4(p.25)).

(f) From (12.2.4(p.71)) and (d) we have $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\mu - s)$, hence $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\mu + s)$ from (b), so $\mathcal{L}(s) = \check{L}(\lambda\beta\mu + s)$, hence $\check{\mathcal{L}}(s) = \mathcal{L}(s)$ from (5.1.5(p.25)).

(g) Since $L(x_L) = 0$ by definition, we have $\check{L}(x_L) = 0$ from (d), hence $\check{L}(x) = 0$ has the solution $x_{\check{L}} = x_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\check{K}(x_K) = 0$ from (e), hence $\check{K}(x) = 0$ has the solution $x_{\check{K}} = x_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\check{\mathcal{L}}(s_{\mathcal{L}}) = 0$ from (f), hence $\check{\mathcal{L}}(x) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$ by definition.

(j) From (12.2.5(p.71)) and (c) with $x = 0$ we have (5.1.6(p.25)). ∎

Definition 12.3.5 (identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.4(p.74) by its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{R}}$. □

12.4 Attribute Vector

Closely looking into the contents of all assertions $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \in \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see `Tom`'s 11.2.1(p.61) and 11.2.2(p.62)), we can immediately see that each assertion is described by using a part or all of the following twelve kinds of elements;

$$a,\ \mu,\ b,\ x_L\ ,\ x_K\ ,\ s_{\mathcal{L}}\ ,\ \kappa,\ T,\ L\ ,\ K\ ,\ \mathcal{L}\ ,\ V_t$$

where V_t represents the sequence $\{V_t,\ t = 1, 2, \cdots\}$ generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.4.1(p.41) (I)). Let us call each element the *attribute element* and the vector of them the *attribute vector*, denoted by

[†] See Lemma 12.1.1(p.70) (b,c).

$$\boldsymbol{\theta}(A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L\ ,\ x_K\ ,\ s_{\mathcal{L}}\ , \kappa, T, L\ , K\ , \mathcal{L}\ , V_t). \tag{12.4.1}$$

In addition, also for the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ we can employ the similar definition, denoted by

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L\ ,\ x_K\ ,\ s_{\mathcal{L}}\ , \kappa, T, L\ , K\ , \mathcal{L}\ , V_t). \tag{12.4.2}$$

12.5 Scenario[ℝ]

In this section we write up a scenario deriving an assertion on $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (buying model with ℝ-mechanism) from a given assertion on $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ (selling model with ℝ-mechanism). Let us refer to this as the scenario of Type ℝ, denoted by Scenario[ℝ].

■ Step 1 (*opening*)

∘ The system of optimality equations for $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ is given by Table 6.4.1(p.41) (I), i.e.,

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu - s,\ V_t = \max\{K(V_{t-1}) + V_{t-1},\ \beta V_{t-1}\},\quad t > 1\}. \tag{12.5.1}$$

∘ Let us consider an assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{\dagger}$ included in **Tom** 11.2.1(p.61) or **Tom** 11.2.2(p.62), which can be written in general as

$$\begin{aligned} A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} &= \{\mathsf{S} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\} \quad (\text{see } (11.3.6(\text{p.65}))) && (12.5.2)\\ &= \{\mathsf{S} \text{ is true on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\} \quad (\text{see } (11.3.8(\text{p.65}))). && (12.5.3) \end{aligned}$$

To facilitate the understanding of the discussion that follows, let us use the following example.[‡]

$$\mathsf{S}\ =\ \langle V_t +\ s_{\mathcal{L}}\ +\ x_L\ + \kappa + a + \mu + b \geq 0,\ \ t > 0\rangle. \tag{12.5.4}$$

∘ The attribute vector of the assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (12.4.1(p.75)), i.e.,

$$\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})\ =\ (a, \mu, b,\ x_L\ ,\ x_K\ ,\ s_{\mathcal{L}}\ , \kappa, T, L\ , K\ , \mathcal{L}\ , V_t). \tag{12.5.5}$$

■ Step 2 (*reverse operation* $\mathcal{R}$)

∘ Applying the reverse operation $\mathcal{R}$ (see Section 12.1.1(p.69)) to (12.5.1(p.75)) produces

$$\begin{aligned} \mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\ t > 1\}\\ &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}\\ &= \{\hat{V}_1 = \beta\hat{\mu} + s,\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}. && (12.5.6) \end{aligned}$$

∘ Applying $\mathcal{R}$ to (12.5.2(p.75)) and (12.5.3(p.75)) yields to

$$\begin{aligned} \mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]\ &=\ \{\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\} && (12.5.7)\\ &=\ \{\mathcal{R}[\mathsf{S}] \text{ is true on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\}. && (12.5.8) \end{aligned}$$

For our example we have:

$$\begin{aligned} \mathcal{R}[\mathsf{S}]\ &=\ \langle -\hat{V}_t +\ s_{\mathcal{L}}\ - \hat{x}_L - \hat{\kappa} - \hat{a} - \hat{\mu} - \hat{b} \geq 0,\ \ t > 0\rangle^{\S}\\ &=\ \langle\ \hat{V}_t -\ s_{\mathcal{L}}\ +\ \hat{x}_L\ + \hat{\kappa} + \hat{a} + \hat{\mu} + \hat{b} \leq 0,\ \ t > 0\rangle. && (12.5.9) \end{aligned}$$

∘ The attribute vector of the assertion $\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (12.5.5(p.75)), i.e.,

$$\begin{aligned} \boldsymbol{\theta}(\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}])\ &\overset{\text{def}}{=}\ \mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] && (12.5.10)\\ &=\ (\hat{a}, \hat{\mu}, \hat{b}, \hat{x}_L\ , \hat{x}_K\ ,\ s_{\mathcal{L}}\ , \hat{\kappa}, \hat{T}, \hat{L}\ , \hat{K}\ , \hat{\mathcal{L}}\ , \hat{V}_t). && (12.5.11) \end{aligned}$$

■ Step 3 (*correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$)

∘ Here let us consider the application of the correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.1(p.72),

$$f(\xi), \hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L\ ,\ \hat{x}_K\ ,\ \ s_{\mathcal{L}}\ ,\ \hat{\kappa},\ \hat{T}(x),\ \hat{L}\,(x),\ \hat{K}\,(x),\ \hat{\mathcal{L}}\,(s) \cdots (1^*),$$

by its right-hand,

$$\check{f}(\hat{\xi}), \check{b},\ \check{\mu},\ \check{a},\ \ x_{\check{\tilde{L}}},\ \ x_{\check{\tilde{K}}},\ \ s_{\check{\tilde{\mathcal{L}}}}\ ,\ \check{\tilde{\kappa}},\ \check{\tilde{T}}\,(\hat{x}),\ \check{\tilde{L}}\,(\hat{x}),\ \check{\tilde{K}}\,(\hat{x}),\ \check{\tilde{\mathcal{L}}}\,(s) \cdots (2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ corresponding to the $F \in \mathscr{F}$.

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† See Def. 11.3.1(p.65) (c) for the symbol "Tom" in $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.

‡ The example is a hypothetical assertion which is not contained in $\mathscr{A}_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$; It is used merely for explanatory convenience.

§ Note Def. 12.3.3(p.73).

- Applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.6(p.75)) leads to

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\check{\mu} + s,\ \hat{V}_t = \min\{\check{\check{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}. \tag{12.5.12}$$

- Applying $\mathcal{C}_{\mathbb{R}}$ to $\mathcal{R}[\mathsf{S}]$ in (12.5.9(p.75)), we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle\, \hat{V}_t - \ s_{\check{\check{\mathcal{L}}}} + \ x_{\check{\check{L}}} + \check{\check{\kappa}} + \check{b} + \check{\mu} + \check{a} \le 0,\ \ t > 0\,\rangle. \tag{12.5.13}$$

Now, let us note here that the application of $\mathcal{C}_{\mathbb{R}}$ inevitably transforms

"$F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$" in (12.5.2(p.75))

into

$$\text{“}\check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}} \textit{ corresponding to } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\text{”} \tag{12.5.14}$$

where

$$\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\} \subseteq \{\check{F} \mid F \in \mathscr{F}\} = \check{\mathscr{F}} \quad \text{(see (12.1.3(p.69)))}. \tag{12.5.15}$$

Hence, applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.7(p.75)) produces

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}} \text{ corresponding to } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{12.5.16}$$

Now, since the phrase "$\check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}$" is *implicitly* accompanied with the phrase "*corresponding to* $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$", the latter phrase becomes redundant. Accordingly, (12.5.16(p.76)) can be rewritten as

$$\begin{aligned}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \\ &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\}\end{aligned} \tag{12.5.17}$$

where

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle = \{(\boldsymbol{p}, \check{F}) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \quad \text{(compare (11.3.3(p.65)))}. \tag{12.5.18}$$

- The attribute vector of $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.10(p.75)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]) &= \mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &= (\check{b}, \check{\mu}, \check{a},\ x_{\check{\check{L}}},\ x_{\check{\check{K}}},\ s_{\check{\check{\mathcal{L}}}}, \check{\check{\kappa}}, \check{\check{T}}, \check{\check{L}}, \check{\check{K}}, \check{\check{\mathcal{L}}}, V_t).\end{aligned} \tag{12.5.19}$$

■ Step 4 (*identity replacement operation* $\mathcal{I}_{\mathbb{R}}$)

- Here let us consider the application of the identity replacement operation $\mathcal{I}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.3(p.73),

$$\check{f}(\xi),\ \check{a},\ \check{\mu},\ \check{b},\ x_{\check{\check{L}}},\ x_{\check{\check{K}}},\ s_{\check{\check{\mathcal{L}}}},\ \check{\check{\kappa}},\ \check{\check{T}}(x),\ \check{\check{L}}(x),\ \check{\check{K}}(x),\ \check{\check{\mathcal{L}}}(s) \cdots (1^*),$$

by its right-hand side,

$$f(\xi),\ a,\ \mu,\ b,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa}\ \tilde{T}(x),\ \tilde{L}(x),\ \tilde{K}(x),\ \tilde{\mathcal{L}}(s) \cdots (2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ which is identical to the $\check{F} \in \mathscr{F}$, i.e., $\check{F} \equiv F \cdots (1)$ (see Lemma 12.1.1(p.70) (c)).

- Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.12(p.76)) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\mu + s,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}. \tag{12.5.20}$$

Now, we have $\hat{V}_1 = \beta\mu + s = V_1$ from (6.4.3(p.41)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, since $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$ from (6.4.4(p.41)), by induction $\hat{V}_t = V_t$ for $t > 0$. Thus (12.5.20(p.76)) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\},$$

which is the same as $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.4.1(p.41) (II)). Thus we have

$$\begin{aligned}\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} &= \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\ \}] && (12.5.21)\\ &= \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1\}. && (12.5.22)\end{aligned}$$

∘ Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.17(p.76)) yields (note $\check{F} \equiv F$ in (1))

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \;=\; \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. \tag{12.5.23}$$

Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.13(p.76)) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle V_t - \ s_{\tilde{\mathcal{L}}}\ +\ x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ \ t > 0\rangle. \tag{12.5.24}$$

Now V_t within $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}]$ is generated from $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (12.5.23(p.77)) can be regarded as the assertion on $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (see Remark 6.1.1(p.29)). Thus, we have

$$\begin{aligned} A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;&=\; \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] && (12.5.25)\\ &=\; \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\}. && (12.5.26)\end{aligned}$$

∘ The attribute vector of $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.19(p.76)), i.e.,

$$\begin{aligned} \boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})]\\ &=\; (b, \mu, a,\ x_{\tilde{L}}, x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t), \end{aligned} \tag{12.5.27}$$

■ Step 5 (*symmetry transformation operation* $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$)

Lining up the four attribute vectors in Steps 1-4, we have the following:

$$\begin{array}{lllr} \text{Step 1:} & \boldsymbol{\theta}(a, \mu, b, & x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t) & (\leftarrow (12.5.5(p.75)))\\ & \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{R}\\ \text{Step 2:} & \boldsymbol{\theta}(\hat{a}, \hat{\mu}, \hat{b}, & \hat{x}_L, \hat{x}_K, s_{\mathcal{L}}, \hat{\kappa}, \hat{T}, \hat{L}, \hat{K}, \hat{\mathcal{L}}, \hat{V}_t) & (\leftarrow (12.5.11(p.75)))\\ & \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{C}_{\mathbb{R}}\\ \text{Step 3:} & \boldsymbol{\theta}(\check{b}, \check{\mu}, \check{a}, & x_{\check{\check{L}}}, x_{\check{\check{K}}}, s_{\check{\check{\mathcal{L}}}}, \check{\check{\kappa}}, \check{\check{T}}, \check{\check{L}}, \check{\check{K}}, \check{\check{\mathcal{L}}}, V_t) & (\leftarrow (12.5.19(p.76)))\\ & \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{I}_{\mathbb{R}}\\ \text{Step 4:} & \boldsymbol{\theta}(b, \mu, a, & x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) & (\leftarrow (12.5.27(p.77))) \end{array} \tag{12.5.28}$$

The above flow can be eventually reduced to

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} a, \mu, b, \ x_{L_{\mathbb{R}}}, x_{K_{\mathbb{R}}}, s_{\mathcal{L}_{\mathbb{R}}}, \kappa_{\mathbb{R}}, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t\\ \wr\ \ \wr\ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\\ b, \mu, a, \ x_{\tilde{L}_{\mathbb{R}}}, x_{\tilde{K}_{\mathbb{R}}}, s_{\tilde{\mathcal{L}}_{\mathbb{R}}}, \tilde{\kappa}_{\mathbb{R}}, \tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, V_t \end{array} \right\} \tag{12.5.29}$$

called the *symmetry transformation operation*, which can be regarded as the successive application of the three operations, i.e., " $\mathcal{R} \to \mathcal{C}_{\mathbb{R}} \to \mathcal{I}_{\mathbb{R}}$ ". Hence, defining

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \;=\; \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}, \tag{12.5.30}$$

we can rewrite (12.5.25(p.77)) as

$$\begin{aligned} A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;&=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]\\ &=\; \{\tilde{\mathsf{S}} \text{ holds on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \} \end{aligned} \tag{12.5.31}$$

where

$$\tilde{\mathsf{S}} \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathsf{S}]. \tag{12.5.32}$$

Then, from (12.5.24(p.77)) we have

$$\tilde{\mathsf{S}} = \langle V_t - \ s_{\tilde{\mathcal{L}}}\ +\ x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ \ t > 0\rangle. \tag{12.5.33}$$

Furthermore, (12.5.21(p.76)) can be rewritten as

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\ \}]. \tag{12.5.34}$$

In addition, (12.5.27(p.77)) can be rewritten as

$$\begin{aligned} \boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] && (12.5.35)\\ &=\; (b, \mu, a,\ x_{\tilde{L}}, x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). && (12.5.36)\end{aligned}$$

From all the above we see that Scenario[$\mathbb{R}$] starting with (12.5.3(p.75)) finally ends up with (12.5.31(p.77)), which can be alternatively rewritten as respectively (see (11.3.5(p.65)))

$$\begin{aligned} &A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle \quad \text{(see (11.3.8(p.65)))},\\ &A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle. \end{aligned} \tag{12.5.37}$$

From the above two results and (12.5.34(p.77)) we eventually obtain the following lemma.

Lemma 12.5.1 *Let* $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$. *Then* $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ *where*

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \Box \tag{12.5.38}$$

■ Step 6 (*Completeness of* $\tilde{\mathcal{T}}\mathtt{om}$)

aggregation scenario

↓

★ Condition Space $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$

Applying Lemma 12.5.1(p.78) to any assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ included in Tom's 11.2.1(p.61) and 11.2.2(p.62), we have $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ corresponding to each $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, which are given by Tom's 12.7.1(p.84) and 12.7.2(p.84). Below let us define

$$\mathtt{Tom}_1 \stackrel{\text{def}}{=} \mathtt{Tom}\ 12.7.1(\text{p.84}) \quad \text{and} \quad \mathtt{Tom}_2 = \mathtt{Tom}\ 12.7.2(\text{p.84}).$$

Furthermore, let

$$\mathtt{Tom} \stackrel{\text{def}}{=} \mathtt{Tom}_1, \mathtt{Tom}_2, \cdots. \tag{12.5.39}$$

Here, as one corresponding to (12.5.18(p.76)), let us define

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}_i}\rangle \;=\; \{(\boldsymbol{p},\check{F}) \mid \boldsymbol{p}\in\mathscr{P}_{A_{\mathtt{Tom}\,i}}\subseteq\mathscr{P}, \check{F}\in\check{\mathscr{F}}_{A\cdot\mathtt{Tom}_i|\boldsymbol{p}}\subseteq\check{\mathscr{F}}\}, \quad i=1,2,\cdots. \tag{12.5.40}$$

In general, let

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \;=\; \{(\boldsymbol{p},\check{F}) \mid \boldsymbol{p}\in\mathscr{P}_{A_{\mathtt{Tom}}}\subseteq\mathscr{P}, \check{F}\in\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\subseteq\check{\mathscr{F}}\}. \tag{12.5.41}$$

In addition, let us define

$$\mathtt{Tom}_i \;\stackrel{\text{def}}{=}\; \{A^1_{\mathtt{Tom}_i}, A^2_{\mathtt{Tom}_i}, \cdots\} = \{A_{\mathtt{Tom}_i}\},$$
$$\tilde{\mathcal{T}}\mathtt{om} \;\stackrel{\text{def}}{=}\; \{\mathtt{Tom}_1, \mathtt{Tom}_2, \cdots\} = \{\mathtt{Tom}\}.$$

Then, as one corresponding to (11.3.17(p.66)), let us define

$$\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle \;\stackrel{\text{def}}{=}\; \cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle = \cup_{A_{\mathtt{Tom}_i}\in\mathtt{Tom}_i}\check{\mathscr{C}}\langle A_{\mathtt{Tom}_i}\rangle, \quad i=1,2,\cdots, \tag{12.5.42}$$

which is the *aggregation* of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$, $j=1,2,\cdots$, into $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$, i.e.,

$$\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle \;\stackrel{\text{def}}{=}\; \{\check{\mathscr{C}}\langle A^1_{\mathtt{Tom}_i}\rangle, \check{\mathscr{C}}\langle A^2_{\mathtt{Tom}_i}\rangle, \cdots\}, \quad i=1,2,\cdots. \tag{12.5.43}$$

□ *Example* **12.5.1** Let $\tilde{\mathcal{T}}\mathtt{om} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, \mathtt{Tom}_3\}$ and $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle = \{\check{\mathscr{C}}\langle A^1_{\mathtt{Tom}_i}\rangle, \check{\mathscr{C}}\langle A^2_{\mathtt{Tom}_i}\rangle, \check{\mathscr{C}}\langle A^3_{\mathtt{Tom}_i}\rangle\}$, $i=1,2,3$. □

Then, the flow of the above *aggregation* can be depicted as in Figure 12.5.1(p.78) below:

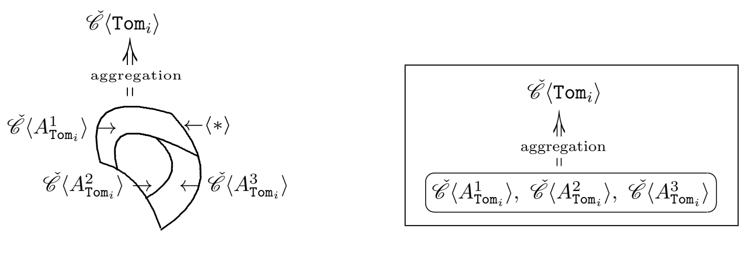


Figure 12.5.1: Aggregation of $\check{\mathscr{C}}\langle A^1_{\mathtt{Tom}_i}\rangle$, $\check{\mathscr{C}}\langle A^2_{\mathtt{Tom}_i}\rangle$,$\check{\mathscr{C}}\langle A^3_{\mathtt{Tom}_i}\rangle$ into $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$

★ Condition Space $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

As one corresponding to (11.3.20(p.67)), let us define

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \overset{\text{def}}{=} \cup_{i=1,2,\cdots}\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle = \cup_{\tilde{\mathtt{T}}\mathtt{om}\in\tilde{\mathcal{T}}\mathtt{om}}\check{\mathscr{C}}\langle\mathtt{Tom}\rangle, \tag{12.5.44}$$

called the *condition space* of $\tilde{\mathcal{T}}\mathtt{om}$, which is the aggregation of $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$, depicted as in Figure 12.5.2(p.79) below (compare Figure 11.3.3(p.67)).

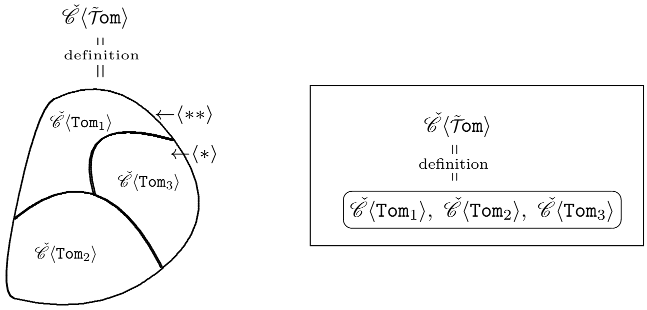


Figure 12.5.2: Condition space $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

In the above figure, the *small* deformed circle $\langle*\rangle$ is the same as the deformed circle $\langle*\rangle$ in Figure 12.5.1(p.78) .

★ Construction of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Using (12.5.42(p.78)), as ones corresponding to (11.3.22(p.67))-(11.3.24(p.67)), from (12.5.44(p.79)) we have

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \cup_{i=1,2,\cdots}\cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle \tag{12.5.45}$$

$$= \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}}\cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}}\rangle \tag{12.5.46}$$

$$= \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}}\cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}}\mathscr{C}\langle A_{\mathtt{Tom}}\rangle \tag{12.5.47}$$

Then, fetching Figure 12.5.1(p.78) in Figure 12.5.2(p.79) , we see that (12.5.45(p.79)) produces Figure 12.5.3(p.79) below, demonstrating the *aggregation* of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$ to $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$.

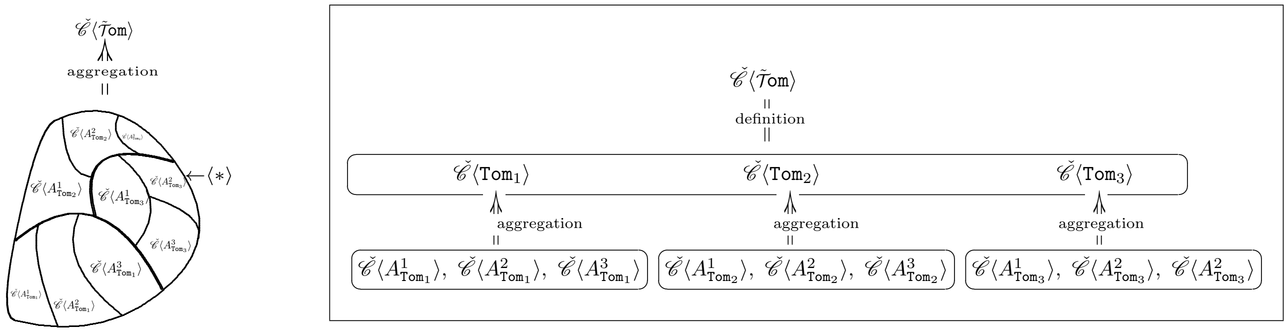


Figure 12.5.3: The aggregation of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

Figure 12.5.3(p.79) above implies that first

"aggregating $\boxed{\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle}$, $j = 1, 2, 3$, for $i = 1, 2, 3$ produces $\boxed{\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle}$"

and then

"aggregating $\boxed{\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle}$, $i = 1, 2, 3$, produces $\boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle}$".

The above two successive aggregating procedures eventually yields

$$\text{“ aggregating } \boxed{\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle} \text{ for } i,j = 1,2,3 \text{ produces } \boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle} \text{ ”}, \tag{12.5.48}$$

Moreover, note that $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is what is aggregated over $\boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle}$, i.e.,

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle\ . \tag{12.5.49}$$

★ Completeness of $\tilde{\mathcal{T}}\mathtt{om}$ on $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$

From (12.5.49(p.79)) and (11.3.25(p.68)) we see that aggregating Lemma 12.5.1(p.78) produces Lemma 12.5.2(p.80) below.

Lemma 12.5.2 *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\;\Box$$

Here note again (11.3.26(p.68)), i.e.

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}. \tag{12.5.50}$$

What is interesting here is that also for $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ we have the same result as above, i.e.,

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}. \tag{12.5.51}$$

● *Proof* Note here that for any given $\check{F}\in\check{\mathscr{F}}$ there exists a $F\in\mathscr{F}$ such that $F\equiv\check{F}\cdots$**(1)** (see Lemma 12.1.1(p.70) (b)) and that for any given $F\in\mathscr{F}$ there exists a $\check{F}\in\check{\mathscr{F}}$ such that $\check{F}\equiv F\cdots$**(2)** (see Lemma 12.1.1(p.70) (c)).

- ∘ From (12.5.18(p.76)) we have $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \subseteq \{(\boldsymbol{p},\check{F}) \mid \boldsymbol{p}\subseteq\mathscr{P}, \check{F}\subseteq\check{\mathscr{F}}\}$ for any $A_{\mathtt{Tom}}$, hence due to (1) we get $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \subseteq \{(\boldsymbol{p},F) \mid \boldsymbol{p}\in\mathscr{P}, F\in\check{\mathscr{F}}\} = \mathscr{P}\times\check{\mathscr{F}} = \mathscr{P}\times\mathscr{F}$ due to $\check{\mathscr{F}}=\mathscr{F}$ from Lemma 12.1.1(p.70) (a). Accordingly, from (12.5.47(p.79)) we obtain $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \subseteq \cup_{\tilde{\mathtt{Tom}}\in\tilde{\mathcal{T}}\mathtt{om}} \cup_{A_{\mathtt{Tom}}\in\tilde{\mathtt{Tom}}} \mathscr{P}\times\mathscr{F} = \mathscr{P}\times\mathscr{F}\cdots$**(3)**.
- ∘ Consider any $(\boldsymbol{p},F)\in\mathscr{P}\times\mathscr{F}\cdots$**(4)**. Then, since $(\boldsymbol{p},F)\in\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ due to (11.3.26(p.68)), we have $(\boldsymbol{p},F)\in\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ for at least one $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ due to (11.3.24(p.67)). Hence, since $F\in\mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ due to (11.3.7(p.65)), we have $\check{F}\in\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ due to (12.1.3(p.69)), hence $(\boldsymbol{p},\check{F})\in\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ due to (12.5.18(p.76)), thus $(\boldsymbol{p},F)\in\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ due to (2), hence $(\boldsymbol{p},F)\in\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ due to (12.5.47(p.79)). Accordingly, from (4) we have $\mathscr{P}\times\mathscr{F}\subseteq\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle\cdots$**(5)**.

From (3) and (5) we obtain $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$. ∎

Let us refer to the equality (12.5.51(p.80)) as the <u>*completeness*</u> of $\tilde{\mathcal{T}}\mathtt{om}$ on $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$. Then (12.5.48(p.79)) can be rewritten as

$$\text{“aggregating } \check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle \text{ for } i,j=1,2,3, \text{ produces } \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}\text{”}, \tag{12.5.52}$$

hence Figure 12.5.3(p.79) can be rewritten as Figure 12.5.4(p.80) below.

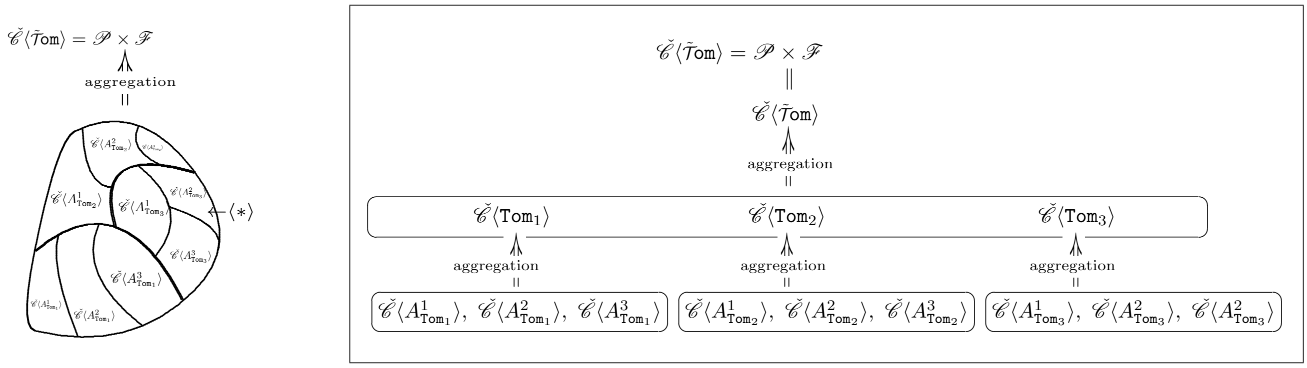


Figure 12.5.4: The aggregation of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$

■ Step 7 (*symmetry theorem* ($\mathbb{R}\to\tilde{\mathbb{R}}$))

From (12.5.50(p.80)) and (12.5.51(p.80)), it follows that Lemma 12.5.2(p.80) can be rewritten as Theorem 12.5.1(p.80) below.

Theorem 12.5.1 (symmetry theorem ($\mathbb{R}\to\tilde{\mathbb{R}}$)) *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\;\Box \tag{12.5.53}$$

Then, clearly the attribute vector of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ becomes as follows (see (12.5.35(p.77)))

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{12.5.54}$$

$$\;=\; (b,\mu,a,\; x_{\tilde{L}},\; x_{\tilde{K}},\; s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \tag{12.5.55}$$

↑

aggregation scenario

■ Step 8 (*summary of* Scenario[$\mathbb{R}$])

At a glance, the symmetry transformation operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ seems to be rather complicated, however it can be simply prescribed as follows.

- Firstly, apply the reverse operation $\mathcal{R}$ to all *reversible* elements (see Defs 12.3.3(p.73)) appearing within the description of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 11.2.1(p.61) and 11.2.2(p.62)).
- Next, replace each of all elements, whether resultant ones (reversible) or non-reversible ones, with the right side of its corresponding equality in Lemma 12.3.1(p.72) (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$).
- Finally, remove the check sign "ˇ" from all the *replaced* symbols (identity replacement operation $\mathcal{I}_{\mathbb{R}}$). □

12.6 Derivation of $\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$, and $\tilde{\kappa}_{\mathbb{R}}$

To begin with, let us note here the fact that Scenario[$\mathbb{R}$] with $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ is applicable for an assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ related to the attribute vector (see Section 12.4(p.74))

$$\boldsymbol{\theta} = (a, \mu, b,\ x_L\ ,\ x_K\ ,\ s_{\mathcal{L}}\ , \kappa, T, L\ , K\ , \mathcal{L}\ , V_t).$$

This fact implies that the scenario can be always applied also to any assertions involved with the attribute vector $\boldsymbol{\theta}$. Accordingly, applying the scenario to any assertions on $T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}$, and $\kappa_{\mathbb{R}}$ yields the corresponding assertions on $\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$ and $\tilde{\kappa}_{\mathbb{R}}$, i.e.,

$$\mathscr{A}\{\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}].$$

Accordingly, we have the following lemma:

Lemma 12.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any* $F \in \mathscr{F}$:

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *strictly increasing on* $(-\infty, b]$.
(f) $\tilde{T}(x) = \mu - x$ *on* $[b, \infty)$ *and* $\tilde{T}(x) < \mu - x$ *on* $(-\infty, b)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $\tilde{T}(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \le \min\{0, \mu - x\}$ *on* $x \in (-\infty, \infty)$.
(i) $\tilde{T}(0) = 0$ *if* $a > 0$ *and* $\tilde{T}(0) = \mu$ *if* $b < 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b > y$, *then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b > \mu$. □

● *Proof by symmetry* The lemma, excluding (a,n), can be easily obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Lemmas 10.1.1(p.55) as shown below.

(a) Evident from the fact that $\min\{\boldsymbol{\xi} - x, 0\}$ in (5.1.11(p.25)) is continuous on $(-\infty, \infty)$.

(b) Lemma 10.1.1(p.55) (b) can be rewritten as $A = \{T(x) \ge \tilde{T}(x')$ for $x < x'\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) \ge -\hat{T}(x')$ for $-\hat{x} < -\hat{x}'\} = \{\hat{T}(\hat{x}) \le \hat{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\tilde{T}}(\hat{x}) \le \check{\tilde{T}}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) \le \tilde{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A = \{\tilde{T}(x) \le \tilde{T}(x')$ for $x > x'\}$, meaning that $\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.

(c-e) Almost the same as the proof of (b)

(f) Let the former half of Lemma 10.1.1(p.55) (f) can by rewritten as $A = \{T(x) = \mu - x$ for $x \le a\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) = -\hat{\mu} + \hat{x}$ for $-\hat{x} \le -\hat{a}\} = \{\hat{T}(x) = \hat{\mu} - \hat{x}$ for $\hat{x} \ge \hat{a}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\tilde{T}}(\hat{x}) = \check{\mu} - \hat{x}$ for $\hat{x} \ge \check{b}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this lead to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) = \mu - \hat{x}$ for $\hat{x} \ge b\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) = \mu - x$ for $x \ge b\} = \{\tilde{T}(x) = \mu - x$ on $[b, \infty)\}$. The proof of the latter half is almost the same as the above.

(g) The former half of Lemma 10.1.1(p.55) (g) can be rewritten by $A = \{T(x) > 0$ for $x < b\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) > 0$ for $-\hat{x} < -\hat{b}\} = \{\hat{T}(x) < 0$ for $\hat{x} > \hat{b}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\tilde{T}}(\hat{x}) < 0$ for $\hat{x} > \check{a}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) < 0$ for $\hat{x} > a\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) < 0$ for $x > a\} = \{\tilde{T}(x) < 0$ on $(a, \infty)\}$. The proof of the latter half is almost the same as the above.

(h) Applying $\mathcal{R}$ to Lemma 10.1.1(p.55) (h) yields $\mathcal{R}[A] = \{-\hat{T}(x) \ge \max\{0, -\hat{\mu} + \hat{x}\}$ for $-\infty < -\hat{x} < \infty\} = \{\hat{T}(x) \le \min\{0, \hat{\mu} - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\tilde{T}}(\hat{x}) \le \min\{0, \check{\mu} - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) \le \min\{0, \mu - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) \le \min\{0, \mu - x\}$ for $\infty > x > -\infty\} = \{\tilde{T}(x) \le \min\{0, \mu - x\}$ on $(-\infty, \infty)\}$.

(i) Immediate from $\tilde{T}(0) = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}I(a \le \boldsymbol{\xi} \le b)]$ from (5.1.11(p.25)) and (2.2.3(p.13))).

(j,k) Almost the same as the proof of (b and c)

(l) Lemma 10.1.1(p.55) (l) can be rewritten as A ={If $x < y$ and $a < y$, then $T(x) + x < T(y) + y$}. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]$={If $-\hat{x} < -\hat{y}$ and $-\hat{a} < -\hat{y}$, then $-\hat{T}(x) - \hat{x} < -\hat{T}(y) - \hat{y}$}={If $\hat{x} > \hat{y}$ and $\hat{a} > \hat{y}$, then $\hat{T}(x)\hat{x} > T(y) + \hat{y}$}, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={If $\hat{x} > \hat{y}$ and $\check{b} > \hat{y}$, then $\check{\hat{T}}(\hat{x}) + \hat{x} > \check{\hat{T}}(\hat{y}) + \hat{y}$}={If $x > y$ and $\check{b} > y$, then $\check{\hat{T}}(x) + x > \check{\hat{T}}(y) + y$}. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={If $x > y$ and $b > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$}.

(m) The former half of Lemma 10.1.1(p.55) (m) can be rewritten as Let $A = \{\lambda\beta T(\lambda\beta\mu - s) - s$ is nonincreasing in $s\}$, which can be rewritten as $A = \{\lambda\beta T(\lambda\beta\mu - s) - s \ge \lambda\beta T(\lambda\beta\mu - s') - s'$ for $s < s'\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) - s \ge -\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') - s'$ for $s < s'\}$=$\{\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) + s \le \lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') + s'$ for $s < s'\}$,† and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\check{\hat{T}}(\overline{-\lambda\beta\check{\mu} - s}) + s \le \lambda\beta\check{\hat{T}}(\overline{-\lambda\beta\check{\mu} - s'}) + s'$ for $s < s'\}$=$\{\lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s) + s \le \lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s') + s'$ for $s < s'\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s \le \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'\}$, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nondecreasing in s. Similarly, the latter half of Lemma 10.1.1(p.55) (m) can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s < \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'\}$, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nonincreasingness in s.

(n) Clear from (2.2.2(p.13)). ∎

• *Direct proof* See the proof of Lemma A 1.1(p.289) . ∎

We have:

$$\tilde{L}(x) \begin{cases} = \lambda\beta\mu + s - \lambda\beta x \text{ on } [b, -\infty) & \cdots (1), \\ < \lambda\beta\mu + s - \lambda\beta x \text{ on } (-\infty, b) & \cdots (2), \end{cases} \tag{12.6.1}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta\mu + s - \delta x & \text{on} \quad [b, \infty) & \cdots (1), \\ < \lambda\beta\mu + s - \delta x & \text{on} \quad (-\infty, b) & \cdots (2). \end{cases} \tag{12.6.2}$$

$$\tilde{K}(x) \begin{cases} < -(1-\beta)x + s & \text{on} \quad (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s & \text{on} \quad (-\infty, a] & \cdots (2), \end{cases} \tag{12.6.3}$$

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{12.6.4}$$

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta\mu + s + (1-\lambda)\beta x \text{ on } & [b, \infty) & \cdots (1), \\ \beta x + s & \text{on} \quad (-\infty, a] & \cdots (2). \end{cases} \tag{12.6.5}$$

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)x_{\tilde{L}} \cdots (1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)x_{\tilde{K}} \cdots (2). \tag{12.6.6}$$

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to (10.2.3(p.57)) - (10.2.8(p.57)). ∎

• *Direct proof* See (A 1.1(p.290))-(A 1.6(p.291)) . ∎

Lemma 12.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *Let $s = 0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\ge)\ 0$.*
(e) *Let $s > 0$.*

1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.*
2. *$(\lambda\beta\mu + s)/\lambda\beta \ge (<)\ b \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta\mu + s)/\lambda\beta \ge (<)\ b$.* □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Lemmas 10.2.1(p.57) ∎

• *Direct proof* See the proof of Lemma A 1.2(p.291) . ∎

Corollary 12.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (\ge)\ 0$.
(b) $x_{\tilde{L}} \le (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$. □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Corollaries 10.2.1(p.57) ∎

• *Direct proof* See the proof of Corollary A 1.1(p.291) . ∎

Lemma 12.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty, \infty)$.*

†Note Def. 12.3.3(p.73)).

(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a,\infty)$.
(d) $\tilde{K}(x)$ *is strictly decreasing on* $(-\infty,\infty)$ *if* $\beta<1$.
(e) $\tilde{K}(x)+x$ *is nondecreasing on* $(-\infty,\infty)$.
(f) $K(x)+x$ *is strictly increasing on* $(-\infty,b]$.
(g) $\tilde{K}(x)+x$ *is strictly increasing on* $(-\infty,\infty)$ *if* $\lambda<1$.
(h) *If* $x>y$ *and* $b>y$, *then* $\tilde{K}(x)+x>\tilde{K}(y)+y$.
(i) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{K}}=a$ *where* $x_{\tilde{K}}<(\geq)\ x \Leftrightarrow \tilde{K}(x)<(=)\ 0 \Rightarrow \tilde{K}(x)<(\geq)\ 0$.
(j) *Let* $\beta<1$ *or* $s>0$.

1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}}<(=(>))\ x \Leftrightarrow \tilde{K}(x)<(=(>))\ 0$.
2. $(\lambda\beta\mu+s)/\delta \geq (<)\ b \Leftrightarrow x_{\tilde{K}}=(<)\ (\lambda\beta\mu+s)/\delta$.
3. *Let* $\tilde{\kappa}<(=(>))\ 0$. *Then* $x_{\tilde{K}}<(=(>))\ 0$. □

- *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Lemmas 10.2.2(p.57). ▮
- *Direct proof* See the proof of Lemma A 1.3(p.291) . ▮

Corollary 12.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}}<(\geq)\ x \Leftrightarrow \tilde{K}(x)<(\geq)\ 0$.
(b) $x_{\tilde{K}}\leq(\geq)\ x \Rightarrow \tilde{K}(x)\leq(\geq)\ 0$. □

- *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Corollaries 10.2.2(p.58). ▮
- *Direct proof* See the proof of Corollary A 1.2(p.292) . ▮

Lemma 12.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}=a$.
(b) *Let* $\beta=1$ *and* $s>0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}$.
(c) *Let* $\beta<1$ *and* $s=0$. *Then* $a<(=(>))\ 0 \Leftrightarrow x_{\tilde{L}}<(=(>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(=))\ 0$.
(d) *Let* $\beta<1$ *and* $s>0$. *Then* $\tilde{\kappa}<(=(>))\ 0 \Leftrightarrow x_{\tilde{L}}<(=(>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(>))\ 0$. □

- *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Lemmas 10.2.3(p.58). ▮
- *Direct proof* See the proof of Lemma A 1.4(p.292) . ▮

Lemma 12.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta<1$.
(b) *Let* $\lambda\beta\mu\leq a$.

1. $x_{\tilde{L}}\geq\lambda\beta\mu+s$.
2. *Let* $s>0$ *and* $\lambda\beta<1$. *Then* $x_{\tilde{L}}>\lambda\beta\mu+s$.

(c) *Let* $\lambda\beta\mu>a$. *Then, there exists a* $s_{\tilde{\mathcal{L}}}>0$ *such that if* $s_{\tilde{\mathcal{L}}}>(\leq)\ s$, *then* $x_{\tilde{L}}<(\geq)\ \lambda\beta\mu+s$. □

- *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Lemmas 10.2.4(p.59). ▮
- *Direct proof* See the proof of Lemma A 1.5(p.293) . ▮

Lemma 12.6.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have*:

(a) $\tilde{\kappa}=\lambda\beta\mu+s$ *if* $b<0$ *and* $\tilde{\kappa}=s$ *if* $a>0$.
(b) *Let* $\beta<1$ *or* $s>0$. *Then* $\tilde{\kappa}<(=(>))\ 0 \Leftrightarrow x_{\tilde{K}}<(=(>))\ 0$. □

- *Proof* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Lemmas 10.3.1(p.59). ▮
- *Direct proof* See the proof of Lemma A 1.6(p.293) . ▮

12.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Lemma 12.7.1 ($\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$) *The optimal initiating time* t^*_τ (OIT) *is not subject to the influence of the symmetry transformation operation* $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))). □

- *Proof* First, let us represent (7.2.5(p.44)) as $D \overset{\text{def}}{=} \{I_\tau^{t^*_\tau} \geq I_\tau^t \text{ for } \tau\geq t\geq t_{qd}\}\cdots(1)$, which can be rewritten as $D=\{\beta^{\tau-t^*_\tau}V_{t^*_\tau}\geq \beta^{\tau-t}V_t \text{ for } \tau\geq t\geq t_{qd}\}$. Next, applying $\mathcal{R}$ to this yields $\mathcal{R}[D]=\{-\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau}\geq -\beta^{\tau-t}\hat{V}_t \text{ for } \tau\geq t\geq t_{qd}\}=\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau}\leq\beta^{\tau-t}\hat{V}_t$ for $\tau\geq t\geq t_{qd}\}$. Then, even if applying $\mathcal{C}_{\mathbb{R}}$ (Lemma 12.3.1(p.72)) to this, no change occurs, i.e., $\mathcal{C}_{\mathbb{R}}\mathcal{R}[D]=\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau}\leq\beta^{\tau-t}\hat{V}_t$ for $\tau\geq t\geq t_{qd}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ (Lemma 12.3.3(p.73)) to this, we have $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]=\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau}\leq\beta^{\tau-t}\hat{V}_t$ for $\tau\geq t\geq t_{qd}\}$. Then, since $\hat{V}_t$ changes into V_t for the same reason as been stated just below (12.5.20(p.76)), so we have $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]=\{\beta^{\tau-t^*_\tau}V_{t^*_\tau}\leq\beta^{\tau-t}V_t$ for $\tau\geq t\geq t_{qd}\}$, i.e., $\{I_\tau^{t^*_\tau}\leq I_\tau^t \text{ for } \tau\geq t\geq t_{qd}\}\cdots(2)$. The above result means that the optimal initiating time is t^*_τ even if $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}(=\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R})$ is applied, hence it follows that the optimal initiating time t^*_τ due to (1) is entirely inherited to t^*_τ due to (2). ▮

□ **Tom 12.7.1** ($\boxdot\mathscr{A}_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

- *Proof by symmetry* Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Tom 11.2.1(p.61). ▮
- *Direct proof* See the proof of Tom A 4.1(p.303) . ▮

□ **Tom 12.7.2** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.*
   - i. *Let $a < 0$ $(\!(\tilde{\kappa} < 0)\!)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let $a = 0$ $(\!(\tilde{\kappa} = 0)\!)$.*
     1. *Let $\beta\mu + s \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let $a > 0$ $(\!(\tilde{\kappa} > 0)\!)$.*
     1. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{ⓢ\blacktriangle}\boxed{◎\ \parallel}$ *is true.* □

- *Proof by symmetry* Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.77))) to Tom 11.2.2(p.62). ▮
- *Direct proof* See the proof of Tom A 4.2(p.304) . ▮

12.8 $\tilde{\text{S}}$cenario[ℝ]

In this section we write up the inverse of Scenario[ℝ](p.75) which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 11.2.1(p.61) and 11.2.2(p.62)) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 12.7.1(p.84) and 12.7.2(p.84)). Let us represent this scenario as $\tilde{\text{S}}$cenario[ℝ].

■ $\tilde{\text{S}}$tep 1 (*opening*)

◦ The system of optimality equation of $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ is given by Table 6.4.1(p.41) (II), i.e.,

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\quad t > 1\}. \tag{12.8.1}$$

◦ Let us consider an assertion $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ in each of Tom's 12.7.1(p.84) and 12.7.2(p.84), which can be rewritten as

$$\begin{aligned} A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} &= \{\tilde{\mathsf{S}} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \text{ with } \boldsymbol{p} \in \mathscr{P}_{A_{\tilde{\texttt{T}}\texttt{om}}} \subseteq \mathscr{F}\} \\ &= \{\tilde{\mathsf{S}} \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\} \quad (\text{see } (12.5.31(\text{p.77}))) \end{aligned} \tag{12.8.2}$$

where

$$\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P},\ F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}.$$

To facilitate the understanding of the discussion that follows let us use the following example.

$$\tilde{\mathsf{S}} = \langle V_t - s_{\tilde{\mathcal{L}}} + x_{\tilde{\mathcal{L}}} + \tilde{\kappa} + b + \mu + a \leq 0,\ t > 0\rangle \quad (\text{see } (12.5.33(\text{p.77}))).$$

◦ The attribute vector of the assertion $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (12.5.36(p.77)), i.e.,

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = (b, \mu, a,\ x_{\tilde{\mathcal{L}}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{12.8.3}$$

■ $\tilde{\text{S}}$tep 2 (*reverse operation $\mathcal{R}$*)

◦ Applying the reverse operation $\mathcal{R}$ to (12.8.1(p.84)) produces

$$\begin{aligned} \mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = \min\{-\hat{\tilde{K}}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\ t > 1\} \\ &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = -\max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}\} \\ &= \{\hat{V}_1 = \beta\hat{\mu} - s,\ \hat{V}_t = \max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}. \end{aligned} \tag{12.8.4}$$

- Applying $\mathcal{R}$ to (12.8.2(p.84)) yields to

$$\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \}. \tag{12.8.5}$$

For our example we have:

$$\begin{aligned}\mathcal{R}[\tilde{\mathsf{S}}] &= \langle -\hat{V}_t - s_{\tilde{\mathcal{L}}} - \hat{x}_{\tilde{L}} - \hat{\tilde{\kappa}} - \hat{b} - \hat{\mu} - \hat{a} \le 0,\ \ t > 0 \rangle \\ &= \langle \hat{V}_t + s_{\tilde{\mathcal{L}}} + \hat{x}_{\tilde{L}} + \hat{\tilde{\kappa}} + \hat{b} + \hat{\mu} + \hat{a} \ge 0,\ \ t > 0 \rangle.\end{aligned} \tag{12.8.6}$$

- The attribute vector of the assertion $\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (12.5.36(p.77)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &\overset{\text{def}}{=} \mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &= (\hat{b}, \hat{\mu}, \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \hat{\kappa}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t).\end{aligned} \tag{12.8.7}$$

■ **$\tilde{\text{S}}$tep 3** (*correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$*)

- Here let us consider the application of the correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.2(p.72).

$$\hat{b},\ \hat{\mu},\ \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}}(x),\ \hat{\tilde{L}}(x),\ \hat{\tilde{K}}(x),\ \hat{\tilde{\mathcal{L}}}(s) \cdots (1^*)$$

by its right-hand side

$$\check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(\hat{x}),\ \check{L}(\hat{x}),\ \check{K}(\hat{x}),\ \check{\mathcal{L}}(s) \cdots (2^*)$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ corresponding to the $F \in \mathscr{F}$.

- Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (12.8.4(p.84)) leads to

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\hat{V}_1 = \beta\check{\mu} - s,\ \hat{V}_t = \max\{\check{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}. \tag{12.8.8}$$

- Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to $\mathcal{R}[\tilde{\mathsf{S}}]$ in (12.8.6(p.85)), we have

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] = \langle \hat{V}_t + s_{\check{\mathcal{L}}} + \check{x}_L + \check{\kappa} + \check{a} + \check{\mu} + \check{b} \le 0,\ \ t > 0 \rangle. \tag{12.8.9}$$

Now, let us note here that the application of $\tilde{\mathcal{C}}_{\mathbb{R}}$ (see Lemma 12.3.2(p.72)) inevitably changes

"*for* $F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$" in (12.8.5(p.85))

into

"*for* $\check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$ *corresponding to any* $F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}$ *with* $\boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P}$"

where

$$\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} = \{\check{F}\ \big|\ F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\} \quad \text{(see (12.1.3(p.69)))}.$$

Hence, applying (12.8.5(p.85)), we have

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F} \tag{12.8.10}$$

$$\text{corresponding to } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F} \text{ with } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P}\}. \tag{12.8.11}$$

Now, since the phrase "$F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$" is implicitly accompanied with the phrase "$\check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$". Accordingly (12.8.11(p.85)) can be rewritten as

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \\ &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \}\end{aligned} \tag{12.8.12}$$

where

$$\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F)\ \big|\ \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{12.8.13}$$

- The attribute vector of $\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (12.8.7(p.85)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &= \tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &= (\check{a}, \check{\mu}, \check{b},\ \check{x}_L, \check{x}_K, s_{\check{\mathcal{L}}}.\check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t).\end{aligned} \tag{12.8.14}$$

■ **$\tilde{\text{S}}$tep 4** (*identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$*)

- Here let us consider the application of the identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$, i.e., the replacement of the left-hand of each equality in Lemma 12.3.4(p.74)

$$\check{F},\ \check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(x),\ \check{L}(x),\ \check{K}(x),\ \check{\mathcal{L}}(s) \cdots (1^*)$$

by its right-hand side

$$F,\ a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T(x),\ L(x),\ K(x),\ \mathcal{L}(s) \cdots (2^*)$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ which is identical to the $F \in \mathscr{F}$, i.e., $\check{F} \equiv F \cdots (1)$.

- Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.8(p.85)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\mu - s,\ \hat{V}_t = \max\{K(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}.$$

Now, we have $\hat{V}_1 = \beta\mu - s = V_1$ from (6.4.5(p.41)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, since $\hat{V}_t = \max\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$ from (6.4.6(p.41)), by induction $\hat{V}_t = V_t$ for $t > 0$. Thus we have

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{V_1 = \beta\mu - s,\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\},$$

which is the same as $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.4.1(p.41) (I)). Thus we have

$$\begin{aligned}\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} &= \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \\ &= \{V_1 = \beta\mu - s,\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1\}.\end{aligned} \tag{12.8.15}$$

- Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.12(p.85)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \ = \ \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. \tag{12.8.16}$$

Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.9(p.85)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] = \langle V_t +\ s_{\mathcal{L}}\ +\ x_L\ + \kappa + a + \mu + b \le 0,\ \ t > 0\rangle. \tag{12.8.17}$$

Now V_t within $\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]$ is generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (12.8.16(p.86)) can be regarded as an assertion as to $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$. Thus, we have

$$\begin{aligned}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \ &= \ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \\ &= \ \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}.\end{aligned} \tag{12.8.18}$$

- The attribute vector of $A_{\tilde{\mathtt{Tom}}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.14(p.85)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \ &= \ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &= \ (a, \mu, b,\ x_L\ , x_K\ ,\ s_{\mathcal{L}}\ , \kappa, T, L\ , K, \mathcal{L}, V_t),\end{aligned} \tag{12.8.19}$$

■ **$\tilde{\textsf{S}}$tep 5** (*symmetry transformation operation $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$*)

Below, letting us line up the attribute vectors given in $\tilde{\textsf{S}}$tep 1 to $\tilde{\textsf{S}}$tep 4, we have the following:

$$\begin{array}{lll} \tilde{\mathsf{S}}\text{tep 1:} & \boldsymbol{\theta}(\ \not{b}, \not{\mu}, \not{a},\ x_{\tilde{L}}\ ,\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}\ , \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}}, V_t\ ) & (\leftarrow (12.8.3(p.84)\ )) \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \ \ \leftarrow\ \mathcal{R} \\ \tilde{\mathsf{S}}\text{tep 2:} & \boldsymbol{\theta}(\ \hat{b}, \hat{\mu},\ \hat{a},\ \hat{\tilde{x}}_L\ ,\ \hat{\tilde{x}}_K,\ s_{\tilde{\mathcal{L}}}\ , \hat{\tilde{\kappa}},\ \hat{\tilde{T}},\ \hat{\tilde{L}},\ \hat{\tilde{K}},\ \hat{\tilde{\mathcal{L}}}, \hat{V}_t\ ) & (\leftarrow (12.8.7(p.85)\ )) \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \ \ \leftarrow\ \tilde{\mathcal{C}}_{\mathbb{R}} \\ \tilde{\mathsf{S}}\text{tep 3:} & \boldsymbol{\theta}(\ \check{a},\ \check{\mu},\ \check{b},\ x_{\check{L}}\ , x_{\check{K}},\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T},\ \check{L},\ \check{K},\ \check{\mathcal{L}}, V_t\ ) & (\leftarrow (12.8.14(p.85)\ )) \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \ \ \leftarrow\ \tilde{\mathcal{I}}_{\mathbb{R}} \\ \tilde{\mathsf{S}}\text{tep 4:} & \boldsymbol{\theta}(\ a,\ \mu,\ b,\ x_L\ , x_K\ ,\ s_{\mathcal{L}}\ , \kappa,\ T,\ \ L,\ K,\ \mathcal{L}, V_t\ ) & (\leftarrow (12.8.19(p.86)\ )) \end{array} \tag{12.8.20}$$

The above flow can be eventually reduced to

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \stackrel{\text{def}}{=} \left\{\begin{array}{l} b,\ \mu,\ a,\ x_{\tilde{L}}\ , x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}\ , \tilde{\kappa}, \tilde{T}, \tilde{L}\ , \tilde{K}, \tilde{\mathcal{L}},\ V_t \\ \ \wr\ \ \wr\ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \downarrow\ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr \\ a,\ \mu,\ b,\ x_L\ , x_K\ , s_{\mathcal{L}}\ , \kappa,\ T, L\ , K, \mathcal{L},\ V_t \end{array}\right\} \tag{12.8.21}$$

called the *symmetry transformation operation*, which can be regarded as the successive application of the three operations, i.e., " $\mathcal{R} \to \tilde{\mathcal{C}}_{\mathbb{R}} \to \tilde{\mathcal{I}}_{\mathbb{R}}$ ". Here let us define

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \ \stackrel{\text{def}}{=}\ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}. \tag{12.8.22}$$

Then (12.8.18(p.86)) can be rewritten as

$$\begin{aligned}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \ &= \ \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \\ &= \ \{\mathsf{S} \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}\end{aligned} \tag{12.8.23}$$

where

$$\mathsf{S}\ =\ \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\tilde{\mathsf{S}}]. \tag{12.8.24}$$

Then, from (12.8.17(p.86)) we have

$$\mathsf{S} = \langle V_t +\ s_{\mathcal{L}}\ +\ x_L\ + \kappa + a + \mu + b \le 0,\ \ t > 0\rangle.$$

Then, (12.8.15(p.86)) can be rewritten as

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\ \}]. \tag{12.8.25}$$

In addition, (12.5.27(p.77)) can be rewritten as

$$\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{12.8.26}$$

$$\;=\; (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \tag{12.8.27}$$

From all the above we see that $\tilde{\text{S}}$cenario[$\mathbb{R}$] starting with (12.8.2(p.84)) finally ends up with (12.8.23(p.86)), which can be rewritten as respectively

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle, \tag{12.8.28}$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle. \tag{12.8.29}$$

From the above two results and (12.8.25(p.87)) we eventually obtain the following lemma.

Lemma 12.8.1 *Let* $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$*. Then* $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ *where*

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \ \square \tag{12.8.30}$$

■ $\tilde{\text{S}}$tep 6 (*aggregation*)

We can construct quite the same procedure as in Step 6 (p.78).

■ $\tilde{\text{S}}$tep 7 (*symmetry theorem* $\mathbb{R} \leftarrow \tilde{\mathbb{R}}$)

Through the procedure in $\tilde{\text{S}}$tep 6 (p.87) we have the following theorem

Theorem 12.8.1 *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \ \square \tag{12.8.31}$$

● *Proof* Immediate for the same reason as in Theorem 12.5.1(p.80). ∎

The attribute vector of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{12.8.32}$$

$$\;=\; (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t) \tag{12.8.33}$$

12.9 Definition of Symmetry

Thus far, the term of *symmetry* has been used in the rather intuitive nuance. In order to make our discussions more clear, below let us provide its strict definition.

Definition 12.9.1

(a) Let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be assertions on models M_1 and M_2 respectively. Then, if $A\{\mathsf{M}_2\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A\{\mathsf{M}_1\}]$ and $A\{\mathsf{M}_1\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A\{\mathsf{M}_2\}$, let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be said to be *symmetrical*, denoted by $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$. Then let us employ the expression of "M_1 and M_2 are symmetrical with respect to A".

(b) For given two assertion systems $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ which are one-to-one correspondent, if $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$ for any pair $(A\{\mathsf{M}_1\}, A\{\mathsf{M}_2\})$ where $A\{\mathsf{M}_1\} \in \mathscr{A}\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\} \in \mathscr{A}\{\mathsf{M}_2\}$, then $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ are said to be *symmetrical*, denoted by $\mathscr{A}\{\mathsf{M}_1\} \sim \mathscr{A}\{\mathsf{M}_2\}$. Then, let us employ the expression of "M_1 and M_2 are symmetrical with respect to $\mathscr{A}$".

(c) Without confusion, let us remove the phrases "with respect to A" and "with respect to $\mathscr{A}$". □

Lemma 12.9.1 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *and* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *are symmetrical, i.e.,*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \sim \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}. \ \square \tag{12.9.1}$$

● *Proof* Immediate from (12.5.53(p.80)) and (12.8.31(p.87)). ∎

12.10 Symmetry-Operation-Free

When no change occurs even if the symmetry operation is applied to a given assertion A, the assertion is said to be *free from* the symmetry operation, called the *symmetry-operation-free assertion.*

Lemma 12.10.1 *Even if the symmetry operation is applied to the symmetry-operation-free assertion, no change occurs.* □

● *Proof* Evident. ∎

12.11 Symmetry between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Here note that the symmetrical relation holds between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see (I) and (II) in Table 6.4.1(p.41)), i.e., $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \sim \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. It is an important point that, due to this very fact, the symmetry theorems (Theorems 12.5.1(p.80) and 12.8.1(p.87)) can be derived. It will be known later on that this symmetrical relation is one of the necessary conditions on which the integrated theory can be successfully constructed.

Chapter 13

Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

In this chapter we present a methodology which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism).

13.1 Preliminary

Lemma 13.1.1 ([47,You])
(a) *Let $x \geq b$. Then $z(x) = b$.*
(b) *Let $x < b$. Then $x < z(x) < b$.*
(c) *$z(x) \geq a$ for any x.* □

● *Proof* (a) Let $x \geq b$. If $z < b \cdots$ (I), then $z < x$, hence $p(z)(z - x) < 0$ due to (5.1.29 (1) (p.26)), and if $b \leq z \cdots$ (III), then $p(z)(z - x) = 0$ due to (5.1.29 (2) (p.26)). Hence $z(x)$ can be given by any $z \geq b$, thus $z(x) = b$ due to Def. 5.1.1(p.26) .

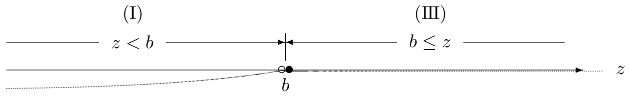


Figure 13.1.1: Case $x \geq b$

(b) Let $x < b$. If $z \leq x \cdots$ (I), then $p(z)(z - x) \leq 0$, if $x < z < b \cdots$ (II), then $p(z)(z - x) > 0$ due to (5.1.29 (1) (p.26)), and if $b \leq z \cdots$ (III), then $p(z)(z - x) = 0$ from (5.1.29 (2) (p.26)). Hence, $z(x)$ is given by z such that $x < z < b$ or equivalently $x < z(x) < b$.

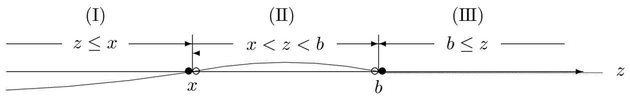


Figure 13.1.2: Case $x < b$

(c) Assume that $z(x) < a$ for a certain x. Then, since $p(z(x)) = 1 = p(a)$ due to (5.1.28 (1) (p.26)), from (5.1.25(p.26)) we have $T(x) = p(z(x))(z(x) - x) = z(x) - x < a - x = p(a)(a - x) \leq T(x)$, which is a contradiction. Hence, it must be that $z(x) \geq a$ for any x. ∎

Corollary 13.1.1 ([47,You]) *$a \leq z(x) \leq b$ for any x.* □
● *Proof* Immediate from Lemma 13.1.1(p.89) . ∎

Lemma 13.1.2 ([47,You]) *$p(z)$ is nonincreasing on $(-\infty, \infty)$ and strictly decreasing in $z \in [a, b]$.* □
● *Proof* The former half is immediate from (5.1.18(p.26)). Let $a \leq z' < z \leq b$. Then $p(z') - p(z) = \Pr\{z' \leq \boldsymbol{\xi}\} - \Pr\{z \leq \boldsymbol{\xi}\} = \Pr\{z' \leq \boldsymbol{\xi} < z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.2.3 (2) (p.13))), hence $p(z') > p(z)$, i.e., $p(z)$ is strictly decreasing on $[a, b]$. ∎

Lemma 13.1.3 ([47,You]) *$z(x)$ is nondecreasing on $(-\infty, \infty)$.* □
● *Proof* From (5.1.25(p.26)), for any x and y we have

$$\begin{aligned}
T(x) &= p(z(x))(z(x) - x) \\
&= p(z(x))(z(x) - y) - (x - y)p(z(x)) \\
&\leq T(y) - (x - y)p(z(x)) \\
&= p(z(y))(z(y) - y) - (x - y)p(z(x)) \\
&= p(z(y))\big(z(y) - x + (x - y)\big) - (x - y)p(z(x)) \\
&= p(z(y))(z(y) - x) + (x - y)(p(z(y)) - p(z(x))) \\
&\leq T(x) + (x - y)(p(z(y)) - p(z(x))).
\end{aligned}$$

‡This is the most important property of the function T, which was proven in [?, 0298].

Hence $0 \le (x-y)(p(z(y)) - p(z(x)))$. Let $x > y$. Then $0 \le p(z(y)) - p(z(x))$, so $p(z(x)) \le p(z(y)) \cdots (1)$. Since $a \le z(x) \le b$ and $a \le z(y) \le b$ from Corollary 13.1.1(p.89), if $z(x) < z(y)$, then $p(z(x)) > p(z(y))$ from Lemma 13.1.2(p.89), which contradicts (1). Hence, it must be that $z(x) \ge z(y)$, i.e., $z(x)$ is nondecreasing in $x \in (-\infty, \infty)$. ∎

Lemma 13.1.4

(a) *$T(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$T(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$T(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$T(x) > 0$ on $(-\infty, b)$ and $T(x) = 0$ on $[b, \infty)$.*
(e) *$T(x) \ge a - x$ on $(-\infty, \infty)$.*
(f) *$T(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(g) *$\beta T(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(h) *$\beta T(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(i) *$T(x) \ge \max\{0, a - x\}$ on $(-\infty, \infty)$.*
(j) *$\lambda\beta T(\lambda\beta a - s) - s$ is nonincreasing in s and is strictly decreasing in s if $\lambda\beta < 1$.* □

• *Proof* (a,b) Immediate from the fact that $p(z)(z - x)$ in (5.1.19(p.26)) is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any z.

(c) Let $x' < x < b$. Then $z(x) < b$ from Lemma 13.1.1(p.89) (b). Accordingly, since $p(z(x)) > 0$ due to (5.1.29 (1) (p.26)) and since $z(x) - x < z(x) - x'$, from (5.1.25(p.26)) we have $T(x) = p(z(x))(z(x) - x) < p(z(x))(z(x) - x') \le T(x')$, implying that $T(x)$ is *strictly* decreasing on $(-\infty, b) \cdots\cdots (1)$. Assume $T(b) = T(x)$ for a given $x < b$, so $b - x > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $b - x > 2\varepsilon > 0$ we have $b > b - \varepsilon > x + \varepsilon > x$, hence $T(b) = T(x) > T(b - \varepsilon) \ge T(b)$ due to the *strict* unceasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, since $T(x) \ne T(b)$ for any $x < b$, we have $T(x) > T(b)$ or $T(x) < T(b)$ for any $x < b$. However, the latter is impossible due to (b), hence only the former is possible. Consequently, it follows that $T(x)$ is strictly decreasing on $(-\infty, b]$ instead of $(-\infty, b)$.

(d) Let $x \ge b$. Then, since $z(x) = b$ from Lemma 13.1.1(p.89) (a), we have $p(z(x)) = 0$ due to (5.1.29 (2) (p.26)), hence $T(x) = p(z(x))(z(x) - x) = 0$ on $[b, \infty)$. Let $x < b$. Then, from (c) we have $T(x) > T(b) = 0$, i.e., $T(x) > 0$ on $(-\infty, b)$.

(e) Since $p(a) = 1$ from (5.1.28 (1) (p.26)), we have $T(x) \ge p(a)(a - x) = a - x$ for any x on $(-\infty, \infty)$.

(f) Let $x < x'$. Then, we have

$$\begin{aligned} T(x) + x &= p(z(x))(z(x) - x) + x \\ &= p(z(x))z(x) + (1 - p(z(x)))x \\ &\le p(z(x))z(x) + (1 - p(z(x)))x' \\ &= p(z(x))(z(x) - x') + x' \le T(x') + x', \end{aligned}$$

implying that $T(x) + x$ is nondecreasing on $(-\infty, \infty)$.

(g) If $\beta = 1$, then $\beta T(x) + x = T(x) + x$, hence the assertion is true from (f).

(h) Since $\beta T(x) + x = \beta(T(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Immediate from the fact that $T(x) \ge a - x$ on $(-\infty, \infty)$ from (e) and $T(x) \ge 0$ on $(-\infty, \infty)$ from (d).

(j) From (5.1.19(p.26)) we have

$$\lambda\beta T(\lambda\beta a - s) - s \;=\; \lambda\beta \max_z p(z)(z - \lambda\beta a + s) - s = \max_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s) - s.$$

Let $s > s'$. Then, we have

$$\begin{aligned} &\lambda\beta T(\lambda\beta a - s) - s - \lambda\beta T(\lambda\beta a - s') + s' \\ &\quad= \max_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s) - \max_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s') - (s - s') \\ &\quad\le \max_z p(z)(s - s')\lambda\beta - (s - s')^\dagger \\ &\quad\le \max_z (s - s')\lambda\beta - (s - s') \quad \text{(due to } p(z) \le 1 \text{ and } s - s' > 0) \\ &\quad= (s - s')\lambda\beta - (s - s') \\ &\quad= -(s - s')(1 - \lambda\beta) \le (<)\ 0 \text{ if } \lambda\beta \le (<)\ 1. \end{aligned}$$

Hence, since $\lambda\beta T(\lambda\beta a - s) - s \le (<)\ \lambda\beta T(\lambda\beta a - s') - s'$ if $\lambda\beta \le (<)\ 1$, it follows that $T(\lambda\beta a - s) - s$ is nonincreasing (strictly decreasing) in s if $\lambda\beta \le (<)\ 1$. ∎

Let us define

$$h(z) = p(z)(z - a)/(1 - p(z)), \quad z > a,$$
$$h^\star = \sup_{a<z} h(z),$$

$^\dagger \max_x g(x) - \max_x h(x) \le \max_x \{g(x) - h(x)\}$.

Below, for a given x let us define the following successive four assertions:

$$\begin{aligned}
A_1(x) &= \langle\!\langle\, z(x) > a \,\rangle\!\rangle, \\
A_2(x) &= \langle\!\langle\, T(a,x) < T(z',x,) \text{ for at least one } z' > a \,\rangle\!\rangle, \\
A_3(x) &= \langle\!\langle\, a - h(z') < x \text{ for at least one } z' > a \,\rangle\!\rangle, \\
A_4(x) &= \langle\!\langle\, \inf_{z>a}\{a - h(z)\} < x \,\rangle\!\rangle.
\end{aligned}$$

Proposition 13.1.1 For any given x we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. □

● *Proof* Letting $T(z,x) \stackrel{\text{def}}{=} p(z)(z-x)$, we can rewrite (5.1.19(p.26)) as $T(x) = \max_z T(z,x) = T(z(x),x)$ (see (5.1.25(p.26))).

1. Let $A_1(x)$ be true for any given x. Suppose $T(a,x) \ge T(z',x)$ for all $z' \ge a$, hence the maximum of $T(z,x)$ for all $z \ge a$ is attained at $z = a$, i.e., $z(x) = a$ (see Def. 5.1.1(p.26)), which contradicts $A_1(x)$. Hence it must be that $T(a,x) < T(z',x)$ for at least one $z' > a$, thus $A_2(x)$ becomes true. Accordingly, we have $A_1(x) \Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any given x. Then, if $z(x) = a$, we have $T(a,x) < T(z',x) \le T(x) = T(z(x),x) = T(a,x)$, which is a contradiction, hence it must be that $z(x) > a$ due to Lemma 13.1.1(p.89) (c). Accordingly, we have $A_2(x) \Rightarrow A_1(x)$. Thus, it follows that $A_1(x) \Leftrightarrow A_2(x)$ for any given x.
2. Since $p(a) = 1$ from (5.1.28 (1) (p.26)), for $z' > a$ (hence $1 > p(z') \cdots$ **(1)** from (5.1.28 (2) (p.26))) we have

$$\begin{aligned}
&T(a,x) - T(z',x) \\
&\quad = p(a)(a-x) - p(z')(z'-x) \\
&\quad = a - x - p(z')(z'-x) \\
&\quad = a - x - p(z')(a - x + z' - a) \\
&\quad = a - x - p(z')(a-x) - p(z')(z'-a) \\
&\quad = (1-p(z'))(a-x) - p(z')(z'-a) \\
&\quad = (1-p(z'))\big(a - x - p(z')(z'-a)/(1-p(z'))\big) \\
&\quad = (1-p(z'))(a - x - h(z')) \\
&\quad = (1-p(z'))(a - h(z') - x).
\end{aligned}$$

   Accordingly, due to (1) we immediately obtain $A_2(x) \Leftrightarrow A_3(x)$ for any given x.
3. Let $A_3(x)$ be true for any given x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x) \Rightarrow A_4(x)$. Let $A_4(x)$ be true for any given x. Then evidently $a - \tilde{h}(z') < x$ for at least one $z' > a$, hence $A_3(x)$ is true, so we have $A_4(x) \Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x) \Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. ▮

Lemma 13.1.5

(a) $0 < h^\star < \infty$.
(b) $x^\star = a - h^\star < a$.
(c) $x^\star < (\ge)\ x \Leftrightarrow z(x) > (=)\ a$.
(d) $a^\star < a$. □

● *Proof* (a) For any infinitesimal $\varepsilon > 0$ such that $a < b - \varepsilon < b \cdots$ (II) we have $0 < p(b-\varepsilon) < 1$ from (5.1.29 (1) (p.26)) and (5.1.28 (2) (p.26)), hence $h(b-\varepsilon) = p(b-\varepsilon)(b-\varepsilon-a)/(1-p(b-\varepsilon)) > 0$. If $b \le z \cdots$ (III), then $p(z) = 0$ due to (5.1.29 (2) (p.26)), hence $h(z) = 0$ for $z \ge b$. From the above we have $h^\star > 0$ (finite) or $h^\star = \infty$.

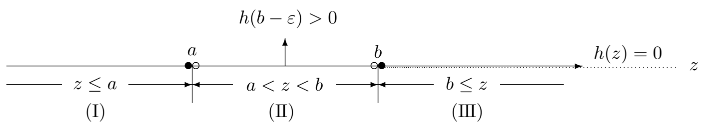


Figure 13.1.3: $h(b-\varepsilon) > 0$ and $h(z) = 0$ for $z \ge b$

Assume that $h^\star = \infty$. Then, there exists at least one z' on $a < z' < b$ such that $h(z') \ge N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$† with any $M > 1$, we have $h(z') \ge M/\underline{f}$ or equivalently $p(z')(z'-a)/(1-p(z')) \ge M/\underline{f}$. Hence, noting (5.1.18(p.26)), we have

$$p(z')(z'-a) \;\ge\; (1-p(z'))M/\underline{f} = (1 - \Pr\{z' \le \boldsymbol{\xi}\})M/\underline{f} = \Pr\{\boldsymbol{\xi} < z'\}M/\underline{f} \cdots (*)$$

†See (2.2.4(p.13))

where $\Pr\{\boldsymbol{\xi} < z'\} = \int_a^{z'} f(w)dw \geq \int_a^{z'} \underline{f}dw = (z'-a)\underline{f}$. Accordingly, since $p(z')(z'-a) \geq (z'-a)\underline{f}M/\underline{f} = (z'-a)M$, we have $p(z') \geq M > 1$ due to $z'-a > 0$, which is a contradiction. Hence, it must follow that $h^\star < \infty$.

(b) Since $A_1(x) \Rightarrow A_4(x)$ due to Proposition 13.1.1, we can rewritten (5.1.27(p.26)) as

$$\begin{aligned} x^\star &= \inf\{x \mid \inf_{z>a}\{a-h(z)\} < x\} \\ &= \inf_{z>a}\{a-h(z)\} \cdots \mathbf{(1)} \\ &= a - \sup_{a<z} h(z) = a - h^\star < a \quad \text{(due to (a))}, \end{aligned}$$

hence (b) holds.

(c) If $x^\star < x$, then $\inf_{z>a}\{a-h(z)\} < x$ from (1), hence $z(x) > a$ due to $A_4(x) \Rightarrow A_1(x)$. If $x^\star \geq x$, then $\inf_{a<z}\{a-h(z)\} \geq x$ from (1). Now, since $\inf_{a<z}\{a-h(z)\} \geq x \Leftrightarrow z(x) \leq a$ due to a contraposition of $A_4(x) \Leftrightarrow A_1(x)$, hence we obtain $z(x) = a$ due to Lemma 13.1.1(p.89) (c).

(d) First note $T(x) \geq p(z')(z'-x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a+\varepsilon < b$ we have $p(a+\varepsilon) > 0$ from (5.1.29 (1) (p.26)), hence $T(a) \geq p(a+\varepsilon)(a+\varepsilon-a) = p(a+\varepsilon)\varepsilon > 0$. Adding a to the inequality yields $T(a)+a > a$. Thus, we have $T(x)+x \geq T(a)+a > a$ for *any* $x \geq a$ due to Lemma 13.1.4(p.90) (f). Accordingly, if $a^\star \geq a$, then since $T(a^\star)+a^\star \geq T(a)+a > a$, from Lemma 13.1.4(p.90) (a) we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ for any sufficiently small $\varepsilon > 0$ or equivalently $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.26))). Therefore, it must be that $a^\star < a$. ∎

Lemma 13.1.6

(a) *$T(x)+x$ is strictly increasing on $[a^\star, \infty)$.*
(b) *$T(x) = a-x$ on $(-\infty, a^\star]$ and $T(x) > a-x$ on $(a^\star, \infty)$.*
(c) *$T(0) = a$ if $a^\star > 0$ and $T(0) = 0$ if $b < 0$.*
(d) *If $x < y$ and $a^\star < y$, then $T(x)+x < T(y)+y$.* □

● *Proof* (a) From (5.1.25(p.26)) we have

$$T(x)+x = p(z(x))(z(x)-x)+x = p(z(x))z(x)+(1-p(z(x)))x. \cdots \mathbf{(1)}$$

- Let $x^\star < x$. Then $z(x) > a$ from Lemma 13.1.5(p.91) (c), hence $p(z(x)) < 1$ due to (5.1.28 (2) (p.26)), so $1-p(z(x)) > 0$. If $x < x'$, from (1) we have

$$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

i.e., $T(x)+x$ is strictly increasing on $(-\infty, \infty)$, hence understandably so also on $[a^\star, \infty)$.

- Let $x^\star \geq x$. Then $z(x) = a$ from Lemma 13.1.5(p.91) (c), hence $p(z(x)) = 1$ from (5.1.28 (1) (p.26)), so $T(x) = p(z(x))(z(x)-x) = a-x \cdots \mathbf{(2)}$. Suppose $a^\star < x^\star$. Then, since $a^\star < a^\star+2\varepsilon < x^\star$ for an infinitesimal $\varepsilon > 0$, we have $a^\star < a^\star+\varepsilon < x^\star-\varepsilon < x^\star$ or equivalently $x^\star > a^\star+\varepsilon$; accordingly, due to (2) we obtain $T(a^\star+\varepsilon) = a-(a^\star+\varepsilon) \cdots \mathbf{(3)}$. Now, due to (5.1.26(p.26)) we have $T(a^\star+\varepsilon) > a-(a^\star+\varepsilon)$, which contradicts (3). Accordingly, it must be that $x^\star \leq a^\star$. Let $x' > x > a^\star$. Then, since $x^\star < x$, we have $z(x) > a$ Lemma 13.1.5(p.91) (c), hence $p(z(x)) < 1$ due to (5.1.28 (2) (p.26)) or equivalently $1-p(z(x)) > 0$. Thus, from (1) we have

$$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

implying that $T(x)+x$ is *strictly increasing*$\cdots \mathbf{(4)}$ *on* $(a^\star, \infty)$. Now, let us assume $T(x)+x = T(a^\star)+a^\star$ on $a^\star < x$, so $x-a^\star > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x-a^\star > 2\varepsilon$ we have $x > x-\varepsilon > a^\star+\varepsilon > a^\star$, hence $T(x)+x = T(a^\star)+a^\star \leq T(a^\star+\varepsilon)+a^\star+\varepsilon < T(x)+x$ due to the nondecreasing in Lemma 13.1.4(p.90) (f) and the *strict increasingness* shown above, which is a contradiction. Thus, it must be that $T(x)+x \neq T(a^\star)+a^\star$ on $a^\star < x$, so we have $T(x)+x > T(a^\star)+a^\star$ or $T(x)+x < T(a^\star)+a^\star$ on $a^\star < x$; however, the latter is impossible due to the nondecreasing in Lemma 13.1.4(p.90) (f), hence it follows that $T(x)+x > T(a^\star)+a^\star$ on $a^\star < x$. From this fact and (4) it inevitably follows that $T(x)+x$ is strictly increasing on $a^\star \leq x$, i.e., $T(x)+x$ is strictly increasing on not $(a^\star, -\infty)$ but $[a^\star, -\infty)$.

Accordingly, whether $x^\star < x$ or $x^\star \geq x$, it follows that $T(x)+x$ is strictly increasing on $[a^\star, \infty)$.

(b) Due to (5.1.26(p.26)) we have $T(x) > a-x$ for $x > a^\star$, i.e., $T(x) > a-x$ on $(a^\star, \infty)$, hence the latter half is true. Since $T(x) \geq a-x$ on $(-\infty, \infty)$ due to Lemma 13.1.4(p.90) (e), we have $T(x)+x \geq a \cdots \mathbf{(5)}$ on $(-\infty, \infty)$. Suppose $T(a^\star)+a^\star > a$. Then, for an infinitesimal $\varepsilon > 0$ we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ due to Lemma 13.1.4(p.90) (a), i.e., $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.26))). Consequently, we have $T(a^\star)+a^\star = a \cdots \mathbf{(6)}$ or equivalently $T(a^\star) = a-a^\star$. Let $x < a^\star$. Then, from Lemma 13.1.4(p.90) (f) we have $T(x)+x \leq T(a^\star)+a^\star = a$. From the result and (5) we have $T(x)+x = a$, hence $T(x) = a-x$ on $(-\infty, a^\star)$. From this and (6) it follows that $T(x) = a-x$ on $(-\infty, a^\star]$. Hence the former half is true.

(c) Let $a^\star > 0$. Then, since $0 \in (-\infty, a^\star]$, we have $T(0) = a$ from the former half of (b). We have $T(0) = \max_z p(z)z \cdots \mathbf{(7)}$ from (5.1.19(p.26)). Let $b < 0$. Then, if $z \geq b$, we have $p(z)z = 0$ from (5.1.29 (2) (p.26)) and if $z < b\,(< 0)$, then $p(z)z < 0$ from (5.1.29 (1) (p.26)), hence $T(0) = 0$ due to (7).

(d) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $T(x)+x \leq T(a^\star)+a^\star < T(y)+y$ due to Lemma 13.1.4(p.90) (f) and (a), and if $a^\star < x$, then $a^\star \leq x < y$, hence $T(x)+x < T(y)+y$ due to (a). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $T(x)+x < \tilde{T}(y)+y$. ∎

13.2 Analogy Replacement Operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$

13.2.1 Three Facts

Let us focus on the three facts below.

★ Fact 1 First, the following lemma can be obtained.

Lemma 13.2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty)$ ← ← Lemma 13.1.4(p.90) (a)
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$ ← ← Lemma 13.1.4(p.90) (b)
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b]$ ← ← Lemma 13.1.4(p.90) (c)
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ ← ← Lemma 13.1.4(p.90) (f)
(e) $T(x) + x$ *is strictly increasing on* $[a^\star, \infty)$ ← ← Lemma 13.1.6(p.92) (a)
(f) $T(x) = a - x$ *on* $(-\infty, a^\star]$ *and* $T(x) > a - x$ *on* $(a^\star, \infty)$ ← ← Lemma 13.1.6(p.92) (b)
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty)$ ← ← Lemma 13.1.4(p.90) (d)
(h) $T(x) \geq \max\{0, a - x\}$ *on* $(-\infty, \infty)$ ← ← Lemma 13.1.4(p.90) (i)
(i) $T(0) = a$ *if* $a^\star > 0$ *and* $T(0) = 0$ *if* $b < 0$ ← ← Lemma 13.1.6(p.92) (c)
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$ ← ← Lemma 13.1.4(p.90) (g)
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$ ← ← Lemma 13.1.4(p.90) (h)
(l) *If* $x < y$ *and* $a^\star < y$, *then* $T(x) + x < T(y) + y$ ← ← Lemma 13.1.6(p.92) (d)
(m) $\lambda\beta T(\lambda\beta a - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$ ← ← Lemma 13.1.4(p.90) (j)
(n) $a^\star < a$ ← ← Lemma 13.1.5(p.91) (d)

Here we shall pay attention to the fact that replacing a and μ in Lemma 10.1.1(p.55) ($\mathscr{A}\{T_{\mathbb{R}}\}$)(p.55) by $a^\star$ and a respectively yields Lemma 13.2.1(p.93) ($\mathscr{A}\{T_{\mathbb{P}}\}$). Let us represent this replacement by

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} = \{a \to a^\star,\ \mu \to a\}. \tag{13.2.1}$$

In other words, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the former lemma leads to the latter lemma, i.e.,

$$\text{Lemma 13.2.1(p.93) } (\mathscr{A}\{T_{\mathbb{P}}\}) \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{ Lemma 10.1.1(p.55) } (\mathscr{A}\{T_{\mathbb{R}}\})]. \tag{13.2.2}$$

Here let us focus on the following fact. The whole description proving Lemma 10.1.1(p.55) is *quite different* from that proving Lemma 13.2.1(p.93); in other words, no relation exists at all between both descriptions. Nevertheless, what is amazing here is that the whole descriptions of both lemmas are joined together by $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$. In the paper, we call $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ the *analogy replacement operation*.

★ Fact 2 Next, note that replacing μ in $\mathcal{L}(s) = L(\lambda\beta\mu - s)$ (see (5.1.5(p.25))) by a yields $\mathcal{L}(s) = L(\lambda\beta a - s)$ (see (5.1.22(p.26))). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the characteristic vector $(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$ (see (5.1.3(p.25)) - (5.1.6(p.25))) produces $(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$ (see (5.1.20(p.26)) - (5.1.23(p.26))), i.e.,

$$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})]. \tag{13.2.3}$$

★ Fact 3 Finally, note that replacing μ in $V_1 = \beta\mu - s$ (see (6.4.1(p.41))) by a yields $V_1 = \beta a - s$ (see (6.4.5(p.41))). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the system of optimality equations SOE{M:1[ℝ][A]} (see Table 6.4.1(p.41) (I)) leads to SOE{M:1[ℙ][A]} (see Table 6.4.1(p.41) (III)), i.e.,

$$\mathtt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}]. \tag{13.2.4}$$

13.2.2 Prefiguration I

Here let us present a prefiguration through which $\mathscr{A}${M:1[ℙ][A]} can be obtained *only* by replacing a and μ appearing $\mathscr{A}${M:1[ℝ][A]} by $a^\star$ and a respectively.

- First, by $F_{(a,\mu,b)}$ let us denote the distribution function with the lower bound a, the expectation μ, and the upper bound b ($a < \mu < b$). For convenience of reference, below let us copy (13.2.2(p.93)) – (13.2.4(p.93)):

For $F_{(a,\mu,b)}$		For $F_{(a,\mu,b)}$
Lemma 13.2.1(p.93) ($\mathscr{A}\{T_{\mathbb{P}}\}$)	$=\ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	Lemma 10.1.1(p.55) ($\mathscr{A}\{T_{\mathbb{R}}\}$)]
$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$	$=\ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	$(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$]
$(1^\star)^l$ SOE{M:1[ℙ][A]}	$=\ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	SOE{M:1[ℝ][A]} $(1^\star)^r$]
Procedure[ℙ]		Procedure[ℝ]

- Next, closely looking at the flow of the proofs of Tom's 11.2.1(p.61)-11.2.2(p.62), we see that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ was derived *only* from the procedure related to the three terms within the box $(1^\star)^r$ above; here let us denote this procedure by Procedure[$\mathbb{R}$]. Now, for quite the same reason as in Procedure[$\mathbb{R}$] we also see that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ will be derived from the procedure related to the three terms within the box $(1^\star)^l$ above, then let us denote this procedure by Procedure[$\mathbb{P}$]. The flow of the above two procedures can be schematized as below.

$$\begin{array}{rcl}
\text{For } F_{(a,\mu,b)} & & \text{For } F_{(a,\mu,b)} \\
\text{Lemma 13.2.1(p.93) } (\mathscr{A}\{T_{\mathbb{P}}\}) & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \text{Lemma 10.1.1(p.55) } (\mathscr{A}\{T_{\mathbb{R}}\}) \;] \\
(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) \;] \\
(1^\star)^l \quad \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \quad (1^\star)^r \;] \\
\downarrow & & \downarrow \\
\text{Procedure}[\mathbb{P}] & & \text{Procedure}[\mathbb{R}] \\
\downarrow & & \downarrow \\
(2^\star)^l \quad \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} & & \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \quad (2^\star)^r
\end{array}$$

- Then, since we have the relation $(1^\star)^l = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(1^\star)^r]$ due to the three Facts in the preceding section, it can be prefigured that this relation will be inherited also between Procedure[$\mathbb{P}$] and Procedure[$\mathbb{R}$], i.e.,

$$\text{Procedure}[\mathbb{P}] = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]],$$

hence also between $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, i.e.

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{13.2.5}$$

In other words, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ can be obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. From the above discussions we see that the above figure can be rewritten as below.

$$\begin{array}{rcl}
\text{For } F_{(a,\mu,b)} & & \\
\text{Lemma 13.2.1(p.93) } (\mathscr{A}\{T_{\mathbb{P}}\}) & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \text{Lemma 10.1.1(p.55) } (\mathscr{A}\{T_{\mathbb{R}}\}) \;] \\
(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) \;] \\
(1^\star)^l \quad \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \quad (1^\star)^r \;] \\
\downarrow & & \downarrow \\
\text{Procedure}[\mathbb{P}] & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \text{Procedure}[\mathbb{R}] \;] \\
\downarrow & & \downarrow \\
(2^\star)^l \quad \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} & = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[& \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \quad (2^\star)^r \;]
\end{array}$$

Put29.6 12.6For $F_{(a,\mu,b)}$

Here note that the above discussions is not *a proof* but *a prefiguration.*

13.2.3 Prefiguration II

Below is another prefiguration through which the validity of (13.2.5(p.94)) will be confirmed.

- First, let us represent the procedure proving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}_{(a,\mu,b)}$ with $F_{(a,\mu,b)}$ by Procedure$[\mathbb{R}]_{(a,\mu,b)}$ (see Section 11.2(p.61)). Now, since $a^\star < a < b$ due to Lemma 13.2.1(p.93) (n), we can express the F with the lower bound $a^\star$, the expectation a, and the upper bound b as $F_{(a^\star,a,b)}$, hence we can define Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, proving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}_{(a^\star,a,b)}$ with $F_{(a^\star,a,b)}$. Here note that Procedure$[\mathbb{R}]_{(a^\star,a,b)}$ is identical to one resulting from replacing a and μ in Procedure$[\mathbb{R}]_{(a,\mu,b)}$ by $a^\star$ and a respectively, i.e.,

$$\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}].$$

- Then, from the three facts in Section 13.2.1(p.93) we can regard Procedure$[\mathbb{P}]_{(a,\mu,b)}$ as *quite* the same as Procedure$[\mathbb{R}]_{(a^\star,a,b)}$ from the viewpoint of symbolic logic,[†] i.e.,

$$\text{Procedure}[\mathbb{P}]_{(a,\mu,b)} \overset{\text{s-logic}}{=} \text{Procedure}[\mathbb{R}]_{(a^\star,a,b)}$$

hence we have

$$\text{Procedure}[\mathbb{P}]_{(a,\mu,b)} \overset{\text{s-logic}}{=} \text{Procedure}[\mathbb{R}]_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}].$$

[†]A logic is regarded as reducing deduction to the process which transforms the expressions by representing propositions, the concept of logic, and so on with symbols such as $+$, $-$, $>$, $<$, $\vee$, $\wedge$, $\Rightarrow$, and so on (Wikipedia)

- The above relation implies that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)}$ proved by Procedure$[\mathbb{P}]_{(a,\mu,b)}$ becomes identical (in the sense of "symbolic logic") to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}$ proved by Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, i.e.,

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)} \overset{\texttt{s}\cdot\texttt{logic}}{=} \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}.$$

In other words, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)}$ can be given by $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}$ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}$ or equivalently from replacing a and μ in $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}$ by $a^\star$ and a respectively, i.e.,

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)} \overset{\texttt{s}\cdot\texttt{logic}}{=} \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}];$$

13.2.4 Strict Proof

In this section, by dividing the *intuitive* prefiguration in Section 13.2.2(p.93) into several stages, we shall *strictly* prove that (13.2.5(p.94)) holds also *theoretically*.

☐ First, let us note that Procedure[$\mathbb{R}$] deriving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Section 11.2(p.61)) can be restated as below.

- First, by applying $\mathscr{A}\{T_\mathbb{R}\}$ (see Lemma 10.1.1(p.55)) to the characteristic vector $(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})$ consisting of (5.1.3(p.25))-(5.1.6(p.25)), we obtain expressions (10.2.3(p.57)) - (10.2.8(p.57)); let us denote these expressions by $\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$.
- Next, by applying the $\mathscr{A}\{T_\mathbb{R}\}$ to the $\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$, we get the assertion system $\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$ (see Lemmas 10.2.1(p.57) - 10.3.1(p.59)).
- Finally, by applying the system of optimality equations $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Table 6.4.1(p.41) (I)) to $\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$, we get the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see `Tom`'s 11.2.1(p.61) and 11.2.2(p.62)).

The above flow of Procedure[$\mathbb{R}$] can be schematized as below.

$$\begin{aligned}\text{Procedure}[\mathbb{R}] \;=\; \langle\!\langle\, & \mathscr{A}\{T_\mathbb{R}\} \Rightarrow (L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}) \to \{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\},\\ & \mathscr{A}\{T_\mathbb{R}\} \Rightarrow \{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\} \to \mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\} \Rightarrow \mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\} \to \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}\,\rangle\!\rangle\end{aligned}$$

☐ Secondarily, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the above flow leads to

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned}$$

☐ Thirdly, due to (13.2.2(p.93))-(13.2.4(p.93)) we can replace

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}],\quad \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})],\quad \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]$$

in the above flow by

$$\mathscr{A}\{T_\mathbb{P}\},\quad (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}),\quad \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$$

respectively. Accordingly, the above flow can be rewritten as follows.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \underline{\mathscr{A}\{T_\mathbb{P}\}} \Rightarrow \underline{(L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{R})} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \underline{\mathscr{A}\{T_\mathbb{P}\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \underline{\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned} \tag{13.2.6}$$

☐ Fourthly, let us focus our attentions on the items without underline in the above flow, i.e.,

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}) \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]},\\ & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle\end{aligned} \tag{13.2.7}$$

Here $(L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P})$ can be describes as follows.

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x \text{ on } (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \lambda\beta x \text{ on } (a^\star, \infty) & \cdots(2),\end{cases} \tag{13.2.8}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x \quad\text{on}\quad (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \delta x \quad\text{on}\quad (a^\star, \infty) & \cdots(2),\end{cases} \tag{13.2.9}$$

$$K(x)\begin{cases} > -(1-\beta)x - s \text{ on } \quad (-\infty, b) & \cdots(1),\\ = -(1-\beta)x - s \text{ on } \quad [b, \infty) & \cdots(2),\end{cases} \tag{13.2.10}$$

$$K(x)+x \geq \beta x - s \quad \text{on} \quad (-\infty,\infty), \tag{13.2.11}$$

$$K(x)+x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x \text{ on} & (-\infty, a^\star] \quad \cdots (1), \\ \beta x - s \quad\quad \text{on} & [b,\infty) \quad\quad \cdots (2), \end{cases} \tag{13.2.12}$$

$$K(\,x_L\,) = -(1-\beta)\,x_L \;\cdots (1), \quad L(\,x_K\,) = (1-\beta)\,x_K \cdots (2), \tag{13.2.13}$$

• *Direct proof* See (A 2.1(p.294))-(A 2.6(p.294)) . ∎

☐ Fifthly, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the relations $\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ (see Lemmas 10.2.1(p.57)-10.3.1(p.59)) yields the relations $\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] = \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{13.2.14}$$

☐ Finally, noting (13.2.14(p.96)), we can rewrite (13.2.7(p.95)) as below.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; & \langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]}, \\ & \texttt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle \end{aligned} \tag{13.2.15}$$

☐ Now we have

$$\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} = \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{13.2.16}$$

Accordingly (13.2.15(p.96)) can be rewritten as below.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; & \langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \texttt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle. \end{aligned} \tag{13.2.17}$$

☐ Applying (13.2.16(p.96)) to Lemmas 10.2.1(p.57) to 10.3.1(p.59) yields the following lemmas and corollaries:

Lemma 13.2.2 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$L(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$L(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *Let $s=0$. Then $x_L = b$ where $x_L > (\leq)\ x \Leftrightarrow L(x) > (=)\ 0 \Rightarrow L(x) > (\leq)\ 0$.*
(e) *Let $s>0$.*
   1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\ x \Leftrightarrow L(x) > (= (<))\ 0$.*
   2. *$(\lambda\beta a - s)/\lambda\beta \leq (>)\ a^\star \Leftrightarrow x_L = (>)\ (\lambda\beta a - s)/\lambda\beta$.* □

• *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.1(p.57). ∎
• *Direct proof* See the proof of Lemma A 2.2(p.294). ∎

Corollary 13.2.1 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$x_L > (\leq)\ x \Leftrightarrow L(x) > (\leq)\ 0$.*
(b) *$x_L \geq (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$.* □

• *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 10.2.1(p.57). ∎
• *Direct proof* See the proof of Corollary A 2.1(p.294). ∎

Lemma 13.2.3 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$K(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty,\infty)$ if $\beta<1$.*
(e) *$K(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(f) *$K(x)+x$ is strictly increasing on $[a^\star,\infty)$.*
(g) *$K(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(h) *If $x<y$ and $a^\star<y$, then $K(x)+x<K(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta<1$ or $s>0$.*

1. *There uniquely exists* x_K *where* $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.
2. $(\lambda\beta a - s)/\delta \le (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.
3. *Let* $\kappa > (= (<))\ 0$. *Then* $x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.2(p.57). ∎
- *Direct proof* See the proof of Lemma A 2.3(p.294). ∎

Corollary 13.2.2 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) $x_K > (\le)\ x \Leftrightarrow K(x) > (\le)\ 0$.
(b) $x_K \ge (\le)\ x \Rightarrow K(x) \ge (\le)\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 10.2.2(p.58). ∎
- *Direct proof* See the proof of Lemma A 2.2(p.295). ∎

Lemma 13.2.4 ($\mathscr{A}\{L_{\mathbb{P}}/K_{\mathbb{P}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_L = x_K = b$.
(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_L = x_K$.
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $b > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (=))\ 0$.
(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\kappa > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.3(p.58). ∎
- *Direct proof* See the proof of Lemma A 2.4(p.295). ∎

Lemma 13.2.5 ($\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$)

(a) $\mathcal{L}(s)$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(b) *Let* $\lambda\beta a \ge b$.
  1. $x_L \le \lambda\beta a - s$.
  2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_L < \lambda\beta a - s$.

(c) *Let* $\lambda\beta a < b$. *Then, there exists a* $s_{\mathcal{L}} > 0$ *such that if* $s_{\mathcal{L}} > (\le)\ s$, *then* $x_L > (\le)\ \lambda\beta a - s$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.4(p.59). ∎
- *Direct proof* See the proof of Lemma A 2.5(p.296). ∎

Lemma 13.2.6 ($\kappa_{\mathbb{P}}$) *We have*:

(a) $\kappa = \lambda\beta a - s$ *if* $a^\star > 0$ *and* $\kappa = -s$ *if* $b < 0$.
(b) *Let* $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.3.1(p.59). ∎
- *Direct proof* See the proof of Lemma A 2.6(p.296). ∎

□ Since the assertion system $\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1[\mathbb{R}][E]}\}}$ in (13.2.17(p.96)) is derived from $\mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, it can be regarded as an assertion system for the model $\mathsf{M{:}1[\mathbb{P}][E]}$ (see Remark 6.1.1(p.29)), i.e., $\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, hence we have

$$\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][A]}\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1[\mathbb{R}][A]}\}] \quad \text{(the same as (13.2.5(p.94)))}. \tag{13.2.18}$$

Thus (13.2.17(p.96)) can be rewritten as follows.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \langle\!\langle\, & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ & \mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}\,\rangle\!\rangle\end{aligned} \tag{13.2.19}$$

□ The whole of the r.h.s. of (13.2.19(p.97)) can be regarded as the procedure deriving $\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, so let us denote it by Procedure⟨ℙ⟩, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \text{Procedure}[\mathbb{P}]. \tag{13.2.20}$$

Accordingly, finally it follows that we have

$$\begin{aligned}\text{Procedure}[\mathbb{P}] = \langle\!\langle\, & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ & \mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}\,\rangle\!\rangle\end{aligned}$$

13.3 Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

From (13.2.5(p.94)) we immediately obtain the following theorem.

Theorem 13.3.1 (analogy ($\mathbb{R} \to \mathbb{P}$)**)** *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ ...

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{13.3.1}$$

Then, from the comparison of (I) and (III) of Tables 6.4.1(p.41) we also get

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{13.3.2}$$

Moreover, from (12.4.2(p.75)) we obtain the following:

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{13.3.3}$$

$$= (a^\star, a, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{13.3.4}$$

The analogy replacement operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is a *mere* replacement of the two symbols, $a \to a^\star$ and $\mu \to a$. Hence, defining its inverse as

$$\mathcal{A}_{\mathbb{P}\to\mathbb{R}} = \{a^\star \to a,\ a \to \mu\}, \tag{13.3.5}$$

we can immediately obtain the inverse of the above theorem becomes true as follows.

Theorem 13.3.2 (analogy ($\mathbb{P} \leftarrow \mathbb{R}$)**)** *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{13.3.6}$$

In addition, as an inverses of (13.3.2(p.98)) and (13.3.3(p.98)) we immediately obtain

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{13.3.7}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{13.3.8}$$

$$= (a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{13.3.9}$$

13.4 Derivation of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

☐ **Tom 13.4.1** ($\boxdot\,\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

- *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 11.2.1(p.61). ∎
- *Direct proof* See the proof of Tom A 4.3(p.305) . ∎

☐ **Tom 13.4.2** ($\boxdot\,\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \geq b$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c) *Let* $\beta a < b$.

1. *Let* $\beta = 1$.
   - i. *Let* $a - s \leq a^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let* $a - s > a^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ ($s > 0$).
   - i. *Let* $b > 0$ ($\kappa > 0$). *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let* $b = 0$ ($\kappa = 0$).
     1. *Let* $\beta a - s \leq a^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let* $\beta a - s > a^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let* $b < 0$ ($\kappa < 0$).
     1. *Let* $\beta a - s \leq a^\star$ *or* $s_{\mathcal{L}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.62) $\boxed{Ⓢ\blacktriangle \mid ⓞ\ \parallel}$ *is true.* □

- *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 11.2.2(p.62). ∎
- *Direct proof* See the proof of Tom A 4.4(p.306) . ∎

13.5 Strict Definition of Analogy

Below let us provide the strict definition for "analogy" that we have indefinitely used so far.

Definition 13.5.1 (analogy)

(a) By $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathfrak{X}]$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathfrak{X}]$) let us denote the assertion defined by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$) to a given $\mathfrak{X}$.

(b) If $A\{\mathfrak{X}_2\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[A\{\mathfrak{X}_1\}]$ and $A\{\mathfrak{X}_1\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[A\{\mathfrak{X}_2\}]$, then $A\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\}$ is said to be *analogous*, denoted by $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$.

(c) For given two assertion systems $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ which are one-to-one correspondent, if $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$ for any pair $(A\{\mathfrak{X}_1\}, A\{\mathfrak{X}_2\})$ where $A\{\mathfrak{X}_1\} \in \mathscr{A}\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\} \in \mathscr{A}\{\mathfrak{X}_2\}$ are correspondent each other, then $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ are said to be *analogous*, denoted by $\mathscr{A}\{\mathfrak{X}_1\} \bowtie \mathscr{A}\{\mathfrak{X}_2\}$. □

13.6 Analogy-Operation-Free

When no change occurs even if the analogy operation is applied to a given assertion A, the assertion is said to be *free from* the analogy operation, called the *analogy-operation-free assertion.*

Lemma 13.6.1 *Even if the analogy operation is applied to the analogy-operation-free assertion, no change occurs.* □

• *Proof* Evident. ▮

13.7 Optimal Price to Propose

Lemma 13.7.1 ($\mathscr{A}\{$M:1[$\mathbb{P}$][A]$\}$) *The optimal price z_t to propose is nondecreasing in $t > 0$.* □

• *Proof* Obvious from (6.2.34(p.31)), `Tom`'s 13.4.1(p.98) (a) and 13.4.2(p.98) (a), and Lemma 13.1.3(p.89). ▮

13.8 Analogy between SOE{M:1[ℝ][A]} and SOE{M:1[ℙ][A]}

Here note that the analogical relation holds between SOE{M:1[$\mathbb{R}$][A]} and SOE{M:1[$\mathbb{P}$][A]} (see (I) and (III) in Table 6.4.1(p.41)), i.e., SOE{M:1[$\mathbb{P}$][A]} $\bowtie$ SOE{M:1[$\mathbb{R}$][A]}. It is an important point that, due to this very fact, the analogy theorems (Theorems 13.3.1(p.98) and 13.3.2(p.98)) can be derived. It will be known later on that the analogical relation is one of the necessary conditions on which the integrated theory can be successfully constructed.

Chapter 14

Symmetry Theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$)

In this chapter we present the methodology deriving $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism).

14.1 Functions $\check{T}$, $\check{L}$, $\check{K}$, and $\check{\mathcal{L}}$ of Type $\mathbb{P}$

Below let us define ones corresponding to the underlying functions that were defined in Section 5.1.3(p.26). First let us define the T-function of Type $\mathbb{P}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ (see (5.1.19(p.26)) and (5.1.18(p.26))) by

$$\check{T}(x) = \max_z \check{p}(z)(z-x) \cdots (1), \qquad \check{p}(z) = \Pr\{z \leq \hat{\boldsymbol{\xi}}\} \cdots (2). \tag{14.1.1}$$

By $\check{z}(x)$ let us define z maximizing $\check{p}(z)(z-x)$ if it exists, i.e.,

$$\check{T}(x) = \check{p}(\check{z}(x))(\check{z}(x) - x). \tag{14.1.2}$$

Furthermore, let us define

$$\check{L}(x) = \lambda\beta\check{T}(x) - s, \tag{14.1.3}$$

$$\check{K}(x) = \lambda\beta\check{T}(x) - (1-\beta)x - s, \tag{14.1.4}$$

$$\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\check{a} - s), \tag{14.1.5}$$

$$\check{\kappa} = \lambda\beta\check{T}(0) - s. \tag{14.1.6}$$

Then, let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by respectively $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ if they exist. If multiple solutions exist for each of $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$, let us employ the *smallest* as its solution (see Sections 5.2(p.27) (a) and 12.2.1(p.71)). Furthermore, let us define (see Figure 12.1.1(p.70) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{a}^\star = \inf\{x \mid \check{T}(x) > \check{a} - x\} \quad \text{(see (5.1.26(p.26))),} \tag{14.1.7}$$

$$\check{x}^\star = \inf\{x \mid \check{z}(x) > \check{a}\} \quad \text{(see (5.1.27(p.26))).} \tag{14.1.8}$$

By $\check{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ let us define $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (see Table 6.4.1(p.41) (III)) we can obtain

$$\mathtt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \{V_1 = \beta\check{a} - s,\, V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{14.1.9}$$

14.2 Functions $\check{\tilde{T}}$, $\check{\tilde{L}}$, $\check{\tilde{K}}$, and $\check{\tilde{\mathcal{L}}}$ of Type $\mathbb{P}$

Below let us define ones corresponding to the underlying functions that were defined in Section 5.1.4(p.26). First, let us define the $\tilde{T}$-function of $\tilde{\text{T}}$ype $\mathbb{P}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ by (see (5.1.32(p.26))).

$$\check{\tilde{T}}(x) = \min_z \check{\tilde{p}}(z)(z-x) \cdots (1), \qquad \check{\tilde{p}}(z) = \Pr\{\hat{\boldsymbol{\xi}} \leq z\} \cdots (2) \tag{14.2.1}$$

where by $\check{\tilde{z}}(x)$ let us define z minimizing $\check{\tilde{p}}(z)(z-x)$ if it exists, i.e.,

$$\check{\tilde{T}}(x) = \check{\tilde{p}}(\check{\tilde{z}}(x))(\check{\tilde{z}}(x) - x). \tag{14.2.2}$$

Let us define

$$\check{\tilde{L}}(x) = \lambda\beta\check{\tilde{T}}(x) + s, \tag{14.2.3}$$

$$\check{\tilde{K}}(x) = \lambda\beta\check{\tilde{T}}(x) - (1-\beta)x + s, \tag{14.2.4}$$

$$\check{\tilde{\mathcal{L}}}(s) = \check{\tilde{L}}(\lambda\beta\check{b} + s), \tag{14.2.5}$$

$$\check{\tilde{\kappa}} = \lambda\beta\check{\tilde{T}}(0) + s \tag{14.2.6}$$

where let us define the solutions of $\check{\tilde{L}}(x) = 0$, $\check{\tilde{K}}(x) = 0$, and $\check{\tilde{\mathcal{L}}}(x) = 0$ by respectively $x_{\check{\tilde{L}}}$, $x_{\check{\tilde{K}}}$, and $s_{\check{\tilde{\mathcal{L}}}}$. If multiple solutions exist for each of $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$, we shall employ the *largest* as its solution (see Sections 5.2(p.27) (b)). Furthermore let us define (see Figure 12.1.1(p.70) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{b}^\star = \sup\{x \mid \check{\tilde{T}}(x) < \check{b} - x\} \quad \text{(see (5.1.39(p.27)))}, \tag{14.2.7}$$

$$\check{\tilde{x}}^\star = \sup\{x \mid \check{z}(x) < \check{b}\} \quad \text{(see (5.1.40(p.27)))}. \tag{14.2.8}$$

By $\check{\tilde{\mathsf{M}}}$:1[$\mathbb{P}$][A] let us define $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for SOE{$\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]} (see Table 6.4.1(p.41) (IV)) we can obtain

$$\texttt{SOE}\{\check{\tilde{\mathsf{M}}}{:}1[\mathbb{P}][\mathtt{A}]\} = \{V_1 = \beta\check{b} + s,\ V_t = \min\{\check{\tilde{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{14.2.9}$$

14.3 List of Underline Functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

The table below is the list of the four kinds of underline functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$ (see Table 12.2.1(p.71)).

Table 14.3.1: List of the underlying functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

Type $\mathbb{P}$	$\check{\text{T}}$ype $\mathbb{P}$
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \max_z p(z)(z - x)$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta a - s)$ See Section 5.1.3(p.26)	$\check{T}(x) = \max_z \check{p}(z)(z - x)$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{a} - s)$ See Section 14.1
$\tilde{T}(x) = \min_z \tilde{p}(z)(z - x)$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta b + s)$ See Section 5.1.4(p.26)	$\check{\tilde{T}}(x) = \min_z \check{\tilde{p}}(z)(z - x)$ $\check{\tilde{L}}(x) = \beta\check{\tilde{T}}(x) + s$ $\check{\tilde{K}}(x) = \beta\check{\tilde{T}}(x) - (1-\beta)x + s$ $\check{\tilde{\mathcal{L}}}(x) = \check{\tilde{L}}(\beta\check{b} + s)$ See Section 14.2

14.4 Two Kinds of Replacements

14.4.1 Correspondence Replacement

Lemma 14.4.1 ($\mathcal{C}_{\mathbb{P}}$) *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to the* F. *Then*:

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}$, $\hat{a}^\star = \check{b}^\star$, $\hat{b} = \check{a}$.
(c) $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.
(d) $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.
(e) $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.
(g) $\hat{x}_L = x_{\check{\tilde{L}}}$.
(h) $\hat{x}_K = x_{\check{\tilde{K}}}$.
(i) $s_{\mathcal{L}} = s_{\check{\tilde{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \check{\tilde{\kappa}}$. □

• *Proof* (a) The same as (12.1.9(p.70)).

(The first and third equalities of (b)) The same as the first and third equalities of (12.1.10(p.70)). The second equality will be proven after the proof of (c).

(c) From (5.1.18(p.26)) and (14.2.1 (2) (p.101)), we obtain

$$p(z) = \Pr\{-\hat{z} \le -\hat{\boldsymbol{\xi}}\} = \Pr\{\hat{\boldsymbol{\xi}} \le \hat{z}\} = \check{\tilde{p}}(\hat{z}), \tag{14.4.1}$$

hence from (5.1.19(p.26)) we have $T(x) = \max_z \check{\tilde{p}}(\hat{z})(-\hat{z} + \hat{x}) = -\min_z \check{\tilde{p}}(\hat{z})(\hat{z} - \hat{x})$. Now, in general "$\min_z = \min_{-\infty<z<\infty} = \min_{-\infty<-\hat{z}<\infty} = \min_{\infty>\hat{z}>-\infty} = \min_{-\infty<\hat{z}<\infty} = \min_{\hat{z}}$", hence we have $T(x) = -\min_{\hat{z}} \check{\tilde{p}}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $T(x) = -\min_z \check{\tilde{p}}(z)(z - \hat{x})$. Accordingly, since $T(x) = -\check{\tilde{T}}(\hat{x})$ from (14.2.1 (1) (p.101)), we obtain $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.

(The second equality of (b)) From (5.1.26(p.26)) we have $a^\star = \inf\{-\hat{x} \mid -\hat{T}(x) > -\hat{a} + \hat{x}\} = -\sup\{\hat{x} \mid \hat{T}(x) < \hat{a} - \hat{x}\} = -\sup\{\hat{x} \mid \check{\tilde{T}}(\hat{x}) < \check{b} - \hat{x}\}$ due to (c) and (b). Without loss of generality, this can be rewritten as $a^\star = -\sup\{x \mid \check{\tilde{T}}(x) < \check{b} - x\}$, hence $a^\star = -\check{b}^\star$ due to (14.2.7(p.102)), so that $\hat{a}^\star = \check{b}^\star$.

(d) From (5.1.20(p.26)) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\tilde{T}}(\hat{x}) - s = -\check{\tilde{L}}(\hat{x})$ from (14.2.3(p.101)), hence $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.

(e) From (5.1.21(p.26)) and (c) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\tilde{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\tilde{K}}(\hat{x})$ from (14.2.4(p.101)), hence $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.

(f) From (5.1.22(p.26)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta a - s) = -\check{\tilde{L}}(\widehat{\lambda\beta a - s})$ due to (d). Then, since $\mathcal{L}(s) = -\check{\tilde{L}}(-\lambda\beta a + s) = -\check{\tilde{L}}(\lambda\beta\hat{a} + s) = -\check{\tilde{L}}(\lambda\beta\check{b} + s)$ due to (b), we have $\mathcal{L}(s) = -\check{\tilde{\mathcal{L}}}(s)$ from (14.2.5(p.101)), hence $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $-\hat{L}(x_L) = 0$, i.e., $\hat{L}(x_L) = 0$, hence $\check{\tilde{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\tilde{L}}(x) = 0$ has the solution $x_{\check{\tilde{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $-\hat{K}(x_K) = 0$, i.e., $\hat{K}(x_K) = 0$, hence $\check{\tilde{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\tilde{K}}(x) = 0$ has the solution $x_{\check{\tilde{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $-\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, i.e., $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, hence $\check{\tilde{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\tilde{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\tilde{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.23(p.26)) we have $\kappa = -\lambda\beta\hat{T}(0) - s = -\lambda\beta\check{\tilde{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\tilde{T}}(0) - s = -\check{\tilde{\kappa}}$ from (14.2.6(p.101)), thus $\hat{\kappa} = \check{\tilde{\kappa}}$. ∎

Definition 14.4.1 (correspondent replacement operation $\mathcal{C}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in Lemma 14.4.1(p.102) by its right-hand side the *correspondence replacement operation* $\mathcal{C}_{\mathbb{P}}$. □

Lemma 14.4.2 ($\tilde{\mathcal{C}}_{\mathbb{P}}$) *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to the* F. *Then*:

(a) $f(\xi) = \check{f}(\hat{\xi})$.

(b) $\hat{a} = \check{b}, \quad \hat{b}^\star = \check{a}^\star, \quad \hat{b} = \check{a}$.

(c) $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(d) $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) $\hat{x}_{\tilde{L}} = \check{x}_L$.

(h) $\hat{x}_{\tilde{K}} = \check{x}_K$.

(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.

(j) $\hat{\tilde{\kappa}} = \check{\kappa}$. □

• *Proof* (27.2.3) The same as (12.1.9(p.70)).

(The first and third equalities of (b)) The same as the first and third equation of (12.1.10(p.70)). The second equality will be proven after the proof of (c).

(c) From (5.1.31(p.26)) and (14.1.1 (2) (p.101)) we obtain

$$\tilde{p}(z) = \Pr\{-\hat{\boldsymbol{\xi}} \le -\hat{z}\} = \Pr\{\hat{\boldsymbol{\xi}} \ge \hat{z}\} = \Pr\{\hat{z} \le \hat{\boldsymbol{\xi}}\} = \check{p}(\hat{z}), \tag{14.4.2}$$

hence from (5.1.32(p.26)) we have $\tilde{T}(x) = \min_z \check{p}(\hat{z})(-\hat{z} + \hat{x}) = -\max_z \check{p}(\hat{z})(\hat{z} - \hat{x})$. Now, in general "$\max_z = \max_{-\infty<z<\infty} = \max_{-\infty<-\hat{z}<\infty} = \max_{\infty>\hat{z}>-\infty} = \max_{-\infty<\hat{z}<\infty} = \max_{\hat{z}}$", hence we have $\tilde{T}(x) = -\max_{\hat{z}} \check{p}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $\tilde{T}(x) = -\max_z \check{p}(z)(z - \hat{x})$. Accordingly, since $\tilde{T}(x) = -\check{T}(\hat{x})$ from (14.1.1 (1) (p.101)), we obtain $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(The second equality of (b)) From (5.1.39(p.27)) we have $b^\star = \sup\{-\hat{x} \mid -\hat{\tilde{T}}(x) < -\hat{b} + \hat{x}\} = -\inf\{\hat{x} \mid \hat{\tilde{T}}(x) > \hat{b} - \hat{x}\} = -\inf\{\hat{x} \mid \check{T}(\hat{x}) > \check{a} - \hat{x}\}$ due to (c) and (b). Without loss of generality, this can be rewritten as $b^\star = -\inf\{x \mid \check{T}(x) > \check{a} - x\}$ we have $b^\star = -\check{a}^\star$ due to (14.1.7(p.101)) or equivalently $-b^\star = \check{a}^\star$, hence $\hat{b}^\star = \check{a}^\star$.

(d) From (5.1.33(p.27)) and (c) we have $\tilde{L}(x) = -\lambda\beta\hat{\tilde{T}}(x) + s = -\lambda\beta\check{T}(\hat{x}) + s = -\check{L}(\hat{x})$ from (14.1.3(p.101)), hence $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) From (5.1.34(p.27)) and (c) we have $\tilde{K}(x) = -\lambda\beta\hat{\tilde{T}}(x) + (1-\beta)\hat{x} + s = -\lambda\beta\check{T}(\hat{x}) + (1-\beta)\hat{x} + s = -\check{K}(\hat{x})$ from (14.1.4(p.101)), hence $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) From (5.1.35(p.27)) we have $\tilde{\mathcal{L}}(s) = -\hat{\tilde{L}}(\lambda\beta b + s)$, hence from (d) we obtain $\tilde{\mathcal{L}}(s) = -\check{L}(\widehat{\lambda\beta b + s}) = -\check{L}(-\lambda\beta b - s) = -\check{L}(\lambda\beta\hat{b} - s) = -\check{L}(\lambda\beta\check{a} - s)$ due to (b). Accordingly, from (14.1.5(p.101)) we obtain $\tilde{\mathcal{L}}(s) = -\check{\mathcal{L}}(s)$, hence $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $-\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, i.e., $\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, hence $\check{L}(\hat{x}_{\tilde{L}}) = 0$ from (d), implying that $\check{L}(x) = 0$ has the solution $x_{\check{L}} = \hat{x}_{\tilde{L}}$ by definition.

(h) Since $\check{K}(x_{\tilde{K}}) = 0$ by definition, we have $-\hat{\check{K}}(x_{\tilde{K}}) = 0$, i.e., $\hat{\check{K}}(x_{\tilde{K}}) = 0$, hence $\check{K}(\hat{x}_{\tilde{K}}) = 0$ from (e), implying that $\check{K}(x) = 0$ has the solution $x_{\check{K}} = \hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $-\hat{\check{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, i.e., $\hat{\check{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, hence $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), implying that $\check{\mathcal{L}}(s) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.36(p.27)) we have $\tilde{\kappa} = -\lambda\beta\hat{\check{T}}(0) + s$, leading to $\tilde{\kappa} = -\lambda\beta\check{T}(\hat{0}) + s$ from (c), hence $\tilde{\kappa} = -\lambda\beta\check{T}(0) + s = -\check{\kappa}$ from (14.1.6(p.101)), thus $\hat{\tilde{\kappa}} = \check{\kappa}$. ∎

Remark 14.4.1 The equality $\hat{\mu} = \check{\mu}$ in Lemmas 12.3.1(p.72) (b) changes into respectively $\hat{a}^\star = \check{b}^\star$ in Lemma 14.4.1(p.102) (b) and the equality $\hat{\mu} = \check{\mu}$ in (12.1.10(p.70)) changes into $\hat{b}^\star = \check{a}^\star$ in Lemma 14.4.2(p.103) (b). □

The definition below is the same as Def. 12.3.3(p.73).

Definition 14.4.2 (reversible element and non-reversible element) It should be noted that the left side of each of the equalities in Lemmas 14.4.1(p.102) (i) and 14.4.2(p.103) (i) is respectively $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ without the hat symbol " ˆ "; in other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reverse. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reversible element* and to each of all the other elements as the *reversible element*. □

Definition 14.4.3 (correspondent replacement operation $\tilde{\mathcal{C}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in the above lemma by its right-hand side the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{P}}$. □

14.4.2 Identity Replacement

Lemma 14.4.3 ($\mathcal{I}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $\check{F} \equiv F \cdots$ [1*].*† *Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots$ [2*] *and* $\check{f}(\xi) = f(\xi) \cdots$ [3*] *for any* ξ,
(b) $\check{a} = a,\ \check{b}^\star = b^\star,\ \check{b} = b$,
(c) $\check{\tilde{T}}(x) = \tilde{T}(x)$,
(d) $\check{\tilde{L}}(x) = \tilde{L}(x)$,
(e) $\check{\tilde{K}}(x) = \tilde{K}(x)$,
(f) $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$,
(g) $x_{\check{\tilde{L}}} = x_{\tilde{L}}$,
(h) $x_{\check{\tilde{K}}} = x_{\tilde{K}}$,
(i) $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$,
(j) $\check{\tilde{\kappa}} = \tilde{\kappa}$. □

● *Proof* (a) Clear from [1*].

(the first and last equalities of (b)) Immediate from (a). The second equality will be proven after the proof of (c).

(c) From (14.2.1 (2) (p.101)) we have $\check{\tilde{p}}(z) = \Pr\{\check{\boldsymbol{\xi}} \le z\} = \int_{-\infty}^{z} \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{\tilde{p}}(z) = \int_{-\infty}^{z} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} \le z\} = \tilde{p}(z)$ from (5.1.31(p.26)). Hence, we have that $\check{\tilde{T}}(x)$ given by (14.2.1 (1) (p.101)) becomes $\check{\tilde{T}}(x) = \min_z \tilde{p}(z)(z-x)$, which is identical to $\tilde{T}(x)$ given by (5.1.32(p.26)), i.e., $\check{\tilde{T}}(x) = \tilde{T}(x)$ for any x.

(the second equality of (b)) From (14.2.7(p.102)) and (c) we have $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < \check{b} - x\}$, hence from (b) we get $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < b - x\} = b^\star$ due to (5.1.39(p.27)).

(d,e) Noting (c), from (14.2.3(p.101)) and (5.1.33(p.27)) we have $\check{\tilde{L}}(x) = \tilde{L}(x)$. Similarly, from (14.2.4(p.101)) and (5.1.34(p.27)) we have $\check{\tilde{K}}(x) = \tilde{K}(x)$.

(f) (14.2.5(p.101)) becomes $\check{\tilde{\mathcal{L}}}(s) = \check{\tilde{L}}(\lambda\beta b + s)$ due to (b). This can be rewritten as $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta b + s)$ due to (d), which is the same as $\tilde{\mathcal{L}}(s)$ given by (5.1.35(p.27)), i.e., $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$.

(g-i) Evident from (d-f).

(j) (14.2.6(p.101)) becomes $\check{\tilde{\kappa}} = \lambda\beta\tilde{T}(0) + s$ due to (c), which is the same as $\tilde{\kappa}$ given by (5.1.36(p.27)). ∎

Definition 14.4.4 (identity replacement operation $\mathcal{I}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma by its right-hand the *identity replacement operation* $\mathcal{I}_{\mathbb{P}}$. □

Lemma 14.4.4 ($\tilde{\mathcal{I}}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $F \equiv \check{F} \cdots$ [1*]. Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots$ [2*] *and* $\check{f}(\xi) = f(\xi) \cdots$ [3*] *for any* ξ,
(b) $\check{a} = a,\ \check{a}^\star = a^\star,\ \check{b} = b$,
(c) $\check{T}(x) = T(x)$,
(d) $\check{L}(x) = L(x)$,

†See Lemma 12.1.1(p.70) (b)

(e) $\check{K}(x) = K(x)$,
(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$,
(g) $x_{\check{L}} = x_L$,
(h) $x_{\check{K}} = x_K$,
(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$,
(j) $\check{\kappa} = \kappa$. □

• *Proof* (a) Clear from [1*].

(The first and last equalities of b)) Immediate form (a). The second equality will be proven after the proof of (c).

(c) From (14.1.1 (2) (p.101)) we have $\check{p}(z) = \Pr\{z \le \hat{\boldsymbol{\xi}}\} = \int_z^\infty \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{p}(z) = \int_z^\infty f(\xi)d\xi = \Pr\{z \le \boldsymbol{\xi}\} = p(z)$ from (5.1.18(p.26)). Hence, we have that $\check{T}(x)$ given by (14.1.1 (1) (p.101)) becomes $\check{T}(x) = \max_z p(z)(z-x)$, which is identical to $T(x)$ given by (5.1.19(p.26)), i.e., $\check{T}(x) = T(x)$ for any x.

(the second equality of (b)) From (14.1.7 (1) (p.101)) and (c) we have $\check{a}^\star = \inf\{x \mid T(x) > \check{a} - x\}$, hence from (b) we get $\check{a}^\star = \inf\{x \mid T(x) > a - x\} = a^\star$ due to (5.1.26(p.26)). Thus, the second equality of (b) is true.

(d,e) Noting (c), from (14.1.3(p.101)) and (5.1.20(p.26)) we have $\check{L}(x) = L(x)$. Similarly, from (14.1.4(p.101)) and (5.1.21(p.26)) we have $\check{K}(x) = K(x)$.

(f) (14.1.5(p.101)) becomes $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta a - s)$ due to (b). This can be rewritten as $\check{\mathcal{L}}(s) = L(\lambda\beta a - s)$ due to (d), which is the same as $\mathcal{L}(s)$ given by (5.1.22(p.26)), i.e., $\check{\mathcal{L}}(s) = \mathcal{L}(s)$.

(g-i) Evident from (d-f).

(j) (14.1.6(p.101)) becomes $\check{\kappa} = \lambda\beta T(0) - s$ due to (c), which is the same as κ given by (5.1.23(p.26)). ∎

Definition 14.4.5 (Identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma by its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{P}}$. □

14.5 Scenario of Type $\mathbb{P}$

14.5.1 Scenario[$\mathbb{P}$]

This section provides the scenario deriving $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism), denoted by Scenario[$\mathbb{P}$].

■ Before moving on, here let us carry out a review of Scenario[$\mathbb{R}$]. For convenience of reference, below let us copy the transformation process of the attribute vectors (see (12.5.28(p.77))) in Scenario[$\mathbb{R}$].

$$
\begin{array}{llll}
\text{Step 1}[\mathbb{R}]: & & \boldsymbol{\theta}(\,a,\ \mu,\ b,\ x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) & = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 2}[\mathbb{R}]: & \mathcal{R}\rightarrow & \boldsymbol{\theta}(\,\hat{a}, \hat{\mu}, \hat{b},\ \hat{x}_L, \hat{x}_K, s_{\mathcal{L}}, \hat{\kappa}, \hat{T}, \hat{L}, \hat{K}, \hat{\mathcal{L}}, \hat{V}_t\,) & \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 3}[\mathbb{R}]: \text{ Lemma 12.3.1(p.72)} & \mathcal{C}_{\mathbb{R}}\rightarrow & \boldsymbol{\theta}(\,\check{b}, \check{\mu}, \check{a},\ x_{\check{\tilde{L}}}, x_{\check{\tilde{K}}}, s_{\check{\tilde{\mathcal{L}}}}, \check{\tilde{\kappa}}, \check{\tilde{T}}, \check{\tilde{L}}, \check{\tilde{K}}, \check{\tilde{\mathcal{L}}}, \hat{V}_t\,) & \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 4}[\mathbb{R}]: \text{ Lemma 12.3.3(p.73)} & \mathcal{I}_{\mathbb{R}}\rightarrow & \boldsymbol{\theta}(\,b,\ \mu,\ a,\ x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t\,) & = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})
\end{array}
\tag{14.5.1}
$$

■ From the above flow of the attribute vectors, we see that Scenario[$\mathbb{P}$] is the same as Scenario[$\mathbb{R}$] only except that

- a and μ in $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})$ is replaced $a^\star$ and a in $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ (see (13.2.1(p.93))) and
- Lemmas 12.3.1(p.72) and 12.3.3(p.73) are changed into Lemmas 14.4.1(p.102) and 14.4.3(p.104) respectively.

Therefore the above flow of attribute vectors can be rewritten as follows.

$$
\begin{array}{llll}
\text{Step 1}[\mathbb{R}]: & & \boldsymbol{\theta}(\,a,\ \mu,\ b,\ x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) & \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\hline
\text{Step 1}[\mathbb{P}]: \ \downarrow \text{Scenario}[\mathbb{P}] & & \boldsymbol{\theta}(\,a^\star, a,\ b,\ x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) & = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 2}[\mathbb{P}]: & \mathcal{R}\rightarrow & \boldsymbol{\theta}(\,\hat{a}^\star, \hat{a},\ \hat{b},\ \hat{x}_L, \hat{x}_K, s_{\mathcal{L}}, \hat{\kappa}, \hat{T}, \hat{L}, \hat{K}, \hat{\mathcal{L}}, \hat{V}_t\,) & \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 3}[\mathbb{P}]: \text{ Lemma 14.4.1(p.102)} & \mathcal{C}_{\mathbb{P}}\rightarrow & \boldsymbol{\theta}(\,\check{b}^\star, \check{b},\ \check{a},\ x_{\check{\tilde{L}}}, x_{\check{\tilde{K}}}, s_{\check{\tilde{\mathcal{L}}}}, \check{\tilde{\kappa}}, \check{\tilde{T}}, \check{\tilde{L}}, \check{\tilde{K}}, \check{\tilde{\mathcal{L}}}, \hat{V}_t\,) & \\
 & & \quad\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow & \\
\text{Step 4}[\mathbb{P}]: \text{ Lemma 14.4.3(p.104)} & \mathcal{I}_{\mathbb{P}}\rightarrow & \boldsymbol{\theta}(\,b^\star, b,\ a,\ x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t\,) & = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}
\end{array}
\tag{14.5.2}
$$

Accordingly, it follows that the operation which transforms $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})$ can be eventually reduced to the operation below:

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \stackrel{\text{def}}{=} \left\{\begin{matrix} a^\star, & a, & b, & x_{L_\mathbb{P}}, & x_{K_\mathbb{P}}, & s_{\mathcal{L}_\mathbb{P}}, & \kappa_\mathbb{P}, & T_\mathbb{P}, & L_\mathbb{P}, & K_\mathbb{P}, & \mathcal{L}_\mathbb{P}, & V_t \\ \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr \\ b^\star, & b, & a, & x_{\tilde{L}_\mathbb{P}}, & x_{\tilde{K}_\mathbb{P}}, & s_{\tilde{\mathcal{L}}_\mathbb{P}}, & \tilde{\kappa}_\mathbb{P}, & \tilde{T}_\mathbb{P}, & \tilde{L}_\mathbb{P}, & \tilde{K}_\mathbb{P}, & \tilde{\mathcal{L}}_\mathbb{P}, & V_t \end{matrix}\right\}.^\dagger \tag{14.5.3}$$

■ Thus, one sees that in Scenario[$\mathbb{P}$] it suffices to change $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_\mathbb{R}\mathcal{C}_\mathbb{R}\mathcal{R}$(see (12.5.30(p.77))) into $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \mathcal{I}_\mathbb{P}\mathcal{C}_\mathbb{R}\mathcal{P}$ above.
■ Moreover, from (III) and (IV) of Table 6.4.1(p.41) it can be easily seen that

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{14.5.4}$$

From all the above discussions it follows that for quite the same reason as that for which Lemma 12.5.1(p.78) was derived we can immediately obtain Lemma 14.5.1(p.106) below.

Lemma 14.5.1 *Let $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. Then $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ where*

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{14.5.5}$$

Finally, also for almost the same reason as that for which Theorem 12.5.1(p.80) is derived from Lemma 12.5.1(p.78) we have Theorem 14.5.1(p.106) below.

Theorem 14.5.1 (symmetry theorem ($\mathbb{P}\to\tilde{\mathbb{P}}$)) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{14.5.6}$$

In addition, we have (see (12.5.54(p.80)))

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{14.5.7}$$
$$= (b^\star, b, a,\ x_{\tilde{L}},\ s_{\tilde{\mathcal{L}}},\ x_{\tilde{K}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{14.5.8}$$

14.5.2 $\tilde{\text{S}}$cenario[$\mathbb{P}$]

This section provides the inverse of Scenario[$\mathbb{R}$], i.e., the scenario deriving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism), denoted by $\tilde{\text{S}}$cenario[$\mathbb{P}$].

■ Before moving on, here let us carry out a review of $\tilde{\text{S}}$cenario[$\mathbb{R}$]. For convenience of reference, below let us copy the transformation process (see (12.8.20(p.86))) of the attribute vectors in Scenario[$\mathbb{R}$].

$$\begin{array}{llrl}
\text{Step 1}[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(\tilde{b}, \mu, a,\ x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 2}[\tilde{\mathbb{R}}]: & & \mathcal{R}\to & \boldsymbol{\theta}(\hat{b}, \hat{\mu}, \hat{a},\ \hat{\tilde{x}}_L, \hat{\tilde{x}}_K, s_{\hat{\tilde{\mathcal{L}}}}, \hat{\tilde{\kappa}}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 3}[\tilde{\mathbb{R}}]: & \text{Lemma 14.4.1(p.102)} & \tilde{\mathcal{C}}_\mathbb{R}\to & \boldsymbol{\theta}(\check{a}, \check{\mu}, \check{b},\ x_{\check{L}}, x_{\check{K}}, s_{\check{\mathcal{L}}}, \check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 4}[\tilde{\mathbb{R}}]: & \text{Lemma 14.4.3(p.104)} & \tilde{\mathcal{I}}_\mathbb{R}\to & \boldsymbol{\theta}(a, \mu, b,\ x_L, x_K, s_\mathcal{L}, \kappa, T, L, K, \mathcal{L}, V_t) = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})
\end{array} \tag{14.5.9}$$

■ From the above we see that $\tilde{\text{S}}$cenario[$\mathbb{P}$] is the same as $\tilde{\text{S}}$cenario[$\mathbb{R}$] only except that

- b and μ in $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})$ is replaced $b^\star$ and b in $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ and
- Lemmas 14.4.1(p.102) and 14.4.3(p.104) used there are changed into Lemmas 14.4.2(p.103) and 14.4.4(p.104) respectively.

Therefore the above flow of attribute vectors can be rewritten as follows.

$$\begin{array}{llrl}
\text{Step 1}[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(b, \mu, b,\ x_L, x_K, s_\mathcal{L}, \kappa, T, L, K, \mathcal{L}, V_t) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 1}[\tilde{\mathbb{P}}] & & & \boldsymbol{\theta}(b^\star, b, a,\ x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 2}[\tilde{\mathbb{P}}] & & \mathcal{R}\to & \boldsymbol{\theta}(\hat{b}^\star, \hat{b}, \hat{a},\ \hat{\tilde{x}}_L, \hat{\tilde{x}}_K, s_{\hat{\tilde{\mathcal{L}}}}, \hat{\tilde{\kappa}}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 3}[\tilde{\mathbb{P}}] & \text{Lemma 14.4.2(p.103)} & \tilde{\mathcal{C}}_\mathbb{P}\to & \boldsymbol{\theta}(\check{a}^\star, \check{a}, \check{b},\ x_{\check{L}}, x_{\check{K}}, s_{\check{\mathcal{L}}}, \check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t) \\
& & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\text{Step 4}[\tilde{\mathbb{P}}] & \text{Lemma 14.4.4(p.104)} & \tilde{\mathcal{I}}_\mathbb{P}\to & \boldsymbol{\theta}(a^\star, a, b,\ x_L, x_K, s_\mathcal{L}, \kappa, T, L, K, \mathcal{L}, V_t) = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}
\end{array} \tag{14.5.10}$$

[†] Compare the dash box ⬚ with that in (12.5.29(p.77)).

Accordingly it follows that the operation which transforms $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ can be eventually reduced to the operation below:

$$\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \overset{\text{def}}{=} \tilde{\mathcal{I}}_{\mathbb{P}}\tilde{\mathcal{C}}_{\mathbb{P}}\mathcal{R} = \left\{\begin{matrix} b^\star, & b, & a, & x_{\tilde{L}_{\mathbb{P}}}, & x_{\tilde{K}_{\mathbb{P}}}, & s_{\tilde{\mathcal{L}}_{\mathbb{P}}}, & \tilde{\kappa}_{\mathbb{P}}, & \tilde{T}_{\mathbb{P}}, & \tilde{L}_{\mathbb{P}}, & \tilde{K}_{\mathbb{P}}, & \tilde{\mathcal{L}}_{\mathbb{P}}, & V_t \\ \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr \\ a^\star, & a, & b, & x_{L_{\mathbb{P}}}, & x_{K_{\mathbb{P}}}, & s_{\mathcal{L}_{\mathbb{P}}}, & \kappa_{\mathbb{P}}, & T_{\mathbb{P}}, & L_{\mathbb{P}}, & K_{\mathbb{P}}, & \mathcal{L}_{\mathbb{P}}, & V_t \end{matrix}\right\}. \tag{14.5.11}$$

- ■ Thus, one sees that in $\tilde{\text{S}}$cenario[$\mathbb{P}$] it suffices to change $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}$(see (14.5.3(p.106))) into $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{P}$ above.
- ■ Moreover, from (III) and (IV) of Table 6.4.1(p.41) it can be easily seen that

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{14.5.12}$$

From all the above discussions it follows that for quite the same reason as that for which Lemma 12.8.1(p.87) was derived we can immediately obtain Lemma 14.5.2(p.107) below.

Lemma 14.5.2 *Let $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. Then $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ where*

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{14.5.13}$$

Finally, for the same reason as the one for which Theorem 12.8.1(p.87) is derived from Lemma 12.8.1(p.87) we have Theorem 14.5.2(p.107) below.

Theorem 14.5.2 (symmetry theorem ($\tilde{\mathbb{P}} \to \mathbb{P}$)) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{14.5.14}$$

From (12.8.32(p.87)) we have

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) \overset{\text{def}}{=} \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{14.5.15}$$

$$= (a^\star, a, b, x_L, s_{\mathcal{L}}, x_K, \kappa, T, L, K, \mathcal{L}, V_t). \tag{14.5.16}$$

14.6 Derivation of $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$

For the same reason as in Section 26.2.2(p.274) we see that applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathscr{A}\{T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$ given by Lemmas 13.2.1(p.93)–13.2.6(p.97) yields $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$.

Lemma 14.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) *$\tilde{T}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a, -\infty)$.*
(d) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(e) *$\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.*
(f) *$\tilde{T}(x) = b - x$ on $[b^\star, \infty)$ and $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$.*
(g) *$\tilde{T}(x) < 0$ on (a, ∞) and $T(x) = 0$ on $(-\infty, a]$.*
(h) *$\tilde{T}(x) \le \min\{0, b - x\}$ on $(-\infty, \infty)$.*
(i) *$\tilde{T}(0) = b$ if $b^\star \le 0$ and $\tilde{T}(0) = 0$ if $a > 0$.*
(j) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(k) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(l) *If $x > y$ and $b^\star > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$.*
(m) *$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing in s and is strictly increasing in s if $\lambda\beta < 1$.*
(n) *$b^\star > b$.*

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.1(p.93). ∎
- *Direct proof* See the proof of Lemma A 3.7(p.300) . ∎

Applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (13.2.8(p.95)) - (13.2.13(p.96)), we obtain the relations below:

$$\tilde{L}(x)\begin{cases} = \lambda\beta b + s - \lambda\beta x \text{ on } [b^\star, -\infty) & \cdots(1), \\ < \lambda\beta b + s - \lambda\beta x \text{ on } (-\infty, b^\star) & \cdots(2), \end{cases} \tag{14.6.1}$$

$$\tilde{K}(x)\begin{cases} = \lambda\beta b + s - \delta x \text{ on } [b^\star, \infty) & \cdots(1), \\ < \lambda\beta b + s - \delta x \text{ on } (-\infty, b^\star) & \cdots(2). \end{cases} \tag{14.6.2}$$

$$\tilde{K}(x)\begin{cases} < -(1-\beta)x + s \text{ on } (a, \infty) & \cdots(1), \\ = -(1-\beta)x + s \text{ on } (-\infty, a] & \cdots(2), \end{cases} \tag{14.6.3}$$

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{14.6.4}$$

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta b + s + (1-\lambda)\beta x \text{ on } [b^\star, \infty) & \cdots(1), \\ \beta x + s \quad \text{on } (-\infty, a] & \cdots(2). \end{cases} \tag{14.6.5}$$

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\,x_{\tilde{L}} \cdots(1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\,x_{\tilde{K}} \cdots(2). \tag{14.6.6}$$

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (13.2.8(p.95))-(13.2.13(p.96)). ∎
- *Direct proof* See the proof of (A 3.1(p.300)) - (A 3.6(p.301)). ∎

Lemma 14.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $\tilde{L}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{L}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{L}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) *Let* $s = 0$. *Then* $x_{\tilde{L}} = a$ *where* $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.
(e) *Let* $s > 0$.
  1. $x_{\tilde{L}}$ *uniquely exists with* $x_{\tilde{L}} > a$ *where* $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.
  2. $(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^{\star} \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta < (\geq)\ b^{\star}$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.2(p.96). ∎

● *Direct proof* See the proof of Lemma A 3.8(p.301) . ∎

Corollary 14.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.
(b) $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 13.2.1(p.96). ∎

● *Direct proof* See the proof of Corollary A 3.2(p.301) . ∎

Lemma 14.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $\tilde{K}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{K}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{K}(x)$ *is strictly decreasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(e) $\tilde{K}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(f) $\tilde{K}(x) + x$ *is strictly increasing on* $(-\infty, b^{\star}]$.
(g) $\tilde{K}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\lambda < 1$.
(h) *If* $x > y$ *and* $b^{\star} > y$, *then* $\tilde{K}(x) + x > \tilde{K}(y) + y$.
(i) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{K}} = a$ *where* $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.
(j) *Let* $\beta < 1$ *or* $s > 0$.
  1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (=>))\ 0$.
  2. $(\lambda\beta b + s)/\delta \geq (<)\ b^{\star} \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$.
  3. *Let* $\tilde{\kappa} < (= (>))\ 0$. *Then* $x_{\tilde{K}} < (= (>))\ 0$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.3(p.96). ∎

● *Direct proof* See the proof of Lemma A 3.9(p.301) . ∎

Corollary 14.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.
(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 13.2.2(p.97). ∎

● *Direct proof* See the proof of Corollary A 3.3(p.302) . ∎

Lemma 14.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}/\tilde{K}_{\mathbb{P}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}} = a$.
(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}}$.
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $a < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (=))\ 0$.
(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (>))\ 0$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.4(p.97). ∎

● *Direct proof* See the proof of Lemma A 3.10(p.302) . ∎

Lemma 14.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$.
(b) *Let* $\lambda\beta b \leq a$.
  1. $x_{\tilde{L}} \geq \lambda\beta b + s$.
  2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_{\tilde{L}} > \lambda\beta b + s$.
(c) *Let* $\lambda\beta b > a$. *Then there exists a* $s_{\tilde{\mathcal{L}}} > 0$ *such that if* $s_{\tilde{\mathcal{L}}} > (\leq)\ s$, *then* $x_{\tilde{L}} < (\geq)\ \lambda\beta b + s$.

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.5(p.97). ∎

● *Direct proof* See the proof of Lemma A 3.11(p.302) . ∎

Lemma 14.6.6 ($\tilde{\kappa}_{\mathbb{P}}$) *We have*:

(a) $\tilde{\kappa} = \lambda\beta b + s$ *if* $b^{\star} < 0$ *and* $\tilde{\kappa} = s$ *if* $a > 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0$. *Then* $x_{\tilde{K}} < (= (>))\ 0$. □

● *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.6(p.97). ∎

● *Direct proof* See the proof of Lemma A 3.12(p.303) . ∎

14.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

☐ **Tom 14.7.1** ($\boxminus\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$. ☐

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 13.4.1(p.98). ▮

● *Direct proof* See the proof of Tom A 4.5(p.308) . ▮

☐ **Tom 14.7.2** ($\boxminus\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 0$ or $s > 0$. Then, for a given starting time $\tau > 1$:*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
   ii. *Let $b + s < b^\star$. Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*
      1. *Let $\beta b + s \ge b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
      1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel}$ *is true.*

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 13.4.2(p.98). ▮

● *Direct proof* See the proof of Tom A 4.6(p.308) . ▮

14.8 Optimal Price to Propose

Lemma 14.8.1 ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *The optimal price to propose z_t is nonincreasing in $t > 0$.* ☐

● *Proof* Obvious from Tom's 14.7.1(p.109) (a) and 14.7.2(p.109) (a) and from (6.2.50(p.32)) and Lemma A 3.3(p.297) . ▮

14.9 Symmetry-Operation-Free

When no change occurs even if the symmetry operation is applied to a given assertion A, the assertion is said to be *free from* the symmetry operation, called the *symmetry-operation-free assertion.*

Lemma 14.9.1 *Even if the symmetry operation is applied to the symmetry-operation-free assertion, no change occurs.* ☐

● *Proof* Evident. ▮

Chapter 15

Analogy Theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)

15.1 Relationship between $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][$\mathtt{A}$] and $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][$\mathtt{A}$]

In this chapter we clarify the interrelationship between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism) and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism).

15.1.1 Assertion System $\mathscr{A}$

First, note the three following relations:

$$\circ\ \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.5.53\text{(p.80)})), \tag{15.1.1}$$

$$\bullet\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (13.3.1\text{(p.98)})), \tag{15.1.2}$$

$$\bullet\ \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.6\text{(p.106)})). \tag{15.1.3}$$

Next, the inverses of the above relations are:

$$\bullet\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.8.31\text{(p.87)})), \tag{15.1.4}$$

$$\circ\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.3.6\text{(p.98)})), \tag{15.1.5}$$

$$\circ\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.14\text{(p.107)})). \tag{15.1.6}$$

Then, from • (15.1.3(p.111)), • (15.1.2(p.111)), and • (15.1.4(p.111)) we obtain the following relation:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{15.1.7}$$

Finally, from ∘ (15.1.1(p.111)), ∘ (15.1.5(p.111)), and ∘ (15.1.6(p.111)) we obtain the following relation:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.8}$$

15.1.2 System of Optimality Equations (SOE)

First, note the following three relations:

$$\circ\ \mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (12.5.34\text{(p.77)})), \tag{15.1.9}$$

$$\bullet\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (13.3.2\text{(p.98)})), \tag{15.1.10}$$

$$\bullet\ \mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.4\text{(p.106)})), \tag{15.1.11}$$

Next, the inverses of the above relations are:

$$\bullet\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (12.8.25\text{(p.87)})), \tag{15.1.12}$$

$$\circ\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \ (\leftarrow (13.3.7\text{(p.98)})), \tag{15.1.13}$$

$$\circ\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.12\text{(p.107)})), \tag{15.1.14}$$

Then, from • (15.1.11(p.111)), • (15.1.10(p.111)), and • (15.1.12(p.111)) we obtain the following relation:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.15}$$

Finally, from ∘ (15.1.9(p.111)), ∘ (15.1.13(p.111)), and ∘ (15.1.14(p.111)) we obtain the following relation:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.16}$$

15.1.3 Attribute Vector $\boldsymbol{\theta}$

First, note the following three relations:

$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (12.5.54\text{(p.80)})), \tag{15.1.17}$$

$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (13.3.3\text{(p.98)})), \tag{15.1.18}$$

$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (14.5.7\text{(p.106)})), \tag{15.1.19}$$

Next, then the inverses of the above relations are:

$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (12.8.32\text{(p.87)})), \tag{15.1.20}$$

$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (13.3.8\text{(p.98)})), \tag{15.1.21}$$

$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (14.5.15\text{(p.107)})), \tag{15.1.22}$$

Then, from • (15.1.19(p.112)), • (15.1.18(p.112)), and • (15.1.20(p.112)) we obtain the following relation:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{15.1.23}$$

$$= (b^\star, b, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (14.5.8\text{(p.106)})). \tag{15.1.24}$$

Finally, from ∘ (15.1.17(p.112)), ∘ (15.1.21(p.112)), and ∘ (15.1.22(p.112)) we obtain the following relation:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{15.1.25}$$

$$= (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (12.5.55\text{(p.80)})). \tag{15.1.26}$$

15.1.4 Symmetry Theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)

Here let us define

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}, \tag{15.1.27}$$

$$\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}. \tag{15.1.28}$$

Then (15.1.7(p.111)) and (15.1.8(p.111)) can be expresses as below.

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.29}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.30}$$

(15.1.29(p.112)) implies that the following theorem holds.

Theorem 15.1.1 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \stackrel{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{15.1.31}$$

Similarly (15.1.30(p.112)) implies that the following theorem (inverse of the above theorem) holds.

Theorem 15.1.2 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \stackrel{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{15.1.32}$$

Then (15.1.15(p.111)) and (15.1.16(p.111)) can be expresses as below.

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.33}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.34}$$

Similarly (15.1.23(p.112)) and (15.1.25(p.112)) can be expresses as below.

$$\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}])], \tag{15.1.35}$$

$$\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}])]. \tag{15.1.36}$$

15.1.5 The Structure of $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$

The operation $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$ given by (15.1.27(p.112)) means that the three operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}} \to \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$. Then, putting this flow in vertically, we have

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l} \cdots(1) \\ \\ \cdots(2) \end{array} \quad (\leftarrow \ (12.8.21(\text{p.86})))$$

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} \overset{\text{def}}{=} \left\{\begin{array}{cc} a, & \mu \\ \downarrow & \downarrow \\ a^\star, & a \end{array}\right\} \begin{array}{l} \cdots(3) \\ \\ \cdots(4) \end{array} \quad (\leftarrow \ (13.2.1(\text{p.93})))$$

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l} \cdots(5) \\ \\ \cdots(6) \end{array} \quad (\leftarrow \ (14.5.3(\text{p.106})))$$

The above flow can be interpreted as follows:

- First, let us focus attention on elements *outside* the dashbox ⬚. Then, we see that first (1)-row changes into (2)-row, next (2)-row is identical to (5)-row, and finally (5)-row changes into (6)-row, which is identical to the original (1)-row. In other words, (1)-row remains unchanged *outside* the dash-box even if these operations are applied.
- Next, let us focus attention on elements *inside* the dashbox ⬚. Then, we see that first (1)-row changes into (2)-row, next (2)-row identical to (5)-row, and finally (5)-row changes into (6)-row. In other words, b and μ in (1)-row change into respectively $b^\star$ and b in (6)-row through the applications of these operations.

From the above we see that the above triple operations can be eventually reduced to the single operation

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \left\{\begin{array}{cc|cccccccccc} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \qquad (15.1.37)$$

Removing the unchanged elements from the above $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, eventually we obtain

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \{b \to b^\star,\ \mu \to b\}. \qquad (15.1.38)$$

Similarly, the operation $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$ given by (15.1.28(p.112)) means that the three operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \to \mathcal{A}_{\mathbb{P}\to\mathbb{R}} \to \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$. Then, putting this flow in vertically, we have

$$\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l} \cdots(1) \\ \\ \cdots(2) \end{array} \quad (\leftarrow \ (14.5.11(\text{p.107})))$$

$$\mathcal{A}_{\mathbb{P}\to\mathbb{R}} \overset{\text{def}}{=} \left\{\begin{array}{cc} a^\star, & a \\ \downarrow & \downarrow \\ a, & \mu \end{array}\right\} \begin{array}{l} \cdots(3) \\ \\ \cdots(4) \end{array} \quad (\leftarrow \ (13.3.5(\text{p.98})))$$

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l} \cdots(5) \\ \\ \cdots(5) \end{array} \quad (\leftarrow \ (12.5.29(\text{p.77})))$$

The above flow can be eventually reduced to as follows.

$$\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} = \{b^\star \to b,\ b \to \mu\}. \qquad (15.1.39)$$

From the comparison of `Tom`'s 12.7.2(p.84) and 14.7.2(p.109) we can easily reconfirm that Theorem 15.1.1(p.112) holds in fact.

15.2 Relationship between $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E] and $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]

It can be easily confirmed that the same as in Section 15.1(p.111) holds also for $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{E}]\}$ and $\mathscr{A}\{\mathsf{M}\text{:}1[\mathbb{P}][\mathtt{E}]\}$. Then we have

Theorem 15.2.1 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{R}][\mathtt{E}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{E}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{E}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{R}][\mathtt{E}]\}]. \quad \square \qquad (15.2.1)$$

Theorem 15.2.2 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{R}][\mathtt{A}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \qquad (15.2.2)$$

It can be easily confirmed that $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$ and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$ are the same as (15.1.38(p.113)) and (15.1.39(p.113)) respectively.

Chapter 16

Integrated Theory

16.1 Flow of Discussions

Let us here again recall Motive 2(p.3) "Does a general theory integrating quadruple-asset-trading-problems exist ?", and this motivation was put an end with a successful construction of the integrated theory, which is summarized as below, which is summarized as below.

⟨1⟩ $\mathscr{A}\{T_{\mathbb{R}}\}$ is *proven* (see Lemma 10.1.1(p.55)).

⟨2⟩ $\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ is *proven* (see Lemmas 10.2.1(p.57) - 10.3.1(p.59)).

⟨3⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is *proven* (see Tom's 11.2.1(p.61) and 11.2.2(p.62)).

⟨4⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is *derived* (see Tom's 12.7.1(p.84) and 12.7.2(p.84)).

⟨5⟩ $\mathscr{A}\{T_{\mathbb{P}}\}$ is *proven* (see Lemma 13.2.1(p.93)).

⟨6⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ is *derived* (see Tom's 13.4.1(p.98) and 13.4.2(p.98)).

⟨7⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is *derived* (see Tom's 14.7.1(p.109) and 14.7.2(p.109)).

⟨8⟩ The analogous relation between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is shown (see Theorems 15.1.1(p.112) and 15.1.2(p.112)).

16.2 Structure of Integrated Theory

The above flow, ⟨1⟩ – ⟨8⟩, can be schematized as in Figure 16.2.1(p.115) below where the three shadow boxes are *directly proven* and the remaining four frame boxes are all *indirectly derived* by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to .

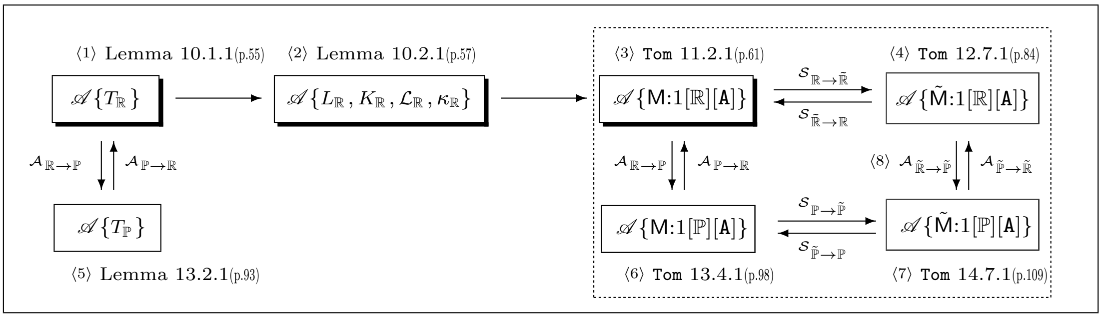


Figure 16.2.1: The whole flow of constructing the integrated theory

16.3 Implications

The interrelationship among the quadruple assertion systems within the dashbox of Figure 16.2.1(p.115) implies the following. First, an assertion system of $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ is *defined* as a *core* within the quadruple-asset-trading-models $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ and then *proven* (see Chap. 11(p.61)). Next, the assertion system for each of the remaining three models is *derived* by sequentially applying the operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ and $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the above *core* assertion system (see Chaps. 12(p.69) and 13(p.89)). Finally, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see Chap. 14(p.101)). Since it is proven that any of the above four operations are reversible, even if any other assertion system within $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ is selected as a *core*, the same flow as the above can be depicted. Let us refer to the whole structure consisting of the quadruple assertion systems in such a fashion as stated above as the *integrated theory*. In the conventional approach, each of the quadruple assertion systems must be defined *separately* and proven *one by one*. On the other hand, in our approach based on the integrated theory, the number of assertion systems which must be defined and proven is *only one* as a core. In Part 3 that follows we try to apply the integrated theory to all of the remaining five quadruple-asset-trading-models in Table 3.2.1(p.17) except for $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ the analyses of which was already ended. From all the above, it will be realized that the integrated theory provides a strong tool for the treatment of asset trading problems.

16.3.1 Limitation of Integration Theory

Here note that the successful construction of the integrated theory is based on the following two premises: one is that price ξ is defined on the total market $(-\infty, \infty)$, the other is that the symmetrical relation between $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ and $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ and the analogy relation between $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ and $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}$ must be satisfied (see ⋆Sections 12.11(p.87) and 13.8(p.99)). However, as seen from ⋆Tables 6.4.3(p.41) - 6.4.6(p.41), although the symmetrical relation always holds between $\texttt{SOE}\{\mathsf{M}{:}1/2/3[\mathbb{R}/\mathbb{P}][\texttt{A/E}]\}$ and $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1/2/3[\mathbb{R}/\mathbb{P}][\texttt{A/E}]\}$ (compare (I) and (II)), the analogical relation between $\texttt{SOE}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2/3[\mathbb{R}][\texttt{A/E}]\}$ and $\texttt{SOE}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2/3[\mathbb{P}][\texttt{A/E}]\}$ does not hold (compare (I) and (III)). In other words, for Models 1/2/3 the symmetry theorems can be always applied; however, the analogy theorems cannot be applied. Accordingly, it follows that the integrated theory are applicable *only* in discussions related to symmetry. For the treatment of the case where the analogy theorem cannot be applied, see Lemma 20.1.1(p.153) and Section 20.1.5(p.166).

Chapter 17

Market Restriction

17.1 Preliminary

As seen from the whole discussions over Chaps. 10(p.55) –15(p.111), the integrated theory is constructed under the premise that prices ξ, whether reservation price or posted price, is defined on the total-DF-space (see (2.2.5(p.13))), i.e.,

$$\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}, \tag{17.1.1}$$

called the *total market*. However, since the prices ξ in a usual market of the real world are positive, i.e., $\xi \in (0, \infty)$, the above premise, permitting a negative price $\xi \in (-\infty, 0)$, must be said to be unrealistic. This chapter proposes a methodology working through this problem.

17.2 Market Restriction

Let us refer to the restriction of the *total market* $\mathscr{F}$ to a given subset

$$\mathscr{F}' \subseteq \mathscr{F} \tag{17.2.1}$$

as the *market restriction* of $\mathscr{F}$ to $\mathscr{F}'$ and to the $\mathscr{F}'$ as the *restricted market*. Throughout this paper let us consider the following three kinds of restricted markets:

$$\mathscr{F}^+ \stackrel{\text{def}}{=} \{F \mid 0 < a < b\} \quad (\textit{positive market}), \tag{17.2.2}$$

$$\mathscr{F}^\pm \stackrel{\text{def}}{=} \{F \mid a \le 0 \le b\} \quad (\textit{mixed market}), \tag{17.2.3}$$

$$\mathscr{F}^- \stackrel{\text{def}}{=} \{F \mid a < b < 0\} \quad (\textit{negative market}) \tag{17.2.4}$$

where clearly

$$\mathscr{F} = \mathscr{F}^+ \cup \mathscr{F}^\pm \cup \mathscr{F}^-. \tag{17.2.5}$$

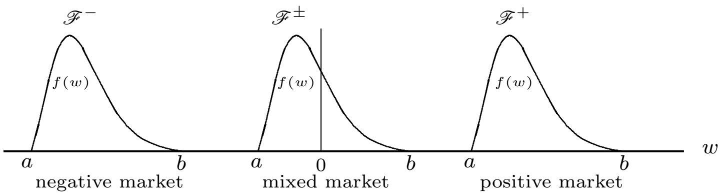


Figure 17.2.1: Three kinds of markets

Definition 17.2.1 In the present paper, let us represent the restriction of $\mathscr{F}$ to the above three restricted markets by the same symbols $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ above, called the *positive market restriction* $\mathscr{F}^+$, the *mixed market restriction* $\mathscr{F}^\pm$, and the *negative market restriction* $\mathscr{F}^-$ respectively. See Section A 7.5(p.317) for an economic implication brought about by the three market restrictions. □

17.3 Market Restricted Models

Throughout the rest of this paper, let us denote the models defined on the restricted markets $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ by Model^+, $\mathsf{Model}^\pm$, and Model^- respectively, called the *market restricted models*. For $x = 1, 2, 3$ and $\mathtt{X} = \mathtt{A}, \mathtt{E}$ let us define the quadruple-asset-trading-models:

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^+\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^+, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^+, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^+, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^+\}, \tag{17.3.1}$$

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^\pm\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^\pm, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^\pm\}, \tag{17.3.2}$$

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^-\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^-, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^-, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^-, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^-\}. \tag{17.3.3}$$

17.4 Inequalities Resulting From Market Restriction

The lemma below will be used to examine what occurs when the market restriction is applied to results derived by using the integrated theory constructed on the total market $\mathscr{F}$.

Lemma 17.4.1 (positive market $\mathscr{F}^+$) *Suppose* $0 < a$. *Then we have:*

[1] [ref.8078] $0 < a < \mu < b$. Proof: Evident from (2.2.2(p.13)).

[2] [ref.9343] $\beta b \le b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < \beta b \le b$ with $\beta = 1$.

[3] [ref.7865] $\beta\mu < b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < \beta\mu < b$ with $\beta = 1$.

[4] [ref.8369] $\beta a < b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < \beta a < b$ with $\beta = 1$.

[5] [ref.9483] $a < \beta\mu$ *and* $\beta\mu \le a$ *are both possible.* Proof: Since $0 < a < \beta\mu$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for any sufficiently small $\beta > 0$.

[6] [ref.6867] $a < \beta b$ *and* $\beta b \le a$ *are both possible.* Proof: Since $0 < a < \beta b$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for any sufficiently small $\beta > 0$.

[7] [ref.6296] $\beta b < b^\star$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < \beta b < b^\star$ with $\beta = 1$ due to Lemma 14.6.1(p.107) (n). □

Lemma 17.4.2 (mixed market $\mathscr{F}^\pm$) *Suppose* $a \le 0 \le b$. *Then we have:*

[8] [ref.8062] $a < \beta\mu < b$ *for* $0 < \beta \le 1$. Proof: Let $\mu = 0$. Then $a < \mu = \beta\mu = 0 < b$ for $0 < \beta \le 1$. Let $\mu \ne 0$. If $a < \mu < 0$, then $a < \beta\mu < 0 \le b$ with $\beta = 1$, hence $a < \beta\mu < 0 \le b$ for $0 < \beta \le 1$ and if $0 < \mu < b$, then $a \le 0 < \beta\mu < b$ with $\beta = 1$, hence $a \le 0 < \beta\mu < b$ for $0 < \beta \le 1$. Accordingly, whether $a < \mu < 0$ or $0 < \mu < b$, we have $a < \beta\mu < b$ for $0 < \beta \le 1$. Thus, whether $\mu = 0$ or $\mu \ne 0$, it follows that $a < \beta\mu < b$ for $0 < \beta \le 1$.

[9] [ref.6907] $\beta a < b$ *for* $0 < \beta \le 1$. Proof: Let $\beta = 1$. Then $\beta a = a < b$. Let $\beta < 1$. If $a = 0$, then $\beta a = a = 0 < b$ and if $a < 0$, then $\beta a < 0 \le b$, hence $\beta a < b$ whether $a = 0$ or $a < 0$. Thus, whether $\beta = 1$ or $\beta < 1$ (i.e., $0 < \beta \le 1$) it follows that we have $\beta a < b$.

[10] [ref.6892] $a < \beta b$ *for* $0 < \beta \le 1$. Proof: If $b > 0$, then $a \le 0 < b = \beta b$ with $\beta = 1$, hence $a \le 0 < \beta b$ for $0 < \beta \le 1$. If $b = 0$, then $a < b = \beta b = 0$ for $0 < \beta \le 1$. Therefore, whether $b > 0$ or $b = 0$, we have $a < \beta b$ for $0 < \beta \le 1$.

[11] [ref.6896] $a^\star < \beta a$ *for* $0 < \beta \le 1$. Proof: Immediate from $a^\star < \beta a \le 0$ with $\beta = 1$ due to Lemma 13.2.1(p.93) (n).

[12] [ref.6298] $\beta b < b^\star$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 \le \beta b < b^\star$ with $\beta = 1$ due to Lemma 14.6.1(p.107) (n). □

Lemma 17.4.3 (negative market $\mathscr{F}^-$) *Suppose* $b < 0$. *Then we have:*

[13] [ref.7486] $a < \mu < b < 0$. Proof: Evident from (2.2.2(p.13)).

[14] [ref.6118] $a \le \beta a$ *for* $0 < \beta \le 1$. Proof: Immediate from $a \le \beta a < 0$ with $\beta = 1$.

[15] [ref.8068] $a < \beta\mu$ *for* $0 < \beta \le 1$. Proof: Immediate from $a < \beta\mu < 0$ with $\beta = 1$.

[16] [ref.7482] $a < \beta b$ *for* $0 < \beta \le 1$. Proof: Immediate from $a < \beta b < 0$ with $\beta = 1$.

[17] [ref.7478] $\beta\mu < b$ *and* $b \le \beta\mu$ *are both possible.* Proof: Since $\beta\mu < b < 0$ with $\beta = 1$, the former is true for a β sufficiently close to $\beta = 1$ and the latter is true for a sufficiently small $\beta > 0$.

[18] [ref.8296] $\beta a < b$ *and* $b \le \beta a$ *are both possible.* Proof: Since $\beta a < b < 0$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for a sufficiently small $\beta > 0$.

[19] [ref.6919] $a^\star < \beta a$ *for* $0 < \beta \le 1$. Proof: Immediate from $a^\star < \beta a < 0$ with $\beta = 1$ due to Lemma 13.2.1(p.93) (n). □

Definition 17.4.1 (market-restriction-free-assertion) When no change occurs even if a market restriction is applied to a given assertion, the assertion is said to be *free from* the market restriction, called the *market-restriction-free assertion.* □

Lemma 17.4.4 *Even if a market restriction is applied to a market-restriction-free assertion, no change occurs.* □

● *Proof* Evident. ∎

17.5 Market Restriction

17.5.1 $\mathscr{A}\{$M:1[$\mathbb{R}$][A]$\}$

17.5.1.1 Positive Restriction

□ `Pom` **17.5.1** ($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\ge t>1_\blacktriangle}$. □

● *Proof* The same as `Tom` 11.2.1(p.61) due to Lemma 17.4.4(p.118). ∎

□ Pom **17.5.2** ($\mathscr{A}${M:1[ℝ][A]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$* (impossible). ______

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$,
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ (see *Numerical Example* 1(p.126)).
   ii. *Let $s \geq \beta\mu$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ (see *Numerical Example* 2(p.126)).

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Let $\beta < 1$ or $s > 0$. Then $\kappa = \beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a) with $\lambda = 1$.

(a) The same as Tom 11.2.2(p.62) (a).

(b,c) Always $\beta\mu < b$ due to [3(p.118)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as Tom 11.2.2(p.62) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 11.2.2(p.62).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$. Then, since $\kappa > 0$ due to (2), it suffices to consider only (c2i) of Tom 11.2.2(p.62).

(c3ii) Let $\beta\mu \leq s$. Then, since $\kappa \leq 0$ due to (2) and since $\beta\mu - s \leq 0 < a$, it suffices to consider only (c2ii1,c2iii1) of Tom 11.2.2(p.62). ∎

17.5.1.2 Mixed Restriction

□ Mim **17.5.1** ($\mathscr{A}${M:1[ℝ][A]$^\pm$}) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

● *Proof* The same as Tom 11.2.1(p.61) due to Lemma 17.4.4(p.118). ∎

□ Mim **17.5.2** ($\mathscr{A}${M:1[ℝ][A]$^\pm$}) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$* (impossible). ______

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\texttt{S}_1$(p.62) $\boxed{ⓢ\blacktriangle \mid ⓞ\parallel}$ *is true.*

● *Proof* Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.

(a) The same as Tom 11.2.2(p.62) (a).

(b,c) Always $\beta\mu < b$ due to [8(p.118)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as Tom 11.2.2(p.62) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of Tom 11.2.2(p.62) and if $b = 0$, then since always $\beta\mu - s = \beta\mu > a$ due to [8], it suffices to consider only (c2ii2) of Tom 11.2.2(p.62). Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from Tom 11.2.2(p.62) (c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.7(p.25)) with $\lambda = 1$. ∎

17.5.1.3 Negative Restriction

☐ Nem **17.5.1** ($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.

● *Proof* The same as Tom 11.2.1(p.61) due to Lemma 17.4.4(p.118). ∎

☐ Nem **17.5.2** ($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* S_1(p.62) $\boxed{ⓢ\blacktriangle \mid ⓞ \parallel}$ *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* S_1(p.62) $\boxed{ⓢ\blacktriangle \mid ⓞ \parallel}$ *is true.*

● *Proof* Suppose $b < 0 \cdots$ **(1)**. Let $\beta < 1$ or $s > 0$. Then, we have $\kappa = -s \cdots$ **(2)** from Lemma 10.3.1(p.59) (a). Moreover, in this case, both $\beta\mu \geq b$ and $\beta\mu < b$ are possible due to [17(p.118)].

(a,b) The same as Tom 11.2.2(p.62) (a,b).

(c) Let $\beta\mu < b$. Then $s_{\mathcal{L}} > 0 \cdots$ **(3)** from Lemma 10.2.4(p.59) (c).

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as Tom 11.2.2(p.62) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii1,c2iii2) of Tom 11.2.2(p.62). Since $\beta\mu - s = \beta\mu > a$ due to [15(p.118)] and since $s = 0 < s_L$ due to (3), we have Tom 11.2.2(p.62) (c2iii2).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of Tom 11.2.2(p.62). ∎

17.5.2 $\mathscr{A}\{\tilde{\text{M}}$:1[$\mathbb{R}$][A]$\}$

17.5.2.1 Positive Restriction

☐ Pom **17.5.3** ($\mathscr{A}\{\tilde{\text{M}}$:1[$\mathbb{R}$][A]$^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.

● *Proof* The same as Tom 12.7.1(p.84) due to Lemma 17.4.4(p.118). ∎

☐ Pom **17.5.4** ($\mathscr{A}\{\tilde{\text{M}}$:1[$\mathbb{R}$][A]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \leq a$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu + s < b$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* S_1(p.62) $\boxed{ⓢ\blacktriangle \mid ⓞ \parallel}$ *is true.*
3. *Let $\beta < 1$ and $s > 0$.*[†]
   i. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* S_1(p.62) $\boxed{ⓢ\blacktriangle \mid ⓞ \parallel}$ *is true* (see Numerical Example 3(p.127)).

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 12.6.6(p.83) (a). Here note that $\mu\beta \leq a$ and $\mu\beta > a$ are both possible due to [5(p.118)].

(a,b) The same as Tom 12.7.2(p.84) (a,b).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(3)** due to Lemma 12.6.5(p.83) (c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have Tom 12.7.2(p.84) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.118)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of Tom 12.7.2(p.84).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of Tom 12.7.2(p.84). ∎

17.5.2.2 Mixed Restriction

□ `Mim` **17.5.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.

● *Proof* The same as `Tom` 12.7.1(p.84) due to Lemma 17.4.4(p.118). ∎

□ `Mim` **17.5.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$* (impossible). ———

(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   - ii. *Let $s = -\beta\tilde{T}(0)$.*
     1. *Let $\beta\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu + s < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   - iii. *Let $s > -\beta\tilde{T}(0)$.*
     1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel}$ *is true.*

● *Proof* Suppose $a \le 0 \le b$.

(a) The same as `Tom` 12.7.2(p.84) (a).

(b,c) Always $\beta\mu > a$ due to [8(p.118)], hence $\beta\mu \le a$ is impossible. Hence $s_{\tilde{\mathcal{L}}} > 0 \cdots (1)$ due to Lemma 12.6.5(p.83) (c).

(c1-c1ii) The same as `Tom` 12.7.2(p.84) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of `Tom` 12.7.2(p.84). Let $a = 0$. Then $\beta\mu + s = \beta\mu < b$ due to [8(p.118)], hence it suffices to consider only (c2ii2) of `Tom` 12.7.2(p.84). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions become true from `Tom` 12.7.2(p.84) (c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.25)). ∎

17.5.2.3 Negative Restriction

□ `Nem` **17.5.3** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.

● *Proof* The same as `Tom` 12.7.1(p.84) due to Lemma 17.4.4(p.118). ∎

□ `Nem` **17.5.4** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$* (impossible). ———

(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta\mu < -s$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   - ii. *Let $\beta\mu \ge -s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

● *Proof* Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\tilde{\kappa} = \beta\mu + s \cdots (3)$ due to Lemma 12.6.6(p.83) (a).

(a) The same as `Tom` 12.7.2(p.84) (a).

(b,c) Always $a < \beta\mu$ due to [15(p.118)], hence $\beta\mu \le a$ is impossible.

(c1-c1ii) The same as the proof of `Tom` 12.7.2(p.84) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c2i) of `Tom` 12.7.2(p.84).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu < -s$, hence $\beta\mu + s < 0$. Hence, since $\tilde{\kappa} < 0$ due to (3), it suffices to consider only (c2i) of Tom 12.7.2(p.84).

(c3ii) Let $\beta\mu \geq -s$, hence $\beta\mu + s \geq 0$. Let $\beta\mu + s = 0$. Then, since $\tilde{\kappa} = 0$ due to (3) and $\beta\mu + s > b$ due to (2), it suffices to consider only (c2iii1) of Tom 12.7.2(p.84). Let $\beta\mu + s > 0$. Then, since $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c2iii) of Tom 12.7.2(p.84). Then, since $\beta\mu + s > 0 > b$ due to (1), it suffices to consider only (c2ii1) of Tom 12.7.2(p.84). Accordingly, whether $\beta\mu + s = 0$ or $\beta\mu + s > 0$, we have the same result. ∎

17.5.3 $\mathscr{A}${M:1[ℙ][A]}

17.5.3.1 Positive Restriction

☐ **Pom 17.5.5** ($\mathscr{A}${M:1[ℙ][A]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 13.4.1(p.98) due to Lemma 17.4.4(p.118). ∎

☐ **Pom 17.5.6** ($\mathscr{A}${M:1[ℙ][A]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$* (impossible). ———

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \leq a^\star$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
   ii. *Let $a - s > a^\star$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
      2. *Let $\beta a - s > a^\star$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathscr{L}} \leq s$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
      2. *Let $\beta a - s > a^\star$ and $s_{\mathscr{L}} > s$. Then* S$_1$(p.62) [Ⓢ▲ | ◎ ‖].

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$.

(a) The same as Tom 13.4.2(p.98) (a).

(b,c) Always $\beta a < b$ due to [4(p.118)], hence $\beta a \geq b$ is impossible.

(c1-c1ii) The same as Tom 13.4.2(p.98) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 13.4.2(p.98).

(c3-c3iii2) Immediate from Tom 13.4.2(p.98) (c2-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.26)) with $\lambda = 1$. ∎

17.5.3.2 Mixed Restriction

☐ **Mim 17.5.5** ($\mathscr{A}${M:1[ℙ][A]$^\pm$}) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* [Ⓢ dOITs$_{\tau}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 13.4.1(p.98) due to Lemma 17.4.4(p.118). ∎

☐ **Mim 17.5.6** ($\mathscr{A}${M:1[ℙ][A]$^\pm$}) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$* (impossible). ———

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \leq a^\star$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
   ii. *Let $a - s > a^\star$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.

ii. *Let* $s = \beta T(0)$.

1. *Let* $\beta a - s \leq a^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
2. *Let* $\beta a - s > a^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_{\blacktriangle}$.

iii. *Let* $s > \beta T(0)$.

1. *Let* $\beta a - s \leq a^\star$ *or* $s_{\mathcal{L}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.62) $\boxed{Ⓢ_\blacktriangle \mid ⓞ_\parallel}$.

● *Proof* Suppose $a \leq 0 \leq b$.

(a) The same as Tom 13.4.2(p.98) (a).

(b,c) Always $\beta a < b$ due to [9(p.118)], hence $\beta a \geq b$ is impossible. .

(c1-c1ii) The same as Tom 13.4.2(p.98) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from Tom 13.4.2(p.98) (c2i) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.118)], hence the assertion become true from Tom 13.4.2(p.98) (c2ii2). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as Tom 13.4.2(p.98) (c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.26))) with $\lambda = 1$. ▮

17.5.3.3 Negative Restriction

☐ **Nem 17.5.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_{\blacktriangle}$.

● *Proof* Immediate from Tom 13.4.1(p.98) due to Lemma 17.4.4(p.118). ▮

☐ **Nem 17.5.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $\geq x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \geq b$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c) *Let* $\beta a < b$.

1. *Let* $\beta = 1$.
   i. *Let* $a - s \leq a^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let* $a - s > a^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_{\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_1$(p.62) $\boxed{Ⓢ_\blacktriangle \mid ⓞ_\parallel}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\beta a - s \leq a^\star$ *or* $s_{\mathcal{L}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.62) $\boxed{Ⓢ_\blacktriangle \mid ⓞ_\parallel}$.

● *Proof* Suppose $b < 0 \cdots$**(1)**, hence $\kappa = \kappa_{\mathbb{P}} = -s \cdots$**(2)** from Lemma 13.2.6(p.97) (a). Then, both $\beta a \geq b$ and $\beta a < b$ are possible due to [18(p.118)]. If $\beta a < b$, then $s_{\mathcal{L}} > 0 \cdots$**(3)** due to Lemma 13.2.5(p.97) (c) with $\lambda = 1$.

(a) The same as Tom 13.4.2(p.98) (a).

(b) Let $\beta a \geq b$. Then, the assertion is true Tom 13.4.2(p.98) (b).

(c) Let $\beta a < b$.

(c1-c1ii) The same as Tom 13.4.2(p.98) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of Tom 13.4.2(p.98). In addition, since $\beta a - s = \beta a > a^\star$ due to [19(p.118)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of Tom 13.4.2(p.98).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ from (2), it suffices to consider only (c2iii) of Tom 13.4.2(p.98). ▮

17.5.4 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

17.5.4.1 Positive Restriction

☐ **Pom 17.5.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_{\blacktriangle}$.

● *Proof* The same as Tom 14.7.1(p.109) due to Lemma 17.4.4(p.118). ▮

☐ **Pom 17.5.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $\beta b \leq a$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau}\langle 1\rangle}_{\parallel}$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   i. *Let $b + s \geq b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ $\rightarrow$
   ii. *Let $b + s < b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_1$(p.62) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
   ii. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then, $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 14.6.6(p.108) (a). In this case, $\beta b \leq a$ and $\beta b > a$ are both possible due to [6(p.118)], and if $\beta b > a$, then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(3)** due to Lemma 14.6.5(p.108) (c) with $\lambda = 1$. In addition, we have

(a,b) The same as Tom 14.7.2(p.109) (a,b).

(c) Let $\beta b > a$.

(c1-c1ii)

The same as Tom 14.7.2(p.109) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of Tom 14.7.2(p.109). In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.118)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of Tom 14.7.2(p.109).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii-c2iii2) of Tom 14.7.2(p.109). ∎

17.5.4.2 Mixed Restriction

□ **Mim 17.5.7** ($\mathscr{A}\{\tilde{\mathrm{M}}{:}1[\mathbb{P}][\mathrm{A}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 14.7.1(p.109) due to Lemma 14.7.1(p.109). ∎

□ **Mim 17.5.8** ($\mathscr{A}\{\tilde{\mathrm{M}}{:}1[\mathbb{P}][\mathrm{A}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \geq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \leq a$* (impossible). ———

(c) *Let $\beta b > a$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $b + s \geq b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
   ii. *Let $b + s < b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \geq b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$.

● *Proof* Let $b \geq 0 \geq a \cdots$ **(1)**.

(a) The same as Tom 14.7.2(p.109) (a).

(b,c) Always $\beta b > a$ due to [10(p.118)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as Tom 14.7.2(p.109) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of Tom 14.7.2(p.109). Let $a < 0$. Then, the assertion is true from Tom 14.7.2(p.109) (c2i). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.118)], it suffices to consider only (c2ii2) of Tom 14.7.2(p.109). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 14.7.2(p.109) (c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.27)) with $\lambda = 1$. ∎

17.5.4.3 Negative Restriction

□ Nem **17.5.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.

● *Proof* The same as Tom 14.7.1(p.109) due to Lemma 17.4.4(p.118). ∎

□ Nem **17.5.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \geq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \leq a$* (impossible). ———

(c) *Let $\beta b > a$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $b + s \geq b^\star$. Then* $\boxed{●\ \mathtt{dOITd}_\tau\langle 1\rangle}_\parallel$.
   ii. *Let $b + s < b^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
   ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \geq b^\star$. Then* $\boxed{●\ \mathtt{dOITd}_\tau\langle 1\rangle}_\parallel$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1}{}_\blacktriangle$.
   iii. *Let $-\beta\tilde{T}(0) < s$.*
      1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{●\ \mathtt{dOITd}_\tau\langle 1\rangle}_\parallel$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.62) $\boxed{Ⓢ\blacktriangle \mid ⊙\ \parallel}$.

● *Proof* Let $b < 0$, hence $a < b < 0 \cdots (1)$.

(a) The same as Tom 14.7.2(p.109) (a).

(b,c) Always $\beta b > a$ due to [16(p.118)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as Tom 14.7.2(p.109) (c1-c1ii).

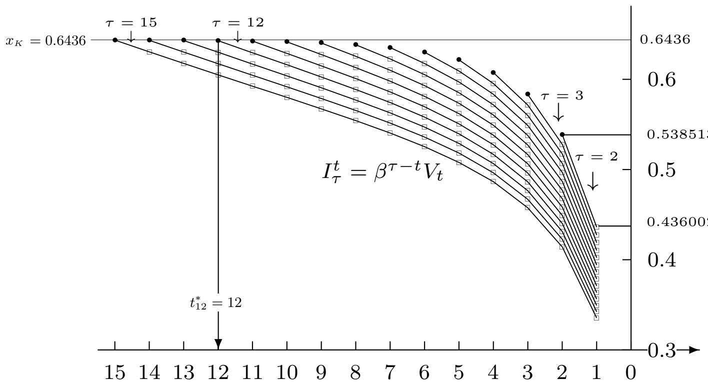
(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 14.7.2(p.109).

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 14.7.2(p.109) (c2-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0)+s$ from (5.1.36(p.27)) with $\lambda = 1$. ∎

17.6 Numerical Example

Numerical Example **1** **($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$\}^+$ (selling model)**

This is the example for $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ in Pom 17.5.2(p.119) (c3i) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.† Then, we have $x_K = 0.6436$ (see Section A 6(p.312)). Figure 17.6.1(p.126) below is the graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$ (see (7.2.4(p.44))). For example, the two points on the line of $\tau = 2$ are given by $V_2 = 0.538513$ (•) and $\beta V_1 = 0.98 \times 0.444900 = 0.436002$ (▫), hence $V_2 > \beta V_1$. Similarly, the three points on the polygonal curve of $\tau = 3$ are given by $V_3 = 0.583152$ (•), $\beta V_2 = 0.98 \times 0.538513 = 0.52774274$ (▫), and $\beta^2 V_1 = 0.98^2 \times 0.4449 = 0.42728196$ (▫), hence $V_3 > \beta V_2 > \beta^2 V_1$. Then, the value of t on the horizontal line corresponding to the bullet • provides the optimal initiating time t^*_τ for each of $\tau = 2, 3, \cdots, 15$, i.e., $\text{OIT}_\tau\langle t^*_\tau\rangle$, so we have $t^*_2 = 2$, $t^*_3 = 3$, $\cdots$, $t^*_{15} = 15$ (see t^*_τ-column of the table below). This result means $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ for $\tau = 2, 3, \cdots, 15$. Since $V_t - \beta V_t > 0$ for $t = 2, 3, \cdots, 15$ (see values of $V_t - \beta V_t$-column in the table below), we have $L(V_{t-1}) > 0$ from (11.1.1(p.61)), meaning $\text{Conduct}_{15\geq t>1\blacktriangle}$ from (11.1.5(p.61)), i.e., it is strictly optimal to conduct the search on $15 \geq t > 1$.



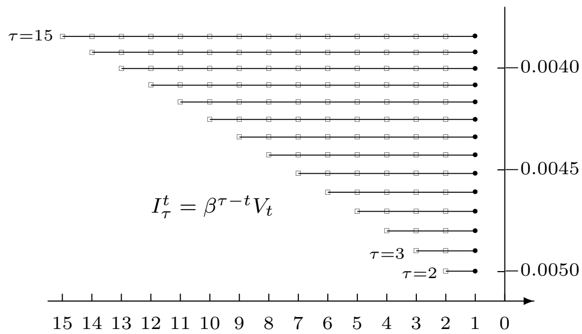
$V_t - \beta V_{t-1}$, t^*_τ (OIT)

t	V_τ	$V_t - \beta V_{t-1}$	t^*_τ	decision
0				
1	0.444900			
2	0.538513	+0.102511	2	Conduct▲
3	0.583152	+0.055409	3	Conduct▲
4	0.607492	+0.036003	4	Conduct▲
5	0.621595	+0.026252	5	Conduct▲
6	0.630035	+0.020871	6	Conduct▲
7	0.635180	+0.017745	7	Conduct▲
8	0.638351	+0.015874	8	Conduct▲
9	0.640318	+0.014734	9	Conduct▲
10	0.641544	+0.014032	10	Conduct▲
11	0.642309	+0.013596	11	Conduct▲
12	0.642788	+0.013325	12	Conduct▲
13	0.643088	+0.013155	13	Conduct▲
14	0.643276	+0.013049	14	Conduct▲
15	0.643393	+0.012983	15	Conduct▲

Figure 17.6.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$ where • represents OIT

Numerical Example **2** **($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$\}^+$ (selling model)**

This is the example for $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ in Pom 17.5.2(p.119) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.50$† The bullet • in each of the 14 horizontal straight lines in Figure 17.6.2(p.126) below shows that the optimal initiating time t^*_τ degenerates to time 1 (i.e., $t^*_\tau = 1$ for $\tau = 2, 3, \cdots, 15$) under Preference Rule 7.2.1(p.45), i.e., $\boxed{\bullet\ \text{dOITd}_{\tau=2,3,\cdots,15}\langle 1\rangle}_{\parallel}$. The result comes from the fact of $V_t - \beta V_t = 0$ for $t = 2, 3, \cdots, 15$ with $t = 2, 3, \cdots, 15$ (see $V_t - \beta V_{t-1}$-column in the table below), leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$ for $\tau = 2, 3, \cdots, 15$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$ for $\tau = 2, 3, \cdots, 15$.



$V_t - \beta V_{t-1}$

t	V_t	$V_t - \beta V_{t-1}$	t^*_τ	$\mathbb{S}_t$
0				
1	−0.005100			
2	−0.004998	0.000000	1	−0.00010200
3	−0.004898	0.000000	1	−0.00021960
4	−0.004800	0.000000	1	−0.00029996
5	−0.004704	0.000000	1	−0.00039600
6	−0.004610	0.000000	1	−0.00049008
7	−0.004517	0.000000	1	−0.00058220
8	−0.004427	0.000000	1	−0.00067334
9	−0.004338	0.000000	1	−0.00076154
10	−0.004252	0.000000	1	−0.00084876
11	−0.004167	0.000000	1	−0.00093304
12	−0.004083	0.000000	1	−0.00101634
13	−0.004002	0.000000	1	−0.00109866
14	−0.003922	0.000000	1	−0.00117804
15	−0.003843	0.000000	1	−0.00125644

Figure 17.6.2: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$ where • represents OIT

Note here that numbers in V_t-column are all negative, meaning that tackling the asset selling problem makes no profits (red ink). Accordingly, if this is of tE-case (see H1(p.8) (a)), you must resign to the red ink and if it is of tA-case (see H1(p.8) (b)), it suffices to pass over the problem without tackling the selling problem itself. Since $0.5 \times (a + b) = 0.505$ and since $V_t < 0 < 0.01 = a$ for $t = 1, 2, \cdots, 15$ (see V_t-column of the above table), from (A 7.2 (1) (p.314)) we have $T(V_t) = 0.505 - V_t$ for $t = 1, 2, \cdots, 15$, hence we have:

$T(V_1) = 0.505 - (-0.005100) = 0.510100$, $T(V_6) = 0.505 - (-0.004610) = 0.509610$, $T(V_{11}) = 0.505 - (-0.004167) = 0.509167$,
$T(V_2) = 0.505 - (-0.004998) = 0.509998$, $T(V_7) = 0.505 - (-0.004517) = 0.509517$, $T(V_{12}) = 0.505 - (-0.004083) = 0.509083$,
$T(V_3) = 0.505 - (-0.004898) = 0.509898$, $T(V_8) = 0.505 - (-0.004427) = 0.509427$, $T(V_{13}) = 0.505 - (-0.004002) = 0.509002$,
$T(V_4) = 0.505 - (-0.004800) = 0.509800$, $T(V_9) = 0.505 - (-0.004338) = 0.509338$, $T(V_{14}) = 0.505 - (-0.003922) = 0.508922$,
$T(V_5) = 0.505 - (-0.004704) = 0.509704$, $T(V_{10}) = 0.505 - (-0.004252) = 0.509252$, $T(V_{15}) = 0.505 - (-0.003843) = 0.508843$.

---

†Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 > 0.05 = s$. Thus, the condition of this assertion is satisfied.

†Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.50 > 0$. In addition, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 < 0.50 = s$. Thus, the condition of the assertion is satisfied.

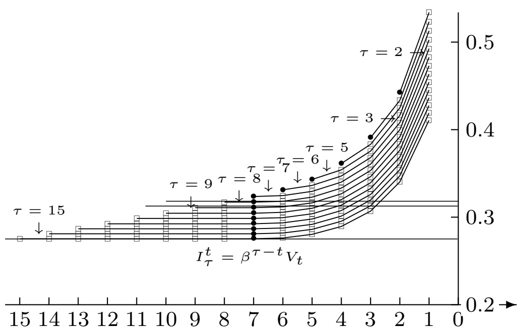
Since $\mathbb{S}_t = 0.98 \times T(V_{t-1}) - 0.5$ from (6.2.13(p.30)), we get

$\mathbb{S}_2 = 0.98 \times 0.510100 - 0.5 = -0.00010200$, $\mathbb{S}_7 = 0.98 \times 0.509610 - 0.5 = -0.00058220$, $\mathbb{S}_{12} = 0.98 \times 0.509167 - 0.5 = -0.00101634$,
$\mathbb{S}_3 = 0.98 \times 0.509998 - 0.5 = -0.00021960$, $\mathbb{S}_8 = 0.98 \times 0.509517 - 0.5 = -0.00067334$, $\mathbb{S}_{13} = 0.98 \times 0.509083 - 0.5 = -0.00109866$,
$\mathbb{S}_4 = 0.98 \times 0.509898 - 0.5 = -0.00029996$, $\mathbb{S}_9 = 0.98 \times 0.509427 - 0.5 = -0.00076154$, $\mathbb{S}_{14} = 0.98 \times 0.509002 - 0.5 = -0.00117804$,
$\mathbb{S}_5 = 0.98 \times 0.509800 - 0.5 = -0.00039600$, $\mathbb{S}_{10} = 0.98 \times 0.509338 - 0.5 = -0.00084876$, $\mathbb{S}_{15} = 0.98 \times 0.508922 - 0.5 = -0.00125644$,
$\mathbb{S}_6 = 0.98 \times 0.509704 - 0.5 = -0.00049008$, $\mathbb{S}_{11} = 0.98 \times 0.509252 - 0.5 = -0.00093304$.

From the results of the above numerical calculation we have $\mathbb{S}_t < 0$ for $15 \geq t > 1$, hence it is *strictly optimal* to skip the search over $15 \geq t > 1$ due to (6.2.9(p.30)), i.e., $\texttt{Skip}_\blacktriangle$. However, since $V_t - \beta V_{t-1} = 0$ for $15 \geq t > 1$ (see $(V_t - \beta V_{t-1})$-column in the above table), we have $V_{15} = \beta V_{14} = \cdots = \beta^{14} V_1$, i.e., the profit attained are indifferent over $15 \geq t > 0$. This is not a contradiction, which is a false feeling caused by confusion from the jumble of intuition and theory (see Alice 2(p.44)).

Numerical Example **3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\}$ **(buying model)**

This is the numerical example for $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau} \langle t^\bullet_\tau \rangle}_{\|}$ in $\texttt{S}_1$(p.62) $\boxed{\circledS_\blacktriangle \,|\, \circledcirc\ \|}$ of Pom 17.5.4(p.120) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.† Then, we have $s_{\tilde{\mathcal{L}}} = 0.323274$ (see Section A 6(p.312)). Hence, the optimal initiating time t^*_τ is given by t attaining $\min_{\tau \geq t > 0} I^t_\tau$ (see (7.2.5(p.44))).‡ The bullet • in Figure 17.6.3(p.127) below shows the optimal initiating time for each of $\tau = 2, 3, \cdots, 15$ (see t^*_τ-column in the table below). From the figure and table we see that $t^*_\tau = \tau$ for $\tau = 2, 3, \cdots, 7$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{7 \geq \tau > 1} \langle \tau \rangle}_\blacktriangle$ (see $\texttt{S}_1$(p.62)(1)) and that $t^*_\tau = 7$ for $\tau = 8, 9, \cdots, 15$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > 7} \langle 7 \rangle}_{\|}$ (see $\texttt{S}_1$(p.62)(2)). In the numerical example, note the fact that $\tilde{\mathbb{S}} = \tilde{L}(V_{\tau-1})$ are all negative (< 0 ($-$), i.e., $\texttt{Skip}_\blacktriangle$) for $t = 2, 3, \cdots, 7$ and positive (> 0 (+), i.e., $\texttt{Conduct}_\blacktriangle$) for $t = 8, 9, \cdots, 15$. Moreover, note that we have $V_t - \beta V_{t-1} = 0$ or equivalently $V_t = \beta V_{t-1}$ for $t = 8, 9, \cdots, 15$ and $V_t - \beta V_{t-1} < 0$ or equivalently $V_t < \beta V_{t-1}$ for $t = 2, 3, \cdots, 7$ (see $V_t - \beta V_{t-1}$-column), hence $V_{15} = \beta V_{14} = \beta^2 V_{13} = \cdots = \beta^8 V_7 < \beta^9 V_6 < \beta^{10} V_5 < \cdots < \beta^{14} V_1$ (see $\beta^{15-t} V_t$-column), so we have $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > 7} \langle 7 \rangle}_{\|}$.



t	β^{15-t}	V_t	$V_t - \beta V_{t-1}$	$\beta^{15-t} V_t$	t^*_τ	$\tilde{\mathbb{S}}_t = \tilde{L}(V_{t-1})$	decision
0							
1	0.753641	0.544900		0.410658			
2	0.769022	0.442388	−0.091614	0.340206	2	−0.091614	$\texttt{Conduct}_\blacktriangle$
3	0.784716	0.391004	−0.042535	0.306827	3	−0.042535	$\texttt{Conduct}_\blacktriangle$
4	0.800731	0.361335	−0.021849	0.289332	4	−0.021849	$\texttt{Conduct}_\blacktriangle$
5	0.817072	0.343013	−0.011094	0.280266	5	−0.011094	$\texttt{Conduct}_\blacktriangle$
6	0.833747	0.331264	−0.004889	0.276190	6	−0.004889	$\texttt{Conduct}_\blacktriangle$
7	0.850763	0.323555	−0.001084	0.275268	7	−0.001084	$\texttt{Conduct}_\blacktriangle$
8	0.868125	0.317084	0.000000	0.275268	7	+0.001338	$\texttt{Skip}_\blacktriangle$
9	0.885842	0.310742	0.000000	0.275268	7	+0.003326	$\texttt{Skip}_\blacktriangle$
10	0.903920	0.304527	0.000000	0.275268	7	+0.005233	$\texttt{Skip}_\blacktriangle$
11	0.922368	0.298437	0.000000	0.275268	7	+0.007064	$\texttt{Skip}_\blacktriangle$
12	0.941192	0.292468	0.000000	0.275268	7	+0.008822	$\texttt{Skip}_\blacktriangle$
13	0.960400	0.286618	0.000000	0.275267	7	+0.010508	$\texttt{Skip}_\blacktriangle$
14	0.980000	0.280886	0.000000	0.275268	7	+0.012127	$\texttt{Skip}_\blacktriangle$
15	1.000000	0.275268	0.000000	0.275268	7	+0.013680	$\texttt{Skip}_\blacktriangle$

Figure 17.6.3: Graphs of $I^t_\tau = \beta^{\tau-t} V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$

†Note that $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01+1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949$, hence $\beta\mu + s = 0.4949 + 0.05 = 0.5449 < 1.00 = b$. In addition, $s_{\tilde{\mathcal{L}}} = 0.323274 > 0.05 = s$. Thus, the conditions for the assertions are satisfied.

‡Note that this is a selling model with cost minimization.

Chapter 18

Conclusions of Part 2 (Integrated Theory)

Below let us summarize the whole discussions over Chaps. 10(p.55) - 17(p.117) .

$\overline{\overline{\mathsf{C}}}$1. **Two preliminary steps**

a. **Proofs of assertions on underlying functions**
The first preliminary step in constructing the integrated theory is to prove assertions on underlying functions (see Chap. 10(p.55)).

b. **Proofs of four theorems**
The second preliminary step is to prove the following four theorems.

1. **Symmetry theorem** ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$)
The concept of symmetry between selling problem and buying problem was first vaguely inspired from the pattern of the *yin-yang principle* in an ancient Chinese philosophy. This rather superstitious and shaky concept was first formalized by the introduction of the *reverse operation* $\mathcal{R}$ (see Section 12.1.1(p.69) and Step 2 (p.75)). After that, through more than twenty years of trial-and-errors, this concept led us to the *correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$ (see Lemma 12.3.1(p.72) and Step 3 (p.75)) and then to *identity replacement operation* $\mathcal{I}_{\mathbb{R}}$ (see Lemma 12.3.3(p.73) and Step 4 (p.76)). Finally, the above three operations were compiled into a single operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}$ (see (12.5.30(p.77))), called the *symmetry transformation operation*, yielding Theorem 12.5.1(p.80) (*symmetry theorem*), which derives $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ in Tom's 11.2.1(p.61) and 11.2.2(p.62) . In addition, we obtained Theorem 12.8.1(p.87) (the inverse of Theorem 12.5.1(p.80)), which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$.
2. **Analogy theorem** ($\mathbb{R} \leftrightarrow \mathbb{P}$)
In the earlier stage of this study, we did not anticipate at all that there would exist a relation between asset trading problem with $\mathbb{R}$-mechanism and asset trading problem with $\mathbb{P}$-mechanism. However, as proceeding with analyses of both problems, we gradually noticed similarities between the two procedures for treating both problems. This realization led us, as if solving the *jigsaw puzzle*, to the existence of an analogous relation between the above two problems. This recognition eventually was materialized by the proof of Lemmas 10.1.1(p.55) and 13.2.1(p.93) , which finally leads to the *analogy replacement operation* $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (13.2.1(p.93))). This finding produced Theorems 13.3.1(p.98) and 13.3.2(p.98) (*analogy theorem*), which combines $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
3. **Symmetry theorem** ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$)
While the two symmetry theorems in $\overline{\overline{\mathsf{C}}}$1b1(p.129) combine $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ nd $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, we can relatively easily obtain the two theorems, Theorems 14.5.1(p.106) and 14.5.2(p.107) , which combine $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$.
4. **Analogy theorem** ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)
In Chap. 15(p.111) it was demonstrated that the two theorems, Theorems 15.1.1(p.112) and 15.1.2(p.112) , combining $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ can be relatively easily derived.

$\overline{\overline{\mathsf{C}}}$2. **Integrated theory**
The highly distinguishing results in the present paper is the successful construction of the integrated theory (see Motive 2(p.3) and Chap. 16(p.115)), by use of which all models included in a given structured-unit-of-models (see Section 3.3(p.18)) can be systematically analyzed. The theory consists of the two symmetry theorems (see Theorems 12.5.1(p.80) and 14.5.1(p.106)) and the two analogy theorems (see Theorems 13.3.1(p.98) and 15.1.1(p.112)). The former two combines the asset selling problem and the asset buying problem and the latter two combines the asset trading problem with $\mathbb{R}$-mechanism and the asset trading problem with $\mathbb{P}$-mechanism. The integrated theory plays a decisively important role in the analysis of not only all models in the present paper but also all variations of the models which will be dealt with in the future (see Chap. 30(p.287)). However, the integrated theory is not always versatile, which has the following two weak points.

a. **Market restriction**
Here, let us note again that the integrated theory can be constructed under the premise that the price ξ, whether $\mathbb{R}$-price or $\mathbb{P}$-price, is defined on the total market $\mathscr{F}$ (see (17.1.1(p.117))). Under the integration theory we clarified that

$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model with $\mathbb{R}$-mechanism) can be derived so as to be *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{R}$-mechanism) and that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling problem with $\mathbb{P}$-mechanism) can be derived so as to be *analogous* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling problem $\mathbb{R}$-mechanism). However, since trading on the normal market in the real world is usually conducted on the positive market $\mathscr{F}^+$ (see (17.2.2(p.117))), it is an open question whether symmetry and analogy on $\mathscr{F}$ are inherited by $\mathscr{F}^+$. To approach this problem, in this paper, we employ the methodology of restricting results obtained on $\mathscr{F}$ to $\mathscr{F}^+$ by using Lemmas 17.4.1(p.118) - 17.4.3(p.118) . Through this methodology, we will show in C2C(p.134) , C3C(p.135) , C2C(p.148) , and C3C(p.149) that the symmetrical relation and the analogouse relation can strikingly collapse on $\mathscr{F}^+$.

b. **Symmetry and/or analogy among SOE's**

As stated in Section 16.3.1(p.116) , the integrated theory has the following imitation. In Model 1, the successful construction of the integrated theory is based on the fact that the symmetrical relation between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and the analogy relation between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ must be satisfied (see Sections 12.11(p.87) and 13.8(p.99)). However, for Models 2/3, from Tables 6.4.3(p.41) -6.4.6(p.41) we see that although the symmetrical relation always holds between $\mathtt{SOE}\{\mathsf{M}{:}1/2/3[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1/2/3[\mathbb{R}][\mathtt{A}]\}$ (compare (I) and (II)), the analogical relation between $\mathtt{SOE}\{\mathsf{M}{:}2/3[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\mathsf{M}{:}2/3[\mathbb{P}][\mathtt{A}]\}$ (compare (I) and (III)) does not hold. In other words, while the symmetry theorems can be applied for Models 1/2/3, the analogy theorems cannot be applied for Models 2/3. Accordingly, it follows that the integrated theory are applicable in discussions *only* related to symmetry. For the treatment of the case where the analogy theorem cannot be applied, see Section 20.1.5(p.166) .

$\overline{\overline{\mathsf{C}}}$3. Summary of operations

For convenience of reference, let us summarize all operations depicted in Figure 16.2.1(p.115) below.

$$(12.5.29(p.77)) \rightarrow \qquad \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \left\{\begin{matrix} a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ b, \mu, a, x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t \end{matrix}\right\}. \tag{18.0.1}$$

$$(12.8.21(p.86)) \rightarrow \qquad \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \left\{\begin{matrix} b, \mu, a, x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t \end{matrix}\right\}. \tag{18.0.2}$$

$$(14.5.3(p.106)) \rightarrow \qquad \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \left\{\begin{matrix} a^\star, a, b, x_L, x_K, \kappa, s_{\mathcal{L}}, T, L, K, \mathcal{L}, V_t \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ b^\star, b, a, x_{\tilde{L}}, x_{\tilde{K}}, \tilde{\kappa}, s_{\tilde{\mathcal{L}}}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t \end{matrix}\right\}. \tag{18.0.3}$$

$$(14.5.11(p.107)) \rightarrow \qquad \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} = \left\{\begin{matrix} b^\star, b, a, x_{\tilde{L}}, x_{\tilde{K}}, \tilde{\kappa}, s_{\tilde{\mathcal{L}}}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ a^\star, a, b, x_L, x_K, \kappa, s_{\mathcal{L}}, T, L, K, \mathcal{L}, V_t \end{matrix}\right\}. \tag{18.0.4}$$

$$(13.2.1(p.93)) \rightarrow \qquad \mathcal{A}_{\mathbb{R}\to\mathbb{P}} = \{a \to a^\star,\ \mu \to a\}. \tag{18.0.5}$$

$$(13.3.5(p.98)) \rightarrow \qquad \mathcal{A}_{\mathbb{P}\to\mathbb{R}} = \{a^\star \to a,\ a \to \mu\}. \tag{18.0.6}$$

$$(15.1.38(p.113)) \rightarrow \qquad \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \{b \to b^\star,\ \mu \to b\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}. \tag{18.0.7}$$

$$(15.1.39(p.113)) \rightarrow \qquad \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \{b^\star \to b,\ b \to \mu\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}. \tag{18.0.8}$$

Part 3

No-Recall-Model

Chapter 19

Analysis of Model 1

19.1 Search-Allowed-Model 1: $\mathscr{Q}\{\mathsf{M}{:}1[\mathtt{A}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

All analyses of the search-Allowed-model 1 already completed in Part 2(p.51). Below, let us summarize the whole conclusions obtained there.

19.1.1 Conclusion 1 (Search-Allowed-Model 1)

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$$\mathscr{Q}\langle \mathsf{M}:1[\mathtt{A}]\rangle = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\},$$

are given by

$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$
↓	↓	↓	↓
Tom's 11.2.1(p.61), 11.2.2(p.62),	12.7.1(p.84), 12.7.2(p.84),	13.4.1(p.98), 13.4.2(p.98),	14.7.1(p.109), 14.7.2(p.109).

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$$\mathscr{Q}\langle \mathsf{M}:1[\mathtt{A}]\rangle^+ = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+, \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+, \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+, \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\},$$

are given by

$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$
↓	↓	↓	↓
Pom's 17.5.1(p.118), 17.5.2(p.119),	17.5.3(p.120), 17.5.4(p.120),	17.5.5(p.122), 17.5.6(p.122),	17.5.7(p.123), 17.5.8(p.123).

■ Closely looking into all the above assertion systems $\mathscr{A}$ leads to the following conclusions.

C1. Mental Conflict

On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$ we have:

a. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ (selling model) is nondecreasing in t as in Figure 7.3.1(p.47) (I) (see Tom's 11.2.1(p.61) (a) and 11.2.2(p.62) (a)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

b. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ (selling model) is nondecreasing in t as in Figure 7.3.1(p.47) (I) (see Lemma 13.7.1(p.99)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

c. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (buying model) is nonincreasing in t as in Figure 7.3.1(p.47) (II) (see Tom's 12.7.1(p.84) (a) and 12.7.2(p.84) (a)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

d. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ (buying model) is nonincreasing in t as in Figure 7.3.1(p.47) (II) (see Lemma 14.8.1(p.109)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

The above results can be summarized as below.

A. On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6).

C2. Symmetry

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Pom } 17.5.3\text{(p.120)} \sim \texttt{Pom } 17.5.1\text{(p.118)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{+} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}),$$
$$\texttt{Pom } 17.5.7\text{(p.123)} \sim \texttt{Pom } 17.5.5\text{(p.122)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{+}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Pom } 17.5.4\text{(p.120)} \not\sim \texttt{Pom } 17.5.2\text{(p.119)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{+} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}) \cdots (s^1),$$
$$\texttt{Pom } 17.5.8\text{(p.123)} \not\sim \texttt{Pom } 17.5.6\text{(p.122)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{+}) \cdots (s^2).$$

b. On $\mathscr{F}^{\pm}$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Mim } 17.5.3\text{(p.121)} \sim \texttt{Mim } 17.5.1\text{(p.119)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}),$$
$$\texttt{Mim } 17.5.7\text{(p.124)} \sim \texttt{Mim } 17.5.5\text{(p.122)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Mim } 17.5.4\text{(p.121)} \sim \texttt{Mim } 17.5.2\text{(p.119)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}),$$
$$\texttt{Mim } 17.5.8\text{(p.124)} \sim \texttt{Mim } 17.5.6\text{(p.122)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm}).$$

c. On $\mathscr{F}^{-}$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Nem } 17.5.3\text{(p.121)} \sim \texttt{Nem } 17.5.1\text{(p.120)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}),$$
$$\texttt{Nem } 17.5.7\text{(p.125)} \sim \texttt{Nem } 17.5.5\text{(p.123)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{-}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Nem } 17.5.4\text{(p.121)} \not\sim \texttt{Nem } 17.5.2\text{(p.120)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}) \cdots (s^3),$$
$$\texttt{Nem } 17.5.8\text{(p.125)} \not\sim \texttt{Nem } 17.5.6\text{(p.123)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{-}) \cdots (s^4).$$

The above results can be summarized as below.

A. On $\mathscr{F}^{\pm}$, for any $\beta \leq 1$ and $s \geq 0$, the symmetry is inherited (see C2b(p.134)).

B. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2a1(p.134)/C2c1(p.134)).

C. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the symmetry collapses (see $(s^1)/(s^2)/(s^3)/(s^4)$).

C3. Analogy

a. On $\mathscr{F}^{+}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Pom } 17.5.5\text{(p.122)} \bowtie \texttt{Pom } 17.5.1\text{(p.118)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}),$$
$$\texttt{Pom } 17.5.7\text{(p.123)} \bowtie \texttt{Pom } 17.5.3\text{(p.120)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Pom } 17.5.6\text{(p.122)} \not\bowtie \texttt{Pom } 17.5.2\text{(p.119)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \not\bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}) \cdots (a^1),$$
$$\texttt{Pom } 17.5.8\text{(p.123)} \bowtie \texttt{Pom } 17.5.4\text{(p.120)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{+} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{+}).$$

b. On $\mathscr{F}^{\pm}$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Mim } 17.5.5\text{(p.122)} \bowtie \texttt{Mim } 17.5.1\text{(p.119)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}),$$
$$\texttt{Mim } 17.5.7\text{(p.124)} \bowtie \texttt{Mim } 17.5.3\text{(p.121)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Mim } 17.5.6\text{(p.122)} \bowtie \texttt{Mim } 17.5.2\text{(p.119)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}),$$
$$\texttt{Mim } 17.5.8\text{(p.124)} \bowtie \texttt{Mim } 17.5.4\text{(p.121)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm}).$$

c. On $\mathscr{F}^{-}$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Nem } 17.5.5\text{(p.123)} \bowtie \texttt{Nem } 17.5.1\text{(p.120)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}),$$
$$\texttt{Nem } 17.5.7\text{(p.125)} \bowtie \texttt{Nem } 17.5.3\text{(p.121)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Nem } 17.5.6\text{(p.123)} \bowtie \texttt{Nem } 17.5.2\text{(p.120)} \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}),$$
$$\texttt{Nem } 17.5.8\text{(p.125)} \not\bowtie \texttt{Nem } 17.5.4\text{(p.121)} \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^{-} \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-}) \cdots (a^2).$$

The above results can be summarized as below.

A. On $\mathscr{F}^{\pm}$, for any $\beta \leq 1$ and $s \geq 0$, the analogy are inherited (see C2b(p.134)).

B. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the analogy is inherited (see C3a1(p.134) /C3c1(p.134)).

C. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the analogy *partially* collapses (see $(a^1)/(a^2)$).

C4. Optimal Initiation Time (OIT)

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 17.5.1(p.118) , **Mim** 17.5.1(p.119) , **Nem** 17.5.1(p.120) ,
Pom 17.5.3(p.120) , **Mim** 17.5.3(p.121) , **Nem** 17.5.3(p.121) ,
Pom 17.5.5(p.122) , **Mim** 17.5.5(p.122) , **Nem** 17.5.5(p.123) ,
Pom 17.5.7(p.123) , **Mim** 17.5.7(p.124) , **Nem** 17.5.7(p.125)

we obtain Table 19.1.1(p.135) below (the symbol "○" in the table below represents "`possible`"):

Table 19.1.1: Possible `OIT` ($\beta = 1$ and $s = 0$)

		$\mathscr{F}^{+}$	$\mathscr{F}^{\pm}$	$\mathscr{F}^{-}$
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\|}$	ⓢ$_{\|}$			
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\triangle}$	ⓢ$_{\triangle}$			
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\blacktriangle}$	ⓢ$_{\blacktriangle}$	○	○	○
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\|}$	◎$_{\|}$			
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\triangle}$	◎$_{\triangle}$			
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\blacktriangle}$	◎$_{\blacktriangle}$			
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\|}$	ⓓ$_{\|}$			
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\triangle}$	ⓓ$_{\triangle}$			
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\blacktriangle}$	ⓓ$_{\blacktriangle}$			

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 17.5.2(p.119) , **Mim** 17.5.2(p.119) , **Nem** 17.5.2(p.120) ,
Pom 17.5.4(p.120) , **Mim** 17.5.4(p.121) , **Nem** 17.5.4(p.121) ,
Pom 17.5.6(p.122) , **Mim** 17.5.6(p.122) , **Nem** 17.5.6(p.123) ,
Pom 17.5.8(p.123) , **Mim** 17.5.8(p.124) , **Nem** 17.5.8(p.125)

we obtain Table 19.1.2(p.135) below:

Table 19.1.2: Possible `OIT` ($\beta < 1$ or $s > 0$)

		$\mathscr{F}^{+}$	$\mathscr{F}^{\pm}$	$\mathscr{F}^{-}$
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\|}$	ⓢ$_{\|}$			
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\triangle}$	ⓢ$_{\triangle}$			
[ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$]$_{\blacktriangle}$	ⓢ$_{\blacktriangle}$	○	○	○
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\|}$	◎$_{\|}$	○	○	○
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\triangle}$	◎$_{\triangle}$			
[◎ $\texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle$]$_{\blacktriangle}$	◎$_{\blacktriangle}$			
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\|}$	ⓓ$_{\|}$	○	○	○
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\triangle}$	ⓓ$_{\triangle}$			
[● $\texttt{dOITd}_\tau\langle 0\rangle$]$_{\blacktriangle}$	ⓓ$_{\blacktriangle}$			

c. The table below is the list of the occurrence rates of ⓢ, ◎, and ⓓ on $\mathscr{F}$ appearing in the primitive **Tom** 11.2.1(p.61) (▣) and **Tom** 11.2.2(p.62) (▣) (see Def. 11.2.2(p.61)).

Table 19.1.3: Occurence rates of ⓢ, ◎, and ⓓ on $\mathscr{F}$

ⓢ			◎			ⓓ		
50.0 % / 5			10.0 % / 1			40.0 % / 4		
ⓢ$_{\|}$	ⓢ$_{\triangle}$	ⓢ$_{\blacktriangle}$	◎$_{\|}$	◎$_{\triangle}$	◎$_{\blacktriangle}$	ⓓ$_{\|}$	ⓓ$_{\triangle}$	ⓓ$_{\blacktriangle}$
–	×	`possible`	`possible`	×	×	`possible`	×	×
– % / –	0.0 % / 0	50.0 % / 5	10.0 % / 1	0.0 % / 0	0.0 % / 0	40.0 % / 4	0.0 % / 0	0.0 % / 0

C5. Null-Time-Zone and Deadline-Engulfing

From Table 19.1.3(p.135) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▴ (strict optimality).

b. As a whole, we have Ⓢ, ◎, and ⓓ at 50.0%, 10.0%, and 40.0% respectively where

1. $Ⓢ_{\|}$ cannot be defined due to Remark 7.2.3(p.45).
2. $◎_{\|}$ is possible (10.0 %).
3. $ⓓ_{\|}$ is possible (40.0 %).
4. $Ⓢ_{\vartriangle}$ never occur (0.0 %).
5. $◎_{\vartriangle}$ never occur (0.0 %).
6. $ⓓ_{\vartriangle}$ never occur (0.0 %).
7. $Ⓢ_{\blacktriangle}$ is possible (50.0 %).
8. $◎_{\blacktriangle}$ never occur (0.0 %).
9. $ⓓ_{\blacktriangle}$ never occur (0.0 %).

From the above results we see that on $\mathscr{F}$:

A. ◎ and ⓓ causing the `null-time-zone` are possible at 50.0% (= 10.0% + 40.0%).

B. $◎_{\blacktriangle}$ *strictly* causing the `null-time-zone` is impossible (0.0%).

C. $ⓓ_{\blacktriangle}$ *strictly* causing the `null-time-zone` is impossible (0.0%), i.e., the deadline-engulfing is impossible.

19.2 Search-Enforced-Model 1: $\mathcal{Q}\{\mathsf{M}{:}1[\mathsf{E}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}], \mathsf{M}{:}1[\mathbb{P}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$

19.2.1 Preliminary

As ones corresponding to Theorems 12.5.1(p.80), 13.3.1(p.98), and 14.5.1(p.106), let us consider the following three theorems:

Theorem 19.2.1 (symmetry[$\mathbb{R} \to \mathbb{R}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{19.2.1}$$

Theorem 19.2.2 (analogy[$\mathbb{R} \to \mathbb{P}$)]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{19.2.2}$$

Theorem 19.2.3 (symmetry[$\mathbb{P} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}]. \quad \square 9039 \tag{19.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}], \tag{19.2.4}$$
$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}], \tag{19.2.5}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}], \tag{19.2.6}$$

corresponding to (12.5.34(p.77)), (13.2.4(p.93)), and (14.5.4(p.106)). Then, for the same reason as in Chap. 15(p.111) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}] \tag{19.2.7}$$

holds (corresponding to (15.1.33(p.112))) and that we have the following theorem, corresponding to Theorem 15.1.1(p.112)

Theorem 19.2.4 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{19.2.8}$$

In fact, from the comparisons of (I) and (II), of (I) and (III), of (III) and (IV), and of (II) and (IV) in Table 6.4.2(p.41) we can easily show that (19.2.4(p.136)) - (19.2.7(p.136)) hold.

19.2.2 M:1[ℝ][E]

19.2.2.1 Analysis

To begin with, let us note that

$$\lambda = 1 \tag{19.2.9}$$

is assumed in the model (see $A2$(p.21)), hence from (10.2.1(p.56)) we have

$$\delta = 1 \tag{19.2.10}$$

☐ **Tom 19.2.1** ($\blacksquare\mathscr{A}\{$M:1[ℝ][E]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.4(p.25)) we have $K(x) = T(x) \geq 0 \cdots (1)$ for any x due to Lemma 10.1.1(p.55) (g).

(a) From (6.4.10(p.41)) with $t = 2$ we have $V_2 = K(V_1) + V_1 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.2.2(p.57) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) From (6.4.9(p.41)) we have $V_1 = \mu < b \cdots (2)$. Suppose $V_{t-1} < b$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (h) we have $V_t < K(b) + b = T(b) + b = b$ due to (1) and Lemma 10.1.1(p.55) (g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.57) (d), thus $L(V_{t-1}) > 0$ for $\tau \geq t > 1$. Then, from (6.4.10(p.41)) and from (5.1.8(p.25)) we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) > 0$ for $\tau \geq t > 1$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Hence, since $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, thus $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. ■

For explanatory simplicity, let us define the statement below:

$$\mathsf{S}_2\,\boxed{\circledS^{\blacktriangle}\,|\,\circledcirc\ {\scriptstyle\parallel}\,|\,\circledcirc\ {\scriptstyle\triangle}\,|\,\circledcirc\ {\scriptstyle\blacktriangle}} = \left\{\begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1)\ \boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_{\blacktriangle}, \\ (2)\ \boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\triangle}, \\ (3)\ \boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\parallel}\ (\!(\ \boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\blacktriangle}\ )\!).^\dagger \end{array}\right\}$$

☐ **Tom 19.2.2** ($\blacksquare\mathscr{A}\{$M:1[ℝ][E]$\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.

(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.*
   - i. *Let $b > 0$ $(\!(\kappa > 0)\!)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   - ii. *Let $b = 0$ $(\!(\kappa = 0)\!)$.*
     1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   - iii. *Let $b < 0$ $(\!(\kappa < 0)\!)$.*
     1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.
     2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\mathsf{S}_2\,\boxed{\circledS^{\blacktriangle}\,|\,\circledcirc\ {\scriptstyle\parallel}\,|\,\circledcirc\ {\scriptstyle\triangle}\,|\,\circledcirc\ {\scriptstyle\blacktriangle}}$ *is true.* $\mapsto$ $\qquad\to \circledcirc_{\blacktriangle}$

● *Proof* Let $\beta < 1$ or $s > 0$. From (6.4.10(p.41)) and (5.1.8(p.25)), we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) \cdots (1)$ for $t > 1$. From (6.4.10(p.41)) with $t = 2$ we have $V_2 - V_1 = K(V_1) \cdots (2)$.

(a) Note that $V_1 = \beta\mu - s$ from (6.4.9(p.41)). Then, from Lemma 10.2.2(p.57) (j2) we have $x_K \geq \beta\mu - s$ due to (19.2.9(p.137)) and (19.2.10(p.137)), hence $x_K \geq V_1 \cdots (3)$. Accordingly, since $K(V_1) \geq 0$ due to Lemma 10.2.2(p.57) (j1), we have $V_1 \leq V_2$ from (2). Suppose $V_{t-1} \leq V_t$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Note (3). Suppose $V_{t-1} \leq x_K$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq K(x_K) + x_K = x_K$. Hence, by induction $V_t \leq x_K$ for $t > 0$, i.e., V_t is upper bounded in t, thus V_t converges to a finite V as $t \to \infty$. Accordingly, from (6.4.10(p.41)) we have $V = K(V) + V$, hence $K(V) = 0$, thus $V = x_K$ due to Lemma 10.2.2(p.57) (j1).

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† The outer side of $(\!(\ )\!)$ is for $s = 0$ and the inner side is for $s > 0$.

(b) Let $\beta\mu \geq b \cdots$ **(4)**. Then $x_L \leq \beta\mu - s$ from Lemma 10.2.4(p.59) (b1), hence $x_L \leq V_1$ from (6.4.9(p.41)), so $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a), hence $L(V_{t-1}) \leq 0 \cdots$ **(5)** for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau \leq \beta V_{\tau-1}$, $V_{\tau-1} \leq \beta V_{\tau-2}$, $\cdots$, $V_2 \leq \beta V_1$, so $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1}V_1$, hence it follows that $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$.

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots$ **(6)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K \cdots$ **(7)** due to Lemma 10.2.3(p.58) (b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(8)**.

(c1i) Let $\mu - s \leq a$. Then, noting (6), (19.2.9(p.137)), and (19.2.10(p.137)), we have $x_K = \mu - s \cdots$ **(9)** from Lemma 10.2.2(p.57) (j2), hence $x_K = V_1$ from (6.4.9(p.41)). Let $V_{t-1} = x_K$. Then, from (6.4.10(p.41)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ from (7). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, thus $L(V_{t-1}) = 0$ for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau = \beta V_{\tau-1}$, $V_{\tau-1} = \beta V_{\tau-2}$, $\cdots$, $V_2 = \beta V_1$, so $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1}V_1$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ (see Preference-Rule 7.2.1(p.45)).

(c1ii) Let $\mu - s > a$. Then, since $V_1 > a$ from (6.4.9(p.41)), we have $V_{t-1} > a$ for $t > 1$ from (a). From (7) and Lemma 10.2.2(p.57) (j2) we have $x_L = x_K > \mu - s = V_1$ from (6.4.9(p.41)). Let $V_{t-1} < x_L$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(x_L) + x_L = x_L$ due to (8), hence by induction $V_{t-1} < x_L$ for $t > 1$. Thus, since $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.57) (e1), for the same reason as in the proof of Tom 19.2.1(p.137) (b) we obtain $\boxed{\text{Ⓢ}\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c2) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c2i) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K \cdots$ **(10)** from Lemma 10.2.3(p.58) (c ⟪d⟫). Now, since $x_K \geq \beta\mu - s$ due to Lemma 10.2.2(p.57) (j2), we have $x_K \geq V_1$ from (6.4.9(p.41)). Suppose $x_K \geq V_{t-1}$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq K(x_K) + x_K = x_K$. Thus, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (10). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 19.2.1(p.137) (b) we obtain $\boxed{\text{Ⓢ}\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c2ii) Let $b = 0$ ⟪$\kappa = 0$⟫. Then $x_L = x_K \cdots$ **(11)** from Lemma 10.2.3(p.58) (c ⟪d⟫), hence $K(x_L) = K(x_K) = 0 \cdots$ **(12)**.

(c2ii1) Let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s \cdots$ **(13)** from Lemma 10.2.2(p.57) (j2), we have $x_K = V_1$ from (6.4.9(p.41)). Let $V_{t-1} = x_K$. Then, from (6.4.10(p.41)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (11). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, for the same reason as in the proof of (c1i) we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$ from (6.4.9(p.41)), we have $V_{t-1} > a$ for $t > 1$ from (a). From (11) and Lemma 10.2.2(p.57) (j2) we have $x_L = x_K > \beta\mu - s = V_1$. Let $V_{t-1} < x_L$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(x_L) + x_L = x_L$ due to (12), hence, by induction $V_{t-1} < x_L$ for $t > 1$. Consequently, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 19.2.1(p.137) (b) we obtain $\boxed{\text{Ⓢ}\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c2iii) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ **(14)** from Lemma 10.2.3(p.58) (c ⟪d⟫).

(c2iii1) Let $\beta\mu - s \leq a$, then $x_L < x_K = \beta\mu - s = V_1$ from (14), Lemma 10.2.2(p.57) (j2) and (6.4.9(p.41)), so $x_L \leq V_1$. Let $s_{\mathcal{L}} \leq s$, then $x_L \leq \beta\mu - s$ due to Lemma 10.2.4(p.59) (c), hence $x_L \leq V_1$. Therefore, whether $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, hence $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$.

(c2iii2) Suppose $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Hence, since $V_1 > a$ from (6.4.9(p.41)), we have $V_{t-1} > a$ for $t > 0$ from (a). Then, since $x_K > x_L > \beta\mu - s = V_1 \cdots$ **(15)** from (14), Lemma 10.2.4(p.59) (c), and (6.4.9(p.41)), we have $K(V_1) > 0$ from Lemma 10.2.2(p.57) (j1), hence $V_2 > V_1$ from (2). Suppose $V_{t-1} < V_t$. Then, from (6.4.10(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (15). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \leq x_L \cdots$ **(16)** from (a). Then, since $V = x_K$ due to (a), we have the contradiction of $V = x_K > x_L \geq V$ due to (14) and (16). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots \quad \cdots \textbf{(17)},$$

from which we have

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t^\bullet_\tau}, \qquad x_L < V_{t-1}, \quad t > t^\bullet_\tau + 1. \tag{19.2.11}$$

Hence, we have

$$
\begin{array}{lll}
L(V_{t-1}) > 0 & \cdots \textbf{(18)}\ t^\bullet_\tau \geq t > 1 & (\leftarrow \text{Corollary 10.2.1(p.57) (a)}) \\
L(V_{t^\bullet_\tau}) \leq 0 & \cdots \textbf{(19)} & (\leftarrow \text{Corollary 10.2.1(p.57) (a)}) \\
L(V_{t-1}) = ⟪< 0⟫^\dagger & \cdots \textbf{(20)}\ t > t^\bullet_\tau + 1 & (\leftarrow \text{Lemma 10.2.1(p.57) (d⟪e1⟫)})
\end{array}
$$

† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

- Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) > 0 \cdots (21)$ for $\tau \geq t > 1$ from (18). Since $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (1) and (21), we have $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$. Therefore, since $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, we obtain $t^*_\tau = \tau$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle \tau \rangle}_\blacktriangle$, thus $\mathtt{S}_2(1)$ is true. Let us note here that when $\tau = t^\bullet_\tau$, we have $V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (22)$.

- Let $\tau = t^\bullet_\tau + 1$. From (1) with $t = t^\bullet_\tau + 1$ and (19) we have $V_{t^\bullet_\tau+1} - \beta V_{t^\bullet_\tau} \leq 0$, hence $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$. Accordingly, from (22) we have

$$V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau} V_1 \cdots (23),$$

  thus $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau \rangle}_\vartriangle$, thus $\mathtt{S}_2(2)$ is true.

- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau+1}) = (\!(<)\!)\ 0$ from (20) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1}$ from (1), hence from (23) we have

$$V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_1$$

  Similarly we have

$$V_{t^\bullet_\tau+3} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^2 V_{t^\bullet_\tau+1} \leq \beta^3 V_{t^\bullet_\tau} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+2} V_1.$$

  By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain

$$V_\tau = (\!(<)\!)\ \beta V_{\tau-1} = (\!(<)\!)\ \cdots = (\!(<)\!)\ \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1. \cdots (24)$$

  - Let $s = 0$. Then (24) can be written as

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} = \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1,$$

    hence we have $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau \rangle}_{||}$ (see Preference Rule 7.2.1(p.45)), hence $\mathtt{S}_2(3)$ is true.

  - Let $s > 0$. Then (24) can be written as

$$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} < \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1, \qquad (19.2.12)$$

    hence we have $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau \rangle}_\blacktriangle$, hence $\mathtt{S}_2(3)$ is true. ▮

19.2.2.2 Market Restriction

19.2.2.2.1 Positive Restriction

□ **Pom 19.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau \rangle}_\blacktriangle$.

● *Proof* The same as Tom 19.2.1(p.137) due to Lemma 17.4.4(p.118). ▮

□ **Pom 19.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$* (impossible).

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*
   1. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1 \rangle}_{||}$.
   2. *Let $\mu - s > a$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau \rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau \rangle}_\blacktriangle$.
3. *Let $\beta < 1$ and $s > 0$.*
   1. *Let $\beta\mu > s$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau \rangle}_\blacktriangle$.
   2. *Let $\beta\mu \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1 \rangle}_\vartriangle$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a) with $\lambda = 1$.

(a) The same as Tom 19.2.2(p.137) (a).

(b,c) Always $\beta\mu < b$ from [3(p.118)], hence $\beta\mu \geq b$ is impossible.

(c1-c1ii) The same as Tom 19.2.2(p.137) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.2(p.137).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$, hence $\kappa > 0$ due to (2). Hence it suffices to consider only (c2i) of Tom 19.2.2(p.137).

(c3ii) Let $\beta\mu \leq s$, hence $\kappa \leq 0$ due to (2). Then, since $\beta\mu - s \leq 0 < a$, it suffices to consider only (c2iii1) of Tom 19.2.2(p.137). ▮

19.2.2.2.2 Mixed Restriction

☐ **Mim 19.2.1** ($\mathscr{A}\{$M:1[ℝ][E]$^\pm\}$) *Suppose $a \le 0 \le 0$. Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

• *Proof* The same as Tom 19.2.1(p.137) due to Lemma 17.4.4(p.118). ▮

☐ **Mim 19.2.2** ($\mathscr{A}\{$M:1[ℝ][E]$^\pm\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*
(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \ge b$* (impossible). ———
(c) *Let $\beta\mu < b$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $\mu - s \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
    ii. *Let $\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < \beta T(0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
    ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta\mu - s \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
    iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta\mu - s \le a$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.
      2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ $\boxed{\circledS^\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*

• *Proof* Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.

(a) The same as Tom 19.2.2(p.137) (a).

(b,c) Always $\beta\mu < b$ due to [8(p.118)], hence $\beta\mu \ge b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$".

(c1i,c1ii) The same as Tom 19.2.2(p.137) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of Tom 19.2.2(p.137) and if $b = 0$, then since always $\beta\mu - s = \beta\mu > a$ due to [8(p.118)], it suffices to consider only (c2ii2) of Tom 19.2.2(p.137). Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from Tom 19.2.2(p.137) (c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.7(p.25)) with $\lambda = 1$. ▮

19.2.2.2.3 Negative Restriction

☐ **Nem 19.2.1** ($\mathscr{A}\{$M:1[ℝ][E]$^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

• *Proof* The same as Tom 19.2.1(p.137) due to Lemma 17.4.4(p.118). ▮

☐ **Nem 19.2.2** ($\mathscr{A}\{$M:1[ℝ][E]$^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*
(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.
(c) *Let $\beta\mu < b$.*
  1. *Let $\beta = 1$.*
    i. *Let $\mu - s \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
    ii. *Let $\mu - s > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
  2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ $\boxed{\circledS^\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $\beta\mu - s \le a$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.
    ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ $\boxed{\circledS^\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*

• *Proof* Suppose $b < 0$, hence $a < \mu < b < 0 \cdots (1)$. Hence $\kappa = -s \cdots (2)$ from Lemma 10.3.1(p.59) (a) with $\lambda = 1$. In addition, $\beta\mu \ge b$ and $\beta\mu < b$ are both possible due to [17(p.118)].

(a,b) The same as Tom 19.2.2(p.137) (a,b).

(c) Let $\beta\mu < b$.

(c1-c1ii) The same as Tom 19.2.2(p.137) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $b < 0$ due to (1), it suffices to consider only (c2iii) of Tom 19.2.2(p.137). In this case, since $\beta\mu - s = \beta\mu > \beta a > a$ due to (1) and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 10.2.4(p.59) (c), it suffices to consider only (c2iii2) of Tom 19.2.2(p.137).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (2). Thus, it suffices to consider only (c2iii1-c2iii2) of Tom 19.2.2(p.137). ▮

19.2.3 $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]

19.2.3.1 Analysis

□ `Tom` **19.2.3** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathrm{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to `Tom` 19.2.1(p.137). ∎

□ `Tom` **19.2.4** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   - ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*†
     1. *Let $\beta\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   - iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
     1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ◎\ \parallel \mid ◎\ \vartriangle \mid ◎\ \blacktriangle}$ *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to `Tom` 19.2.2(p.137). ∎

19.2.3.2 Market Restriction

19.2.3.2.1 Positive Restriction

□ `Pom` **19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.3(p.141) due to Lemma 17.4.4(p.118). ∎

□ `Pom` **19.2.4** ($\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ◎\ \parallel \mid ◎\ \vartriangle \mid ◎\ \blacktriangle}$ *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
   - ii. *Let $\beta\mu + s < b$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ◎\ \parallel \mid ◎\ \vartriangle \mid ◎\ \blacktriangle}$ *is true* (see *Numerical Example* 4(p.147)).

● *Proof* Suppose $a > 0 \cdots (1)$, hence $\tilde{\kappa} = s \cdots (2)$ from Lemma 12.6.6(p.83) (a). Here note that $\mu\beta \le a$ and $\mu\beta > a$ are both possible due to [5(p.118)].

(a,b) The same as `Tom` 19.2.4(p.141) (a,b).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots (3)$ due to Lemma 12.6.5(p.83) (c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have `Tom` 19.2.4(p.141) (c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.118)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of `Tom` 19.2.4(p.141).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of `Tom` 19.2.4(p.141). ∎

19.2.3.2.2 Mixed Restriction

☐ `Mim` **19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.3(p.141) due to Lemma 17.4.4(p.118). ∎

☐ `Mim` **19.2.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \ge b$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \ge b$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
      2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel \mid ◎\ \vartriangle \mid ◎\ \blacktriangle}$ *is true.*

● *Proof* Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.

(a) The same as `Tom` 19.2.4(p.141) (a).

(b,c) Always $\beta\mu > a$ due to [8(p.118)], hence $\beta\mu \le a$ is impossible. Then $s_{\tilde{\mathcal{L}}} > 0$ due to Lemma 12.6.5(p.83) (c).

(c1-c1ii) The same as `Tom` 19.2.4(p.141) (c-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of `Tom` 19.2.4(p.141). Let $a = 0$. Now, in this case, since $\beta\mu + s = \beta\mu < b$ due to [8(p.118)], it suffices to consider only (c2ii2) of `Tom` 19.2.4(p.141). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions become true from `Tom` 19.2.4(p.141) (c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.25)). ∎

19.2.3.2.3 Negative Restriction

☐ `Nem` **19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.4(p.141) due to Lemma 17.4.4(p.118). ∎

☐ `Nem` **19.2.4** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \ge b$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu < -s$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   ii. *Let $\beta\mu \ge -s$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

● *Proof* Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\tilde{\kappa} = \beta\mu + s \cdots (3)$ due to Lemma 12.6.6(p.83) (a).

(a) The same as `Tom` 19.2.4(p.141) (a).

(b,c) Always $a < \beta\mu$ due to [15(p.118)], hence $\beta\mu \le a$ is impossible.

(c1-c1ii) The same as the proof of `Tom` 19.2.4(p.141) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c2i) of `Tom` 19.2.4(p.141).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu < -s$, hence $\beta\mu + s < 0$. Then, since $\tilde{\kappa} < 0$ due to (3), it suffices to consider only (c2i) of `Tom` 19.2.4(p.141).

(c3ii) Let $\beta\mu \ge -s$, hence $\beta\mu + s \ge 0$. Let $\beta\mu + s = 0$. Then, since $\tilde{\kappa} = 0$ due to (3) and since $\beta\mu + s = 0 > b$ due to (2), it suffices to consider only (c2ii1) of `Tom` 19.2.4(p.141). Let $\beta\mu + s > 0$. Then, since $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c2iii) of `Tom` 19.2.4(p.141). Then, since $\beta\mu + s > 0 > b$ due to (1), it suffices to consider only (c2iii1) of `Tom` 19.2.4(p.141). Accordingly, whether $\beta\mu + s = 0$ or $\beta\mu + s > 0$, we have the same result. ∎

19.2.4 M:1[ℙ][E]

19.2.4.1 Analysis

□ Tom **19.2.5** (⊡ $\mathscr{A}${M:1[ℙ][E]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof by analogy* The same as Tom 19.2.1(p.137) due to Lemma 13.6.1(p.99). ∎

□ Tom **19.2.6** (⊡ $\mathscr{A}${M:1[ℙ][E]}) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \ge b$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c) *Let $\beta a < b$.*

1. *Let $\beta = 1$.*
   i. *Let $a - s \le a^\star$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
   ii. *Let $a - s > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   ii. *Let $b = 0$ ($\kappa = 0$).*
      1. *Let $\beta a - s \le a^\star$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   iii. *Let $b < 0$ ($\kappa < 0$).*
      1. *Let $\beta a - s \le a^\star$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* S$_2$ [ⓢ▲ | ⓞ ‖ | ⓞ △ | ⓞ ▲] *is true.* □

● *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom 19.2.2(p.137). ∎

Corollary 19.2.1 (optimal price to propose) *The optimal price to propose z_t is nondecreasing in $t > 0$.* □

● *Proof* Immediate from Tom's 19.2.5(p.143) (a) and 19.2.6(p.143) (a) and from (6.2.34(p.31)) and Lemma 13.1.3(p.89). ∎

19.2.4.2 Market Restriction

19.2.4.2.1 Positive Restriction

□ Pom **19.2.5** ($\mathscr{A}${M:1[ℙ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof* The same as Tom 19.2.5(p.143) due to Lemma 17.4.4(p.118). ∎

□ Pom **19.2.6** ($\mathscr{A}${M:1[ℙ][E]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \ge b$* (impossible).

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \le a^\star$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
   ii. *Let $a - s > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \le a^\star$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \le a^\star$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{● \ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
      2. *Let $\beta a - s > a^\star$ and $s < s_{\mathcal{L}}$. Then* S$_2$ [ⓢ▲ | ⓞ ‖ | ⓞ △ | ⓞ ▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots$ **(1)**.

(a) The same as Tom 19.2.6(p.143) (a).

(b,c) Always $\beta a < b$ from [4(p.118)], hence $\beta a \ge b$ is impossible.

(c1-c1ii) The same as Tom 19.2.6(p.143) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.6(p.143).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i-c3iii2) Immediate from Tom 19.2.6(p.143) (c2i-c2iii2) due to (2) with $\kappa = \beta T(0) - s \cdots$ **(2)** from (5.1.23(p.26)). ∎

19.2.4.2.2 Mixed Restriction

☐ **Mim 19.2.5** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof* The same as Tom 19.2.5(p.143) due to Lemma 17.4.4(p.118). ∎

☐ **Mim 19.2.6** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $\beta a \ge b$* (impossible). ———

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \le a^\star$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $a - s > a^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \le a^\star$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \le a^\star$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* S_2 [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true* ☐

● *Proof* Suppose $a \le 0 \le b$.

(a) The same as Tom 19.2.6(p.143) (a).

(b,c) Always $\beta a < b$ due to [9(p.118)], hence $\beta a \ge b$ is impossible. .

(c1-c1ii) The same as Tom 19.2.6(p.143) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from Tom 19.2.6(p.143) (c2i) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.118)], hence the assertion become true from Tom 19.2.6(p.143) (c2ii2). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as Tom 19.2.6(p.143) (c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.26))) with $\lambda = 1$. ∎

19.2.4.2.3 Negative Restriction

☐ **Nem 19.2.5** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof* The same as Tom 19.2.5(p.143) due to Lemma 17.4.4(p.118). ∎

☐ **Nem 19.2.6** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \ge b$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c) *Let $\beta a < b$.*

1. *Let $\beta = 1$.*
   i. *Let $a - s \le a^\star$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $a - s > a^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* S_2 [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta a - s \le a^\star$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{\text{● dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
   ii. *Let $\beta a - s > a^\star$ and $s < s_{\mathcal{L}}$. Then* S_2 [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

● *Proof* Suppose $b < 0$. Then, $\kappa = -s \cdots (1)$ from Lemma 13.2.6(p.97) (a). In addition, $\beta a \ge b$ and $\beta a < b$ are both possible due to [18(p.118)].

(a,b) The same as Tom 19.2.6(p.143) (a,b).

(c) Let $\beta a < b$.

(c1-c1ii) The same as Tom 19.2.6(p.143) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2iii-c2iii2) of Tom 19.2.6(p.143). In this case, since $\beta a - s = \beta a > a^\star$ due to [19(p.118)] and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 13.2.5(p.97) (c), it suffices to consider only (c2iii2) of Tom 19.2.6(p.143).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (1). Hence, it suffices to consider only (c2iii1,c2iii2) of Tom 19.2.6(p.143). ∎

19.2.5 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$

19.2.5.1 Analysis

□ `Tom` **19.2.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) to `Tom` 19.2.5(p.143). ∎

□ `Tom` **19.2.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_\vartriangle$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_{||}$.
   - ii. *Let $b + s < b^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   - i. *Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   - ii. *Let $a = 0$ ⦅$\tilde{\kappa} = 0$⦆.*
     1. *Let $\beta b + s \ge b^\star$.*[†] *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_{||}$.
     2. *Let $\beta b + s < b^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   - iii. *Let $a > 0$ ⦅$\tilde{\kappa} > 0$⦆.*
     1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_\vartriangle$.
     2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\texttt{S}_2$ $\boxed{ⓢ^\blacktriangle \,|\, ⦾\ || \,|\, ⦾\ \vartriangle \,|\, ⦾\ \blacktriangle}$ *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) to `Tom` 19.2.6(p.143). ∎

Corollary 19.2.2 (optimal price to propose) *The optimal price to propose z_t is nonincreasing in $t > 0$.* □

● *Proof* Immediate from `Tom`'s 19.2.7(p.145) (a) and 19.2.8(p.145) (a) and from (6.2.50(p.32)) and Lemma A 3.3(p.297)). ∎

19.2.5.2 Market Restriction

19.2.5.2.1 Positive Restriction

□ `Pom` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

● *Proof* The same as `Tom` 19.2.7(p.145) due to Lemma 17.4.4(p.118). ∎

□ `Pom` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_\vartriangle$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_{||}$.
   - ii. *Let $b + s < b^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_2$ $\boxed{ⓢ^\blacktriangle \,|\, ⦾\ || \,|\, ⦾\ \vartriangle \,|\, ⦾\ \blacktriangle}$ *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle1\rangle}_\vartriangle$.
   - ii. *Let $\beta b + s < b^\star$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\texttt{S}_2$ $\boxed{ⓢ^\blacktriangle \,|\, ⦾\ || \,|\, ⦾\ \vartriangle \,|\, ⦾\ \blacktriangle}$ *is true.*

● *Proof* Suppose $a > 0 \cdots (1)$. Then, $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a). In addition, $\beta b \le a$ and $\beta b > a$ are both possible due to [6(p.118)].

(a,b) The same as `Tom` 19.2.8(p.145) (a,b).

(c) Let $\beta b > a$.

(c1-c1ii) The same as `Tom` 19.2.8(p.145) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii-c2iii2) of `Tom` 19.2.8(p.145). In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.118)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from Lemma 14.6.5(p.108) (c) with $\lambda = 1$, it suffices to consider only (c2iii2) of `Tom` 19.2.8(p.145).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\tilde{\kappa} > 0$ due to (2). Hence, it suffices to consider only (c2iii1,c2iii2) of `Tom` 19.2.8(p.145). ∎

19.2.5.2.2 Mixed Restriction

□ `Mim` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^{\pm}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.7(p.145) due to Lemma 17.4.4(p.118). ▮

□ `Mim` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \ge x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \le a$* (impossible). ———
(c) *Let $\beta b > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
    ii. *Let $b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
    ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
    iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{E}}} \le s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{E}}} > s$. Then* $\texttt{S}_2$ $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \parallel \mid \circledcirc\ \vartriangle \mid \circledcirc\ \blacktriangle}$ *is true.*

● *Proof* Let $b \ge 0 \ge a \cdots$ **(1)**.

(a) The same as `Tom` 19.2.8(p.145) (a).

(b,c) Always $\beta b > a$ due to [10(p.118)], hence $\beta b \le a$ is impossible.

(c1-c1ii) The same as `Tom` 19.2.8(p.145) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of `Tom` 19.2.8(p.145). Let $a < 0$. Then, the assertion is true from `Tom` 19.2.8(p.145) (c2i). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.118)], it suffices to consider only (c2ii2) of `Tom` 19.2.8(p.145). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from `Tom` 19.2.8(p.145) (c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.27)) with $\lambda = 1$. ▮

19.2.5.2.3 Negative Restriction

□ `Nem` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.7(p.145) due to Lemma 17.4.4(p.118). ▮

□ `Nem` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \ge x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \le a$* (impossible). ———
(c) *Let $\beta b > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
    ii. *Let $b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
    ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
    iii. *Let $-\beta\tilde{T}(0) < s$.*
      1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{E}}} \le s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{E}}} > s$. Then* $\texttt{S}_2$ $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \parallel \mid \circledcirc\ \vartriangle \mid \circledcirc\ \blacktriangle}$ *is true.*

• *Proof* Let $b < 0$, hence $a < b < 0 \cdots (1)$.

(a) The same as Tom 19.2.8(p.145) (a).

(b,c) Always $\beta b > a$ due to [16(p.118)], hence $\beta b \le a$ is impossible.

(c1-c1ii) The same as Tom 19.2.8(p.145) (c1-c1ii).

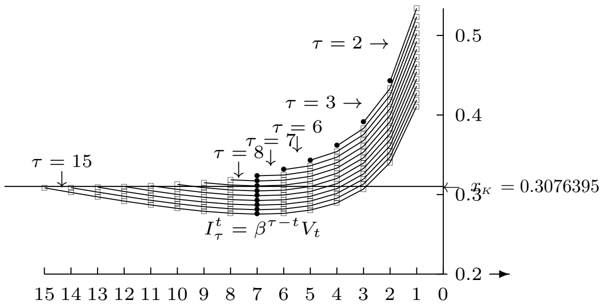
(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.8(p.145).

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 19.2.8(p.145) (c2-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0)+s$ from (5.1.36(p.27)) with $\lambda = 1$. ∎

19.2.6 Numerical Calculation

Numerical Example **4** **($\mathscr{A}\{\tilde{\mathsf{M}}$:1[ℝ][E]$^+\}$ (buying model)**

This is the example for ⓢ▲ and ◎▲ of S$_2$(p.137) ⓢ▲ | ◎ ∥ | ◎ △ | ◎ ▲ in Pom 19.2.4(p.141) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$ where $x_{\tilde{K}} = 0.3076395$ and $s_{\tilde{\mathcal{L}}} = 0.3232736$.† Note that the example is for the model of a buying problem with the cost minimization. The figure below is the graph of $I^t_\tau = \beta^{\tau-t}V_t$ where the symbol • shows the optimal initiating time (OIT) for each $\tau = 2, 3, \cdots, 15$ (see t^*-column in the table below). In addition, note that each of polygonal curves for $\tau = 2, 3, \cdots, 7$ is strictly decreasing in $t = 1, 2, \cdots, 7$ and that each of polygonal curves for $\tau = 8, 9, \cdots, 15$ is *strictly* decreasing in $t = 1, 2, \cdots, 7$ and *strictly* increasing in $t = 7, 8, \cdots, 15$. The fact implies that the optimal initiating time t^*_τ degenerates to the starting time $\tau = 2, 3, \cdots, 7$, i.e., ⓢ dOITs$_\tau\langle\tau\rangle_\blacktriangle$ and that it is given by $t^*_\tau = 7$ (non-degenerate) for each of $\tau = 8, 9, \cdots, 15$, i.e., ◎ ndOIT$_\tau\langle 7\rangle_\blacktriangle$ (see t^*−column in the table below). Finally, note here that the leftmost point V_t in each curves converges to $x_{\tilde{K}} = 0.3076395$ as $\tau \to \infty$ (see Pom 19.2.4(p.141) (a)).



t	V_t	$\Delta_\beta V_t$	t^*_τ
0			
1	0.544900		
2	0.442388	−0.091614	2
3	0.391004	−0.042535	3
4	0.361335	−0.021849	4
5	0.343013	−0.011094	5
6	0.331264	−0.004889	6
7	0.323555	−0.001084	7
8	0.318422	+0.001338	7
9	0.314972	+0.002918	7
10	0.312638	+0.003965	7
11	0.311053	+0.004667	7
12	0.309973	+0.005141	7
13	0.309236	+0.005462	7
14	0.308732	+0.005681	7
15	0.308388	+0.005830	7

$\Delta_\beta V_t = V_t - \beta V_{t-1}$

Figure 19.2.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$

19.2.7 Conclusion 2 (Search-Enforced-Model 1)

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}$:1[ℝ][E]$\}$ of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$$\mathcal{Q}\langle\mathsf{M}:1[\mathsf{E}]\rangle = \{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}], \mathsf{M}{:}1[\mathbb{P}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$$

are given by

$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$
↓	↓	↓	↓
Tom's 19.2.1(p.137), 19.2.2(p.137),	19.2.3(p.141), 19.2.4(p.141),	19.2.5(p.143), 19.2.6(p.143),	19.2.7(p.145), 19.2.8(p.145).

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}$:1[ℝ][E]$^+\}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$$\mathcal{Q}\langle\mathsf{M}:1[\mathsf{E}]\rangle^+ = \{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^+, \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^+, \mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]^+, \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]^+\}$$

are given by

$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]^+\}$
↓	↓	↓	↓
Pom's 19.2.1(p.139), 19.2.2(p.139),	19.2.3(p.141), 19.2.4(p.141),	19.2.5(p.143), 19.2.6(p.143),	19.2.7(p.145), 19.2.8(p.145).

■ Closely looking into all the above assertion systems $\mathscr{A}$ leads to the following conclusions.

C1. Mental Conflict

On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$ we have:

a. The opt-ℝ-price V_t in M:1[ℝ][E] (selling model) is nondecreasing in t as in Figure 7.3.1(p.47) (I) (see Tom's 19.2.1(p.137) (a) and 19.2.2(p.137) (a), hence we have the normal conflict (see Remark 7.3.1(p.47)).

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†Since $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$, we have $\mu = (0.01 + 1.00)/2 = 0.525$, $\beta\mu = 0.98 \times 0.525 = 0.5145 > 0.01 = a$, $\beta\mu + s = 0.5145 + 0.05 = 0.5645 < 1.00 = b$, and $s = 0.05 < 0.3232736 = s_{\tilde{\mathcal{L}}}$. Thus, the condition of this assertion is satisfied.

b. The opt-$\mathbb{P}$-price z_t in M:1[$\mathbb{P}$][E] is nondecreasing (selling model) in t as in Figure 7.3.1(p.47) (I) (see Corollary 19.2.1(p.143)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

c. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] (buying model) is nonincreasing in t as in Figure 7.3.1(p.47) (II) (see `Tom`'s 19.2.3(p.141) (a) and 19.2.4(p.141) (a), hence we have the normal conflict (see Remark 7.3.1(p.47)).

d. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E] (buying model) is nonincreasing in t as in Figure 7.3.1(p.47) (II) (see Corollary 19.2.2(p.145)), hence we have the normal conflict (see Remark 7.3.1(p.47)).

The above results can be summarized as below.

A. On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict, which coincides with ***expectations*** in *Example*s 1.4.1(p.5) - 1.4.4(p.6) .

C2. Symmetry

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

`Pom` 19.2.3(p.141) $\backsim$ `Pom` 19.2.1(p.139) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^+ \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^+$),

`Pom` 19.2.7(p.145) $\backsim$ `Pom` 19.2.5(p.143) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^+ \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^+$).

2. Let $\beta < 1$ or $s > 0$. Then we have:

`Pom` 19.2.4(p.141) $\not\backsim$ `Pom` 19.2.2(p.139) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^+ \not\backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^+$) $\cdots (s^1)$,

`Pom` 19.2.8(p.145) $\not\backsim$ `Pom` 19.2.6(p.143) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^+ \not\backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^+$) $\cdots (s^2)$.

b. On $\mathscr{F}^\pm$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

`Mim` 19.2.3(p.142) $\backsim$ `Mim` 19.2.1(p.140) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^\pm \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^\pm$),

`Mim` 19.2.7(p.146) $\backsim$ `Mim` 19.2.5(p.144) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^\pm \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^\pm$).

2. Let $\beta < 1$ or $s > 0$. Then we have:

`Mim` 19.2.4(p.142) $\backsim$ `Mim` 19.2.2(p.140) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^\pm \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^\pm$),

`Mim` 19.2.8(p.146) $\backsim$ `Mim` 19.2.6(p.144) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^\pm \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^\pm$).

c. On $\mathscr{F}^-$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

`Nem` 19.2.3(p.142) $\backsim$ `Nem` 19.2.1(p.140) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^- \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^-$),

`Nem` 19.2.7(p.146) $\backsim$ `Nem` 19.2.5(p.144) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^- \backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^-$).

2. Let $\beta < 1$ or $s > 0$. Then we have:

`Nem` 19.2.4(p.142) $\not\backsim$ `Nem` 19.2.2(p.140) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^- \not\backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^-$) $\cdots (s^3)$,

`Nem` 19.2.8(p.146) $\not\backsim$ `Nem` 19.2.6(p.144) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^- \not\backsim \mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^-$) $\cdots (s^4)$.

The above results can be summarized as below.

A. On $\mathscr{F}^\pm$, for any $\beta \le 1$ and $s \ge 0$, the symmetry is inherited (see C3b(p.148)).

B. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2a1(p.148) /C2c1(p.148)).

C. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta < 1$ or $s > 0$, the symmetry collapses (see $(s^1)/(s^2)/(s^3)/(s^4)$).

C3. Analogy

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

`Pom` 19.2.5(p.143) $\bowtie$ `Pom` 19.2.1(p.139) ($\mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^+ \bowtie \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^+$),

`Pom` 19.2.7(p.145) $\bowtie$ `Pom` 19.2.3(p.141) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^+$).

2. Let $\beta < 1$ or $s > 0$. Then we have:

`Pom` 19.2.6(p.143) $\not\bowtie$ `Pom` 19.2.2(p.139) ($\mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^+$) $\cdots (a^1)$,

`Pom` 19.2.8(p.145) $\bowtie$ `Pom` 19.2.4(p.141) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^+$).

b. On $\mathscr{F}^\pm$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

`Mim` 19.2.5(p.144) $\bowtie$ `Mim` 19.2.1(p.140) ($\mathscr{A}\{\mathsf{M}$:1[$\mathbb{P}$][E]$\}^\pm \bowtie \mathscr{A}\{\mathsf{M}$:1[$\mathbb{R}$][E]$\}^\pm$),

`Mim` 19.2.7(p.146) $\bowtie$ `Mim` 19.2.3(p.142) ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]$\}^\pm \bowtie \mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]$\}^\pm$).

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Mim}\ 19.2.6(\text{p.144}) \bowtie \texttt{Mim}\ 19.2.2(\text{p.140}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}^{\pm} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}^{\pm}),$$

$$\texttt{Mim}\ 19.2.8(\text{p.146}) \bowtie \texttt{Mim}\ 19.2.4(\text{p.142}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^{\pm}).$$

c. On $\mathscr{F}^-$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Nem}\ 19.2.5(\text{p.144}) \bowtie \texttt{Nem}\ 19.2.1(\text{p.140}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}^{-}),$$

$$\texttt{Nem}\ 19.2.7(\text{p.146}) \bowtie \texttt{Nem}\ 19.2.3(\text{p.142}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^{-} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^{-}).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Nem}\ 19.2.6(\text{p.144}) \bowtie \texttt{Nem}\ 19.2.2(\text{p.140}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}^{-}),$$

$$\texttt{Nem}\ 19.2.8(\text{p.146}) \not\bowtie \texttt{Nem}\ 19.2.4(\text{p.142}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^{-} \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^{-}) \cdots (a^2).$$

The above results can be summarized as below.

A. On $\mathscr{F}^{\pm}$, for any $\beta \leq 1$ and $s \geq 0$, the analogy is inherited (see C3b(p.148)).

B. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, then the analogy is inherited (see C3a1(p.148) /C3c1(p.149)).

C. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, then the analogy *partially* collapses (see $(a^1)/(a^2)$).

C4. Optimal initiating time (OIT)

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 19.2.1(p.139), **Mim** 19.2.1(p.140), **Nem** 19.2.1(p.140),
Pom 19.2.3(p.141), **Mim** 19.2.3(p.142), **Nem** 19.2.3(p.142),
Pom 19.2.5(p.143), **Mim** 19.2.5(p.144), **Nem** 19.2.5(p.144),
Pom 19.2.7(p.145), **Mim** 19.2.7(p.146), **Nem** 19.2.7(p.146)

we obtain the following table (the symbol "○" in the table below represents "possible"):

Table 19.2.1: Possible OIT ($\beta = 1$ and $s = 0$)

		$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ dOITs$_\tau\langle\tau\rangle_{\parallel}$	Ⓢ$_{\parallel}$			
Ⓢ dOITs$_\tau\langle\tau\rangle_{\vartriangle}$	Ⓢ$_{\vartriangle}$			
Ⓢ dOITs$_\tau\langle\tau\rangle_{\blacktriangle}$	Ⓢ$_{\blacktriangle}$	○	○	○
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_{\parallel}$	◎$_{\parallel}$			
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$	◎$_{\vartriangle}$			
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$	◎$_{\blacktriangle}$			
● dOITd$_\tau\langle 0\rangle_{\parallel}$	ⓓ$_{\parallel}$			
● dOITd$_\tau\langle 0\rangle_{\vartriangle}$	ⓓ$_{\vartriangle}$			
● dOITd$_\tau\langle 0\rangle_{\blacktriangle}$	ⓓ$_{\blacktriangle}$			

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 19.2.2(p.139) , Mim 19.2.2(p.140) , Nem 19.2.2(p.140) ,
Pom 19.2.4(p.141) , Mim 19.2.4(p.142) , Nem 19.2.4(p.142) ,
Pom 19.2.6(p.143) , Mim 19.2.6(p.144) , Nem 19.2.6(p.144) ,
Pom 19.2.8(p.145) , Mim 19.2.8(p.146) , Nem 19.2.8(p.146)

we obtain the following table:

Table 19.2.2: Possible OIT ($\beta < 1$ or $s > 0$)

		$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
[ⓢ dOITs$_\tau\langle\tau\rangle$]$_{\|\|}$	ⓢ$_{\|\|}$			
[ⓢ dOITs$_\tau\langle\tau\rangle$]$_\triangle$	ⓢ$_\triangle$			
[ⓢ dOITs$_\tau\langle\tau\rangle$]$_\blacktriangle$	ⓢ$_\blacktriangle$	○	○	○
[⊚ ndOIT$_\tau\langle t^\bullet_\tau\rangle$]$_{\|\|}$	⊚$_{\|\|}$	○	○	○
[⊚ ndOIT$_\tau\langle t^\bullet_\tau\rangle$]$_\triangle$	⊚$_\triangle$	○	○	○
[⊚ ndOIT$_\tau\langle t^\bullet_\tau\rangle$]$_\blacktriangle$	⊚$_\blacktriangle$	○	○	○
[● dOITd$_\tau\langle 0\rangle$]$_{\|\|}$	🅓$_{\|\|}$	○	○	○
[● dOITd$_\tau\langle 0\rangle$]$_\triangle$	🅓$_\triangle$	○	○	○
[● dOITd$_\tau\langle 0\rangle$]$_\blacktriangle$	🅓$_\blacktriangle$			

c. The table below is the list of the occurrence rates of ⓢ , ⊚ , and 🅓 on $\mathscr{F}$ appearing in the primitive Tom 19.2.1(p.137) (▣) and Tom 19.2.2(p.137) (▣) (see Def. 11.2.2(p.61)).

Table 19.2.3: Occurence rates of ⓢ , ⊚ , and 🅓 on $\mathscr{F}$

ⓢ			⊚			🅓		
41.7 % / 5			25.0 % / 3			33.3 % / 4		
ⓢ$_{\|\|}$	ⓢ$_\triangle$	ⓢ$_\blacktriangle$	⊚$_{\|\|}$	⊚$_\triangle$	⊚$_\blacktriangle$	🅓$_{\|\|}$	🅓$_\triangle$	🅓$_\blacktriangle$
−	×	possible	possible	possible	possible	possible	possible	×
−%/−	0.0 %/ 0	41.7 %/ 5	8.3 %/ 1	8.3 %/ 1	8.3 %/ 1	16.7 %/ 2	16.7 %/ 2	0.0 %/ 0

C5. Null-time-zone and deadline-engulfing

From Table 19.2.3(p.150) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▲ (strict optimality).

b. As a whole we have ⓢ , ⊚ , and 🅓 at 41.7%, 25.0%, and 33.3% respectively where

1. ⓢ$_{\|\|}$ cannot be defined due to Remark 7.2.3(p.45) .
2. ⊚$_{\|\|}$ is possible (8.3 %).
3. 🅓$_{\|\|}$ is possible (16.7 %).
4. ⓢ$_\triangle$ never occur (0.0 %).
5. ⊚$_\triangle$ is possible (3.8 %).
6. 🅓$_\triangle$ is possible (16.7 %).
7. ⓢ$_\blacktriangle$ is possible (41.7 %).
8. ⊚$_\blacktriangle$ is possible (8.3 %).
   - See Tom 19.2.2(p.137) (c2iii2)
9. 🅓$_\blacktriangle$ never occur (0.0 %).

From the above results we see that on $\mathscr{F}$:

A. ⊚ and 🅓 causing the null-time-zone are possible at 58.3% (= 25.0% + 33.3%).

B. ⊚$_\blacktriangle$ *strictly* causing the null-time-zone is possible at 8.3%.

C. 🅓$_\blacktriangle$ *strictly* causing the null-time-zone is impossible (0.0%), i.e., the deadline-engulfing is impossible.

19.3 Conclusions of Model 1

Conclusions 1 (p.133) and 2 (p.147) can be summarized as below.

$\overline{\text{C}}$1. Mental Conflict

On $\mathscr{F}$, from C1A(p.133) and C1A(p.148), for any $\beta \le 1$ and $s \ge 0$, whether search-**A**llowed-model od search-**E**nforced-model, whether selling problem or buying problem, and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6).

$\overline{\text{C}}$2. Symmetry

a. On $\mathscr{F}^{\pm}$, for any $\beta \le 1$ and $s \ge 0$, the symmetry is inherited (see C2A(p.134) and C2A(p.148)).

b. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2B(p.134) and C2B(p.148)).

c. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the symmetry *may* collapse on $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$ (see C2C(p.134) and C2C(p.148)).

$\overline{\text{C}}$3. Analogy

a. On $\mathscr{F}^{\pm}$, for any $\beta \le 1$ and $s \ge 0$, the analogy are inherited (see C2A(p.134) and C2A(p.148)).

b. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the analogy are inherited (see C2B(p.134) and C2B(p.148)).

c. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the analogy *may* collapse on $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$ (see C2C(p.134) and C2C(p.148)).

$\overline{\text{C}}$4. Optimal initiating time (`OIT`)

On $\mathscr{F}^{+}$, $\mathscr{F}^{\pm}$, and $\mathscr{F}^{-}$, we have:

a. Let $\beta = 1$ and $s = 0$. Then only ⓢ$_{\blacktriangle}$ is possible (see Figures 19.1.1(p.135) and 19.2.1(p.149)).

b. Let $\beta < 1$ or $s > 0$. Then:

1. For `sA`-model we have only ⓢ$_{\blacktriangle}$, ◎$_{\|}$, and 🅓$_{\|}$ (see Figure 19.1.2(p.135)).
2. For `sE`-model we have ⓢ$_{\blacktriangle}$, ◎$_{\|}$, ◎$_{\vartriangle}$, ◎$_{\blacktriangle}$, 🅓$_{\|}$, and 🅓$_{\vartriangle}$ (see Figure 19.2.2(p.150)).

c. Joining Tables 19.1.3(p.135) and 19.2.3(p.150) produces the following table:

Table 19.3.1: Occurance rates of ⓢ, ◎, and 🅓 on $\mathscr{F}$

ⓢ			◎			🅓		
45.5 % / 10			18.2 % / 4			36.3 % / 8		
ⓢ$_{\|}$	ⓢ$_{\vartriangle}$	ⓢ$_{\blacktriangle}$	◎$_{\|}$	◎$_{\vartriangle}$	◎$_{\blacktriangle}$	🅓$_{\|}$	🅓$_{\vartriangle}$	🅓$_{\blacktriangle}$
−	×	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	×
−%/−	0.0 %/ 0	45.5 %/ 10	9.0 %/ 2	4.6 %/ 1	4.6 %/ 1	27.3 %/ 6	9.0 %/ 2	0.0 %/ 0

$\overline{\text{C}}$5. Null-time-zone and deadline-engulfing

From Table 19.3.1(p.151) above we see that on $\mathscr{F}$

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol $\blacktriangle$ (<u>strict</u> optimality).

b. As a whole we have ⓢ, ◎, and 🅓 at 45.5%, 18.2%, and 36.3% respectively where

1. ⓢ$_{\|}$ cannot be defined due to Remark 7.2.3(p.45).
2. ◎$_{\|}$ is possible (9.0 %).
3. 🅓$_{\|}$ is possible (27.3 %).
4. ⓢ$_{\vartriangle}$ never occur (0.0 %).
5. ◎$_{\vartriangle}$ is possible (4.6 %).
6. 🅓$_{\vartriangle}$ is possible (9.0 %).
7. ⓢ$_{\blacktriangle}$ is possible (45.5%),
8. ◎$_{\blacktriangle}$ is possible(4.6%).
   - `Tom` 19.2.2(p.137) (c2iii2)
9. 🅓$_{\blacktriangle}$ never occur (0.0%).

From the above results we see that:

A. ◎ and 🅓 causing the `null-time-zone` are possible at 54.5% (= 18.2% + 36.3%).

B. ◎$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is possible at 4.6%.

C. 🅓$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is impossible (0.0%), i.e., the deadline-engulfing is impossible.

Chapter 20

Analysis of Model 2

20.1 Search-Allowed-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$

20.1.1 Preliminary

As ones corresponding to Theorems 12.5.1(p.80), 13.3.1(p.98), and 14.5.1(p.106), let us consider the following three theorems:

Theorem 20.1.1 (symmetry$[\mathbb{R} \to \tilde{\mathbb{R}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.1}$$

Theorem 20.1.2 (analogy$[\mathbb{R} \to \mathbb{P}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.2}$$

Theorem 20.1.3 (symmetry$[\mathbb{P} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{20.1.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}], \tag{20.1.6}$$

corresponding to (12.5.34(p.77)), (13.2.4(p.93)), and (14.5.4(p.106)). Then, for the same reason as in Chap. 15(p.111) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}] \tag{20.1.7}$$

holds (corresponding to (15.1.33(p.112))) and that we have the following theorem, corresponding to Theorem 15.1.1(p.112)

Theorem 20.1.4 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.8}$$

In fact, from the comparisons of (I) and (II), of (I) and (III), of (III) and (IV), and of (II) and (IV) in Table 6.4.3(p.41) we can easily show that (20.1.4(p.153)) - (20.1.7(p.153)) hold.

20.1.2 A Lemma

The following lemma provides the conditions which determine if each of Theorems 20.1.1(p.153), 20.1.2(p.153), and 20.1.3(p.153) holds by testing whether or not each of (20.1.4(p.153)), (20.1.5(p.153)), and (20.1.6(p.153)) is true.

Lemma 20.1.1 ($\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$)

(a) Theorem 20.1.1(p.153) *holds.*
(b) Theorem 20.1.3(p.153) *holds.*
(c) *If* $\rho \le a^\star$ *or* $b \le \rho$*, then* Theorem 20.1.2(p.153) *holds.*
(d) *If* $a^\star < \rho < b$*, then* Theorem 20.1.2(p.153) *does not always hold.* $\square$

• *Proof* (a) From Table 6.4.3(p.41) (I) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\} = \{V_0 = \rho,\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 0\} \tag{20.1.9}$$

First, applying the operation $\mathcal{R}$ (see Step 2 (p.75)) to this leads to

$$\begin{aligned}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] &= \{-\hat{V}_0 = \rho,\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 0\} \\ &= \{-\hat{V}_0 = \rho,\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\ &= \{\hat{V}_0 = -\rho,\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\ &= \{\hat{V}_0 = \hat{\rho},\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}\end{aligned} \tag{20.1.10}$$

Then, applying $\mathcal{C}_{\mathbb{R}}$ (see Step 3 (p.75)) to this yields

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\check{\hat{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}. \tag{20.1.11}$$

Finally, applying $\mathcal{I}_{\mathbb{R}}$ (see Step 4 (p.76)) to this produces

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}. \tag{20.1.12}$$

Since this holds for any $\rho \in (-\infty, \infty)$, it holds also for $\hat{\rho} \in (-\infty, \infty)$, hence holds also for the $\hat{\hat{\rho}}$, i.e.,

$$\begin{aligned}\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] &= \{\hat{V}_0 = \hat{\hat{\rho}},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\ &= \{\hat{V}_0 = \rho,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}\end{aligned} \tag{20.1.13}$$

due to $\rho = \hat{\hat{\rho}}$. Now, we have $\hat{V}_0 = \rho = V_0$ from (6.4.17(p.41)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, the second term in the r.h.s. of (20.1.13(p.154)) can be rewritten as $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Thus, by induction $\hat{V}_t = V_t$ for $t \geq 0$. Accordingly (20.1.13(p.154)) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{V_0 = \rho,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 0, \tag{20.1.14}$$

which is identical to $\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\texttt{A}]\}$ (see Table 6.4.3(p.41) (II)), i.e.,

$$\begin{aligned}\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\texttt{A}]\} &= \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] \\ &= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] \quad \text{(see (12.5.30(p.77)))}.\end{aligned} \tag{20.1.15}$$

Hence, since (20.1.4(p.153)) holds, it follows that Theorem 20.1.1(p.153) holds.

(b) From Table 6.4.3(p.41) (III) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\} = \left\{\begin{array}{l} V_0 = \rho, \\ V_1 = \max\{\lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 1 \end{array}\right\}$$

Applying the operation $\mathcal{R}$ to this leads to

$$\begin{aligned}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = \max\{\lambda\beta\max\{0, -\hat{a} - \rho\} + \beta\rho - s, \beta\rho\}, \\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = \max\{-\lambda\beta\min\{0, \hat{a} + \rho\} + \beta\rho - s, \beta\rho\}, \\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = -\min\{\lambda\beta\min\{0, \hat{a} + \rho\} - \beta\rho + s, -\beta\rho\}, \\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} \hat{V}_0 = -\rho, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a} + \rho\} + \beta\rho - s, \beta\rho\}, \\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a} - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.\end{aligned}$$

Applying $\mathcal{C}_{\mathbb{P}}$ to this yields

$$\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \check{b} - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\check{\tilde{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.$$

Applying $\mathcal{I}_{\mathbb{P}}$ to this produces

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, b - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.$$

For the same reason as in the proof of (a), we can replace $\hat{\rho}$ by ρ, hence we obtain.

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} V_0 = \rho, \\ V_1 = \min\{\lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 1 \end{array}\right\},$$

which is the same as $\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\texttt{A}]\}$ given by Table 6.4.3(p.41) (IV), hence we have

$$\begin{aligned} \texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\texttt{A}]\} &= \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] &(20.1.16)\\ &= \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] \quad \text{(see (14.5.3(p.106)))}. &(20.1.17)\end{aligned}$$

Hence, since (20.1.6(p.153)) holds, it follows that Theorem 20.1.3(p.153) holds.

(c) Let $\rho \leq a^\star$ or $b \leq \rho$.

1. Let $\rho \leq a^\star$. Then, since $\rho \leq a^\star < a$ due to Lemma 13.2.1(p.93) (n), we have $\max\{0, a-\rho\} = a - \rho \cdots$ **(1)**. In addition, since $T_{\mathbb{R}}(\rho) = \mu - \rho$ from Lemma 10.1.1(p.55) (f) and since $T_{\mathbb{P}}(\rho) = a - \rho$ from Lemma 13.2.1(p.93) (f), we have
$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mu - \rho] = a - \rho = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots \textbf{(2)} \quad \text{(due to (1))}$$
2. Let $b \leq \rho$. Then, since $a < b < \rho$, we have $\max\{0, a-\rho\} = 0 \cdots$ **(3)**. In addition, since $T_{\mathbb{R}}(\rho) = 0$ from Lemma 10.1.1(p.55) (g) and since $T_{\mathbb{P}}(\rho) = 0$ from Lemma 13.2.1(p.93) (g), we have
$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = 0 = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots \textbf{(4)} \quad \text{(due to (3))}.$$

From (2) and (4), whether $\rho \leq a^\star$ or $b \leq \rho$, we have

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\}, \tag{20.1.18}$$

hence from (5.1.4(p.25)) we have

$$\begin{aligned} \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\lambda\beta T_{\mathbb{R}}(\rho) - (1-\beta)\rho - s] \\ &= \lambda\beta\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] - (1-\beta)\rho - s \\ &= \lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s. \end{aligned} \tag{20.1.19}$$

Accordingly, we have

$$\begin{aligned} &\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.4.18(p.41)) \text{ with } t = 1] \\ &\quad= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(V_0) + V_0, \beta V_0\}\}] \\ &\quad= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(\rho) + \rho, \beta\rho\}\}] \\ &\quad= \{V_1 = \max\{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] + \rho, \beta\rho\}\} \\ &\quad= \{V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s + \rho, \beta\rho\}\} \quad \text{(due to (20.1.19(p.155)))} \\ &\quad= \{V_1 = \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}\} \\ &\quad= \{(6.4.22(p.41))\}. \end{aligned}$$

The above result means that $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.4.18(p.41))$ with "$t > 0$" is separated into the two cases, (6.4.22(p.41)) with "$t = 1$" and (6.4.23(p.41)) "with "$t > 1$". This fact implies that

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}]. \tag{20.1.20}$$

Accordingly, since (20.1.5(p.153)) holds whether $\rho \leq a^\star$ or $b \leq \rho$, it follows that Theorem 20.1.2(p.153) holds.

(d) Let $a^\star < \rho < b$. Then, since the same reasoning as in the proof of (c) does not always hold, it follows that Theorem 20.1.2(p.153) does not always hold. ∎

Remark 20.1.1 (pseudo-reversible element ρ) Let us recall here that $\mathcal{R}$ is an operation applied *only* to attribute elements which depend on the distribution function F (see Section 12.1.1(p.69)). Accordingly, the operation cannot be applied to the constant ρ which is not related to F, implying that the $\hat{\rho}$ in the proofs of (a,b) is one resulting from *merely rearranging* the expression $-\hat{V}_1 = \rho$ as $\hat{V}_1 = -\rho \to \hat{V}_1 = \hat{\rho}$. However, superficially this transformation $\rho \to \hat{\rho}$ seems to be the application of the reversible operation $\mathcal{R}$ defined in Section 12.1.1(p.69). For this reason, regarding this ρ, which is *originally* a non-reversible element, as a *reversible element* of a sort (see Def. 12.3.3(p.73)), let us call it the *pseudo-reversible element.* □

20.1.3 M:2[ℝ][A]

20.1.3.1 Preliminary

From (6.4.18(p.41)) and (5.1.8(p.25)) we have

$$V_t = \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}$$
$$= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0, \tag{20.1.21}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 0. \tag{20.1.22}$$

Then, for $t > 0$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \quad \text{(see (5.1.9(p.25)))}, \tag{20.1.23}$$
$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{20.1.24}$$

Finally, from (6.2.75(p.33)) and from (6.2.71(p.33)) and (6.2.73(p.33)) we have

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}), \quad t > 0, \tag{20.1.25}$$
$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}), \quad t > 0. \tag{20.1.26}$$

20.1.3.2 Analysis

20.1.3.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 20.1.1** (▣ $\mathscr{A}$ {M:2[ℝ][A]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < b$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$, hence $x_L = x_K = b \cdots$ **(1)** from Lemmas 10.2.3(p.58) (a). Then, since $K(x) = \lambda T(x) \cdots$ **(2)** for any x from (5.1.4(p.25)), due to Lemma 10.1.1(p.55) (g) we have $K(x) \geq 0 \cdots$ **(3)** for any x and $K(b) = 0 \cdots$ **(4)**.

(a) From (6.4.18(p.41)) we have $V_t \geq K(V_{t-1}) + V_{t-1}$ for $t > 0$, hence $V_t \geq V_{t-1}$ for $t > 0$ due to (3). Thus V_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq b$, hence $\rho \geq x_L$ due to (1). Then, since $V_0 \geq x_L$ from (6.4.17(p.41)), we have $V_{t-1} \geq x_L$ for $t > 0$ from (a). Hence, since $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 10.2.1(p.57) (d), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.156)), thus $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$, i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$. Hence, since $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, we have $t^*_\tau = 0$ for $\tau > 0$ due to Preference Rule 7.2.1(p.45), i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) Let $\rho < b$. Then $V_0 < b$ from (6.4.17(p.41)). Suppose $V_{t-1} < b$. Then, from Lemma 10.2.2(p.57) (h) and (6.4.18(p.41)) with $\beta = 1$ we have $V_t < \max\{K(b) + b, b\} = \max\{b, b\}$ due to (4), hence $V_t < b$. Accordingly, by induction $V_{t-1} < b \cdots$ **(5)** for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ due to (1). Thus, since $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (d), we have $L(V_{t-1}) > 0 \cdots$ **(6)** for $\tau \geq t > 0$. Accordingly, from (20.1.22(p.156)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 0$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^\tau V_0$. Accordingly, we have $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 0$ due to (6) and (20.1.26(p.156)). ∎

20.1.3.2.2 Case of $\beta <$ or $s > 0$

For explanatory simplicity, let us define

$$\texttt{S}_3\boxed{Ⓢ\blacktriangle \mid ⓞ\ \parallel} = \left\{\begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 0 \text{ such that} \\ (1)\ \boxed{Ⓢ\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 0}\langle \tau\rangle}_{\blacktriangle} \text{ where } \texttt{Conduct}_{\tau\geq t>0\blacktriangle}, \\ (2)\ \boxed{ⓞ\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\parallel} \text{ where } \texttt{Conduct}_{t^\bullet_\tau \geq t>0\blacktriangle}. \end{array}\right\}.$$

□ **Tom 20.1.2** (▣ $\mathscr{A}$ {M:2[ℝ][A]}) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{Ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $a < \rho$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho \leq a$.*

1. *Let* $(\lambda\mu - s)/\lambda \le a$.
   i. *Let* $\lambda = 1$. *Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
   ii. *Let* $\lambda < 1$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .
   i. *Let* $a < \rho$.
      1. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆ . *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
      2. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|}$ *is true.*
   ii. *Let* $\rho \le a$.
      1. *Let* $(\lambda\beta\mu - s)/\delta \le a$.
         i. *Let* $\lambda = 1$.
            1. *Let* $b > 0$ ⦅$\kappa > 0$⦆ . *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
            2. *Let* $b \le 0$ ⦅$\kappa \le 0$⦆ . *Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
         ii. *Let* $\lambda < 1$.
            1. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆ . *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
            2. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|}$ *is true.*
      2. *Let* $(\lambda\beta\mu - s)/\delta > a$.
         i. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆ . *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
         ii. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|}$ *is true.* □

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots$**(1)**. Then $V_0 < x_K \cdots$**(2)** from (6.4.17(p.41)) and $K(\rho) > 0 \cdots$**(3)** due to Lemma 10.2.2(p.57) (j1). Accordingly, from (6.4.18(p.41)) with $t = 1$ we have $V_1 - V_0 = V_1 - \rho = \max\{K(\rho), \beta\rho - \rho\} \ge K(\rho) > 0$ due to (3), hence $V_1 > V_0 \cdots$**(4)**.

(a) Note (4), hence $V_0 \le V_1$. Suppose $V_{t-1} \le V_t$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \le \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \ge V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \ge 0$. Again note (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 10.2.2(p.57) (f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, and if $a < \rho$, then $a < V_0$ from (6.4.17(p.41)), hence $a < V_t$ for $t \ge 0$ due to (a), thus from Lemma 10.2.2(p.57) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \ge 0$. Consider a sufficiently large $M > 0$ with $\rho \le M$ and $b \le M$, hence $V_0 \le M$ from (6.4.17(p.41)). Suppose $V_{t-1} \le M$. Then, from Lemma 10.2.2(p.57) (e) and (6.4.18(p.41)) we have $V_t \le \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (10.2.7 (2) (p.57)), hence $V_t \le \max\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Thus, by induction $V_t \le M$ for $t \ge 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta V\}$ from (6.4.18(p.41)), we have $0 = \max\{K(V), -(1-\beta)V\}$, hence $K(V) \le 0$, so $V \ge x_K$ due to Lemma 10.2.2(p.57) (j1).

(b) Let $x_L \le \rho$. Then, since $x_L \le V_0$ from (6.4.17(p.41)), we have $x_L \le V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a). Accordingly, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.156)), we have $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 0$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \ge t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, implying that $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) Let $\rho < x_L \cdots$**(5)**, hence $V_0 < x_L \cdots$**(6)** from (6.4.17(p.41)).

(c1) Since $L(V_0) = L(\rho) > 0 \cdots$**(7)** due to (5) and Corollary 10.2.1(p.57) (a), we have $V_1 = L(V_0) + \beta V_0 \cdots$**(8)** due to (20.1.23(p.156)) with $t = 1$, hence $V_1 > \beta V_0 \cdots$**(9)**. Accordingly, we have $t^*_1 = 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle} \cdots$**(10)** and $\mathtt{Conduct}_{1\blacktriangle} \cdots$**(11)** due to (7) and (20.1.26(p.156)) with $t = 1$. Below let $\tau > 1$.

(c2) Let $\beta = 1$, hence $s > 0 \cdots$**(12)** due to the assumption "$\beta < 1$ or $s > 0$". Then $\delta = \lambda \cdots$**(13)** from (10.2.1(p.56)) and $x_L = x_K \cdots$**(14)** from Lemma 10.2.3(p.58) (b), hence $K(x_L) = K(x_K) = 0 \cdots$**(15)**. Then, from (5) and (14) we have $\rho < x_K \cdots$**(16)**.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.4.17(p.41)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.4.18(p.41)) with $\beta = 1$ and Lemma 10.2.2(p.57) (g) we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (14). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (e1), for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ and $\mathtt{CONDUCT}_{\tau\ge t>0\blacktriangle}$.

(c2ii) Let $\rho \le a \cdots$**(17)**. Then $V_0 \le a \cdots$**(18)** from (6.4.17(p.41)). Here note that (8) can be rewritten as $V_1 = K(V_0) + V_0 = K(\rho) + \rho \cdots$**(19)** due to (5.1.9(p.25)). Then, from (17) and (10.2.7 (1) (p.57)) with $\beta = 1$ we have $V_1 = \lambda\mu - s + (1-\lambda)\rho \cdots$**(20)**

(c2ii1) Let $(\lambda\mu - s)/\lambda \le a$. Then $x_K = (\lambda\mu - s)/\lambda \le a \cdots$**(21)** from Lemma 10.2.2(p.57) (j2) and (13). Hence $K(a) \le 0 \cdots$**(22)** from Lemma 10.2.2(p.57) (j1). Note (18). Suppose $V_{t-1} \le a$. Then, from Lemma 10.2.2(p.57) (e) and (6.4.18(p.41)) with $\beta = 1$ we have $V_t \le \max\{K(a) + a, a\} = a$ due to (22), hence by induction $V_{t-1} \le a$ for $t > 0$. Accordingly, from (6.4.18(p.41)) with $\beta = 1$ and (10.2.7 (1) (p.57)) we have $V_t = \max\{\lambda\mu - s + (1-\lambda)V_{t-1}, V_{t-1}\} \cdots$**(23)** for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, since $x_K = \mu - s$ from (21), we have $V_1 = \mu - s = x_K \cdots$ **(24)** from (20). In addition, from (23) we have $V_t = \max\{\mu - s, V_{t-1}\} = \max\{x_K, V_{t-1}\}$ for $t > 0$. Note (24). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} = x_L$ for $t > 1$ due to (14). Hence $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, so $L(V_{t-1}) = 0 \cdots$ **(25)** for $\tau \geq t > 1$. Then, from (20.1.22(p.156)) we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$, i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$. From this and (9) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\shortparallel}$. Then, from (7) and (20.1.26(p.156)) with $t = 1$ we have $\mathtt{Conduct}_{1\blacktriangle}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, since $L(V_{t-1}) > 0$ due to Lemma 10.2.1(p.57) (e1), from (20.1.23(p.156)) and Lemma 10.2.2(p.57) (f) we have $V_t = K(V_{t-1}) + V_{t-1} < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, so $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (e1). Hence, for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a \cdots$ **(26)** from Lemma 10.2.2(p.57) (j2). Note (6). Suppose $V_{t-1} < x_L$. Then $L(V_{t-1}) > 0$ from Lemma 10.2.1(p.57) (e1), hence $V_t = K(V_{t-1}) + V_{t-1}$ from (20.1.23(p.156)). Now, since $a < x_K = x_L$ due to (26) and (14), from Lemma 10.2.2(p.57) (h) we have $V_t < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L \cdots$ **(27)** for $t > 0$, thus $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (e1). Hence, for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3) Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.

(c3i) Let $a < \rho \cdots$ **(28)**. Then, we have $a < V_0$ from (6.4.17(p.41)), hence $a < V_{t-1} \cdots$ **(29)** for $t > 0$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.57) (g) and (6.4.18(p.41)) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_{t-1} is *strictly increasing* in $t > 0 \cdots$ **(30)**.

(c3i1) Let $b \geq 0$ $(\!(\kappa \geq 0)\!)$. Then, $x_L \geq x_K \geq 0 \cdots$ **(31)** from Lemma 10.2.3(p.58) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (29), Lemma 10.2.2(p.57) (g), and (6.4.18(p.41)) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (31), we have $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a). Consequently, for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ † and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3i2) Let $b < 0$ $(\!(\kappa < 0)\!)$. Then $x_L < x_K \cdots$ **(32)** from Lemma 10.2.3(p.58) (c (d)). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$ due to (a). Then, since $x_K \leq V \cdots$ **(33)** due to (a), we have the contradiction of $V \leq x_L < x_K \leq V$ from (32). Accordingly, it is impossible that $V_{t-1} \leq x_L$ for *all* $t > 0$. Therefore, from (6) and (30) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots.$$

Hence, for almost the same reason as in the proof of Tom 11.2.2(p.62) (c2iii2) we immediately see that $\mathtt{S}_3$ is true.‡

(c3ii) Let $\rho \leq a \cdots$ **(34)**, hence $V_0 \leq a$ from (6.4.17(p.41)). Then, from (19) and (10.2.7 (1) (p.57)) we have $V_1 = \lambda\beta\mu - s + (1 - \lambda)\beta\rho$.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \leq a$. Then, since $x_K = (\lambda\beta\mu - s)/\delta \leq a \cdots$ **(35)** from Lemma 10.2.2(p.57) (j2), we have $\delta x_K = \lambda\beta\mu - s$, hence $V_1 = \delta x_K + (1 - \lambda)\beta\rho \cdots$ **(36)**.

(c3ii1i) Let $\lambda = 1$. Then, since $\delta = 1$ from (10.2.1(p.56)), we have $x_K = \beta\mu - s \leq a$ from (35) and $V_1 = x_K \leq a \cdots$ **(37)** from (36).

(c3ii1i1) Let $b > 0$ $(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots$ **(38)** due to Lemma 10.2.3(p.58) (c (d)). Note (37). Suppose $V_{t-1} = x_K$. Then, from (6.4.18(p.41)) we have $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Hence, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} < x_L$ for $t > 1$ due to (38). Accordingly $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to (7). Therefore, for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1i2) Let $b \leq 0$ $(\!(\kappa \leq 0)\!)$. Then, since $x_L \leq x_K$ due to Lemma 10.2.3(p.58) (c (d)), from (37) we have $V_1 \geq x_L$, hence $V_{t-1} \geq x_L$ for $t > 1$ from (a), so $V_{t-1} \geq x_L$ for $\tau \geq t > 1$. Accordingly, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from Corollary 10.2.1(p.57) (a), we obtain $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.22(p.156)) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$. From this and (9) we obtain $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\shortparallel}$. Then, we have $\mathtt{Conduct}_{1\blacktriangle}$ from (7) and (20.1.26(p.156)) with $t = 1$.

(c3ii1ii) Let $\lambda < 1$. Note (4). Suppose $V_{t-1} < V_t$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_t is *strictly increasing* in $t \geq 0 \cdots$ **(39)**.

(c3ii1ii1) Let $b \geq 0$ $(\!(\kappa \geq 0)\!)$. Then $x_L \geq x_K \geq 0 \cdots$ **(40)** from Lemma 10.2.3(p.58) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (f) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to

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†Note that we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ instead of $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ due to (c1).

‡Note the fine difference between $\mathtt{S}_3$ and $\mathtt{S}_1$(p.62).

$x_K \geq 0$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (40). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1ii2) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K$ from Lemma 10.2.3(p.58) (c (d)). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$. Then, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} \leq x_L$ for all $t > 0$. Therefore, from (6) and (39) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

hence for almost the same reason as in the proof of Tom 11.2.2(p.62) (c2iii2) we have $\texttt{S}_3$[‡] is true.

(c3ii2) Let $(\lambda\beta\mu - s)/\lambda > a \cdots$ (41). Then $x_K > (\lambda\beta\mu - s)/\delta > a \cdots$ (42) from Lemma 10.2.2(p.57) (j2). Let us note here that:

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \geq 0$ for the same reason as in the proof of (c3ii1ii).
2. Let $\lambda = 1$. Then $\beta\mu - s > a \cdots$ (43) from (41). Now, since $K(\rho) + \rho = \beta\mu - s$ from (10.2.7 (1) (p.57)) and (34), we have $V_1 = \beta\mu - s$ from (19), hence $V_1 > a$ from (43), so $V_{t-1} > a$ for $t > 1$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t > 0$.

Consequently, whether $\lambda < 1$ or $\lambda = 1$, it follows that V_t is *strictly increasing* in $t > 0 \cdots$ (44).

(c3ii2i) Let $b \geq 0$ ($\kappa \geq 0$). Then $x_L \geq x_K \geq 0 \cdots$ (45) from Lemma 10.2.3(p.58) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.4.18(p.41)) and from (42) and Lemma 10.2.2(p.57) (h) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (45), so $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a). Hence, for almost the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii2ii) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots$ (46) from Lemma 10.2.3(p.58) (c (d)). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots$ (47). Now, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. Therefore, from (44) and (6) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

hence for almost the same reason as in the proof of Tom 11.2.2(p.62) (c2iii2) we have $\texttt{S}_3$ is true. ∎

□ **Tom 20.1.3** (▣ $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*

1. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let $b \leq 0$ ($\kappa \leq 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. □

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then $V_0 = x_K \cdots$ (1) from (6.4.17(p.41)), hence $K(V_0) = K(x_K) = 0 \cdots$ (2).

(a) We obtain $V_1 \geq K(V_0) + V_0 = V_0 \cdots$ (3) from (6.4.18(p.41)) with $t = 1$ and (2). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.2.2(p.57) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K$ from Lemma 10.2.3(p.58) (b). Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.4.18(p.41)) we have $V_t = \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} = x_L$ for $t > 0$, so $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$. Accordingly, for the same reason as in the proof of Tom 20.1.1(p.156) (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) Let $\beta < 1$ and $s = 0$ ($s > 0$).

(c1) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots$ (4) from Lemma 10.2.3(p.58) (c (d)). Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.4.18(p.41)) we have $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (4), so $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a). Therefore, for the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c2) Let $b \leq 0$ ($\kappa \leq 0$). Then, since $x_L \leq x_K$ from Lemma 10.2.3(p.58) (c (d)), we have $x_L \leq V_0$ from (1), hence $x_L \leq V_{t-1}$ for $t > 0$ from (a), so $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a). Then, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from(20.1.22(p.156)), for the same reason as in the proof of Tom 20.1.1(p.156) (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ∎

‡Note the fine difference between $\texttt{S}_3$ and $\texttt{S}_1$(p.62).

$$\mathsf{S}_4\boxed{ⓢ\blacktriangle \mid \bullet\| \mid \mathrm{C{\leadsto}S}\vartriangle \mid \mathrm{C{\leadsto}S}\blacktriangle} = \left\{\begin{array}{l} \text{There exist } t^\bullet_\tau \text{ and } t^\circ_\tau\ (t^\bullet_\tau > t^\circ_\tau \geq 0) \text{ such that} \\ (1)\ \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau \geq \tau > 0}\langle 0\rangle}_{\|}, \\ (2)\ \boxed{ⓢ\ \mathtt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau \geq t > t^\bullet_\tau\,\blacktriangle} \quad \cdots (1^*) \text{ and} \\ \qquad \text{where } \mathtt{C{\leadsto}S}_{\,t^\bullet_\tau \geq t > t^\circ_\tau\,\vartriangle} \quad \cdots (2^*) \text{ and} \\ \qquad \mathtt{C{\leadsto}S}_{\,t^\circ_\tau \geq t > 0\,\vartriangle}\ (\!(\mathtt{C{\leadsto}S}_{\,t^\circ_\tau \geq t > 0\,\blacktriangle})\!) \cdots (3^*).^\dagger \end{array}\right\}$$

□ **Tom 20.1.4** (▣ $\mathscr{A}$\{M:2[ℝ][A]\}) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*[‡]

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $x_L > \rho$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ $(\!(s > 0)\!)$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
2. *Let $b \leq 0$ $(\!(\kappa \leq 0)\!)$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $b > 0$ $(\!(\kappa > 0)\!)$.*
   i. *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho = x_L$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\|}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let $x_L < \rho$. Then* $\mathsf{S}_4\boxed{ⓢ\blacktriangle \mid \bullet\| \mid \mathrm{C{\leadsto}S}\vartriangle \mid \mathrm{C{\leadsto}S}\blacktriangle}$ *is true.* ↦ → $\boxed{\mathrm{C{\leadsto}S}}_{\blacktriangle}$

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ $(\!(s > 0)\!)$.*

1. *V_t is nondecreasing in $t \geq 0$.*
2. *Let $b \leq 0$ $(\!(\kappa \leq 0)\!)$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $b > 0$ $(\!(\kappa > 0)\!)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho > x_K \cdots$ **(1)**. Hence $V_0 > x_K \cdots$ **(2)** from (6.4.17(p.41)) and $K(\rho) < 0 \cdots$ **(3)** due to Lemma 10.2.2(p.57) (j1). Note that $V_0 \geq x_K$. Suppose $V_{t-1} \geq x_K$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \geq K(V_{t-1}) + V_{t-1} \geq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \geq x_K \cdots$ **(4)** for $t > 0$. From (6.4.18(p.41)) with $t = 1$ we have

$$V_1 - V_0 = V_1 - \rho = \max\{K(V_0) + V_0, \beta V_0\} - \rho = \max\{K(\rho) + \rho, \beta\rho\} - \rho = \max\{K(\rho), -(1-\beta)\rho\} \cdots \textbf{(5)}.$$

(a) Let $\beta = 1$ or $\rho = 0$.

(a1) Then, since $-(1-\beta)\rho = 0$, due to (3) we have $V_1 - V_0 = 0$ from (5), i.e., $V_0 = V_1$. Suppose $V_{t-1} = V_t$. Then, from (6.4.18(p.41)) we have $V_t = \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Thus, by induction $V_{t-1} = V_t$ for $t > 0$, i.e., $V_0 = V_1 = V_2 = \cdots$, hence $V_t = V_0 = \rho$ for $t \geq 0$.

(a2) Let $x_L \leq \rho$. Then, since $x_L \leq V_{t-1}$ for $t > 0$ from (a1), we have $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), hence $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.156)). Accordingly, for the same reason as in the proof of Tom 20.1.1(p.156) (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(a3) Let $x_L > \rho$. Then, since $x_L > V_{t-1}$ for $t > 0$ from (a1), we have $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), hence for the same reason as in the proof of Tom 20.1.1(p.156) (c) we obtain $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) Let $\beta < 1 \cdots$ **(6)** and $\rho > 0 \cdots$ **(7)** and let $s = 0$ $(\!(s > 0)\!)$. Then, since $-(1-\beta)\rho < 0 \cdots$ **(8)**, from (5) and (3) we have $V_1 - V_0 < 0$, so $V_1 > V_0$, hence $\rho = V_0 > V_1 \cdots$ **(9)** from (6.4.17(p.41)).

(b1) We have $V_0 \geq V_1$ from (9). Suppose $V_{t-1} \geq V_t$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \geq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 0$, i.e., V_t is nonincreasing in $t \geq 0$. In addition, since V_t is lower bounded in t due to (4), it follows that V_t converges to a finite V as $t \to \infty$. Accordingly, from (4) we have $V \geq x_K$.

(b2) Let $b \leq 0$ $(\!(\kappa \leq 0)\!)$. Then, since $x_L \leq x_K$ due to Lemma 10.2.3(p.58) (c (d)), from (4) we have $V_{t-1} \geq x_L$ for $t > 0$. Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.156)), hence for the same reason as in the proof of Tom 20.1.1(p.156) (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(b3) Let $b > 0$ $(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots$ **(10)** from Lemma 10.2.3(p.58) (c (d)).

(b3i) Let $\rho < x_L$. Then, since $V_0 < x_L$ from (6.4.17(p.41)), we have $V_{t-1} < x_L$ for $t > 0$ due to (b1). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ and $\mathtt{CONDUCT}_{\tau\geq t>0\blacktriangle}$.

[†]See Def. 2.2.1(p.12) for the definition of the symbol C⇝S.

[†]The inverse of the condition "$\beta = 1$ or $\rho = 0$" is "$\beta < 1$ and $\rho \neq 0$", which is classified into the two cases of "$\beta < 1$ and $\rho > 0$" and "$\beta < 1$ and $\rho < 0$", leading to the conditions in (b) and (c) that follows.

(b3ii) Let $\rho = x_L \cdots (11)$. Then, since $V_0 = x_L$ from (6.4.17(p.41)), we have $L(V_0) = L(x_L) = 0 \cdots (12)$, hence from (20.1.24(p.156)) with $t = 1$ we have $V_1 = \beta V_0 \cdots (13)$, so $t_1^* = 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$. Below let $\tau > 1$. From (9) and (11) we have $V_1 < V_0 = x_L$. Accordingly, since $V_{t-1} < x_L$ for $t > 1$ from (b1), we have $L(V_{t-1}) > 0 \cdots (14)$ for $t > 1$ from Corollary 10.2.1(p.57) (a), hence $L(V_{t-1}) > 0 \cdots (15)$ for $\tau \geq t > 1$. Therefore, $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (20.1.22(p.156)), hence $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, so that $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. From this and (13) we obtain $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. Then $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (15) and (20.1.26(p.156)).

(b3iii) Let $x_L < \rho$, hence $x_L < V_0 \cdots (16)$ from (6.4.17(p.41)), so $x_L \leq V_0$. Suppose $x_L \leq V_{t-1} \cdots (17)$ for *all* $t > 0$. Then, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), we have $V_t = \beta V_{t-1}$ for $t > 0$ from (20.1.24(p.156)), hence $V_t = \beta^t V_0 = \beta^t \rho > 0$ for $t \geq 0$ due to (7). Then, since $\lim_{t\to\infty} V_t = 0$ due to (6), from (10) we have $x_L > x_K > V_t > 0$ for a sufficiently large t, which contradicts (17). Hence, it is impossible that $x_L \leq V_{t-1}$ for *all* $t > 0$. Accordingly, from (16) and (b1) we see that there exist t_τ° and $t_\tau^\bullet$ ($t_\tau^\circ < t_\tau^\bullet$) such that

$$V_0 \geq V_1 \geq \cdots \geq V_{t_\tau^\circ-1} > V_{t_\tau^\circ} = V_{t_\tau^\circ+1} = \cdots = V_{t_\tau^\bullet-1} = x_L > V_{t_\tau^\bullet} \geq V_{t_\tau^\bullet+1} \geq \cdots\cdots (18)$$

Hence, we have

$$x_L > V_{t_\tau^\bullet},\ \ x_L > V_{t_\tau^\bullet+1},\ \cdots,$$
$$V_{t_\tau^\circ} = x_L,\ V_{t_\tau^\circ+1} = x_L,\ \cdots,\ V_{t_\tau^\bullet-1} = x_L,$$
$$V_0 > x_L,\ V_1 > x_L,\ \cdots,\ V_{t_\tau^\circ-1} > x_L,$$

or equivalently

$$x_L > V_{t-1} \cdots (19),\qquad t > t_\tau^\bullet,$$
$$V_{t-1} = x_L \cdots (20),\qquad t_\tau^\bullet \geq t > t_\tau^\circ,$$
$$V_{t-1} > x_L \cdots (21),\qquad t_\tau^\circ \geq t > 0.$$

Accordingly, we have:

1. Let $t_\tau^\bullet \geq \tau > 0$. Then, since $V_{t-1} \geq x_L$ for $\tau \geq t > 0$ from (20) and (21), we have $L(V_{t-1}) \leq 0 \cdots (22)$ for $\tau \geq t > 0$ from Corollary 10.2.1(p.57) (a), hence $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$ from (20.1.22(p.156)), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0 \cdots (23)$, hence $t_\tau^* = 0$ for $t_\tau^\bullet \geq \tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet \geq \tau > 0}\langle 0\rangle}_{\parallel}$. Accordingly, $\mathtt{S}_4(1)$ is true. Then, from (23) with $\tau = t_\tau^\bullet$ we have $V_{t_\tau^\bullet} = \beta V_{t_\tau^\bullet-1} = \cdots = \beta^{t_\tau^\bullet} V_0 \cdots (24)$,
2. Let $\tau > t_\tau^\bullet$. Then, since $x_L > V_{t-1}$ for $\tau \geq t > t_\tau^\bullet$ from (19), we have $L(V_{t-1}) > 0 \cdots (25)$ for $\tau \geq t > t_\tau^\bullet$ from Corollary 10.2.1(p.57) (a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t_\tau^\bullet$ from (20.1.22(p.156)), i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > t_\tau^\bullet$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} \cdots (26)$. From this and (24) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t_\tau^\bullet} V_{t_\tau^\bullet} = \beta^{\tau-t_\tau^\bullet+1} V_{t_\tau^\bullet-1} = \cdots = \beta^\tau V_0,$$

hence $t_\tau^* = \tau$ for $\tau > t_\tau^\bullet$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t_\tau^\bullet}\langle\tau\rangle}_{\blacktriangle}$, so the former half of $\mathtt{S}_4(2)$ is true.

(i) We have $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t_\tau^\bullet \cdots (27)$ form (25) and (20.1.26(p.156)). Hence the latter half (1^*) of $\mathtt{S}_4(2)$ is true.

Below let us show the latter half (2^*) and (3^*) of $\mathtt{S}_4(2)$.

(ii) If $t_\tau^\bullet \geq t > t_\tau^\circ$, then $L(V_{t-1}) = L(x_L) = 0$ from (20), hence we have $\mathtt{Skip}_{t\vartriangle}$ from (20.1.25(p.156)), implying $\mathtt{C{\leadsto}S}_{t\vartriangle}$ (see Figure 7.2.1(p.44) (II)) or equivalently $\mathtt{C{\leadsto}S}_{t_\tau^\bullet \geq t > t_\tau^\circ\vartriangle}$. Hence the latter half (2^*) of $\mathtt{S}_4(2)$ is true.

(iii) If $t_\tau^\circ \geq t > 0$, then $L(V_{t-1}) = (\!(<)\!)\ 0$ ‡ from (21) and Lemma 10.2.1(p.57) (d (!(e1)!)), hence we have $\mathtt{Skip}_{t\vartriangle}$ (!($\mathtt{Skip}_{t\blacktriangle}$)!) from (20.1.25(p.156)) (!((20.1.26(p.156)))!) , implying $\mathtt{C{\leadsto}S}_{t\vartriangle}$ (!($\mathtt{C{\leadsto}S}_{t\blacktriangle}$)!) or equivalently $\mathtt{C{\leadsto}S}_{t_\tau^\circ \geq t > 0\vartriangle}$ (!($\mathtt{C{\leadsto}S}_{t_\tau^\circ \geq t > 0\blacktriangle}$)!) . Hence the latter half (3^*) of $\mathtt{S}_4(2)$ is true..

(c) Let $\beta < 1$ and $\rho < 0 \cdots (28)$ and let $s = 0$ (!($s > 0$)!) .

(c1) Since $-(1-\beta)\rho > 0$, from (5) we have $V_1 - V_0 > 0$, i.e., $V_0 < V_1$, hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from (6.4.18(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(c2) Let $b \leq 0$ (!($\kappa \leq 0$)!) . Then $x_L \leq x_K$ due to Lemma 10.2.3(p.58) (c (!(d)!)), hence from (4) we have $V_{t-1} \geq x_L$ for $t > 0$. Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.156)), hence for the same reason as in the proof of Tom 20.1.1(p.156) (b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c3) Let $b > 0$ (!($\kappa > 0$)!) . Then $x_L > x_K > 0 \cdots (29)$ from Lemma 10.2.3(p.58) (c (!(d)!)). Then, since $\rho < 0 < x_K$ from (28) and (29), we have $V_0 < x_K$ from (6.4.17(p.41)), hence $V_0 \leq x_K$. Suppose $V_{t-1} \leq x_K$, hence $V_{t-1} < x_L$ form (29), thus $L(V_{t-1}) > 0$ from Corollary 10.2.1(p.57) (a). Accordingly, from (20.1.23(p.156)) and Lemma 10.2.2(p.57) (e) we have $V_t = K(V_{t-1}) + V_{t-1} \leq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \leq x_K$ for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ from (29). Therefore, since $L(V_{t-1}) > 0 \cdots (30)$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.1.1(p.156) (c) we have $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. ∎

‡ If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

20.1.3.3 Market Restriction

20.1.3.3.1 Positive Restriction

□ Pom **20.1.1** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* The same as Tom 20.1.1(p.156) due to Lemma 17.4.4(p.118). ▮

□ Pom **20.1.2** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a \leq \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
2. *Let $\beta = 1$, hence $s > 0$.*
   - i. *Let $a \leq \rho$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho < a$.*
     1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
4. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $a < \rho$.*
     1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $\lambda\beta\mu < s$. Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \parallel}$ *is true.*
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $\beta\mu > s$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
          2. *Let $\beta\mu \leq s$. Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
          2. *Let $\lambda\beta\mu < s$. Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \parallel}$ *is true.*
     2. *Let $(\lambda\beta\mu - s)/\delta > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0 \cdots$ **(2)**. Then, we have $\kappa = \lambda\beta\mu - s \cdots$ **(3)** from Lemma 10.3.1(p.59) (a).

(a-c2ii2) The same as Tom 20.1.2(p.156) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c3i1,c3ii1i1, c3ii1ii1,c3ii2i) of Tom 20.1.2(p.156).

(c4) Let $\beta < 1$ and $s > 0$.

(c4i-c4ii1ii2) Immediate from (3) and Tom 20.1.2(p.156) (c3i-c3ii1ii2) with κ.

(c4ii2) Let $(\lambda\beta\mu - s)/\delta > a$. Then, since$(\lambda\beta\mu - s)/\delta > a > 0$ due to (1), we have $\lambda\beta\mu - s > 0$, so that $\kappa > 0$ due to (3). Hence, it suffices to consider only (c3ii2i) of Tom 20.1.2(p.156). ▮

□ Pom **20.1.3** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $s = 0$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $\lambda\beta\mu > s$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let $\lambda\beta\mu \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

• *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then, we have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a).

(a,b) The same as Tom 20.1.3(p.159) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.3(p.159).

(d) Let $\beta < 1$ and $s > 0$.

(d1,d2) Immediate from (2) and Tom 20.1.3(p.159) (c1,c2) with κ. ∎

□ **Pom 20.1.4** ($\mathscr{A}\{$M:2[ℝ][A]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $x_L > \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\rho = x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
4. *Let $x_L < \rho$. Then* $\texttt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true.*

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s > 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $\lambda\beta\mu \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $\lambda\beta\mu > s$.*
   i. *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho = x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle\ \triangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $x_L < \rho$. Then* $\texttt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true* (see *Numerical Example* 5(p.187)).

(d) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$.*

1. *V_t is nondecreasing in $t \geq 0$.*
2. $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let $\beta < 1$ and $\rho < 0$ and let $s > 0$.*

1. *V_t is nondecreasing in $t \geq 0$.*
2. *Let $\lambda\beta\mu \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $\lambda\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

• *Proof* Suppose $a > 0$, hence $b > \mu > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a).

(a-a3) The same as Tom 20.1.4(p.160) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. First, (b1) is the same as Tom 20.1.4(p.160) (b1). Next, due to (1) it suffices to consider only (b3i-b3iii) of Tom 20.1.4(p.160).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. First, (c1) is the same as Tom 20.1.4(p.160) (b1). Next, due to (1) it suffices to consider only (b3i-b3iii) of Tom 20.1.4(p.160).

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. First, (d1) is the same as Tom 20.1.4(p.160) (c1). Next, since $\kappa = \lambda\beta\mu > 0$ due to (2) and (1), it suffices to consider only (c3) of Tom 20.1.4(p.160).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. First, (e1) is the same as Tom 20.1.4(p.160) (c1). Next, (e2,e3) are the same as Tom 20.1.4(p.160) (c2,c3) with κ. ∎

20.1.3.3.2 Mixed Restriction

Omitted.

20.1.3.3.3 Negative Restriction

Omitted.

20.1.4 $\tilde{\text{M}}$:2[ℝ][A]

20.1.4.1 Preliminary

Due to Lemma 20.1.1(p.153) (a), we see that the following Tom's 20.1.5(p.164) – 20.1.8(p.164) can be obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom's 20.1.1(p.156) – 20.1.4(p.160) (see Theorem 20.1.1(p.153)).

20.1.4.2 Analysis

20.1.4.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 20.1.5** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *Let $\rho \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.1.1(p.156). ∎

20.1.4.2.2 Case of $\beta < 1$ or $s > 0$

□ **Tom 20.1.6** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$ or $b > \rho$, and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \ge \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\circledS\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\,\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $b > \rho$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
   - ii. *Let $\rho \ge b$.*
     1. *Let $(\lambda\mu + s)/\lambda \ge b$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{\odot\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\,\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
     2. *Let $(\lambda\mu + s)/\lambda < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ $(s > 0)$.*
   - i. *Let $b > \rho$.*
     1. *Let $a \le 0$ $(\tilde{\kappa} \le 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
     2. *Let $a > 0$ $(\tilde{\kappa} > 0)$. Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS^{\blacktriangle} \mid \odot\ \parallel}$ *is true.*
   - ii. *Let $\rho \ge b$.*
     1. *Let $(\lambda\beta\mu + s)/\delta \ge b$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $a < 0$ $(\tilde{\kappa} < 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
          2. *Let $a \ge 0$ $(\tilde{\kappa} \ge 0)$. Then* $\boxed{\odot\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\,\blacktriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $a \le 0$ $(\tilde{\kappa} \le 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
          2. *Let $a > 0$ $(\tilde{\kappa} > 0)$. Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS^{\blacktriangle} \mid \odot\ \parallel}$ *is true.*
     2. *Let $(\lambda\beta\mu + s)/\delta < b$.*
        - i. *Let $a \le 0$ $(\tilde{\kappa} \le 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
        - ii. *Let $a > 0$ $(\tilde{\kappa} > 0)$. Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS^{\blacktriangle} \mid \odot\ \parallel}$ *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.1.2(p.156). ∎

□ **Tom 20.1.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $s = 0$ $(s > 0)$.*

1. *Let $a < 0$ $(\tilde{\kappa} < 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
2. *Let $a \ge 0$ $(\tilde{\kappa} \ge 0)$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.1.3(p.159). ∎

□ **Tom 20.1.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_{\tilde{L}} \ge \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ $(s > 0)$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $a < 0$ ($\tilde{\kappa} < 0$).*
   - i. *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\|}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - iii. *Let $x_{\tilde{L}} > \rho$. Then* $\mathtt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true.*

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($s > 0$).*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see to Tom 20.1.4(p.160)). ▮

20.1.4.3 Market Restriction

20.1.4.3.1 Positive Restriction

□ **Pom 20.1.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\rho > a$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* The same as Tom 20.1.5(p.164) due to Lemma 17.4.4(p.118). ▮

□ **Pom 20.1.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$ or $b \geq \rho$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $b > \rho$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho \geq b$.*
     1. *Let $(\lambda\mu + s)/\lambda \geq b$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $(\lambda\mu + s)/\lambda < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then we have* $\mathtt{S}_3$(p.156) | ⓢ▲ | ⓞ ‖ |.
4. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $b > \rho$. Then* $\mathtt{S}_3$(p.156) | ⓢ▲ | ⓞ ‖ | *is true.*
   - ii. *Let $\rho \geq b$.*
     1. *Let $(\lambda\beta\mu + s)/\delta \geq b$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\mathtt{S}_3$(p.156) | ⓢ▲ | ⓞ ‖ | *is true.*
     2. *Let $(\lambda\beta\mu + s)/\delta < b$. Then* $\mathtt{S}_3$(p.156) | ⓢ▲ | ⓞ ‖ | *is true.* □

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0 \cdots$ **(2)** and $\tilde{\kappa} = s \cdots$ **(3)** from Lemma 12.6.6(p.83) (a).

(a-c2ii2) The same as Tom 20.1.6(p.164) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu + s)/\delta \geq b$. Then, since $\lambda\beta\mu/\delta \geq b$, we have $\lambda\beta\mu \geq \delta b$, hence $\lambda\beta\mu \geq \delta b \geq \lambda b$ due to (2) and (10.2.2 (1) (p.56)), so that $\beta\mu \geq b$, which contradicts [3(p.118)]. Thus, it must be that $(\lambda\beta\mu + s)/\delta < b$. From this and (1) it suffices to consider only (c3ii2ii) of Tom 20.1.6(p.164).

(c4-c4ii2) If $\beta < 1$ and $s > 0$, then $\kappa > 0$ due to (3), hence it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) with κ. ▮

□ **Pom 20.1.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *We have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. □

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 12.6.6(p.83) (a).

(a) The same as `Tom` 20.1.7(p.164) (a).

(2b) Let $\beta = 1$. Then it suffices to consider only (b) of `Tom` 20.1.7(p.164). Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (c2) of `Tom` 20.1.7(p.164) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of `Tom` 20.1.7(p.164), thus, whether $s = 0$ or $s > 0$ we have the same result. Accordingly, whether $\beta = 1$ or $\beta < 1$, it follows that we have the same result. ▮

□ **`Pom` 20.1.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau \geq t > 0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 12.6.6(p.83) (a).

(a-a3) The same as `Tom` 20.1.8(p.164) (a-a3).

(b) Let $\beta < 1$ and $\rho < 0$.

(b1) The same as `Tom` 20.1.8(p.164) (b1).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of `Tom` 20.1.8(p.164) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b2) of `Tom` 20.1.8(p.164). Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $\beta < 1$ and $\rho > 0$.

(c1) The same as `Tom` 20.1.8(p.164) (c1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of `Tom` 20.1.8(p.164) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of `Tom` 20.1.8(p.164). Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ▮

20.1.4.3.2 Mixed Restriction

Omitted.

20.1.4.3.3 Negative Restriction

Omitted.

20.1.5 M:2[ℙ][A]

20.1.5.1 Preliminary

From (6.4.23(p.41)) and from (5.1.21(p.26)) and (5.1.20(p.26)) we have

$$\begin{aligned} V_t &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{20.1.27}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 1. \tag{20.1.28}$$

Then, for $t > 1$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \tag{20.1.29}$$

$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{20.1.30}$$

Now, from (6.2.107(p.35)) and from (6.2.103(p.35)) and (6.2.105(p.35)) we have, for $t > 1$,

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \mathtt{Conduct}_{t\vartriangle}(\mathtt{Skip}_{t\vartriangle}), \tag{20.1.31}$$

$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}(\mathtt{Skip}_{t\blacktriangle}). \tag{20.1.32}$$

From (6.4.22(p.41)) we have

$$V_1 = \max\{\lambda\beta \max\{0, a-\rho\} - s, 0\} + \beta\rho, \tag{20.1.33}$$

hence

$$V_1 - \beta V_0 = V_1 - \beta\rho = \max\{\lambda\beta \max\{0, a-\rho\} - s, 0\} \geq 0. \tag{20.1.34}$$

From the comparison of the two terms within { } in the r.h.s. of (20.1.33(p.166)) it can be seen that

$$\mathbb{S}_1 = \lambda\beta \max\{0, a-\rho\} \geq (\leq)\ s \Rightarrow \mathtt{Conduct}_{1\vartriangle}(\mathtt{Skip}_{1\vartriangle}), \tag{20.1.35}$$

$$\mathbb{S}_1 = \lambda\beta \max\{0, a-\rho\} > (<)\ s \Rightarrow \mathtt{Conduct}_{1\blacktriangle}(\mathtt{Skip}_{1\blacktriangle}). \tag{20.1.36}$$

20.1.5.2 Analysis

20.1.5.2.1 Case of $\beta = 1$ and $s = 0$

20.1.5.2.1.1 Preliminary

Let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.26)), (5.1.20(p.26)), and Lemma 13.2.1(p.93) (g) we have

$$K(x) = L(x) = \lambda T(x) \geq 0 \quad \text{for any } x. \tag{20.1.37}$$

In addition, from (20.1.28(p.166)) we have

$$V_t - \beta V_{t-1} = \max\{\lambda T(V_{t-1}), 0\} = \lambda T(V_{t-1}) \geq 0, \quad t > 1. \tag{20.1.38}$$

Finally, from (20.1.33(p.166)) we have

$$V_1 = \max\{\lambda \max\{0, a-\rho\}, 0\} + \rho \tag{20.1.39}$$
$$= \lambda \max\{0, a-\rho\} + \rho \quad (\text{due to } \lambda \max\{0, a-\rho\} \geq 0) \tag{20.1.40}$$
$$= \max\{\rho, \lambda a + (1-\lambda)\rho\}. \tag{20.1.41}$$

20.1.5.2.1.2 Case of $\rho \leq a^\star$

In this case, due to Lemma 20.1.1(p.153) (c), we can obtain Tom 20.1.1(p.167) below by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom 20.1.1(p.156) with the condition $\rho \leq a^\star$ (see Theorem 20.1.2(p.153)).

Proposition 20.1.1 ($\rho \leq a^\star$) Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.1.1(p.156) (a).

(b) Due to the assumption $\rho \leq a^\star$ we have $\rho \leq a^\star < a < b$ from Lemma 13.2.1(p.93) (n). Hence it suffices to consider only (c) of Tom 20.1.1(p.156). ∎

20.1.5.2.1.3 Case of $b \leq \rho$

In this case, due to Lemma 20.1.1(p.153) (c), we can obtain Tom 20.1.2(p.167) below by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom 20.1.1(p.156) with the condition $b \leq \rho$ (see Theorem 20.1.2(p.153)).

Proposition 20.1.2 ($b \leq \rho$) Assume $b \leq \rho$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. □

● *Proof* Assume $b \leq \rho \cdots$ **(1)** and let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.1.1(p.156) (a).

(b) Due to (1) it suffices to consider only (b) of Tom 20.1.1(p.156). ∎

20.1.5.2.1.4 Case of $a^\star < \rho < b$

In this case, Theorem 20.1.2(p.153) does not always hold due to Lemma 20.1.1(p.153) (d), hence $\mathscr{A}\{\text{M:2}[\mathbb{P}][\text{A}]\}$ must be directly found.

Proposition 20.1.3 ($a^\star < \rho < b$) Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho$. Then $\boxed{\bullet\ \text{dOITd}_1\langle 0\rangle}_{\|}$ and $\boxed{\text{Ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ and $\texttt{C}\rightsquigarrow\texttt{S}_{1\vartriangle}$.

(c) Let $\rho < a$. Then $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Assume $a^\star < \rho < b \cdots$ **(1)** and let $\beta = 1$ and $s = 0$. Then, from (5.1.20(p.26)) and (5.1.21(p.26)) we have $L(x) = K(x) = \lambda T(x) \geq 0 \cdots$ **(2)** for any x from Lemma 13.2.1(p.93) (g). Then, since $\rho < b$ and $a < b$, from (20.1.41(p.167)) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ due to (1), from (6.4.23(p.41)) with $\beta = 1$ we have $V_t < \max\{K(b)+b, b\}$ from Lemma 13.2.3(p.96) (h), hence $V_t < \max\{\beta b - s, b\}$ from (13.2.12 (2) (p.96)), so $V_{t-1} < \max\{b, b\} = b$ due to the assumption "$\beta = 1$ and $s = 0$". Accordingly, by induction we have $V_{t-1} < b \cdots$ **(3)** for $t > 1$, hence $T(V_{t-1}) > 0 \cdots$ **(4)** for $t > 1$ from Lemma 13.2.1(p.93) (g). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (20.1.38(p.167)), i.e., $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots$ **(5)** for $\tau > 1$. In addition, since $L(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots$ **(6)** for $\tau \geq t > 1$ due to (4), we have $\texttt{Conduct}_{\tau\geq t>1\blacktriangle} \cdots$ **(7)** from (20.1.32(p.166)).

(a) From (20.1.40(p.167)) and (6.4.21(p.41)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a-\rho\} \geq 0$, hence $V_1 \geq V_0 \cdots$ **(8)**. Since $V_2 \geq K(V_1) + V_1$ from (6.4.23(p.41)) with $t = 2$, we have $V_2 - V_1 \geq K(V_1) \geq 0$ due to (2), hence $V_2 \geq V_1 \cdots$ **(9)**. Suppose $V_t \geq V_{t-1}$.

Then from (6.4.23(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_{t+1} = \max\{K(V_t) + V_t, \beta V_t\} \geq \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 1$. From this and (8) we have $V_t \geq V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho \cdots$ (10), hence $V_1 = \rho$ from (20.1.40(p.167)), so $V_1 < b$ due to (1). Then $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$ from (6.4.21(p.41)), hence $V_1 = \beta V_0 \cdots$ (11), so $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$. Below let $\tau > 1$. Then, from (5) and (11) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. Here note $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ from (7). In addition, since $\lambda \max\{0, a - \rho\} = 0$ due to (10), we have $\lambda \max\{0, a - \rho\} = 0 \leq s$ for any $s \geq 0$, hence $\texttt{Skip}_{1\vartriangle}$ due to (20.1.35(p.166)). Accordingly, it follows that we have $\texttt{C}\rightsquigarrow\texttt{S}_{1\vartriangle}$ (see Remark 7.2.1(p.44)).

(c) Let $\rho < a \cdots$ (12), hence $V_1 = \lambda(a - \rho) + \rho$ due to (20.1.40(p.167)). Then, from (6.4.21(p.41)) we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a - \rho) > 0$, i.e., $V_1 > \beta V_0 \cdots$ (13), hence $t_1^* = 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_\blacktriangle \cdots$ (14). Below let $\tau > 1$. Then, from (5) and (13) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. From the result and (14) we have $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$. Since $a - \rho > 0$ due to (12), we have $\lambda \max\{0, a - \rho\} > 0 = s$, implying that we have $\texttt{Conduct}_{1\blacktriangle}$ due to (20.1.36(p.166)). From this and (7) it follows that $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. ▮

20.1.5.2.1.5 Integration of Propositions 20.1.1(p.167) − 20.1.3(p.167)

□ **Tom 20.1.9** (▣$\mathscr{A}$\{M:2[ℙ][A]\}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \leq a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c) *Let $b \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ *and* $\texttt{C}\rightsquigarrow\texttt{S}_{1\vartriangle}$.
2. *Let $\rho < a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* (a) The same as Tom's 20.1.1(p.167) (a), 20.1.2(p.167) (a), and 20.1.3(p.167) (a).

(b) The same as Tom 20.1.1(p.167) (b).

(c) The same as Tom 20.1.2(p.167) (b).

(d-d2) The same as Tom 20.1.3(p.167) (b,c). ▮

Corollary 20.1.1 *Let $\beta = 1$ and $s = 0$. Then, the optimal price to propose z_t is nondecreasing in t.* □

● *Proof* Immediate from Lemma 20.1.9(p.168) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ▮

20.1.5.2.2 Case of $\beta < 1$ or $s > 0$

20.1.5.2.2.1 Case of $\rho \leq a^\star$

In this case, Theorem 20.1.2(p.153) holds due to Lemma 20.1.1(p.153) (c), hence Tom's 20.1.10(p.168)–20.1.12(p.169) below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom's 20.1.2(p.156)–20.1.4(p.160). In the proofs below, let us represent what results from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to a given Tom by Tom′, i.e.,

$$\texttt{Tom}' = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{Tom}]. \tag{20.1.42}$$

□ **Tom 20.1.10** (□$\mathscr{A}$\{M:2[ℙ][A]\}) *Assume $\rho \leq a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{⊚\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *and* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
      1. *Let $\lambda = 1$.*
         i. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
         ii. *Let $b \leq 0$ ($\kappa \leq 0$). Then* $\boxed{⊚\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.

2. *Let* $\lambda < 1$.
   i. *Let* $b \geq 0$ ⦅$\kappa \geq 0$⦆ . *Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $\texttt{S}_3$(p.156) $\boxed{ⓢ^{\blacktriangle} \mid ⓞ \parallel}$ *is true.*

ii. *Let* $(\lambda\beta a - s)/\delta > a^\star$.
   1. *Let* $b \geq 0$ ⦅$\kappa \geq 0$⦆ . *Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   2. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $\texttt{S}_3$(p.156) $\boxed{ⓢ^{\blacktriangle} \mid ⓞ \parallel}$ *is true.* □

● *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.1.2(p.156). Then "$a < \rho$" in Tom 20.1.2(p.156) (c2i,c3i) changes into "$a^\star < \rho$" in the Tom′, which contradicts the assumption $\rho \leq a^\star$. Accordingly, removing all assertions with "$a^\star < \rho$" from the Tom′ leads to Tom 20.1.10 above. ∎

Corollary 20.1.2 ($\mathscr{A}${M:2[ℙ][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho < x_K$. *Then, the optimal price to propose* z_t *is nondecreasing in* $t \geq 0$. ∎

● *Proof* Immediate from Tom 20.1.10(p.168) (24.2.43) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

□ **Tom 20.1.11** (⊡$\mathscr{A}${M:2[ℙ][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho = x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\text{● dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .
   1. *Let* $b > 0$ ⦅$\kappa > 0$⦆ . *Then* $\boxed{\text{ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   2. *Let* $b \leq 0$ ⦅$\kappa \leq 0$⦆ . *Then* $\boxed{\text{● dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof by analogy* The same as Tom 20.1.3(p.159) due to Lemma 13.6.1(p.99). ∎

Corollary 20.1.3 ($\mathscr{A}${M:2[ℙ][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho = x_K$. *Then, the optimal price to propose* z_t *is nondecreasing in* $t \geq 0$. ∎

● *Proof* Immediate from Tom 20.1.11(p.169) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

□ **Tom 20.1.12** (⊡$\mathscr{A}${M:2[ℙ][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.
   1. $V_t = \rho$ *for* $t \geq 0$.
   2. *Let* $x_L \leq \rho$. *Then* $\boxed{\text{● dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
   3. *Let* $x_L > \rho$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ ⦅$s > 0$⦆ .
   1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
   2. *Let* $b \leq 0$ ⦅$\kappa \leq 0$⦆ . *Then* $\boxed{\text{● dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
   3. *Let* $b > 0$ ⦅$\kappa > 0$⦆ .
      i. *Let* $\rho < x_L$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      ii. *Let* $\rho = x_L$. *Then* $\boxed{\text{● dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      iii. *Let* $\rho > x_L$. *Then* $\texttt{S}_4$ $\boxed{ⓢ^{\blacktriangle} \mid ●\parallel \mid \text{c⇝s}\triangle \mid \text{c⇝s}\blacktriangle}$ *is true.*

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ ⦅$s > 0$⦆ .
   1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
   2. *Let* $b \leq 0$ ⦅$\kappa \leq 0$⦆ . *Then* $\boxed{\text{● dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
   3. *Let* $b > 0$ ⦅$\kappa > 0$⦆ . *Then* $\boxed{\text{ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by analogy* The same as Tom 20.1.4(p.160) (see Lemma 13.6.1(p.99)). ∎

Corollary 20.1.4 ($\mathscr{A}${M:2[ℙ][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$. *Then* $z_t = z(\rho)$ *for* $t \geq 0$, *i.e., constant in* $t \geq 0$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ ⦅$s > 0$⦆. *Then* z_t *is nonincreasing in* $t \geq 0$.

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ ⦅$s > 0$⦆. *Then* z_t *is nondecreasing in* $t \geq 0$. □

● *Proof* Immediate from Tom 20.1.12(p.169) (a1,b1,c1) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

20.1.5.2.2.2 Case of $b \le \rho$

In this case, Theorem 20.1.2(p.153) holds due to Lemma 20.1.1(p.153) (c), hence the following Tom's 20.1.13(p.170) –20.1.15(p.170) can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom's 20.1.2(p.156) –20.1.4(p.160) :

☐ **Tom 20.1.13** (⊡$\mathscr{A}${M:2[ℙ][A]}) *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$.

(c) *Let $\rho < x_L$.*

1. [ⓢ dOITs$_1\langle 1\rangle$]$_\blacktriangle$ *where* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   i. *Let $b \ge 0$ ⦅$\kappa \ge 0$⦆. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
   ii. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_3$(p.156) [ⓢ$^\blacktriangle$ | ⓞ ‖] *is true.* ☐

● *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.1.2(p.156) . Then "$\rho \le a$" in Tom 20.1.2(p.156) (c2i,c3i) changes into "$\rho \le a^\star$" in the Tom′, hence $\rho \le a^\star < a < b$ due to Lemma 13.2.1(p.93) (n), which contradicts the assumption $b \le \rho$. Accordingly, removing all assertions with "$\rho \le a$" from the Tom′ leads to Tom 20.1.13 above. ∎

Corollary 20.1.5 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$. Then z_t is nondecreasing in $t \ge 0$.* ☐

● *Proof* Immediate from Tom 20.1.13(p.170) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89) . ∎

☐ **Tom 20.1.14** (⊡$\mathscr{A}${M:2[ℙ][A]}) *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*

1. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
2. *Let $b \le 0$ ⦅$\kappa \le 0$⦆. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$. ☐

● *Proof by analogy* The same as Tom 20.1.3(p.159) due to Lemma 13.6.1(p.99) . ∎

Corollary 20.1.6 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$. Then z_t is nondecreasing in $t \ge 0$.* ☐

● *Proof* Immediate from Tom 20.1.14(p.170) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89) . ∎

☐ **Tom 20.1.15** (⊡$\mathscr{A}${M:2[ℙ][A]}) *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_L \le \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$.
3. *Let $x_L > \rho$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⦅$s > 0$⦆.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $b \le 0$ ⦅$\kappa \le 0$⦆. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$.
3. *Let $b > 0$ ⦅$\kappa > 0$⦆.*
   i. *Let $\rho < x_L$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
   ii. *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_{\|}$ *and* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
   iii. *Let $x_L < \rho$. Then* S$_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⦅$s > 0$⦆.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $b \le 0$ ⦅$\kappa \le 0$⦆. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$.
3. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$. ☐

● *Proof by analogy* The same as Tom 20.1.4(p.160) due to Lemma 13.6.1(p.99) . ∎

Corollary 20.1.7 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then $z_t = z(\rho)$ for $t \ge 0$.*

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⦅$s > 0$⦆. Then z_t is nonincreasing in $t \ge 0$.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⦅$s > 0$⦆. Then z_t is nondecreasing in $t \ge 0$.* ☐

● *Proof* Immediate from Tom 20.1.15(p.170) (a1,b1,c1) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89) . ∎

20.1.5.2.2.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.1.2(p.153) does not always hold due to Lemma 20.1.1(p.153) (d), hence $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ must be directly found. For convenience of reference, below let us copy (20.1.33(p.166))

$$V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - s, 0\} + \beta\rho. \tag{20.1.43}$$

Lemma 20.1.2

(a) *Let $V_1 \le x_K$. Then V_t is nondecreasing in $t > 0$.*

(b) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$. Then $V_t = V_1$ for $t > 0$.*
2. *Let $\beta < 1$ and $V_1 > 0$. Then V_t is nonincreasing in $t > 0$.*
3. *Let $\beta < 1$ and $V_1 < 0$. Then V_t is nondecreasing in $t > 0$.* □

● *Proof* (a) Let $V_1 \le x_K$. Then, $K(V_1) \ge 0$ due to Corollary 13.2.2(p.97) (b), hence from (6.4.23(p.41)) with $t = 2$ we have $V_2 \ge K(V_1) + V_1 \ge V_1$. Suppose $V_{t-1} \le V_t$. Then, from (6.4.23(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_t \le \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \le V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Let $V_1 > x_K$. Then $K(V_1) \le 0 \cdots (1)$ due to Corollary 13.2.2(p.97) (a). Hence, from (6.4.23(p.41)) with $t = 2$, hence $V_2 - V_1 = \max\{K(V_1) + V_1, \beta V_1\} - V_1 = \max\{K(V_1), -(1-\beta)V_1\} \cdots (2)$.

(b1) Let $\beta = 1$ or $V_1 = 0$. Then, since $-(1-\beta)V_1 = 0$, from (2) we have $V_2 - V_1 = \max\{K(V_1), 0\} = 0$ due to (1), hence $V_2 = V_1$. Suppose $V_{t-1} = V_1$. Then from (6.4.23(p.41)) we have $V_t = \max\{K(V_1) + V_1, \beta V_1\} = V_2 = V_1$. Hence, by induction we have $V_t = V_1$ for $t > 0$.

Below note that $\overline{\beta = 1 \text{ or } V_1 = 0}$ (the negation of $\beta = 1$ or $V_1 = 0$) is "$\beta < 1$ and $V_1 \ne 0$", which can be classified into the two cases, "$\beta < 1$ and $V_1 > 0$" and "$\beta < 1$ and $V_1 > 0$".

(b2) Let $\beta < 1$ and $V_1 > 0$. Then, since $-(1-\beta)V_1 < 0$, from (2) we have $V_2 - V_1 \le 0$ due to (1), hence $V_2 \le V_1$. Suppose $V_{t-1} \le V_{t-2}$. Then, from (6.4.23(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_t \le \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \le V_{t-1}$ for $t > 1$, thus V_t nonincreasing in $t > 0$.

(b3) Let $\beta < 1$ and $V_1 < 0$. Then, since $-(1-\beta)V_1 > 0$, from (2) we have $V_2 - V_1 > 0$ or equivalently $V_2 > V_1$, so $V_2 \ge V_1$. Suppose $V_{t-1} \ge V_{t-2}$. Then from (6.4.23(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \ge V_{t-1}$ for $t > 1$, thus V_t nondecreasing in $t > 0$. ∎

Let us define:

$$\mathsf{S}_5\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|} = \left\{\begin{array}{l} \text{There exists } t^\bullet_\tau > 1 \text{ such that:} \\ (1)\ \ t^\bullet_\tau \ge \tau > 1 \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\ge t>1\,\blacktriangle} \\ (2)\ \ \tau > t^\bullet_\tau \Rightarrow \boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\|} \text{ where } \mathtt{Conduct}_{t^\bullet_\tau\ge t>1\,\blacktriangle}. \end{array}\right\}$$

$$\mathsf{S}_6\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|}\boxed{\bullet\|}\boxed{\mathtt{C}{\rightsquigarrow}\mathtt{S}\vartriangle}\boxed{\mathtt{C}{\rightsquigarrow}\mathtt{S}\blacktriangle} = \left\{\begin{array}{l} \text{There exists } t^{\bullet\,\dagger}_\tau \text{ and } t^\circ_\tau\ (t^\bullet_\tau > t^\circ_\tau > 1) \text{ such that:} \\ (1)\ \ t^\bullet_\tau \ge \tau > 1 \Rightarrow \text{If } \lambda\beta\max\{0, a-\rho\} \le s, \text{ then } \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle}_{\|}. \\ \qquad\qquad\qquad\qquad \text{If } \lambda\beta\max\{0, a-\rho\} > s, \text{ then } \boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle}_{\|} \text{ where } \mathtt{Conduct}_{1\,\blacktriangle}. \\ (2)\ \ \tau > t^\bullet_\tau \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\ge t>t^\bullet_\tau\,\blacktriangle}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{where } \mathtt{pSKIP}_{t^\bullet_\tau\ge\tau>t^\circ_\tau\,\vartriangle}\ (\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\bullet_\tau\ge t>t^\circ_\tau\,\vartriangle}), \text{ and} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{where } \mathtt{pSKIP}_{t^\circ_\tau\ge t>1\,\vartriangle}\ \langle\!\langle\mathtt{pSKIP}_{t^\circ_\tau\ge t>1\,\blacktriangle}\rangle\!\rangle\ (\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\circ_\tau\ge t>1\,\vartriangle}\ \langle\!\langle\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\circ_\tau\ge t>1}\rangle\!\rangle). \end{array}\right\}$$

$$\mathsf{S}_7\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \|}\boxed{\bullet\|}\boxed{\mathtt{C}{\rightsquigarrow}\mathtt{S}\vartriangle} = \left\{\begin{array}{l} \text{There exists } t^\bullet_\tau > 1 \text{ such that:} \\ (1)\ \ t^\bullet_\tau \ge \tau > 1 \Rightarrow \text{If } \lambda\beta\max\{0, a-\rho\} \le s, \text{ then } \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle}_{\|}. \\ \qquad\qquad\qquad\qquad \text{If } \lambda\beta\max\{0, a-\rho\} > s, \text{ then } \boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle}_{\|} \text{ where } \mathtt{Conduct}_{1\,\blacktriangle}. \\ (2)\ \ \tau > t^\bullet_\tau \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\ge t>t^\bullet_\tau\,\blacktriangle} \text{ and where } \mathtt{pSKIP}_{t^\bullet_\tau\ge\tau>1\,\vartriangle}. \end{array}\right\}$$

Remark 20.1.2 For explanatory convenience, let us represent "$\beta = 1$ or $V_1 = 0$" as $\{\beta = 1 \cup V_1 = 0\}$. Then, its negation $\overline{\{\beta = 1 \cup V_1 = 0\}}$ can be written as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap V_1 \ne 0\} = \{\beta < 1 \cap V_1 > 0\} \cup \{\beta < 1 \cap V_1 < 0\}.$$

Without loss of generality, this can be further expressed as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap s \ge 0 \cap V_1 > 0\} \cup \{\beta < 1 \cap s \ge 0 \cap V_1 < 0\}.$$

Furthermore, since $\{s \ge 0\}$ can be denoted by $\{s = 0\,\langle\!\langle s > 0\rangle\!\rangle\}$, it follows that the above expression can be rewritten as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap \{s = 0\,\langle\!\langle s > 0\rangle\!\rangle\} \cap \{V_1 > 0\}\} \cup \{\beta < 1 \cap \{s = 0\,\langle\!\langle s > 0\rangle\!\rangle\} \cap \{V_1 < 0\}\}. \quad \square$$

□ **Tom 20.1.16** ($\blacksquare\mathscr{A}\{\mathsf{M{:}2[\mathbb{P}][A]}\}$) *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \le s$, then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*

(b) *Let $V_1 \le x_K$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $V_1 \ge x_L$. Then, if $\lambda\beta\max\{0, a-\rho\} \le s$, we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.
   - ii. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
     1. *Let $b > 0\,(\!(\kappa > 0)\!)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.
     2. *Let $b \le 0\,(\!(\kappa \le 0)\!)$. Then $\mathtt{S}_5$* $\boxed{\circledS\blacktriangle \mid \circledcirc\ \|}$ *is true.*

(c) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$.*
   - i. *$V_t = V_1$ for $t > 0$.*
   - ii. *If $\lambda\max\{0, a-\rho\} \le s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$* (see Remark 20.1.2(p.171) above)
   - i. *Let $V_1 > 0$.*
     1. *V_t is nonincreasing in $t \ge 0$ and converges to $V \ge x_K$ as $t \to \infty$.*
     2. *Let $b > 0\,(\!(\kappa > 0)\!)$. Then*
        - i. *Let $V_1 > x_L$. Then $\mathtt{S}_6$* $\boxed{\circledS\blacktriangle \mid \circledcirc\ \| \mid \bullet\| \mid \mathtt{c{\leadsto}s}\triangle \mid \mathtt{c{\leadsto}s}\blacktriangle}$ *is true.* $\mapsto$ $\quad\to\boxed{\mathtt{C{\leadsto}S}}_{\blacktriangle}$
        - ii. *Let $V_1 = x_L$. Then $\mathtt{S}_7$* $\boxed{\circledS\blacktriangle \mid \circledcirc\ \| \mid \bullet\| \mid \mathtt{c{\leadsto}s}\triangle}$ *is true.* $\mapsto$ $\quad\to\boxed{\mathtt{C{\leadsto}S}}_{\blacktriangle}$
        - iii. *Let $V_1 < x_L$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.
     3. *Let $b \le 0\,(\!(\kappa \le 0)\!)$. If $\lambda\beta\max\{0, a-\rho\} \le s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
   - ii. *Let $V_1 < 0$.*
     1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
     2. *Let $b > 0\,(\!(\kappa > 0)\!)$.*
        - i. *Let $V_1 \ge x_L$. If $\lambda\beta\max\{0, a-\rho\} \le s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $V_1 < x_L$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.
     3. *Let $b \le 0\,(\!(\kappa \le 0)\!)$. If $\lambda\beta\max\{0, a-\rho\} \le s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. □

● *Proof* Assume $a^\star < \rho < b \cdots (1)$ and let $\beta < 1$ or $s > 0$.

(a)

i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, since $\lambda\beta\max\{0, a-\rho\} - s \le 0$, we have $V_1 - \beta V_0 = 0$ from (20.1.34(p.166)), i.e., $V_1 = \beta V_0 \cdots (2)$, hence $t_1^* = 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\|}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, since $\lambda\beta\max\{0, a-\rho\} - s > 0$, we have $V_1 - \beta V_0 > 0$ from (20.1.34(p.166)), i.e., $V_1 > \beta V_0 \cdots (3)$, hence $t_1^* = 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. Then, since $\lambda\beta\max\{0, a-\rho\} - s > 0$, from the comparison of the two terms within { } in the r.h.s. of (20.1.33(p.166)) it follows that conducting the search is *strictly* optimal at time $t = 1$, i.e., $\mathtt{Conduct}_{1\blacktriangle} \cdots (4)$.

Below let $\tau > 1$.

(b) Let $V_1 \le x_K \cdots (5)$.

(b1) V_t is nondecreasing in $t > 0$ due to Lemma 20.1.2(p.171) (a). Consider a sufficiently large $M > 0$ with $b \le M$ and $V_1 \le M$. Suppose $V_{t-1} \le M$. Then, from (6.4.23(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_t \le \max\{K(M)+M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (13.2.12 (2) (p.96)), hence $V_t \le \max\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Accordingly, by induction $V_t \le M$ for $t > 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta M\} \cdots (6)$ from (6.4.23(p.41)), we have $0 = \max\{K(V), -(1-\beta)V\} \cdots (7)$, hence $K(V) \le 0$, so $V \ge x_K$ due to Lemma 13.2.3(p.96) (j1).

(b2) Let $V_1 \ge x_L$. Then, since $V_{t-1} \ge x_L$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.1.28(p.166)), i.e., $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (8)$.

i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, from (8) and (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t_\tau^* = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (8) and (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$. In addition, we have $\mathtt{Conduct}_{1\blacktriangle}$ from (4).

(b3) Let $V_1 < x_L \cdots (9)$.

(b3i) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$", thus $x_L = x_K \cdots (10)$ from Lemma 13.2.4(p.97) (b). Now, since $V_1 \ge \beta\rho$ from (6.4.22(p.41)), we have $V_1 \ge \rho$ due to the assumption $\beta = 1$, hence $a^\star < V_1$ due to (1). Accordingly, it follows that $a^\star \le V_{t-1}$ for $t > 1$ due to (b1). Note $V_1 < x_K$ from (9) and (10). Suppose $V_{t-1} < x_K$. Then, from Lemma 13.2.3(p.96) (f) and (6.4.23(p.41)) with $\beta = 1$ we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$.

Accordingly, by induction $V_{t-1} < x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (10), so $L(V_{t-1}) > 0$ for $t > 1$ from Lemma 13.2.2(p.96) (e1). Then, since $L(V_{t-1}) > 0 \cdots$ (11) for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (20.1.28(p.166)), i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. In addition, since $V_1 \geq \beta V_0$ from (20.1.34(p.166)), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., ⓢ $\texttt{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$. Then, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ from (11) and (20.1.32(p.166)).

(b3ii) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(b3ii1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots$ (12) from Lemma 13.2.4(p.97) (c (d)). Here note (9) and (b1). Then suppose there exists a t' such that $V_{t-1} \geq x_L$ for $t \geq t'$. Then $L(V_{t-1}) \leq 0$ for $t \geq t'$ from Corollary 13.2.1(p.96) (a), hence $V_t = \beta V_{t-1}$ for $t \geq t'$ due to (20.1.30(p.166)). Therefore, we have $V_t = \beta^{t-t'+1} V_{t'-1}$ for $t \geq t'$, leading to $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (12), which contradicts $V \geq x_K$ in (b1). Accordingly, it follows that $V_{t-1} < x_L$ for all $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a). Thus, for the same reason as in the proof of (b3i) we have ⓢ $\texttt{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.

(b3ii2) Let $b \leq 0\,(\!(\kappa \leq 0)\!)$.

- Let $b = 0\,(\!(\kappa = 0)\!)$. Then $x_L = x_K = 0 \cdots$ (13) from Lemma 13.2.4(p.97) (c (d)), hence $V \geq x_K = x_L = 0$ from (b1). Here assume $V > x_K = 0$. Then, since $-(1-\beta)V < 0$, we have $K(V) = 0$ from (7), leading to the contradiction $V = x_K$ due to Lemma 13.2.3(p.96) (j1). Thus it must be that $V = x_K = 0$. Accordingly, due to (b1) and due to $V_1 < x_L = x_K = V$ from (9) and (13) it follows that there exists a $t^\bullet_\tau > 1$ such that
$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau-1} < x_L = x_K = V_{t^\bullet_\tau} = V_{t^\bullet_\tau+1} = \cdots,^\dagger$$
where $t^\bullet_\tau$ might be infinity (i.e., $t^\bullet_\tau = \infty$). Hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $V_{t-1} = x_L$ for $t > t^\bullet_\tau$. Thus, from Corollary 13.2.1(p.96) (a) we have
$$L(V_{t-1}) > 0 \text{ for } t^\bullet_\tau \geq t > 1 \text{ and } L(V_{t-1}) = 0 \text{ (hence } L(V_{t-1}) \leq 0) \text{ for } t > t^\bullet_\tau \cdots (14).$$
- Let $b < 0\,(\!(\kappa < 0)\!)$. Then $x_L < x_K$ from Lemma 13.2.4(p.97) (c (d)). Since $V_1 < x_L$ from (9) and since $x_L < x_K \leq V$ from (b1), there exists $t^\bullet_\tau$ such that
$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau-1} < x_L \leq V_{t^\bullet_\tau} \leq V_{t^\bullet_\tau+1} \leq \cdots,$$
hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \leq V_{t-1}$ for $t > t^\bullet_\tau$. Accordingly, from Corollary 13.2.1(p.96) (a) we have
$$L(V_{t-1}) > 0 \text{ for } t^\bullet_\tau \geq t > 1 \text{ and } L(V_{t-1}) \leq 0 \text{ for } t > t^\bullet_\tau \cdots (15).$$

From (14) and (15) we have, whether $b = 0\,(\!(\kappa = 0)\!)$ or $b < 0\,(\!(\kappa < 0)\!)$ (or equivalently $b \leq 0\,(\!(\kappa \leq 0)\!)$),
$$L(V_{t-1}) > 0 \cdots (16) \text{ for } t^\bullet_\tau \geq t > 1,$$
$$L(V_{t-1}) \leq 0 \cdots (17) \text{ for } t > t^\bullet_\tau.$$

Accordingly, from (20.1.28(p.166)) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet_\tau \geq t > 1$ due to (16) and $V_t - \beta V_{t-1} = 0$ for $t > t^\bullet_\tau$ due to (17) or equivalently
$$V_t > \beta V_{t-1} \cdots (18), \quad t^\bullet_\tau \geq t > 1, \qquad V_t = \beta V_{t-1} \cdots (19), \quad t > t^\bullet_\tau.$$

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $V_t > \beta V_{t-1} \cdots$ (20) for $\tau \geq t > 1$ due to (18), for the same reason as in the proof of (b3i) we have ⓢ $\texttt{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$. Hence (1) of $\texttt{S}_5$ holds. Then, since (20) with $\tau = t^\bullet_\tau$ can be rewritten as $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, we have
$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau-1} V_1 \cdots (21).$$
2. Let $\tau > t^\bullet_\tau$. Then $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ due to (19), hence
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (22).$$
Hence, from (22) and (21) and from the fact that $V_1 \geq \beta V_0$ due to (2) and (3) we obtain
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$
so we have $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$, i.e., ⓢ $\texttt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\parallel$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (16) and (20.1.32(p.166)). From the above we see that (2) of $\texttt{S}_5$ holds.

(c) Let $V_1 > x_K \cdots$ (23).

(c1) Let $\beta = 1$ or $V_1 = 0$.

(c1i) The same as Lemma 20.1.2(p.171) (b1).

(c1ii) Since $V_\tau = V_{\tau-1} = \cdots = V_1$ for $\tau > 0$ from (c1i), we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots$ (24).

† Since $V_t \leq V$ for any $t > 0$ due to (b1), if $V \leq V_t$ for a t, then $V = V_t$.

i. Let $\lambda\max\{0,a-\rho\} \le s$. Then, from (2) and (24) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\max\{0,a-\rho\} > s$. Then, from (3) and (24) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ where $\mathtt{Conduct}_{1\blacktriangle}$ from (4).

(c2) Let $\beta < 1 \cdots$ (25) and $s = 0$ ⟪$s > 0$⟫ .

(c2i) Let $V_1 > 0$.

(c2i1) The former half is the same as Lemma 20.1.2(p.171) (b2). The latter half can be proven as follows. Note (23), hence $V_1 \ge x_K$. Suppose $V_{t-1} \ge x_K$. Then from (6.4.23(p.41)) we have $V_t \ge K(V_{t-1}) + V_{t-1} \ge K(x_K) + x_K$ due to Lemma 13.2.3(p.96) (e), hence $V_t \ge x_K$ since $K(x_K) = 0$. Accordingly, by induction $V_t \ge x_K$ for $t > 0$, i.e., V_t is lower bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta V\}$ from (6.4.23(p.41)), we have $0 = \max\{K(V), -(1-\beta)V\}$, hence $K(V) \le 0$, so $V \ge x_K$ due to Lemma 13.2.3(p.96) (j1).

(c2i2) Let $b > 0$ ⟪$\kappa > 0$⟫ . Then $x_L > x_K > 0 \cdots$ (26) from Lemma 13.2.4(p.97) (c ⟪d⟫).

(c2i2i) Let $V_1 > x_L \cdots$ (27), hence $V_1 \ge x_L$. Suppose $V_{t-1} \ge x_L$ for *all* $t > 1$. Then, since $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.1.28(p.166)), i.e., $V_t = \beta V_{t-1}$ for all $t > 1$, hence $V_t = \beta^{t-1}V_1$. Accordingly, we have $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (26), which contradicts $V \ge x_K$ in (c2i1). Thus it is impossible that $x_L \le V_{t-1}$ for *all* $t > 0$. Accordingly, due to (27) and (c2i1) it follows that there exist $t^\bullet_\tau$ and t°_τ ($t^\bullet_\tau > t^\circ_\tau > 0$) such that

$$V_1 \ge V_2 \ge \cdots \ge V_{t^\circ_\tau-1} > x_L = V_{t^\circ_\tau} = V_{t^\circ_\tau+1} = \cdots = V_{t^\bullet_\tau-1} > V_{t^\bullet_\tau} \ge V_{t^\bullet_\tau+1} \ge \cdots .$$

Hence, we have

$$x_L > V_{t^\bullet_\tau},\ x_L > V_{t^\bullet_\tau+1}, \cdots,$$
$$V_{t^\circ_\tau} = x_L,\ V_{t^\circ_\tau+1} = x_L, \cdots, V_{t^\bullet_\tau-1} = x_L,$$
$$V_1 > x_L,\ V_2 > x_L, \cdots, V_{t^\circ_\tau-1} > x_L,$$

or equivalently

$$x_L > V_{t-1} \cdots (28), \qquad t > t^\bullet_\tau,$$
$$V_{t-1} = x_L \cdots (29), \qquad t^\bullet_\tau \ge t > t^\circ_\tau,$$
$$V_{t-1} > x_L \cdots (30), \qquad t^\circ_\tau \ge t > 1.$$

Accordingly, we have:

1. Let $t^\bullet_\tau \ge \tau > 1$. Then, since $V_{t-1} \ge x_L$ for $\tau \ge t > 1$ from (29) and (30), we have $L(V_{t-1}) \le 0 \cdots$ (31) for $\tau \ge t > 1$ from Corollary 13.2.1(p.96) (a), hence $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 1$ from (20.1.28(p.166)), i.e., $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 \cdots$ (32).

   i. Let $\lambda\max\{0,a-\rho\} \le s$. Then, from (2) and (32) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle}_{\parallel}$.

   ii. Let $\lambda\max\{0,a-\rho\} > s$. Then, from (3) and (32) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle}_{\parallel}$ where $\mathtt{Conduct}_{1\blacktriangle}$ from (4).

   Accordingly $\mathtt{S}_6(1)$ holds. From (32) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau-1} = \cdots = \beta^{t^\bullet_\tau-1}V_1 \cdots$ (33).

2. Let $\tau > t^\bullet_\tau$. Then, since $x_L > V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$ from (28), due to Corollary 13.2.1(p.96) (a) we have $L(V_{t-1}) > 0 \cdots$ (34) for $\tau \ge t > t^\bullet_\tau$. Accordingly, from (20.1.28(p.166)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet_\tau}V_{t^\bullet_\tau}$. From this and (33) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau-t^\bullet_\tau+1}V_{t^\bullet_\tau-1} = \cdots = \beta^{\tau-1}V_1. \cdots (35).$$

   Since $V_1 \ge \beta V_0$ due to (2) and (3), from (35) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau-t^\bullet_\tau+1}V_{t^\bullet_\tau-1} = \cdots = \beta^{\tau-1}V_1 \ge \beta^\tau V_0.$$

   Hence, we have $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle}$, thus the former half of $\mathtt{S}_6(2)$ holds. The latter half can be proven as follows.

   (i) If $\tau \ge t > t^\bullet_\tau$, then $\mathtt{Conduct}_{t\blacktriangle}$ from (34) and (20.1.32(p.166)).

   (ii) If $t^\bullet_\tau \ge t > t^\circ_\tau$, then $V_{t-1} = x_L$ from (29), hence $L(V_{t-1}) = L(x_L) = 0$, so $\mathtt{Skip}_{t\vartriangle}$ from (20.1.31(p.166)), implying that we have $\mathtt{C{\rightsquigarrow}S}_{t^\bullet_\tau\ge t>t^\circ_\tau\vartriangle}$ (see Figure 7.2.1(p.44) (II).

   (iii) If $t^\circ_\tau \ge t > 1$, then $V_{t-1} > x_L$ from (30), hence $L(V_{t-1}) =$ ⟪<⟫ 0[‡] from Lemma 13.2.2(p.96) (d ⟪e1⟫); i.e., $\mathtt{Skip}_{t\vartriangle}$ ⟪$\mathtt{Skip}_{t\blacktriangle}$⟫ due to (20.1.31(p.166)) ⟪(20.1.32(p.166))⟫, implying that we have $\mathtt{C{\rightsquigarrow}S}_{t^\circ_\tau\ge t>1\vartriangle}$ ⟪$\mathtt{C{\rightsquigarrow}S}_{t^\circ_\tau\ge t>1}$⟫.

‡If $s = 0$, then "$= 0$" , or else "< 0".

From the above results we see that the latter half of $\texttt{S}_6(2)$ holds.

(c2i2ii) Let $V_1 = x_L$. Suppose $V_{t-1} = x_L$ for *all* $t > 1$. Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t - \beta V_{t-1} = 0$ for *all* $t > 1$ from (20.1.28(p.166)), i.e., $V_t = \beta V_{t-1}$ for all $t > 1$, hence $V_t = \beta^{t-1}V_1$. Then $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (26), which contradicts $V \geq x_K$ in (c2i1). Hence, since V_{t-1} is not equal to x_L for *all* $t > 1$, due to (c2i1) it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau - 1} = x_L > V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

or equivalently $V_{t-1} = x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Thus, due to Corollary 13.2.1(p.96) (a) we have

$$L(V_{t-1}) = L(x_L) = 0 \cdots (36), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) > 0 \cdots (37), \quad t > t^\bullet_\tau.$$

Accordingly, we have:

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, from (36) and (20.1.28(p.166)) we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, from which we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$.
   i. Let $\lambda\beta\max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_{\parallel}$.
   ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_{\parallel}$. In addition, we have $\texttt{Conduct}_{1\blacktriangle}$ from (4).

   Accordingly, it follows that $\texttt{S}_7(1)$ holds.

2. Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) > 0 \cdots (38)$ for $\tau \geq t > t^\bullet_\tau$ from (37), hence due to (20.1.28(p.166)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} \cdots (39)$. In addition, since $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \geq t > 1$ from (36) and (20.1.28(p.166)), we have $V_t = \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, leading to
$$V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1}V_1 \cdots (40).$$

   From (39) and (40) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1.$$

   In addition, since $V_1 \geq \beta^\tau V_0$ from (2) and (3), we eventually obtain

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1 \geq \beta^\tau V_0 \cdots (41).$$

   Thus $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\blacktriangle}$, hence the former half of $\texttt{S}_7(2)$ holds. Then, we have that $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t^\bullet_\tau$ due to (38) and (20.1.32(p.166)). Moreover, we have $\texttt{Skip}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (36) and (20.1.31(p.166)), so it follows that we have $\texttt{pSKIP}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ (see Figure 7.2.1(p.44))(II)) or equivalently $\texttt{pSKIP}_{t^\bullet_\tau \geq t > 1\vartriangle}$. Hence the latter half of $\texttt{S}_7(2)$ holds.

(c2i2iii) Let $V_1 < x_L$. Then $V_{t-1} < x_L$ for $t > 1$ due to (c2i1), hence $L(V_{t-1}) > 0 \cdots (42)$ for $t > 1$ from Corollary 13.2.1(p.96) (a). Accordingly, since $L(V_{t-1}) > 0 \cdots (43)$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (20.1.28(p.166)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1.$$

Since $V_1 \geq \beta V_0$ from (2) and (3), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \geq \beta^\tau V_0,$$

hence we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > 1}\langle \tau\rangle}_{\blacktriangle}$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (43) and (20.1.32(p.166)).

(c2i3) Let $b \leq 0$ ($\kappa \leq 0$), hence $x_L \leq x_K \cdots (44)$ from Lemma 13.2.4(p.97) (c (d)). Then, from (23) and (c2i1) we have $V_{t-1} \geq x_K$ for all $t > 1$, hence $V_{t-1} \geq x_L$ for all $t > 1$ due to (44), thus $L(V_{t-1}) \leq 0$ for all $t > 1$ from Corollary 13.2.1(p.96) (a). Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.28(p.166)) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, hence

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1.$$

i. Let $\lambda\beta\max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau > 1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > 1}\langle 1\rangle}_{\parallel}$. Then $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii) Let $V_1 < 0$.

(c2ii1) The same as the proof of (c2i1).

(c2ii2) Let $b > 0$ ⟪$\kappa > 0$⟫, hence $x_L > x_K > 0 \cdots$ **(**45**)** from Lemma 13.2.4(p.97) (c ⟪d⟫).

(c2ii2i) Let $V_1 \geq x_L$. Then, since $V_{t-1} \geq x_L$ for $t > 1$ due to (c2ii1), we have $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), hence $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Thus $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.28(p.166)), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, so

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1.$$

i. Let $\lambda\beta \max\{0, a - \rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\mathtt{dOIT}_{\tau>1}\langle 0\rangle_{\parallel}$.

ii. Let $\lambda\beta \max\{0, a - \rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{◎\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$. Then $\mathtt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii2ii) Let $V_1 < x_L$. Suppose that there exists $t' > 1$ such that $x_L \leq V_{t-1}$ for $t > t'$. Then, since $L(V_{t-1}) \leq 0$ for $t > t'$ from Corollary 13.2.1(p.96) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > t'$ due to (20.1.28(p.166)), hence $V_t = \beta V_{t-1}$ for $t > t'$, so

$$V_t = \beta V_{t-1} = \beta^2 V_{t-2} = \cdots = \beta^{t-t'} V_{t'}.$$

Accordingly $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (45), which contradicts $V \geq x_K$ in (c2ii1), hence it must be that $V_{t-1} < x_L$ for $t > 1$. Then, since $V_{t-1} < x_L$ for $\tau \geq t > 1$, we have $L(V_{t-1}) > 0 \cdots$ **(**46**)** for $\tau \geq t > 1$ from Corollary 13.2.1(p.96) (a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (20.1.28(p.166)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, thus

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1.$$

Since $V_1 \geq \beta V_0$ from (2) and (3), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$. From (46) and (20.1.32(p.166)) we have $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2ii3) Let $b \leq 0$ ⟪$\kappa \leq 0$⟫, hence $x_L \leq x_K \cdots$ **(**47**)** from Lemma 13.2.4(p.97) (c ⟪d⟫). Then, due to (23) and (c2ii1) we have $V_{t-1} > x_K$ for $t > 1$, hence $V_{t-1} > x_L$ for $t > 1$ from (47), thus $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a). Accordingly, the assertion is true for the same reason as in the proof of (c2ii2i). ∎

Corollary 20.1.8 *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $V_1 \leq x_K$. Then z_t is nondecreasing in $t > 0$.*

(b) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$. Then $z_t = z(V_1)$ for $t > 0$.*
2. *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*
   - i. *Let $V_1 > 0$. Then z_t is nonincreasing in $t > 0$.*
   - ii. *Let $V_1 < 0$. Then z_t is nondecreasing in $t > 0$.* □

● *Proof* Immediate from Tom 20.1.16(p.172) (b1,c1i,c2i1,c2ii1) and from (6.2.94(p.35))
and Lemma 13.1.3(p.89). ∎

20.1.5.3 Market Restriction

20.1.5.3.1 Positive Restriction

20.1.5.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.1.9** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \leq a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c) *Let $b \leq \rho$. Then* $\boxed{●\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \leq \rho$. Then* $\boxed{●\ \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ *and* $\mathtt{pSKIP}_1$ (⟮C⇝S⟯)
2. *Let $\rho < a$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* The same as Lemma 20.1.9(p.168) due to Lemma 17.4.4(p.118). ∎

20.1.5.3.1.2 Case of $\beta < 1$ or $s > 0$

20.1.5.3.1.2.1 Case of $\rho \le a^\star$

□ Pom 20.1.10 ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda a - s)/\lambda \le a^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
   ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *and* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta a - s)/\delta \le a^\star$.*
      1. *Let $\lambda = 1$.*
         i. *Let $s < \lambda\beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
         ii. *Let $s \ge \lambda\beta T(0)$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$.*
         i. *Let $s \le \lambda\beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
         ii. *Let $s > \lambda\beta T(0)$. Then* $\texttt{S}_3$(p.156) | ⓢ▲ | ◎ ‖ | *is true.*
   ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
      1. *Let $s \ge \lambda\beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
      2. *Let $s < \lambda\beta T(0)$. Then* $\texttt{S}_3$(p.156) | ⓢ▲ | ◎ ‖ | *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a-c2ii) The same as Tom 20.1.10(p.168) (24.2.43-c2ii).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of Tom 20.1.10(p.168).

(c4-c4ii2) The same as Tom 20.1.10(p.168) (c3-c3ii2) with κ. ∎

□ Pom 20.1.11 ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $s < \beta\mu T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
2. *Let $s \ge \beta\mu T(0)$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau}\langle 0\rangle}_{\parallel}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a,b) The same as Tom 20.1.11(p.169) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.11(p.169).

(d-d2) The same as Tom 20.1.11(p.169) (c1,c2) with κ. ∎

□ Pom 20.1.12 ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_L > \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite V as $t \to \infty$.*
2. *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
3. *Let $\rho = x_L$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\vartriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
4. *Let $x_L < \rho$. Then* $\texttt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true.*

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s > 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite V as $t \to \infty$.*

2. *Let* $s \geq \beta\mu T(0)$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $s < \beta\mu T(0)$.
   i. *Let* $\rho < x_L$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $\rho = x_L$. *Then* $\boxed{\bullet\,\texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\vartriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ $\boxed{\circledS\blacktriangle \mid \bullet\parallel \mid \text{c}\rightsquigarrow\text{s}\vartriangle \mid \text{c}\rightsquigarrow\text{s}\blacktriangle}$ *is true.*

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.
1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \to \infty$.
2. $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.
1. V_t *is nondecreasing in* t $(\tau \geq t \geq 0)$ *and converges to a finite* V *as* $t \to \infty$.
2. *Let* $s \geq \beta\mu T(0)$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $s < \beta\mu T(0)$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a-a3) The same as Tom 20.1.12(p.169) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of Tom 20.1.12(p.169).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as Tom 20.1.12(p.169) (b1-b3iii) with κ.

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of Tom 20.1.12(p.169).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as Tom 20.1.12(p.169) (c1-c3) with κ. ∎

20.1.5.3.1.2.2 Case of $b \leq \rho$

□ Pom **20.1.13** ($\mathscr{A}\{\text{M:2}[\mathbb{P}][\texttt{A}]^+\}$) *Suppose* $a > 0$. *Assume* $b \leq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \leq \rho$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\rho < x_L$.
1. $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s \leq \lambda\beta T(0)$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $s > \lambda\beta T(0)$. *Then* $\texttt{S}_3$(p.156) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel}$ *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a-c2) The same as Tom 20.1.13(p.170) (a-c2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i) of Tom 20.1.13(p.170).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then, we have the same as Tom 20.1.13(p.170) (c3i,c3ii) with κ. ∎

□ Pom **20.1.14** ($\mathscr{A}\{\text{M:2}[\mathbb{P}][\texttt{A}]^+\}$) *Suppose* $a > 0$. *Assume* $b \leq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(d) *Let* $\beta < 1$ *and* $s > 0$.
1. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let* $s \geq \lambda\beta T(0)$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a,b) The same as Tom 20.1.14(p.170) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.14(p.170).

(d-d2) Let $\beta < 1$ and $s > 0$. Then, we have the same as Tom 20.1.14(p.170) (c1,c2) with κ. ∎

□ Pom **20.1.15** ($\mathscr{A}\{\text{M:2}[\mathbb{P}][\texttt{A}]^+\}$) *Suppose* $a > 0$. *Assume* $b \leq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.
1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_L \leq \rho$. *Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

3. *Let* $x_L > \rho$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $\rho < x_L$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\|$ *and* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
4. *Let* $x_L < \rho$. *Then* $\mathtt{S}_4$ $\boxed{\text{ⓢ}\blacktriangle \mid \bullet\| \mid c\rightsquigarrow s\triangle \mid c\rightsquigarrow s\blacktriangle}$ *is true.*

(c) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s > 0$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $s \geq \lambda\beta T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$.
3. *Let* $s < \lambda\beta T(0)$.
   i. *Let* $\rho < x_L$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\|$ *and* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let* $x_L < \rho$. *Then* $\mathtt{S}_4$ $\boxed{\text{ⓢ}\blacktriangle \mid \bullet\| \mid c\rightsquigarrow s\triangle \mid c\rightsquigarrow s\blacktriangle}$ *is true.*

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.

1. V_t *is nondecreasing in* t $(\tau \geq t \geq 0)$.
2. *Let* $s \geq \lambda\beta T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$.
3. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

• *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a-a3) The same as Tom 20.1.15(p.170) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of Tom 20.1.15(p.170).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as Tom 20.1.15(p.170) (b1-b3iii) with κ.

(d,d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of Tom 20.1.15(p.170).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as Tom 20.1.15(p.170) (c1-c3) with κ. ∎

20.1.5.3.1.2.3 Case of $a^\star < \rho < b$

☐ **Pom 20.1.16** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Assume* $a^\star \leq \rho < b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\max\{0, a-\rho\} \leq s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\|$, *or else* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$.

(b) *Let* $V_1 \leq x_K$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $V_1 \geq x_L$. *Then, if* $\lambda\beta\max\{0, a-\rho\} \leq s$, *we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\|$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
3. *Let* $V_1 < x_L$.
   i. *Let* $\beta = 1$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $\beta < 1$ *and* $s > 0$.
      1. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
      2. *Let* $s \geq \lambda\beta T(0)$. *Then* $\mathtt{S}_5$ $\boxed{\text{ⓢ}\blacktriangle \mid \circledcirc\|}$ *is true.*

(c) *Let* $V_1 > x_K$.

1. *Let* $\beta = 1$ *or* $V_1 = 0$.
   i. $V_t = V_1$ *for* $t > 0$.
   ii. *If* $\lambda\max\{0, a-\rho\} \leq s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\|$, *or else* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$.
   i. *Let* $V_1 > 0$.
      1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
      2. *Let* $V_1 > x_L$. *Then* $\mathtt{S}_6$ $\boxed{\text{ⓢ}\blacktriangle \mid \circledcirc\| \mid \bullet\| \mid c\rightsquigarrow s\triangle \mid c\rightsquigarrow s\blacktriangle}$ *is true.*
      3. *Let* $V_1 = x_L$. *Then* $\mathtt{S}_7$ $\boxed{\text{ⓢ}\blacktriangle \mid \circledcirc\| \mid \bullet\| \mid c\rightsquigarrow s\triangle}$ *is true.*

4. *Let* $V_1 < x_L$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

ii. *Let* $V_1 < 0$.

1. *Then* V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $V_1 \ge x_L$. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
3. *Let* $V_1 < x_L$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s > 0$.

i. *Let* $V_1 > 0$.

1. V_t *is nonincreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $s < \lambda\beta T(0)$.
   i. *Let* $V_1 > x_L$. *Then* $\texttt{S}_6$ $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \parallel}\boxed{\bullet\parallel}\boxed{\text{c}\rightsquigarrow\text{s}\,\triangle}\boxed{\text{c}\rightsquigarrow\text{s}\blacktriangle}$ *is true.*
   ii. *Let* $V_1 = x_L$. *Then* $\texttt{S}_7$ $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \parallel}\boxed{\bullet\parallel}\boxed{\text{c}\rightsquigarrow\text{s}\,\triangle}$ *is true.*
   iii. *Let* $V_1 < x_L$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
3. *Let* $s \ge \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$

ii. *Let* $V_1 < 0$.

1. *Then* V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $s < \lambda\beta T(0)$.
   i. *Let* $V_1 \ge x_L$. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
   ii. *Let* $V_1 < x_L$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
3. *Let* $s \ge \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.26)).

(a-b3i) The same as `Tom` 20.1.16(p.172) (a-b3i).

(b3ii) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (b3ii1) of `Tom` 20.1.16(p.172).

(b3iii-b3iii2) Let $\beta < 1$ and $s > 0$. Then, the two assertions are immediate from `Tom` 20.1.16(p.172) (b3ii1,b3ii2) with κ.

(c-c1ii) The same as `Tom` 20.1.16(p.172) (c-c1ii).

(c2-c2i4) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i-c2i1,c2i2i-c2i2iii) of `Tom` 20.1.16(p.172).

(c2ii-c2ii3) Due to (1) it suffices to consider only (c2ii,c2ii1,c2ii2i,c2ii2ii) of `Tom` 20.1.16(p.172).

(c3-c3i3) Let $\beta < 1$ and $s > 0$. Then, we have the same as `Tom` 20.1.16(p.172) (c2-c2i1,c2i2i-c2i2iii) with κ.

(c3ii-c3ii3) We have the same as `Tom` 20.1.16(p.172) (c2ii-c2ii2ii) with κ. ∎

20.1.5.3.2 Mixed Restriction

Omitted.

20.1.5.3.3 Negative Restriction

Omitted.

20.1.6 $\tilde{\mathsf{M}}$:2[ℙ][A]

20.1.6.1 Preliminary

Since Theorem 20.1.3(p.153) holds due to Lemma 20.1.1(p.153) (b), we can derive $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) to $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$.

20.1.6.2 Analysis

20.1.6.2.1 Case of $\beta = 1$ and $s = 0$

□ **`Tom` 20.1.17** ($\boxdot\,\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \ge 0$.

(b) *Let* $\rho \ge b^\star$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(c) *Let* $a \ge \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let* $b^\star > \rho > a$.

1. *Let* $b \ge \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$ *and* $\texttt{pSKIP}_{1\triangle}$.
2. *Let* $\rho > b$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 20.1.9(p.168). ∎

Corollary 20.1.9 *Let* $\beta = 1$ *and* $s = 0$. *Then* z_t *is nonincreasing in* $t \ge 0$. □

● *Proof* Immediate from `Tom` 20.1.17(p.180) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

20.1.6.2.2 Case of $\beta < 1$ or $s > 0$

20.1.6.2.2.1 Case of $\rho \geq b^{\star\dagger}$

$\square$ `Tom` **20.1.18** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
     1. *Let $\lambda = 1$. Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
     2. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ($\!($s > 0$)\!$).*
   - i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
     1. *Let $\lambda = 1$.*
        - i. *Let $a < 0$ ($\!(\tilde{\kappa} < 0)\!$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        - ii. *Let $a \geq 0$ ($\!(\tilde{\kappa} \geq 0)\!$). Then* $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
     2. *Let $\lambda < 1$.*
        - i. *Let $a \leq 0$ ($\!(\tilde{\kappa} \leq 0)\!$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        - ii. *Let $a > 0$ ($\!(\tilde{\kappa} > 0)\!$). Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel}$ *is true.*
   - ii. *Let $(\lambda\beta b + s)/\delta < b^\star$.*
     1. *Let $a \leq 0$ ($\!(\tilde{\kappa} \leq 0)\!$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $a > 0$ ($\!(\tilde{\kappa} > 0)\!$). Then* $\mathtt{S}_3$(p.156) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel}$ *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.10(p.168). ∎

Corollary 20.1.10 *Assume $\rho \geq b^\star$, let $\beta < 1$ or $s > 0$, and let $\rho > x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

● *Proof* Immediate from `Tom` 20.1.18(p.181) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

$\square$ `Tom` **20.1.19** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then, for a given starting time $\tau > 0$:*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $s = 0$ ($\!(s > 0)\!$).*

1. *Let $a < 0$ ($\!(\tilde{\kappa} < 0)\!$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *and* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let $a \geq 0$ ($\!(\tilde{\kappa} \geq 0)\!$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof by symmetry* Clear from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.11(p.169). ∎

Corollary 20.1.11 *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

● *Proof* Immediate from `Tom` 20.1.19(p.181) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

$\square$ `Tom` **20.1.20** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($\!(s > 0)\!$).*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ($\!(\tilde{\kappa} \geq 0)\!$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $a < 0$ ($\!(\tilde{\kappa} < 0)\!$).*
   - i. *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$ *where* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - iii. *Let $\rho < x_{\tilde{L}}$. Then* $\mathtt{S}_4$ $\boxed{\circledS\blacktriangle \mid \bullet\parallel \mid \mathrm{c}{\rightsquigarrow}\mathrm{s}\triangle \mid \mathrm{c}{\rightsquigarrow}\mathrm{s}\blacktriangle}$ *is true.*

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†The condition of $\rho \geq b^\star$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition $\rho \leq a^\star$ in Section 20.1.5.2.2.1(p.168).

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ $\llparenthesis s > 0 \rrparenthesis$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ $\llparenthesis \tilde{\kappa} \geq 0 \rrparenthesis$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $a < 0$ $\llparenthesis \tilde{\kappa} < 0 \rrparenthesis$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.12(p.169). ▮

Corollary 20.1.12 *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then z_t is constant in t ($z_t = z(\rho)$ for $t \geq 0$).*
(b) *Let $\beta < 1$ and $\rho > 0$. Then z_t is nondecreasing in $t \geq 0$ for any $s \geq 0$.*
(c) *Let $\beta < 1$ and $\rho < 0$. Then z_t is nonincreasing in $t \geq 0$ for any $s \geq 0$.* □

● *Proof by symmetry* Evident from Tom 20.1.20(p.181) (a1,b1,c1) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

20.1.6.2.2.2 Case of $a \geq \rho$[†]

□ Tom **20.1.21** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\textcircled{s}\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ $\llparenthesis s > 0 \rrparenthesis$.*
   i. *Let $a \leq 0$ $\llparenthesis \tilde{\kappa} \leq 0 \rrparenthesis$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $a > 0$ $\llparenthesis \tilde{\kappa} > 0 \rrparenthesis$. Then* $\mathtt{S}_3$(p.156) [ⓢ▲ | ⊚ ‖] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.13(p.170). ▮

Corollary 20.1.13 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

● *Proof* Evident from Tom 20.1.21(p.182) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

□ Tom **20.1.22** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let $\beta < 1$ and $s = 0$ $\llparenthesis s > 0 \rrparenthesis$.*

1. *Let $a < 0$ $\llparenthesis \tilde{\kappa} < 0 \rrparenthesis$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *and* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let $a \geq 0$ $\llparenthesis \tilde{\kappa} \geq 0 \rrparenthesis$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.14(p.170). ▮

Corollary 20.1.14 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

● *Proof* Evident from Tom 20.1.22(p.182) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

□ Tom **20.1.23** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ $\llparenthesis s > 0 \rrparenthesis$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ $\llparenthesis \tilde{\kappa} \geq 0 \rrparenthesis$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let $a < 0$ $\llparenthesis \tilde{\kappa} < 0 \rrparenthesis$.*
   i. *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\|}$ *where* $\boxed{\textcircled{s}\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let $x_{\tilde{L}} > \rho$. Then* $\mathtt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ $\llparenthesis s > 0 \rrparenthesis$.*

[†] The condition of $a \geq \rho$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $b \leq \rho$ in Section 20.1.5.2.2.2(p.170).

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $a < 0$ $(\tilde{\kappa} < 0)$. *Then* $\boxed{ⓢ\, \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.15(p.170). ▮

Corollary 20.1.15 *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$. *Then* $z_t = z(\rho)$ *for* $t \geq 0$.
(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $(s > 0)$. *Then* z_t *is nondecreasing in* $t \geq 0$.
(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $(s > 0)$. *Then* z_t *is nonincreasing in* $t \geq 0$. □

● *Proof* Evident from Tom 20.1.23(p.182) (a1,b1,c1) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

20.1.6.2.2.3 Case of $b^\star > \rho > a$[†]

Let us here note that (20.1.43(p.171)) changes as follows.

$$V_1 = \min\{\lambda\beta\min\{0, b-\rho\} + s, 0\} + \beta\rho.^{\dagger} \tag{20.1.44}$$

□ **Tom 20.1.24** ($\square\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume* $b^\star > \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\, \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$, *or else* $\boxed{ⓢ\, \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$.
(b) *Let* $V_1 \geq x_{\tilde{K}}$.
  1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
  2. *Let* $V_1 \leq x_{\tilde{L}}$. *Then, if* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *we have* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{◎\, \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
  3. *Let* $V_1 > x_{\tilde{L}}$.
    i. *Let* $\beta = 1$. *Then* $\boxed{ⓢ\, \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
    ii. *Let* $\beta < 1$ *and* $s = 0$ $(s > 0)$.
      1. *Let* $a < 0$ $(\tilde{\kappa} < 0)$. *Then* $\boxed{ⓢ\, \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
      2. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *Then* $\mathtt{S}_5$ $\boxed{ⓢ\blacktriangle \mid ◎\parallel}$ *is true.*

(c) *Let* $V_1 < x_{\tilde{K}}$.
  1. *Let* $\beta = 1$ *or* $V_1 = 0$.
    i. $V_t = V_1$ *for* $t > 0$.
    ii. *If* $\lambda\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{◎\, \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
  2. *Let* $\beta < 1$ *and* $s = 0$ $(s > 0)$.[†]
    i. *Let* $V_1 < 0$.
      1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $\tau \to \infty$.
      2. *Let* $a < 0$ $(\tilde{\kappa} < 0)$. *Then*
        i. *Let* $V_1 < x_{\tilde{L}}$. *Then* $\mathtt{S}_6$ $\boxed{ⓢ\blacktriangle \mid \bullet\parallel \mid ◎\parallel \mid \mathrm{c}{\rightsquigarrow}\mathrm{s}\vartriangle \mid \mathrm{c}{\rightsquigarrow}\mathrm{s}\blacktriangle}$ *is true.*
        ii. *Let* $V_1 = x_{\tilde{L}}$. *Then* $\mathtt{S}_7$ $\boxed{ⓢ\blacktriangle \mid \bullet\parallel \mid ◎\parallel \mid \mathrm{c}{\rightsquigarrow}\mathrm{s}\vartriangle}$ *is true.*
        iii. *Let* $V_1 > x_{\tilde{L}}$. *Then* $\boxed{ⓢ\, \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      3. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{◎\, \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
    ii. *Let* $V_1 > 0$.
      1. *Then* V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $\tau \to \infty$.
      2. *Let* $a < 0$ $(\tilde{\kappa} < 0)$. *Then*
        i. *Let* $V_1 \leq x_{\tilde{L}}$. *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{◎\, \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        ii. *Let* $V_1 > x_{\tilde{L}}$. *Then* $\boxed{ⓢ\, \mathtt{dOITs}_{\tau>}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
      3. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{◎\, \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. □

[†] The condition of $b^\star > \rho > a$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $a^\star < \rho < b$ in Section 20.1.5.2.2.3(p.171).

[†]
$$\begin{aligned}
-\hat{V}_1 &= \max\{\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} - s, 0\} - \beta\hat{\rho} \quad \text{(apply the reverse to (20.1.43(p.171)))}\\
\hat{V}_1 &= -\max\{\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} - s, 0\} + \beta\hat{\rho} \quad \text{(multiply the above by } -1)\\
&= \min\{-\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} + s, 0\} + \beta\hat{\rho} \quad \text{(arrangement the above)}\\
&= \min\{\lambda\beta\min\{0, \hat{a}-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} \quad \text{(arrangement the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, \check{b}-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} \quad \text{(apply } \mathcal{I}_{\mathbb{R}} \text{ to the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, b-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} \quad \text{(apply } \mathcal{C}_{\mathbb{R}} \text{ to the above)}\\
V_1 &= \min\{\lambda\beta\min\{0, b-\rho\} + s, 0\} + \beta\rho \quad \text{(remove the hat symbol } \hat{\ })
\end{aligned}$$

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.16(p.172). ∎

Corollary 20.1.16 *Assume $b^\star > \rho > a$. Let $\beta < 1$ or $s > 0$:*

(a) *Let $V_1 \geq x_{\tilde{K}}$. Then z_t is nonincreasing in $t > 0$.*

(b) *Let $V_1 < x_{\tilde{K}}$. Then*

1. *Let $\beta = 1$ or $V_1 = 0$. Then z_t is constant in $t > 0$ ($z_t = z(V_1)$ for $t > 0$).*
2. *Let $\beta < 1$.*
   i. *Let $V_1 < 0$. Then z_t is nondecreasing in $t > 0$ for any $s \geq 0$.*
   ii. *Let $V_1 > 0$. Then z_t is nonincreasing in $t > 0$ for any $s \geq 0$.* □

● *Proof* Immediate from Tom 20.1.24(p.183) (b1,c1i,c2i1,c2ii1) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

20.1.6.3 Market Restriction

20.1.6.3.1 Positive Restriction

20.1.6.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.1.17** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{P}][\mathtt{A}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \geq b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c) *Let $a \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let $b^\star > \rho > a$.*

1. *Let $b \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ *and* $\mathtt{C}\rightsquigarrow\mathtt{S}_{1\vartriangle}$.
2. *Let $\rho > b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* The same as Tom 20.1.17(p.180) due to Lemma 17.4.4(p.118). ∎

20.1.6.3.1.2 Case of $\beta < 1$ or $s > 0$

20.1.6.3.1.2.1 Case of $\rho \geq b^\star$

□ **Pom 20.1.18** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{P}][\mathtt{A}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{ⓢ\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *and* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{◎\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$. Then we have* $\mathtt{S}_3$(p.156) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$.
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{◎\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* $\mathtt{S}_3$(p.156) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$ *is true.*
   ii. *Let $(\lambda\beta b + s)/\delta < b^\star$. Then* $\mathtt{S}_3$(p.156) $\boxed{ⓢ\blacktriangle \mid ◎\ \parallel}$ *is true.* □

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0 \cdots$ **(2)** and $b^\star > 0 \cdots$ **(3)** from Lemma 14.6.1(p.107) (n) and (2). Then we have $\tilde{\kappa} = s \cdots$ **(4)** from Lemma 14.6.6(p.108) (a).

(a-c2ii) The same as Tom 20.1.18(p.181) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta b + s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (10.2.2 (1) (p.56)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (3), so that $\beta b \geq b^\star$, which contradicts [7(p.118)]. Thus it must be that $(\lambda\beta b + s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of Tom 20.1.18(p.181).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\kappa > 0$ due (2), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 20.1.18(p.181); accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

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† See Remark 20.1.2(p.171).

□ **Pom 20.1.19** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *We have* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof* Suppose $a > 0$. Then $\tilde{\kappa} = s \cdots (1)$ from Lemma 14.6.6(p.108) (a).

(a) The same as Tom 20.1.19(p.181) (a).

(b) Let $\beta = 1$. Then, we have $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ from Tom 20.1.19(p.181) (b). Let $\beta < 1$. Then, if $s = 0$, it suffices to consider only (c2) of Tom 20.1.19(p.181) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (1), hence it suffices to consider only (c2) of Tom 20.1.19(p.181); accordingly, whether $s = 0$ or $s > 0$, we have the same results. Therefore, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

□ **Pom 20.1.20** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{\circledS\, \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a-a3) The same as Tom 20.1.20(p.181) (a-a3).

(b-b2) Let $\beta < 1$ and $\rho > 0$. First, we have the same as Pom 20.1.20(b1). Next, if $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 20.1.20(p.181) and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (b2) of Tom 20.1.20(p.181). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c-c2) Let $\beta < 1$ and $\rho < 0$. First, we have the same as Pom 20.1.20(c1). Next, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.1.20(p.181) and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (c2) of Tom 20.1.20(p.181). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.6.3.1.2.2 Case of $a \geq \rho$

□ **Pom 20.1.21** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\circledS\, \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $\boxed{\circledS\, \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$. Then* S_3(p.156) $\boxed{\circledS\blacktriangle \mid \circledcirc \parallel}$ *is true.* □

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a-c2) The same as Tom 20.1.21(p.182) (a-c2).

(c3) Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c3ii) of Tom 20.1.21(p.182) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 20.1.21(p.182). Thus, whether $s = 0$ or $s > 0$, we have the result. ∎

□ **Pom 20.1.22** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$, and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *We have* $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a) The same as Tom 20.1.22(p.182) (a).

(b) Let $\beta = 1$. Then $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ from Tom 20.1.22(p.182) (b). Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.1.22(p.182), and if $s > 0$, then $\tilde{\kappa} \geq 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.1.22(p.182) with $\tilde{\kappa}$; accordingly, whether $s = 0$ or $s > 0$, we have $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. Thus, whether $\beta = 1$ or $\beta < 1$, we have $\boxed{\bullet\, \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. ∎

□ $\mathtt{Pom}$ **20.1.23** $(\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{P}][\mathtt{A}]^{+}\})$ *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.*
3. *Let $x_{\tilde{L}} < \rho$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.*

(b) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *$\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.*

(c) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *$\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.* □

● *Proof* Suppose $a > 0 \cdots (1)$, hence $b > a > 0$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a-a3) The same as $\mathtt{Tom}$ 20.1.23(p.182) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as $\mathtt{Pom}$ 20.1.23(b1).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of $\mathtt{Tom}$ 20.1.23(p.182) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (b2) of $\mathtt{Tom}$ 20.1.23(p.182). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as $\mathtt{Pom}$ c1(b1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of $\mathtt{Tom}$ 20.1.23(p.182) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (c2) of $\mathtt{Tom}$ 20.1.23(p.182). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.6.3.1.2.3 Case of $b^\star > \rho > b$

□ $\mathtt{Pom}$ **20.1.24** $(\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{P}][\mathtt{A}]^{+}\})$ *Suppose $a > 0$. Assume $b^\star > \rho > b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then $\boxed{\bullet\,\mathtt{dOITd}_{1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledS\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{1\blacktriangle}$. Below let $\tau > 1$.*

(b) *Let $V_1 \geq x_{\tilde{K}}$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \leq x_{\tilde{L}}$. Then, if $\lambda\beta\max\{0, \rho - b\} \leq s$, we have $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
3. *Let $V_1 > x_{\tilde{L}}$.*
   1. *Let $\beta = 1$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.*
   2. *Let $\beta < 1$. Then $\mathtt{S}_5$ $\boxed{\circledS_{\blacktriangle} \mid \circledcirc_{\|}}$ is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *Let $\beta = 1$ or $V_1 = 0$. Then:*
   1. *$V_t = V_1$ for $t > 0$.*
   2. *If $\lambda\min\{0, \rho - b\} \geq -s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
2. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*[†]
   1. *Let $V_1 < 0$.*
      1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
      2. *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
   2. *Let $V_1 > 0$.*
      1. *Then V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$ where $V = x_{\tilde{K}}$ if the immediate initiation is strictly optimal for any $\tau \gg 0$.*
      2. *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a-b3i) The same as $\mathtt{Tom}$ 20.1.24(p.183) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (b3ii2) of $\mathtt{Tom}$ 20.1.24(p.183) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii2) of $\mathtt{Tom}$ 20.1.24(p.183). Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $V_1 < x_{\tilde{K}}$.

(c1-c1ii) The same as $\mathtt{Tom}$ 20.1.24(p.183) (c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(c2i,c2i1) The same as $\mathtt{Tom}$ 20.1.24(p.183) (c2i,c2i1).

(c2i2) The same as $\mathtt{Tom}$ 20.1.24(p.183) (c2i3).

[†] See Remark 20.1.2(p.171).

(c2ii,c2ii1) The same as Tom 20.1.24(p.183) (c2ii,c2ii1).

(c2ii2) If $s = 0$, then due to (1) it suffices to consider only (c2ii3) of Tom 20.1.24(p.183) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence t suffices to consider only (c2ii3) of Tom 20.1.24(p.183). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.6.3.1.2.4 Mixed Restriction

Omitted.

20.1.6.3.1.2.5 Negative Restriction

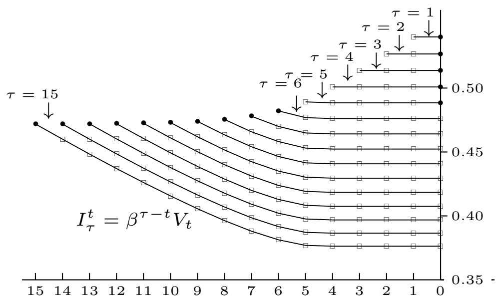
Omitted.

20.1.7 Numerical Calculation

Numerical Example **5** ($\mathscr{A}$\{M:2[ℝ][A]$^+$\} **(selling model)**

This is the example for [c⇝s▲] of S_4 [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] in Pom 20.1.4(p.163) (c3iii) in which $a > 0$, $\rho > x_K$, $\beta < 1$, $\rho > 0$, $s > 0$, and $x_L < \rho$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$where $x_L = 0.462767$.† The graph below is for $I^t_\tau = \beta^{\tau-t}V_t$, $\tau = 1, 2, \cdots, 15$ and $t = 0, 1, \cdots, \tau$, where • represents the optimal initiating time (OIT) for each $\tau = 1, 2, \cdots, 15$ (see t^*_τ - column in the table below).

1. Since $\Delta_\beta V_1 = \Delta_\beta V_2 = \Delta_\beta V_3 = \Delta_\beta V_4 = 0$ (see $\Delta_\beta V_t$-column in the table below), we have $V_4 = \beta V_3$, $V_3 = \beta V_2$, $V_2 = \beta V_1$, and $V_1 = \beta V_0$, implying that it becomes *indifferent* to skip the search up to the deadline $t_d = 0$ on $t = 4, 3, 2, 1$ (see Preference Rule 7.2.1(p.45)), i.e., [● dOITd$_{\tau=4,3,2,1}\langle 0\rangle$]$_\blacktriangle$. On the other hand, since $L(V_{t-1}) < 0$ for $1 \le t \le 4$ (see $L(V_{t-1})$-column in the table below), it follows that it is *strictly optimal* to skip the search up to the deadline 0 (see (20.1.26(p.156))) for $1 \le t \le \tau = 4$, i.e., [● dOITd$_{\tau=4,3,2,1}\langle 0\rangle$]$_\blacktriangle$. Although the above two results "*indifferent*" and "*strictly optimal*" seem to contradict each other at a glance, it is what is caused by the jumble of intuition and theory (see Alice 2(p.44)).
2. Each of the graphs for $\tau = 6, 7, \cdots, 15$ shows that the optimal initiating time is *strictly*, i.e., [ⓢ dOITs$_{6\le\tau\le 15}\langle\tau\rangle$]$_\blacktriangle$, meaning that the immediate initiation is strictly optimal and that conducting the search is *strictly optimal* at time $t = 6, 7, \cdots, 15$ (Conduct$_\blacktriangle$) and skipping the search becomes *strictly optimal* at time $t = 5, 4, 3, 2, 1$ after that (see $L(V_{t-1})$-column in the table below), implying that we have C⇝S$_\blacktriangle$ (see Remark 7.2.1(p.44)) occurs.



t	V_t	$\Delta_\beta V_t = V_t - \beta V_{t-1}$	t^*_τ	$L(V_{\tau-1})$
0	0.5000000			
1	0.4900000	0.0000000	0	−0.0133838 (SKIP▲)
2	0.4802000	0.0000000	0	−0.0098846 (SKIP▲)
3	0.4705960	0.0000000	0	−0.0063880 (SKIP▲)
4	0.4611841	0.0000000	0	−0.0028969 (SKIP▲)
5	0.4525469	+0.0005865	5	+0.0005865 (Conduct▲)
6	0.4473331	+0.0038371	6	+0.0038371 (Conduct▲)
7	0.4442109	+0.0058244	7	+0.0058244 (Conduct▲)
8	0.4423501	+0.0070235	8	+0.0070235 (Conduct▲)
9	0.4412444	+0.0077413	9	+0.0077413 (Conduct▲)
10	0.4405885	+0.0081690	10	+0.0081690 (Conduct▲)
11	0.4401998	+0.0084231	11	+0.0084231 (Conduct▲)
12	0.4399696	+0.0085738	12	+0.0085738 (Conduct▲)
13	0.4398333	+0.0086631	13	+0.0086631 (Conduct▲)
14	0.4397527	+0.0087160	14	+0.0087160 (Conduct▲)
15	0.4397049	+0.0087473	15	+0.0087473 (Conduct▲)

Figure 20.1.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \ge \tau > 1, \tau \ge t > 0)$

20.1.8 Conclusion 3 (Search-Allowed-Model 2)

■ The assertion systems $\mathscr{A}$\{M/M̃:2[ℝ][A]\} of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$$\mathscr{Q}\langle \text{M:2[A]}\rangle = \{\text{M:2[ℝ][A]}, \tilde{\text{M}}\text{:2[ℝ][A]}, \text{M:2[ℙ][A]}, \tilde{\text{M}}\text{:2[ℙ][A]}\}$$

are given by

$\mathscr{A}$\{M:2[ℝ][A]\}

↓

Tom's 20.1.1(p.156), 20.1.2(p.156), 20.1.3(p.159), 20.1.4(p.160),

$\mathscr{A}$\{M̃:2[ℝ][A]\}

↓

Tom's 20.1.5(p.164), 20.1.6(p.164), 20.1.7(p.164), 20.1.8(p.164),

$\mathscr{A}$\{M:2[ℙ][A]\}

↓

Tom's 20.1.9(p.168), 20.1.10(p.168), 20.1.11(p.169), 20.1.12(p.169), 20.1.13(p.170), 20.1.14(p.170), 20.1.15(p.170), 20.1.16(p.172),

$\mathscr{A}$\{M̃:2[ℙ][A]\}

↓

Tom's 20.1.17(p.180), 20.1.18(p.181), 20.1.19(p.181), 20.1.20(p.181), 20.1.21(p.182), 20.1.22(p.182), 20.1.23(p.182), 20.1.24(p.183),

---

†Note that $a = 0.01 > 0$, $\rho = 0.5 > 0$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. In addition, since $\mu = (1.00 + 0.01)/2 = 0.505$, we have $\lambda\beta\mu = 0.34643 > 0.1 = s$. Furthermore, we have $x_L = 0.4627674 < 0.5 = \rho$. Thus the condition of the assertion is satisfied.

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$ of the quadruple-asset-trading-models for Model 2 on the positive market $\mathscr{F}^+$

$$\mathcal{Q}\langle \mathsf{M}:2[\mathtt{A}]\rangle^+ = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+, \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+, \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+, \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+$$

are given by

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$$
$$\downarrow$$

Pom's 20.1.1(p.162), 20.1.2(p.162), 20.1.3(p.162), 20.1.4(p.163),

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$$
$$\downarrow$$

Pom's 20.1.5(p.165), 20.1.6(p.165), 20.1.7(p.165), 20.1.8(p.166),

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$$
$$\downarrow$$

Pom's 20.1.9(p.176), 20.1.10(p.177), 20.1.11(p.177), 20.1.12(p.177), 20.1.13(p.178), 20.1.14(p.178), 20.1.15(p.178), 20.1.16(p.179),

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$$
$$\downarrow$$

Pom's 20.1.17(p.184), 20.1.18(p.184), 20.1.19(p.185), 20.1.20(p.185), 20.1.21(p.185), 20.1.22(p.185), 20.1.23(p.186), 20.1.24(p.186),

■ Closely looking into all the assertion systems above leads to the following conclusions.

C1. Mental Conflict

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$.

1. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ (selling model) is nondecreasing in $t^{\blacktriangle a}$ as in Figure 7.3.1(p.47) (I), hence we have the normal conflict (see Remark 7.3.1(p.47)).
2. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$ (selling model) is nondecreasing in $t^{\blacktriangle b}$ as in Figure 7.3.1(p.47) (I), hence we have the normal conflict (see Remark 7.3.1(p.47)).
3. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$ (buying model) is nonincreasing in $t^{\blacktriangledown c}$ as in Figure 7.3.1(p.47) (II), hence we have the normal conflict (see Remark 7.3.1(p.47)).
4. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$ (buying model) is nonincreasing in $t^{\blacktriangledown d}$ as in Figure 7.3.1(p.47) (II), hence we have the normal conflict (see Remark 7.3.1(p.47)).

- $^{\blacktriangle a}$ ← Tom's 20.1.1(p.156) (a).
- $^{\blacktriangle b}$ ← Corollaries 20.1.1(p.168).
- $^{\blacktriangledown c}$ ← Tom's 20.1.5(p.164) (a).
- $^{\blacktriangledown d}$ ← Corollaries 20.1.9(p.180).

b. Let $\beta < 1$ or $s > 0$.

1. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ (selling model) is nondecreasing$^{\blacktriangle a}$, constant $^{\parallel a}$, or nonincreasing in $t^{\blacktriangledown a}$ as in Figure 7.3.2(p.48) (I), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).
2. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$ (selling model) is nondecreasing$^{\blacktriangle b}$, constant $^{\parallel b}$, or nonincreasing in $t^{\blacktriangledown b}$ as in Figure 7.3.2(p.48) (I), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).
3. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$ (buying model) is nondecreasing$^{\blacktriangle c}$, constant $^{\parallel c}$, or nonincreasing in $t^{\blacktriangledown c}$ as in Figure 7.3.2(p.48) (II), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).
4. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$ (buying model) is nondecreasing$^{\blacktriangle d}$, constant $^{\parallel d}$, or nonincreasing in $t^{\blacktriangledown d}$ as in Figure 7.3.2(p.48) (II), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).

- $^{\blacktriangle a}$ ← 20.1.2(p.156) (a), 20.1.3(p.159) (a), 20.1.4(p.160) (c1).
  $^{\parallel a}$ ← Tom 20.1.4(p.160) (a1)).
  $^{\blacktriangledown a}$ ← Tom 20.1.4(p.160) (b1).
- $^{\blacktriangle b}$ ← 20.1.2(p.169), 20.1.3(p.169), 20.1.4(p.169) (c),
  20.1.5(p.170), 20.1.6(p.170), 20.1.7(p.170) (c), 20.1.8(p.176) (a,b2ii).
  $^{\parallel b}$ ← Corollary 20.1.4(p.169) (a), 20.1.7(p.170) (a), 20.1.8(p.176) (b1).
  $^{\blacktriangledown b}$ ← Corollaries 20.1.4(p.169) (b), 20.1.7(p.170) (b), 20.1.8(p.176) (b2i).
- $^{\blacktriangle c}$ ← Tom 20.1.8(p.164) (b1).
  $^{\parallel c}$ ← Tom 20.1.8(p.164) (a1).
  $^{\blacktriangledown c}$ ← 20.1.6(p.164) (a), 20.1.7(p.164) (a), 20.1.8(p.164) (c1).
- $^{\blacktriangle d}$ ← Corollaries 20.1.12(p.182) (b), 20.1.15(p.183) (b),20.1.16(p.184) (b2i).
  $^{\parallel d}$ ← Corollaries 20.1.15(p.183) (a), 20.1.16(p.184) (b1).
  $^{\blacktriangledown d}$ ← 20.1.10(p.181), 20.1.11(p.181), 20.1.12(p.182) (c),
  20.1.13(p.182), 20.1.14(p.182), 20.1.15(p.183) (c), 20.1.16(p.184) (a),b2ii).

The above results can be summarized as below.

A. If $\beta = 1$ and $s = 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6) .

B. If $\beta < 1$ or $s > 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the abnormal mental conflict, which does not coincide with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6) .

C2. Symmetry

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 20.1.5(p.165) $\sim$ Pom 20.1.1(p.162) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.17(p.184) $\sim$ Pom 20.1.9(p.176) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$.

b. Let $\beta < 1$ or $s > 0$. Then we have

Pom 20.1.6(p.165) $\not\sim$ Pom 20.1.2(p.162) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.7(p.165) $\not\sim$ Pom 20.1.3(p.162) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$,
Pom 20.1.8(p.166) $\not\sim$ Pom 20.1.4(p.163) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.18(p.184) $\not\sim$ Pom 20.1.10(p.177) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$,
Pom 20.1.19(p.185) $\not\sim$ Pom 20.1.11(p.177) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.20(p.185) $\not\sim$ Pom 20.1.12(p.177) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$,
Pom 20.1.21(p.185) $\not\sim$ Pom 20.1.13(p.178) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.22(p.185) $\not\sim$ Pom 20.1.14(p.178) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$,
Pom 20.1.23(p.186) $\not\sim$ Pom 20.1.15(p.178) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.24(p.186) $\not\sim$ Pom 20.1.16(p.179) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$.

The above results can be summarized as below.

A. Let $\beta = 1$ and $s = 0$. Then the symmetry is inherited.

B. Let $\beta < 1$ or $s > 0$. Then the symmetry collapses.

C3. Analogy

On $\mathscr{F}^+$, for any $\beta \leq 1$ and $s \geq 0$ we have:

a. We have:

Pom 20.1.9(p.176) $\bowtie$ Pom 20.1.1(p.162) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.10(p.177) $\bowtie$ Pom 20.1.2(p.162) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$,
Pom 20.1.17(p.184) $\bowtie$ Pom 20.1.5(p.165) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+)$,
Pom 20.1.18(p.184) $\bowtie$ Pom 20.1.6(p.165) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+)$.

The above results can be summarized as below.

A. The analogy collapses.

C4. Optimal initiating time (OIT)

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 20.1.1(p.162) , Pom 20.1.5(p.165) , Pom 20.1.9(p.176) , Pom 20.1.17(p.184) ,

we have the following table:

Table 20.1.1: Possible OIT ($\beta = 1$ and $s = 0$)

		$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
Ⓢ dOITs$_\tau\langle\tau\rangle_{\parallel}$	Ⓢ$_{\parallel}$				
Ⓢ dOITs$_\tau\langle\tau\rangle_{\triangle}$	Ⓢ$_{\triangle}$				
Ⓢ dOITs$_\tau\langle\tau\rangle_{\blacktriangle}$	Ⓢ$_{\blacktriangle}$	○	○	○	○
◎ ndOIT$_\tau\langle t^\bullet_\tau\rangle_{\parallel}$	◎$_{\parallel}$				
◎ ndOIT$_\tau\langle t^\bullet_\tau\rangle_{\triangle}$	◎$_{\triangle}$				
◎ ndOIT$_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	◎$_{\blacktriangle}$				
● dOITd$_\tau\langle 0\rangle_{\parallel}$	ⓓ$_{\parallel}$	○	○	○	○
● dOITd$_\tau\langle 0\rangle_{\triangle}$	ⓓ$_{\triangle}$				
● dOITd$_\tau\langle 0\rangle_{\blacktriangle}$	ⓓ$_{\blacktriangle}$				

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 20.1.2(p.162) ,	Pom 20.1.3(p.162) ,	Pom 20.1.4(p.163) ,	Pom 20.1.5(p.165) ,	Pom 20.1.6(p.165) ,
Pom 20.1.7(p.165) ,	Pom 20.1.8(p.166) ,	Pom 20.1.10(p.177) ,	Pom 20.1.11(p.177) ,	Pom 20.1.12(p.177) ,
Pom 20.1.13(p.178) ,	Pom 20.1.14(p.178) ,	Pom 20.1.15(p.178) ,	Pom 20.1.16(p.179) ,	Pom 20.1.19(p.185) ,
Pom 20.1.20(p.185) ,	Pom 20.1.21(p.185) ,	Pom 20.1.22(p.185) ,	Pom 20.1.23(p.186) ,	Pom 20.1.24(p.186) ,

we have the following table:

Table 20.1.2: Possible OIT ($\beta <$ or $s > 0$)

		$\mathscr{A}${M:2[ℝ][A]$^+$}	$\mathscr{A}${M̃:2[ℝ][A]$^+$}	$\mathscr{A}${M:2[ℙ][A]$^+$}	$\mathscr{A}${M̃:2[ℙ][A]$^+$}
Ⓢ dOITs$_\tau\langle\tau\rangle_{\|}$	Ⓢ$_{\|}$				
Ⓢ dOITs$_\tau\langle\tau\rangle_\triangle$	Ⓢ$_\triangle$				
Ⓢ dOITs$_\tau\langle\tau\rangle_\blacktriangle$	Ⓢ$_\blacktriangle$	○	○	○	○
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_{\|}$	◎$_{\|}$	○	○	○	○
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_\triangle$	◎$_\triangle$				
◎ ndOIT$_\tau\langle t_\tau^\bullet\rangle_\blacktriangle$	◎$_\blacktriangle$				
● dOITd$_\tau\langle 0\rangle_{\|}$	🅓$_{\|}$	○	○	○	○
● dOITd$_\tau\langle 0\rangle_\triangle$	🅓$_\triangle$				
● dOITd$_\tau\langle 0\rangle_\blacktriangle$	🅓$_\blacktriangle$				

c. The table below is the list of the occurrence rates of Ⓢ, ◎, and 🅓 on $\mathscr{F}$ (See the primitive Tom's 20.1.1(p.156) (▣), 20.1.2(p.156) (▣), 20.1.3(p.159) (▣), 20.1.4(p.160) (▣), and 20.1.16(p.172) (▣)).

Table 20.1.3: Occurence rates of Ⓢ, ◎, and 🅓 on $\mathscr{F}^+$

Ⓢ			◎			🅓		
47.5 % / 29			21.3 % / 13			31.2 % / 19		
Ⓢ$_{\|}$	Ⓢ$_\triangle$	Ⓢ$_\blacktriangle$	◎$_{\|}$	◎$_\triangle$	◎$_\blacktriangle$	🅓$_{\|}$	🅓$_\triangle$	🅓$_\blacktriangle$
—	×	possible	possible	×	×	possible	×	×
– %/ –	0.0 %/ 0	47.5 %/ 29	21.3 %/ 13	0.0 %/ 0	0.0 %/ 0	31.2 %/ 19	0.0 %/ 0	0.0 %/ 0

C5. Null-time-zone and deadline-engulfing

From Table 20.1.3(p.190) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▲ (strict optimality).

b. As a whole we have Ⓢ, ◎, and 🅓 at 47.5%, 21.3%, and 31.2% respectively where

1. Ⓢ$_{\|}$ cannot be defined due to Remark 7.2.3(p.45) .
2. ◎$_{\|}$ is possible (21.3 %).
3. 🅓$_{\|}$ is possible (31.2 %).
4. Ⓢ$_\triangle$ never occur (0.0 %).
5. ◎$_\triangle$ never occur (0.0 %).
6. 🅓$_\triangle$ never occur (0.0 %).
7. Ⓢ$_\blacktriangle$ is possible (47.5 %).
8. ◎$_\blacktriangle$ never occurs (0.0 %).
9. 🅓$_\blacktriangle$ never occurs (0.0 %).

From the above results we see that:

A. ◎ and 🅓 causing the **null-time-zone** are possible at 52.5% (= 21.3% + 31.2%).

B. ◎$_\blacktriangle$ *strictly* causing the **null-time-zone** is impossible (0.0%).

C. 🅓$_\blacktriangle$ *strictly* causing the **null-time-zone** is impossible (0.0%), i.e., the deadline-engulfing is impossible.

C6. C⇝S On $\mathscr{F}^+$, we have (see (A5b(p.12))):

Let $\beta < 1$ or $s > 0$. Then from Pom's 20.1.4(p.163) , 20.1.12(p.177) , 20.1.15(p.178) , and 20.1.16(p.179) we have the following table:

Table 20.1.4: C⇝S ($\beta < 1$ or $s > 0$)

	$\mathscr{A}${M:2[ℝ][A]$^+$}	$\mathscr{A}${M̃:2[ℝ][A]$^+$}	$\mathscr{A}${M:2[ℙ][A]$^+$}	$\mathscr{A}${M̃:2[ℙ][A]$^+$}
(a) C⇝S$_\triangle$	○		○	
(b) C⇝S$_\blacktriangle$	○		○	

a. $\texttt{C}\rightsquigarrow\texttt{S}_{\triangle}$ occurs only for $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+$ and $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+$ (both are a selling model).

b. $\texttt{C}\rightsquigarrow\texttt{S}_{\blacktriangle}$ occurs only for $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+$ and $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+$ (both are a selling model).

- Tom 20.1.4(p.160) (b3iii),
- Tom 20.1.16(p.172) (c2i2i),
- Tom 20.1.16(p.172) (c2i2ii).

20.2 Search-Enforced-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{E}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}], \mathsf{M}{:}2[\mathbb{P}][\mathtt{E}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\}$

20.2.1 Theorems

As ones corresponding to Theorems 19.2.1(p.136), 19.2.2(p.136), and 19.2.3(p.136), let us consider here the following three theorems:

Theorem 20.2.1 (symmetry[$\mathbb{R} \to \mathbb{R}$])) *Let $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}]. \quad \square \tag{20.2.1}$$

Theorem 20.2.2 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}]. \quad \square \tag{20.2.2}$$

Theorem 20.2.3 (symmetry[$\mathbb{P} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\}]. \quad \square \tag{20.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}], \tag{20.2.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}], \tag{20.2.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\}], \tag{20.2.6}$$

corresponding to (19.2.4(p.136)), (19.2.5(p.136)), and (19.2.6(p.136)). Then, for the same reason as in Chap. 15(p.111) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}] \tag{20.2.7}$$

holds (corresponding to (19.2.7(p.136))) and that we have the following theorem, corresponding to Theorem 19.2.4(p.136).

Theorem 20.2.4 (analogy [$\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}$]) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\}]. \quad \square \tag{20.2.8}$$

In fact, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.4.4(p.41) it can be easily shown that (20.2.4(p.191)) and (20.2.6(p.191)) hold; however, from the comparison of (I) and (III) in Table 6.4.4(p.41) we can immediately see that (20.2.5(p.191)) does not always hold.

20.2.2 A Lemma

The following lemma provides the conditions on which whether each of Theorems 20.2.1(p.191), 20.2.2(p.191), and 20.2.3(p.191) holds or not.

Lemma 20.2.1

(a) Theorem 20.2.1(p.191) *always hold.*

(b) Theorem 20.2.3(p.191) *always hold.*

(c) *Let $\rho \le a^\star$ or $b \le \rho$. Then* Theorem 20.2.2(p.191) *holds.*

(d) *Let $a^\star < \rho < b$. Then* Theorem 20.2.2(p.191) *does not always hold.* $\square$

• *Proof* (a,b) From the comparisons of (I) and (II) in Table 6.4.4(p.41) and that of (III) and (IV) in Table 6.4.4(p.41) we see that (20.2.4(p.191)) and (20.2.6(p.191)) hold, hence Theorems 20.2.1(p.191) and 20.2.3 hold.

(c,d) From the comparison of (I) and (III) in Table 6.4.4(p.41) we see that (20.2.5(p.191)) does not always hold, hence it follows that Theorem 20.2.2(p.191) does not always hold. The proofs for the two assertions (c,d) are the same as those of Lemma 20.1.1(p.153) (c,d). ∎

20.2.3 $\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]$

20.2.3.1 Preliminary

From (6.4.28(p.41)) and (5.1.8(p.25)) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 0. \tag{20.2.9}$$

20.2.3.2 Analysis

20.2.3.2.1 Case of $\beta = 1$ and $s = 0$

□ Tom 20.2.1 (■$\mathscr{A}${M:2[ℝ][E]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\rho < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x) \cdots (1)$ from (5.1.4(p.25)), we have $K(x) \geq 0 \cdots (2)$ for any x due to Lemma 10.1.1(p.55) (g).

(a) From (6.4.28(p.41)) and (2) we obtain $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq b$. Then, since $b \leq V_0$ from (6.4.27(p.41)), we have $b \leq V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 10.2.1(p.57) (d), thus $V_t = \beta V_{t-1}$ for $t > 0$ from (20.2.9(p.191)). Then, since $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ (see Preference Rule 7.2.1(p.45)).

(c) Let $\rho < b$. Then $V_0 < b \cdots (3)$ from (6.4.27(p.41)). Let $V_{t-1} < b$. Then, since $V_t < K(b) + b$ from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (h), we have $V_t < \beta b - s = b$ from (10.2.7 (2) (p.57)) and the assumptions "$\beta = 1$ and $s = 0$". Hence, by induction $V_{t-1} < b$ for $t > 0$, so $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (d). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 0$ from (20.2.9(p.191)) or equivalently $V_t > \beta V_{t-1}$ for $t > 0$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. ∎

20.2.3.2.2 Case of $\beta < 1$ or $s > 0$

Let us define

$$\mathtt{S}_8\,\boxed{ⓢ\blacktriangle\,|\,◎\parallel\,|\,◎\vartriangle\,|\,◎\blacktriangle} = \left\{\begin{array}{l}\text{For any } \tau > 0 \text{ there exists } t^\bullet_\tau > 0 \text{ such that}\\ (1)\ \boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 0}\langle \tau\rangle}_{\blacktriangle},\\ (2)\ \boxed{◎\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\vartriangle},\\ (3)\ \boxed{◎\ \texttt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\parallel}\ (\boxed{◎\ \texttt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\blacktriangle}).\end{array}\right\}.$$

Remark 20.2.1 $\mathtt{S}_8$ is the same as $\mathtt{S}_2$(p.137) except that the inequalities of $\tau > 1$, $t^\bullet_\tau > 1$, and $t^\bullet_\tau \geq \tau > 1$ in $\mathtt{S}_2$ changes into $\tau > 0$, $t > 0$, and $t^\bullet_\tau \geq \tau > 0$ respectively in $\mathtt{S}_8$. □

□ Tom 20.2.2 (■$\mathscr{A}${M:2[ℝ][E]}) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $a < \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
     2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0\,(s > 0)$.*
   - i. *Let $a < \rho$.*
     1. *Let $b \geq 0\,(\kappa \geq 0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
     2. *Let $b < 0\,(\kappa < 0)$. Then $\mathtt{S}_8$* $\boxed{ⓢ\blacktriangle\,|\,◎\parallel\,|\,◎\vartriangle\,|\,◎\blacktriangle}$ *is true.* ↦ → ◎$_\blacktriangle$
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $b > 0\,(\kappa > 0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
          2. *Let $b \leq 0\,(\kappa \leq 0)$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $b \geq 0\,(\kappa \geq 0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
          2. *Let $b < 0\,(\kappa < 0)$. Then $\mathtt{S}_8$* $\boxed{ⓢ\blacktriangle\,|\,◎\parallel\,|\,◎\vartriangle\,|\,◎\blacktriangle}$ *is true.* ↦ → ◎$_\blacktriangle$

2. *Let* $(\lambda\beta\mu - s)/\delta > a$.

i. *Let* $b \geq 0\,(\kappa \geq 0)$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$

ii. *Let* $b < 0\,(\kappa < 0)$. *Then* $\texttt{S}_8$ $\boxed{\circledS\blacktriangle \mid \circledcirc\ \| \mid \circledcirc\ \vartriangle \mid \circledcirc\ \blacktriangle}$ *is true.* $\square \mapsto$ $\rightarrow \circledcirc_{\blacktriangle}$

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots (1)$. Then $V_0 < x_K \cdots (2)$ from (6.4.27(p.41)) and $K(\rho) > 0$ due to Lemma 10.2.2(p.57) (j1). Since $V_1 = K(\rho) + \rho \cdots (3)$ from (6.4.28(p.41)) with $t = 1$, we have $V_1 - V_0 = V_1 - \rho = K(\rho) > 0$, hence $V_1 > V_0 \cdots (4)$.

(a) Note (4), hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, due to Lemma 10.2.2(p.57) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$ from (6.4.28(p.41)). Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$. Note again (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 10.2.2(p.57) (f) we have $V_t < K(V_t) + V_t = V_{t+1}$. If $a < \rho$, then $a < V_0$ from (6.4.27(p.41)), hence $a < V_{t-1}$ for $t > 0$ due to the nondecreasing of V_t, so from Lemma 10.2.2(p.57) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction we have $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \geq 0$. Consider a sufficiently large $M > 0$ with $\rho \leq M$ and $b \leq M$, hence from (6.4.27(p.41)) we have $V_0 \leq M$. Suppose $V_{t-1} \leq M$. Then, from Lemma 10.2.2(p.57) (e) we have $V_t \leq K(M) + M = \beta M - s$ due to (10.2.7 (2) (p.57)), hence $V_t \leq M$ due to the assumptions "$\beta \leq 1$ and $s \geq 0$". Accordingly, by induction $V_t \leq M$ for $t \geq 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Thus $V = K(V) + V$ from (6.4.28(p.41)), hence $K(V) = 0$, so $V = x_K$ due to Lemma 10.2.2(p.57) (j1).

(b) Let $x_L \leq \rho$. Then, since $x_L \leq V_0$ from (6.4.27(p.41)), we have $x_L \leq V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), thus $V_t - \beta V_{t-1} \leq 0$ for $t > 0$ from (20.2.9(p.191)) or equivalently $V_t \leq \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) Let $\rho < x_L \cdots (5)$. Then $V_0 < x_L \cdots (6)$ from (6.4.27(p.41)), hence $L(V_0) > 0 \cdots (7)$ due to Corollary 10.2.1(p.57) (a).

(c1) Since $V_1 - \beta V_0 = L(V_0) > 0$ from (20.2.9(p.191)) with $t = 1$ and (7), we have $V_1 > \beta V_0$, hence $t^*_1 = 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle} \cdots (8)$. Below let $\tau > 1 \cdots (9)$.

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $\delta = \lambda$ from (10.2.1(p.56)) and $x_L = x_K \cdots (10)$ from Lemma 10.2.3(p.58) (b), hence $K(x_L) = K(x_K) = 0 \cdots (11)$.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.4.27(p.41)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (10), we have $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (e1), hence for the same reason as in the proof of Tom 20.2.1(p.192) (c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2ii) Let $\rho \leq a$, hence $V_0 \leq a \cdots (12)$ from (6.4.27(p.41)). Then, from (3) and (10.2.7 (1) (p.57)) we have $V_1 = \lambda\mu - s + (1-\lambda)\rho$.

(c2ii1) Let $(\lambda\mu - s)/\lambda \leq a$. Then $x_K = (\lambda\mu - s)/\lambda \leq a \cdots (13)$ from Lemma 10.2.2(p.57) (j2). Hence $K(a) \leq 0$ from Lemma 10.2.2(p.57) (j1). Note (12). Suppose $V_{t-1} \leq a$. Then, from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \leq K(a) + a \leq a$, hence by induction $V_{t-1} \leq a$ for $t > 0$. Accordingly, from (6.4.28(p.41)) and (10.2.7 (1) (p.57)) we have $V_t = \lambda\mu - s + (1-\lambda)V_{t-1} \cdots (14)$ for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, we have $x_K = \mu - s$ from (13) and $V_t = \mu - s$ for $t > 0$ from (14), hence $V_t = x_K$ for $t > 0$, so $V_{t-1} = x_K$ for $t > 1$. Accordingly, $V_{t-1} = x_L$ for $t > 1$ due to (10). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.2.9(p.191)) or equivalently $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau = \beta V_{\tau-1} \cdots = \beta^{\tau-1} V_1$ for $\tau > 1$. From this and (4) we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from Lemma 10.2.2(p.57) (f) and (11). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.57) (e1). Thus, for the same reason as in the proof of Tom 20.2.1(p.192) (c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a$ from Lemma 10.2.2(p.57) (j2), hence $x_L > a$ from (10). Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from Lemma 10.2.2(p.57) (h) and (11). Accordingly, by induction $V_{t-1} < x_L \cdots (15)$ for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ due to Lemma 10.2.1(p.57) (e1). Consequently, for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3) Let $\beta < 1$ and $s = 0\,(s > 0)$.

(c3i) Let $a < \rho \cdots (16)$. Then, since $a < V_0$ from (6.4.27(p.41)), we have $a < V_{t-1}$ for $t > 0$ due to (a).

(c3i1) Let $b \geq 0\,(\kappa \geq 0)$. Then $x_L \geq x_K \cdots (17)$ from Lemma 10.2.3(p.58) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (g) we have $V_t < K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (17). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3i2) Let $b < 0\,(\kappa < 0)$. Then $x_L < x_K \cdots (18)$ from Lemma 10.2.3(p.58) (c (d)). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L$. Now, since $V = x_K$ due to (a), we have $x_L < V$ due to (18), which is a contradiction. Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. In addition, from (6) and the *strict increasingness* of V_t due to (a), it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau -1} < x_L \le V_{t^\bullet_\tau} < V_{t^\bullet_\tau +1} < V_{t^\bullet_\tau +2} < \cdots .$$

from which we have

$$V_{t-1} < x_L ,\ t^\bullet_\tau \ge t > 0, \qquad x_L \le V_{t^\bullet_\tau}, \qquad x_L < V_{t-1},\ t > t^\bullet_\tau + 1. \tag{20.2.10}$$

Hence, we have

$$L(V_{t-1}) > 0 \quad \cdots (19), \quad t^\bullet_\tau \ge t > 0 \quad \text{(due to Corollary 10.2.1(p.57) (a))}$$

$$L(V_{t^\bullet_\tau}) \le 0 \quad \cdots (20), \quad \text{(due to Corollary 10.2.1(p.57) (a))}$$

$$L(V_{t-1}) = (\!(< 0)\!)^\dagger \cdots (21), \quad t > t^\bullet_\tau + 1 \quad \text{(due to Lemma 10.2.1(p.57) (d(\!(e1)\!)))}$$

- Let $t^\bullet_\tau \ge \tau > 0$. Then $L(V_{t-1}) > 0 \cdots (22)$ for $\tau \ge t > 0$ from (19). Hence, for the same reason as in Tom 20.2.1(p.192) (c) we obtain $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $t^\bullet_\tau \ge \tau > 0$. Accordingly, $\mathtt{S}_8(1)$ is true. Now, since $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > 0$ from (20.2.9(p.191)) and (22), we have $V_t > \beta V_{t-1}$ for $\tau \ge t > 0$, hence
$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0.$$
Accordingly, when $\tau = t^\bullet_\tau$, we have
$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau -1} > \cdots > \beta^{t^\bullet_\tau} V_0 \cdots (23)$$
- Let $\tau = t^\bullet_\tau + 1$. From (20.2.9(p.191)) with $t = t^\bullet_\tau + 1$ and (20) we have $V_{t^\bullet_\tau+1} - \beta V_{t^\bullet_\tau} = L(t^\bullet_\tau) \le 0$, hence $V_{t^\bullet_\tau+1} \le \beta V_{t^\bullet_\tau}$. Accordingly, from (23) we have
$$V_{t^\bullet_\tau+1} \le \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_0 \cdots (24),$$
thus $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_\vartriangle$, so that $\mathtt{S}_8(2)$ is true.
- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau+1}) = (\!(<)\!)\ 0$ from (21) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1}$ from (20.2.9(p.191)), hence from (24) we have
$$V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1} \le \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+2} V_0$$
Similarly we have
$$V_{t^\bullet_\tau+3} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^2 V_{t^\bullet_\tau+1} \le \beta^3 V_{t^\bullet_\tau} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+3} V_0.$$
By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain
$$V_\tau = (\!(<)\!)\ \beta V_{\tau-1} = (\!(<)\!)\ \cdots = (\!(<)\!)\ \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = (\!(<)\!)$$
$$\beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \le \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau-1} > \cdots > \beta^\tau V_0. \cdots (25)$$
  - Let $s = 0$. Then (25) can be written as
  $$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \le \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau-1} > \cdots > \beta^\tau V_0,$$
  hence $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_\parallel$ (see Preference Rule 7.2.1(p.45)), hence $\mathtt{S}_8(3)$ is true.
  - Let $s > 0$. Then (25) can be written as
  $$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} < \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \le \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau-1} > \cdots > \beta^\tau V_0, \tag{20.2.11}$$
  hence $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\circ\rangle}_\blacktriangle$, hence $\mathtt{S}_8(3)$ is true.

(c3ii) Let $\rho \le a$, hence $V_0 \le a$ from (6.4.27(p.41)). Then, from (3) and (10.2.7 (1) (p.57)) we have $V_1 = \lambda\beta\mu - s + (1-\lambda)\beta\rho$.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \le a$. Then $x_K = (\lambda\beta\mu - s)/\delta \le a \cdots (26)$ from Lemma 10.2.2(p.57) (j2(p.58)). Hence $V_1 = \delta x_K + (1-\lambda)\beta\rho \cdots (27)$.

(c3ii1i) Let $\lambda = 1$, hence $\delta = 1$ from (10.2.1(p.56)). Thus, from (26) and (27) we have $x_K = \beta\mu - s \le a$ and $V_1 = x_K \le a \cdots (28)$.

(c3ii1i1) Let $b > 0\ (\!(\kappa > 0)\!)$. Then $x_L > x_K \cdots (29)$ due to Lemma 10.2.3(p.58) (c (\!(d)\!)). Note (28). Suppose $V_{t-1} = x_K$. Then, from (6.4.28(p.41)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (29), thus $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a). Hence, from (7) we obtain $L(V_{t-1}) > 0$ for $t > 0$. Accordingly, for almost the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c3ii1i2) Let $b \le 0\ (\!(\kappa \le 0)\!)$. Then, since $x_L \le x_K$ from Lemma 10.2.3(p.58) (c (\!(d)\!)), we have $V_1 \ge x_L$ from (28), hence $V_{t-1} \ge x_L$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a), we have $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$, thus $V_t - \beta V_{t-1} \le 0$ for $\tau \ge t > 1$ from (20.2.9(p.191)), i.e., $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$. Hence $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \cdots (30)$. Now, from (6.4.27(p.41)), (4), (28), and (29) we have $\rho = V_0 < V_1 = x_K < x_L$, hence $L(\rho) > 0$ from

† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

Corollary 10.2.1(p.57) (a). In addition, from (3) and (6.4.27(p.41)) we have $V_1-\beta V_0 = V_1-\beta\rho = K(\rho)+\rho-\beta\rho = K(\rho)+(1-\beta)\rho = L(\rho) > 0$ from (5.1.8(p.25)), hence $V_1 > \beta V_0$. Accordingly, from (30) we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1}V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c3ii1ii) Let $\lambda < 1$.

(c3ii1ii1) Let $b \ge 0\,(\!(\kappa \ge 0)\!)$. Then $x_L \ge x_K \cdots$ **(31)** from Lemma 10.2.3(p.58) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from Lemma 10.2.2(p.57) (f) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ due to (31). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c3ii1ii2) Let $b < 0\,(\!(\kappa < 0)\!)$. Then $x_L < x_K \cdots$ **(32)** from Lemma 10.2.3(p.58) (c (d)). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \le x_L$ due to (a). Now, since $V = x_K$ from (a), we have the contradiction $x_L < V$ from (32). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From this and the strict increasingness of V_t due to (a), it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau-1} < x_L \le V_{t^\bullet_\tau} < V_{t^\bullet_\tau+1} < V_{t^\bullet_\tau+2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we have $\texttt{S}_8$ $\boxed{ⓢ\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$.

(c3ii2) Let $(\lambda\beta\mu - s)/\delta > a \cdots$ **(33)**. Then $x_K > (\lambda\beta\mu - s)/\delta > a$ from Lemma 10.2.2(p.57) (j2).

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \ge 0$ due to (a).
2. Let $\lambda = 1$, hence $\delta = 1$ from (10.2.1(p.56)), so $\beta\mu - s > a$ from (33). Now $K(x) \ge \beta\mu - s - x$ for any x from (10.2.4(p.57)) or equivalently $K(x) + x \ge \beta\mu - s$ for any x, so $V_1 \ge \beta\mu - s > a$ from (3). Accordingly $V_{t-1} > a$ for $t > 1$ due to (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.57) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \ge 0$.

From the above, whether $\lambda < 1$ or $\lambda = 1$, we see that V_t is *strictly increasing* in $t > 0$.

(c3ii2i) Let $b \ge 0\,(\!(\kappa \ge 0)\!)$. Then $x_L \ge x_K \cdots$ **(34)** from Lemma 10.2.2(p.57) (c (d)). From the above strict increasingness of V_t in $t \ge 0$ and (a) we have $V_{t-1} < V = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (34). Thus, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c3ii2ii) Let $b < 0\,(\!(\kappa < 0)\!)$. Then $x_L < x_K \cdots$ **(35)** from Lemma 10.2.3(p.58) (c (d)). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \le x_L$. Now, since $V = x_K$ from (a), we have $x_L < V$ from (35), which is a contradiction. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From this, (6), and the above strict increasingness of V_t in $t \ge 0$ it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau-1} < x_L \le V_{t^\bullet_\tau} < V_{t^\bullet_\tau+1} < V_{t^\bullet_\tau+2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we can immediately see that the assertion holds true. ∎

□ **Tom 20.2.3** ($\blacksquare\mathscr{A}\{\mathsf{M{:}2[\mathbb{R}][E]}\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(c) *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

1. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.
2. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$. □

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Hence $V_0 = \rho = x_K \cdots$ **(1)** from (6.4.27(p.41)).

(a) Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.4.28(p.41)) we have $V_t = K(x_K) + x_K = x_K$. Hence, by induction $V_t = x_K = \rho$ for $t \ge 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K$ from Lemma 10.2.3(p.58) (b). Accordingly, since $V_{t-1} = x_L$ for $t > 0$ from (a), we have $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$, hence for the same reason as in the proof of Tom 20.2.1(p.192) (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(c) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(c1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then, since $x_L > x_K$ from Lemma 10.2.3(p.58) (c (d)), we have $x_L > x_K = V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), thus for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.

(c2) Let $b \le 0\,(\!(\kappa \le 0)\!)$. Then $x_L \le x_K$ from Lemma 10.2.3(p.58) (c (d)). Hence, since $x_L \le x_K = V_{t-1}$ for $t > 0$ from (a), we have $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), hence $V_t - \beta V_{t-1} \le 0$ for $t > 0$ from (20.2.9(p.191)) or equivalently $V_t \le \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 0$, we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^\tau V_0$, thus $t^*_\tau = 0$ for $\tau > 0$, i.e., $\texttt{dOIT}_{\tau>0}\langle 0\rangle_\vartriangle$. ∎

$$\mathtt{S}_9\boxed{\circledS\triangle \mid \bullet\triangle \mid \bullet\blacktriangle} = \left\{\begin{array}{l}\text{For any } \tau > 0 \text{ there exists } t^\bullet > 0 \text{ such that}\\ (1)\ \boxed{\bullet\ \mathtt{dOITd}_{\tau=1}\langle 0\rangle}_{||}\ (\boxed{\bullet\ \mathtt{dOITd}_{\tau=1}\langle 0\rangle}_\blacktriangle),\\ (2)\ \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet}\langle\tau\rangle}_\triangle \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau>t^\bullet}\langle 0\rangle}_\triangle,\\ (3)\ \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet\geq\tau>1}\langle 0\rangle}_\triangle\ (\!(\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet\geq\tau>1}\langle 0\rangle}_\blacktriangle)\!).\end{array}\right\}$$

□ **Tom 20.2.4** (■ $\mathscr{A}$ {M:2[ℝ][E]}) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as to $\rightarrow \infty$.*

(b) *Let $\rho < x_L$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.

(c) *Let $\rho = x_L$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{||}$ *and* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(d) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$.
2. *Let $\beta < 1$ and $s = 0$ (\!($s > 0$)\!).*
   i. *Let $b \leq 0$ (\!($\kappa \leq 0$)\!). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$ (\!($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$)\!) . ↦ → ⓓ▲
   ii. *Let $b > 0$ (\!($\kappa > 0$)\!). Then* $\mathtt{S}_9\boxed{\circledS\triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* □↦ → ⓓ▲

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then $V_0 > x_K \cdots$ **(1)** from (6.4.27(p.41)) and $K(\rho) < 0 \cdots$ **(2)** from Lemma 10.2.2(p.57) (j1). From (6.4.28(p.41)) with $t = 1$ and from (6.4.27(p.41)) we have $V_1 - V_0 = K(V_0) = K(\rho) < 0$, hence $V_1 < V_0 \cdots$ **(3)**. In addition, from (20.2.9(p.191)) with $t = 1$ we have $V_1 - \beta V_0 = L(V_0) = L(\rho) \cdots$ **(4)** from (6.4.27(p.41)).

(a) Note (3), hence $V_0 \geq V_1$. Suppose $V_{t-1} \geq V_t$. Then, from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \geq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 0$, i.e., V_t is nonincreasing in $t \geq 0$. Let $\lambda < 1$. Note again (3). Suppose $V_{t-1} > V_t$. Then, from Lemma 10.2.2(p.57) (f) we have $V_t > K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} > V_t$ for $t > 0$, i.e., V_t is *strictly decreasing* in $t \geq 0$. Note (1), hence $V_0 \geq x_K$. Suppose $V_{t-1} \geq x_K$. Then, from (6.4.28(p.41)) and Lemma 10.2.2(p.57) (e) we have $V_t \geq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \geq x_K \cdots$ **(5)** for $t > 0$, i.e., V_t is lower bounded in t. Thus, it follows that V_t converges to a finite V as $t \rightarrow \infty$. Hence, since $V = K(V) + V$ from (6.4.28(p.41)), we have $K(V) = 0$, thus $V = x_K$ due to Lemma 10.2.2(p.57) (j1).

(b) Let $\rho < x_L$. Then, since $V_0 < x_L$ from (6.4.27(p.41)), we have $V_{t-1} < x_L$ for $t > 0$ due to (a). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.57) (a), for the same reason as in the proof of Tom 20.2.1(p.192) (c) we obtain $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.

(c) Let $\rho = x_L \cdots$ **(6)**. Then, since $L(\rho) = L(x_L) = 0$, we have $V_1 - \beta V_0 = 0$ from (4) or equivalently $V_1 = \beta V_0 \cdots$ **(7)**, hence $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{||}$. Below, let $\tau > 1$. Now, since $V_1 = K(\rho) + \rho < \rho$ from (6.4.28(p.41)) with $t = 1$ and (2), we have $V_{t-1} < \rho$ for $t > 1$ from (a), hence $V_{t-1} < x_L$ for $t > 1$ due to (6), so $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.57) (a). Accordingly, since $L(V_{t-1}) > 0$ for $\tau \geq t > 1$, we have $V_t - \beta V_t > 0$ for $\tau \geq t > 1$ due to (20.2.9(p.191)) or equivalently $V_t > \beta V_t$ for $\tau \geq t > 1$, from which we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. Hence, from (7) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0.$$

Accordingly, we obtain $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(d) Let $x_L < \rho \cdots$ **(8)**, hence $x_L < V_0 \cdots$ **(9)** from (6.4.27(p.41)). Thus, if $s =$ (\!(>)\!) 0, then $L(V_0) =$ (\!(<)\!) $0 \cdots$ **(10)** from Lemma 10.2.1(p.57) (d(\!(e1)\!)), hence $V_1 - \beta V_0 =$ (\!(<)\!) 0 from (4) or equivalently $V_1 =$ (\!(<)\!) $\beta V_0 \cdots$ **(11)**.

(d1) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $L(V_0) < 0$ from (10), hence $V_1 < \beta V_0 \cdots$ **(12)** from (20.2.9(p.191)). Now, since $x_L = x_K$ due to Lemma 10.2.3(p.58) (b), from (5) we have $V_{t-1} \geq x_L$ for $t > 0$, hence $L(V_{t-1}) \leq 0$ for $t > 0$ due to Lemma 10.2.1(p.57) (e1), thus $V_t - \beta V_{t-1} \leq 0$ for $t > 0$ from (20.2.9(p.191)). Then, since $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 0$, we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 0$, leading to

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \leq \beta^\tau V_0.$$

Hence we have $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$.

(d2) Let $\beta < 1$ and $s = 0$ (\!($s > 0$)\!).

(d2i) Let $b \leq 0$ (\!($\kappa \leq 0$)\!). Then $x_L \leq x_K$ due to Lemma 10.2.3(p.58) (c(\!(d)\!)). Hence, from (5) we have $V_{t-1} \geq x_L$ for $t > 0$, hence $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 10.2.1(p.57) (a), so $V_t - \beta V_{t-1} \leq 0$ for $t > 0$ from (20.2.9(p.191)). Then, since $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 0$, we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 0$, leading to

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \leq \beta^\tau V_0.$$

Due to (11) the inequality can be rewritten as

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 = (\!(<)\!)\ \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$ (\!($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$)\!) .

(d2ii) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K > 0 \cdots$ **(13)** from Lemma 10.2.3(p.58) (c ⟪d⟫). Hence, from (3) and (9) and from the nonincreasingness of V_t and the convergency of V_t to $V = x_K$ due to (a) we see that there exists $t^\bullet > 0$ such that

$$V_0 > V_1 \geq V_2 \geq \cdots \geq V_{t^\bullet - 1} \geq x_L > V_{t^\bullet} \geq V_{t^\bullet + 1} \geq \cdots \to x_K \cdots \mathbf{(14)}$$

or equivalently $V_0 > x_L$, $V_{t-1} \geq x_L$ for $t^\bullet \geq t > 1$, and $x_L > V_{t-1}$ for $t > t^\bullet$. Hence, we have

$$\begin{array}{lll} L(V_{t-1}) > 0, & t > t^\bullet, & \text{due to Corollary 10.2.1(p.57) (a),} \\ L(V_{t-1}) \leq 0, & t^\bullet \geq t > 1, & \text{due to Corollary 10.2.1(p.57) (a),} \\ L(V_0) = ⟪<⟫\ 0 & & \text{due to Lemma 10.2.1(p.57) (d⟪e1⟫).} \end{array}$$

Hence, from (20.2.9(p.191)) we have

$$V_t > \beta V_{t-1} \cdots \mathbf{(15)}, \quad t > t^\bullet, \qquad V_t \leq \beta V_{t-1} \cdots \mathbf{(16)}, \quad t^\bullet \geq t > 1, \qquad V_1 = ⟪<⟫\ \beta V_0 \cdots \mathbf{(17)}.$$

⟨A⟩ Let $\tau = 1$. Then, since $V_1 = ⟪<⟫\ \beta V_0$ due to (17), we have $\boxed{\bullet\ \mathtt{dOITd}_{\tau=1}\langle 0\rangle}_{\parallel}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau=1}\langle 0\rangle}_{\blacktriangle}$), hence (1) of $\mathtt{S}_9$ holds.

⟨B⟩ Let $t^\bullet \geq \tau > 1$. Then, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$ from (16), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1,$$

hence

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 = ⟪<⟫\ \beta^\tau V_0 \cdots \mathbf{(18)}, \quad t^\bullet \geq \tau > 0,$$

from (17) or equivalently

$$I_\tau^\tau \leq I_\tau^{\tau-1} \leq \cdots \leq I_\tau^1 = ⟪<⟫\ I_\tau^0 \cdots \mathbf{(19)}, \quad t^\bullet \geq \tau > 0.$$

Thus $t_\tau^* = 0$ for $t^\bullet \geq \tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet \geq \tau > 1}\langle 0\rangle}_{\vartriangle}$ ⟪$\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet \geq \tau > 1}\langle 0\rangle}_{\blacktriangle}$⟫, hence (2) of $\mathtt{S}_9$ holds. Now, from (18) with $\tau = t^\bullet$ we have

$$V_{t^\bullet} \leq \beta V_{t^\bullet - 1} \leq \cdots \leq \beta^{t^\bullet - 1} V_1 = ⟪<⟫\ \beta^{t^\bullet} V_0 \cdots \mathbf{(20)}.$$

⟨C⟩ Let $\tau > t^\bullet$ (> 0), hence $\tau > 1$. From (15) with $\tau \geq t > t^\bullet$ we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet - 1} V_{t^\bullet + 1} > \beta^{\tau - t^\bullet} V_{t_\tau^\bullet} \cdots \mathbf{(21)}, \quad \tau > t^\bullet.$$

Combining (21) and (20) leads to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet - 1} V_{t^\bullet + 1} > \beta^{\tau - t^\bullet} V_{t^\bullet} \leq \beta^{\tau - t^\bullet + 1} V_{t^\bullet - 1} \leq \cdots \leq \beta^{\tau-1} V_1 = ⟪<⟫\ \beta^\tau V_0, \quad \tau > t^\bullet,$$

or equivalently

$$I_\tau^\tau > I_\tau^{\tau-1} > I_\tau^{\tau-2} > \cdots > I_\tau^{t^\bullet + 1} > I_\tau^{t^\bullet} \leq I_\tau^{t^\bullet - 1} \leq \cdots \leq I_\tau^1 = ⟪<⟫\ I_\tau^0 \cdots \mathbf{(22)}, \quad \tau > t^\bullet.$$

Hence we have $\boxed{ⓢ\ \mathtt{dOITs}_{\tau > t^\bullet}\langle \tau\rangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet}\langle 0\rangle}$, thus (3) of $\mathtt{S}_9$ holds. ∎

20.2.3.3 Market Restriction

20.2.3.3.1 Positive Restriction

20.2.3.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.2.1** ($\mathscr{A}\{$M:2[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\rho < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

● *Proof* The same as Tom 20.2.1(p.192) due to Lemma 17.4.4(p.118). ∎

20.2.3.3.1.2 Case of $\beta < 1$ or $s > 0$

□ **Pom 20.2.2** ($\mathscr{A}\{$M:2[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a \leq \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $a \leq \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
   - ii. *Let $\rho < a$.*
     1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
     2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
4. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $a \leq \rho$.*
     1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.*IvsD*
     2. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.192) | ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲ | *is true.*
   - ii. *Let $\rho < a$.*
     1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
          2. *Let $\beta\mu \leq s$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
          2. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.192) | ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲ | *is true.*
     2. *Let $(\lambda\beta\mu - s)/\delta > a$.*
        - i. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
        - ii. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.192) | ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲ | *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a).

(a-c2ii2) The same as Tom 20.2.2(p.192) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1,c3ii1i1,c3ii1ii1,c3ii2i) of Tom 20.2.2(p.192).

(c4-c4ii2ii) Let $\beta < 1$ and $s < 0$. Then, due to (2) it suffices to consider only (c3-c3ii2ii) of Tom 20.2.2(p.192) with κ. ∎

□ **Pom 20.2.3** ($\mathscr{A}\{$M:2[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t > 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $\lambda\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
2. *Let $\lambda\beta\mu \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a).

(a,b) The same as Tom 20.2.3(p.195) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.2.3(p.195).

(d,d2) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (c1,c2) of Tom 20.2.3(p.195). ∎

□ Pom **20.2.4** ($\mathscr{A}\{$M:2[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho < x_L$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_▲$ → → ⓢ

(c) *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_{\|\|}$ *and* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_▲$ *for $\tau > 1$.*

(d) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_△$.
2. *Let $\beta < 1$ and $s = 0$. Then* S$_9$(p.196) [ⓢ△ | ●△ | ●▲] *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\lambda\beta\mu \leq s$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_▲$.
   ii. *Let $\lambda\beta\mu > s$. Then* S$_9$(p.196) [ⓢ△ | ●△ | ●▲] *is true* (see *Numerical Example* 6(p.219))

● *Proof* Suppose $a > 0$. Then $b > a > 0 \cdots (1)$. We have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.59) (a).

(a-d1) The same as Tom 20.2.4(p.196) (a-d1).

(d2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (d2ii) of Tom 20.2.4(p.196).

(d3,d3ii) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (d2i,d2ii) of Tom 20.2.4(p.196) with κ. ∎

20.2.3.3.2 Mixed Restriction

Omitted.

20.2.3.3.3 Negative Restriction

Omitted.

20.2.4 M̃:2[ℝ][E]

Due to Lemma 20.2.1(p.191) (a), we see that the following Tom's 20.2.5(p.199) – 20.2.8(p.200) can be obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom's 20.2.1(p.192) – 20.2.4(p.196) (see Theorem 20.2.1(p.191)).

20.2.4.1 Analysis

20.2.4.1.1 Case of $\beta = 1$ and $s = 0$

□ Tom **20.2.5** (⊡$\mathscr{A}\{$M̃:2[ℝ][E]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \leq a$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|\|}$.

(c) *Let $\rho > a$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_▲$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.2.1(p.192). ∎

20.2.4.1.2 Case of $\beta < 1$ or $s > 0$

□ Tom **20.2.6** (⊡$\mathscr{A}\{$M̃:2[ℝ][E]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$ or $b > \rho$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_△$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. [ⓢ dOITs$_1\langle 1\rangle$]$_▲$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $b > \rho$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_▲$.
   ii. *Let $\rho \geq b$.*
      1. *Let $(\lambda\mu + s)/\lambda \geq b$.*
         i. *Let $\lambda = 1$. Then* [⊚ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|\|}$.
         ii. *Let $\lambda < 1$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_▲$.
      2. *Let $(\lambda\mu + s)/\lambda < b$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_▲$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $b > \rho$.*
      1. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_▲$.

2. *Let* $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. *Then* $\texttt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

ii. *Let* $\rho \geq b$.

1. *Let* $(\lambda\beta\mu + s)/\delta \geq b$.

i. *Let* $\lambda = 1$.

1. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

2. *Let* $a \geq 0\,(\!(\tilde{\kappa} \geq 0)\!)$. *Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

ii. *Let* $\lambda < 1$.

1. *Let* $a \leq 0\,(\!(\tilde{\kappa} \leq 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

2. *Let* $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. *Then* $\texttt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

2. *Let* $(\lambda\beta\mu + s)/\delta < b$.

i. *Let* $a \leq 0\,(\!(\tilde{\kappa} \leq 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

ii. *Let* $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. *Then* $\texttt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.2.2(p.192). ▮

□ Tom **20.2.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

1. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

2. *Let* $a \geq 0\,(\!(\tilde{\kappa} \geq 0)\!)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.2.3(p.195). ▮

□ Tom **20.2.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.

1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

i. *Let* $a \geq 0\,(\!(\tilde{\kappa} \geq 0)\!)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $(\!(\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle})\!)$.

ii. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then* $\texttt{S}_9$ [ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 20.2.4(p.196). ▮

20.2.4.2 Market Restriction

20.2.4.2.1 Positive Restriction

20.2.4.2.1.1 Case of $\boldsymbol{\beta = 1}$ and $\boldsymbol{s = 0}$

□ Pom **20.2.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.

(b) *Let* $\rho \leq a$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\rho > a$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as Tom 20.2.5(p.199) due to Lemma 17.4.4(p.118). ▮

20.2.4.2.1.2 Case of $\boldsymbol{\beta < 1}$ or $\boldsymbol{s > 0}$

□ Pom **20.2.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$ *or* $b \geq \rho$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau}\langle 0\rangle}_{\vartriangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.

1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.

2. *Let* $\beta = 1$.

i. *Let* $b \geq \rho$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau}\langle\tau\rangle}_{\blacktriangle}$.

ii. *Let* $\rho > b$.

1. *Let* $(\lambda\mu + s)/\lambda \geq b$.
   i. *Let* $\lambda = 1$. *Then* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let* $\lambda < 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
2. *Let* $(\lambda\mu + s)/\lambda < b$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s = 0$. *Then we have* $\texttt{S}_8$(p.192) $\boxed{Ⓢ\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$.
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $b > \rho$. *Then* $\texttt{S}_8$(p.192) $\boxed{Ⓢ\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$ *is true.*
   ii. *Let* $\rho \geq b$.
      1. *Let* $(\lambda\beta\mu + s)/\delta \geq b$.
         i. *Let* $\lambda = 1$. *Then* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
         ii. *Let* $\lambda < 1$. *Then* $\texttt{S}_8$(p.192) $\boxed{Ⓢ\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$ *is true.*
      2. *Let* $(\lambda\beta\mu + s)/\delta < b$. *Then* $\texttt{S}_8$(p.192) $\boxed{Ⓢ\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$ *is true.* □

● *Proof* Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$. Then $\tilde{\kappa} = s \cdots (3)$ from Lemma 12.6.6(p.83) (a).

(a-c2ii2) The same as Tom 20.2.6(p.199) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu + s)/\delta \geq b$. Then, since $\lambda\beta\mu/\delta \geq b$, we have $\lambda\beta\mu \geq \delta b$ from (10.2.2 (1) (p.56)), hence $\lambda\beta\mu \geq \delta b \geq \lambda b$ due to (2), so $\beta\mu \geq b$, which contradicts [3(p.118)]. Thus, it must be that $(\lambda\beta\mu + s)/\delta < b$. From this it suffices to consider only (c3i2,c3ii2ii) of Tom 20.2.6(p.199).

(c4-c4ii2) Let $\beta < 1$ and $s > 0$. Then it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of Tom 20.2.2(p.192) with κ. ▮

□ Pom **20.2.7** $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]^+\})$ *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 12.6.6(p.83) (a).

(a,b) The same as Tom 20.2.7(p.200) (a,b).

(c) If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.2.7(p.200) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.2.7(p.200) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ▮

□ Pom **20.2.8** $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]^+\})$ *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.
   1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
   2. *Let* $\beta < 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $(\!(\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle})\!)$.

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 12.6.6(p.83) (a).

(a-d1) The same as Tom 20.2.8(p.200) (a-d1).

(d2) If $s = 0$, due to (1) it suffices to consider only (d2i) of Tom 20.2.8(p.200) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of Tom 20.2.8(p.200) (d2i) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ▮

20.2.4.2.2 Mixed Restriction

Omitted.

20.2.4.2.3 Negative Restriction

Omitted.

20.2.5 M:2[$\mathbb{P}$][E]

20.2.5.1 Preliminary

From (6.4.33(p.41)) and from (5.1.21(p.26)) and (5.1.20(p.26)) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 1. \tag{20.2.12}$$

From (6.4.32(p.41)) we have

$$V_1 - \beta V_0 = V_1 - \beta\rho = \lambda\beta \max\{0, a - \rho\} - s. \tag{20.2.13}$$

20.2.5.2 Analysis

20.2.5.2.1 Case of $\beta = 1$ and $s = 0$

Let $\beta = 1$ and $s = 0$. Then, from (20.2.12(p.201)) and (5.1.20(p.26)) we have

$$V_t - \beta V_{t-1} = \lambda T(V_{t-1}) \geq 0, \quad t > 1, \tag{20.2.14}$$

due to Lemma 13.2.1(p.93) (g). From (6.4.32(p.41)) we have

$$\begin{aligned} V_1 &= \lambda \max\{0, a - \rho\} + \rho & (20.2.15) \\ &= \max\{\rho, \lambda a + (1-\lambda)\rho\}. & (20.2.16) \end{aligned}$$

20.2.5.2.1.1 Case of $\rho \leq a^\star$

In this case, Theorem 20.2.2(p.191) holds due to Lemma 20.2.1(p.191) (c). Hence, Proposition 20.2.1 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom 20.2.1(p.192).

Proposition 20.2.1 ($\rho \leq a^\star$) Assume $\rho \leq a^\star$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$. □

● *Proof by analogy* Assume $\rho \leq a^\star$. Let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.2.1(p.192) (a).

(b) Since (b,c) of Tom 20.2.1(p.192) have none of a and μ, even if $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied the two assertions, no change occurs (see Lemma 13.6.1(p.99)). However, since $\rho \leq a^\star < a < b$ due to the assumption $\rho \leq a^\star$ and Lemma 13.2.1(p.93) (n), it follows that only (c) of Tom 20.2.1(p.192) holds. ∎

20.2.5.2.1.2 Case of $b \leq \rho$

In this case, Theorem 20.2.2(p.191) holds due to Lemma 20.2.1(p.191) (c). Hence, Proposition 20.2.2 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom 20.2.1(p.192).

Proposition 20.2.2 ($b \leq \rho$) Assume $b \leq \rho$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

● *Proof by analogy* Assume $b \leq \rho$. Let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.2.1(p.192) (a).

(b) Due to the assumption $b \leq \rho$, only (b) of Tom 20.2.1(p.192) holds. ∎

20.2.5.2.1.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.2.2(p.191) does not always hold due to Lemma 20.2.1(p.191) (d). Hence, Proposition 20.2.3 below must be directly proven.

Proposition 20.2.3 ($a^\star < \rho < b$) Assume $a^\star < \rho < b$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho$. Then $\boxed{\bullet\ \text{dOITd}_1\langle 0\rangle}_{\parallel}$ and $\boxed{\text{Ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c) Let $\rho < a$. Then $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$. □

● *Proof* Assume $a^\star < \rho < b \cdots$ **(1)** and let $\beta = 1$ and $s = 0$. Then $L(x) = K(x) = \lambda T(x) \geq 0 \cdots$ **(2)** for any x from (5.1.20(p.26)) and (5.1.21(p.26)) and from Lemma 13.2.1(p.93) (g). Since $V_0 < b$ from (1) and (6.4.31(p.41)), we have $L(V_0) = \lambda T(V_0) = \lambda T(\rho) > 0 \cdots$ **(3)** from (2) and Lemma 13.2.1(p.93) (g). Then, since $\rho < b$ and $a < b$, from (20.2.16(p.202)) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ from (1), we have $V_t < K(b) + b$ from (6.4.33(p.41)) and Lemma 13.2.3(p.96) (h), hence $V_t < \beta b - s$ from (13.2.12 (2) (p.96)), so $V_{t-1} < b$ due to the assumption "$\beta = 1$ and $s = 0$". Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $T(V_{t-1}) > 0 \cdots$ **(4)** for $t > 1$ from Lemma 13.2.1(p.93) (g). Thus $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (20.2.14(p.202)) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots \textbf{(5)}, \qquad \tau > 1.$$

In addition, from (2) we have $L(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots$ **(6)** for $t > 1$ due to (4), so $L(V_{t-1}) > 0$ for $t > 0$ due to (3).

(a) From (20.2.15(p.202)) and (6.4.31(p.41)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a - \rho\} \geq 0$, hence $V_1 \geq V_0 \cdots$ **(7)**. From (6.4.33(p.41)) with $t = 2$ we have $V_2 - V_1 = K(V_1) > 0$ due to (6) with $t = 2$, hence $V_2 > V_1$, so $V_2 \geq V_1 \cdots$ **(8)**. Suppose $V_t \geq V_{t-1}$. Then from (6.4.33(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_{t+1} = K(V_t) + V_t \geq K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 1$. From this and (7) we have $V_t \geq V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \geq 0$.

(b) Let $a \le \rho$, hence $V_1 = \lambda \max\{0, a-\rho\} + \rho = \rho$ from (6.4.32(p.41)), so $V_1 < b$ due to (1). Then, since $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$, we have $V_1 = \beta V_0 \cdots$ (9), hence $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$. Below let $\tau > 1$. Then, from (5) and (9) we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0,$$

hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c) Let $\rho < a$. Then, since $V_1 = \lambda(a-\rho) + \rho$ due to (6.4.32(p.41)), we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a-\rho) > 0$, i.e., $V_1 > \beta V_0$, hence $t_1^* = 1 \cdots$ (10). Let $\tau > 1$. Then, from (5) we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0, \qquad \tau > 1,$$

hence $t_\tau^* = \tau$ for $\tau > 1$, hence $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. From this and (10) we have $t_\tau^* = \tau$ for $\tau > 0$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$. ∎

20.2.5.2.1.4 Integration of Propositions 20.2.1(p.202) –20.2.3(p.202)

□ **Tom 20.2.9** ($\mathscr{A}$\{M:2[ℙ][E]\}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\rho \le a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let $b \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\rho < a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$. □

● *Proof* (a) The same as Propositions 20.2.1(p.202) (a), 20.2.2(p.202) (a), and 20.2.3(p.202) (a).

(b) The same as Proposition 20.2.1(p.202) (b).

(c) The same as Proposition 20.2.2(p.202) (b).

(d-d2) The same as Proposition 20.2.3(p.202) (b,c). ∎

Corollary 20.2.1 (M:2[ℙ][E]) *Let $\beta = 1$ and $s = 0$. Then, z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Lemma 20.2.9(p.203) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

20.2.5.2.2 Case of $\beta < 1$ or $s > 0$

20.2.5.2.2.1 Case of $\rho \le a^\star$

In this case, Theorem 20.2.2(p.191) holds due to Lemma 20.2.1(p.191) (c), hence Tom's 20.2.10(p.203) –20.2.12(p.204) below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.130))) to Tom's 20.2.2(p.192)–20.2.4(p.196). In the proofs below, let us represent what results from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to a given Tom by Tom′ (see (20.1.42(p.168))).

□ **Tom 20.2.10** (⊡$\mathscr{A}$\{M:2[ℙ][E]\}) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $(\lambda a - s)/\lambda \le a^\star$.*
     1. *Let $\lambda = 1$. Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   - ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $(\lambda\beta a - s)/\delta \le a^\star$.*
     1. *Let $\lambda = 1$.*
        - i. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
        - ii. *Let $b \le 0$ ($\kappa \le 0$). Then* $\boxed{◎\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
     2. *Let $\lambda < 1$.*
        - i. *Let $b \ge 0$ ($\kappa \ge 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
        - ii. *Let $b < 0$ ($\kappa < 0$). Then* $\texttt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
   - ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
     1. *Let $b \ge 0$ ($\kappa \ge 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
     2. *Let $b < 0$ ($\kappa < 0$). Then* $\texttt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.2.2(p.192). Then "$a < \rho$" in Tom 20.2.2(p.192) (c2i,c3i) changes into "$a^\star < \rho$" in the Tom′, which contradicts the assumption $\rho \le a^\star$. Accordingly, removing all assertions with "$a^\star < \rho$" from the Tom′ leads to Tom 20.2.10 above. ∎

Corollary 20.2.2 (M:2[ℙ][E]) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$. Then, z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.10(p.203) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

□ **Tom 20.2.11** (⊡𝒜{M:2[ℙ][E]}) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆.

1. *Let* $b > 0$ ⦅$\kappa > 0$⦆. *Then* [ⓢ $\mathtt{dOITs}_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$.
2. *Let* $b \le 0$ ⦅$\kappa \le 0$⦆. *Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\vartriangle}$. □

● *Proof by analogy* The same as Tom 20.2.3(p.195) due to Lemma 13.6.1(p.99). ∎

Corollary 20.2.3 (M:2[ℙ][E]) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then, the optimal price to propose is given by $z_t = z(\rho)$ for $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.11(p.204) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

□ **Tom 20.2.12** (⊡𝒜{M:2[ℙ][E]}) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.

(b) *Let* $\rho < x_L$. *Then* [ⓢ $\mathtt{dOITs}_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$.

(c) *Let* $\rho = x_L$. *Then* [● $\mathtt{dOITd}_{1}\langle 0\rangle$]$_{\vartriangle}$ *and* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$.

(d) *Let* $\rho > x_L$.

1. *Let* $\beta = 1$. *Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\vartriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆.
   i. *Let* $b \le 0$ ⦅$\kappa \le 0$⦆. *Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\vartriangle}$ ⦅[● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\blacktriangle}$⦆.
   ii. *Let* $b > 0$ ⦅$\kappa > 0$⦆. *Then* $\mathtt{S}_9$ [ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by analogy* The same as Tom 20.2.4(p.196) due to Lemma 13.6.1(p.99). ∎

Corollary 20.2.4 (M:2[ℙ][E]) *Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then, the optimal price to propose z_t is nonincreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.12(p.204) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

20.2.5.2.2.2 Case of $b \le \rho$

In this case, Theorem 20.2.2(p.191) holds due to Lemma 20.2.1(p.191) (c). Hence Tom's 20.2.13-20.2.15 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom's 20.2.2(p.192)-20.2.4(p.196).

□ **Tom 20.2.13** (⊡𝒜{M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \le \rho$. *Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\vartriangle}$.

(c) *Let* $\rho < x_L$.

1. [ⓢ $\mathtt{dOITs}_{1}\langle 1\rangle$]$_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆.
   i. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆. *Then* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$.
   ii. *Let* $b < 0$ ⦅$\kappa < 0$⦆. *Then* $\mathtt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.2.2(p.192). Then "$\rho \le a$" in (c2ii,c3ii) of Tom 20.2.2(p.192) changes into "$\rho \le a^\star$" in the Tom′, hence $\rho \le a^\star < a < b$ due to Lemma 13.2.1(p.93) (n), which contradicts the assumption $b \le \rho$. Accordingly, removing all assertions with "$\rho \le a$" from the Tom′ leads to Tom 20.2.13 above. ∎

Corollary 20.2.5 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$. Then, the optimal price to propose z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.13(p.204) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

□ **Tom 20.2.14** ($\boxdot\mathscr{A}${M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

1. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. □

● *Proof by analogy* The same as Tom 20.2.3(p.195) due to Lemma 13.6.1(p.99). ▮

Corollary 20.2.6 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then, the optimal price to propose is given by $z_t = z(\rho)$ for $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.14(p.205) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ▮

□ **Tom 20.2.15** ($\boxdot\mathscr{A}${M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let* $\rho > x_L$.

1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
   i. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $(\!(\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle})\!)$.
   ii. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\texttt{S}_9$ | ⓢ△ | ●△ | ●▲ | *is true.* □

● *Proof by analogy* Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. In this case, even if $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to Tom 20.2.4(p.196), it can be easily confirmed that no change occurs (see Lemma 13.6.1(p.99)). However, if the condition $\rho < x_L$ is added, we encounter the following contradiction. Then we have $b \le \rho < x_L \cdots$ **(1)**. Now, since $0 = L(x_L) = \lambda\beta T(x_L) - s$ and $T(x_L) = 0$ from Lemma 13.2.1(p.93) (g), we have $0 = -s$, hence $s = 0$, so we have $x_L = b$ due to Lemma 13.2.2(p.96) (d), which is a contradicts (1). Accordingly, the condition $\rho < x_L$ becomes impossible. This result implies that the assertion (b) with $\rho \ge x_L$ in Tom 20.2.4(p.196) must be omitted; accordingly, it follows that we have Tom 20.2.15 above. ▮

Corollary 20.2.7 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then, the optimal price to propose z_t is nonincreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.2.15(p.205) (a) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ▮

20.2.5.2.2.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.2.2(p.191) does not always hold due to Lemma 20.2.1(p.191) (d). Hence, Tom 20.2.16(p.206) below must be directly proven. For explanatory convenience, let us define:

$\texttt{S}_{10}$ | ⓢ△ | ●△ | =
- We have:
- (1) Let $\lambda\max\{0, a-\rho\} < s$. Then $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
- (2) Let $\lambda\max\{0, a-\rho\} \ge s$. Then $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.

$\texttt{S}_{11}$ | ⓢ△ | ⓢ▲ | ⊚△ | ●△ | =
- There exists $t^\bullet_\tau > 1$ such that:
- (1) If $\lambda\beta\max\{0, a-\rho\} < s$, then
  - i. $\boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau\ge\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle}_{\vartriangle}$,
  - ii. $\boxed{⊚\ \texttt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>t^\bullet_\tau}\langle 0\rangle}_{\vartriangle}$.
- (2) If $\lambda\beta\max\{0, a-\rho\} \ge s$, then
  - i. $\boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau\ge\tau>1}\langle\tau\rangle}_{\blacktriangle}$,
  - ii. $\boxed{⊚\ \texttt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\vartriangle}$.

$\texttt{S}_{12}$ | ⓢ△ | ⓢ▲ | ⊚△ | ●△ | ●▲ | =
- There exists $t^\bullet_\tau > 1$ such that:
- (1) If $\lambda\beta\max\{0, a-\rho\} < s$, then
  - i. $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau\ge\tau>0}\langle 0\rangle}_{\blacktriangle}$,
  - ii. $\boxed{ⓢ\ \texttt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>t^\bullet_\tau}\langle 0\rangle}_{\vartriangle}$.
- (2) If $\lambda\beta\max\{0, a-\rho\} \ge s$, then
  - i. $\boxed{⊚\ \texttt{ndOIT}_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle}_{\parallel}$,
  - ii. $\boxed{ⓢ\ \texttt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle}$.

$$\mathtt{S}_{13}\ \boxed{ⓢ\triangle \mid ⓞ\ \triangle \mid \bullet\triangle \mid \bullet\blacktriangle} = \left\{ \begin{array}{l} \text{There exists } t^\bullet_\tau > 1 \text{ and } t^\star_\tau > 1 \text{ such that:} \\ (1)\ \text{If } \lambda\beta\max\{0, a-\rho\} < s, \text{ then} \\ \quad \text{i. } \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_\blacktriangle, \\ \quad \text{ii. } \boxed{ⓢ\ \mathtt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\triangle \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle}_\triangle. \\ (2)\ \text{If } \lambda\beta\max\{0, a-\rho\} \geq s, \text{ then} \\ \quad \text{i. } \boxed{ⓞ\ \mathtt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_\triangle, \\ \quad \text{ii. } \boxed{ⓢ\ \mathtt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\triangle \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle}_\triangle. \end{array} \right\}$$

For convenience of reference, below let us copy (6.4.32(p.41))

$$V_1 = \lambda\beta\max\{0, a-\rho\} + \beta\rho - s. \tag{20.2.17}$$

□ **Tom 20.2.16** ($\blacksquare\mathscr{A}${M:2[ℙ][E]}) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\triangle$, *or else* $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let $\tau > 1$.*

(b) *Let $V_1 \leq x_K$.*

1. *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $V_1 \geq x_L$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\triangle$, *or else* $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$.
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then* $\mathtt{S}_{10}\ \boxed{ⓢ\triangle \mid \bullet\triangle}$ *is true.*
   - ii. *Let $\beta < 1$ and $s = 0\,(s > 0)$.*
     1. *Let $b > 0\,(\kappa > 0)$. Then* $\mathtt{S}_{10}\ \boxed{ⓢ\triangle \mid \bullet\triangle}$ *is true.*
     2. *Let $b = 0\,(\kappa = 0)$. If $\lambda\beta\max\{0, a-\rho\} < s$, then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_\triangle$ *or* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\triangle$, *or else* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_\triangle$.
     3. *Let $b < 0\,(\kappa < 0)$. Then* $\mathtt{S}_{11}\ \boxed{ⓢ\triangle \mid ⓢ\blacktriangle \mid ⓞ\ \triangle \mid \bullet\triangle}$ *is true.*

(c) *Let $V_1 > x_K$.*

1. *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $\beta = 1$. If $\lambda\max\{0, a-\rho\} < s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\blacktriangle$, *or else* $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$. ↦ → ⓓ$_\blacktriangle$
3. *Let $\beta < 1$ and $s = 0\,(s > 0)$.*
   - i. *Let $b > 0\,(\kappa > 0)$.*
     1. *Let $V_1 < x_L$. Then* $\mathtt{S}_{10}\ \boxed{ⓢ\triangle \mid \bullet\triangle}$ *is true.*
     2. *Let $V_1 = x_L$. Then* $\mathtt{S}_{12}\ \boxed{ⓢ\triangle \mid ⓢ\blacktriangle \mid ⓞ\ \triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* ↦ → ⓓ$_\blacktriangle$
     3. *Let $V_1 > x_L$. Then* $\mathtt{S}_{13}\ \boxed{ⓢ\triangle \mid ⓞ\ \triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* ↦ → ⓓ$_\blacktriangle$
   - ii. *Let $b \leq 0\,(\kappa \leq 0)$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\triangle$, *or else* $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$. ∎

● *Proof* Assume $a^\star < \rho < b \cdots (1)$ and let $\beta < 1$ or $s > 0$.

(a) If $\lambda\beta\max\{0, a-\rho\} \leq s$, then $V_1 \leq \beta V_0$ from (20.2.13(p.201)) or equivalently $V_1 \leq \beta V_0 \cdots (2)$, hence $t^*_1 = 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\triangle \cdots (3)$, or else $V_1 > \beta V_0 \cdots (4)$, hence $t^*_1 = 1$, i.e., $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle \cdots (5)$. Below let $\tau > 1$.

(b) Let $V_1 \leq x_K \cdots (6)$, hence $K(V_1) \geq 0 \cdots (7)$ from Lemma 13.2.3(p.96) (j1).

(b1) From (6.4.33(p.41)) with $t = 2$ we have $V_2 = K(V_1) + V_1 \geq V_1$ due to (7). Suppose $V_t \geq V_{t-1}$. Then $V_{t+1} \geq K(V_{t-1}) + V_{t-1} = V_t$ from Lemma 13.2.3(p.96) (e), hence by induction $V_t \geq V_{t-1}$ for $t > 1$, so V_t is nondecreasing in $t > 0$. Note (6). Suppose $V_{t-1} \leq x_K$. Then, from (6.4.33(p.41)) and Lemma 13.2.3(p.96) (e) we have $V_t \leq K(x_K) + x_K = x_K$. Hence, by induction $V_t \leq x_K \cdots (8)$ for $t > 0$, i.e., V_t is upper bounded in t, hence V_t converges to a finite V as $t \to \infty$. Then, since $V = K(V) + V$ as $\tau \to \infty$ from (6.4.33(p.41)), we have $V = K(V) + V$, hence $K(V) = 0$ thus $V = x_K$ from Lemma 13.2.3(p.96) (j1).

(b2) Let $V_1 \geq x_L$. Then, since $x_L \leq V_{t-1}$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), thus $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$ from (20.2.12(p.201)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \cdots (9), \qquad \tau > 1.$$

(1) Let $\lambda\beta\max\{0, a-\rho\} \leq s$. Then, from (2) and (9) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1} V_1 \leq \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\triangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (4) and (9) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1} V_1 > \beta^\tau V_0,$$

hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{ⓞ\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$.

(b3) Let $V_1 < x_L \cdots (10)$.

(b3i) Let $\beta = 1 \cdots (11)$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K \cdots (12)$ from Lemma 13.2.4(p.97) (b), hence $V_{t-1} \leq x_L$ for $t > 1$ due to (8). Accordingly, since $V_{t-1} \leq x_L$ for $\tau \geq t > 1$, we have $L(V_{t-1}) \geq 0$ for $\tau \geq t > 1$ from Lemma 13.2.2(p.96) (e1), hence $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$ from (20.2.12(p.201)), so

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \cdots (13), \quad \tau > 1.$$

(A) Let $\lambda \max\{0, a-\rho\} < s$, hence $\lambda\beta \max\{0, a-\rho\} < s$ due to (11). Then $V_1 - \beta V_0 < 0 \cdots (14)$ from (20.2.13(p.201)) or equivalently $V_1 < \beta V_0 \cdots (15)$. Hence, from (13) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 < \beta^\tau V_0 \cdots (16), \quad \tau > 1.$$

Thus, we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$, hence (1) of $\texttt{S}_{10}$ is true.

(B) Let $\lambda \max\{0, a-\rho\} \geq s$, hence $\lambda\beta \max\{0, a-\rho\} \geq s$ due to (11). Then $V_1 - \beta V_0 \geq 0$ from (20.2.13(p.201)) or equivalently $V_1 \geq \beta V_0$ from (20.2.13(p.201)). Then, from (13) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$, thus (2) of $\texttt{S}_{10}$ holds.

(b3ii) Let $\beta < 1 \cdots (17)$ and $s = 0\,(\!(s > 0)\!)$.

(b3ii1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots (18)$ from Lemma 13.2.4(p.97) (c (d)). Accordingly, from (8) we have $V_{t-1} \leq x_K < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), thus $L(V_{t-1}) > 0$ for $\tau \geq t > 1$. Accordingly, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$ from (20.2.12(p.201)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (19) \quad \tau > 1.$$

(1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then for the same reason as in (A) we have (1) of $\texttt{S}_{10}$.

(2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then for the same reason as in (B) we have (2) of $\texttt{S}_{10}$.

(b3ii2) Let $b = 0\,(\!(\kappa = 0)\!)$. Then $x_L = x_K$ from Lemma 13.2.4(p.97) (c (d)). Accordingly, from (6) and (b1) we have $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} \leq x_K = x_L$ for $\tau \geq t > 1$. Therefore, from Corollary 13.2.1(p.96) (b) we have $L(V_{t-1}) \geq 0 \cdots (20)$ for $\tau \geq t > 1$, hence $V_t - \beta V_{t-1} \geq 0$ for $\tau \geq t > 1$ from (20.2.12(p.201)) or equivalently $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$, leading to

$$V_t \geq \beta V_{t-1} \geq \cdots \geq \beta^{t-1} V_1.$$

(1) Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, since $V_1 \leq \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \leq \beta^\tau V_0,$$

hence $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$.

(2) Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 > \beta^\tau V_0,$$

hence $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.

(b3ii3) Let $b < 0\,(\!(\kappa < 0)\!)$, hence $x_L < x_K \leq 0 \cdots (21)$ from Lemma 13.2.4(p.97) (c (d)). Then, from (10) we have $V_1 < x_L < x_K = V$ due to (b1). Accordingly, due to the nondecreasing of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} \leq V_{t^\bullet_\tau + 1} \leq \cdots.$$

Hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \leq V_{t-1}$ for $t > t^\bullet_\tau$. Therefore, from Corollary 13.2.1(p.96) (a) we have

$$L(V_{t-1}) > 0 \cdots (22), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \leq 0 \cdots (23), \quad t > t^\bullet_\tau.$$

- Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0$ for $\tau \geq t > 1$ from (22), we have $V_t - \beta V_{t-1} >$ for $\tau \geq t > 1$ from (20.2.12(p.201)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, so

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (24).$$

(1) Let $\lambda\beta \max\{0, a-\rho\}\rho < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0$$

from (24), hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau>1}\langle 0\rangle}_{\vartriangle}$. Accordingly (1i) of $\texttt{S}_{11}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\}\rho \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \geq \beta^\tau V_0$$

from (24), hence $t^*_\tau = \tau$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle$. Accordingly (2i) of $\mathtt{S}_{11}$ holds.

∘ Let $\tau > t^\bullet_\tau$. Since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (23), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (20.2.12(p.201)), hence

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (25), \quad \tau > t^\bullet_\tau.$$

From (22) and (20.2.12(p.201)) we have $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau-1} V_1 \cdots (26).$$

From (25) and (26) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} > \cdots > \beta^{\tau-1}V_1. \cdots (27)$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} > \cdots > \beta^{\tau-1}V_1 < \beta^\tau V_0,$$

Hence, we have $t^*_\tau = t^\bullet_\tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_\vartriangle$. Accordingly (1ii) of $\mathtt{S}_{11}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.201)), from (27) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} > \cdots > \beta^{\tau-1}V_1 \geq \beta^\tau V_0,$$

hence $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle$. Accordingly (2ii) of $\mathtt{S}_{11}$ holds.

(c) Let $V_1 > x_K \cdots (28)$, hence $K(V_1) < 0 \cdots (29)$ due to Lemma 13.2.3(p.96) (j1).

(c1) From (6.4.33(p.41)) with $t = 2$ we have $V_2 = K(V_1) + V_1 < V_1 \cdots (30)$ due to (29), hence $V_2 \leq V_1$. Suppose $V_t \leq V_{t-1}$. Then, from Lemma 13.2.3(p.96) (e) we have $V_{t+1} = K(V_t) + V_t \leq K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \leq V_{t-1}$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Note (28), hence $V_1 \geq x_K$. Suppose $V_{t-1} \geq x_K$. Then, since $V_t \geq K(x_K) + x_K = x_K$ from Lemma 13.2.3(p.96) (e), by induction we have $V_t \geq x_K \cdots (31)$ for $t > 0$, i.e., V_t is lower bounded in t, hence V_t converges to a finite V. Then, we have $V = x_K$ for the same reason as in the proof of (b1).

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then, since $x_L = x_K \cdots (32)$ from Lemma 13.2.4(p.97) (b), we have $V_{t-1} \geq x_L$ for $t > 1$ from (31). Accordingly $L(V_{t-1}) \leq 0$ for $t > 1$ from Lemma 13.2.2(p.96) (e1), hence $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, so $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$ from (20.2.12(p.201)), leading to $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1$.

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau > 1}\langle 0\rangle}_\blacktriangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.201)) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \geq \beta^\tau V_0,$$

hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau > 1}\langle 1\rangle}_\vartriangle$.

(c3) Let $\beta < 1 \cdots (33)$ and $s = 0$ ($s > 0$).

(c3i) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots (34)$ from Lemma 13.2.4(p.97) (c (d)).

(c3i1) Let $V_1 < x_L$, hence $x_L > V_{t-1}$ for $t > 1$ from (c1). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), we have $V_t - \beta V_{t-1} > 0$ for $t > 1$ due to (20.2.12(p.201)) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$, hence $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, leading to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \cdots (35).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then for the same reason as in (A(p.207)) we have (1) of $\mathtt{S}_{10}$.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then for the same reason as in (B(p.207)) we have (2) of $\mathtt{S}_{10}$.

(c3i2) Let $V_1 = x_L$. Then, since $V_1 = x_L > x_K = V$ from (34) and (c1), there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau - 1} = x_L > V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

i.e., $V_{t-1} = x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 13.2.1(p.96) (a) we have

$$L(V_{t-1}) = L(x_L) = 0 \cdots (36), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) > 0 \cdots (37), \quad t > t^\bullet_\tau.$$

Accordingly, from (20.2.12(p.201)) we have $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \geq t > 1$ and $V_t - \beta V_{t-1} > 0$ for $t > t^\bullet_\tau$ or equivalently

$$V_t = \beta V_{t-1} \cdots (38), \quad t^\bullet_\tau \geq t > 1, \qquad V_t > \beta V_{t-1} \cdots (39), \quad t > t^\bullet_\tau.$$

∘ Let $t^\bullet_\tau \geq \tau > 1$. Then, we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$ from (38), leading to

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (40).$$

(1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_\blacktriangle$, hence (1i) of $\mathtt{S}_{12}$ holds.

(2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$$

for $t^\bullet_\tau \geq \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_{\parallel}$, hence (2i) of $\mathtt{S}_{12}$ holds.

From (40) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1} V_1 \cdots (41).$$

∘ Let $\tau > t^\bullet_\tau$. Then, we have $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (39), leading to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (42).$$

From this and (41) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1.$$

(1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\triangle$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_\triangle$, thus (1ii) of $\mathtt{S}_{12}$ holds.

(2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 \geq \beta^\tau V_0$$

for $\tau > t^\bullet_\tau$, hence $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\blacktriangle$, hence (2ii) of $\mathtt{S}_{12}$ holds.

(c3i3) Let $V_1 > x_L \cdots (43)$. Then, since $V_1 > x_L > x_K = V$ from (34) and (c1), due to the nonincreasingness of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \geq V_2 \geq \cdots \geq V_{t^\bullet_\tau - 1} > x_L \geq V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

from which $V_{t-1} > x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \geq V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 13.2.1(p.96) (a) we have

$$L(V_{t-1}) \leq 0 \cdots (44), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \geq 0 \cdots (45), \quad t > t^\bullet_\tau.$$

∘ Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from (44), hence $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (20.2.12(p.201)), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$. Hence

$$V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1} V_1 \cdots (46).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle}_\blacktriangle$, so (1i) of $\mathtt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \ge s$. Then, since $V_1 \ge \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1 \ge \beta^\tau V_0$$

for $t^\bullet_\tau \ge \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle}_\vartriangle$, hence (2i) of $\mathtt{S}_{13}$ holds.

From (46) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} \le \beta V_{t^\bullet_\tau - 1} \le \cdots \le \beta^{t^\bullet_\tau - 1}V_1 \cdots (47).$$

∘ Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) \ge 0$ for $\tau \ge t > t^\bullet_\tau$ from (45), hence $V_t - \beta V_{t-1} \ge 0$ for $\tau \ge t > t^\bullet_\tau$ from (20.2.12(p.201)) or equivalently $V_t \ge \beta V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$, leading to

$$V_\tau \ge \beta V_{\tau-1} \ge \cdots \ge \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}.$$

Hence, from (47) we have

$$V_\tau \ge \beta V_{\tau-1} \ge \cdots \ge \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \le \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} \le \cdots \le \beta^{\tau-1}V_1 \cdots (48).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Since $V_1 - \beta V_0 < 0 \cdots (49)$ from (20.2.13(p.201)) or equivalently $V_1 < \beta V_0 \cdots (50)$. Then, from (48) and (50) we have

$$V_\tau \ge \beta V_{\tau-1} \ge \cdots \ge \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \le \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} \le \cdots \le \beta^{\tau-1}V_1 < \beta^\tau V_0.$$

hence Thus, we obtain $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$, hence (1ii) of $\mathtt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \ge s$. Then $V_1 - \beta V_0 \ge 0$ from (20.2.13(p.201)), hence $V_1 \ge \beta V_0$. Then, from (48) we have

$$V_\tau \ge \beta V_{\tau-1} \ge \cdots \ge \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \le \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} \le \cdots \le \beta^{\tau-2}V_2 \le \beta^{\tau-1}V_1 \ge \beta^\tau V_0.$$

Thus, we have $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$, hence (2ii) of $\mathtt{S}_{13}$ holds.

(c3ii) Let $b \le 0$ ⟪$\kappa \le 0$⟫. Then, since $x_L \le x_K$ from Lemma 13.2.4(p.97) (c ⟪d⟫), we have $V_1 > x_K \ge x_L$ from (28), hence $V_{t-1} \ge x_K \ge x_L$ for $t > 1$ due to (c1). Accordingly $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.96) (a), hence $V_t - \beta V_{t-1} \le 0$ for $t > 1$ from (20.2.12(p.201)) or equivalently $V_t \le \beta V_{t-1}$ for $t > 1$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$, we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1 \cdots (51).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, since $V_1 \le \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1 \le \beta^\tau V_0,$$

from (51), hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (20.2.13(p.201)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1 > \beta^\tau V_0,$$

from (51), hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$. ∎

Corollary 20.2.8 (M:2[ℙ][E]) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.* :

(a) *Let $x_K \ge V_1$. Then z_t is nondecreasing in $t > 0$.*

(b) *Let $x_K < V_1$. Then z_t is nonincreasing in $t > 0$.* ∎

● *Proof* Immediate from Tom 20.2.16(p.206) (b1,c1) and from (6.2.94(p.35)) and Lemma 13.1.3(p.89). ∎

20.2.5.3 Market Restriction

20.2.5.3.1 Positive Restriction

20.2.5.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.2.9** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\rho \le a^\star$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.

(c) *Let $b \le \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \le \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\parallel$ *and* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\rho < a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.

● *Proof* The same as Tom 20.2.9(p.203) due to Lemma 17.4.4(p.118). ∎

20.2.5.3.1.2 Case of $\beta < 1$ or $s > 0$

20.2.5.3.1.2.1 Case of $\rho \leq a^\star$

☐ **Pom 20.2.10** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a^\star < \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$.

(c) *Let $\rho < x_L$.*

1. $\boxed{Ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
     1. *Let $\lambda = 1$. Then* $\boxed{⊚\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\parallel$.
     2. *Let $\lambda < 1$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
   - ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
4. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
     1. *Let $\lambda = 1$.*
        - i. *Let $s < \lambda\beta T(0)$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
        - ii. *Let $s \geq \lambda\beta T(0)$. Then* $\boxed{⊚\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$.
     2. *Let $\lambda < 1$.*
        - i. *Let $s \leq \lambda\beta T(0)$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
        - ii. *Let $s > \lambda\beta T(0)$. Then* $\texttt{S}_8$(p.192) | Ⓢ▲ | ⊚ ‖ | ⊚ △ | ⊚ ▲ | *is true*
   - ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
     1. *Let $s \leq \lambda\beta T(0)$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
     2. *Let $s > \lambda\beta T(0)$. Then* $\texttt{S}_8$(p.192) | Ⓢ▲ | ⊚ ‖ | ⊚ △ | ⊚ ▲ | *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a-c2ii) The same as Tom 20.2.10(p.203) (a-c2ii).

(c3) Due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of Tom 20.2.10(p.203).

(c4-c4ii2) Immediate from (2) and Tom 20.2.10(p.203) (c3-c3ii2) with κ due to (2). ∎

☐ **Pom 20.2.11** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \geq 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(c) *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $s < \lambda\beta T(0)$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.
2. *Let $s \geq \lambda\beta T(0)$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a,b) The same as Tom 20.2.11(p.204) (a,b).

(c) Due to (1) it suffices to consider only (c1) of Tom 20.2.11(p.204).

(d-d2) Immediate from (2) and Tom 20.2.11(p.204) (c1,c2) with κ. ∎

☐ **Pom 20.2.12** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho < x_L$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$.

(c) *Let $\rho = x_L$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_\triangle$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_\blacktriangle$.

(d) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_9$(p.196) | Ⓢ△ | ●△ | ●▲ | *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s \geq \lambda\beta T(0)$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\triangle$ ($\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$).
   - ii. *Let $s < \lambda\beta T(0)$. Then* $\texttt{S}_9$(p.196) | Ⓢ△ | ●△ | ●▲ | *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a-d1) The same as Tom 20.2.12(p.204) (a-d1).

(d2) Due to (1) it suffices to consider only (d2ii) of Tom 20.2.12(p.204).

(d3,d3ii) Immediate from (2) and Tom 20.2.12(p.204) (d2i,d2ii) with κ. ∎

20.2.5.3.1.2.2 Case of $b \le \rho$

☐ **Pom 20.2.13** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$.

(c) *Let $\rho < x_L$.*

1. [Ⓢ dOITs$_1\langle 1\rangle$]$_\blacktriangle$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.
3. *Let $\beta < 1$ and $s = 0$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s \le \lambda\beta T(0)$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.
   ii. *Let $s > \lambda\beta T(0)$. Then* S$_8$(p.192) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a-c2) The same as Tom 20.2.13(p.204) (a-c2).

(c3) Due to (1) it suffices to consider only (c3i) of Tom 20.2.13(p.204).

(c4-c4ii) Immediate from (2) and Tom 20.2.13(p.204) (c3i,c3ii) with κ. ∎

☐ **Pom 20.2.14** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \ge 0$.*

(b) *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\parallel$.

(c) *Let $\beta < 1$ and $s = 0$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $s < \lambda\beta T(0)$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.
2. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a,b) The same as Tom 20.2.14(p.205) (a,b).

(c) Due to (1) it suffices to consider only (c1) of Tom 20.2.14(p.205).

(d-d2) Immediate from (2) and Tom 20.2.14(p.205) (c1,c2) with κ. ∎

☐ **Pom 20.2.15** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_\triangle$ *and* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.

(c) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$.
2. *Let $\beta < 1$ and $s = 0$. Then* S$_9$(p.196) [Ⓢ△ | ●△ | ●▲] *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$ ⟪[● dOITd$_{\tau>0}\langle 0\rangle$]$_\blacktriangle$⟫.
   ii. *Let $s < \lambda\beta T(0)$. Then* S$_9$(p.196) [Ⓢ△ | ●△ | ●▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a-c1) The same as Tom 20.2.15(p.205) (a-c1).

(c2) Due to (1) it suffices to consider only (c2ii) of Tom 20.2.15(p.205).

(c3-c3ii) Immediate from (2) and Tom 20.2.15(p.205) (c2i,c2ii) with κ. ∎

20.2.5.3.1.2.3 Case of $a^\star < \rho < b$

☐ **Pom 20.2.16** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $a^\star \le \rho < a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a - \rho\} < s$, then* [● dOITd$_1\langle 0\rangle$]$_\triangle$, *or else* [Ⓢ dOITs$_1\langle 1\rangle$]$_\blacktriangle$. *Below let $\tau > 1$.*

(b) *Let $x_K \ge V_1$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_K$ as $t \to \infty$*

2. *Let* $x_L \leq V_1$. *If* $\lambda\beta\max\{0, a-\rho\} \leq s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
3. *Let* $x_L > V_1$.
   i. *Let* $\beta = 1$. *Then* $\texttt{S}_{10}$(p.205) $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ *is true.*
   ii. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\texttt{S}_{10}$(p.205) $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ *is true.*
   iii. *Let* $\beta < 1$ *and* $s > 0$.
      1. *Let* $s < \lambda\beta T(0)$. *Then* $\texttt{S}_{10}$(p.205) $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ *is true.*
      2. *Let* $s = \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ *or* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.
      3. *Let* $s > \lambda\beta T(0)$. *Then* $\texttt{S}_{11}$(p.205) $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledcirc\vartriangle \mid \bullet\vartriangle}$ *is true.*

(c) *Let* $x_K < V_1$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\beta = 1$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\blacktriangle}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$.
   i. *Let* $x_L > V_1$. *Then* $\texttt{S}_{10}$(p.205) $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ *is true.*
   ii. *Let* $x_L = V_1$. *Then* $\texttt{S}_{12}$(p.205) $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.*
   iii. *Let* $x_L < V_1$. *Then* $\texttt{S}_{8}$(p.192) $\boxed{\circledS\blacktriangle \mid \circledcirc\parallel \mid \circledcirc\vartriangle \mid \circledcirc\blacktriangle}$ *is true.*
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < \lambda\beta T(0)$.
      1. *Let* $x_L > V_1$. *Then* $\texttt{S}_{10}$(p.205) $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ *is true.*
      2. *Let* $x_L = V_1$. *Then* $\texttt{S}_{12}$(p.205) $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.*
      3. *Let* $x_L < V_1$. *Then* $\texttt{S}_{8}$(p.192) $\boxed{\circledS\blacktriangle \mid \circledcirc\parallel \mid \circledcirc\vartriangle \mid \circledcirc\blacktriangle}$ *is true.*
   ii. *Let* $s \geq \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

• *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then, we have $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.26)).

(a-b3i) The same as `Tom` 20.2.16(p.206) (a-b3i).

(b3ii) Due to (1) it suffices to consider only (b3ii1) of `Tom` 20.2.16(p.206).

(b3iii-b3iii3) The same as `Tom` 20.2.16(p.206) (b3ii1-b3ii3).

(c-c2) Immediate from (2) and `Tom` 20.2.16(p.206) (c-c2).

(c3-c3iii) Due to (1) it suffices to consider only (c3i1-c3i3) of `Tom` 20.2.16(p.206).

(c4-c4ii) Immediate from (2) and `Tom` 20.2.16(p.206) (c3i-c3ii). ∎

20.2.5.3.2 Mixed Restriction

Omitted.

20.2.5.3.3 Negative Restriction

Omitted.

20.2.6 $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]

20.2.6.1 Preliminary

Since Theorem 20.2.3(p.191) holds due to Lemma 20.2.1(p.191) (b), we can derive $\mathscr{A}\{\tilde{\mathsf{M}}\text{:2}[\mathbb{P}][\text{E}]\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) to $\mathscr{A}\{\mathsf{M}\text{:2}[\mathbb{P}][\text{E}]\}$.

20.2.6.2 Analysis

20.2.6.2.1 Case of $\beta = 1$ and $s = 0$

□ **`Tom` 20.2.17** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}\text{:2}[\mathbb{P}][\text{E}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.

(b) *Let* $\rho \geq b^\star$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let* $a \geq \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let* $b^\star > \rho > a$.

1. *Let* $b \geq \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let* $\rho > b$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$. □

• *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 20.2.9(p.203). ∎

Corollary 20.2.9 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Let* $\beta = 1$ *and* $s = 0$. *Then,* z_t *is nonincreasing in* $t \geq 0$. □

• *Proof* Immediate from `Tom` 20.2.17(p.213) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

20.2.6.2.2 Case of $\beta < 1$ or $s > 0$

20.2.6.2.2.1 Case of $\rho \ge b^\star$

□ `Tom` **20.2.18** (⊡$\mathscr{A}${M̃:2[ℙ][E]}) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \ge \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. [ⓢ dOITs$_1\langle 1\rangle$]$_\blacktriangle$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $(\lambda b + s)/\lambda \ge b^\star$.*
     1. *Let $\lambda = 1$. Then* [◎ ndOIT$_{\tau>1}\langle 1\rangle$]$_\parallel$.
     2. *Let $\lambda < 1$. Then* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.
   - ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $(\lambda\beta b + s)/\delta \ge b^\star$.*
     1. *Let $\lambda = 1$.*
        - i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.
        - ii. *Let $a \ge 0$ ($\tilde{\kappa} \ge 0$). Then* [◎ ndOIT$_{\tau>1}\langle 1\rangle$]$_\triangle$.
     2. *Let $\lambda < 1$.*
        - i. *Let $a \le 0$ ($\tilde{\kappa} \le 0$). Then* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.
        - ii. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* $\mathtt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
   - ii. *Let $(\lambda\beta b + s)/\delta < b^\star$.*
     1. *Let $a \le 0$ ($\tilde{\kappa} \le 0$). Then* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.
     2. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* $\mathtt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.10(p.203). ▮

Corollary 20.2.10 (M̃:2[ℙ][E]) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then, z_t is nonincreasing in $t \ge 0$.* □

● *Proof* Immediate from `Tom` 20.2.18(p.214) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

□ `Tom` **20.2.19** (⊡$\mathscr{A}${M̃:2[ℙ][E]}) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \ge 0$.*

(b) *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\parallel$.

(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*

1. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* [ⓢ dOITs$_{\tau>0}\langle \tau\rangle$]$_\blacktriangle$.
2. *Let $a \ge 0$ ($\tilde{\kappa} \ge 0$). Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.11(p.204). ▮

Corollary 20.2.11 (M̃:2[ℙ][E]) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then, $z_t = z(\rho)$ for $t \ge 0$.* □

● *Proof* Immediate from `Tom` 20.2.19(p.214) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

□ `Tom` **20.2.20** (⊡$\mathscr{A}${M̃:2[ℙ][E]}) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho > x_{\tilde{L}}$. Then* [ⓢ dOITs$_{\tau>0}\langle \tau\rangle$]$_\blacktriangle$.

(c) *Let $\rho = x_{\tilde{L}}$. Then* [● dOITd$_1\langle 0\rangle$]$_\triangle$ *and* [ⓢ dOITs$_{\tau>1}\langle \tau\rangle$]$_\blacktriangle$.

(d) *Let $\rho < x_{\tilde{L}}$.*

1. *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a \ge 0$ ($\tilde{\kappa} \ge 0$). Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\triangle$ ([● dOITd$_{\tau>0}\langle 0\rangle$]$_\blacktriangle$).
   - ii. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\mathtt{S}_9$ [ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.12(p.204). ▮

Corollary 20.2.12 (M̃:2[ℙ][E]) *Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$. Then, z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from `Tom` 20.2.20(p.214) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

20.2.6.2.2.2 Case of $a \geq \rho$

☐ $\mathtt{Tom}$ **20.2.21** ($\boxminus\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\circledS\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
   - ii. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* $\mathtt{S}_8$ $\boxed{\circledS\blacktriangle \mid \circledcirc \parallel \mid \circledcirc \vartriangle \mid \circledcirc \blacktriangle}$ *is true.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathtt{Tom}$ 20.2.13(p.204).[†] ∎

Corollary 20.2.13 ($\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then, z_t is nonincreasing in $t \geq 0$.*

● *Proof* Immediate from $\mathtt{Tom}$ 20.2.21(p.215) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

☐ $\mathtt{Tom}$ **20.2.22** ($\boxminus\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*

1. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
2. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathtt{Tom}$ 20.2.14(p.205). ∎

Corollary 20.2.14 ($\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then, $z_t = z(\rho)$ for $t \geq 0$.* □

● *Proof* Immediate from $\mathtt{Tom}$ 20.2.22(p.215) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

☐ $\mathtt{Tom}$ **20.2.23** ($\boxminus\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

(c) *Let $\rho < x_{\tilde{L}}$.*

1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$).
   - ii. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\mathtt{S}_9$ $\boxed{\circledS\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathtt{Tom}$ 20.2.15(p.205).[‡] ∎

Corollary 20.2.15 ($\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$. Then, z_t is nondecreasing in $t \geq 0$.* □

● *Proof* Immediate from $\mathtt{Tom}$ 20.2.23(p.215) (a) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ∎

20.2.6.2.2.3 Case of $b^\star > \rho > a$

By applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.2.3(p.191), we see that $\mathtt{S}_{10}$(p.205) −24.1.12 change as follows respectively:

$\mathtt{S}_{14}$ $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ = {
We have:
(1) Let $\lambda\min\{0, \rho - b\} > -s$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
(2) Let $\lambda\min\{0, \rho - b\} \leq -s$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\vartriangle}$. }

$\mathtt{S}_{15}$ $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledcirc \vartriangle \mid \bullet\vartriangle}$ = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta\min\{0, \rho - b\} > -s$, then
i. $\boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau\geq\tau>1}\langle \tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet_\tau\geq\tau>1}\langle 0\rangle}_{\vartriangle}$,
ii. $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>t^\bullet_\tau}\langle 0\rangle}_{\vartriangle}$.
(2) If $\lambda\beta\min\{0, \rho - b\} \leq -s$, then
i. $\boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau\geq\tau>1}\langle \tau\rangle}_{\blacktriangle}$,
ii. $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\vartriangle}$. }

[†] $\mathtt{S}_8$ does not change by the application of the operation.
[‡] $\mathtt{S}_9$ does not change by the application of the operation.

$$\mathtt{S}_{16}\ \boxed{\circledS\vartriangle \mid \circledS\blacktriangle\circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle} = \left\{\begin{array}{l}\text{There exists } t^{\bullet}_{\tau} > 1 \text{ such that:}\\ \text{(1) If } \lambda\beta\min\{0,\rho-b\} > -s, \text{ then}\\ \quad\text{i. } \boxed{\bullet\ \mathtt{dOITd}_{t^{\bullet}_{\tau}\ge\tau>0}\langle 0\rangle}_{\blacktriangle},\\ \quad\text{ii. } \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^{\bullet}_{\tau}}\langle\tau\rangle}_{\vartriangle} \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau>t^{\bullet}_{\tau}}\langle 0\rangle}_{\vartriangle}.\\ \text{(2) If } \lambda\beta\min\{0,\rho-b\} \le -s, \text{ then}\\ \quad\text{i. } \boxed{\circledcirc\ \mathtt{ndOIT}_{t^{\bullet}_{\tau}\ge\tau>1}\langle 1\rangle}_{\parallel},\\ \quad\text{ii. } \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^{\bullet}_{\tau}}\langle\tau\rangle}_{\blacktriangle}.\end{array}\right\}$$

$$\mathtt{S}_{17}\ \boxed{\circledS\vartriangle \mid \circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle} = \left\{\begin{array}{l}\text{There exists } t^{\bullet}_{\tau} > 1 \text{ and } t^{\star}_{\tau} > 1 \text{ such that:}\\ \text{(1) If } \lambda\beta\min\{0,\rho-b\} > -s, \text{ then}\\ \quad\text{i. } \boxed{\bullet\ \mathtt{dOITd}_{t^{\bullet}_{\tau}\ge\tau>1}\langle 0\rangle}_{\blacktriangle},\\ \quad\text{ii. } \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^{\bullet}_{\tau}}\langle\tau\rangle}_{\vartriangle} \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau>t^{\bullet}_{\tau}}\langle t^{\star}_{\tau}\rangle}_{\vartriangle}.\\ \text{(2) If } \lambda\beta\min\{0,\rho-b\} \le -s, \text{ then}\\ \quad\text{i. } \boxed{\circledcirc\ \mathtt{ndOIT}_{t^{\bullet}_{\tau}\ge\tau>1}\langle 1\rangle}_{\vartriangle},\\ \quad\text{ii. } \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^{\bullet}_{\tau}}\langle\tau\rangle}_{\vartriangle} \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau>t^{\bullet}_{\tau}}\langle t^{\star}_{\tau}\rangle}_{\vartriangle}.\end{array}\right\}$$

Moreover, note that (20.2.17(p.206)) can be changed into

$$V_1 = \lambda\beta\min\{0,\rho-b\} + \beta\rho + s. \tag{20.2.18}$$

□ **Tom 20.2.24** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$) *Assume $b^{\star} \ge \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\min\{0,\rho-b\} \ge -s$, then $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. Below let $\tau > 1$.*

(b) *Let $V_1 \ge x_{\tilde{K}}$.*[†]

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \le x_{\tilde{L}}$. If $\lambda\beta\min\{0,\rho-b\} \ge -s$, then $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.*
3. *Let $V_1 > x_{\tilde{L}}$.*
   - i. *Let $\beta = 1$. Then $\mathtt{S}_{14}\ \boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ is true.*
   - ii. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
     1. *Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. Then $\mathtt{S}_{14}\ \boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ is true.*
     2. *Let $a = 0\,(\!(\tilde{\kappa} = 0)\!)$. If $\lambda\beta\min\{0,\rho-b\} > -s$, then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.*
     3. *Let $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. Then $\mathtt{S}_{15}\ \boxed{\circledS\vartriangle \mid \circledS\blacktriangle\circledcirc\vartriangle \mid \bullet\vartriangle}$ is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $\beta = 1$. If $\lambda\beta\min\{0,\rho-b\} > -s$, then $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\blacktriangle}$.*
3. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
   - i. *Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$.*
     1. *Let $V_1 \ge x_{\tilde{L}}$. Then $\mathtt{S}_{14}\ \boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ is true.*
     2. *Let $V_1 = x_{\tilde{L}}$. Then $\mathtt{S}_{16}\ \boxed{\circledS\vartriangle \mid \circledS\blacktriangle\circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ is true.*
     3. *Let $V_1 < x_{\tilde{L}}$. Then $\mathtt{S}_{17}\ \boxed{\circledS\vartriangle \mid \circledcirc\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ is true.*
   - ii. *Let $a \ge 0\,(\!(\tilde{\kappa} \ge 0)\!)$. If $\lambda\beta\min\{0,\rho-b\} > -s$, then $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledcirc\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.2.16(p.206). ▮

Corollary 20.2.16 ($\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]$) *Assume $b^{\star} \ge \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $V_1 \ge x_{\tilde{K}}$. Then z_t is nonincreasing in $t > 0$.*

(b) *Let $V_1 < x_{\tilde{K}}$. Then z_t is nondecreasing in $t > 0$.* □

● *Proof* Immediate from Tom 20.2.24(p.216) (b1,c1) and from (6.2.111(p.36)) and Lemma A 3.3(p.297). ▮

[†] $V_1 = \lambda\beta\min\{0,b-\rho\} + \beta\rho + s$ (see (6.4.25(p.41))).

20.2.6.3 Market Restriction

20.2.6.3.1 Positive Restriction

20.2.6.3.1.1 $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$

20.2.6.3.1.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.2.17** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \geq b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let $a \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(d) *Let $b^\star > \rho > a$.*

1. *Let $b \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $\rho > b$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

● *Proof* The same as Tom 20.2.17(p.213) due to Lemma 17.4.4(p.118). ∎

20.2.6.3.1.1.2 Case of $\beta < 1$ or $s > 0$

20.2.6.3.1.1.2.1 Case of $\rho \geq b^\star$

□ **Pom 20.2.18** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *and* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

   ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_8$ $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau}\langle 1\rangle}_{\triangle}$.
      2. *Let $\lambda < 1$. Then* $\texttt{S}_8$(p.192) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*

   ii. *Let $(\lambda\beta b + s)/\delta < b^\star$. Then* $\texttt{S}_8$(p.192) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \parallel \mid \circledcirc\ \triangle \mid \circledcirc\ \blacktriangle}$ *is true.*

● *Proof* Suppose $a > 0 \cdots (1)$, hence $b^\star > b > a > 0 \cdots (2)$ from Lemma 14.6.1(p.107) (n). Then we have $\tilde{\kappa} = s \cdots (3)$ from Lemma 14.6.6(p.108) (a).

(a-c2ii) The same as Tom 20.2.18(p.214) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$, hence $\tilde{\kappa} = 0$ due to (3). Assume $(\lambda\beta b + s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (10.2.2 (1) (p.56)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (2), so $\beta b \geq b^\star$, which contradicts [7(p.118)]. Thus it must be that $(\lambda\beta b + s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of Tom 20.2.18(p.214).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} > 0$ from (3), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 20.2.18(p.214) with κ. ∎

□ **Pom 20.2.19** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*

(b) *We have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

● *Proof* Let $a > 0 \cdots (1)$, then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a) The same as Tom 20.2.19(p.214) (a).

(b) Let $\beta = 1$. Then we have Tom 20.2.19(p.214) (a). Let $\beta < 1$. Then, if $s = 0$, due to (1) it suffices to consider only (c2) of Tom 20.2.19(p.214) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (c2 of Tom 20.2.19(p.214). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ **Pom 20.2.20** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\triangle}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.

1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
2. *Let* $\beta < 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$).

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 14.6.6(p.108) (a).

(a-d1) The same as Tom 20.2.20(p.214) (a-d1).

(d2) If $s = 0$, due to (1) it suffices to consider only (d2i) of Tom 20.2.20(p.214) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of Tom 20.2.20(p.214). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.2.6.3.1.1.2.2 Case of $a \geq \rho$

☐ Pom **20.2.21** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.

1. $\boxed{Ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
3. *Let* $\beta < 1$. *Then* $\mathtt{S}_8$(p.192) $\boxed{Ⓢ\blacktriangle \mid ◎\ \| \mid ◎\ \vartriangle \mid ◎\ \blacktriangle}$ *is true.*

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a-c2) The same as Tom 20.2.21(p.215) (a-c2).

(c3) If $s = 0$, due to (1) it suffices to consider only (c3ii) of Tom 20.2.21(p.215) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 20.2.21(p.215). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ Pom **20.2.22** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let* $\beta < 1$. *Then we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.108) (a).

(a) The same as Tom 20.2.22(p.215) (a).

(b) The same as Tom 20.2.22(p.215) (b).

(c) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (c2) of Tom 20.2.22(p.215). If $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.2.22(p.215). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ Pom **20.2.23** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

(c) *Let* $\rho < x_{\tilde{L}}$.

1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
2. *Let* $\beta < 1$ *and let* $s = 0 (s > 0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$).

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 14.6.6(p.108) (a).

(a,b) The same as Tom 20.2.23(p.215) (a,b).

(c) Let $\rho < x_{\tilde{L}}$.

(c1) Let $\beta = 1$. Then we have $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ from Tom 20.2.23(p.215) (c1).

(c2) Let $\beta < 1$. If $s = 0$, then due to (2) it suffices to consider only (c2i) of Tom 20.2.23(p.215) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2i) of Tom 20.2.23(p.215). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.2.6.3.1.1.2.3 Case of $b^\star > \rho > a$

☐ Pom **20.2.24** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Assume* $b^\star \geq \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta \max\{0, \rho - b\} \leq s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{Ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.

(b) *Let* $V_1 \geq x_{\tilde{K}}$.[†]

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $V_1 \geq x_{\tilde{L}}$. *If* $\lambda\beta \max\{0, \rho - b\} \leq s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{◎\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
3. *Let* $V_1 > x_{\tilde{L}}$.

[†] $V_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s$ (see (6.4.25(p.41))).

i. *Let $\beta = 1$. Then* $\mathtt{S}_{14}$(p.215) [Ⓢ▵ | ●▵] *is true.*

ii. *Let $\beta < 1$. Then* $\mathtt{S}_{15}$(p.215) [Ⓢ▵ | Ⓢ▴ | ◎ ▵ | ●▵] *is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *If $\lambda\beta \max\{0, \rho - b\} < s$, then* [● $\mathtt{dOITd}_{\tau>1}\langle 0\rangle$]$_\blacktriangle$, *or else* [◎ $\mathtt{ndOIT}_{\tau>1}\langle 1\rangle$]$_\vartriangle$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0$. Then $\tilde{\kappa} = s \cdots$ **(2)** due to Lemma 14.6.6(p.108) (a).

(a-b3i) The same as Tom 20.2.24(p.216) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (b3ii3) of Tom 20.2.24(p.216) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii3) of Tom 20.2.24(p.216). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as Tom 20.2.24(p.216) (c1).

(c2) If $\beta = 1$, then it suffices to consider only (c2) of Tom 20.2.24(p.216) and if $\beta < 1$, whether $s = 0$ or $s > 0$, it suffices to consider only (c3ii) of Tom 20.2.24(p.216). Accordingly, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

20.2.6.3.2 Mixed Restriction

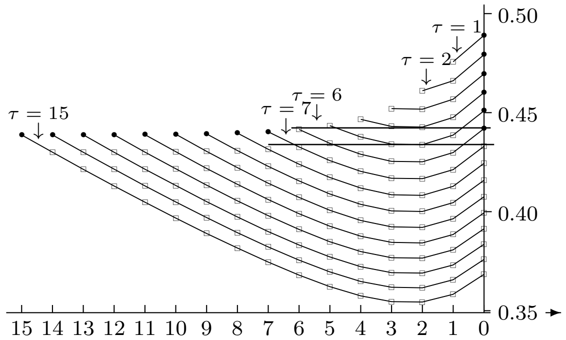
Omitted.

20.2.6.3.3 Negative Restriction

Omitted.

20.2.6.4 Numerical Calculation

Numerical Example **6** **($\mathscr{A}$\{M:2[ℝ][E]$^+$\} (selling model)** This example is for the assertion in Pom 20.2.4(p.199) (d3ii) in which $a > 0$, $\rho > x_K$, $\rho > x_L$, $\beta < 1$, $s > 0$, and $\lambda\beta\mu > s$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$.† where $x_L = 0.462767$ and $x_K = 0.439640$. The symbols • in the figure below shows the optimal initiating times $t^*_{15 \ge \tau \ge 1}$ (see the t^*_τ-column in the table of Figure 20.2.2(p.219) below).



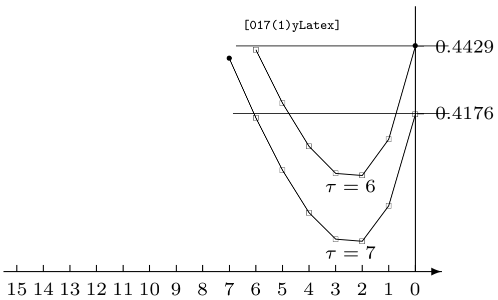
Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $15 \ge \tau > 0$ and $\tau \ge t \ge 0$

t	V_t	$\Delta_\beta V_t$	t^*_τ
0	0.5000000		
1	0.4766162	−0.0133838	1
2	0.4619911	−0.0050927	1
3	0.4530367	+0.0002854	1
4	0.4476274	+0.0036514	1
5	0.4443866	+0.0057117	1
6	0.4424547	+0.0069558	1
7	0.4413065	+0.0077009	7
8	0.4406253	+0.0081449	8
9	0.4402216	+0.0084088	9
10	0.4399825	+0.0085653	10
11	0.4398410	+0.0086581	11
12	0.4397572	+0.0087130	12
13	0.4397076	+0.0087456	13
14	0.4396783	+0.0087648	14
15	0.4396609	+0.0087762	15

$\Delta_\beta V_t = V_t - \beta V_{t-1}$

Figure 20.2.2: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $15 \ge \tau \ge 2$ and $\tau \ge t \ge 1$

Scaling up the graphs for $\tau = 6$ and $\tau = 7$ in the above figure, we have the figure below. This figure demonstrates that the optimal initiating time *shifts* from 0 to 7 when the starting time changes from $\tau = 6$ to $\tau = 7$.



Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 6, 7$

$\tau = 6$: t	$\beta^{6-t}V_t$	$\tau = 7$: t	$\beta^{7-t}V_t$
0	0.4429212	0	0.4340628
1	0.4308233	1	0.4222069
2	0.4261259	2	0.4176034
3	0.4263946	3	0.4178667
4	0.4299014	4	0.4213034
5	0.4354989	5	0.4267889
6	0.4424547	6	0.4336056
		7	0.4413065

Figure 20.2.3: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 6$ and $\tau = 7$

† We have $\rho = 0.5 > 0.462767 = x_L$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. Since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\lambda\beta\mu = 0.7 \times 0.98 \times 0.505 = 0.34634 > 0.1 = s$. Thus the condition of this assertion is confirmed.

20.2.6.5 Conclusion 4 (Search-Enforced-Model 2)

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$ of the quadruple-asset-trading-models for Model 2 on the total market $\mathscr{F}$

$$\mathcal{Q}\langle \mathsf{M}:2[\mathrm{E}]\rangle = \{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}], \mathsf{M}{:}2[\mathbb{P}][\mathrm{E}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$$

are given by

$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}$
↓
Tom's 20.2.1(p.192), 20.2.2(p.192), 20.2.3(p.195), 20.2.4(p.196),

$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$
↓
Tom's 20.2.5(p.199), 20.2.6(p.199), 20.2.7(p.200), 20.2.8(p.200),

$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]\}$
↓
Tom's 20.2.9(p.203), 20.2.10(p.203), 20.2.11(p.204), 20.2.12(p.204), 20.2.13(p.204), 20.2.14(p.205), 20.2.15(p.205), 20.2.16(p.206),

$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$
↓
Tom's 20.2.17(p.213), 20.2.18(p.214), 20.2.19(p.214), 20.2.20(p.214), 20.2.21(p.215), 20.2.22(p.215), 20.2.23(p.215), 20.2.24(p.216),

■ The assertion systems $\mathscr{A}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^+\}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$$\mathcal{Q}\langle \mathsf{M}:2[\mathrm{E}]\rangle^+ = \{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]^+, \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^+, \mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]^+, \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}^+$$

are given by

$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]^+\}$
↓
Pom's 20.2.1(p.198), 20.2.2(p.198), 20.2.3(p.198), 20.2.4(p.199),

$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^+\}$
↓
Pom's 20.2.5(p.200), 20.2.6(p.200), 20.2.7(p.201), 20.2.8(p.201),

$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]^+\}$
↓
Pom's 20.2.9(p.210), 20.2.10(p.211), 20.2.11(p.211), 20.2.12(p.211), 20.2.13(p.212), 20.2.14(p.212), 20.2.15(p.212), 20.2.16(p.212),

$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]^+\}$
↓
Pom's 20.2.17(p.217), 20.2.18(p.217), 20.2.19(p.217), 20.2.20(p.217), 20.2.21(p.218), 20.2.22(p.218), 20.2.23(p.218), 20.2.24(p.218),

■ Closely looking into all the assertion systems above leads to the following conclusions.

C1. Mental Conflict

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$.

1. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]$ (selling model) is nondecreasing in t ▲[a] as in Figure 7.3.1(p.47) (I), hence we have the normal conflict (see Remark 7.3.1(p.47)).
2. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]$ (selling model) is nondecreasing in t ▲[b] as in Figure 7.3.1(p.47) (I), hence we have the normal conflict (see Remark 7.3.1(p.47)).
3. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]$ (buying model) is nonincreasing in t ▼[c] as in Figure 7.3.1(p.47) (II), hence we have the normal conflict (see Remark 7.3.1(p.47)).
4. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]$ (buying model) is nonincreasing in t as in Figure 7.3.1(p.47) (II), hence we have the normal conflict (see Remark 7.3.1(p.47)), ▼[d].

· ▲[a] ← Tom 20.2.1(p.192) (a)
· ▲[b] ← Corollary 20.2.1(p.203)
· ▼[c] ← Tom 20.2.5(p.199) (a)
· ▼[d] ← Corollary 20.2.9(p.213).

b. Let $\beta < 1$ or $s > 0$.

1. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]$ (selling model) is nondecreasing in t ▲[a], constant ‖[a], or nonincreasing in t ▼[a] as in Figure 7.3.2(p.48) (I), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).
2. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]$ (selling model) is nondecreasing in t ▲[b], constant ‖[b], or nonincreasing in t ▼[b] as in Figure 7.3.2(p.48) (I), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).

3. The opt-ℝ-price V_t in $\tilde{\mathsf{M}}$:2[ℝ][E] (buying model) is nondecreasing in $t^{\blacktriangle c}$, constant $\|^{c}$, or nonincreasing in $t^{\blacktriangledown c}$ as in Figure 7.3.2(p.48) (II), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).
4. The opt-ℙ-price z_t in $\tilde{\mathsf{M}}$:2[ℙ][E] (buying model) is nondecreasing in $t^{\blacktriangle d}$, constant $\|^{d}$, or nonincreasing in $t^{\blacktriangledown d}$ as in Figure 7.3.2(p.48) (II), hence we have the abnormal conflict (see Remark 7.3.2(p.48)).

· $\blacktriangle^{a}$ ← Tom 20.2.1(p.192) (a), 20.2.2(p.192) (a).
  $\|^{a}$ ← Tom 20.2.3(p.195) (a)).
  $\blacktriangledown^{a}$ ← Tom 20.2.4(p.196) (a).

· $\blacktriangle^{b}$ ← Corollary 20.2.1(p.203) , 20.2.2(p.204) , 20.2.5(p.204) ,20.2.8(p.210) (a).
  $\|^{b}$ ← Corollary 20.2.3(p.204) , 20.2.6(p.205) .
  $\blacktriangledown^{b}$ ← Corollary 20.2.4(p.204) , 20.2.7(p.205) , 20.2.8(p.210) (b).

· $\blacktriangle^{c}$ ← Tom 20.2.8(p.200) (a).
  $\|^{c}$ ← Tom 20.2.7(p.200) (a).
  $\blacktriangledown^{c}$ ← Tom 20.2.5(p.199) (a), 20.2.6(p.199) (a).

· $\blacktriangle^{d}$ ← Corollary 20.2.12(p.214) , 20.2.15(p.215) , 20.2.16(p.216) (b).
  $\|^{c}$ ← Corollary 20.2.11(p.214) , 20.2.14(p.215) .
  $\blacktriangledown^{d}$ ← Corollary 20.2.9(p.213) , 20.2.10(p.214) , 20.2.13(p.215) , 20.2.16(p.216) (a).

The above results can be summarized as below.

A. If $\beta = 1$ and $s = 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether ℝ-model or ℙ-model, we have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6) .

B. If $\beta < 1$ or $s > 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether ℝ-model or ℙ-model, we have the abnormal mental conflict, which does not coincide with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6) .

C2. Symmetry

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 20.2.5(p.200)	$\sim$ Pom 20.2.1(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$),
Pom 20.2.17(p.217)	$\sim$ Pom 20.2.9(p.210)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℙ][E]$\}^+ \sim \mathscr{A}\{\mathsf{M}$:2[ℙ][E]$\}^+$).

b. Let $\beta < 1$ or $s > 0$. Then we have:

Pom 20.2.6(p.200)	$\not\sim$ Pom 20.2.2(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.7(p.201)	$\not\sim$ Pom 20.2.3(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.8(p.201)	$\not\sim$ Pom 20.2.4(p.199)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.18(p.217)	$\not\sim$ Pom 20.2.10(p.211)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.19(p.217)	$\not\sim$ Pom 20.2.11(p.211)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.20(p.217)	$\not\sim$ Pom 20.2.12(p.211)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.21(p.218)	$\not\sim$ Pom 20.2.13(p.212)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.22(p.218)	$\not\sim$ Pom 20.2.14(p.212)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.23(p.218)	$\not\sim$ Pom 20.2.15(p.212)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.24(p.218)	$\not\sim$ Pom 20.2.16(p.212)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\sim \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)

The above results can be summarized as below.

A. Let $\beta = 1$ and $s = 0$. Then the symmetry is always <u>inherited</u> (see C2a(p.221)).

B. Let $\beta < 1$ or $s > 0$. Then the symmetry always <u>collapses</u> (see C2b(p.221)).

C3. Analogy

a. On $\mathscr{F}^+$, for any $\beta \le 1$ and $s \ge 0$ we have:

Pom 20.2.9(p.210)	$\not\bowtie$ Pom 20.2.1(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.10(p.211)	$\not\bowtie$ Pom 20.2.2(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.11(p.211)	$\bowtie$ Pom 20.2.3(p.198)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$) ⋯ (∗)
Pom 20.2.12(p.211)	$\not\bowtie$ Pom 20.2.4(p.199)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.17(p.217)	$\not\bowtie$ Pom 20.2.5(p.200)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.18(p.217)	$\not\bowtie$ Pom 20.2.6(p.200)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.19(p.217)	$\not\bowtie$ Pom 20.2.7(p.201)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.20(p.217)	$\not\bowtie$ Pom 20.2.8(p.201)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.21(p.218)	$\not\bowtie$ Pom 20.2.6(p.200)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)
Pom 20.2.22(p.218)	$\bowtie$ Pom 20.2.7(p.201)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$) ⋯ (∗∗)
Pom 20.2.23(p.218)	$\not\bowtie$ Pom 20.2.8(p.201)	($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}$:2[ℝ][E]$\}^+$)

The above results can be summarized as below.

A. The analogy <u>collapses</u> except (∗) and (∗∗).

C4. Optimal initiating time (OIT)

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 20.2.1(p.198) , Pom 20.2.5(p.200) , Pom 20.2.9(p.210) , Pom 20.2.17(p.217) ,

we obtain the following table.

Table 20.2.3: Possible OIT on $\mathscr{F}^+$ ($\beta = 1$ and $s = 0$)

		$\mathscr{A}\{$M:2[ℝ][E]$^+\}$	$\mathscr{A}\{\tilde{\text{M}}$:2[ℝ][E]$^+\}$	$\mathscr{A}\{$M:1[ℙ][E]$^+\}$	$\mathscr{A}\{\tilde{\text{M}}$:2[ℙ][E]$^+\}$
ⓢ dOITs$_\tau\langle\tau\rangle$ $_{\|\|}$	ⓢ$_{\|\|}$				
ⓢ dOITs$_\tau\langle\tau\rangle$ $_\triangle$	ⓢ$_\triangle$				
ⓢ dOITs$_\tau\langle\tau\rangle$ $_\blacktriangle$	ⓢ$_\blacktriangle$	○	○	○	○
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_{\|\|}$	⊙$_{\|\|}$				
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_\triangle$	⊙$_\triangle$				
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_\blacktriangle$	⊙$_\blacktriangle$				
● dOITd$_\tau\langle 0\rangle$ $_{\|\|}$	ⓓ$_{\|\|}$	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ $_\triangle$	ⓓ$_\triangle$				
● dOITd$_\tau\langle 0\rangle$ $_\blacktriangle$	ⓓ$_\blacktriangle$				

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 20.2.4(p.199) , Pom 20.2.12(p.211) , Pom 20.2.15(p.212) , Pom 20.2.16(p.212) , Pom 20.2.24(p.218) ,
Pom 20.2.2(p.198) , Pom 20.2.3(p.198) , Pom 20.2.4(p.199) , Pom 20.2.6(p.200) , Pom 20.2.8(p.201) ,
Pom 20.2.10(p.211) , Pom 20.2.11(p.211) , Pom 20.2.13(p.212) , Pom 20.2.14(p.212) , Pom 20.2.16(p.212) ,
Pom 20.2.18(p.217) , Pom 20.2.20(p.217) , Pom 20.2.23(p.218) , Pom 20.2.16(p.212) , Pom 20.2.21(p.218) ,
Pom 20.2.7(p.201) , Pom 20.2.19(p.217) , Pom 20.2.22(p.218) , Pom 20.2.22(p.218) ,

we obtain the following table:

Table 20.2.4: Possible OIT on $\mathscr{F}^+$ ($\beta < 1$ or $s > 0$)

		$\mathscr{A}\{$M:2[ℝ][E]$^+\}$	$\mathscr{A}\{\tilde{\text{M}}$:2[ℝ][E]$^+\}$	$\mathscr{A}\{$M:1[ℙ][E]$^+\}$	$\mathscr{A}\{\tilde{\text{M}}$:2[ℙ][E]$^+\}$
ⓢ dOITs$_\tau\langle\tau\rangle$ $_{\|\|}$	ⓢ$_{\|\|}$				
ⓢ dOITs$_\tau\langle\tau\rangle$ $_\triangle$	ⓢ$_\triangle$	○	○	○	○
ⓢ dOITs$_\tau\langle\tau\rangle$ $_\blacktriangle$	ⓢ$_\blacktriangle$	○	○	○	○
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_{\|\|}$	⊙$_{\|\|}$	○	○	○	○
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_\triangle$	⊙$_\triangle$	○	○	○	○
⊙ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ $_\blacktriangle$	⊙$_\blacktriangle$	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ $_{\|\|}$	ⓓ$_{\|\|}$	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ $_\triangle$	ⓓ$_\triangle$	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ $_\blacktriangle$	ⓓ$_\blacktriangle$	○	○	○	○

c. The table below is the list of the occurrence rates of ⓢ , ⊙ , and ⓓ on $\mathscr{F}$ (see Tom's 20.2.1(p.192) (▣), 20.2.2(p.192) (▣), 20.2.3(p.195) (▣), 20.2.4(p.196) (▣), 20.2.3(p.202) (▣), and 20.2.16(p.206) (▣)).

Table 20.2.5: Occurance rates of ⓢ , ⊙ , and ⓓ on $\mathscr{F}^+$

ⓢ			⊙			ⓓ		
41.4 % / 29			24.3 % / 17			34.3 % / 24		
ⓢ$_{\|\|}$	ⓢ$_\triangle$	ⓢ$_\blacktriangle$	⊙$_{\|\|}$	⊙$_\triangle$	⊙$_\blacktriangle$	ⓓ$_{\|\|}$	ⓓ$_\triangle$	ⓓ$_\blacktriangle$
—	possible	possible	possible	possible	possible	possible	possible	possible
– % / –	12.9 % / 9	28.5 % / 20	5.7 % / 4	14.3 % / 10	4.3 % / 3	5.7 % / 4	21.5 % / 15	7.1 % / 5

C5. Null-time-zone and deadline-engulfing

From Table 20.2.5(p.222) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▴ (<u>strict</u> optimality).

b. As a whole we have ⓢ, ⊚, and 🅓 occur at 41.4%, 24.3%, and 34.3% respectively where

1. ⓢ$_{\|}$ cannot be defined due to Preference Rule 7.2.1(p.45).
2. ⊚$_{\|}$ is possible (5.7 %).
3. 🅓$_{\|}$ is possible (5.7 %).
4. ⓢ$_{\vartriangle}$ is possible (12.9 %).
5. ⊚$_{\vartriangle}$ is possible (14.3 %).
6. 🅓$_{\vartriangle}$ is possible (21.5 %).
7. ⓢ$_{\blacktriangle}$ is possible (28.5 %).
8. ⊚$_{\blacktriangle}$ is possible (4.3 %).
   - `Tom` 20.2.2(p.192) (c3i2,c3ii1ii2,c3ii2i).
9. 🅓$_{\blacktriangle}$ is possible (7.1 %).
   - `Tom` 20.2.4(p.196) (d2i,d2ii).
   - `Tom` 20.2.16(p.206) (c2,c3i2,c3i3).

From the above results we see that:

A. ⊚ and 🅓 causing the `null-time-zone` are possible at 58.6% (= 24.3% + 34.3%).

B. ⊚$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is possible at 4.3%.

C. 🅓$_{\blacktriangle}$ *strictly* causing the `null-time-zone` are possible at 7.1%, i.e., the deadline-engulfing is possible.

20.3 Conclusions of Model 2

Conclusions 3(p.187) and 4(p.220) can be summed up as below.

$\overline{\text{C}}1.$ Mental Conflict

On $\mathscr{F}^+$, from C1A(p.189) and C1B(p.189) and from C1A(p.221) and C1B(p.221). we have:

A. If $\beta = 1$ and $s = 0$, then, on $\mathscr{F}^+$, whether search-**A**llowed-model or search-**E**nforced-model, whether selling problem or buying problem, and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict, which coincides with ***expectations*** in *Example*s 1.4.1(p.5) - 1.4.4(p.6).

B. If $\beta < 1$ or $s > 0$, then, on $\mathscr{F}^+$, whether search-**A**llowed-model or search-**E**nforced-model, whether selling problem or buying problem, and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the abnormal mental conflict, which does not coincide with ***expectations*** in *Example*s 1.4.1(p.5) - 1.4.4(p.6).

$\overline{\text{C}}2.$ Symmetry

On $\mathscr{F}^+$, we have:

a. If $\beta = 1$ and $s = 0$, the symmetry is always inherited (see C2A(p.189) and C2A(p.221)).

b. if $\beta < 1$ or $s > 0$, the symmetry always collapses (see C2B(p.189) and C2B(p.221)).

$\overline{\text{C}}3.$ Analogy

On $\mathscr{F}^+$, we have:

a. For any $\beta \le 1$ and $s \ge 0$, the analogy collapse (see C3A(p.189) and C3A(p.222)) except (∗) and (∗∗) of C3(p.221).

$\overline{\text{C}}4.$ Optimal Initiating Time (`OIT`)

a. Let $\beta = 1$ and $s = 0$. Then we have ⓢ$_{\blacktriangle}$ and 🅓$_{\|}$ on $\mathscr{F}^+$ (see Tables 20.1.1(p.189) and 20.2.3(p.222)).

b. Let $\beta < 1$ or $s > 0$.

1. For sA-model we have ⓢ$_{\blacktriangle}$, ⊚$_{\|}$, and 🅓$_{\|}$ (see Table 20.1.2(p.190)).
2. For sE-model we have ⓢ$_{\vartriangle}$, ⓢ$_{\blacktriangle}$, ⊚$_{\|}$, ⊚$_{\vartriangle}$, ⊚$_{\blacktriangle}$, 🅓$_{\|}$, 🅓$_{\vartriangle}$, and 🅓$_{\blacktriangle}$ (see Table 20.2.4(p.222)).

c. Joining Tables 20.1.3(p.190) and 20.2.5(p.222) produces the following table:

Table 20.3.1: Occurence rates of ⓢ, ⊚, and 🅓 on $\mathscr{F}^+$

ⓢ			⊚			🅓		
44.2 % / 58			23.0 % / 30			32.8 % / 43		
ⓢ$_{\|}$	ⓢ$_{\vartriangle}$	ⓢ$_{\blacktriangle}$	⊚$_{\|}$	⊚$_{\vartriangle}$	⊚$_{\blacktriangle}$	🅓$_{\|}$	🅓$_{\vartriangle}$	🅓$_{\blacktriangle}$
−	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`
− % / −	6.8 % / 9	37.4 % / 49	13.2 % / 17	7.5 % / 10	2.3 % / 3	17.5 % / 23	11.5 % / 15	3.8 % / 5

C̄5. Null-time-zone and deadline-engulfing

On $\mathscr{F}^+$, from Table 20.3.1(p.223) above we see that:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▲ (<u>strict</u> optimality).

b. As a whole, we have ⓢ, ◎, and 🅓 at respectively 44.2%, 23.0%, and 32.8% where

1. ⓢ$_{\|}$ cannot be defined due to Preference Rule 7.2.1(p.45).
2. ◎$_{\|}$ is possible (13.2 %).
3. 🅓$_{\|}$ is possible (17.5 %).
4. ⓢ$_{\triangle}$ is possible (6.8 %).
5. ◎$_{\triangle}$ is possible (7.5 %).
6. 🅓$_{\triangle}$ is possible (11.5 %).
7. ⓢ$_{\blacktriangle}$ is possible (37.4 %).
8. ◎$_{\blacktriangle}$ is possible (2.3 %).
   - `Tom` 20.2.2(p.192) (c3i2,c3ii1ii2,c3ii2i).
9. 🅓$_{\blacktriangle}$ is possible (3.8 %).
   - `Tom` 20.2.4(p.196) (d2i,d2ii).
   - `Tom` 20.2.16(p.206) (c2,c3i2,c3i3).

From the above results we see that:

A. ◎ and 🅓 causing the `null-time-zone` are possible at 55.8% (= 23.0% + 32.8%).

B. ◎$_{\blacktriangle}$ and 🅓$_{\blacktriangle}$ *strictly* causing the `null-time-zone` are possible at 2.3% and 3.8% respectively.

Chapter 21

Analysis of Model 3

21.1 Reduction

Definition 21.1.1 (reduction)

(a) If it is always optimal to reject the intervening quitting penalty price ρ in Model 3, then it follows that Model 3 is substantively reduced to Model 2 in which the ρ is not defined, schematized as

$$\text{Model 3} \twoheadrightarrow \text{Model 2}. \tag{21.1.1}$$

Let us represent this model reduction as the `model-running-back`; in other words, Model 3 in "downstream" runs back to Model 2 in "upstream".

(b) Let us define

$$\texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop} \overset{\text{def}}{=} \{\text{Accept the intervening quitting penalty price } \rho \text{ at any given time point on } t \geq 0 \ \} \\ \text{and stop the process}\}. \tag{21.1.2}$$

Let us represent the reduction of this optimal decision rule (`odr`) as `odr` $\mapsto \texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop}$.

(c) Let us schematize the above two reductions as

$$\text{Reduction} \begin{cases} \text{model reduction} \ \rightarrow \ \texttt{model-running-back} & (\twoheadrightarrow) \\ \texttt{odr} \text{ reduction} \ \rightarrow \ \texttt{odr} \mapsto \texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop} & (\mapsto) \ \square \end{cases} \tag{21.1.3}$$

Lemma 21.1.1 *Let* $\texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop}$ *holds. Then*

(a) *Let* $\beta = 1$. *Then we have* $\text{ⓓ}_{\parallel}$ *for any* ρ.

(b) *Let* $\beta < 1$ *and* $\rho < 0$. *Then we have* $\text{ⓓ}_{\blacktriangle}$.

(c) *Let* $\beta < 1$ *and* $\rho = 0$. *Then we have* $\text{ⓓ}_{\parallel}$.

(d) *Let* $\beta < 1$ *and* $\rho > 0$. *Then we have* $\text{ⓢ}_{\blacktriangle}$.

(e) *Let* $\rho \geq 0$. *Then we have* $\text{ⓢ}_{\vartriangle}$. $\square$

● *Proof* If $\texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop}$ holds, then we have $V_t = \rho$ for $t > 0$ from (6.4.38(p.41)), (6.4.44(p.41)), (6.4.52(p.41)), and (6.4.58(p.41)), we have $I_\tau^t = \beta^{\tau-t}\rho$ for $t > 0$ from (7.2.4(p.44)).

(a) Let $\beta = 1$. Then $\beta^0\rho = \beta^1\rho = \cdots = \beta^\tau\rho = \rho$ for any ρ, hence $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^0 = \rho$, so $t_\tau^* = 0$, i.e., $\text{ⓓ}_{\parallel}$.

(b) Let $\beta < 1$ and $\rho < 0$. Then $\beta^0\rho < \beta^1\rho < \cdots < \beta^\tau\rho$, hence $I_\tau^\tau < I_\tau^{\tau-1} < \cdots < I_\tau^0$, so $t_\tau^* = 0$, i.e., $\text{ⓓ}_{\blacktriangle}$.

(c) Let $\beta < 1$ and $\rho = 0$. Then $\beta^0\rho = \beta^1\rho = \cdots = \beta^\tau\rho = 0$, hence $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^0$, so $t_\tau^* = \tau = 0$, i.e., $\text{ⓓ}_{\parallel}$.

(d) Let $\beta < 1$ and $\rho > 0$. Then $\beta^0\rho > \beta^1\rho > \cdots > \beta^\tau\rho$, hence $I_\tau^\tau > I_\tau^{\tau-1} > \cdots > I_\tau^0$, so $t_\tau^* = \tau$, i.e., $\text{ⓢ}_{\blacktriangle}$.

(e) Let $\rho \geq 0$. Then $\beta^0\rho \geq \beta^1\rho \geq \cdots \geq \beta^\tau\rho$ for any $0 < \beta \leq 1$, hence $I_\tau^\tau \geq I_\tau^{\tau-1} \geq \cdots \geq I_\tau^0$, so $t_\tau^* = \tau$, i.e., $\text{ⓢ}_{\vartriangle}$. ∎

21.2 Search-Allowed-Model 3: $\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\} = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}3[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

21.2.1 Theorems

As ones corresponding to Theorems 12.5.1(p.80), 13.3.1(p.98), and 14.5.1(p.106) let us consider the following three theorems:

Theorem 21.2.1 (symmetry$[\mathbb{R} \rightarrow \tilde{\mathbb{R}}]$) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\rightarrow\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \ \square \tag{21.2.1}$$

Theorem 21.2.2 (analogy$[\mathbb{R} \rightarrow \mathbb{P}]$) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\rightarrow\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \ \square \tag{21.2.2}$$

Theorem 21.2.3 (symmetry($\mathbb{P} \to \tilde{\mathbb{P}}$)) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{21.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.4}$$
$$\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.5}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.6}$$

corresponding to (12.5.34(p.77)), (13.2.4(p.93)), and (14.5.4(p.106)). Now, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.4.5(p.41) it can be easily shown that (21.2.4(p.226)) and (21.2.6(p.226)) hold. However, from the comparison of (I) and (III) in Table 6.4.5(p.41) we can immediately see that (21.2.5(p.226)) does not always hold, hence it follows that also Theorem 21.2.2(p.225) does not always hold.

21.2.2 A Lemma

The following lemma determines if Theorem 21.2.2(p.225) holds by testing whether or not each of (21.2.5(p.226)) is true.

Lemma 21.2.1

(a) Theorem 21.2.1(p.225) *always hold.*

(b) Theorem 21.2.3(p.226) *always hold.*

(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$*. Then* Theorem 21.2.2(p.225) *holds.*

(d) *Let* $a^\star < \rho < b$*. Then* Theorem 21.2.2(p.225) *does not always hold.* $\square$

● *Proof* Almost the same as the proof of Lemma 20.1.1(p.153). ▮

21.2.3 $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$

□ **Tom 21.2.1** (▣$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$)

(a) *Let* $\rho \le x_K$ *or* $\rho \le 0$*. Then* $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) *Let* $\rho \ge x_K$ *and* $\rho \ge 0$*. Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. $\square$

● *Proof* From (6.4.39(p.41)) with $t = 1$ and (6.4.37(p.41)) we have $U_1 = \max\{K(V_0)+\rho, \beta V_0)\} = \max\{K(\rho)+\rho, \beta\rho\} \cdots$ **(1)**, hence $U_1 - \rho = \max\{K(\rho), -(1-\beta)\rho\} \cdots$ **(2)**. From (6.4.38(p.41)) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.4.39(p.41)) with $t = 2$ and Lemma 10.2.2(p.57) (e) we have $U_2 = \max\{K(V_1) + V_1, \beta V_1\} = \max\{K(V_0) + V_0, \beta V_0\} = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence from (6.4.38(p.41)) we have $V_{t-1} = \max\{\rho, U_{t-1}\} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$. Then, from (6.4.39(p.41)) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$ due to Lemma 10.2.2(p.57) (e). Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., we have that U_t is nondecreasing in $t > 0 \cdots$ **(3)**.

(a) Let $\rho \le x_K$ or $\rho \le 0$. Suppose $\rho \le x_K$, hence $K(\rho) \ge 0 \cdots$ **(4)** from Corollary 10.2.2(p.58) (b). Then, from (1) we have $U_1 \ge K(\rho) + \rho \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (3). Suppose $\rho \le 0$, hence $-(1-\beta)\rho \ge 0$. Then, noting (4), from (2) we have $U_1 - \rho \ge 0$, i.e., $U_1 \ge \rho$, so $U_t \ge \rho$ for $t > 0$ due to (3). Accordingly, whether $\rho \le x_K$ or $\rho \le 0$, we have $U_t \ge \rho$ for $t > 0$, meaning that it is always optimal to reject the intervening quitting penalty price ρ for any $t > 0$. This fact is the same as the event "the intervening quitting penalty price ρ does not exist on any time $t > 0$"; in other words, it follows that $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$ is substantially reduced to $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ which has not an intervening quitting penalty price ρ, i.e., $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) Let $\rho \ge x_K$ and $\rho \ge 0 \cdots$ **(5)**, hence $K(\rho) \le 0 \cdots$ **(6)** from Corollary 10.2.2(p.58) (a) and $-(1-\beta)\rho \le 0$. Then, since $U_1 - \rho \le 0$ from (2), we have $U_1 \le \rho \cdots$ **(7)**. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.4.38(p.41)), hence from (6.4.39(p.41)) we have $U_t = \max\{K(\rho) + \rho, \beta\rho\} = U_1 \le \rho$ due to (1) and (7). Accordingly, by induction $U_t \le \rho$ for $t > 0$, meaning that it is always optimal to accept the intervening quitting penalty price ρ at all time $t \ge 0$ and stop the process. Hence we have $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. ▮

21.2.4 $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]$

□ **Tom 21.2.2** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$)

(a) *Let* $\rho \ge x_{\tilde{K}}$ *or* $\rho \ge 0$*. Then* $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) *Let* $\rho \ge x_{\tilde{K}}$ *and* $\rho \ge 0$*. Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. $\square$

● *Proof by symmetry* Immediately from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom 21.2.1(p.226) due to Lemma 21.2.1(p.226) (a). ▮

21.2.5 M:3[ℙ][A]

21.2.5.1 Case of $\rho \le a^\star$ or $b \le \rho$

□ **Tom 21.2.3** (⊡$\mathscr{A}${M:3[ℙ][A]}) *Assume $\rho \le a^\star$ or $b \le \rho$. Then:*

(a) *Let $\rho \le x_K$ or $\rho \le 0$. Then* M:3[ℙ][A] ↠M:2[ℙ][A].

(b) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then we have* odr ↦ $\texttt{Accept}_{t\ge 0}(\rho)$ ▷ Stop. □

● *Proof by analogy* The same as Tom 21.2.1(p.226) due to Lemma 13.6.1(p.99). ∎

21.2.5.2 Case of $a^\star < \rho < b$

□ **Tom 21.2.4** (⊡$\mathscr{A}${M:3[ℙ][A]}) *Assume $a^\star < \rho < b$. Let $\beta = 1$ and $s = 0$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A]. □

● *Proof by analogy* Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.26)) we have $K(x) = \lambda T(x) \ge 0 \cdots$ **(1)** for any x due to Lemma 13.2.1(p.93) (g). From (6.4.45(p.41)) we have $U_1 \ge \beta\rho = \rho$. Suppose $U_{t-1} \ge \rho$. Then, from (6.4.44(p.41)) we have $V_{t-1} = U_{t-1} \ge \rho$, hence from (6.4.46(p.41)) we obtain $U_t \ge \beta V_{t-1} = V_{t-1} \ge \rho$. Thus, by induction $U_t \ge \rho$ for $t > 0$. Accordingly, for the same reason as in the proof of Tom 21.2.1(p.226) (a) we have M:3[ℙ][A] ↠ M:2[ℙ][A]. ∎

□ **Tom 21.2.5** (▣$\mathscr{A}${M:3[ℙ][A]}) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ or $-(1-\beta)\rho \ge 0$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A].

(b) *Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ and $-(1-\beta)\rho \le 0$.*

1. *Let $\tau = 1$. Then we have* odr ↦ $\texttt{Accept}_1(\rho)$ ▷ Stop.†
2. *Let $\tau > 1$. Then:*
   i. *Let $\rho \le x_K$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A]
   ii. *Let $\rho \ge x_K$. Then we have* odr ↦ $\texttt{Accept}_{t\ge 0}(\rho)$ ▷ Stop.† □

● *Proof* Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.4.45(p.41)) we have

$$U_1 - \rho = \max\{\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s, -(1-\beta)\rho\} \cdots \mathbf{(1)}.$$

(a) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ or $-(1-\beta)\rho \ge 0$, hence $U_1 - \rho \ge 0$ from (1) or equivalently $U_1 \ge \rho \cdots$ **(2)**. Then, since $V_1 = U_1 \cdots$ **(3)** from (6.4.44(p.41)) with $t = 1$, from (6.4.46(p.41)) with $t = 2$ we have $U_2 = \max\{K(V_1) + V_1, \beta V_1\} = \max\{K(U_1) + U_1, \beta U_1\} \cdots$ **(4)**. Hence, from (2), Lemma 13.2.3(p.96) (e), and (5.1.21(p.26)) we have

$$\begin{aligned} U_2 &\ge \max\{K(\rho) + \rho, \beta\rho\} \\ &= \max\{\lambda\beta T(\rho) - (1-\beta)\rho - s + \rho, \beta\rho\} \\ &= \max\{\lambda\beta T(\rho) + \beta\rho - s, \beta\rho\}. \end{aligned}$$

Then, from Lemma 13.2.1(p.93) (h) we have $U_2 \ge \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\} = U_1$ due to (6.4.45(p.41)). Suppose $U_{t-1} \ge U_{t-2}$, so $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.4.44(p.41)). Hence, from (6.4.46(p.41)) and Lemma 13.2.3(p.96) (e) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$. Accordingly, by induction $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t \ge \rho$ for $t > 0$. Therefore, for almost the same reason as in the proof of Tom 21.2.1(p.226) (a) we have M:3[ℙ][A] ↠ M:2[ℙ][A].

(b) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ and $-(1-\beta)\rho \le 0 \cdots$ **(5)**. Then $U_1 - \rho \le 0$ from (1), i.e., $U_1 \le \rho \cdots$ **(6)**.

(b1) Let $\tau = 1$. Then (6) implies that it is optimal to accept the intervening quitting penalty price ρ at $t = 1$ and stop the process, i.e., odr ↦ $\texttt{Accept}_1(\rho)$ ▷ Stop.

(b2) Let $\tau > 1$. Due to (6) we have $V_1 = \rho$ from (6.4.44(p.41)) with $t = 1$, hence $U_2 = \max\{K(\rho) + \rho, \beta\rho\} \cdots$ **(7)** from (6.4.46(p.41)) with $t = 2$.

(b2i) Let $\rho \le x_K$. Then $K(\rho) \ge 0$ from Lemma 13.2.3(p.96) (j1), hence from (7) we have $U_2 \ge K(\rho) + \rho \ge \rho$. Suppose $U_{t-1} \ge \rho$, hence $V_{t-1} = U_{t-1} = \rho$ from (6.4.44(p.41)). Then, from (6.4.46(p.41)) and Lemma 13.2.3(p.96) (e) we have $U_t \ge \max\{K(\rho) + \rho, \beta\rho\} \ge K(\rho) + \rho \ge \rho$. Accordingly, by induction we have $U_t \ge \rho$ for $t > 1$. Thus the assertion holds for the same reason as in the proof of Lemma 21.2.1(p.226) (a).

(b2ii) Let $\rho \ge x_K$, hence $K(\rho) < 0$ from Lemma 13.2.3(p.96) (j1). Then, from (7) we have $U_2 \le \max\{\rho, \beta\rho\} \cdots$ **(8)**. If $\beta < 1$, then $\rho \ge 0$ from (5), hence $U_2 \le \max\{\rho, \rho\} = \rho$ and if $\beta = 1$, then $U_2 \le \max\{\rho, \rho\} = \rho$. Accordingly, whether $\beta < 1$ or $\beta = 1$, we have $U_2 \le \rho$ for $t > 0$. Suppose $U_{t-1} \le \rho$, hence $V_{t-1} = \rho$ from (6.4.44(p.41)). Then, from (6.4.46(p.41)) we have $U_t = \max\{K(\rho) + \rho, \beta\rho\} = U_2 \le \rho$. Accordingly, by induction we have $U_t \le \rho$ for $t > 1$. Hence, from (6) we have $U_t \le \rho$ for $t > 0$. Thus, for the same reason as in the proof of Tom 21.2.1(p.226) (b) it follows that the assertion holds. ∎

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†In this case, we have four possibilities for the optimal initiating time (OIT): ⓓ‖, ⓓ▲, ⓢ▲, and ⓢ△.

21.2.6 $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A]

21.2.6.1 Case of $\rho \geq b^\star$ or $a \geq \rho$

☐ Tom **21.2.6** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A]$\}$) *Assume $\rho \geq b^\star$ or $a \geq \rho$.*

(a) *Let $\rho \geq x_{\tilde{K}}$ or $\rho \geq 0$. Then* $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A] ↠ $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A].

(b) *Let $\rho \leq x_{\tilde{K}}$ and $\rho \leq 0$. Then we have* odr ↦ $\mathtt{Accept}_\tau(\rho)$ ▷ Stop. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) due to Lemma 21.2.1(p.226) (b). ∎

21.2.6.2 Case of $b^\star > \rho > a$

☐ Tom **21.2.7** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A]$\}$) *Assume $b^\star > \rho > b$. Let $\beta = 1$ and $s = 0$. Then* $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A] ↠ $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) due to Lemma 21.2.1(p.226) (b). ∎

☐ Tom **21.2.8** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A]$\}$) *Assume $b^\star > \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \geq 0$ or $(1-\beta)\rho \geq 0$. Then* $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A] ↠ $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A].

(b) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho < s$ and $(1-\beta)\rho < 0$.*

  1. *Let $\tau = 1$. Then we have* odr ↦ $\mathtt{Accept}_1(\rho)$ ▷ Stop.
  2. *Let $\tau > 1$.*
     i. *Let $\rho > x_{\tilde{K}}$. Then* $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A] ↠ $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A].
     ii. *Let $\rho \leq x_{\tilde{K}}$. Then we have* odr ↦ $\mathtt{Accept}_{t\geq 0}(\rho)$ ▷ Stop. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.130))) due to Lemma 21.2.1(p.226) (b). ∎

21.2.7 Conclusion 5 (Search-Allowed-Model 3)

Model 3 (search-Allowed-model) is reduced to either of the following two cases (see (21.1.3(p.225))):

Case A M/$\tilde{\mathsf{M}}$:3[$\mathbb{R}$/$\mathbb{P}$][A] ↠ M/$\tilde{\mathsf{M}}$:2[$\mathbb{R}$/$\mathbb{P}$][A] where

1. M:3[$\mathbb{R}$][A] ↠ rM:2[$\mathbb{R}$][A]; see Tom 21.2.1(p.226) (a),
2. $\tilde{\mathsf{M}}$:3[$\mathbb{R}$][A] ↠ r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]; see Tom 21.2.2(p.226) (a),
3. M:3[$\mathbb{P}$][A] ↠ rM:2[$\mathbb{P}$][A]; see Tom 21.2.3(p.227) (a), 21.2.4(p.227) , and 21.2.5(p.227) (a,b2i),
4. $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A] ↠ r$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]; see Tom 21.2.6(p.228) (a), 21.2.7(p.228) , and 21.2.8(p.228) (a,b2i).

Case B odr ↦ $\mathtt{Accept}_{t\geq 0}(\rho)$ ▷ Stop where

1. For M:3[$\mathbb{R}$][A], see Tom 21.2.1(p.226) (b),
2. For $\tilde{\mathsf{M}}$:3[$\mathbb{R}$][A], see Tom 21.2.2(p.226) (b),
3. For M:3[$\mathbb{P}$][A], see Tom 21.2.3(p.227) (b),21.2.5(p.227) (b1,b2ii),
4. For $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A], see Tom 21.2.6(p.228) (b),21.2.8(p.228) (b1.b2ii).

21.3 Search-Enforced-Model 3: $\mathcal{Q}\{$M:3[E]$\}$ = $\{$M:3[$\mathbb{R}$][E], $\tilde{\mathsf{M}}$:3[$\mathbb{R}$][E], M:3[$\mathbb{P}$][E], $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][E]$\}$

21.3.1 Preliminary

As the ones corresponding to Theorems 21.2.1(p.225) , 21.2.2(p.225) , and 21.2.3(p.226) let us consider the following three theorems:

Theorem 21.3.1 (symmetry[$\mathbb{R}\to\mathbb{R}$]) *Let $\mathscr{A}\{$M:3[$\mathbb{R}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}$:3[$\mathbb{R}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}\text{:3}[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}\text{:3}[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{21.3.1}$$

Theorem 21.3.2 (analogy[$\mathbb{R}\to\mathbb{P}$]) *Let $\mathscr{A}\{$M:3[$\mathbb{R}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{$M:3[$\mathbb{P}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}\text{:3}[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}\text{:3}[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{21.3.2}$$

Theorem 21.3.3 (symmetry[$\mathbb{P}\to\mathbb{P}$])) *Let $\mathscr{A}\{$M:3[$\mathbb{P}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}$:3[$\mathbb{P}$][E]$\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}\text{:3}[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}\text{:3}[\mathbb{P}][\mathrm{E}]\}]. \quad \square \tag{21.3.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}\text{:3}[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}\text{:3}[\mathbb{R}][\mathrm{E}]\}], \tag{21.3.4}$$

$$\mathtt{SOE}\{\mathsf{M}\text{:3}[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}\text{:3}[\mathbb{R}][\mathrm{E}]\}], \tag{21.3.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}\text{:3}[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}\text{:3}[\mathbb{P}][\mathrm{E}]\}], \tag{21.3.6}$$

corresponding to (21.2.4(p.226)), (21.2.5(p.226)), and (21.2.6(p.226)). Now, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.4.6(p.41) it can be easily shown that (21.3.4(p.228)) and (21.3.6(p.228)) hold. However, from the comparison of (I) and (III) in Table 6.4.6(p.41) we can immediately see that (21.3.5(p.228)) does not hold, hence it follows that also Theorem 21.3.2(p.228) does not always hold.

21.3.2 A Lemma

Lemma 21.3.1

(a) Theorem 21.3.1(p.228) *always hold.*
(b) Theorem 21.3.3(p.228) *always hold.*
(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$. *Then* Theorem 21.3.2(p.228) *holds.*
(d) *Let* $a^\star < \rho < b$. *Then* Theorem 21.3.2(p.228) *does not always hold.* □

● *Proof* Almost the same as the proof of Lemma 20.1.1(p.153). ▮

21.3.3 M:3[ℝ][E]

□ Tom **21.3.1** (▣$\mathscr{A}${M:3[ℝ][E]})

(a) *Let* $\rho \le x_K$. *Then* M:3[ℝ][E] ↠ M:2[ℝ][E].

(b) *Let* $\rho \ge x_K$. *Then we have* odr ↦ $\texttt{Accept}_{\tau \ge t \ge 0}(\rho)$ ▷ Stop.[†] □

● *Proof* From (6.4.53(p.41)) with $t = 1$ and (6.4.51(p.41)) we have $U_1 = K(\rho) + \rho \cdots (1)$ and from (6.4.52(p.41)) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.4.53(p.41)) with $t = 2$ and Lemma 10.2.2(p.57) (e) we have $U_2 = K(V_1) + V_1 \ge K(\rho) + \rho = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.4.52(p.41)). Then from (6.4.53(p.41)) we have $U_t = K(V_{t-1}) + V_{t-1} \ge K(V_{t-2}) + V_{t-2} = U_{t-1}$ due to Lemma 10.2.2(p.57) (e) Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0 \cdots (2)$.

(a) Let $\rho \le x_K$, hence $K(\rho) \ge 0$ from Corollary 10.2.2(p.58) (b). Then, from (1) we have $U_1 \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (2). Accordingly, for almost the same reason as in the proof of Tom 21.2.1(p.226) (a) we have M̃:3[ℝ][E] ↠ M̃:2[ℝ][E].

(b) Let $\rho \ge x_K$, hence $K(\rho) \le 0 \cdots (3)$ from Corollary 10.2.2(p.58) (a). Then, from (1) we have $U_1 \le \rho$. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.4.52(p.41)), hence from (6.4.53(p.41)) we have $U_t = K(\rho) + \rho \le \rho$ due to (3). Accordingly, by induction $U_t \le \rho$ for $t > 0$, so we have odr ↦ $\texttt{Accept}_{\tau \ge t \ge 0}(\rho)$ ▷ Stop for the same reason as in Tom 21.2.1(p.226) (b).

21.3.4 M̃:3[ℝ][E]

□ Tom **21.3.2** (◫$\mathscr{A}${M̃:3[ℝ][E]}) *For any* $\beta \le 1$ *and* $s \ge 0$ *we have*:

(a) *Let* $\rho \le x_{\tilde{K}}$. *Then* M̃:3[ℝ][E] ↠ M̃:2[ℝ][E].

(b) *Let* $\rho \le x_{\tilde{K}}$. *Then we have* odr ↦ $\texttt{Accept}_{\tau \ge t \ge 0}(\rho)$ ▷ Stop. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R} \to \tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) due to Lemma 21.3.1(p.229) (a). ▮

21.3.5 M:3[ℙ][E]

21.3.5.1 Case of $\rho \le a^\star$ or $b \le \rho$

In this case, we can use Lemma 21.3.1(p.229) (c) to prove Tom 21.3.3(p.229) below.

□ Tom **21.3.3** (◫$\mathscr{A}${M:3[ℙ][E]}) *Assume* $\rho \le a^\star$ *or* $b \le \rho$.

(a) *Let* $\rho \le x_K$. *Then* M:3[ℙ][E] ↠M:2[ℙ][E].

(b) *Let* $\rho \ge x_K$. *Then we have* odr ↦ $\texttt{Accept}_{\tau \ge t \ge 0}(\rho)$ ▷ Stop. □

● *Proof by analogy* The same as Tom 21.3.1(p.229) due to Lemma 13.6.1(p.99). ▮

21.3.5.2 Case of $a^\star < \rho < b$

In this case, Tom's 21.3.4(p.229) and 21.3.5(p.229) below must be directly proven due to Lemma 21.3.1(p.229) (d).

□ Tom **21.3.4** (▣$\mathscr{A}${M:3[ℙ][E]}) *Assume* $a^\star < \rho < b$ *and let* $\beta = 1$ *and* $s = 0$. *Then we have* M:3[ℙ][E] ↠ M:2[ℙ][E]. □

● *Proof* Suppose $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. From (5.1.21(p.26)) we have $K(x) = \lambda T(x) \ge 0 \cdots (1)$ for any x due to Lemma 13.2.1(p.93) (g). Now, from (6.4.59(p.41)) we have $U_1 = \lambda \max\{0, a - \rho\} + \rho \ge \rho$ due to $\max\{0, a - \rho\} \ge 0$. Suppose $U_{t-1} \ge \rho$. Then, since $V_{t-1} = U_{t-1}$ due to (6.4.58(p.41)), from (6.4.60) we have $U_t = K(U_{t-1}) + U_{t-1} \ge U_{t-1}$ due to (1), hence $U_t \ge \rho$. Accordingly, by induction $U_t \ge \rho$ for $t > 0$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Thus, for almost the same as in the proof of Tom 21.2.1(p.226) (a) we have M:3[ℙ][E] ↠ M:2[ℙ][E]. ▮

□ Tom **21.3.5** (▣$\mathscr{A}${M:3[ℙ][E]}) *Assume* $a^\star < \rho < b$ *and let* $\beta < 1$ *or* $s > 0$.

(a) *Let* $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \ge s$. *Then* M:3[ℙ][E] ↠ M:2[ℙ][E].
(b) *Let* $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \le s$.

[†] In this case, we have four possibilities for the optimal initiating time (OIT): ⓓ$_{\|}$, ⓓ$_{\blacktriangle}$, ⓢ$_{\blacktriangle}$, and ⓢ$_{\triangle}$ (see Lemma 21.1.1(p.225)).

1. *Let $\tau = 1$. Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{t=1}(\rho) \triangleright \texttt{Stop}$.
2. *Let $\tau > 1$. Then*
   i. *Let $\rho \leq x_K$. Then* $\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]$.
   ii. *Let $\rho \geq x_K$. Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \texttt{Stop}$.

● *Proof* Suppose $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.4.59(p.41)) we have

$$U_1 - \rho = \lambda\beta \max\{0, a-\rho\} - (1-\beta)\rho - s \cdots \textbf{(1)}.$$

(a) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \geq s$, hence $U_1 \geq \rho \cdots \textbf{(2)}$ from (1). Then, since $V_1 = U_1 \cdots \textbf{(3)}$ from (6.4.58(p.41)) with $t = 1$, we have $U_2 = K(U_1) + U_1 \cdots \textbf{(4)}$ from (6.4.60(p.41)) with $t = 2$. Hence, from (2), Lemma 13.2.3(p.96) (e), and (5.1.21(p.26)) we have $U_2 \geq K(\rho) + \rho = \lambda\beta T(\rho) - (1-\beta)\rho - s + \rho = \lambda\beta T(\rho) + \beta\rho - s$. Then, from Lemma 13.2.1(p.93) (h) we have $U_2 \geq \lambda\beta\max\{0, a-\rho\} + \beta\rho - s = U_1$ due to (6.4.59(p.41)). Suppose $U_{t-1} \geq U_{t-2}$, hence $V_{t-1} \geq \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.4.58(p.41)). Then, from Lemma 13.2.3(p.96) (e) we have $U_t \geq K(V_{t-2}) + V_{t-2} = U_{t-1}$. Accordingly, by induction $U_t \geq U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t \geq \rho$ for $t > 0$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Therefore, for the same as in the proof of $\texttt{Tom}$ 21.2.1(p.226) (a) we have $\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]$.

(b) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \leq s \cdots \textbf{(5)}$. Then $U_1 - \rho \leq 0$ from (1), i.e., $U_1 \leq \rho \cdots \textbf{(6)}$.

(b1) Let $\tau = 1$. Now (6) implies that it is optimal to accept the intervening quitting penalty price ρ at the starting time $t = 1$ and the process stops, hence we have $\texttt{odr} \mapsto \texttt{Accept}_{t=1}(\rho) \triangleright \texttt{Stop}$.

(b2) Let $\tau > 1$. Now, due to (6) we have $V_1 = \rho$ from (6.4.58(p.41)) with $t = 1$, thus $U_2 = K(\rho) + \rho \cdots \textbf{(7)}$ from (6.4.60(p.41)) with $t = 2$.

(b2i) Let $\rho \leq x_K$, hence $K(\rho) \geq 0$ from Lemma 13.2.3(p.96) (j1). Then, from (7) we have $U_2 \geq \rho$. Suppose $U_{t-1} \geq \rho$, hence $V_{t-1} = U_{t-1}$ from (6.4.58(p.41)). Then, from (6.4.60(p.41)) and Lemma 13.2.3(p.96) (e) we have $U_t = K(U_{t-1}) + U_{t-1} \geq K(\rho) + \rho \geq \rho$. Hence, by induction $U_t \geq \rho$ for $t > 1$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Thus, for almost the same as in the proof of Lemma 21.2.1(p.226) (a) we have $\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]$.

(b2ii) Let $\rho \geq x_K$. Then $K(\rho) \leq 0 \cdots \textbf{(8)}$ from Lemma 13.2.3(p.96) (j1). Hence $U_2 \leq \rho$ from (7). Suppose $U_{t-1} \leq \rho$, hence $V_{t-1} = \rho$ from (6.4.58(p.41)). Then, from (6.4.60(p.41)) we have $U_t = K(\rho) + \rho \leq \rho \cdots \textbf{(9)}$ due to (8). Thus, by induction $U_t \leq \rho$ for $t > 1$. From this and (6) we have $U_t \leq \rho$ for $t > 0$, hence we have $\texttt{odr} \mapsto \texttt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \texttt{Stop}$ for the same reason as in the proof of $\texttt{Tom}$ 21.2.1(p.226) (b) we have that the assertion holds. ▮

21.3.6 $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]$

21.3.6.1 Case of $\rho \geq b^\star$ or $a \geq \rho$

□ **Tom 21.3.6** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\}$) *Assume $\rho \geq b^\star$ or $a \geq \rho$ and let $\beta \leq 1$ and $s \geq 0$.*

(a) *Let $\rho \geq x_{\tilde{K}}$. Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]$.

(b) *Let $\rho \leq x_{\tilde{K}}$. Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \texttt{Stop}$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P} \to \tilde{\mathbb{P}}}$ (see (18.0.2(p.130))) to $\texttt{Tom}$ 21.3.3(p.229). ▮

21.3.6.2 Case of $b^\star > \rho > a$

□ **Tom 21.3.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\}$) *Assume $b^\star > \rho \geq b$ and let $\beta = 1$ and $s = 0$. Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}] \mapsto \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P} \to \tilde{\mathbb{P}}}$ (see (18.0.2(p.130))) to $\texttt{Tom}$ 21.3.4(p.229). ▮

□ **Tom 21.3.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\}$) *Assume $b^\star > \rho > a$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \geq s$. Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]$.

(b) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \leq s$.*

1. *Let $\tau = 1$. Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{t=1}(\rho) \triangleright \texttt{Stop}$.
2. *Let $\tau > 1$. Then*
   i. *Let $\rho > x_{\tilde{K}}$. Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]$
   ii. *Let $\rho \leq x_{\tilde{K}}$. Then* $\texttt{odr} \mapsto \texttt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \texttt{Stop}$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P} \to \tilde{\mathbb{P}}}$ (see (18.0.2(p.130))) to $\texttt{Tom}$ 21.3.5(p.229). ▮

21.3.7 Conclusion 6 (Search-Enforced-Model 3)

This model (search-Enforced-model) is reduced to either of the following two cases (see (21.1.3(p.225))):

Case A $\mathsf{M}/\tilde{\mathsf{M}}{:}3[\mathbb{R}/\mathbb{P}][\mathtt{E}] \twoheadrightarrow \mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}/\mathbb{P}][\mathtt{E}]$ where

1. $\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}] \twoheadrightarrow \mathrm{r}\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]$; see Tom 21.3.1(p.229) (a),
2. $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}] \twoheadrightarrow \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]$; see Tom 21.3.2(p.229) (a),
3. $\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}] \twoheadrightarrow \mathrm{r}\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]$; see Tom 21.3.3(p.229) (a), 21.3.4(p.229), and 21.3.5(p.229) (a,b2i),
4. $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}] \twoheadrightarrow \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]$; see Tom 21.3.6(p.230) (a), 21.3.7(p.230), and 21.3.8(p.230) (a,b2i).

Case B $\mathtt{odr} \mapsto \mathtt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \mathtt{Stop}$ where

1. For $\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]$, see Tom 21.3.1(p.229) (b),
2. For $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]$, see Tom 21.3.2(p.229) (b),
3. For $\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]$, see Tom 21.3.3(p.229) (b),21.3.5(p.229) (b1,b2ii),
4. For $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]$, see Tom 21.3.6(p.230) (b),21.3.8(p.230) (b1,b2ii).

21.4 Conclusions of Model 3

This model (whether search-Enforced-model or search-Allowed-model) is reduced to either of the following two cases (see Conclusions 5 (p.228) and 6 (p.231)):

$\overline{\mathsf{C}}$1. $\mathsf{M}/\tilde{\mathsf{M}}{:}3[\mathbb{R}/\mathbb{P}][\mathtt{A/E}] \twoheadrightarrow \mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}/\mathbb{P}][\mathtt{A/E}]$.

$\overline{\mathsf{C}}$2. $\mathtt{odr} \mapsto \mathtt{Accept}_{\tau \geq t \geq 0}(\rho) \triangleright \mathtt{Stop}$.

Chapter 22

Conclusions of Part 3 (No-Recall-Model)

Below is the summary of Sections 19.3(p.151), 20.3(p.223), and 21.4(p.231).

22.1 Models 1/2

$\overline{\overline{\mathsf{C}}}$1. Mental Conflict

Here the adverb "always" means "whether search-**A**llowed-model or search-**E**nforced-model, whether selling model or buying model, and whether $\mathbb{R}$-model or $\mathbb{P}$-model". Then, $\overline{\mathsf{C}}$1(p.151) and $\overline{\mathsf{C}}$1(p.223) can be rewritten as follows.

a. Model 1. On $\mathscr{F}^+$,

Let $\beta \leq 1$ and $s \geq 0$. Then we always have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6).

b. Model 2. On $\mathscr{F}^+$,

1. Let $\beta = 1$ and $s = 0$. Then we always have the normal mental conflict, which coincides with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6).
2. Let $\beta < 1$ or $s > 0$. Then we always have the abnormal mental conflict, which does not coincide with ***expectations*** in *Examples* 1.4.1(p.5) - 1.4.4(p.6).

$\overline{\overline{\mathsf{C}}}$2. Symmetry

a. On $\mathscr{F}^+$:

1. Let $\beta = 1$ and $s = 0$. Then, for Models 1/2 the symmetry is inherited (see $\overline{\mathsf{C}}$2b(p.151) and $\overline{\mathsf{C}}$2a(p.223)).
2. Let $\beta < 1$ or $s > 0$. Then the symmetry *may* collapse for Model 1 (see $\overline{\mathsf{C}}$2c(p.151)) and *always* collapse for Model 2 (see $\overline{\mathsf{C}}$2b(p.223)).

$\overline{\overline{\mathsf{C}}}$3. Analogy

a. Model 1. On $\mathscr{F}^+$:

1. Let $\beta = 1$ and $s = 0$. Then the analogy is inherited (see C5b3(p.151)).
2. Let $\beta < 1$ or $s > 0$. Then analogy is *may* collapses (see C3c(p.151)).

b. Model 2. On $\mathscr{F}^+$:

1. For any $\beta \leq 1$ and $s \geq 0$, the analogy *may* collapse (see C3a(p.223)).

$\overline{\overline{\mathsf{C}}}$4. Optimal Initiating Time (OIT)

On $\mathscr{F}^+$:

a. Let $\beta = 1$ and $s = 0$.

1. For Model 1, only ⓢ$_{\blacktriangle}$ is possible (see Tables 19.1.1(p.135) and 19.2.1(p.149)).
2. For Model 2, only ⓢ$_{\blacktriangle}$ and 🅓$_{\|}$ are possible (see Tables 20.1.1(p.189) and 20.2.3(p.222)).
3. What is remarkable here is that 🅓$_{\|}$ (deadline-engulfing) occurs even in the simplest case of "$\beta = 1$ and $s = 0$" (see $\overline{\mathsf{C}}$4a(p.223)).

b. Let $\beta < 1$ or $s > 0$.

1. For Model 1, ⓢ$_{\blacktriangle}$, ⊚$_{\|}$, ⊚$_{\triangle}$, ⊚$_{\blacktriangle}$, 🅓$_{\|}$, and 🅓$_{\triangle}$ are possible (see Tables 19.1.2(p.135) and 19.2.2(p.150)).
2. For Model 2, ⓢ$_{\triangle}$, ⓢ$_{\blacktriangle}$, ⊚$_{\|}$, ⊚$_{\triangle}$, ⊚$_{\blacktriangle}$, 🅓$_{\|}$, 🅓$_{\triangle}$, and 🅓$_{\blacktriangle}$ are possible (see Tables 20.1.2(p.190) and 20.2.4(p.222)).

c. Joining Tables 19.3.1(p.151) and 20.3.1(p.223) produces the following table:

Table 22.1.1: Occurance rates of ⓢ, ⊚, and 🅓 on $\mathscr{F}$

ⓢ			⊚			🅓		
44.4 % / 68			22.2 % / 34			33.4 % / 51		
ⓢ$_{\|}$	ⓢ$_{\triangle}$	ⓢ$_{\blacktriangle}$	⊚$_{\|}$	⊚$_{\triangle}$	⊚$_{\blacktriangle}$	🅓$_{\|}$	🅓$_{\triangle}$	🅓$_{\blacktriangle}$
—	possible	possible	possible	possible	possible	possible	possible	possible
– % / –	5.9 % / 9	38.6 % / 59	12.4 % / 19	7.2 % / 11	2.6 % / 4	19.0 % / 29	11.1 % / 17	3.2 % / 5

C̿5. Null-time-zone and deadline-engulfing

From Table 22.1.1(p.234) above, we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.45) for the noteworthy implication of the symbol ▲ (strict optimality).

b. As a whole, we have ⓢ, ⊚, and 🅓 at 44.4%, 22.2%, and 33.4% respectively where

1. ⓢ$_{\|}$ cannot be defined due to Preference Rule 7.2.1(p.45).
2. ⊚$_{\|}$ is possible (12.4 %).
3. 🅓$_{\|}$ is possible (19.0 %).
4. ⓢ$_{\triangle}$ never occur (5.9 %).
5. ⊚$_{\triangle}$ is possible (7.2 %).
6. 🅓$_{\triangle}$ is possible (11.1 %).
7. ⓢ$_{\blacktriangle}$ is possible (38.6%) (see Remark 7.2.2(p.45)),
8. ⊚$_{\blacktriangle}$ is possible(2.6%).
   - `Tom` 19.2.2(p.137) (c2iii2)
   - `Tom` 20.2.2(p.192) (c3i2,c3ii1ii2,c3ii2i).
9. 🅓$_{\blacktriangle}$ is possible (3.2%).
   - `Tom` 20.2.4(p.196) (d2i,d2ii).
   - `Tom` 20.2.16(p.206) (c2,c3i2,c3i3).

The following three are especially noteworthy findings:

A. ⊚ and 🅓 causing the null-time-zone occur at 55.6% (= 22.2% + 33.4%).

B. 🅓 causing the deadline-engulfing occurs at 33.4%.

C. ⊚ $_{\blacktriangle}$ and 🅓 $_{\blacktriangle}$ causing the deadline-engulfing occurs at 2.6% and 3.2% respectively.

D. 🅓$_{\|}$ causing the deadline-engulfing occurs even in the simplest case of "$\beta = 1$ and $s = 0$" (see C̿4a3(p.233)).

C̿6. `C⇝S` (`Conduct⇝Skip`) (see Def. 2.2.1(p.12) and Remark 7.2.1(p.44))

It is *only* for M:2[ℝ][`A`]$^+$ and M:2[ℙ][`A`]$^+$ with $\beta < 1$ or $s > 0$ (see Table 20.1.4(p.190)) that we have observed `C⇝S`. It is usual to assume that once conducting a search is optimal, it will become optimal to continue conducting the search afterward. However, we demonstrated that this expectation does not always hold. In other words, it can become optimal to skip the search after initially continuing it for a while.

22.2 Models 3

C̿9. Reduction

Model 3 is reduced to the following two cases (see Section 21.4(p.231)):

a. model-running-back M/M̃:3[ℝ/ℙ][`A/E`] ↠ M/M̃:2[ℝ/ℙ][`A/E`].

b. `odr`-reduction `odr` ↦ $\texttt{Accept}_{\tau\geq t\geq 0}(\rho)$ ▷ `Stop`.

Part 4

Recall-Model

Chapter 23

Definitions of Models

23.1 Future Subjects

F.S. 1 (future subject) *In the recall-model with $\mathbb{R}$-mechanism it suffices to memorize only the best of prices which have been rejected so far. Against this, in the recall-model with $\mathbb{P}$-mechanism it is hard to define the best price itself. For this reason, in this chapter we exclude the application of the integrated-theory to the latter model, which is left as a subject to be tackled in the future* (see F2(p.288)). □

For convenience of reference, below let us copy Table 3.2.2(p.18) where ——— represents the model excluded for the above reason.

Table 23.1.1: The 24 recall-models

	ASP[$\mathbb{R}$]	ABP[$\mathbb{R}$]	ASP[$\mathbb{P}$]	ABP[$\mathbb{P}$]
$\mathcal{Q}${rM:1[A]} = {	rM:1[$\mathbb{R}$][A],	r$\tilde{\text{M}}$:1[$\mathbb{R}$][A],	~~rM:1[$\mathbb{P}$][A]~~,	~~r$\tilde{\text{M}}$:1[$\mathbb{P}$][A]~~ }
$\mathcal{Q}${rM:1[E]} = {	rM:1[$\mathbb{R}$][E],	r$\tilde{\text{M}}$:1[$\mathbb{R}$][E],	~~rM:1[$\mathbb{P}$][E]~~,	~~r$\tilde{\text{M}}$:1[$\mathbb{P}$][E]~~ }
$\mathcal{Q}${rM:2[A]} = {	rM:2[$\mathbb{R}$][A],	r$\tilde{\text{M}}$:2[$\mathbb{R}$][A],	~~rM:2[$\mathbb{P}$][A]~~,	~~r$\tilde{\text{M}}$:2[$\mathbb{P}$][A]~~ }
$\mathcal{Q}${rM:2[E]} = {	rM:2[$\mathbb{R}$][E],	r$\tilde{\text{M}}$:2[$\mathbb{R}$][E],	~~rM:2[$\mathbb{P}$][E]~~,	~~r$\tilde{\text{M}}$:2[$\mathbb{P}$][E]~~ }
$\mathcal{Q}${rM:3[A]} = {	rM:3[$\mathbb{R}$][A],	r$\tilde{\text{M}}$:3[$\mathbb{R}$][A],	~~rM:3[$\mathbb{P}$][A]~~,	~~r$\tilde{\text{M}}$:3[$\mathbb{P}$][A]~~ }
$\mathcal{Q}${rM:3[E]} = {	rM:3[$\mathbb{R}$][E],	r$\tilde{\text{M}}$:3[$\mathbb{R}$][E],	~~rM:3[$\mathbb{P}$][E]~~,	~~r$\tilde{\text{M}}$:3[$\mathbb{P}$][E]~~ }

23.2 Model 1

23.2.1 Search-Enforced-Model 1: $\mathcal{Q}${rM:1[E]} = {**r**M:1[$\mathbb{R}$][E], **r**$\tilde{\text{M}}$:1[$\mathbb{R}$][E], ~~**r**M:1[$\mathbb{P}$][E], **r**$\tilde{\text{M}}$:1[$\mathbb{P}$][E]~~}

23.2.1.1 rM:1[$\mathbb{R}$][E]

This is the most basic model of the selling model with recall, which is identical to M:1[$\mathbb{R}$][E] (see Section 4.1.1.1.1(p.21)) except that the price to be accepted is the best among the prices rejected so far.

23.2.1.2 r$\tilde{\text{M}}$:1[$\mathbb{R}$][E]

This is the most basic model of the buying model with recall, which is the same as $\tilde{\text{M}}$:1[$\mathbb{R}$][E] (see Section 4.1.1.1.2(p.22)) except that the price to be accepted is the best of prices rejected so far.

23.2.2 Search-Allowed-Model 1: $\mathcal{Q}${rM:1[A]} = {**r**M:1[$\mathbb{R}$][A], **r**$\tilde{\text{M}}$:1[$\mathbb{R}$][A], ~~**r**M:1[$\mathbb{P}$][A], **r**$\tilde{\text{M}}$:1[$\mathbb{P}$][A]~~}

This is the same model as the one described in Section 23.2.1(p.237), except that the search is allowed.

23.3 Model 2

This model is defined by adding the terminal quitting penalty price ρ to Model 1 as described in Section 23.2(p.237).

23.4 Model 3

This model is defined by adding the intervening quitting penalty price ρ to Model 2 as described in Section 23.3(p.237).

23.5 Best Price

Definition 23.5.1 (best price)

(a) In the selling model M (buying model $\tilde{\text{M}}$) let us refer to the highest y of buying prices (the highest y of selling prices) which have been offered and rejected as the *best price* y.

(b) By $\texttt{Accept}_t\langle y\rangle$ ($\texttt{Reject}_t\langle y\rangle$) let us denote "Accept (Reject) the best price y at time t". □

Remark 23.5.1 When the process initiates at a given time t, there exist no best price since no search activity is conducted before that. □

Chapter 24

Systems of Optimality Equations

For this model we consider only $\mathbb{R}$-model (see F.S 1(p.237)).

24.1 Model 1

24.1.1 Search-Allowed-Model 1

24.1.1.1 rM:1[$\mathbb{R}$][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = y, \tag{24.1.1}$$
$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{24.1.2}$$
$$V_1 = \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s, \tag{24.1.3}$$
$$V_t = \max\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s, \beta V_{t-1}\} \quad t > 1, \tag{24.1.4}$$

where $V_t(y)$ is the maximum total expected present discounted *profit* from rejecting the best price y, expressed as

$$V_t(y) = \max\{\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.1.5}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\texttt{A}]\} = \{(24.1.1(\text{p.239})) - (24.1.5(\text{p.239}))\}. \tag{24.1.6}$$

For convenience let us define

$$V_0(y) = y. \tag{24.1.7}$$

Then (24.1.2(p.239)) holds for $t \geq 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0. \tag{24.1.8}$$

From (24.1.4(p.239)) and (24.1.5(p.239)) with $t = 1$ we have respectively

$$V_1(y) = \max\{\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s, \beta y\} \tag{24.1.9}$$
$$= \max\{K(y) + y, \beta y\} \quad (\text{from } (5.1.10(\text{p.25})) \text{ with } \lambda = 1) \tag{24.1.10}$$
$$= \max\{L(y) + \beta y, \beta y\} \quad (\text{from } (5.1.9(\text{p.25}))). \tag{24.1.11}$$
$$= \max\{L(y), 0\} + \beta y. \tag{24.1.12}$$

Let us here define

$$\mathbb{S}_t = \beta(\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 1. \tag{24.1.13}$$

Then, (24.1.4(p.239)) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{24.1.14}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 1. \tag{24.1.15}$$

More strictly

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.1.16}$$
$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.1.17}$$
$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.1.18}$$

Furthermore let us define

$$\mathbb{S}_t(y) \;=\; \beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) - s, \quad t > 0. \tag{24.1.19}$$

Then (24.1.5(p.239)) can be rewritten as

$$V_t(y) = \max\{\mathbb{S}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{24.1.20}$$

implying that

$$\mathbb{S}_t(y) \geq (\leq)\; 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{24.1.21}$$

More strictly

$$\mathbb{S}_t(y) \geq (\leq)\; 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.1.22}$$

$$\mathbb{S}_t(y) = (=)\; 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.1.23}$$

$$\mathbb{S}_t(y) > (<)\; 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.1.24}$$

From the comparison of the two terms within { } in the right-hand side of (24.1.2(p.239)) we see that the decision "whether or not to accept the best price y" can be prescribed as follows:

$$\left.\begin{array}{lcl} y \geq V_t(y) & \Rightarrow & \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare \\ y \leq V_t(y) & \Rightarrow & \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t{}^{\dagger} \end{array}\right\} \quad t > 0 \tag{24.1.25}$$

24.1.1.2 r$\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) \;=\; y, \tag{24.1.26}$$

$$v_t(y) \;=\; \min\{y, V_t(y)\}, \quad t > 0, \tag{24.1.27}$$

$$V_1 \;=\; \beta\,\mathbf{E}[\boldsymbol{\xi}] + s = \beta\mu + s, \tag{24.1.28}$$

$$V_t \;=\; \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s, \beta V_{t-1}\} \quad t > 1, \tag{24.1.29}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \min\{\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.1.30}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{r}\tilde{\mathsf{M}}\text{:}1[\mathbb{R}][\texttt{A}]\} = \{(24.1.26(\text{p.}240)) - (24.1.30(\text{p.}240))\}. \tag{24.1.31}$$

For convenience let us define

$$V_0(y) = y. \tag{24.1.32}$$

Then (24.1.27(p.240)) holds for $t \geq 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0. \tag{24.1.33}$$

Let us define

$$\tilde{\mathbb{S}}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 1. \tag{24.1.34}$$

Then (24.1.29(p.240)) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{24.1.35}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 1. \tag{24.1.36}$$

More strictly

$$\tilde{\mathbb{S}}_t \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.1.37}$$

$$\tilde{\mathbb{S}}_t = (=)\; 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.1.38}$$

$$\tilde{\mathbb{S}}_t < (>)\; 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.1.39}$$

Let us define

$$\tilde{\mathbb{S}}_t(y) \;=\; \beta(\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) + s, \quad t > 0. \tag{24.1.40}$$

Then (24.1.30(p.240)) can be rewritten as, for any y,

$$V_t(y) = \min\{\tilde{\mathbb{S}}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{24.1.41}$$

implying that

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{24.1.42}$$

More strictly

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\; 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.1.43}$$

$$\tilde{\mathbb{S}}_t(y) = (=)\; 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.1.44}$$

$$\tilde{\mathbb{S}}_t(y) < (>)\; 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.1.45}$$

---

†The symbol "/" means "or", i.e., "$\texttt{CONDUCT}_t$ or $\texttt{SKIP}_t$".

From the comparison of the two terms within { } in the right-hand side of (24.1.27(p.240)) we see that the decision “whether or not to accept the best price y” can be prescribed as follows:

$$\left.\begin{array}{lll} y \le V_t(y) & \Rightarrow & \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare \\ y \ge V_t(y) & \Rightarrow & \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{array}\right\} \quad t>0 \tag{24.1.46}$$

24.1.2 Search-Enforced-Model 1

24.1.2.1 rM:1[ℝ][E]

This is the most basic model with recall [44,Sak1961], the system of optimality equations of which is given as below. By $v_t(y)$ $(t \ge 0)$ and V_t $(t > 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = y, \tag{24.1.47}$$

$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{24.1.48}$$

$$V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s, \quad t > 0, \tag{24.1.49}$$

where $V_t(y)$ is the maximum total expected present discounted *profit* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s, \quad t > 0. \tag{24.1.50}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\} = \{(24.1.47(p.241)) - (24.1.50(p.241))\}. \tag{24.1.51}$$

For convenience let us define

$$V_0(y) = y. \tag{24.1.52}$$

Then (24.1.48(p.241)) holds for $t \ge 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \ge 0. \tag{24.1.53}$$

From (24.1.49(p.241)) and (24.1.50(p.241)) with $t = 1$ we have respectively

$$V_1 \;=\; \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s, \tag{24.1.54}$$

$$V_1(y) \;=\; \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s \tag{24.1.55}$$

$$= K(y) + y \quad \text{(from (5.1.10(p.25)) with } \lambda = 1) \tag{24.1.56}$$

$$= L(y) + \beta y \quad \text{(from (5.1.9(p.25)))}. \tag{24.1.57}$$

From the comparison of the two terms within { } in the right-hand side of (24.1.48(p.241)) we see that the decision “whether or not to accept the best price y” can be prescribed as follows:

$$\left.\begin{array}{lll} y \ge V_t(y) & \Rightarrow & \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare \\ y \le V_t(y) & \Rightarrow & \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \quad t>0. \tag{24.1.58}$$

24.1.2.2 r$\tilde{\text{M}}$:1[ℝ][E]

By $v_t(y)$ $(t \ge 0)$ and V_t $(t > 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = y, \tag{24.1.59}$$

$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{24.1.60}$$

$$V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s, \quad t > 0, \tag{24.1.61}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s, \quad t > 0. \tag{24.1.62}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \{(24.1.59(p.241)) - (24.1.62(p.241))\}. \tag{24.1.63}$$

For convenience let us define

$$V_0(y) = y. \tag{24.1.64}$$

Then (24.1.60(p.241)) holds for $t \ge 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \ge 0. \tag{24.1.65}$$

From the comparison of the two terms within { } in the right-hand side of (24.1.60(p.241)) we see that the decision “whether or not to accept the best price y” can be prescribed as follows:

$$\left.\begin{array}{lll} y \le V_t(y) & \Rightarrow & \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare \\ y \ge V_t(y) & \Rightarrow & \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \quad t>0. \tag{24.1.66}$$

24.2 Mode 2

24.2.1 Search-Allowed-Model 2

24.2.1.1 rM:2[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{24.2.1}$$
$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{24.2.2}$$
$$V_0 = \rho, \tag{24.2.3}$$
$$V_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0, \tag{24.2.4}$$

where $V_t(y)$ $(t > 0)$ is the maximum total expected present discounted *profit* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.2.5}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:2}[\mathbb{R}][\texttt{A}]\} = \{(24.2.1(\text{p.242})) - (24.2.5(\text{p.242}))\}. \tag{24.2.6}$$

For convenience let us define

$$V_0(y) = \rho. \tag{24.2.7}$$

Then (24.2.2(p.242)) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0, \tag{24.2.8}$$

From (24.2.4(p.242)) and (24.2.5(p.242)) with $t = 1$ we have respectively

$$\begin{aligned}
V_1 \;&=\; \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} && (24.2.9)\\
&=\; \max\{K(\rho) + \rho, \beta\rho\} \quad \text{(see (5.1.10(p.25)))} && (24.2.10)\\
&=\; \max\{L(\rho) + \beta\rho, \beta\rho\} \quad \text{(see (5.1.9(p.25)))} && (24.2.11)\\
&=\; \max\{L(\rho), 0\} + \beta\rho, && (24.2.12)\\
V_1(y) \;&=\; \max\{\lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} && (24.2.13)\\
&=\; \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} && \\
&=\; \max\{K(\max\{y, \rho\}) + \max\{y, \rho\}, \beta\max\{y, \rho\}\} \quad \text{(see (5.1.10(p.25)))} && (24.2.14)\\
&=\; \max\{L(\max\{y, \rho\}) + \beta\max\{y, \rho\}, \beta\max\{y, \rho\}\} \quad \text{(see (5.1.9(p.25)))} && (24.2.15)\\
&=\; \max\{L(\max\{y, \rho\}), 0\} + \beta\max\{y, \rho\}. && (24.2.16)
\end{aligned}$$

Now let us define

$$\mathbb{S}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \tag{24.2.17}$$

Then, (24.2.4(p.242)) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{24.2.18}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{24.2.19}$$

More strictly

$$\begin{aligned}
\mathbb{S}_t \geq (\leq)\ 0 &\Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). && (24.2.20)\\
\mathbb{S}_t = (=)\ 0 &\Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). && (24.2.21)\\
\mathbb{S}_t > (<)\ 0 &\Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). && (24.2.22)
\end{aligned}$$

In addition, let us define

$$\mathbb{S}_t(y) \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) - s, \quad t > 0. \tag{24.2.23}$$

Then (24.2.5(p.242)) can be rewritten as, for any y,

$$V_t(y) = \max\{\mathbb{S}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{24.2.24}$$

implying that

$$\mathbb{S}_t(y) \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{24.2.25}$$

More strictly

$$\begin{aligned}
\mathbb{S}_t(y) \geq (\leq)\ 0 &\Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). && (24.2.26)\\
\mathbb{S}_t(y) = (=)\ 0 &\Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). && (24.2.27)\\
\mathbb{S}_t(y) > (<)\ 0 &\Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). && (24.2.28)
\end{aligned}$$

From the comparison of the two terms within { } in the right-hand side of (24.2.2(p.242)) we see that the decision "whether or not to accept the best price y" can be prescribed as follows:

$$\left.\begin{aligned}
y \geq V_t(y) \quad &\Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare\\
y \leq V_t(y) \quad &\Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t
\end{aligned}\right\}\quad t > 0 \tag{24.2.29}$$

24.2.1.2 r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{24.2.30}$$
$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{24.2.31}$$
$$V_0 = \rho, \tag{24.2.32}$$
$$V_t = \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\}, \quad t > 0, \tag{24.2.33}$$

where $V_t(y)$ $(t > 0)$ is the minimum total expected present discounted *cost* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.2.34}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\texttt{A}]\} = \{(24.2.30(\text{p.243})) - (24.2.34(\text{p.243}))\}. \tag{24.2.35}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{24.2.36}$$

Then (24.2.31(p.243)) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{24.2.37}$$

Let us define

$$\tilde{\mathbb{S}}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{24.2.38}$$

Then (24.2.33(p.243)) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{24.2.39}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{24.2.40}$$

More strictly

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.2.41}$$
$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.2.42}$$
$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.2.43}$$

In addition, let us define

$$\tilde{\mathbb{S}}_t(y) \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) + s, \quad t > 0. \tag{24.2.44}$$

Then (24.2.34(p.243)) can be rewritten as, for any y,

$$V_t(y) = \min\{\tilde{\mathbb{S}}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{24.2.45}$$

implying that

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \tag{24.2.46}$$

More strictly

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{24.2.47}$$
$$\tilde{\mathbb{S}}_t(y) = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{24.2.48}$$
$$\tilde{\mathbb{S}}_t(y) < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{24.2.49}$$

From the comparison of the two terms within { } in the right-hand side of (24.2.31(p.243)) we see that the decision "whether or not to accept the best price y" can be prescribed as follows:

$$\left.\begin{array}{lcl} y \leq V_t(y) & \Rightarrow & \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare \\ y \geq V_t(y) & \Rightarrow & \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{array}\right\}\quad t > 0 \tag{24.2.50}$$

24.2.2 Search-Enforced-Model 2

24.2.2.1 rM:2[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{24.2.51}$$

$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{24.2.52}$$

$$V_0 = \rho, \tag{24.2.53}$$

$$V_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \quad t > 0, \tag{24.2.54}$$

where $V_t(y)$ $(t > 0)$ is the maximum total expected present discounted *profit* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \quad t > 0. \tag{24.2.55}$$

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{rM}{:}2[\mathbb{R}][\mathrm{E}]\} = \{(24.2.51(\text{p.244})) - (24.2.55(\text{p.244}))\}. \tag{24.2.56}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{24.2.57}$$

Then (24.2.52(p.244)) holds for $t \geq 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0. \tag{24.2.58}$$

From (24.2.54(p.244)) and (24.2.55(p.244)) with $t = 1$ we have respectively

$$\begin{aligned} V_1 &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s \\ &= K(\rho) + \rho \quad (\text{from } (5.1.10(\text{p.25}))) && (24.2.59) \\ &= L(\rho) + \beta\rho \quad (\text{from } (5.1.9(\text{p.25}))), && (24.2.60) \end{aligned}$$

$$\begin{aligned} V_1(y) &= \lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s \\ &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}\}] + (1-\lambda)\beta\max\{y, \rho\} - s \\ &= K(\max\{y, \rho\}) + \max\{y, \rho\} \quad (\text{from } (5.1.10(\text{p.25}))) && (24.2.61) \\ &= L(\max\{y, \rho\}) + \beta\max\{y, \rho\} \quad (\text{from } (5.1.9(\text{p.25}))). && (24.2.62) \end{aligned}$$

From the comparison of the two terms within { } in the right-hand side of (24.2.52(p.244)) we see that the decision "whether or not to accept the best price y" can be prescribed as follows:

$$\left.\begin{array}{lll} y \geq V_t(y) & \Rightarrow & \mathtt{Accept}_t\langle y\rangle \text{ and the process stops} \blacksquare \\ y \leq V_t(y) & \Rightarrow & \mathtt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \quad t > 0 \tag{24.2.63}$$

24.2.2.2 r$\tilde{\mathrm{M}}$:2[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{24.2.64}$$

$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{24.2.65}$$

$$V_0 = \rho, \tag{24.2.66}$$

$$V_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \quad t > 0, \tag{24.2.67}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the best price y, expressed as

$$V_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0. \tag{24.2.68}$$

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{r}\tilde{\mathrm{M}}{:}2[\mathbb{R}][\mathrm{E}]\} = \{(24.2.64(\text{p.244})) - (24.2.68(\text{p.244}))\}. \tag{24.2.69}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{24.2.70}$$

Then (24.2.65(p.244)) holds for $t \geq 1$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{24.2.71}$$

From the comparison of the two terms within { } in the right-hand side of (24.2.65(p.244)) we see that the decision "whether or not to accept the best price y" can be prescribed as follows:

$$\left.\begin{array}{lll} y \leq V_t(y) & \Rightarrow & \mathtt{Accept}_t\langle y\rangle \text{ and the process stops} \blacksquare \\ y \geq V_t(y) & \Rightarrow & \mathtt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \quad t > 0 \tag{24.2.72}$$

24.3 Mode : 3

24.3.1 Search-Allowed-Model 3

24.3.1.1 rM:3[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{24.3.1}$$

$$v_t(y) = \max\{y, \rho, U_t(y)\}, \quad t > 0, \tag{24.3.2}$$

$$V_0 = \rho, \tag{24.3.3}$$

$$V_t = \max\{\rho, U_t\}, \quad t > 0. \tag{24.3.4}$$

where $U_t(y)$ in (24.3.2(p.245)) is the maximum total expected present discounted *profit* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \max\{\lambda\beta\, \mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.3.5}$$

where U_t in (24.3.4(p.245)) is the maximum total expected present discounted *profit* from rejecting ρ, expressed as

$$U_t = \max\{\lambda\beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0, \tag{24.3.6}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:3}[\mathbb{R}][\texttt{A}]\} = \{(24.3.1(\text{p.245})) - (24.3.6(\text{p.245}))\}. \tag{24.3.7}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{24.3.8}$$

Then (24.3.2(p.245)) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \max\{\rho, U_t\} \cdots (2), \quad t \geq 0. \tag{24.3.9}$$

24.3.1.2 r$\tilde{\text{M}}$:3[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{24.3.10}$$

$$v_t(y) = \min\{y, \rho, U_t(y)\}, \quad t > 0, \tag{24.3.11}$$

$$V_0 = \rho, \tag{24.3.12}$$

$$V_t = \min\{\rho, U_t\}, \quad t > 0, \tag{24.3.13}$$

where $U_t(y)$ in (24.3.11(p.245)) is the minimum total expected present discounted *cost* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \min\{\lambda\beta\, \mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{24.3.14}$$

and where U_t in (24.3.13(p.245)) is the minimum total expected present discounted *cost* from rejecting ρ, expressed as

$$U_t = \min\{\lambda\beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\}, \quad t > 0, \tag{24.3.15}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:3}[\mathbb{R}][\texttt{A}]\} = \{(24.3.10(\text{p.245})) - (24.3.15(\text{p.245}))\}. \tag{24.3.16}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{24.3.17}$$

Then (24.3.11(p.245)) and (24.3.13(p.245)) hold for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \min\{y, U_t\} \cdots (2), \quad t \geq 0. \tag{24.3.18}$$

24.3.2 Search-Enforced-Model 3

24.3.2.1 rM:3[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \max\{y, \rho\}, \tag{24.3.19}$$
$$v_t(y) = \max\{y, \rho, U_t(y)\}, \quad t > 0, \tag{24.3.20}$$
$$V_0 = \rho, \tag{24.3.21}$$
$$V_t = \max\{\rho, U_t\}, \quad t > 0. \tag{24.3.22}$$

where $U_t(y)$ in (24.3.20(p.246)) is the maximum total expected present discounted *profit* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \quad t > 0. \tag{24.3.23}$$

and where U_t in (24.3.22(p.246)) is the maximum total expected present discounted *profit* from rejecting ρ, expressed as

$$U_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \quad t > 0. \tag{24.3.24}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathsf{rM}{:}3[\mathbb{R}][\mathrm{E}]\} = \{(24.3.19(\text{p.246})) - (24.3.24(\text{p.246}))\}. \tag{24.3.25}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{24.3.26}$$

Then (24.3.20(p.246)) and (24.3.22(p.246)) hold for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \max\{\rho, U_t\} \cdots (2), \quad t \geq 0. \tag{24.3.27}$$

24.3.2.2 r$\tilde{\mathsf{M}}$:3[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the best price y and with no best price respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{24.3.28}$$
$$v_t(y) = \min\{y, \rho, U_t(y)\}, \quad t > 0, \tag{24.3.29}$$
$$V_0 = \rho, \tag{24.3.30}$$
$$V_t = \min\{\rho, U_t\}. \tag{24.3.31}$$

where $U_t(y)$ in (24.3.29(p.246)) is the minimum total expected present discounted *cost* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0. \tag{24.3.32}$$

and where U_t in (24.3.31(p.246)) is the minimum total expected present discounted *cost* from rejecting ρ, expressed as

$$U_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \quad t > 0, \tag{24.3.33}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathsf{r}\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\} = \{(24.3.28(\text{p.246})) - (24.3.33(\text{p.246}))\}. \tag{24.3.34}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{24.3.35}$$

Then (24.3.29(p.246)) and (24.3.31(p.246)) hold for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \min\{y, U_t\} \cdots (2), \quad t \geq 0. \tag{24.3.36}$$

24.4 Reservation Value

⟨a⟩ t-`reservation-value` (no-recall-model).

Consider the selling model with no recall. Here recall (7.2.3(p.44)), i.e.,

$$w \geq (\leq)\ V_t \Rightarrow \texttt{Accept}_t\langle w\rangle\ \ (\texttt{Reject}_t\langle w\rangle), \tag{24.4.1}$$

meaning that the reservation value of the model is given by V_t, which depends on t. Then we say that V_t is the t-dependent `reservation-value` or t-`reservation-value` for short.

⟨b⟩ t-`reservation-value` (recall-model).

Consider the selling model with recall. Here, by $\mathtt{A}_t(y)$ let us represent the profit from accepting the best price y at a given time t, so $\mathtt{A}_t(y) = y$, and by $\mathtt{R}_t(y)$ the profit from rejecting the best price y at a given time t, so $\mathtt{R}_t(y) = V_t(y)$ (see (24.1.48(p.241))). Here let us define

$$\mathtt{AR}_t(y) \overset{\text{def}}{=} \mathtt{A}_t(y) - \mathtt{B}_t(y) = y - V_t(y). \tag{24.4.2}$$

Then suppose that there exists y_t^* such that

$$\mathtt{AR}_t(y) \geq (\leq)\ 0 \Leftrightarrow y \geq (\leq)\ V_t(y) \Leftrightarrow y \geq (\leq)\ y_t^* \Rightarrow \texttt{Accept}_t\langle y\rangle\ (\texttt{Reject}_t\langle y\rangle) \quad \text{(see (24.1.58(p.241)))}, \tag{24.4.3}$$

implying that the reservation value of the model is given by y_t^*, which depends on t. Then we say that y_t^* is the t-`reservation-value`.

⟨c⟩ `c-reservation-value`.

If V_t and y_t^* are constant in t, then we say that each of V_t and y_t^* is the constant `reservation-value` or the `c-reservation-value` for short.

24.5 Systems of Optimality Equations

Below are the systems of optimality equations for the 12 models.

rM:1[$\mathbb{R}$][A] → Section 24.1.1.1(p.239),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][A] → Section 24.1.1.2(p.240),
rM:1[$\mathbb{R}$][E] → Section 24.1.2.1(p.241),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][E] → Section 24.1.2.2(p.241),
rM:1[$\mathbb{R}$][A] → Section 24.2.1.1(p.242),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][A] → Section 24.2.1.2(p.243),
rM:1[$\mathbb{R}$][E] → Section 24.2.2.1(p.244),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][E] → Section 24.2.2.2(p.244),
rM:1[$\mathbb{R}$][A] → Section 24.3.1.1(p.245),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][A] → Section 24.3.1.2(p.245),
rM:1[$\mathbb{R}$][E] → Section 24.3.2.1(p.246),	$\mathsf{r\tilde{M}}$:1[$\mathbb{R}$][E] → Section 24.3.2.2(p.246),

Chapter 25

Analysis of Model 1

25.1 Search-Allowed-Model 1

25.1.1 rM:1[$\mathbb{R}$][A]

25.1.1.1 Lemmas

25.1.1.1.1 Preliminary

Lemma 25.1.1 (rM:1[$\mathbb{R}$][A]) *We have* $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_\vartriangle$. □

• *Proof* Since $V_t \geq \beta V_{t-1}$ for $t > 1$ from (24.1.4(p.239)), we have $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau \geq \beta V_{\tau-1}$, $V_{\tau-1} \geq \beta V_{\tau-2}$, $\cdots$, $V_2 \geq \beta V_1$, leading to $V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-1} \geq \cdots \geq \beta^{\tau-1} V_1$. Thus, we have $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{\text{Ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_\vartriangle$. ∎

Lemma 25.1.2 (rM:1[$\mathbb{R}$][A])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \geq 0$.*
(b) *$v_t(y)$ and $V_t(y)$ are nondecreasing in $t \geq 0$ and $t > 0$ respectively.*[†]
(c) *V_t is nondecreasing in $t > 0$.* □

• *Proof* (a) $v_0(y)$ is nondecreasing in y from (24.1.1(p.239)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (24.1.5(p.239)), hence $v_t(y)$ is nondecreasing in y from (24.1.8(p.239)). Accordingly, by induction $v_t(y)$ is nondecreasing in y for $t \geq 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is nondecreasing in y for $t > 0$ from (24.1.5(p.239)). In addition, $V_0(y)$ is nondecreasing in y from (24.1.7(p.239)), hence it follows that $V_t(y)$ is nondecreasing in y for $t \geq 0$

(b) Clearly $v_1(y) \geq y = v_0(y)$ for any y from (24.1.2(p.239)) with $t = 1$ and (24.1.1(p.239)). Suppose $v_{t-1}(y) \geq v_{t-2}(y)$ for any y. Then, from (24.1.5(p.239)) we have $V_t(y) \geq \max\{\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s, \beta v_{t-2}(y)\} = V_{t-1}(y)$ for any y. Hence, from (24.1.8(p.239)) we have $v_t(y) \geq \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y)$ is nondecreasing in $t \geq 0$ for any y. Since $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y, it follows that $V_t(y)$ is nondecreasing in $t > 0$ for any y from (24.1.5(p.239)).

(c) From (24.1.4(p.239)) with $t = 2$ we have $V_2 \geq \beta\,\mathbf{E}[v_1(\boldsymbol{\xi})] - s$. In addition, since $v_1(\boldsymbol{\xi}) \geq \boldsymbol{\xi}$ for any $\boldsymbol{\xi}$ from (24.1.2(p.239)) with $t = 1$, we have $V_2 \geq \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s = V_1$ due to (24.1.3(p.239)). Suppose $V_{t-1} \geq V_{t-2}$. Now, since $V_{t-1}(\boldsymbol{\xi}) \geq V_{t-2}(\boldsymbol{\xi})$ from (b), we have $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}(\boldsymbol{\xi})\} \geq \max\{\boldsymbol{\xi}, V_{t-2}(\boldsymbol{\xi})\} = v_{t-2}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}$ due to (24.1.8), hence from (24.1.4(p.239)) we have $V_t \geq \max\{\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] - s, \beta V_{t-2}\} = V_{t-1}$. Thus, by induction $V_t \geq V_{t-1}$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. ∎

Since $1 = \mathbf{E}[1] = \mathbf{E}[I(\boldsymbol{\xi} > y) + I(\boldsymbol{\xi} \leq y)]$, we can rewrite (24.1.19(p.240)) as follows.

$$\begin{aligned}
\mathbb{S}_t(y) &= \beta\big(\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} > y) + v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \leq y)] - v_{t-1}(y)(\mathbf{E}[I(\boldsymbol{\xi} > y) + I(\boldsymbol{\xi} \leq y)])\big) - s \\
&= \beta\big(\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} > y) + v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \leq y)] - \mathbf{E}[v_{t-1}(y)I(\boldsymbol{\xi} > y) + v_{t-1}(y)I(\boldsymbol{\xi} \leq y)]\big) - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(\boldsymbol{\xi} > y) + (v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(\boldsymbol{\xi} \leq y)] - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y) + (v_{t-1}(y) - v_{t-1}(y))I(\boldsymbol{\xi} \leq y)] - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y)] - s, \quad t > 0. \qquad (25.1.1)
\end{aligned}$$

Note here that

$$\begin{aligned}
\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} &= \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}(I(\boldsymbol{\xi} > y) + I(\boldsymbol{\xi} \leq y)) \\
&= \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} > y) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} \leq y).
\end{aligned}$$

Now, due to Lemma 25.1.2(p.249) (a), if $\boldsymbol{\xi} > y$, then $v_{t-1}(\boldsymbol{\xi}) \geq v_{t-1}(y)$ or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \geq 0$ and if $\boldsymbol{\xi} \leq y$, then $v_{t-1}(\boldsymbol{\xi}) \leq v_{t-1}(y)$ or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \leq 0$. Hence we have

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[†]From (24.1.10(p.239)) and (24.1.7(p.239)) we have $V_1(y) - V_0(y) = \max\{K(y), -(1-\beta)y\}$. Let $x_K < y$ and $\beta < 1$. Then $K(y) < 0$ due to Lemma 10.2.2(p.57) (j1) and $-(1-\beta)y < 0$ for a $y > 0$, hence $V_1(y) - V_0(y) < 0$, i.e., $V_1(y) < V_0(y)$. Thus $V_t(y)$ does not become nondecreasing in $t \geq 0$ for any y.

$$\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} = (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y).$$

Thus (25.1.1(p.249)) can be rewritten as

$$\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] - s, \quad t > 0. \tag{25.1.2}$$

Then, we have

$$\begin{aligned}\mathbb{S}_1(y) &= \beta\,\mathbf{E}[\max\{v_0(\boldsymbol{\xi}) - v_0(y), 0\}] - s \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}] - s \qquad (\leftarrow\ (24.1.1(\text{p.239}))) \\ &= \beta T(y) - s \qquad (\leftarrow\ (5.1.1(\text{p.25}))) \\ &= L(y) \qquad (\leftarrow\ (5.1.3(\text{p.25}))\text{ with }\lambda = 1).\end{aligned} \tag{25.1.3}$$

Lemma 25.1.3 (rM:1[ℝ][A])
(a) $\mathbb{S}_t(y)$ *is nonincreasing in* y *for* $t > 0$.
(b) $\mathbb{S}_t(y) \le L(y)$ *for any* $t > 0$ *and* y.
(c) *Let* $x_L \le y$. *Then* $\mathbb{S}_t(y) \le 0$ *for* $t > 0$. □

● *Proof* (a) Immediate from (25.1.2(p.250)) and Lemma 25.1.2(p.249) (a).

(b) First, (25.1.2(p.250)) can be rewritten as

$$\begin{aligned}\mathbb{S}_t(y) &= \beta\,\mathbf{E}\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi}) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] - s \\ &= \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] - s \cdots \textbf{(1)}.\end{aligned}$$

Next, we have:

- Let $y \le \boldsymbol{\xi} \cdots \textbf{(2)}$.† Now $v_0(\boldsymbol{\xi}) - v_0(y) = \boldsymbol{\xi} - y \le \boldsymbol{\xi} - y$ from (24.1.1(p.239)). Suppose

$$v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le \boldsymbol{\xi} - y \cdots \textbf{(3)} \quad \text{(induction hypothesis)}.$$

From (24.1.8(p.239)) we have

$$v_t(\boldsymbol{\xi}) - v_t(y) \le \max\{\boldsymbol{\xi} - y, V_t(\boldsymbol{\xi}) - V_t(y)\} \cdots \textbf{(4)}.$$

Then, from (24.1.5(p.239)) we have

$$\begin{aligned}V_t(\boldsymbol{\xi}) - V_t(y) &= \max\left\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\})] - s, \beta v_{t-1}(\boldsymbol{\xi})\right\} - \max\left\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', y\})] - s, \beta v_{t-1}(y)\right\}^{\ddagger} \\ &\le \max\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\})], \beta(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))\} \\ &= \beta\max\{\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\})], v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y)\}.\end{aligned}$$

Here from (3) we have

$$v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\}) \le \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - \max\{\boldsymbol{\xi}', y\} \le \max\{0, \boldsymbol{\xi} - y\}.$$

From this and (3) we obtain

$$\begin{aligned}V_t(\boldsymbol{\xi}) - V_t(y) &\le \beta\max\{\,\mathbf{E}_{\xi'}[\max\{0, \boldsymbol{\xi} - y\}], \boldsymbol{\xi} - y\} \\ &= \beta\max\{\max\{0, \boldsymbol{\xi} - y\}, \boldsymbol{\xi} - y\} \\ &= \beta\max\{\boldsymbol{\xi} - y, 0\}.\end{aligned}$$

In addition, since $\boldsymbol{\xi} - y \ge 0$ due to (2), we have

$$V_t(\boldsymbol{\xi}) - V_t(y) \le \beta(\boldsymbol{\xi} - y) \le \boldsymbol{\xi} - y.$$

Hence, from (4) we have $v_t(\boldsymbol{\xi}) - v_t(y) \le \boldsymbol{\xi} - y$. Accordingly, by induction it follows that $v_t(\boldsymbol{\xi}) - v_t(y) \le \boldsymbol{\xi} - y$ for $t \ge 0$, so $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le \boldsymbol{\xi} - y$ for $t > 1$. Thus we have

$$\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi})] \le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(y \le \boldsymbol{\xi})] \cdots \textbf{(5)}.$$

- Let $\boldsymbol{\xi} < y$. Then $v_{t-1}(\boldsymbol{\xi}) \le v_{t-1}(y)$ from Lemma 25.1.2(p.249) (a) or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le 0 = \max\{\boldsymbol{\xi} - y, 0\}$, hence

$$\begin{aligned}\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] &\le \beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi} - y, 0\}, 0\}I(\boldsymbol{\xi} < y)] \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] \cdots \textbf{(6)}.\end{aligned}$$

From (1) and from (5) and (6) we have

$$\begin{aligned}\mathbb{S}_t(y) &\le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(y \le \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] - s \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}(I(y \le \boldsymbol{\xi}) + I(\boldsymbol{\xi} < y))] - s \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}] - s \\ &= \beta T(y) - s \quad (\text{see }(5.1.1(\text{p.25}))) \\ &= L(y) \quad (\text{see }(5.1.3(\text{p.25}))).\end{aligned}$$

(c) If $x_L \le y$, then $L(y) \le 0$ from Corollary 10.2.1(p.57) (a), hence $\mathbb{S}_t(y) \le 0$ from (b). ∎

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†Note here that this inequality means a group of all pairs $(\boldsymbol{\xi}, y)$ satisfying this inequality itself. Hence, if $\max\{\boldsymbol{\xi}', y\} \le \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}$, the pair $(\max\{\boldsymbol{\xi}', y\}, \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\})$ is also an element of the group.

‡ $\mathbf{E}_{\xi'}$ represent the expectation as to $\boldsymbol{\xi}'$.

25.1.1.1.2 Case of $s = 0$

Lemma 25.1.4 (rM:1[ℝ][A]) *Let $s = 0$. Then $\mathbb{S}_t(y) \geq 0$ for all y and $t > 0$.* □

• *Proof* If $s = 0$, from (25.1.2(p.250)) we have $\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] \geq 0$ for all y and $t > 0$. ∎

25.1.1.1.3 Case of $\beta = 1$ and $s > 0$

Lemma 25.1.5 (rM:1[ℝ][A]) *Let $\beta = 1$ and $s > 0$.*

(a) *Let $y \geq x_K$. Then $y = V_t(y)$ for $t \geq 0$.*
(b) *Let $y \leq x_K$. Then $y \leq V_t(y)$ for $t \geq 0$.*
(c) *$y \leq V_t(y)$ for any y and $t > 0$.* □

• *Proof* Let $\beta = 1$ and $s > 0$.

(a,b) Evident for $t = 0$ from (24.1.7(p.239)). Suppose that $y \geq (\leq)\ x_K \Rightarrow y = (\leq)\ V_{t-1}(y)$ (induction hypothesis).

- Let $y \geq x_K$, hence $K(y) \leq 0 \cdots$ **(1)** from Lemma 10.2.2(p.57) (j1). Due to the induction hypothesis we have $v_{t-1}(y) = y \cdots$ **(2)** from (24.1.2(p.239)). Then, from Lemma 25.1.3(p.250) (b) we have $\mathbb{S}_t(y) \leq L(y) = T(y) - s = K(y)$ from (5.1.3(p.25))) and (5.1.4(p.25)) due to the assumptions $\beta = 1$ and $\lambda = 1$, so $\mathbb{S}_t(y) \leq 0$ due to (1). Hence, from (24.1.20(p.240)) we have $V_t(y) = \beta v_{t-1}(y) = v_{t-1}(y)$, thus $V_t(y) = y$ from (2). This completes the induction.
- Let $y \leq x_K$, hence $K(y) \geq 0 \cdots$ **(3)** from Lemma 10.2.2(p.57) (j1). From (24.1.5(p.239)) we have $V_t(y) \geq \mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$. Since $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \geq \max\{\boldsymbol{\xi}, y\}$ for any $\boldsymbol{\xi}$ and y from (24.1.8(p.239)), we get $V_t(y) \geq \mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y$ from (5.1.10(p.25)) with $\beta = 1$ and $\lambda = 1$. Thus, we obtain $V_t(y) \geq y$ due to (3). This completes the induction.

(c) Immediate from (a,b). ∎

25.1.1.1.4 Case of $\beta < 1$ and $s > 0$

25.1.1.1.4.1 Case of $\kappa > 0$

Lemma 25.1.6 ($\mathscr{A}${rM:1[ℝ][A]}) *Let $\beta < 1$ and $s > 0$ and let $\kappa > 0$.*

(a) *Let $y \geq x_K$. Then $y \geq V_t(y)$ for $t \geq 0$.*
(b) *Let $y \leq x_K$. Then $x_K \geq V_t(y) \geq y$ for $t \geq 0$.* □

• *Proof* Let $\beta < 1$ and $s > 0$ and let $\kappa > 0$. Then, from Lemma 10.2.3(p.58) (d) we have $x_L > x_K > 0 \cdots$ **(1)**.

(a,b) The two assertions are evident for $t = 0$ from (24.1.7(p.239)). Suppose that

$$y \geq (\leq)\ x_K \Rightarrow y \geq V_{t-1}(y) \cdots \textbf{(2)}\ (y \leq V_{t-1}(y) \leq x_K \cdots \textbf{(3)})\ \text{(induction hypothesis)},$$

hence $y \geq (\leq)\ x_K \Rightarrow v_{t-1}(y) = y \cdots$ **(4)** ($v_{t-1}(y) = V_{t-1}(y) \cdots$ **(5)**) from (24.1.2(p.239)).

- Let $y \geq x_K \cdots$ **(6)**, hence $0 < y \cdots$ **(7)** due to (1). Then $v_{t-1}(y) = y \cdots$ **(8)** due to (4).

  1. Let $x_L \geq y\ (\geq x_K) \cdots$ **(9)**. Then $L(y) \geq 0 \cdots$ **(10)** due to Lemma 10.2.1(p.57) (e1) and $K(y) \leq 0 \cdots$ **(11)** due to Lemma 10.2.2(p.57) (j1). Now, since $\mathbb{S}_t(y) \leq L(y) \cdots$ **(12)** for any y from Lemma 25.1.3(p.250) (b), from (24.1.20(p.240)) and from (12), (4), and (10) we have $V_t(y) \leq \max\{L(y), 0\} + \beta y = L(y) + \beta y = K(y) + y \leq y$ due to (5.1.9(p.25)) and (11).
  2. Let $y \geq x_L\ (> x_K) \cdots$ **(13)**, hence $L(y) \leq 0 \cdots$ **(14)** due to Lemma 10.2.1(p.57) (e1). Then we have $\mathbb{S}_t(y) \leq L(y) \leq 0 \cdots$ **(15)** from Lemma 25.1.3(p.250) (b), hence from (24.1.20(p.240)) we have $V_t(y) = \beta v_{t-1}(y) = \beta y \leq y$ due to (4) and (7).

  From the above, if $y \geq x_K$, then whether for $x_L \geq y$ or for $y \geq x_L$, we have $y \geq V_t(y)$ for $t \geq 0$. This completes the induction, i.e., it follows that (a) holds.

- Let $y \leq x_K \cdots$ **(16)**, hence $K(y) \geq 0 \cdots$ **(17)** from Lemma 10.2.2(p.57) (j1). Since $V_t(y) \geq \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$ from (24.1.5(p.239)) and since $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \geq \max\{\boldsymbol{\xi}, y\}$ from (24.1.8(p.239)), we have $V_t(y) \geq \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y$ from (5.1.10(p.25))) with $\lambda = 1$, hence $V_t(y) \geq y$ due to (17). Since $\max\{\boldsymbol{\xi}, y\} \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (16), from Lemma 25.1.2(p.249) (a) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \cdots$ **(18)** for any $\boldsymbol{\xi}$. Furthermore, since $\max\{\boldsymbol{\xi}, x_K\} \geq x_K$ for any $\boldsymbol{\xi}$, due to (2) we have $V_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, hence from (24.1.8(p.239)) we have $v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, so from (18) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$. In addition, since $v_{t-1}(y) = V_{t-1}(y) \leq x_K$ due to (5) and (3), from (24.1.5(p.239)) we have $V_t(y) \leq \max\{\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s, \beta x_K\}$, hence from (5.1.10(p.25)) with $\lambda = 1$ we have $V_t(y) \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ since $x_K > 0$ due to (1). This completes the induction. ∎

25.1.1.1.4.2 Case of $\kappa \le 0$

Lemma 25.1.7 ($\mathscr{A}\{$rM:1[$\mathbb{R}$][A]$\}$) *Let $\beta < 1$ and $s > 0$ and let $\kappa \le 0$.*

(a) *Let $y \ge 0$. Then $y \ge V_t(y)$ for $t \ge 0$.*
(b) *Let $y \le 0$. Then $y \le V_t(y)$ for $t \ge 0$.* □

● *Proof* Let $\beta < 1$ and $s > 0$ and let $\kappa \le 0$. Then, from Lemma 10.2.3(p.58) (d) we have $x_L \le x_K \le 0 \cdots$ **(1)**. Due to (24.1.7(p.239)) the two assertions clearly hold for $t = 0$. Suppose that $y \ge (\le)\ 0 \Rightarrow V_{t-1}(y) \le (\ge)\ y$ (induction hypothesis), hence $v_{t-1}(y) = y$ ($v_{t-1}(y) = V_{t-1}(y)$).

(a) Let $y \ge 0 \cdots$ **(2)**. Then, since $x_L \le y$ from (1), we have $\mathbb{S}_t(y) \le 0$ for $t > 0$ due to Lemma 25.1.3(p.250) (c). Therefore, from (24.1.14(p.239)) we obtain $V_t(y) = \beta V_{t-1}(y)$, hence due to the induction hypothesis we have $V_t(y) \le \beta y \le y$ due to $\beta < 1$ and (2). This completes the induction.

(b) Let $y \le 0 \cdots (3)$. Now, since $V_t(y) \ge \beta v_{t-1}(y)$ from (24.1.5(p.239)) and since $v_{t-1}(y) \ge y$ from (24.1.8(p.239))), we have $V_t(y) \ge \beta y \ge y$ due to $\beta < 1$ and (3). This completes the induction. ∎

25.1.1.2 Analysis

□ **Tom 25.1.1** (■$\mathscr{A}\{$rM:1[$\mathbb{R}$][A]$\}$)

(a) *Let $s = 0$. Then* rM:1[$\mathbb{R}$][A] ↬ rM:1[$\mathbb{R}$][E].
(b) *Let $s > 0$.*
  1. *We have* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\triangle$.†
  2. ♣ *Let $\beta = 1$. Then $y \le V_t(y)$ for any y and $t \ge 0$.*
  3. *Let $\beta < 1$.*
     i. *Let $\kappa > 0$.*
        1. ♠ *Let $y \ge x_K$. Then $y \ge V_t(y)$ for $t \ge 0$.*
        2. ♠ *Let $y \le x_K$. Then $y \le V_t(y)$ for $t \ge 0$.*
     ii. *Let $\kappa \le 0$.*
        1. ◇ *Let $y \ge 0$* (i.e., $\mathscr{F}^+$). *Then $y \ge V_t(y)$ for $t \ge 0$.*
        2. ♣ *Let $y \le 0$* (i.e., $\mathscr{F}^-$). *Then $y \le V_t(y)$ for $t \ge 0$.* □

● *Proof* (a) Let $s = 0$. Then, from Lemma 25.1.4(p.251) we have $\mathbb{S}_t(y) \ge 0$ for all y and $t > 0$, hence it is optimal to Conduct$_t$ for all y and $t > 0$ due to (24.1.21(p.240)). This fact implies that rM:1[$\mathbb{R}$][A] which is originally a search-Allowed-model migrates (↬) over to rM:1[$\mathbb{R}$][E] (see Def. 11.2.3(p.63)) which is a search-Enforced-model.

(b) Let $s > 0$.

(b1) The same as Lemma 25.1.1(p.249).†

(b2) The same as Lemma 25.1.5(p.251) (c).

(b3) Let $\beta < 1$.

(b3i-b3i2) The same as Lemma 25.1.6(p.251).

(b3ii-b3ii2) The same as Lemma 25.1.7(p.252). ∎

25.1.1.3 Flow of Optimal Decision Rules

♣ **Flow-ODR 1** (rM:1[$\mathbb{R}$][A]) (Accept$_0(y)$ ▹ Stop) *Let $s > 0$ and $\beta = 1$* (see Tom 25.1.1(p.252) (♣b2)) *or let $s > 0$, $\beta < 1$, $\kappa \le 0$, and $y \le 0$* (see Tom 25.1.1(p.252) (♣b3ii2) ($\mathscr{F}^-$)). *Then, the inequality $y \le V_t(y)$ for any t and y means that even if the process is initiated at any time t, it is optimal to reject the best price y at that time. Accordingly, it follows that each time a price ξ is proposed, the current best price y continues to be enlarged to $y \overset{\text{def}}{=} \max\{y, \xi\}$, and the process terminates by accepting the best price y at the deadline $t = 0$, i.e.,* Accept$_0(y)$ ▹ Stop.

♠ **Flow-ODR 2** (r$\tilde{\text{M}}$:1[$\mathbb{R}$][A]) (c-reservation-price) *From* Tom 25.1.1(p.252) (♠b3i1,♠b3i2) *and* (24.1.25(p.240)) *we have the following relations for $\tau \ge t \ge 0$:*

$$\begin{cases} y \ge x_K & \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \,\rule{1pt}{8pt} \\ y \le x_K & \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{cases}$$

Namely, the optimal reservation value is given by x_K, which is constant in t.

◇ **Flow-ODR 3** (rM:1[$\mathbb{R}$][A]) (Accept$_{t^*_\tau}(y)$ ▹ Stop) *Let $s > 0$, $\beta < 1$, $\kappa \le 0$, and $y \ge 0$* (see Tom 25.1.1(p.252) (◇b3ii1) ($\mathscr{F}^+$)). *Then the inequality $y \ge V_t(y)$ for $t \ge 0$ implies that when the process initiates at the optimal initiating time t^*_τ, it is optimal to accept the best price y at that time and stop the process.* □

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†Note that we have [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\triangle$ also for any $s \ge 0$.
†This is true also for $s = 0$.

Definition 25.1.1 (reduction) In Tom 25.1.1(p.252) (a) we demonstrated an example that a search-Allowed-model migrates over to a search-Enforced-model, represented as

$$\text{rM:1}[\mathbb{R}][\mathtt{A}] \looparrowright \text{rM:1}[\mathbb{R}][\mathtt{E}]. \tag{25.1.4}$$

Accordingly, adding "model-migration" and "odr-Accept/Stop" to "model-running-back" and " odr-Accept/Stop" in (21.1.3(p.225)), we have

$$\text{Reduction}\begin{cases}\texttt{model reduction} & \begin{cases}\texttt{model-running-back} & (\twoheadrightarrow)\\ \texttt{model-migration} & (\looparrowright)\end{cases}\\ \texttt{odr reduction} & \begin{cases}\texttt{odr-Accept/Stop} & (\mapsto)\end{cases}\end{cases} \tag{25.1.5}$$

25.1.1.4 Market Restriction

25.1.1.4.1 Positive Restriction

□ **Pom 25.1.1** ($\mathscr{A}${rM:1[ℝ][A]$^+$}) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then* rM:1[ℝ][A]$^+$ ↬ rM:1[ℝ][E]$^+$.

(b) *Let $s > 0$.*

1. *We have* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\Delta$ → → ⓢ
2. *Let $\beta = 1$. Then we have* odr ↦ Accept$_0(y)$ ▷ Stop.
3. *Let $\beta < 1$.*
   i. *Let $\beta\mu > s$. Then we have* c-reservation-price.
   ii. *Let $\beta\mu \le s$. Then we have* [● dOITd$_{\tau>0}\langle 1\rangle$]$_{||}$ → → ⓓ

● *Proof* Suppose $a > 0$, hence it suffices to consider y such that $0 < a < y < b \cdots$ **(1)**. Then $\kappa = \beta\mu - s \cdots$ **(2)** from Lemma 10.3.1(p.59) (a) with $\lambda = 1$.

(a) The same as Tom 25.1.1(p.252) (a).

(b) Let $s > 0$.

(b1) The same as Tom 25.1.1(p.252) (b1).

(b2) Evident from Tom 25.1.1(p.252) (b2) and ♣ Flow-ODR 1.

(b3) Let $\beta < 1$.

(b3i) Let $\beta\mu > s$, hence $\kappa > 0$ due to (2). Thus, it suffices to consider only Tom 25.1.1(p.252) (♠ b3i1,♠ b3i2), hence we have ♣ Flow-ODR 2.

(b3ii) Let $\beta\mu \le s$, hence $\kappa \le 0 \cdots$ **(3)** due to (2). In this case, due to (1) it suffices to consider only Tom 25.1.1(p.252) (◇ b3ii1). Then, since it suffices to consider ξ such that $0 < a < \xi < b$, we have $\xi \ge V_{t-1}(\xi)$ for $t > 1$, hence $v_{t-1}(\xi) = \xi$ from (24.1.8(p.239)). Thus, from (24.1.4(p.239)) we have $V_t = \max\{\beta\,\mathbf{E}[\boldsymbol{\xi}] - s, \beta V_{t-1}\} = \max\{\beta\mu - s, \beta V_{t-1}\} = \max\{\kappa, \beta V_{t-1}\}$ for $t > 1$. First $V_1 = \beta\mu - s = \kappa \le 0$ from (24.1.3(p.239)) and (3) or equivalently $V_1 = \beta^0\kappa \le 0$. Suppose $V_{t-1} = \beta^{t-2}\kappa \le 0$. Then $V_t = \max\{\kappa, \beta\beta^{t-2}\kappa\} = \max\{\kappa, \beta^{t-1}\kappa\} = \beta^{t-1}\kappa \le 0$ due to (3). Thus by induction we have $V_t = \beta^{t-1}\kappa \le 0$ for $t > 1$. Accordingly, we have $V_t - \beta V_{t-1} = \beta^{t-1}\kappa - \beta\beta^{t-2}\kappa = \beta^{t-1}\kappa - \beta^{t-1}\kappa = 0$, hence $V_t = \beta V_{t-1}$ for $t > 1$. Accordingly, we get $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1}V_1$, i.e., $t^*_\tau = 1$ for $\tau > 1$ or equivalently [● dOITd$_{\tau>1}\langle 1\rangle$]. ∎

25.1.1.4.2 Mixed Restriction

Omitted.

25.1.1.4.3 Negative Restriction

Omitted.

25.1.2 r$\tilde{\text{M}}$:1[ℝ][A]

25.1.2.1 Preliminary

For almost the same reason as in Section 25.2.2.1(p.261) it can be confirmed that SOE{r$\tilde{\text{M}}$:1[ℝ][A]} (see (24.1.31(p.240))) is symmetrical to SOE{rM:1[ℝ][A]} (see (24.1.6(p.239))). Hence it follows that Scenario[ℝ](p.75) can be applied also to $\mathscr{A}${rM:1[ℝ][A]}.

25.1.2.2 Derivation of $\mathscr{A}${r$\tilde{\text{M}}$:1[ℝ][A]}

□ **Tom 25.1.2** (⊡ $\mathscr{A}${r$\tilde{\text{M}}$:1[ℝ][A]})

(a) *Let $s = 0$. Then* r$\tilde{\text{M}}$:1[ℝ][A] ↬ r$\tilde{\text{M}}$:1[ℝ][E].

(b) *Let $s > 0$.*

1. *We have* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\Delta$ → → ⓢ
2. ♣ *Let $\beta = 1$. Then $y \ge V_t(y)$ for $t \ge 0$ and any t.*
3. *Let $\beta < 1$.*
   i. *Let $\tilde{\kappa} < 0$.*

1. ♠ *Let* $y \leq x_{\tilde{K}}$. *Then* $y \leq V_t(y)$ *for* $t \geq 0$.
2. ♠ *Let* $y \geq x_{\tilde{K}}$. *Then* $y \geq V_t(y)$ *for* $t \geq 0$.

ii. *Let* $\tilde{\kappa} \geq 0$.

1. ◇ *Let* $y \leq 0$ (i.e., $\mathscr{F}^-$). *Then* $y \leq V_t(y)$ *for* $t \geq 0$.
2. ♣ *Let* $y \geq 0$ (i.e., $\mathscr{F}^+$). *Then* $y \geq V_t(y)$ *for* $t \geq 0$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see in (18.0.1(p.130))) to Tom 25.1.1(p.252). ▮

25.1.2.3 Flow of Optimal Decision Rules

♣ `Flow-ODR` **4 (rM:1[ℝ][A])** ($\texttt{Accept}_0(y) \triangleright \texttt{Stop}$) *Let* $s > 0$ *and* $\beta = 1$ (see Tom 25.1.2(p.253) (♣b2)) *or let* $s > 0$, $\beta < 1$, $\tilde{\kappa} \leq 0$, *and* $y \leq 0$ (see Tom 25.1.2(p.253) (♣b3ii2) ($\mathscr{F}^+$)). *Then, the inequality* $y \geq V_t(y)$ *for any* t *and* y *means that even if the process is initiated at any time* t*, it is optimal to reject all prices proposed. Accordingly, it follows that each time a price* ξ*, the current best price* y *continues to be reduced to* $y \overset{\text{def}}{=} \max\{y, \xi\}$ ($\min\{y, \xi\}$)*, and the process terminates by accepting the best price* y *at the deadline* $t = 0$*, i.e.,* $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$.

♠ `Flow-ODR` **5 (r$\tilde{\text{M}}$:1[ℝ][A])** (`c-reservation-price`) *From* Tom 25.1.2(p.253) (♠ b3i1,♠ b3i2) *and* (24.1.46(p.241)) *we have the following relations for* $\tau \geq t \geq 0$*:*

$$\begin{cases} y \leq x_{\tilde{K}} \;\Rightarrow\; \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \,\mathbf{I} \\ y \geq x_{\tilde{K}} \;\Rightarrow\; \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{cases}$$

Namely, the optimal reservation value is given by $x_{\tilde{K}}$*, which is constant in* t.

◇ `Flow-ODR` **6 (rM:1[ℝ][A])** ($\texttt{Accept}_t(y) \triangleright \texttt{Stop}$) *Let* $s > 0$, $\beta < 1$, $\tilde{\kappa} \geq 0$, *and* $y \leq 0$ (see Tom 25.1.2(p.253) (◇b3ii1) ($\mathscr{F}^-$)). *Then the inequality* $y \leq V_t(y)$ *for* $t \geq 0$ *implies that when the process initiates at the optimal initiating time* t^*_τ*, it is optimal to accept the best price* y *at that time and stop the process.* □

25.1.2.4 Market Restriction

25.1.2.4.1 Positive Restriction

□ **Pom 25.1.2 ($\mathscr{A}${r$\tilde{\text{M}}$:1[ℝ][A]$^+$})** *Suppose* $a > 0$.

(a) *Let* $s = 0$. *Then* r$\tilde{\text{M}}$:1[ℝ][A]$^+$ ↬ r$\tilde{\text{M}}$:1[ℝ][E]$^+$.

(b) *Let* $s > 0$.

1. *We have* [Ⓢ $\texttt{dOITs}_{\tau>1}\langle\tau\rangle$]$_\triangle$ → → Ⓢ
2. *We have* `odr` ↦ $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$. □

● *Proof* Suppose $a > 0$. Below consider only y with $0 < a \leq y \leq b$, hence $y \geq 0 \cdots$ **(1)**. Moreover, since $\tilde{\kappa} = s$ from Lemma 12.6.6(p.83) (a), we have $\tilde{\kappa} \geq 0 \cdots$ **(2)** for any $s \geq 0$.

(a) The same as Tom 25.1.2(p.253) (a).

(b) Let $s > 0$.

(b1) The same as Tom 25.1.2(p.253) (b1).

(b2) If $\beta = 1$, then $y \geq V_t(y)$ for $t \geq 0$ from Tom 25.1.2(p.253) (b2). If $\beta < 1$, then due to (2) and (1) it suffices to consider only Tom 25.1.2(p.253) (♣ b3ii2), hence we have $y \geq V_t(y)$ for $t \geq 0$. Accordingly, whether $\beta = 1$ or $\beta < 1$, we have $y \geq V_t(y)$ for $t \geq 0$. Thus, it follows that we have $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$ (♣ `Flow-ODR` 4). ▮

25.1.2.4.2 Mixed Restriction

Omitted.

25.1.2.4.3 Negative Restriction

Omitted.

25.1.3 Conclusion 7 (Search-Allowed-Model 1)

■ The assertion systems $\mathscr{A}$ of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$$\mathcal{Q}\{\text{rM:1[A]}\} = \{\text{rM:1[ℝ][A], r}\tilde{\text{M}}\text{:1[ℝ][A], } \sout{\text{rM:1[ℙ][A], r}\tilde{\text{M}}\text{:1[ℙ][A]}}\}$$

are given by

$\mathscr{A}${rM:1[ℝ][A]} $\mathscr{A}${r$\tilde{\text{M}}$:1[ℝ][A]}
↓ ↓
Tom's 25.1.1(p.252), 25.1.2(p.253),

■ The assertion systems $\mathscr{A}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$$\mathcal{Q}\{\text{rM:1[A]}\}^+ = \{\text{rM:1}[\mathbb{R}][\text{A}]^+,\ \text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]^+,\ \text{rM:1}[\mathbb{P}][\text{A}]^+,\ \text{r}\tilde{\text{M}}\text{:1}[\mathbb{P}][\text{A}]^+\}$$

are given by

$$\begin{array}{cc} \mathscr{A}\{\text{rM:1}[\mathbb{R}][\text{A}]^+\} & \mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]^+\} \\ \downarrow & \downarrow \\ \text{Pom's } 25.1.1\text{(p.253)}, & 25.1.2\text{(p.254)}, \end{array}$$

■ Closely looking into all the assertion systems above leads to the conclusions below.

C1 We have $\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]\}^+ \not\sim \mathscr{A}\{\text{rM:1}[\mathbb{R}][\text{A}]\}^+$.
C2 We have $\text{rM}/\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]^+ \nleftrightarrow \text{rM}/\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]^+$.
C3 We have `odr` $\mapsto$ `Accept`$_0(y) \triangleright$ `Stop` for $\text{rM}/\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]^+$.
C4 We have Ⓢ$_\vartriangle$ for $\text{rM}/\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{A}]^+$.
C5 We have ⓓ$_\parallel$ for $\text{rM:1}[\mathbb{R}][\text{A}]^+$.
C6 We have `c-reservation-price` for $\text{rM:1}[\mathbb{R}][\text{A}]^+$. □

C1 Compare Pom 25.1.2(p.254) and Pom 25.1.1(p.253).
C2 See Pom 25.1.1(p.253) (a) and Pom 25.1.2(p.254) (a).
C3 See Pom 25.1.1(p.253) (b2) and Pom 25.1.2(p.254) (b2).
C4 See Pom 25.1.1(p.253) (b1) and Pom 25.1.2(p.254) (b1).
C5 See Pom 25.1.1(p.253) (b3ii).
C6 See Pom 25.1.1(p.253) (b3i). ∎

25.2 Search-Enforced-Model 1

25.2.1 rM:1[ℝ][E]

Below let us define

$$\mathbb{V}_t \stackrel{\text{def}}{=} V_t - \beta V_{t-1}, \quad t > 1. \tag{25.2.1}$$

25.2.1.1 Some Lemmas

Lemma 25.2.1 (rM:1[ℝ][E])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \geq 0$.*
(b) *$v_t(y)$ and $V_t(y)$ are nondecreasing in $t \geq 0$ and $t > 0$[†] respectively for any y.*
(c) *V_t is nondecreasing in $t > 0$.* □

● *Proof* (a) $v_0(y)$ is nondecreasing in y from (24.1.47(p.241)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (24.1.50(p.241)), hence $v_t(y)$ is nondecreasing in y from (24.1.48(p.241)). Thus, by induction $v_t(y)$ is nondecreasing in y and $t \geq 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is also nondecreasing in y for $t > 0$ from (24.1.50(p.241)). In addition, $V_0(y)$ is nondecreasing in y from (24.1.52(p.241)), hence it follows that $V_t(y)$ is nondecreasing in y for $t \geq 0$.

(b) Clearly $v_1(y) \geq y = v_0(y)$ for any y from (24.1.48(p.241)) with $t = 1$ and (24.1.47(p.241)). Suppose $v_{t-1}(y) \geq v_{t-2}(y)$ for any y. Then, from (24.1.50(p.241)) we have $V_t(y) \geq \beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s = V_{t-1}(y)$ for any y. Hence, from (24.1.48(p.241)) we have $v_t(y) \geq \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y) \geq v_{t-1}(y)$ for $t > 0$ and any y, i.e., $v_t(y)$ is nondecreasing in $t \geq 0$ for any y. Accordingly, since $v_{t-1}(y) \geq v_{t-2}(y)$ for $t > 1$ and any y, from (24.1.50(p.241)) we have $V_t(y) \geq \beta\,\mathbf{E}[v_{t-2}(y)] - s = V_{t-1}(y)$ for $t > 1$ and any y, hence $V_t(y)$ is nondecreasing in $t > 0$ for any y.

(c) We have $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y due to (b), hence V_t is nondecreasing in $t > 0$ from (24.1.49(p.241)). ∎

Lemma 25.2.2 (rM:1[ℝ][E])

(a) *Let $x_K \leq y$. Then $V_t(y) \leq y$ for $t \geq 0$.*
(b) *Let $y \leq x_K$. Then $y \leq V_t(y) \leq x_K$ for $t \geq 0$.* □

● *Proof* [‡] (a) Let $x_K \leq y$. Then $K(y) \leq 0 \cdots$ **(1)** from Corollary 10.2.2(p.58) (a). Now, from (24.1.52(p.241)) we clearly have $V_0(y) \leq y$. Suppose $V_{t-1}(y) \leq y$, hence $v_{t-1}(y) = y$ from (24.1.48(p.241)). Then, since $x_K \leq y \leq \max\{\boldsymbol{\xi}, y\}$ for any $\boldsymbol{\xi}$, we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) = \max\{\boldsymbol{\xi}, y\}$. Accordingly, from (24.1.50(p.241)) we have $V_t(y) = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y \cdots$ **(2)** due to (5.1.10(p.25)) with $\lambda = 1$, hence $V_t(y) \leq y$ due to (1). This completes the induction.

(b) Let $y \leq x_K \cdots$ **(3)**. Then $K(y) \geq 0 \cdots$ **(4)** from Corollary 10.2.2(p.58) (b). Now, from (24.1.53(p.241)) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \geq \max\{\boldsymbol{\xi}, y\}$ for any $t > 0$, $\boldsymbol{\xi}$, and y, hence from (24.1.50(p.241)) and (5.1.10(p.25)) with $\lambda = 1$ we have $V_t(y) \geq \beta[\max\{\boldsymbol{\xi}, y\}] - s =$

[†]It cannot be always guaranteed that $V_1(y) \geq V_0(y)$. For example, let $\beta < 1$ or $s > 0$ and let $y > x_K$. Then, from (24.1.56(p.241)) and (24.1.52(p.241)) we have $V_1(y) - V_0(y) = K(y) < 0$ due to Lemma 10.2.2(p.57) (j1), i.e., $V_1(y) < V_0(y)$.

[‡]Although (a) and (b) are already proven in [44,Sakaguchi,1961], we anew prove herein the two by using properties of the underlying function $K(x)$.

$K(y) + y$ for $t > 0$, so $V_t(y) \geq y$ for $t > 0$ due to (4). In addition, since $V_0(y) \geq y$ from (24.1.52(p.241)), it follows that $V_t(y) \geq y$ for $t \geq 0$. Now, since $\max\{\boldsymbol{\xi}, y\} \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (3), from Lemma 25.2.1(p.255) (a) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \cdots$ **(5)** for any $\boldsymbol{\xi}$ and $t > 0$. Since $x_K \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, due to (a) we have $V_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$, hence $v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) = \max\{\max\{\boldsymbol{\xi}, x_K\}, V_t(\max\{\boldsymbol{\xi}, x_K\})\} = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$ from (24.1.53(p.241)), so from (5) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$. Thus, from (24.1.50(p.241)) and (5.1.10(p.25)) with $\lambda = 1$ we have $V_t(y) \leq \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s = K(x_K) + x_K = x_K$ for $t > 0$. ▮

Since $V_t(y)$ is nondecreasing in $t > 0$ from Lemma 25.2.1(p.255) (b) and is upper bounded in t from Lemma 25.2.2(p.255) (a,b), it converges to a finite $V(y)$ as $t \to \infty$, hence so also do $v_t(y)$, V_t, and $\mathbb{V}_t$ (see (25.2.1(p.255))). Then, defining these limits by $v(y)$, V, and $\mathbb{V}$, from (24.1.50(p.241)), (24.1.48(p.241)), (24.1.49(p.241)), and (25.2.1(p.255)) we have:

$$V(y) = \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})] - s, \tag{25.2.2}$$

$$v(y) = \max\{y, V(y)\}, \tag{25.2.3}$$

$$V = \beta\,\mathbf{E}[v(\boldsymbol{\xi})] - s, \tag{25.2.4}$$

$$\mathbb{V} = (1-\beta)V. \tag{25.2.5}$$

Lemma 25.2.3 (rM:1[ℝ][E])

(a) *Let $x_K \leq y$. Then $V(y) \leq y$.*

(b) *Let $y \leq x_K$. Then $y \leq V(y) \leq x_K$.* □

● *Proof* Immediate from Lemma 25.2.2(p.255). ▮

Lemma 25.2.4 (rM:1[ℝ][E]) *Let $\beta < 1$.*

(a) *Let $y \leq x_K$. Then $V(y) = x_K$.*

(b) *$v(y) = \max\{y, x_K\}$ for any y.*

(c) *$V = x_K$.*

(d) *Let $\kappa > (= (<))\ 0$. Then $\mathbb{V} > (= (<))\ 0$.* □

● *Proof* Let $\beta < 1$.

(a) Let $y \leq x_K \cdots$ **(1)**. Now, (25.2.2(p.256)) can be rewritten as

$$V(y) = \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \leq x_K)] - s \cdots \textbf{(2)}.$$

If $x_K < \boldsymbol{\xi}$, then $y < \boldsymbol{\xi}$ from (1), hence $x_K < \boldsymbol{\xi} = \max\{\boldsymbol{\xi}, y\}$. Thus, from Lemma 25.2.3(p.256) (a) we have $V(\max\{\boldsymbol{\xi}, y\}) \leq \max\{y, \boldsymbol{\xi}\} = \boldsymbol{\xi}$, so from (25.2.3(p.256)) we have $v(\max\{\boldsymbol{\xi}, y\}) = \max\{\max\{\boldsymbol{\xi}, y\}, V(\max\{\boldsymbol{\xi}, y\})\} = \max\{y, \boldsymbol{\xi}\} = \boldsymbol{\xi}$ due to . Therefore, (2) can rewritten as

$$V(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \leq x_K)] - s \cdots \textbf{(3)}.$$

In addition, since $v(\max\{\boldsymbol{\xi}, y\}) = \max\{\max\{\boldsymbol{\xi}, y\}, V(\max\{\boldsymbol{\xi}, y\})\}$ from (25.2.3(p.256)) for ξ and y, we can rewrite (3) as

$$V(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, V(\max\{\boldsymbol{\xi}, y\})\}I(\boldsymbol{\xi} \leq x_K)] - s. \cdots \textbf{(4)}$$

To prove (a) it suffices to show the following two:

1. *Any given function $V'(y) = x_K \cdots$ **(5)** with $y \leq x_K$ is a solution of the functional equation* (4), i.e.,

$$V'(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, V'(\max\{\boldsymbol{\xi}, y\})\}I(\boldsymbol{\xi} \leq x_K)] - s. \cdots \textbf{(6)}$$

To prove this, first let us show that substituting the equality $V'(y) = x_K$ with $y \leq x_K$ for the r.h.s. of (6) yields x_K, hence, as a result, its l.h.s. becomes equal to x_K, i.e., $V'(y) = x_K$, implying that (5) is a solution of the functional equation (6). Below let us show this.

Let $\boldsymbol{\xi} \leq x_K$. Then $\max\{y, \boldsymbol{\xi}\} \leq \max\{x_K, x_K\} = x_K \cdots$ **(7)** due to (1), hence $V'(\max\{y, \boldsymbol{\xi}\}) = x_K$ due to (5). Consequently, we get

$$\begin{aligned}
\text{r.h.s of (6)} &= \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[x_K I(\boldsymbol{\xi} \leq x_K)] - s \\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}I(\boldsymbol{\xi} \leq x_K)] - s \\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s \\
&= K(x_K) + x_K \quad \text{(See (5.1.10(p.25))) with } \lambda = 1 \\
&= x_K.
\end{aligned}$$

Accordingly, it follows that $V'(y) = x_K$ with $y \leq x_K$ is a solution of the functional equation (4).

2. *The solution is unique* Suppose there exists *another* solution $Z(y)$ with $y \le x_K$ where $V'(y) \ne Z(y)$ for at least one $y \le x_K$. Then, let $z(y) \overset{\text{def}}{=} \max\{y, Z(y)\} \cdots (8)$ with $y \le x_K$ (see (25.2.3(p.256))). Accordingly, we have (see (3))

$$Z(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[z(\max\{\boldsymbol{\xi}, y\}) I(\boldsymbol{\xi} \le x_K)] - s. \cdots (9)$$

Hence, from (3) and (9) we have

$$\begin{aligned} |V'(y) - Z(y)| &= \left|\beta\,\mathbf{E}[(v'(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\})) I(\boldsymbol{\xi} \le x_K)]\right| \\ &\le \beta\,\mathbf{E}[\left|v'(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\})\right| I(\boldsymbol{\xi} \le x_K)]. \cdots (10). \end{aligned}$$

Now, in general

$$|v'(y) - z(y)| = |\max\{y, V'(y)\} - \max\{y, Z(y)\}| \le \max\{0, |V'(y) - Z(y)|\} = |V'(y) - Z(y)|$$

for any y, hence we have

$$\left|v'(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\})\right| \le \left|V'(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})\right| \cdots (11).$$

for any y and $\boldsymbol{\xi}$. Thus, from (10) we have

$$|V'(y) - Z(y)| \le \beta\,\mathbf{E}[\left|V'(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})\right| I(\boldsymbol{\xi} \le x_K)] \cdots (12).$$

Let $\nu = \max_{y \le x_K} |V'(y) - Z(y)| \cdots (13)$ where $\nu > 0 \cdots (14)$, hence $|V'(y) - Z(y)| \le \nu \cdots (15)$ for $y \le x_K$. If $\boldsymbol{\xi} \le x_K$, then $\max\{\boldsymbol{\xi}, y\} \le \max\{x_K, x_K\} = x_K \cdots (16)$, hence $|V'(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})| \le \nu$ due to (15). Accordingly, from (12) we have

$$|V'(y) - Z(y)| \le \beta\,\mathbf{E}[\nu I(\boldsymbol{\xi} \le x_K)] = \beta\nu\,\mathbf{E}[I(\boldsymbol{\xi} \le x_K)] = \beta\nu \Pr\{\boldsymbol{\xi} \le x_K\} = \beta\nu F(x_K).$$

Thus, we have $\nu \le \beta\nu F(x_K) \cdots (17)$ due to the definition (13). In addition, since $\beta\nu F(x_K) \le \beta\nu$ due to $F(x_K) \le 1$, we have $\nu \le \beta\nu$ from (17), leading to the contradiction $1 \le \beta$ due to (14). Accordingly, the solution of (4) must be unique. Since the original $V(y)$ satisfy (4), it eventually follows that $V(y) = x_K$ with $y \le x_K$ must be the unique solution of (4).

(b) If $x_K \le y$, from Lemma 25.2.3(p.256) (a) and (25.2.3(p.256)) we have $v(y) = y = \max\{y, x_K\}$. If $y \le x_K$, then from Lemma 25.2.3(p.256) (b) and (25.2.3(p.256)) we have $v(y) = V(y)$ and from (a) we have $V(y) = x_K$, hence it follows that $v(y) = V(y) = x_K = \max\{y, x_K\}$. Thus, whether $x_K \le y$ or $y \le x_K$, we have $v(y) = \max\{y, x_K\}$.

(c) Since $v(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (b), from (25.2.4(p.256)) we have $V = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s = K(x_K) + x_K = x_K$ (see (5.1.10(p.25))).

(d) Let $\kappa > (= (<))\ 0$. Then, since $x_K > (= (<))\ 0$ due to Lemma 10.3.1(p.59) (b), from (c) we have $V > (= (<))\ 0$, hence the assertion becomes true from (25.2.5(p.256)). ∎

Here, let us define

$$\ell_t(y) \overset{\text{def}}{=} v_t(y) - \beta v_{t-1}(y), \quad t > 0. \tag{25.2.6}$$

Then, from (25.2.1(p.255)) and (24.1.49(p.241)) we have

$$\begin{aligned} \mathbb{V}_t &= \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s - \beta(\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] - s) && (25.2.7) \\ &= \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi}) - \beta v_{t-2}(\boldsymbol{\xi})] - (1-\beta)s && (25.2.8) \\ &= \beta\,\mathbf{E}[\ell_{t-1}(\boldsymbol{\xi})] - (1-\beta)s, \quad t > 1. && (25.2.9) \end{aligned}$$

Here, for any y let us define

$$A(y) \overset{\text{def}}{=} \ell_2(y) - \ell_1(y). \tag{25.2.10}$$

Lemma 25.2.5 (rM:1[ℝ][E])

(a) *Let $x_K \le y$. Then $A(y) = 0$.*
(b) *Let $y \le x_K$. Then $A(y)$ is nondecreasing in y.*
(c) *$A(y) \le 0$ for any y.* □

● *Proof* (a) Let $x_K \le y$. Then $V_2(y) \le y$ and $V_1(y) \le y$ from Lemma 25.2.2(p.255) (a), hence from (24.1.53(p.241)) we have $v_2(y) = v_1(y) = y$. In addition, $v_0(y) = y$ from (24.1.47(p.241)). Thus, since $\ell_2(y) = v_2(y) - \beta v_1(y) = (1-\beta)y$ and $\ell_1(y) = v_1(y) - \beta v_0(y) = (1-\beta)y$, we have $A(y) = 0 \cdots (1)$.

(b) Let $y \le x_K \cdots (2)$. Now, from Lemma 25.2.2(p.255) (b) with $t = 1, 2$ and (24.1.48(p.241)) with $t = 1, 2$ we have

$$\begin{aligned} v_1(y) &= V_1(y) = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s && \text{(see (24.1.55(p.241)))} && (25.2.11) \\ &= K(y) + y && \text{(see (5.1.10(p.25)) with } \lambda = 1), && (25.2.12) \\ v_2(y) &= V_2(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s && \text{(see (24.1.50(p.241)) with } t = 2). && (25.2.13) \end{aligned}$$

Hence, we have

$$\ell_1(y) = v_1(y) - \beta v_0(y) = v_1(y) - \beta y \quad \text{(see (24.1.47(p.241)))},$$
$$\ell_2(y) = v_2(y) - \beta v_1(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s - \beta v_1(y),$$

from which we obtain

$$A(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s - (1+\beta)v_1(y) + \beta y,$$

which can be rewritten as

$$A(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + v_1(\max\{\boldsymbol{\xi}, y\})I(x_K \le \boldsymbol{\xi})] - s - (1+\beta)v_1(y) + \beta y. \tag{25.2.14}$$

If $\boldsymbol{\xi} < x_K$, due to (2) we have $\max\{\boldsymbol{\xi}, y\} \le \max\{x_K, x_K\} = x_K$, hence from (25.2.12(p.257)) we have

$$v_1(\max\{\boldsymbol{\xi}, y\}) = K(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}. \tag{25.2.15}$$

If $x_K \le \boldsymbol{\xi}$, then since $x_K \le \boldsymbol{\xi} \le \max\{\boldsymbol{\xi}, y\}$ for any y, from Lemma 25.2.2(p.255) (a) we have $V_1(\max\{\boldsymbol{\xi}, y\}) \le \max\{\boldsymbol{\xi}, y\}$, hence from (24.1.48(p.241)) with $t = 1$ we obtain

$$v_1(\max\{\boldsymbol{\xi}, y\}) = \max\{\boldsymbol{\xi}, y\}. \tag{25.2.16}$$

Accordingly, from (25.2.14(p.258)), (25.2.15(p.258)), and (25.2.16(p.258)) we have

$$\begin{aligned} A(y) &= \beta\,\mathbf{E}[\big(K(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}\big)I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}I(x_K \le \boldsymbol{\xi})] - s - (1+\beta)v_1(y) + \beta y \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}(I(\boldsymbol{\xi} < x_K) + I(x_K \le \boldsymbol{\xi}))] - s - (1+\beta)v_1(y) + \beta y \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}] - s - (1+\beta)v_1(y) + \beta y^{\dagger} \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s - (1+\beta)v_1(y) + \beta y. \end{aligned} \tag{25.2.17}$$

Using (25.2.11(p.257)), we can rewrite the above as

$$\begin{aligned} A(y) &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] + v_1(y) - (1+\beta)v_1(y) + \beta y \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] - \beta(v_1(y) - y). \end{aligned} \tag{25.2.18}$$

Furthermore, since $v_1(y) - y = K(y)$ due to (25.2.12(p.257)), we can rewrite (25.2.18(p.258)) above as

$$\begin{aligned} A(y) &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] - \beta K(y) \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) - K(y)] \\ &= \beta\,\mathbf{E}[B(\boldsymbol{\xi}, y)] \end{aligned} \tag{25.2.19}$$

where

$$B(\boldsymbol{\xi}, y) \overset{\text{def}}{=} K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) - K(y). \tag{25.2.20}$$

Now we have:

1 Let $x_K \le \boldsymbol{\xi}$. Then, since $I(\boldsymbol{\xi} < x_K) = 0$, we have $B(\boldsymbol{\xi}, y) = -K(y)$, which is nondecreasing in $y \le x_K$ from Lemma 10.2.2(p.57) (b).

2 Let $\boldsymbol{\xi} < x_K$. Then, since $I(\boldsymbol{\xi} < x_K) = 1$, we have $B(\boldsymbol{\xi}, y) = K(\max\{\boldsymbol{\xi}, y\}) - K(y)$ for $y \le x_K$. Thus, if $y \le \boldsymbol{\xi}$, then $B(\boldsymbol{\xi}, y) = K(\boldsymbol{\xi}) - K(y)$, which is nondecreasing in $y \le \boldsymbol{\xi}$ due to Lemma 10.2.2(p.57) (b) and if $\boldsymbol{\xi} < y$, then since $\boldsymbol{\xi} < x_K$ due to (2), we have $I(\boldsymbol{\xi} < x_K) = 1$, hence $B(\boldsymbol{\xi}, y) = K(y) - K(y) = 0$ for $y \le x_K$, which can be regarded as nondecreasing in $y > \boldsymbol{\xi}$. Therefore, whether $y \le \boldsymbol{\xi}$ or $\boldsymbol{\xi} < y$ it follows that $B(\boldsymbol{\xi}, y)$ is nondecreasing in $y \le x_K$.

From the above two results, whether $x_K \le \boldsymbol{\xi}$ or $\boldsymbol{\xi} < x_K$ it follows that $B(\boldsymbol{\xi}, y)$ is nondecreasing in $y \le x_K$. Hence, from (25.2.19(p.258)) we see that $A(y)$ is nondecreasing in $y \le x_K$.

(c) Immediate from (a,b) and the fact that $A(y)$ is continuous on $(-\infty, \infty)$. ∎

Lemma 25.2.6 (rM:1[ℝ][E])

(a) *$\ell_t(y)$ is nonincreasing in $t > 0$ for any y.*

(b) *$\mathbb{V}_t$ is nonincreasing in $t \ge 1$.* □

• *Proof* (a) From Lemma 25.2.5(p.257) (c) and (25.2.10(p.257)) we have $\ell_2(y) \le \ell_1(y)$ for any y. Suppose that $\ell_{t-1}(y) \le \ell_{t-2}(y)$ for any y (induction hypothesis).

1. Let $x_K \le y$. Then, since $V_t(y) \le y$ for $t \ge 0$ due to Lemma 25.2.2(p.255) (a), we have $V_{t-1}(y) \le y$ for $t \ge 1$, hence $v_t(y) = y$ for $t \ge 0$ and $v_{t-1}(y) = y$ for $t \ge 1$ from (24.1.53(p.241)). Thus, from (25.2.6(p.257)) we have $\ell_t(y) = (1-\beta)y$ for $t \ge 1$, hence $\ell_{t-1}(y) = (1-\beta)y$ for $t \ge 2$, so $\ell_t(y) = \ell_{t-1}(y)$ for $t \ge 2$, thus $\ell_t(y) \le \ell_{t-1}(y)$ for $t \ge 2$. Accordingly, it follows that $\ell_t(y)$ is nonincreasing in $t \ge 1$ or equivalently in $t > 0$ on $x_K \le y$.

† $I(\boldsymbol{\xi} < x_K) + I(x_K \le \boldsymbol{\xi}) = 1$.

2. Let $y \leq x_K$. Then, since $y \leq V_t(y)$ for $t \geq 0$ and $y \leq V_{t-1}(y)$ for $t > 0$ from Lemma 25.2.2(p.255) (b), we have $v_t(y) = V_t(y)$ for $t \geq 0$ and $v_{t-1}(y) = V_{t-1}(y)$ for $t \geq 1$ from (24.1.53(p.241)), hence from (25.2.6(p.257)) and (24.1.50(p.241)) we have

$$
\begin{aligned}
\ell_t(y) &= V_t(y) - \beta V_{t-1}(y) \\
&= \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s - \beta\big(\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s\big) \\
&= \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - \beta v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s \\
&= \beta\,\mathbf{E}[\ell_{t-1}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s, \quad t \geq 1.
\end{aligned}
$$

Thus, we have

$$\ell_{t-1}(y) = \beta\,\mathbf{E}[\ell_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s, \quad t \geq 2.$$

Here, since $\ell_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq \ell_{t-2}(\max\{\boldsymbol{\xi}, y\})$ due to the induction hypothesis, we have

$$\ell_t(y) \leq \beta\,\mathbf{E}[\ell_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s = \ell_{t-1}(y), \quad t > 1.$$

Accordingly, by induction we have $\ell_t(y) \leq \ell_{t-1}(y)$ for $t \geq 2$ on $y \leq x_K$, i.e., $\ell_t(y)$ is nonincreasing in $t \geq 1$ on $y \leq x_K$.

From the above two results, whether $x_K \leq y$ or $y \leq x_K$ it follows that $\ell_t(y)$ is nonincreasing in $t > 0$.

(b) Immediate from (a(p.258)) and (25.2.9(p.257)). ∎

25.2.1.2 Analysis

From (24.1.49(p.241)) with $t = 2$ we have

$$
\begin{aligned}
V_2 &= \beta\,\mathbf{E}[v_1(\boldsymbol{\xi})] - s \\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, V_1(\boldsymbol{\xi})\}] - s \qquad \text{(see (24.1.48(p.241)) with } t = 1) \\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, K(\boldsymbol{\xi}) + \boldsymbol{\xi}\}] - s \quad \text{(see (24.1.56(p.241)) with } y = \boldsymbol{\xi}) \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\} + \boldsymbol{\xi}] - s \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\,\mathbf{E}[\boldsymbol{\xi}] - s \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\mu - s.
\end{aligned}
$$

Then (25.2.1(p.255)) with $t = 2$ can be rewritten as

$$
\begin{aligned}
\mathbb{V}_2 &= V_2 - \beta V_1 \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\mu - s - \beta(\beta\mu - s) \quad \text{(see (24.1.54(p.241)))} \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + (1-\beta)(\beta\mu - s) \\
&= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}I(\boldsymbol{\xi} < x_K) + \max\{0, K(\boldsymbol{\xi})\}I(x_K \leq \boldsymbol{\xi})] + (1-\beta)(\beta\mu - s).
\end{aligned}
$$

Due to Corollary 10.2.2(p.58) (a) we have $K(\boldsymbol{\xi}) > 0$ for $\boldsymbol{\xi} < x_K$ and $K(\boldsymbol{\xi}) \leq 0$ for $x_K \leq \boldsymbol{\xi}$, hence we have

$$\mathbb{V}_2 = \beta\,\mathbf{E}[K(\boldsymbol{\xi})I(\boldsymbol{\xi} < x_K)] + (1-\beta)(\beta\mu - s). \tag{25.2.21}$$

Let us define

$$\mathtt{S}_{18}\boxed{Ⓢ\blacktriangle \,|\, ◎\,\vartriangle \,|\, ◎\,\blacktriangle} = \left\{ \begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t_\tau^\bullet\ (t_\tau^\circ \geq t_\tau^\bullet > 1) \text{ such that} \\ \boxed{Ⓢ\ \mathtt{dOITs}_{t_\tau^\bullet \geq \tau > 1}\langle\tau\rangle}_\blacktriangle,\ \boxed{◎\ \mathtt{ndOIT}_{t_\tau^\circ \geq \tau > t_\tau^\bullet}\langle t_\tau^\bullet\rangle}_\vartriangle, \text{ and } \boxed{◎\ \mathtt{ndOIT}_{\tau > t_\tau^\bullet}\langle t_\tau^\bullet\rangle}_\blacktriangle. \end{array} \right\}$$

□ **Tom 25.2.1** (▣ $\mathscr{A}$\{rM:1[ℝ][E]\}) *For any* $\tau > 1$:

(a) *We have:*

1. ♣ *Let* $y \geq x_K$. *Then* $y \geq V_t(y)$ *for* $t \geq 0$.
2. ♣ *Let* $y \leq x_K$. *Then* $y \leq V_t(y)$ *for* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$.

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu - s \geq 0$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$.
2. *Let* $\beta\mu - s < 0$ *and* $\beta\mu - s < a$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$.
3. *Let* $\beta\mu - s < 0$ *and* $\beta\mu - s \geq a$ *(hence* $a < 0$*).*
   i. *Let* $\mathbb{V}_2 \leq 0$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
   ii. *Let* $\mathbb{V}_2 > 0$.
      1. *Let* $\kappa \geq 0$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$.
      2. *Let* $\kappa < 0$. *Then we have* $\mathtt{S}_{18}$(p.259) $\boxed{Ⓢ\blacktriangle \,|\, ◎\,\vartriangle \,|\, ◎\,\blacktriangle}$. ↦ $\rightarrow ◎_\blacktriangle$

● *Proof* Since $\lambda = 1$ is assumed in the model, we have $\delta = 1$ (See (10.2.1(p.56))), hence $(\lambda\beta\mu - s)/\delta = \beta\mu - s \cdots$ **(1)** and $K(a) = \beta\mu - s - a \cdots$ **(2)** from (10.2.4 (1) (p.57)).

(a1,a2) The same as Lemma 25.2.2(p.255) (a,b).

(b) Let $\beta = 1$. Then, from (25.2.1(p.255)) we have $\mathbb{V}_t = V_t - \beta V_{t-1} = V_t - V_{t-1}$ for $t > 1$, hence $\mathbb{V}_t \geq 0$ for $t > 1$ due to Lemma 25.2.1(p.255) (c) or equivalently $V_t \geq \beta V_{t-1}$ for $t > 1$. Thus, since $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau \geq \beta V_{\tau-1}$, $V_{\tau-1} \geq \beta V_{\tau-2}, \cdots, V_2 \geq \beta V_1$, hence $V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1$, so $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.

(c) Let $\beta < 1$.

(c1) Let $\beta\mu - s \geq 0$, hence $V_1 \geq 0$ from (24.1.54(p.241)). Then $V_t \geq V_{t-1} \geq V_1 \geq 0$ for $t > 1$ from Lemma 25.2.1(p.255) (c). Hence, from (25.2.1(p.255)) we have $\mathbb{V}_t = V_t - \beta V_{t-1} \geq V_{t-1} - \beta V_{t-1} = (1-\beta)V_{t-1} \geq 0$ for $t > 1$. Then, since $V_t \geq \beta V_{t-1}$ for $t > 1$, for the same reason as in the proof of (b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.

(c2) Let $\beta\mu - s < 0 \cdots$ **(3)** and $\beta\mu - s < a$. Then, from (2) we have $K(a) < 0$, hence $x_K < a$ from Lemma 10.2.2(p.57) (j1). Below it suffices to consider only $y \in [a, b]$ such that $x_K < a \leq y$. Then, since $V_t(y) \leq y$ for $t \geq 0$ from Lemma 25.2.2(p.255) (a), we have $v_t(y) = y$ for $t \geq 0$ from (24.1.53(p.241)), hence $v_{t-1}(y) = y$ for $t > 0$, so from (25.2.6(p.257)) we have $\ell_t(y) = v_t(y) - \beta v_{t-1}(y) = y - \beta y = (1-\beta)y$ for $t > 0$. Accordingly, since $\ell_{t-1}(\boldsymbol{\xi}) = (1-\beta)\boldsymbol{\xi}$ for $t > 1$ and $\boldsymbol{\xi} \in [a, b]$, from (25.2.9(p.257)) we obtain $\mathbb{V}_t = V_t - \beta V_{t-1} = \beta\,\mathbf{E}[(1-\beta)\boldsymbol{\xi}] - (1-\beta)s = \beta(1-\beta)\,\mathbf{E}[\boldsymbol{\xi}] - (1-\beta)s = \beta(1-\beta)\mu - (1-\beta)s = (1-\beta)(\beta\mu - s) < 0$ for $t > 1$ due to (3). Then, since $V_t < \beta V_{t-1}$ for $t > 1$, we have $V_t < \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly, since $V_\tau < \beta V_{\tau-1}$, $V_{\tau-1} < \beta V_{\tau-2}$, $\cdots$, $V_2 < \beta V_1$, we have $V_\tau < \beta V_{\tau-1} < \beta^2 V_{\tau-2} < \cdots < \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$.

(c3) Let $\beta\mu - s < 0 \cdots$ **(4)** and $\beta\mu - s \geq a$, hence $a < 0$. Then, since $K(a) \geq 0$ from (2), we have $a \leq x_K \cdots$ **(5)** from Lemma 10.2.2(p.57) (j1).

(c3i) Let $\mathbb{V}_2 \leq 0$. Then, since $\mathbb{V}_t \leq 0$ for $t > 1$ from Lemma 25.2.6(p.258) (b), we have $\mathbb{V}_t \leq 0$ for $\tau \geq t > 1$. Hence, since $V_\tau - \beta V_{\tau-1} \leq 0$ for $\tau \geq t > 1$ from (25.2.1(p.255)), we have $V_\tau \leq \beta V_{\tau-1}$ for $\tau \geq t > 1$. Accordingly, since $V_\tau \leq \beta V_{\tau-1}$, $V_{\tau-1} \leq \beta V_{\tau-2}, \cdots, V_2 \leq \beta V_1$, we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1$, so $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c3ii) Let $\mathbb{V}_2 > 0 \cdots$ **(6)**.

(c3ii1) Let $\kappa \geq 0$. Then $\mathbb{V} \geq 0$ due to Lemma 25.2.4(p.256) (d). Hence, from (6) and Lemma 25.2.6(p.258) (b) we have $\mathbb{V}_t \geq 0$ for $t > 1$, hence we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ for the same reason as in the proof of (c1).

(c3ii2) Let $\kappa < 0$. Then $\mathbb{V} < 0$ due to Lemma 25.2.4(p.256) (d). Hence, from (6), and Lemma 25.2.6(p.258) (b) it follows that there exist t°_τ and $t^\bullet_\tau$ ($t^\circ_\tau \geq t^\bullet_\tau > 1$) such that

$$\mathbb{V}_2 \geq \cdots \geq \mathbb{V}_{t^\bullet_\tau-1} \geq \mathbb{V}_{t^\bullet_\tau} > 0 \geq \mathbb{V}_{t^\bullet_\tau+1} \geq \mathbb{V}_{t^\bullet_\tau+1} \geq \cdots \geq \mathbb{V}_{t^\circ_\tau} > \mathbb{V}_{t^\circ_\tau+1} \geq \cdots$$

or equivalently

$$\mathbb{V}_t > 0 \cdots (1^*), \quad t^\bullet_\tau \geq t > 1, \qquad 0 \geq \mathbb{V}_t \cdots (2^*), \quad t^\circ_\tau \geq t > t^\bullet_\tau, \qquad 0 > \mathbb{V}_t \cdots (3^*), \quad t > t^\circ_\tau.$$

[1] Let $t^\bullet_\tau \geq \tau > 1$. Then, since $\mathbb{V}_t > 0$ for $\tau \geq t > 1$ due to (1^*), for almost the same reason as in the proof of (b) we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots$ **(7)**, hence $t^*_\tau = \tau$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \cdots$ **(8)**. From (7) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau-1} > \beta^2 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau-1} V_1.$$

[2] Since $\mathbb{V}_{t^\bullet_\tau+1} \leq 0$ due to (2^*), we have $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$ from (25.2.1(p.255)). Hence

$$V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau} V_1 \cdots \textbf{(9)},$$

so $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$ or equivalently $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(10)**. Since $\mathbb{V}_{t^\bullet_\tau+2} \leq 0$ due to (2^*), we have $V_{t^\bullet_\tau+2} \leq \beta V_{t^\bullet_\tau+1}$. Hence, from (9) we have

$$V_{t^\bullet_\tau+2} \leq \beta V_{t^\bullet_\tau+1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_1,$$

so $t^*_{t^\bullet_\tau+2} = t^\bullet_\tau$ or equivalently we have $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+2}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(11)**. Similarly we obtain $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+3}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(12)**, $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\bullet_\tau+4}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(13)**, $\cdots$. Since $\mathbb{V}_{t^\circ_\tau} \leq 0$ due to (2^*), we have $V_{t^\circ_\tau} \leq \beta V_{t^\circ_\tau-1}$. Hence

$$V_{t^\circ_\tau} \leq \beta V_{t^\circ_\tau-1} \leq \cdots \leq \beta^{t^\circ_\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{t^\circ_\tau - t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\circ_\tau-1} V_1 \cdots \textbf{(14)},$$

so $t^*_{t^\circ_\tau} = t^\bullet_\tau$ or equivalently $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\circ_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(15)**. Hence, from (10), (11), (12), (13), $\cdots$, (15) we have $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\circ_\tau \geq \tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots$ **(16)**.

[3] Since $\mathbb{V}_{t^\circ_\tau+1} < 0$ due to (3), we have $V_{t^\circ_\tau+1} < \beta V_{t^\circ_\tau}$, hence from (14) we get

$$V_{t^\circ_\tau+1} < \beta V_{t^\circ_\tau} \leq \beta^2 V_{t^\circ_\tau-1} \leq \cdots \leq \beta^{t^\circ_\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{t^\circ_\tau - t^\bullet_\tau+1} V_{t^\bullet_\tau} > \beta^{t^\circ_\tau - t^\bullet_\tau+2} V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\circ_\tau} V_1,$$

so $t^*_{t^\bullet+1} = t^\bullet_\tau$ or equivalently $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\circ_\tau+1}\langle t^\bullet_\tau\rangle}_\blacktriangle$. Similarly, since $\mathbb{V}_{t^\circ_\tau+2} < 0$, we have $\boxed{\circledcirc\ \texttt{ndOIT}_{t^\circ_\tau+3}\langle t^\bullet_\tau\rangle}_\blacktriangle$. In general, we have $\boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\circ_\tau}\langle t^\bullet_\tau\rangle}_\blacktriangle \cdots$ **(17)**.

From [1]-[3] above we see that (8), (16), and (17) can be summarized as $\texttt{S}_{18}$(p.259) $\boxed{\circledS\blacktriangle}\boxed{\circledcirc\ \vartriangle}\boxed{\circledcirc\ \blacktriangle}$. ∎

25.2.1.3 Flow of Optimal Decision Rules

♣ `Flow-ODR` **7** (rM:1[ℝ][E]) (`c-reservation-price`) *From* `Tom` 25.2.1(p.259) (♣ a1,♣ a2) *and* (24.1.58(p.241)) *we have the following decision rule for* $\tau \geq t > 0$:

$$\begin{cases} y \geq x_K & \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \mathbf{I} \\ y \leq x_K & \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{cases}$$

Namely, the optimal reservation value is given by x_K, which is constant in t.

Definition 25.2.1 (myopic property) `c-reservation-price` implies that the optimal decision of any point in time $t > 1$ is identical to that of time 1 at which the process terminates a period hence, i.e., the deadline, implying that the optimal decision is the same as "behave as if the process terminates a period hence", called the *myopic property*. □

25.2.1.4 Market Restriction

25.2.1.4.1 Positive Restriction

□ `Pom` **25.2.1** ($\mathscr{A}\{$rM:1[ℝ][E]$^+\}$) *Suppose* $a > 0$.

(a) *We have* `c-reservation-price` (♣ `Flow-ODR` 7).

(b) *Let* $\beta = 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu - s \geq 0$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$
2. *Let* $\beta\mu - s < 0$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\kappa = \beta\mu - s$ from Lemma 10.3.1(p.59) (a).

(a) Clear from Lemma 25.2.1(p.259) (♣a1,♣a2) and ♣ `Flow-ODR` 7.

(b) The same as `Tom` 25.2.1(p.259) (b).

(c) Let $\beta < 1$.

(c1) The same as `Tom` 25.2.1(p.259) (c1).

(c2) Let $\beta\mu - s < 0$. Then, since $\beta\mu - s < a$ due to (1), we have `Tom` 25.2.1(p.259) (c2). ∎

25.2.1.4.2 Mixed Restriction

Omitted.

25.2.1.4.3 Negative Restriction

Omitted.

25.2.2 r$\tilde{\text{M}}$:1[ℝ][E]

25.2.2.1 Symmetry of SOE{rM:1[ℝ][E]} and SOE{r$\tilde{\text{M}}$:1[ℝ][E]}

Here let us show that SOE{r$\tilde{\text{M}}$:1[ℝ][E]} (see (24.1.63(p.241))) is symmetrical to SOE{rM:1[ℝ][E]} (see (24.1.51(p.241))), which is a necessary condition under which $\mathscr{A}\{$r$\tilde{\text{M}}$:1[ℝ][E]$\}$ can be derived by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to $\mathscr{A}\{$rM:1[ℝ][E]$\}$ given by `Tom` 25.2.1(p.259).

1. For convenience of reference, below let us copy (24.1.47(p.241))-(24.1.50(p.241)):

   (1^*): $v_0(y) = y$, (2^*): $v_t(y) = \max\{y, V_t(y)\}$, (3^*): $V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s$,
   (4^*): $V_t(y) = \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$. Then we have

   $$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\} = \{(1^*),\ (2^*),\ (3^*),\ (4^*)\}.$$

2. Applying the reverse operation $\mathcal{R}$ to the above four equalities yields:

   $(1^*)' : -\hat{v}_0(-\hat{y}) = -\hat{y}$, $(2^*)' : -\hat{v}_t(-\hat{y}) = \max\{-\hat{y}, -\hat{V}_t(-\hat{y})\} = -\min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)' : -\hat{V}_t = \beta\,\mathbf{E}[-\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] - s$,
   $(4^*)' : -\hat{V}_t(-\hat{y}) = \beta\,\mathbf{E}[-\hat{v}_{t-1}(\max\{-\hat{\boldsymbol{\xi}}, -\hat{y}\})] - s = \beta\,\mathbf{E}[-\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] - s$,

   which can be rearranged as:

   $(1^*)'$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)'$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)'$: $\hat{V}_t = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] + s$,
   $(4^*)'$: $\hat{V}_t(-\hat{y}) = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] + s$. Then we have

   $$\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)',\ (2^*)',\ (3^*)',\ (4^*)'\}.$$

3. We have $\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] = \mathbf{E}[\hat{v}_{t-1}(\boldsymbol{\xi})] = \int_{-\infty}^{\infty} \hat{v}_{t-1}(\xi)f(\xi)d\xi = \int_{-\infty}^{\infty} \hat{v}_{t-1}(\xi)\check{f}(\hat{\xi})d\xi$ (see Lemma 12.3.1(p.72) (a): the application of the correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$). Let $\eta \overset{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then $\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] = -\int_{\infty}^{-\infty} \hat{v}_{t-1}(-\eta)\check{f}(\eta)d\eta = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\eta)\check{f}(\eta)d\eta = \check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\eta})] \cdots (\blacklozenge)$. Similarly we have $\mathbf{E}[\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] = \check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta}, \hat{y}\})]$. Hence $(1^*)'$ - $(4^*)'$ can be rewritten as:

   $(1^*)''$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)''$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)''$: $\hat{V}_t = \beta\, \check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\eta})] + s$,
   $(4^*)''$: $\hat{V}_t(-\hat{y}) = \beta\, \check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta}, \hat{y}\})] + s$, so we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)'', (2^*)'', (3^*)'', (4^*)''\}.$$

4. Let us replace $\check{f}(\eta)$ by $f(\eta)$ in $(\blacklozenge)$ (see Lemma 12.3.3(p.73) (a); the application of the identity replacement operation $\mathcal{I}_{\mathbb{R}}$). Then, $(\blacklozenge)$ can be rearranged as $\check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\eta})] = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\eta)f(\eta)d\eta = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\xi)f(\xi)d\xi^\dagger = \mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})]$. Similarly $\check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta}, \hat{y}\})] + s = \mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi}, \hat{y}\})] + s$. Accordingly $(1^*)''$ - $(4^*)''$ can be rewritten as;

   $(1^*)'''$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)'''$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)'''$: $\hat{V}_t = \beta\, \mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})] + s$,
   $(4^*)'''$: $\hat{V}_t(-\hat{y}) = \beta\, \mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi}, \hat{y}\})] + s$. Then we have

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)''', (2^*)''', (3^*)''', (4^*)'''\}.$$

5. Since $(1^*)'''$ - $(4^*)'''$ hold for any given $y \in (-\infty, \infty)$, they holds also for $\hat{y} \in (-\infty, \infty)$, hence $(1^*)'''$ - $(4^*)'''$ hold for $\hat{\hat{y}} \in (-\infty, \infty)$. Accordingly, since $\hat{\hat{y}} = y$, it follows that they hold also for any given y. Thus, we obtain the following:

   $(1^*)''''$: $\hat{v}_0(-y) = y$, $(2^*)''''$: $\hat{v}_t(-y) = \min\{y, \hat{V}_t(-y)\}$, $(3^*)''''$: $\hat{V}_t = \beta\, \mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})] + s$,
   $(4^*)''''$: $\hat{V}_t(-y) = \beta\, \mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi}, y\})] + s$. Then we have

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)'''', (2^*)'''', (3^*)'''', (4^*)''''\}. \quad (25.2.22)$$

6. Note here that $\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$ can be given by (24.1.59(p.241))-(24.1.62(p.241)), i.e.,

   $(1^*)'''''$: $v_0(y) = y$, $(2^*)'''''$: $v_t(y) = \min\{y, V_t(y)\}$, $(3^*)'''''$: $V_t = \beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s$,
   $(4^*)'''''$: $V_t(y) = \beta\, \mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s$. Then we have

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \{(1^*)''''', (2^*)''''', (3^*)''''', (4^*)'''''\}. \quad (25.2.23)$$

7. From $(1^*)''''$ and $(1^*)'''''$ we have $\hat{v}_0(-y) = y = v_0(y)$ for any y, i.e., $(1^*)'''' = (1^*)'''''$ for $t = 0$. Suppose $\hat{v}_{t-1}(-y) = v_{t-1}(y)$ for any y. Thus $(3^*)'''' = (3^*)'''''$. Then, from $(4^*)''''$ we have $\hat{V}_t(-y) = \beta\, \mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s = V_t(y)$, so $(4^*)'''' = (4^*)'''''$for any y. Hence, from $(2^*)''''$ we have $\hat{v}_t(-y) = \min\{y, V_t(y)\} = v_t(y)$, so $(2^*)'''' = (2^*)'''''$. Accordingly, by induction $\hat{v}_{t-1}(-y) = v_{t-1}(y)$ for any $t > 0$, so $(1^*)'''' = (1^*)'''''$. Thus it follows that (25.2.22(p.262)) is identical to (25.2.23(p.262)), so we have

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{A}]\}] = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{A}]\}] \quad \text{(see (12.5.30(p.77)))},$$

   meaning that $\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$ is symmetrical to $\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}$

25.2.2.2 Derivation of $\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$

As it was demonstrated that $\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$ is symmetrical to $\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}$, we see that $\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$ can be obtained by applying Scenario[$\mathbb{R}$](p.75) to $\mathscr{A}\{\text{rM:1}[\mathbb{R}][\text{E}]\}$ given by `Tom` 25.2.1(p.259). Before conducting its application, let us apply $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to $\mathbb{V}_2$ given by (25.2.21(p.259)). First let us apply the reverse operation $\mathcal{R}$ to $\mathbb{V}_2$ given by (25.2.21(p.259)). Here note that (25.2.21(p.259)) is expressed as

$$\mathbb{V}_2 \;=\; \beta\int_{-\infty}^{\infty} K(\xi)I(\xi < x_K)f(\xi)d\xi + (1-\beta)(-\beta\mu + s).$$

Hence we have

$$\begin{aligned}\mathcal{R}[\mathbb{V}_2] = \hat{\mathbb{V}}_2 = -\mathbb{V}_2 \;&=\; \beta\int_{-\infty}^{\infty} -K(\xi)I(-\xi > -x_K)f(\xi)d\xi + (1-\beta)(-\beta\mu + s)\\ &=\; \beta\int_{-\infty}^{\infty} \hat{K}(\xi)I(\hat{\xi} > \hat{x}_K)f(\xi)d\xi + (1-\beta)(\beta\hat{\mu} + s)\cdots(*).\end{aligned}$$

Then, applying the replacement $\eta = \hat{\xi} = -\xi$ (hence $d\eta = -d\xi$), $\hat{\mu} = \check{\mu}$, $\hat{K}(\xi) = \check{\hat{K}}(\hat{\xi})$, and $\hat{x}_K = x_{\check{\hat{K}}}$ (see Lemma 12.3.1(p.72) (b,e,h)) to $(*)$ leads to

$$\begin{aligned}\mathcal{R}[\mathbb{V}_2] \;&=\; -\beta\int_{\infty}^{-\infty} \check{\hat{K}}(\hat{\xi})I(\eta > x_{\check{\hat{K}}})\check{f}(\eta)d\eta + (1-\beta)(\beta\check{\mu} + s)\\ &=\; \beta\int_{-\infty}^{\infty} \check{\hat{K}}(\eta)I(\eta > x_{\check{\hat{K}}})\check{f}(\eta)d\eta + (1-\beta)(\beta\check{\mu} + s)\\ &=\; \beta\int_{-\infty}^{\infty} \check{\hat{K}}(\xi)I(\xi > x_{\check{\hat{K}}})\check{f}(\xi)dx + (1-\beta)(\beta\check{\mu} + s) \quad \text{(without loss of generality)}\end{aligned}$$

Since the above replacement means the application of $\mathcal{C}_{\mathbb{R}}$ to $\mathcal{R}[\mathbb{V}_2]$, i.e., $\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] = \mathcal{R}[\mathbb{V}_2]$, we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] \;=\; \beta\int_{-\infty}^{\infty} \check{\hat{K}}(\xi)I(\xi > x_{\check{\hat{K}}})\check{f}(\xi)d\xi + (1-\beta)(\beta\check{\mu} + s).$$

---

† without loss of generality

Furthermore, applying the identity replacement operation $\mathcal{I}_{\mathbb{R}}$ to this (see Lemma 12.3.3(p.73) (e,h)) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] = \beta \int_{-\infty}^{\infty} \tilde{K}(\xi) I(\xi > x_{\tilde{K}}) f(\xi) d\xi + (1-\beta)(\beta\mu + s))$$
$$= \beta \mathbf{E}[\tilde{K}(\boldsymbol{\xi}) I(\boldsymbol{\xi} > x_{\tilde{K}})] + (1-\beta)(\beta\mu + s).$$

Noting (12.5.30(p.77)), we can rewrite the above as

$$\tilde{\mathbb{V}}_2 \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathbb{V}_2] = \beta \mathbf{E}[\tilde{K}(\boldsymbol{\xi}) I(\boldsymbol{\xi} > x_{\tilde{K}})] + (1-\beta)(\beta\mu + s).$$

Then we have the following `Tom`.

□ **`Tom` 25.2.2** (⧈$\mathscr{A}${**r**$\tilde{\mathsf{M}}$:1[ℝ][E]})

(a) *We have:*

1. ♣ *Let* $y \le x_{\tilde{K}}$*. Then* $y \le V_t(y)$ *for* $t \ge 0$.
2. ♣ *Let* $y \ge x_{\tilde{K}}$*. Then* $y \ge V_t(y)$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$*. Then* [ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle$]$_\triangle$.

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu + s \le 0$*. Then* [ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle$]$_\triangle$.
2. *Let* $\beta\mu + s > 0$ *and* $\beta\mu + s > b$*. Then* [● `dOITd`$_{\tau>1}\langle 1\rangle$]$_\blacktriangle$.
3. *Let* $\beta\mu + s > 0$ *and* $\beta\mu + s \le b$ *(hence* $b > 0$*).*
   - i. *Let* $\tilde{\mathbb{V}}_2 \ge 0$*. Then* [● `dOITd`$_{\tau>1}\langle 1\rangle$]$_\triangle$.
   - ii. *Let* $\tilde{\mathbb{V}}_2 < 0$.
     1. *Let* $\tilde{\kappa} \le 0$*. Then* [ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle$]$_\triangle$.
     2. *Let* $\tilde{\kappa} > 0$*. Then we have* $\mathtt{S}_{18}$(p.259) [ⓢ▲ | ◎ △ | ◎ ▲]. □

● *Proof by symmetry* Immediately obtained from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 25.2.1(p.259). ∎

25.2.2.3 Flow of Optimal Decision Rules

♣ **`Flow-ODR` 8** (**r**$\tilde{\mathsf{M}}$:1[ℝ][E]) (`c-reservation-price`) *From* `Tom` 25.2.2(p.263) (♣ a1,♣ a2) *and* (24.1.66(p.241)) *we have the following decision rule for* $\tau \ge t > 0$.

$$\left.\begin{array}{lll} y \le x_{\tilde{K}} & \Rightarrow & \mathtt{Accept}_t\langle y\rangle \text{ and the process stops} \,\mathbf{I} \\ y \ge x_{\tilde{K}} & \Rightarrow & \mathtt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \quad t > 0. \tag{25.2.24}$$

Namely, the optimal reservation value is given by $x_{\tilde{K}}$*, which is constant in* t.

25.2.2.4 Market Restriction

25.2.2.4.1 Positive Restriction

□ **`Pom` 25.2.2** ($\mathscr{A}${**r**$\tilde{\mathsf{M}}$:1[ℝ][E]$^+$}) *Suppose* $a > 0$.

(a) *We have* `c-reservation-price`.

(b) *Let* $\beta = 1$*. Then* [ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle$]$_\triangle$

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu + s > b$*. Then* [● `dOITd`$_{\tau>1}\langle 1\rangle$]$_\blacktriangle$
2. *Let* $\beta\mu + s \le b$.
   - i. *Let* $\tilde{\mathbb{V}}_2 \ge 0$*. Then* [● `dOITd`$_{\tau>1}\langle 1\rangle$]$_\triangle$
   - ii. *Let* $\tilde{\mathbb{V}}_2 < 0$.
     1. *Let* $s = 0$*. Then* [ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle$]$_\triangle$
     2. *Let* $s > 0$*. Then we have* $\mathtt{S}_{18}$(p.259) [ⓢ▲ | ◎ △ | ◎ ▲]

● *Proof* Suppose $a > 0$. Then $\mu > a > 0$, hence $\beta\mu > 0$, so $\beta\mu + s > 0 \cdots \mathbf{(1)}$ for any $s \ge 0$. Then $\tilde{\kappa} = s$ from Lemma 12.6.6(p.83) (a).

(a) Clear from Lemma 25.2.2(p.263) (♣a1,♣a2) and ♣ `Flow-ODR` 8.

(b) The same as `Tom` 25.2.2(p.263) (b).

(c) Let $\beta < 1$.

(c1) Let $\beta\mu + s > b$. Then, due to (1) we have `Tom` 25.2.2(p.263) (c2).

(c2-c2ii2) Let $\beta\mu + s \le b$. Then, due to (1) we have `Tom` 25.2.2(p.263) (c3i-c3ii2). ∎

25.2.2.4.2 Mixed Restriction

Omitted.

25.2.2.4.3 Negative Restriction

Omitted.

25.2.3 Conclusion 8 (Search-Enforced-Model 1)

■ The assertion systems $\mathscr{A}$ of the quadruple-asset-trading-models the total market $\mathscr{F}$

$$\mathcal{Q}\{\mathrm{rM}{:}1[\mathrm{E}]\} = \{\mathrm{rM}{:}1[\mathbb{R}][\mathrm{E}],\ \mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}],\ $$ ~~$\mathrm{rM}{:}1[\mathbb{P}][\mathrm{E}],\ \mathrm{r\tilde{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$~~

are given by

$$\begin{array}{cc} \mathscr{A}\{\mathrm{rM}{:}1[\mathbb{R}][\mathrm{E}]\} & \mathscr{A}\{\mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]\} \\ \downarrow & \downarrow \\ \texttt{Tom's}\ 25.2.1(\text{p.259}), & 25.2.2(\text{p.263}), \end{array}$$

■ The assertion systems $\mathscr{A}$ of the quadruple-asset-trading-models for Model 1 on the positive market $\mathscr{F}^+$

$$\mathcal{Q}\{\mathrm{rM}{:}1[\mathrm{E}]\}^+ = \{\mathrm{rM}{:}1[\mathbb{R}][\mathrm{E}]^+,\ \mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+$$ ~~$,\ \mathrm{rM}{:}1[\mathbb{P}][\mathrm{E}]^+,\ \mathrm{r\tilde{M}}{:}1[\mathbb{P}][\mathrm{E}]^+$~~ $\}$

are given by

$$\begin{array}{cc} \mathscr{A}\{r\mathrm{M}{:}1[\mathbb{R}][\mathrm{E}]^+\} & \mathscr{A}\{r\mathrm{\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\} \\ \downarrow & \downarrow \\ \texttt{Pom's}\ 25.2.1(\text{p.261}), & 25.2.2(\text{p.263}), \end{array}$$

■ Closely looking into all the assertion systems above leads to the conclusions below.

C1. We have $\mathscr{A}\{\mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathrm{rM}{:}1[\mathbb{R}][\mathrm{E}]\}^+$.
C2. We have ⓢ$_{\triangle\blacktriangle}$ for $\mathrm{rM/\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+$.
C3. We have ◎$_{\triangle\blacktriangle}$ for $\mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+$.
C4. We have ❶$_{\triangle\blacktriangle}$ for $\mathrm{rM/\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+$.
C5. We have `c-reservation-price` for $\mathrm{rM/\tilde{M}}{:}1[\mathbb{R}][\mathrm{E}]^+$. □

C1 Compare `Pom` 25.2.2(p.263) and `Pom` 25.2.1(p.261).
C2 See `Pom` 25.2.1(p.261) (b,c1) and `Pom` 25.2.2(p.263) (b,c2ii1,c2ii2).
C3 See `Pom` 25.2.2(p.263) (c2ii2).
C4 See `Pom` 25.2.1(p.261) (c2) and `Pom` 25.2.2(p.263) (c1,c2i).
C5 See `Pom` 25.2.1(p.261) (a) and `Pom` 25.2.2(p.263) (a). ▮

Chapter 26

Analysis of Model 2

26.1 Search-Allowed-Model 2

26.1.1 rM:2[ℝ][A]

26.1.1.1 Preliminary

Let us define

$$V_t^\diamond(y) \stackrel{\text{def}}{=} V_t(y) - y, \quad t \geq 0, \quad (\text{see } (24.2.7\text{(p.242)}) \text{ and } (24.2.5\text{(p.242)})) \tag{26.1.1}$$

$$v_t^\diamond(y) \stackrel{\text{def}}{=} v_t(y) - y = \max\{0, V_t^\diamond(y)\}, \quad t \geq 0, \quad (\text{see } (24.2.8\text{(p.242)})) \tag{26.1.2}$$

where

$$V_0^\diamond(y) \;=\; V_0(y) - y = \rho - y \qquad (\text{see } (24.2.7\text{(p.242)})), \tag{26.1.3}$$

$$v_0^\diamond(y) \;=\; v_0(y) - y = \max\{0, \rho - y\} \quad (\text{see } (24.2.1\text{(p.242)})). \tag{26.1.4}$$

Then, from (24.2.5(p.242)) we have

$$\begin{aligned}
V_t^\diamond(y) &= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta(v_{t-1}^\diamond(y) + y) - s, \beta(v_{t-1}^\diamond(y) + y)\} - y \\
&= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta y - s, \beta v_{t-1}^\diamond(y) + \beta y\} - y \\
&= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + K(y) + y, \beta v_{t-1}^\diamond(y) + \beta y\} - y \quad (\text{see } (5.1.10\text{(p.25)})) \\
&= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + K(y), \beta v_{t-1}^\diamond(y) - (1-\beta)y\}, \quad t > 0.
\end{aligned} \tag{26.1.5}$$

By $y_t^\diamond$ let us denote the solution of the equation $V_t^\diamond(y) = 0$ for $t \geq 0$ if it exists, i.e.,

$$V_t^\diamond(y_t^\diamond) = 0, \quad t > 0. \tag{26.1.6}$$

If multiple solutions exist, it is defined to be the *smallest* of them. Let us define

$$\mathbb{V}_t \;\stackrel{\text{def}}{=}\; V_t - \beta V_{t-1}, \quad t > 0. \tag{26.1.7}$$

Then, from (24.2.12(p.242)) and (24.2.3(p.242)) we have

$$\mathbb{V}_1 \;=\; V_1 - \beta V_0 = \max\{L(\rho), 0\}. \tag{26.1.8}$$

From (24.2.1(p.242)) and (24.2.3(p.242)) we have $v_0(\boldsymbol{\xi}) - V_0 = \max\{\boldsymbol{\xi}, \rho\} - \rho = \max\{\boldsymbol{\xi} - \rho, 0\}$, hence from (24.2.17(p.242)) with $t = 1$ we get

$$\begin{aligned}
\mathbb{S}_1 &= \lambda\beta\,\mathbf{E}[v_0(\boldsymbol{\xi}) - V_0] - s \\
&= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - \rho, 0\}] - s \\
&= \lambda\beta T(\rho) - s = L(\rho) \quad (\text{see } (5.1.1\text{(p.25)}) \text{ and } (5.1.3\text{(p.25)})).
\end{aligned} \tag{26.1.9}$$

Now (24.2.23(p.242)) can be rewritten as

$$\begin{aligned}
\mathbb{S}_t(y) &= \lambda\beta\,\mathbf{E}[(v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))] - s \\
&= \lambda\beta\,\mathbf{E}[(v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(y < \boldsymbol{\xi}) + (v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(\boldsymbol{\xi} \leq y)] - s \\
&= \lambda\beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y < \boldsymbol{\xi}) + (v_{t-1}(y) - v_{t-1}(y))I(\boldsymbol{\xi} \leq y)] - s \\
&= \lambda\beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y < \boldsymbol{\xi})] - s.
\end{aligned} \tag{26.1.10}$$

From (24.2.1(p.242)) we have $v_0(\boldsymbol{\xi}) - v_0(y) = \max\{\boldsymbol{\xi}, \rho\} - \max\{y, \rho\} \le \max\{\boldsymbol{\xi} - y, 0\}$ for any $\boldsymbol{\xi}$ and y, hence from (26.1.10(p.265)) with $t = 1$ we have

$$\mathbb{S}_1(y) = \lambda\beta\,\mathbf{E}[(v_0(\boldsymbol{\xi}) - v_0(y))I(y < \boldsymbol{\xi})] - s \le \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(y < \boldsymbol{\xi})] - s.$$

Then, since $\max\{\boldsymbol{\xi} - y, 0\} \ge 0$ and $I(y < \boldsymbol{\xi}) \le 1$, we get $\max\{\boldsymbol{\xi} - y, 0\}I(y < \boldsymbol{\xi}) \le \max\{\boldsymbol{\xi} - y, 0\}$, hence

$$\mathbb{S}_1(y) \;\le\; \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}] - s \tag{26.1.11}$$
$$\;=\; \lambda\beta T(y) - s = L(y) \quad \text{(see (5.1.1(p.25)) and (5.1.3(p.25)))}. \tag{26.1.12}$$

26.1.1.2 Preliminary

Lemma 26.1.1 (rM:2[$\mathbb{R}$][A])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \ge 0$.*
(b) *$V_t^\diamond(y)$ is nonincreasing in y for $t \ge 0$.* □

• *Proof* (a) Clearly $v_0(y)$ is nondecreasing in y from (24.2.1(p.242)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (24.2.5(p.242)), hence $v_t(y)$ is nondecreasing in y from (24.2.8(p.242)). Thus by induction $v_t(y)$ is nondecreasing in y for $t \ge 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is also nondecreasing in y for $t > 0$ from (24.2.5(p.242)). In addition, since $V_0(y)$ can be regarded as nondecreasing in y from (24.2.7(p.242)), it follows that $V_t(y)$ is nondecreasing in y for $t \ge 0$.

(b) $V_0^\diamond(y)$ is nonincreasing in y from (26.1.3(p.265)). Suppose $V_{t-1}^\diamond(y)$ is nonincreasing in y, hence $v_{t-1}^\diamond(y)$ is also nonincreasing in y from (26.1.2(p.265)). In addition, since $K(y)$ and $-(1-\beta)y$ are both nonincreasing in y (see Lemma 10.2.2(p.57) (b)), it follows from (26.1.5(p.265)) that $V_t^\diamond(y)$ is also nonincreasing in y. Thus, by induction $V_t^\diamond(y)$ is also nonincreasing in y for $t \ge 0$. ∎

If $y < (\ge)\ \boldsymbol{\xi}$, then $v_{t-1}(\boldsymbol{\xi}) \ge (\le)\ v_{t-1}(y)$ for $t > 0$ due to Lemma 26.1.1(p.266) (a) or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \ge (\le)\ 0$ for $t > 0$. Then, since

$$\begin{aligned}
&\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} \\
&\quad= \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y < \boldsymbol{\xi}) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \ge \boldsymbol{\xi}) \\
&\quad= (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y < \boldsymbol{\xi}) + 0 \times I(y \ge \boldsymbol{\xi}) \\
&\quad= (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y < \boldsymbol{\xi}),
\end{aligned}$$

we can rewrite (26.1.10(p.265)) as

$$\mathbb{S}_t(y) \;=\; \lambda\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] - s, \quad t > 0. \tag{26.1.13}$$

Lemma 26.1.2 (rM:2[$\mathbb{R}$][A]) *Let $\beta = 1$ or $s = 0$.*

(a) *Let $\beta = 1$. Then $y \le V_t(y)$ for any y and $t > 0$.*
(b) *Let $s = 0$. Then $\mathbb{S}_t(y) \ge 0$ for any y and $t > 0$.* □

• *Proof* (a) If $\beta = 1$, from (24.2.5(p.242)) and (24.2.2(p.242)) we have $V_t(y) \ge \beta v_{t-1}(y) = v_{t-1}(y) \ge y$ for any y and any $t > 0$.

(b) If $s = 0$, from (26.1.13(p.266)) we have $\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] \ge 0$ for any y and $t > 0$. ∎

Lemma 26.1.3 (rM:2[$\mathbb{R}$][A]) *Let $\beta < 1$ and $s > 0$.*

(a) $\lim_{y\to-\infty} V_t^\diamond(y) = \infty$ *for $t \ge 0$.*
(b) $\lim_{y\to\infty} V_t^\diamond(y) = -\infty$ *for $t > 0$.*
(c) *The solution $y_t^\diamond$ exists for $t > 0$ such that*
  1. *Let $y \ge y_t^\diamond$. Then $V_t(y) \le y$ for $t > 0$.*
  2. *Let $y \le y_t^\diamond$. Then $V_t(y) \ge y$ for $t > 0$.* □

• *Proof* Let $\beta < 1$ and $s > 0$.

(a) Obviously $V_0^\diamond(y) \to \infty$ as $y \to -\infty$ from (26.1.3(p.265)). Suppose $V_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$. Then $v_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$ from (26.1.2(p.265)). Hence, from (26.1.5(p.265)) we have $V_t^\diamond(y) \to \infty$ as $y \to -\infty$ due to the facts that $K(y) \to \infty$ as $y = -\infty$ due to (10.2.4 (1) (p.57)) and that $-(1-\beta)y \to \infty$ as $y \to -\infty$. Thus, by induction $V_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$ for $t \ge 0$, i.e., $\lim_{y\to-\infty} V_t^\diamond(y) = \infty$ for $t \ge 0$.

(b) Evidently $v_0^\diamond(y) \to 0$ as $y \to \infty$ from (26.1.4(p.265)). Suppose $v_{t-1}^\diamond(y) \to 0$ as $y \to \infty$. Noting that $K(y) \to -\infty$ as $y \to \infty$ from (10.2.5 (2) (p.57)) and that $-(1-\beta)y \to -\infty$ as $y \to \infty$, from (26.1.5(p.265)) we have $V_t^\diamond(y) \to -\infty$ for $t \ge 0$ as $y \to \infty$. Hence, from (26.1.2(p.265)) we have $v_t^\diamond(y) \to 0$ as $y \to \infty$. Thus, by induction $v_t^\diamond(y) \to 0$ for any $t \ge 0$ as $y \to \infty$, hence $v_{t-1}^\diamond(y) \to 0$ for any $t > 0$ as $y \to \infty$. Then, for the same reason as just above we have $V_t^\diamond(y) \to -\infty$ for $t > 0$ as $y \to \infty$, i.e., $\lim_{y\to\infty} V_t^\diamond(y) = -\infty$ for $t > 0$.

(c) From (a,b) and Lemma 26.1.1(p.266) (b) we see that there exists the solution $y_t^\diamond$, and then clearly we have $\ge (\le)\ y_t^\diamond \Rightarrow V_t^\diamond(y) \le (\ge)\ 0 \Leftrightarrow V_t(y) \le (\ge)\ y$ for $t > 0$ from (26.1.1(p.265)). ∎

Lemma 26.1.4 (**r**M:2[ℝ][A]) *Let $\beta < 1$ and $s > 0$.*

(a) *Let $y \le 0$. Then $V_t(y) \ge y$ for $t > 0$.*

(b) *Let $y > 0$.*

1. *Let $y \ge y_t^\diamond$. Then $V_t(y) \le y$ for $t > 0$,*
2. *Let $y \le y_t^\diamond$. Then $V_t(y) \ge y$ for $t > 0$*

where $y_t^\diamond \ge 0$ for $t > 0$. □

● *Proof* Let $\beta < 1$ and $s > 0$. Since $V_1(y) \ge K(\max\{y,\rho\}) + \max\{y,\rho\}$ for any y from (24.2.14(p.242)) and since $\max\{y,\rho\} \ge y$ for any y, we obtain $V_1(y) \ge K(y) + y \cdots$ (1) for any y due to Lemma 10.2.2(p.57) (e).

(a) Let $y \le 0 \cdots$ (2). Since $V_t(y) \ge \beta v_{t-1}(y)$ for $t > 0$ from (24.2.5(p.242)) and since $v_{t-1}(y) \ge y$ for $t > 0$ from (24.2.2(p.242)), we have $V_t(y) \ge \beta v_{t-1}(y) \ge \beta y$ for $t > 0$. Then, since $\beta y \ge y$ due to (2), we have $V_t(y) \ge y$ for $t > 0$.

(b) Let $y > 0 \cdots$ (3).

(b1,b2) The same as Lemma 26.1.3(p.266) (c1,b1). ∎

26.1.1.3 Analysis

□ `Tom` **26.1.1** (■ $\mathscr{A}$\{**r**M:2[ℝ][A]\})

(a) *Let $s = 0$. Then rM:2[ℝ][A] ↬ rM:2[ℝ][E].*

(b) *Let $\beta = 1$.*

1. ♣ *We have $y \le V_t(y)$ for any y and $t \ge 0$.*
2. *We have the future-subject* [F.S.] 2 (*the conditions for* ⓢ, ◎, *and* ⓓ)

(c) *Let $\beta < 1$ and $s > 0$.*

1. *We have* [ⓢ `dOITs`$_{\tau \ge 0}\langle\tau\rangle$]$_\triangle$.
2. ♣ *Let $y \le 0$. Then $y \le V_t(y)$ for $t \ge 0$.*
3. *Let $y \ge 0$.*
   i. ♠ *Let $y \ge y_t^\diamond$. Then $V_t(y) \le y$ for $t \ge 0$.*
   ii. ♠ *Let $y \le y_t^\diamond$. Then $y \le V_t(y)$ for $t \ge 0$.* □

● *Proof* (a) Let $s = 0$. Then, from Lemma 26.1.2(p.266) (b) we see that it is always optimal to `Conduct`$_t$ the search due to (24.2.25(p.242)), implying that rM:2[ℝ][A], which is originally a search-Allowed-model, is substantially reduced to rM:2[ℝ][E], which is a search-Enforced-model. In other words, rM:2[ℝ][A] migrates to rM:2[ℝ][E], represented as rM:2[ℝ][A] ↬ rM:2[ℝ][E] (see Def. 11.2.3(p.63)).

(b) Let $\beta = 1$.

(b1) The same as Lemma 26.1.2(p.266) (a).

(b2) The subject of future study ——

(c) Let $\beta < 1$ and $s > 0$.

(c1) From (24.2.4(p.242)) we have $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 0$, hence $V_\tau \ge \beta V_{\tau-1}$, $V_{\tau-1} \ge \beta V_{t-2}$, $\cdots$, $V_1 \ge \beta V_0$, so $V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{t-2} \ge \cdots \ge \beta^\tau V_0$. Accordingly, we have $t_\tau^* = \tau$ for $\tau \ge 0$, i.e., [ⓢ `dOITs`$_{\tau \ge 0}\langle\tau\rangle$]$_\triangle$.

(c2) The same as Lemma 26.1.4(p.267) (a).

(c3-c3ii) The same as Lemma 26.1.4(p.267) (b-b2). ∎

26.1.1.4 Flow of Optimal Decision Rules

♣ `Flow-ODR` **9** (**r**M:2[ℝ][A]) (`Accept`$_0(y) \triangleright$`Stop`) *Let $\beta = 1$. Then, the inequality $y \le V_t(y)$ for any y and $t \ge 0$* (see `Tom` 26.2.2(p.274) (♣a1)) *means that even if the process is initiated at any time t, it is optimal to reject the best price y at that time. Accordingly, it follows that each time a price ξ is proposed, the current best price y continues to be enlarged to $y \overset{\text{def}}{=} \max\{y,\xi\}$, and the process terminates by accepting the best price y at the deadline $t = 0$, i.e.,* `Accept`$_0(y) \triangleright$`Stop`. □

♠ `Flow-ODR` **10** (**r**M:2[ℝ][A]) (*t*-`reservation-price`) *Let $\beta < 1$ or $s > 0$. Then, from* `Tom` 26.1.1(p.267) (♠c3i,♠c3ii) *and* (24.2.29(p.242)) *we have the following relations for $\tau \ge t \ge 0$:*

$$\left.\begin{array}{l} y \ge y_t^\diamond \;\Rightarrow\; \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \,\blacksquare \\ y \le y_t^\diamond \;\Rightarrow\; \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{array}\right\} \quad (26.1.14)$$

Namely, the optimal reservation value is given by $y_t^\diamond$, which is constant in t. □

26.1.1.5 Market Restriction

26.1.1.5.1 Positive Restriction

☐ Pom **26.1.1** ($\mathscr{A}${rM:2[ℝ][A]}$^+$) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then r*M:2[ℝ][A]$^+$ ↬ M:2[ℝ][E]$^+$.

(b) *Let $\beta = 1$.*

1. *We have* `odr` ↦ `Accept`$_0(y)$ ▷ `Stop`.
2. *We have the same unsolved subject as* [F.S] 2(p.267) (*the conditions for* ⓢ, ◎, *and* ⓓ).

(c) *Let $\beta < 1$ and $s > 0$.*

1. *We have* [ⓢ `dOITs`$_{\tau \geq 0}\langle \tau \rangle$] → → ⓢ
2. *We have t-*`reservation-price`.

● *Proof* Suppose $a > 0$. Then it suffices to consider only y with $y > a > 0$.

(a) The same as `Tom` 26.1.1(p.267) (a).

(b) Let $\beta = 1$.

(b1) Clear from ♣ `Flow-ODR` 9.

(b2) The subject of future study ——

(c) Let $\beta < 1$ and $s > 0$.

(c1) The same as `Tom` 26.1.1(p.267) (c1).

(c2) Clear from `Tom` 26.1.1(p.267) (♠c3i,♠c3ii). ∎

26.1.1.5.2 Mixed Restriction

Omitted.

26.1.1.5.3 Negative Restriction

Omitted.

26.1.2 r$\tilde{\text{M}}$:2[ℝ][A]

26.1.2.1 Derivation of $\mathscr{A}${r$\tilde{\text{M}}$:2[ℝ][A]}

For almost the same reason as in Section 25.2.2.1(p.261) it can be confirmed that `SOE`{r$\tilde{\text{M}}$:2[ℝ][A]} (see (24.2.35(p.243))) is symmetrical to `SOE`{rM:2[ℝ][A]} (see (24.2.6(p.242))). This results implies that applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to `Tom` 26.1.1(p.267) for rM:2[ℝ][E] (selling model) yields lemmas for r$\tilde{\text{M}}$:2[ℝ][E] (buying model).

☐ `Tom` **26.1.2** (⊡ $\mathscr{A}${r$\tilde{\text{M}}$:2[ℝ][A]})

(a) *Let $s = 0$. Then r*$\tilde{\text{M}}$:2[ℝ][A] ↬ r$\tilde{\text{M}}$:2[ℝ][E].

(b) *Let $\beta = 1$.*

1. ♣ *We have $y \geq V_t(y)$ for $t \geq 0$.*
2. *We have the same unsolved subject as* [F.S] 2(p.267).

(c) *Let $\beta < 1$ and $s > 0$.*

1. *We have* [ⓢ `dOITs`$_{\tau \geq 0}\langle \tau \rangle$]$_\triangle$.
2. ♣ *Let $y \geq 0$. Then $y \geq V_t(y)$ for $t \geq 0$.*
3. *Let $y \leq 0$.*
   i. ♠ *Let $y \leq \tilde{y}_t^\circ$. Then $y \leq V_t(y)$ for $t \geq 0$.*
   ii. ♠ *Let $y \geq \tilde{y}_t^\circ$. Then $y \geq V_t(y)$ for $t \geq 0$.* □

● *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 26.1.1(p.267). ∎

26.1.2.2 Flow of Optimal Decision Rules

♣ `Flow-ODR` **11** (rM:2[ℝ][A]) (`Accept`$_0(y)$ ▷ `Stop`) *Let* $\beta = 1$ (see `Tom` 26.1.2(p.268) (♣b1)). *Then, the inequality $y \geq V_t(y)$ for any y and $t \geq 0$ means that even if the process is initiated at any time t, it is optimal to reject the best price y at that time. Accordingly, it follows that each time a price ξ is proposed, the current best price y continues to be reduced to* $y \overset{\text{def}}{=} \min\{y, \xi\}$, *and the process terminates by accepting the best price y at the deadline $t = 0$, i.e.,* `Accept`$_0(y)$ ▷ `Stop`. □

♠ `Flow-ODR` **12** (rM:2[ℝ][A]) (t-`reservation-price`) *Let $\beta < 1$ and $s > 0$ and let $y \leq 0$. Then, from* `Tom` 26.1.2(p.268) (♠c3i,♠c3ii) *and* (24.2.50(p.243)) *we have the following relations for $\tau \geq t \geq 0$:*

$$\left.\begin{array}{l} y \leq \tilde{y}_t^\circ \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \\ y \geq \tilde{y}_t^\circ \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{array}\right\}. \tag{26.1.15}$$

Namely, the optimal reservation value is given by $\tilde{y}_t^\circ$, which is constant in t. □

26.1.2.3 Market Restriction

26.1.2.3.1 Positive Restriction

☐ **Pom 26.1.2** ($\mathscr{A}\{\mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]^+\}$) *Suppose* $a > 0$.

(a) *Let* $s = 0$. *Then* $r\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+ \looparrowright r\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]^+$.

(b) *Let* $\beta = 1$.

1. ♣ *We have* odr $\mapsto$ $\mathtt{Accept}_0(y) \triangleright \mathtt{Stop}$.
2. *We have the same unsolved subject as* F.S 2(p.267) .

(c) *Let* $\beta < 1$ *and* $s > 0$.

1. *We have* ⓢ $\mathtt{dOITs}_{\tau \geq 0}\langle \tau \rangle_{\triangle}$ → → ⓢ
2. ♣ *We have* odr $\mapsto$ $\mathtt{Accept}_0(y) \triangleright \mathtt{Stop}$.

● *Proof* Suppose $a > 0$. Then it suffices to consider only $y > a > 0$.

(a) The same as Tom 26.1.2(p.268) (a).

(b) Let $\beta = 1$.

(b1) Immediate from Tom 26.1.2(p.268) (♣b1) and ♣ Flow-ODR 11(p.268) .

(b2) The subject of future study ——

(c) Let $\beta < 1$ and $s > 0$.

(c1) The same as Tom 26.1.2(p.268) (c1).

(c2) Immediate Tom 26.1.2(p.268) (♣c2) and ♣Flow-ODR 9. ∎

26.1.2.3.2 Mixed Restriction

Omitted.

26.1.2.3.3 Negative Restriction

Omitted.

26.1.3 Conclusion 9 (Search-Allowed-Model 2)

■ The assertion systems $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$$\mathscr{Q}\{\mathrm{rM}{:}2[\mathtt{A}]\} = \{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}],\ \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}],\ \mathrm{rM}{:}2\sout{[\mathbb{P}][\mathtt{A}],\ \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]}\}$$

are given by

$\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]\}$
↓
Tom's 26.1.1(p.267)

$\mathscr{A}\{\mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$
↓
Tom's 26.1.2(p.268)

■ The assertion systems $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$$\mathscr{Q}\{\mathrm{rM}{:}2[\mathtt{A}]\}^+ = \{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+,\ \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+,\ \mathrm{rM}{:}2\sout{[\mathbb{P}][\mathtt{A}]^+,\ \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+}\}$$

are given by

$\mathscr{A}\{r\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$
↓
Pom's 26.1.1(p.268)

$\mathscr{A}\{r\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$
↓
Pom's 26.1.2(p.269)

■ Closely looking into all the assertion systems above leads to the conclusions below.

C1 We have $\mathscr{A}\{\text{r}\tilde{\text{M}}{:}2[\mathbb{R}][\texttt{A}]\}^+ \curvearrowright \mathscr{A}\{\text{rM}{:}2[\mathbb{R}][\texttt{A}]\}^+$.

C2 We have $\text{rM}/\tilde{\text{M}}{:}2[\mathbb{R}][\texttt{A}]^+ \nleftrightarrow \text{rM}/\tilde{\text{M}}{:}2[\mathbb{R}][\texttt{E}]^+$.

C3 We have Ⓢ$_\vartriangle$ for $\text{rM}/\tilde{\text{M}}{:}2[\mathbb{R}][\texttt{A}]^+$.

C4 We have $\texttt{odr} \mapsto \texttt{Accept}_0(y) \triangleright \texttt{Stop}$ for $\text{rM}/\tilde{\text{M}}{:}2[\mathbb{R}][\texttt{A}]^+$ (i.e., ⓓ).

C5 We have t-`reservation-price` for $\text{rM}{:}2[\mathbb{R}][\texttt{A}]^+$.

C6 We have the future subject F.S 2.

C1 Compare `Pom` 26.1.2(p.269) and `Pom` 26.1.1(p.268).

C2 See `Pom` 26.1.1(p.268) (a) and `Pom` 26.1.2(p.269) (a).

C3 See `Pom` 26.1.1(p.268) (c1) and `Pom` 26.1.2(p.269) (c1).

C4 See `Pom` 26.1.1(p.268) (b1) and `Pom` 26.1.2(p.269) (b1,c2).

C5 See `Pom` 26.1.1(p.268) (c2).

C6 See `Pom` 26.1.1(p.268) (b2) and `Pom` 26.1.2(p.269) (b2). ∎

26.2 Search-Enforced-Model 2

26.2.1 rM:2[ℝ][E]

26.2.1.1 Preliminary

Let us define

$$v_t^\diamond(y) = v_t(y) - y, \quad t \ge 0, \tag{26.2.1}$$

$$V_t^\diamond(y) = V_t(y) - y, \quad t \ge 0. \tag{26.2.2}$$

Then, from (24.2.58(p.244)) we have

$$v_t^\diamond(y) = \max\{0, V_t^\diamond(y)\} \ge 0, \quad t \ge 0, \tag{26.2.3}$$

where

$$v_0^\diamond(y) = v_0(y) - y = \max\{0, \rho - y\} \quad \text{(see (24.2.51(p.244))),} \tag{26.2.4}$$

$$V_0^\diamond(y) = V_0(y) - y = \rho - y \quad \text{(see (24.2.57(p.244)))} \tag{26.2.5}$$

Furthermore, from (24.2.55(p.244)) we have

$$\begin{aligned} V_t^\diamond(y) &= \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta(v_{t-1}^\diamond(y) + y) - s - y \\ &= \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta y - s - y \\ &= \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + K(y) + y - y \quad t > 0 \quad (\leftarrow \text{(5.1.10(p.25))}) \\ &= \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + K(y), \quad t \ge 0. \end{aligned} \tag{26.2.6}$$

By $y_t^\diamond$ let us denote the solution of the equation $V_t^\diamond(y) = 0$ if it exists, i.e.,

$$V_t^\diamond(y_t^\diamond) = 0, \quad t \ge 0. \tag{26.2.7}$$

If multiple solutions exist, it is defined to be the *smallest* of them.

26.2.1.2 Lemmas

Lemma 26.2.1 (rM:2[ℝ][E])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \ge 0$.*

(b) *$V_t^\diamond(y)$ is nonincreasing in y for $t \ge 0$.* □

• *Proof* (a) $v_0(y)$ is nondecreasing in y from (24.2.51(p.244)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (24.2.55(p.244)), hence $v_t(y)$ is also nondecreasing in y from (24.2.58(p.244)). Thus, by induction $v_t(y)$ is nondecreasing in y for $t \ge 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is also nondecreasing in y for $t > 0$ from (24.2.55(p.244)). In addition, since $V_0(y)$ can be regarded as nondecreasing in y from (24.2.57(p.244)), it follows that $V_t(y)$ is nondecreasing in y for $t \ge 0$.

(b) $V_0^\diamond(y)$ is nonincreasing in y from (26.2.4(p.270)). Suppose $V_{t-1}^\diamond(y)$ is nonincreasing in y, hence $v_{t-1}^\diamond(y)$ is also nonincreasing in y from (26.2.3(p.270)). Accordingly, from (26.2.6(p.270)) and Lemma 10.2.2(p.57) (b)) we see that $V_t^\diamond(y)$ is also nonincreasing in y. This completes the induction. ∎

Lemma 26.2.2 (rM:2[ℝ][E]) *Let $\beta = 1$ and $s = 0$. Then $V_t(y) \ge y$ for any y and $t > 0$.* □

• *Proof* Let $\beta = 1$ and $s = 0$, hence $K(y) = \lambda T(y)$ from (5.1.4(p.25)). Then, from (26.2.6(p.270)) we have $V_t^\diamond(y) = \lambda\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)v_{t-1}^\diamond(y) + \lambda T(y)$ for $t \geq 0$. Now, for any $\boldsymbol{\xi}$ and y we have that $v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) \geq 0$ and $v_{t-1}^\diamond(y) \geq 0$ for $t > 0$ from (26.2.3(p.270)) and that $T(y) \geq 0$ due to Lemma 10.1.1(p.55) (g), hence it follows that $V_t^\diamond(y) \geq 0$ for any y and $t > 0$ or equivalently $V_t(y) \geq y$ for any y and $t > 0$ from (26.2.2(p.270)). ∎

Lemma 26.2.3 (rM:2[ℝ][E]) *Let $\beta < 1$ or $s > 0$.*

(a) $\lim_{y\to-\infty} V_t^\diamond(y) = \infty$ *for* $t \geq 0$.
(b) $\lim_{y\to\infty} V_t^\diamond(y) < 0$ *for* $t > 0$.
(c) *The sequence* $y_1^\diamond, y_2^\diamond, \cdots$ *exists where*

$$y \leq (\geq)\ y_t^\diamond \ \Rightarrow\ V_t^\diamond(y) \geq (\leq)\ 0. \quad \square \tag{26.2.8}$$

• *Proof* Let $\beta < 1$ or $s > 0$.

(a) We have $V_0^\diamond(y) \to \infty$ as $y \to -\infty$ from (26.2.5(p.270)). Suppose $V_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$. Then $v_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$ from (26.2.3(p.270)). In addition, since $K(y) \to \infty$ as $y = -\infty$ due to (10.2.4 (1) (p.57)), from (26.2.6(p.270)) we see that $V_t^\diamond(y) \to \infty$ as $y \to -\infty$. This completes the induction.

(b) We have $v_0^\diamond(y) \to 0$ as $y \to \infty$ from (26.2.4(p.270)). Suppose $v_{t-1}^\diamond(y) \to 0$ as $y \to \infty$. Then, the first and second terms of the right-hand side of (26.2.6(p.270)) converge to 0 as $y \to \infty$. In addition, due to (10.2.5 (2) (p.57)), if $\beta = 1$, then $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$", hence $K(y) = -s < 0$ for any y and if $\beta < 1$, then $K(y) \to -\infty < 0$ as $y \to \infty$, so $\lim_{y\to\infty} K(y) < 0$ whether $\beta = 1$ or $\beta < 1$. Hence, it follows that $\lim_{y\to\infty} V_t^\diamond(y) < 0$. Thus, from (26.2.3(p.270)) we have $v_t^\diamond(y) \to 0$ as $y \to \infty$. Hence, by induction we have $v_t^\diamond(y) \to 0$ as $y \to \infty$ for $t \geq 0$. Accordingly, since $v_{t-1}^\diamond(y) \to 0$ as $y \to \infty$ for $t > 0$, for quite the same reason as the above we have $\lim_{y\to\infty} V_{t-1}^\diamond(y) < 0$ for $t > 0$.

(c) Immediate from (a,b) and Lemma 26.2.1(p.270) (b). ∎

Lemma 26.2.4 (rM:2[ℝ][E]) *Let $\rho \leq x_K$. Then for any $y \in [a, b]$ we have:*

(a) $v_t(y)$ *and* $V_t(y)$ *are nondecreasing in* $t \geq 0$.
(b) $v_t(y)$ *and* $V_t(y)$ *converges to finite* $v(y)$ *and* $V(y)$ *respectively as* $t \to \infty$.
(c) $V_t^\diamond(y)$ *is nondecreasing in* $t \geq 0$.
(d) $y_t^\diamond$ *is nondecreasing in* $t > 0$.
(e) V_t *is nondecreasing in* $t \geq 0$.

• *Proof* Let $\rho \leq x_K$ and consider only $y \in [a, b] \cdots$ **(1)**. Then $K(\rho) \geq 0 \cdots$ **(2)** from Corollary 10.2.2(p.58) (b).

(a) Since $\max\{y, \rho\} \geq \rho$ for any y, from (24.2.61(p.244)) and Lemma 10.2.2(p.57) (e) we have $V_1(y) \geq K(\rho) + \rho \geq \rho \cdots$ **(3)** due to (2). Hence, from (24.2.52(p.244)) with $t = 1$ we have $v_1(y) = \max\{y, V_1(y)\} \geq \max\{y, \rho\} = v_0(y)$ for any y from (24.2.51(p.244)). Suppose $v_{t-1}(y) \geq v_{t-2}(y)$ for any y. Then, from (24.2.55(p.244)) we have $V_t(y) \geq \lambda\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-2}(y) - s = V_{t-1}(y)$ for any y. Hence, from (24.2.58(p.244)) we have $v_t(y) \geq \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y)$ is nondecreasing in $t \geq 0$ for any y. Then $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y, hence $V_t(y)$ is nondecreasing in $t > 0$ for any y from (24.2.55(p.244)). From (3) and (24.2.57(p.244)) we have $V_1(y) \geq V_0(y)$. Accordingly, it follows that $V_t(y)$ is nondecreasing in $t \geq 0$ for any y.

(b) Below let us consider only $y \in [a, b]$ and $\boldsymbol{\xi} \in [a, b]$†; in addition, consider a sufficiently large $M > 0$ such that $b \leq M$ and $\rho \leq M$. Then we have $V_0(y) \leq M$ from (24.2.57(p.244)). Suppose $V_{t-1}(y) \leq M \cdots$ **(4)** for any $y \in [a, b]$, hence from (24.2.52(p.244)) we have $v_{t-1}(y) \leq \max\{M, M\} = M$. Then, since $\max\{\boldsymbol{\xi}, y\} \leq \max\{M, M\} = M$ and $\max\{\boldsymbol{\xi}, y\} \in [a, b]$, we have $V_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq M$ due to (4). Thus, from (24.2.52(p.244)) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) = \max\{\max\{\boldsymbol{\xi}, y\}, V_{t-1}(\max\{\boldsymbol{\xi}, y\})\} \leq \max\{M, M\} = M$. Hence, from (24.2.55(p.244)) we have $V_t(y) \leq \lambda\beta\,\mathbf{E}[M] + (1-\lambda)\beta M - s = \lambda\beta M + (1-\lambda)\beta M - s = \beta M - s \leq M$, i.e., $V_t(y)$ is upper bounded in t. Accordingly, due to (a) it follows that $V_t(y)$ converge to a finite $V(y)$ as $t \to \infty$.

(c) Immediate from (26.2.2(p.270)) and (a).

(d) Evident from Lemma 26.2.1(p.270) (b), Lemma 26.2.4(p.271) (c), and Lemma 26.2.3(p.271) (c) (see Figure A 7.2(p.314) (I)).

(e) From (24.2.59(p.244)) and (2) we have $V_1 \geq \rho = V_0$ from (24.2.53(p.244)). Suppose $V_{t-1} \geq V_{t-2}$. Since $v_{t-1}(\boldsymbol{\xi}) \geq v_{t-2}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}$ due to (a), from (24.2.54(p.244)) we have $V_t \geq \lambda\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-2} - s = V_{t-1}$. This completes the induction. ∎

Lemma 26.2.5 (rM:2[ℝ][E]) *Let $\beta < 1$ or $s > 0$.*

(a) *Let* $y \geq y_t^\diamond$. *Then* $y \geq V_t(y)$ *for* $t > 0$.
(b) *Let* $y \leq y_t^\diamond$. *Then* $y \leq V_t(y)$ *for* $t > 0$. □

• *Proof* The same as Lemma 26.2.3(p.271) (c) and (26.2.2(p.270)). ∎

† $a \leq y \leq b \leq M$ and $a \leq \boldsymbol{\xi} \leq b \leq M$.

From (24.2.63(p.244)) and the two inequalities in `Tom` 26.2.5(p.271) (a,b) we have the following decision rule:

$$\left. \begin{array}{l} y \ge y_t^\diamond \ \Rightarrow y \ge V_t(y) \ \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \\ y \le y_t^\diamond \ \Rightarrow y \le V_t(y) \ \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array} \right\} \quad (26.2.9)$$

26.2.1.3 Analysis

□ **`Tom` 26.2.1** (■ $\mathscr{A}$ {**r**M:2[ℝ][E]})

(a) *Let $\beta = 1$ and $s = 0$.*

1. ♣ *We have $y \le V_t(y)$ for any y and $t \ge 0$.*
2. *We have the future subject* F.S. 3 (*the conditions for* ⓢ, ◎, *and* ⓓ).

(b) *Let $\beta < 1$ or $s > 0$.*

1. ♠ *Let $y \ge y_t^\diamond$. Then $y \ge V_t(y)$ for $t \ge 0$.*
2. ♠ *Let $y \le y_t^\diamond$. Then $y \le V_t(y)$ for $t \ge 0$.*
3. *We have the future subject* F.S. 4 (*the conditions for* ⓢ, ◎, *and* ⓓ). □

● *Proof* (a) Let $\beta = 1$ and $s = 0$.

(a1) The same as Lemma 26.2.2(p.270).

(a2) The subject of future study ——

(b) Let $\beta < 1$ or $s > 0$.

(b1,b2) The same as Lemma 26.2.5(p.271).

(b3) The subject of future study —— ▮

26.2.1.4 Flow of Optimal Decision Rules

♣ **`Flow-ODR` 13** (**r**M:2[ℝ][E]) ($\texttt{Accept}_0(y) \triangleright \texttt{Stop}$) *Let $\beta = 1$ and $s = 0$* (see `Tom` 26.2.1(p.272) (♣a1)). *Then, the inequality $y \le V_t(y)$ for any y and $t \ge 0$ means that even if the process is initiated at any time t, it is optimal to reject the best price y at that time. Accordingly, it follows that each time a price ξ is proposed, the current best price y continues to be enlarged to $y \overset{\text{def}}{=} \max\{y, \xi\}$, and the process terminates by accepting the best price y at the deadline $t = 0$, i.e.,* $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$. □

♠ **`Flow-ODR` 14** (**r**M:2[ℝ][E]) (t-`reservation-price`) *Let $\beta < 1$ or $s > 0$. Then, from* `Tom` 26.2.1(p.272) (♠b1,♠b2) *and* (24.1.25(p.240)) *we have the following relations for $\tau \ge t \ge 0$:*

$$\left\{ \begin{array}{l} y \ge y_t^\diamond \ \Rightarrow \ \texttt{Accept}_t\langle y\rangle \textit{ and the process stops} \\ y \le y_t^\diamond \ \Rightarrow \ \texttt{Reject}_t\langle y\rangle \textit{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{array} \right.$$

Namely, the optimal reservation value is given by $y_t^\diamond$, which is constant in t.

26.2.1.5 Market Restriction

26.2.1.5.1 Positive Restriction

□ **`Pom` 26.2.1** ($\mathscr{A}$ {**r**M:2[ℝ][E]}$^+$) *Suppose $a > 0$.*

(a) *Let $\beta = 1$ and $s = 0$.*

1. *We have* $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$ (see ♣`Flow-ODR` 13).
2. *We have the same unsolved subject as* F.S 3(p.272).

(b) *Let $\beta < 1$ or $s > 0$.*

1. *We have* t-`reservation-price` (see ♣`Flow-ODR` 14).
2. *We have the same unsolved subject as* F.S 4(p.272). □

● *Proof* Suppose $a > 0$.

(a) Let $\beta = 1$ and $s = 0$.

(a1) Obvious from `Tom` 26.2.1(p.272) (♣a1).

(a2) The subject of future study ——

(b) Let $\beta < 1$ or $s > 0$.

(b1) Evident from `Tom` 26.2.1(p.272) (♠b1,♠b2). ▮

(b2) The subject of future study —— ▮

26.2.1.5.2 Mixed Restriction

Omitted.

26.2.1.5.3 Negative Restriction

Omitted.

26.2.2 r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]

26.2.2.1 Preliminary

Let us define

$$\tilde{v}_t^\diamond(y) \;=\; v_t(y) - y, \quad t \geq 0, \tag{26.2.10}$$

$$\tilde{V}_t^\diamond(y) \;=\; V_t(y) - y, \quad t \geq 0. \tag{26.2.11}$$

Then, from (24.2.71(p.244)) we have

$$\tilde{v}_t^\diamond(y) \;=\; \min\{0, \tilde{V}_t^\diamond(y)\}, \quad t \geq 0. \tag{26.2.12}$$

By $\tilde{y}_t^\diamond$ let us denote the solution of the equation $\tilde{V}_t^\diamond(y) = 0$, $t > 0$, it exists, i.e.,

$$\tilde{V}_t^\diamond(\tilde{y}_t^\diamond) = 0. \tag{26.2.13}$$

If multiple solutions exist, it is defined to be the *largest* of them. Now, we have

$$\tilde{v}_0^\diamond(y) \;=\; \min\{0, \rho - y\} \quad (\leftarrow (24.2.64(\text{p.244}))), \tag{26.2.14}$$

$$\tilde{V}_0^\diamond(y) \;=\; \rho - y \quad (\leftarrow (24.2.70(\text{p.244}))). \tag{26.2.15}$$

Lemma 26.2.6 (r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]) *We have* $\tilde{y}_t^\diamond = \hat{y}_t^\diamond$ $(= -y_t^\diamond)$ *for* $t > 0$ (see (26.2.7(p.270)) for $y_t^\diamond$). □

● *Proof* First, note that (24.2.68(p.244)) can be rewritten as follows.

$$V_t(y) \;=\; \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{\xi, y\}) f(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0.$$

Next, replacing $f(\xi)$ in the above expression by $\check{f}(\hat{\xi})$ (see (12.1.9(p.70))) leads to

$$\begin{aligned} V_t(y) &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{\xi, y\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{-\hat{\xi}, -\hat{y}\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\hat{\xi}, \hat{y}\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0. \end{aligned}$$

Then, let $\eta \overset{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then, the above expression can be rearranged as

$$\begin{aligned} V_t(y) &= -\lambda\beta \int_{\infty}^{-\infty} v_{t-1}(-\max\{\eta, \hat{y}\}) \check{f}(\eta) d\eta + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\eta, \hat{y}\}) \check{f}(\eta) d\eta + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\xi, \hat{y}\}) \check{f}(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \quad \text{(without loss of generality).} \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\xi, \hat{y}\}) f(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \quad \text{(see (12.1.11(p.70))).} \end{aligned}$$

Applying the reverse operation $\mathcal{R}$ to the above expression yields

$$\begin{aligned} -\hat{V}_t(-\hat{y}) &= -\lambda\beta \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\max\{\xi, \hat{y}\}) f(\xi) d\xi - (1-\lambda)\beta \hat{v}_{t-1}(-\hat{y}) + s \\ &= -\lambda\beta\, \mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, \hat{y}\})] - (1-\lambda)\beta \hat{v}_{t-1}(-\hat{y}) + s, \quad t > 0. \end{aligned}$$

Multiplying the above expression by -1 yields

$$\hat{V}_t(-\hat{y}) \;=\; \lambda\beta\, \mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, \hat{y}\})] + (1-\lambda)\beta \hat{v}_{t-1}(-\hat{y}) - s, \quad t > 0. \cdots \textbf{(1)}.$$

Now, since (1) holds for any y with $-\infty < y < \infty$, it holds also for $\hat{y}$ since $\infty > \hat{y} > -\infty$ or equivalently $-\infty < \hat{y} < \infty$, hence we have

$$\hat{V}_t(-\hat{\hat{y}}) \;=\; \lambda\beta\, \mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, \hat{\hat{y}}\})] + (1-\lambda)\beta \hat{v}_{t-1}(-\hat{\hat{y}}) - s, \quad t > 0. \cdots \textbf{(2)}.$$

Since $\hat{\hat{y}} = y$, we can rewrite (2) as

$$\hat{V}_t(-y) \;=\; \lambda\beta\, \mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta \hat{v}_{t-1}(-y) - s \cdots \textbf{(3)}.$$

◦ Below let us temporarily represent the symbols "v" and "V" used in rM:2[$\mathbb{R}$][E] in Section 24.2.2.1(p.244) by "z" and "Z" respectively. Then (24.2.51(p.244)), (24.2.52(p.244)), (24.2.57(p.244)), and (24.2.55(p.244)) can be rewritten as respectively

$$\begin{aligned} z_0(y) &= \max\{y, \rho\} \cdots \textbf{(4)}, \\ z_t(y) &= \max\{y, Z_t(y)\} \cdots \textbf{(5)}, \quad t > 0, \\ Z_0(y) &= \rho \cdots \textbf{(6)}, \\ Z_t(y) &= \lambda\beta\, \mathbf{E}[z_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta z_{t-1}(y) - s \cdots \textbf{(7)}, \quad t > 0. \end{aligned}$$

In addition, let $Z_t^\diamond(y) \overset{\text{def}}{=} Z_t(y) - y \cdots \textbf{(8)}$ and $z_t^\diamond(y) \overset{\text{def}}{=} z_t(y) - y = \max\{0, Z_t^\diamond(y)\}$. Then we have $Z_t^\diamond(y_t^\diamond) = 0$ and $z_t(y_t^\diamond) - y_t^\diamond = 0$ (see (26.2.7(p.270))).

- Since $V_0(y) = \rho \cdots$ (9) from (24.2.70(p.244)), we have $-\hat{V}_0(-\hat{y}) = -\hat{\rho}$, hence $\hat{V}_0(-\hat{y}) = \hat{\rho}$. Since the equality holds for any $y \in (-\infty, \infty)$ and any $\rho \in (-\infty, \infty)$, so also does for $\hat{y} \in (-\infty, \infty)$ and $\hat{\rho} \in (-\infty, \infty)$. Hence since $\hat{V}_0(-\hat{\hat{y}}) = \hat{\hat{\rho}}$, we have $\hat{V}_0(-y) = \rho \cdots$ (10).

- From (10) and (6) we have $\hat{V}_0(-y) = \rho = Z_0(y)$. Suppose $\hat{V}_{t-1}(-y) = Z_{t-1}(y)$. Then, from (24.2.71(p.244)) we have

$$v_{t-1}(y) = \min\{-\hat{y}, -\hat{V}_{t-1}(-\hat{y})\} = -\max\{\hat{y}, \hat{V}_{t-1}(-\hat{y})\} = -\max\{\hat{y}, Z_{t-1}(\hat{y})\} = -z_{t-1}(\hat{y})$$

due to (5). Hence, since $\hat{v}_{t-1}(y) = z_{t-1}(\hat{y})$, we have

$$\hat{v}_{t-1}(-y) = \hat{v}_{t-1}(\hat{y}) = z_{t-1}(\hat{\hat{y}}) = z_{t-1}(y),$$

hence, since $\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, y\}) = z_{t-1}(\max\{\boldsymbol{\xi}, y\})$. Accordingly, (3) can be rewritten as

$$\hat{V}_t(-y) \ = \ \lambda\beta\,\mathbf{E}[z_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta z_{t-1}(y) - s = Z_t(y) \quad (\text{see } (7)).$$

Hence, since $-V_t(-y) = Z_t(y)$, we have $V_t(-y) = -Z_t(y)$. Since the equality holds for any $y \in (-\infty, \infty)$, so also does for $\hat{y} \in (-\infty, \infty)$, hence $V_t(-\hat{y}) = -Z_t(\hat{y})$, so $V_t(y) = -Z_t(\hat{y})$. Now, from (26.2.13(p.273)) and (26.2.11(p.273)) we have

$$0 = \tilde{V}_t(\tilde{y}_t^\diamond) = V_t(\tilde{y}_t^\diamond) - \tilde{y}_t^\diamond = -Z_t(\hat{\tilde{y}}_t^\diamond) - \tilde{y}_t^\diamond = -Z_t(\hat{\tilde{y}}_t^\diamond) + \hat{\tilde{y}}_t^\diamond = -(Z_t(\hat{\tilde{y}}_t^\diamond) - \hat{\tilde{y}}_t^\diamond) = -Z_t^\diamond(\hat{\tilde{y}}_t^\diamond)$$

due to (8) or equivalently $Z_t^\diamond(\hat{\tilde{y}}_t^\diamond) = 0$. Hence, we have $y_t^\diamond = \hat{\tilde{y}}_t^\diamond$ by definition, or equivalently $\hat{\tilde{y}}_t^\diamond = y_t^\diamond$, so $-\tilde{y}_t^\diamond = y_t^\diamond$, hence $\tilde{y}_t^\diamond = -y_t^\diamond = \hat{y}_t^\diamond$. ∎

26.2.2.2 Derivation of $\mathscr{A}$\{r$\tilde{\mathsf{M}}$:2[ℝ][E]\}

For almost the same reason as in Section 25.2.2.1(p.261) it can be confirmed that SOE\{r$\tilde{\mathsf{M}}$:2[ℝ][E]\} (see (24.2.69(p.244))) is symmetrical to SOE\{rM:2[ℝ][E]\} (see (24.2.56(p.244))). This results implies that applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom 26.2.1(p.272) for rM:2[ℝ][E] yields lemmas for r$\tilde{\mathsf{M}}$:2[ℝ][E].

□ **Tom 26.2.2** (⊡$\mathscr{A}$\{r$\tilde{\mathsf{M}}$:2[ℝ][E]\})

(a) *Let $\beta = 1$ and $s = 0$.*

1. ♣ *We have $y \geq V_t(y)$ for $t \geq 0$ and any y.*
2. *We have the same unsolved subject as* F.S 3(p.272).

(b) *Let $\beta < 1$ or $s > 0$.*

1. ♠ *Let $y \leq \tilde{y}_t^\diamond$. Then $V_t(y) \geq y$ for $t \geq 0$.*
2. ♠ *Let $y \geq \tilde{y}_t^\diamond$. Then $y \geq V_t(y)$ for $t \geq 0$.*
3. *We have the same unsolved subject as* F.S 4(p.272). □

- *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom 26.2.1(p.272). ∎

From (24.2.63(p.244)) and the two inequalities in Tom 26.2.5(p.271) (a,b) we have the following decision rule:

$$\left.\begin{array}{l} y \leq \tilde{y}_t^\diamond \ \Rightarrow y \geq V_t(y) \ \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \mathbf{I} \\ y \geq \tilde{y}_t^\diamond \ \Rightarrow y \leq V_t(y) \ \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{array}\right\} \tag{26.2.16}$$

26.2.2.3 Flow of Optimal Decision Rules

♣ **Flow-ODR 15** (rM:2[ℝ][E]) (Accept$_0(y)$ ▷ Stop) *Let $\beta = 1$ and $s = 0$ (see* Tom 26.2.2(p.274) (♣a1)*). Then, the inequality $y \leq V_t(y)$ for any y and $t \geq 0$ means that even if the process is initiated at any time t, it is optimal to reject the best price y at that time. Accordingly, it follows that each time a price ξ is proposed, the current best price y continues to be reduced to $y \stackrel{\text{def}}{=} \min\{y, \xi\}$, and the process terminates by accepting the best price y at the deadline $t = 0$, i.e.,* Accept$_0(y)$ ▷ Stop. □

♠ **Flow-ODR 16** (rM:2[ℝ][E]) (t-reservation-price) *Let $\beta < 1$ or $s > 0$. Then, from* Tom 26.2.2(p.274) (♠b1,♠b2) *and* (24.1.25(p.240)) *we have the following relations for $\tau \geq t \geq 0$:*

$$\begin{cases} y \leq \tilde{y}_t^\diamond \ \Rightarrow \ \texttt{Accept}_t\langle y\rangle \textit{ and the process stops } \mathbf{I} \\ y \geq \tilde{y}_t^\diamond \ \Rightarrow \ \texttt{Reject}_t\langle y\rangle \textit{ and } \texttt{Conduct}_t/\texttt{Skip}_t \end{cases}$$

Namely, the optimal reservation value is given by $\tilde{y}_t^\diamond$, which is constant in t. □

26.2.2.4 Market Restriction

26.2.2.4.1 Positive Restriction

☐ Pom **26.2.2** ($\mathscr{A}\{$r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$\}^+$) *Assume $a > 0$.*

(a) *Let $\beta = 1$ and $s = 0$.*
  1. ♣ *We have* Accept$_0(y)\triangleright$Stop.
  2. *We have the same unsolved subject as* [F.S] 3(p.272) .

(b) *Let $\beta < 1$ or $s > 0$.*
  1. ♠ *We have t*-reservation-price.
  2. *We have the same unsolved subject as* [F.S] 4(p.272) .

● *Proof* Suppose $a > 0$.

(a) Let $\beta = 1$ and $s = 0$..

(a1) Obvious from Tom 26.2.2(p.274) (♣a) and ♣Flow-ODR 15.

(a2) The subject of future study——

(b) Let $\beta < 1$ or $s > 0$.

(b1) Evident from Tom 26.2.2(p.274) (♠b1,♠b2) and ♠Flow-ODR 16.

(b2) The subject of future study—— ▮

26.2.2.4.2 Mixed Restriction

Omitted.

26.2.2.4.3 Negative Restriction

Omitted.

26.2.3 Conclusion 10 (Search-Enforced-Model 2)

■ The assertion systems $\mathscr{A}\{$M:2[$\mathbb{R}$][E]$\}$ of the quadruple-asset-trading-models on the total market $\mathscr{F}$

$\mathcal{Q}\{$rM:2[E]$\}$ = $\{$rM:2[$\mathbb{R}$][E], r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E], rM:2~~[$\mathbb{P}$][E], r$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$\}$~~

are given by

$\mathscr{A}\{$rM:2[$\mathbb{R}$][E]$\}$
↓
Tom's 25.2.21(p.259)

$\mathscr{A}\{$r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$\}$
↓
Tom's 26.2.2(p.274)

■ The assertion systems $\mathscr{A}\{$M:2[$\mathbb{R}$][E]$^+\}$ of the quadruple-asset-trading-models on the positive market $\mathscr{F}^+$

$\mathcal{Q}\{$rM:2[E]$\}^+$ = $\{$rM:2[$\mathbb{R}$][E]$^+$, r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$^+$, rM:2~~[$\mathbb{P}$][E]$^+$, r$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$^+$~~$\}$

are given by

$\mathscr{A}\{r$M:2[$\mathbb{R}$][E]$^+\}$
↓
Pom's 26.2.1(p.272)

$\mathscr{A}\{r\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$^+\}$
↓
Pom's 26.2.2(p.275)

■ Closely looking into all the assertion systems above leads to the conclusions below.

C1 We have $\mathscr{A}\{$r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$\}^+ \sim \mathscr{A}\{$rM:2[$\mathbb{R}$][E]$\}^+$.

C2 We have odr ↦ Accept$_0(y)\triangleright$Stop for rM/$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$^+$ (i.e., ⓓ).

C3 We have t-reservation-price for rM/$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$^+$.

C4 We have the same unsolved subject as [F.S] 3(p.272) and [F.S] 4(p.272) for rM/$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]$^+$. ☐

C1 Compare Pom 26.2.2(p.275) and 26.2.1(p.272) .

C2 See Pom 26.2.1(p.272) (a1) and Pom 26.2.2(p.275) (a1).

C3 See Pom 26.2.1(p.272) (b1) and Pom 26.2.2(p.275) (b1).

C4 See Pom 26.2.1(p.272) (a2,b2) and Pom 26.2.2(p.275) (a2,b2).

Chapter 27

Analysis of Model 3

27.1 Search-Allowed-Model 3

Lemma 27.1.1 *We have*

(a) *$v_t(y)$ is nondecreasing in $t \geq 0$ for any y.*
(b) *Let $\rho \leq 0$. Then U_t is nondecreasing in $t \geq 0$.*
(c) *Let $\rho \geq x_K$ and $\rho \geq 0$. Then $U_t \leq \rho$ for $t \geq 0$ and $v_t(y) \leq \max\{y, \rho\}$ for $t \geq 0$.* ▮

● *Proof* (a) From (24.3.2(p.245)) with $t = 1$ and (24.3.1(p.245)) we have $v_1(y) \geq \max\{y, \rho\} = v_0(y)$ for any y. Suppose $v_{t-1}(y) \geq v_{t-2}(y)$ for any y. Then, from (24.3.5(p.245)) we have

$$U_t(y) \ \geq \ \max\{\lambda\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-2}(y) - s, \beta v_{t-2}(y)\} = U_{t-1}(y)$$

for any y, so from (24.3.2(p.245)) we have $v_t(y) \geq \max\{y, \rho, U_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction we have $v_t(y) \geq v_{t-1}(y)$ for any y and $t > 0$. Accordingly, it follows that $v_t(y)$ is nondecreasing in $t \geq 0$.

(b) Let $\rho \leq 0$. From (24.3.6(p.245)) with $t = 1$ and (24.3.3(p.245)) we have $U_1 \geq \beta V_0 = \beta\rho \geq \rho = U_0$ from (24.3.8 (2) (p.245)). Suppose $U_t \geq U_{t-1}$. Then, since $v_{t-1}(\xi) \geq v_{t-2}(\xi)$ for any ξ from (a) and since $V_t \geq \max\{\rho, U_{t-1}\} = V_{t-1}$ from (24.3.4(p.245)), we have

$$U_t \geq \max\{\lambda\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-2} - s, \beta V_{t-2}\} = U_{t-1}$$

from (24.3.6(p.245)). This completes the induction.

(c) Let $\rho \geq x_K$ and $\rho \geq 0 \cdots$ **(1)**. Then, we have $K(\rho) \leq 0 \cdots$ **(2)** from Corollary 10.2.2(p.58) (a) and we have $K(\max\{y, \rho\}) \leq 0 \cdots$ **(3)** for any y due to $\max\{y, \rho\} \geq \rho \geq x_K$. Clearly, we have $U_0 \leq \rho$ from (24.3.8 (2) (p.245)) and $v_0(y) \leq \max\{y, \rho\}$ for any y from (24.3.1(p.245)). Suppose $U_{t-1} \leq \rho$ and $v_{t-1}(y) \leq \max\{y, \rho\} \cdots$ **(4)** for any y, hence $V_{t-1} = \rho$ from (24.3.4(p.245)). Then, from (24.3.6(p.245)) we have

$$U_t \leq \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} = \max\{K(\rho) + \rho, \beta\rho\}$$

from (5.1.10(p.25)), hence $U_t \leq \max\{\rho, \beta\rho\} = \rho$ due to (2) and (1). Since $v_{t-1}(\max\{\boldsymbol{\xi}, \rho\}) \leq \max\{\boldsymbol{\xi}, \rho\}$ for any $\boldsymbol{\xi}$ and y due to (4), from (24.3.5(p.245)) we have

$$\begin{aligned}
U_t(y) \ &\leq \ \max\{\lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} \\
&= \ \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} \\
&= \ \max\{K(\max\{y, \rho\}) + \max\{y, \rho\}, \beta\max\{y, \rho\}\}
\end{aligned}$$

from (5.1.10(p.25)). Hence $U_t(y) \leq \max\{\max\{y, \rho\}, \beta\{\max\{y, \rho\}\} = \max\{y, \rho\}$ due to (3) and $\max\{y, \rho\} \geq \rho \geq 0$ for any y. Accordingly, from (24.3.2(p.245)) we have $v_t(y) \leq \max\{y, \rho, \max\{y, \rho\}\} = \max\{y, \rho\}$. This complete the inductions. ▮

□ **Tom 27.1.1** (▣ $\mathscr{A}$\{**r**M:3[ℝ][A]\})

(a) *Let $\rho \leq x_K$ or $\rho \leq 0$. Then we have r*M:3[ℝ][A] ↠ *r*M:2[ℝ][A].
(b) *Let $\rho \geq x_K$ and $\rho \geq 0$. Then we have* Ⓢ$_\Delta$ *where* `odr` ↦ `Accept`$_\tau(\rho)$ ▷ `Stop`. □

● *Proof* From (24.3.6(p.245)) with $t = 1$, (24.3.1(p.245)), and (24.3.3(p.245)) we have

$$U_1 = \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} = \max\{K(\rho) + \rho, \beta\rho\} \cdots \textbf{(1)}$$

due to (5.1.10(p.25)).

(a) Let $\rho \leq x_K$, hence $K(\rho) \geq 0 \cdots$ **(2)** from Corollary 10.2.2(p.58) (b). Since $v_{t-1}(\boldsymbol{\xi}) \geq \max\{\boldsymbol{\xi}, \rho\}$ for any $\boldsymbol{\xi}$ and $t > 0$ from (24.3.2(p.245))) and since $V_t \geq \rho$ for $t > 0$ from (24.3.4(p.245))), from
(24.3.6(p.245))) and (5.1.10(p.25)) we have

$$U_t \ \geq\ \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi},\rho\}]+(1-\lambda)\beta\rho-s,\beta\rho\}=\max\{K(\rho)+\rho,\beta\rho\}\geq K(\rho)+\rho\geq\rho$$

for any $t>0$ due to (2). Let $\rho\leq 0$, hence $-(1-\beta)\rho\geq 0$. From (1) we have $U_1-\rho=\max\{K(\rho),-(1-\beta)\rho\}\geq 0$, so $U_1\geq\rho$; accordingly, we have $U_t\geq\rho$ for $t>0$ from Lemma 27.1.1(p.277) (b). Consequently, whether $\rho\leq x_K$ or $\rho\leq 0$, it follows that $U_t\geq\rho$ for $t>0$. This fact means that "Reject the intervening quitting penalty price ρ for all $t>0$", implying "Behave as if there does not exist the intervening quitting penalty price ρ"; in other words, it eventually follows that rM:3[ℝ][A] is reduced to the model without the intervening quitting penalty price ρ, i.e., rM:2[ℝ][A].

(b) Let $\rho\geq x_K$ and $\rho\geq 0$. Then, we have $U_t\leq\rho$ for $\tau\geq t\geq 0$ from Lemma 27.1.1(p.277) (c), meaning "Accept the intervening quitting penalty price ρ and the process stops" for $\tau\geq t>0$; in other words, we have odr $\mapsto$ $\texttt{Accept}_t(\rho)\triangleright\texttt{Stop}$ for $\tau\geq\tau>0$ (see (21.1.2(p.225))). The proof of ⓢ$_\vartriangle$ is the same as the proof of Tom 27.2.1(p.278) (b2) for $\rho\geq 0$. ∎

27.1.1 r$\tilde{\text{M}}$:3[ℝ][A]

In the same way as in Section 25.2.2.1(p.261) we can easily verify that SOE{r$\tilde{\text{M}}$:3[ℝ][A]} $=\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$[SOE{rM:3[ℝ][A]}] (see (24.3.16(p.245)) and (24.3.7(p.245))). Hence, we can apply $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.130))) to Tom 27.1.1(p.277) , yielding the following Tom (see Lemma 12.10.1(p.87)).

☐ **Tom 27.1.2** (⊡$\mathscr{A}${r$\tilde{\text{M}}$:3[ℝ][A]})

(a) *Let $\rho\geq x_{\tilde{K}}$ or $\rho\geq 0$. Then we have* r$\tilde{\text{M}}$:3[ℝ][A] ↠r$\tilde{\text{M}}$:2[ℝ][A].

(b) *Let $\rho\leq x_{\tilde{K}}$ and $\rho\leq 0$. Then we have* ⓢ$_\vartriangle$ *where* odr $\mapsto$ $\texttt{Accept}_t(\rho)\triangleright\texttt{Stop}$. ☐

27.1.2 Conclusion 11 (Search-Allowed-Model 3)

The following two cases are possible:

C1. We have rM/$\tilde{\text{M}}$:3[ℝ][A] ↠rM/$\tilde{\text{M}}$:2[ℝ][A].

C2. We have odr $\mapsto$ $\texttt{Accept}_\tau(\rho)\triangleright\texttt{Stop}$ where ⓢ$_\vartriangle$ for rM/$\tilde{\text{M}}$:3[ℝ][A]. ☐

C1 See Tom 27.1.1(p.277) (a) and Tom 27.1.2(p.278) (a).

C2 See Tom 27.1.1(p.277) (b) and Tom 27.1.2(p.278) (b). ∎

27.2 Search-Enforced-Model 3

27.2.1 rM:3[ℝ][E]

Lemma 27.2.1 *Let $\rho\geq x_K$. Then $U_t\leq\rho$ and $v_t(y)\leq\max\{y,\rho\}$ for any y and $t\geq 0$.* ∎

● *Proof* Let $\rho\geq x_K$, hence $\max\{y,\rho\}\geq\rho\geq x_K$ for any y. Then, from Corollary 10.2.2(p.58) (a) we have $K(\rho)\leq 0\cdots$**(1)** and $K(\max\{y,\rho\})\leq 0\cdots$**(2)** for any y. Now $U_0\leq\rho$ from (24.3.26 (2) (p.246)) and $v_0(y)\leq\max\{y,\rho\}$ for any y from (24.3.19(p.246)). Suppose $U_{t-1}\leq\rho$ and $v_{t-1}(y)\leq\max\{y,\rho\}$ for any y, hence $V_{t-1}=\rho$ from (24.3.22(p.246)) and $v_{t-1}(\max\{\xi,y\})\leq\max\{\max\{\xi,y\},\rho\}$ for any ξ and y. Then, from (24.3.24(p.246)) we have

$$U_t\leq\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi},\rho\}]+(1-\lambda)\beta\rho-s=K(\rho)+\rho$$

due to (5.1.10(p.25)), hence $U_t\leq\rho$ due to (1). In addition, from (24.3.23(p.246)) we have

$$\begin{aligned}U_t(y)\ &\leq\ \lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi},y\},\rho\}]+(1-\lambda)\beta\max\{y,\rho\}-s\\ &=\ \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi},\max\{y,\rho\}\}]+(1-\lambda)\beta\max\{y,\rho\}-s\\ &=\ K(\max\{y,\rho\})+\max\{y,\rho\}\end{aligned}$$

from (5.1.10(p.25)), hence $U_t(y)\leq\max\{y,\rho\}$ from (2). Accordingly, from (24.3.20(p.246)) we have $v_t(y)\leq\max\{y,\rho,\max\{y,\rho\}\}=\max\{y,\rho\}$. This complete the inductions. ∎

☐ **Tom 27.2.1** (▣$\mathscr{A}${rM:3[ℝ][E]})

(a) *Let $\rho\leq x_K$. Then we have* rM:3[ℝ][E] ↠ rM:2[ℝ][E].

(b) *Let $\rho\geq x_K$.*

1. *We have* odr $\mapsto$ $\texttt{Accept}_t(\rho)\triangleright\texttt{Stop}$ *for $\tau\geq t\geq 0$.*
2. *Let $\rho\geq 0$ $(\rho\leq 0)$. Then we have* ⓢ$_\vartriangle$ (ⓓ$_\vartriangle$).

● *Proof* (a) Let $\rho\leq x_K$, hence $K(\rho)\geq 0\cdots$**(1)** from Corollary 10.2.2(p.58) (b). Since $V_{t-1}\geq\rho$ for $t>0$ from (24.3.22(p.246))) and since $v_{t-1}(y)\geq\max\{y,\rho\}$ for any y, ρ, and $t>0$ from (24.3.20(p.246))), from (24.3.24(p.246))) we have

$$U_t\geq\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi},\rho\}]+(1-\lambda)\beta\rho-s=K(\rho)+\rho,\quad t>0$$

from (5.1.10(p.25)), hence $U_t\geq\rho$ for $t>0$ from (1). This fact means that "Reject the intervening quitting penalty price ρ for all $t>0$", implying "Behave as if there does not exist the intervening quitting penalty price ρ"; in other words, it follows that rM:3[ℝ][E] is reduced to the model without the intervening quitting penalty ρ", i.e., rM:2[ℝ][E].

(b) Let $\rho \geq x_K$.

(b1) Then, we have $U_t \leq \rho$ for $\tau \geq t \geq 0$ from Lemma 27.2.1(p.278), meaning that "Always accept the intervening quitting penalty ρ and the process stops" is optimal for $\tau \geq t > 0$; in other words, we have odr $\mapsto$ $\text{Accept}_t(\rho) \triangleright \text{Stop}$ for $\tau \geq t > 0$ (see (21.1.2(p.225))). Then since $V_t = \rho$ for $\tau \geq t \geq 0$ from (24.3.22(p.246)), we have $I_\tau^t = \beta^{\tau-t} V_t = \beta^{\tau-t}\rho$ for $\tau \geq t \geq 0$ from (7.2.4(p.44)).

(b2) If $\rho \geq 0$, then since $\beta^0\rho \geq \beta^1\rho \geq \cdots \geq \beta^\tau\rho$, we have $I_\tau^\tau \geq I_\tau^{\tau-1} \geq \cdots \geq I_\tau^0$, hence ⓢ $\text{dOITs}_\tau\langle\tau\rangle_\triangle$ and if $\rho \leq 0$, then since $\beta^0\rho \leq \beta^1\rho \leq \cdots \leq \beta^\tau\rho$, we have $I_\tau^\tau \leq I_\tau^{\tau-1} \leq \cdots \leq I_\tau^0$, hence ● $\text{dOITd}_\tau\langle 0\rangle_\triangle$. ∎

27.2.2 r$\tilde{\text{M}}$:3[ℝ][E]

In the same way as in Section 25.2.2.1(p.261) we can easily verify that SOE{r$\tilde{\text{M}}$:3[ℝ][E]} $= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$[SOE{rM:3[ℝ][E]}] (see (24.3.34(p.246)) and (24.3.25(p.246))). Hence we can apply $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 27.2.1(p.278), yielding the following Tom.

☐ **Tom 27.2.2** (◫ $\mathscr{A}${r$\tilde{\text{M}}$:3[ℝ][E]})

(a) *Let $\rho \geq x_{\tilde{K}}$. Then we have r$\tilde{\text{M}}$:3[ℝ][E] ↠ r$\tilde{\text{M}}$:2[ℝ][E].*

(b) *Let $\rho \leq x_{\tilde{K}}$.*

1. *We have* odr $\mapsto$ $\text{Accept}_t(\rho) \triangleright \text{Stop}$ *for $\tau \geq t \geq 0$.*
2. *If $\rho \leq 0$ ($\rho \geq 0$), then we have* ⓢ$_\triangle$ (ⓓ$_\triangle$).

27.2.3 Conclusion 12 (Search-Enforced-Model 3)

The following three cases are possible:

C1. We have rM/$\tilde{\text{M}}$:3[ℝ][E] ↠ rM/$\tilde{\text{M}}$:2[ℝ][E].

C2. We have odr $\mapsto$ $\text{Accept}_\tau(\rho) \triangleright \text{Stop}$ if $\rho \geq 0$ (i.e., ⓢ$_\triangle$) and $\text{Accept}_0(\rho) \triangleright \text{Stop}$ if $\rho < 0$ (i.e., ⓓ$_\triangle$). □

C1 See Tom 27.2.1(p.278) (a) and Tom 27.2.2(p.279) (a).

C2 See Tom 27.2.2(p.279) (b1,b2). ∎

Chapter 28

Conclusion of Part 4 (Recall-Model)

For details, see Conclusions 7 (p.254), 8 (p.264), 9 (p.269), 10 (p.275), 11 (p.278), and 12 (p.279).

28.1 Models 1/2

$\overline{\overline{\mathsf{C}}}$1. Myopic Property

a. For rModel 1 we have the c-reservation-price (see C6(p.255) and C5(p.264)), which yields *myopic property*. This property mental conflict (see Def. 25.2.1(p.261)).

b. For rModel 2 we have the t-reservation-price (see C5(p.270) and C3(p.275)).

c. It was already shown in [44,Sak1961] that rModel 1 (sE-model) has the myopic property (c-reservation-price). After that, any variations have not been posed and examined to date. For this reason, we have continued to think as if this property is a general one for all recall models. However, we demonstrated above that this property does not hold in rModel 2, i.e., it follows that this is not always a property holding for all recall-models.

$\overline{\overline{\mathsf{C}}}$2. Symmetry

a. For rModel 1, the symmetry collapses for both sA-model (see C1(p.255)) and sE-model (see C1(p.264)).

b. For rModel 2, the symmetry collapses for sA-model (see C1(p.270)) but is inherited for sE-model (see C1(p.275)).

$\overline{\overline{\mathsf{C}}}$3. Optimal Initiating Time

For rModel 1 (recall model) with a more complicated structure than Model 1 (no-recall-model), at the beginning we imagined that it would be rather difficult to mathematically (analytically) examine conditions for ⓢ, ⓞ, and ⓓ. However, fortunately we succeeded in finding out the conditions for ⓢ (see C4(p.255) and C2(p.264)), for ⓞ (see C3(p.264)), and for ⓓ (see C5(p.255) and C4(p.264)). What should be noted here is that also ⓞ$_\blacktriangle$ and ⓓ$_\blacktriangle$ (strictness) exist (see C3(p.264) and C4(p.264)).

$\overline{\overline{\mathsf{C}}}$4. Future study

In rModel 2 we did not succeed in finding out the conditions for ⓞ and ⓓ. Mathematical examinations of these conditions are left as a future study (see [F.S] 2(p.267), [F.S] 3(p.272), and [F.S] 4(p.272)).

$\overline{\overline{\mathsf{C}}}$5. Reduction

Model 1/2 is reduced to the following two cases (see Section 21.4(p.231)):

a. mode-migration $\mathsf{rM}/\tilde{\mathsf{M}}{:}1/2[\mathbb{R}][\mathtt{A}]^+ \looparrowright \mathsf{rM}/\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]^+$ (see C2(p.255) / C2(p.270)).

b. odr-reduction $\mathtt{odr} \mapsto \mathtt{Accept}_0(y) \triangleright \mathtt{Stop}$ for sA-model 1/2 (see C3(p.255) / C4(p.270)).

28.2 Models 3

$\overline{\overline{\mathsf{C}}}$6. Reduction

Model 3 is reduced to the following two cases (see Section 21.4(p.231)):

a. model-running-back $\mathsf{rM}/\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A/E}] \twoheadrightarrow \mathsf{rM}/\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A/E}]$ (see C1(p.278) and C1(p.279)).

b. odr-reduction $\mathtt{odr} \mapsto \mathtt{Accept}_\tau(\rho) \triangleright \mathtt{Stop}$ (see C2(p.278) and C2(p.279)).

Part 5

Conclusion

Chapter 29

General Overview of This Paper

29.1 List of Conclusions

Below we summarize the conclusions presented in Chaps. 8(p.49), 18(p.129), 22(p.233), and 28(p.281).

29.1.1 Conclusion of Part 1 (Introduction)

The contents of Chap. 8(p.49) can be summarized as follows.

$\overline{\overline{\text{C}}}$1. Two motives This study was initiated by the two naive motives (see Section 1.2(p.3)).

$\overline{\overline{\text{C}}}$2. Philosophical background The philosophical foundation of this paper roots in the concept of "decision theory as physics" (see Section 1.3(p.4)), which supports all aspects of this study.

$\overline{\overline{\text{C}}}$3. Time concepts All physical phenomena are not alien to the time concepts. This physical recognition inevitable led us to the existence of the following five time points; recognizing time t_r, starting time τ, initiating time t_i, stopping time t_s, and deadline t_d (see H1(p.8) and Section 7.1(p.43)).

$\overline{\overline{\text{C}}}$4. Optimal initiating time The best of conceivable initiating times is called the *optimal initiating time* (`OIT`), represented by t^*_τ $(t_r \geq t_i \geq \tau)$ (see (7.2.5(p.44))). If $t^*_\tau = \tau$, then it is denoted by ⓢ (degeneration to the starting time), if $t_s > t^*_\tau > 0$, then ◎ (non-degeneration), and if $t^*_\tau = 0$, then ⓓ (degeneration to the deadline) (see Section 7.2.4.3(p.45)).

$\overline{\overline{\text{C}}}$5. Null-Time-Zone and Deadline-Engulfing The introduction of the optimal initiating time yields the concepts of the *null-time-zone* (see Sections 7.2.4.5(p.46)) and the *deadline-engulfing* (see 7.2.4.6(p.46)), which are the most significant discoveries in this study. This fact suggests the need for the comprehensive re-examination of conclusions in nearly all conventional studies that have been conducted by many researchers so far.

$\overline{\overline{\text{C}}}$6. Structured-unit-of-models In Section 1.4(p.4) we provided an overview for the four kinds of asset trading problems, referred to as the *quadruple-asset-trading-problems*. In addition, we defined a set consisting of the six kinds of quadruple-asset-trading-problems, called the *structured-unit-of-problems* (see Section 3.3(p.18)). Our key focus in this paper is not on analyzing each model *independently* but on clarifying the *interconnectedness* among these problems.

$\overline{\overline{\text{C}}}$7. Assumptions Among the eleven assumptions in Section 2.2(p.11), the three, A5(p.12), A7(p.12), and A11(p.13), are unique in the sense that they opened a new dimensions in the theory of decision processes.

$\overline{\overline{\text{C}}}$8. Discount factor While a selling problem is framed as a *profit* maximization problem, a buying problem is a *cost* minimization problem. Although the managerial and economical implications of the discount factor for *profit* have been well-documented in many standard textbooks, a persuasive explanation of the implication for *cost* is not found. In this paper, we offered a novel viewpoint for this situation (see Section 2.3(p.13)).

$\overline{\overline{\text{C}}}$9. Underlying function The underlying functions T, L, K, and $\mathcal{L}$ defined in Chap. 5(p.25) are essential for analyzing all the models discussed in this paper. While the function T has been defined and used in existing literature so far, the other three functions are first introduced in this paper.

$\overline{\overline{\text{C}}}$10. Mental conflict As illustrated in *Examples* 1.4.1(p.5)-1.4.4(p.6), the *normal* mental conflict experienced by a leading trader (see Remark 7.3.1(p.47)) can be intuitively understood. However, the *abnormal* mental conflict (see Remark 7.3.2(p.48)) is hard to immediately grasp; it is possible in fact as presented in $\overline{\overline{\text{C}}}$1b2(p.233).

29.1.2 Conclusion of Part 2 (Integrated Theory)

The contents of Chap. 18(p.129) can be summarized as follows.

$\overline{\overline{\text{C}}}$11. Two preliminary steps

a. Proofs of assertions on underlying functions The first preliminary step in constructing the integrated theory is to prove assertions on underlying functions (see Chap. 10(p.55)).

b. Proofs of four theorems The second preliminary step is to prove the following theorems:

- ○ symmetry theorems Theorems 12.5.1(p.80) ($\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$) and 12.8.1(p.87) ($\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$),
- ○ analogy theorems Theorems 13.3.1(p.98) ($\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$) and 13.3.2(p.98) ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$),
- ○ symmetry theorems Theorems 14.5.1(p.106) ($\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$) and 14.5.2(p.107) ($\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$),
- ○ analogy theorems Theorems 15.1.1(p.112) ($\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$) and 15.1.2(p.112) ($\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$).

For more details, see $\overline{\overline{\text{C}}}$1b1(p.129) - $\overline{\overline{\text{C}}}$1b4(p.129).

$\overline{\overline{\text{C}}}$12. Integrated theory The integrated theory (see Chap. 16(p.115)) is constructed through a dual-directional connection of the above eight theorems, which is schematized as in Figure 16.2.1(p.115). The integrated theory is not always versatile, which has the following two weak points.

a. Market restriction The integrated theory is constructed on the premise that the price ξ is defined on $\xi \in (-\infty, \infty)$, implying that the price ξ can become negative, which is irrational from a practical standpoint. To avoid this irrationality, the price must be defined on $\xi \in (0, \infty)$. We refer to the restriction of $(-\infty, \infty)$ to $(0, \infty)$ as the *market restriction* (see Chap. 17.2(p.117)). This restriction leads to the collapse of symmetry and analogy (see $\overline{\overline{\text{C}}}$2a2(p.233), $\overline{\overline{\text{C}}}$3a2(p.233), and $\overline{\overline{\text{C}}}$3b1(p.233)).

b. Symmetry and analogy among `SOE`'s As stated in Section 16.3.1(p.116), the integrated theory has a limitation that the symmetrical and analogical relationships must hold over all `SOE`'s. In fact, the analogical relationships collapse for Models 2/3 (see Table 6.4.3(p.41) - Table 6.4.6(p.41)), implying that the analogy theorem cannot be applied. See Section 20.1.5(p.166) for the treatment of the case where the analogy theorem cannot be applied.

$\overline{\overline{\text{C}}}$13. Summary of operations For convenience of reference, we listed the above eight operations in $\overline{\overline{\text{C}}}$3(p.130).

29.1.3 Conclusion of Part 3 (No-Recall-Model)

The contents of Chap. 22(p.233) can be summarized as follows.

$\overline{\overline{\text{C}}}$15. Mental Conflict It is only for Model 2 with $\beta < 1$ and $s > 0$ that we have the abnormal mental conflict (see $\overline{\overline{\text{C}}}$1b2(p.233)). For all other cases we have the normal mental conflict (see $\overline{\overline{\text{C}}}$1a(p.233) and $\overline{\overline{\text{C}}}$1b1(p.233)).

$\overline{\overline{\text{C}}}$16. Symmetry On $\mathscr{F}^+$, for Models 1/2, the symmetry is inherited only when $\beta = 1$ and $s = 0$ (see $\overline{\overline{\text{C}}}$2a1(p.233)). When $\beta < 1$ or $s > 0$, it *may* collapses for Model 1 and *always* collapses for Model 1 (see $\overline{\overline{\text{C}}}$2a2(p.233)).

$\overline{\overline{\text{C}}}$17. Analogy On $\mathscr{F}^+$, it is only for Model 1 with $\beta = 1$ and $s = 0$ that the analogy is inherited (see $\overline{\overline{\text{C}}}$3a1(p.233)); it is *may* collapses for all other cases (see $\overline{\overline{\text{C}}}$3a2(p.233) and $\overline{\overline{\text{C}}}$3b1(p.233)).

$\overline{\overline{\text{C}}}$18. Optimal initiating time Ⓢ, ◎, and ❶ are all possible for Models 1/2 with any $\beta \le 1$ and $s \ge 0$ (see $\overline{\overline{\text{C}}}$4(p.233)). What is remarkable here is that $❶_{\|}$ (deadline-engulfing) occurs even in the simplest case of "$\beta = 1$ and $s = 0$" (see $\overline{\overline{\text{C}}}$4a3(p.233)).

$\overline{\overline{\text{C}}}$19. Null-time-zone and deadline-engulfing For Models 1/2, ◎ and ❶ causing the null-time-zone occur at a rather high rate of 55.6% (see $\overline{\overline{\text{C}}}$5A(p.234)) and ❶ causing the dead-engulfing occur at a rather high rate of 33.4% (see $\overline{\overline{\text{C}}}$5B(p.234)).

$\overline{\overline{\text{C}}}$20. `C⇝S` (`Conduct⇝Skip`) It is only for M:2[ℝ][`A`]$^+$ and M:2[ℙ][`A`]$^+$ with $\beta < 1$ or $s > 0$ that this rare event becomes possible (see $\overline{\overline{\text{C}}}$6(p.234)).

$\overline{\overline{\text{C}}}$21. Reduction Model 3 is reduced to the following two cases (see $\overline{\overline{\text{C}}}$9(p.234)):

a. model-running-back M/M̃:3[ℝ/ℙ][`A/E`] ↠ M/M̃:2[ℝ/ℙ][`A/E`].

b. `odr`-reduction `odr` ↦ $\texttt{Accept}_{\tau\ge t\ge 0}(\rho) \triangleright \texttt{Stop}$.

29.1.4 Conclusion of Part 4 (Recall-Model)

The contents of Chap. 28(p.281) can be summarized as follows.

$\overline{\overline{\text{C}}}$23. Myopic property It is only for rModel 1 that we have `c-reservation-price` (see $\overline{\overline{\text{C}}}$1a(p.281)), leading to the *myopic property* (see Def. 25.2.1(p.261) and $\overline{\overline{\text{C}}}$1c(p.281)). For Model 2 we have t`-reservation-price` (see $\overline{\overline{\text{C}}}$1b(p.281)). See $\overline{\overline{\text{C}}}$1c(p.281) for further interesting suggestions.

$\overline{\overline{\text{C}}}$24. Symmetry For rModel 1, the symmetry collapses for both sA-model and sE-model (see $\overline{\overline{\text{C}}}$2a(p.281)). For rModel 2, it collapses for sA-model but is inherited for sE-model (see $\overline{\overline{\text{C}}}$2b(p.281)).

$\overline{\overline{\text{C}}}$25. Optimal initiating time It is only for rM:1[ℝ][E] (see `Tom` 25.2.1(p.259)) that the analytical discussions for Ⓢ, ◎, and ❶ become possible (see $\overline{\overline{\text{C}}}$3(p.281)).

$\overline{\overline{\text{C}}}$26. Future studies In rModel 2 we did not succeed in finding out the conditions for ◎ and ❶. Mathematical examinations of these conditions are left as a future study (see [F.S] 2(p.267), [F.S] 3(p.272), and [F.S] 4(p.272)).

$\overline{\overline{\text{C}}}$27. Reduction The same as $\overline{\overline{\text{C}}}$5(p.281) and $\overline{\overline{\text{C}}}$6(p.281).

reduction =

model-migration	rM/M̃:1/2[ℝ][`A`]$^+$ ↬ rM/M̃:1[ℝ][`E`]$^+$	for Model 1/2	(see $\overline{\overline{\text{C}}}$5a(p.281), $\overline{\overline{\text{C}}}$6a(p.281))
model-running-back	rM/M̃:3[ℝ][`A/E`] ↠ rM/M̃:2[ℝ][`A/E`]	for Model 3	(see $\overline{\overline{\text{C}}}$5b(p.281))
`odr`-reduction	`odr` ↦ $\texttt{Accept}_0(y) \triangleright \texttt{Stop}$	for Model 1/2/3	(see $\overline{\overline{\text{C}}}$6b(p.281))

29.2 Final Main Points of This Paper

The conclusions in Section 29.1(p.283) are what we finally reached in this paper. The most essential points which indwell within these conclusions are summarized as below.

P1. **Philosophical Background**

On *March 31, 1966*, the original theme of this paper was proposed by my (Ikuta, the first author of this paper) academic supervisor Prof. Shizuo Senju who has PhD (Eng.) (see the episode in the title page p.i.). Enlightened by his thought background, before long, I also obtained PhD (Eng.) under his research guidance, and more than 20 years later since then, Kang (the second author of this paper) obtained PhD (Mgt. Sci.&Eng.) under my research guidance. In time, we who have the above background found ourself down the middle of the philosophy of "decision theory as physics" (see Section 1.3(p.4)). This philosophical background exerted considerable influence on the whole writing of this paper; in other words, it is no exaggeration to say that this study would not complete at all without it.

P2. **Quadruple-Asset-Trading-Problems**

A trading problem can be classified into the following four types: a selling problem and a buying problem, each of which can be categorized as a problem with a reservation price mechanism (where the counter trader proposes the trading price) and a problem with a posted price mechanism (where the leading trader proposes the trading price). Let us refer to the group of the four problems as the *quadruple-asset-trading-problems* (see Section 1.4.5(p.7)). While these problems have been treated one-by-one and independently so far without touching upon any relationships each other, in the present paper we aim to clarify the *interconnectedness* among these problems.

P3. **Two Motives**

This study was triggered by the following two naive motives (see Section 1.2(p.3)):

Motive 1: Is a buying problem always symmetrical to a selling problem ?

Motive 2: Does a general theory integrating quadruple-asset-trading-problems exist ?

P4. **Integrated Theory**

We have unknowingly assumed, without strong evidence, that the existence of the symmetrical relationship between selling problem and buying problem is enough predictable, and in fact we succeeded in theoretically proving it in the process of constructing the integrated theory. In this theory we derived the two symmetry theorems (Theorems 12.5.1(p.80) and 12.8.1(p.87)), which connect the above two problems by the operation defined by (12.5.29(p.77)). On the other hand, at the earlier stage of this study, we did not anticipate at all the existence of a relationship between trading problem with $\mathbb{R}$-mechanism and trading problem with $\mathbb{P}$-mechanism. However, through countless arrangements and rearrangements, as if solving a jigsaw puzzle, we noticed similarities between the above two problems and finally reached Lemmas 10.1.1(p.55) and 13.2.1(p.93), which are connected by the operation defined by (13.2.1(p.93)). This fact led to the derivation of the two analogy theorems (Theorems 13.3.1(p.98) and 13.3.2(p.98)) combining the above two problems. Finally we reached the conclusion that the integrated theory that we aimed to construct is given by the structure schematized by the quadrangular bi-directional connection of the above four theorems (see Figure 16.2.1(p.115)). This accomplished the aim of the objective in Motive 2(p.3).

P5. **Collapse of Symmetry and Analogy**

When we started this study, we were grappling with the conflict between mathematical thinking and physical thinking; "Should the price ξ be defined on which of $(-\infty,\infty)$ and $(0,\infty)$". It goes without saying that defining on $(-\infty,\infty)$ makes the mathematical treatment easier than on $(0,\infty)$. For this reason we tried to construct the integrated theory on $(-\infty,\infty)$ and fortunately succeeded in the construction of the integrated theory under this premise. However, it should be defined on $(0,\infty)$ in the usual transaction market of the actual world so that a negative price does not occur. Then, we brought the solution to this problem by formulating the methodology of transforming results obtained on $(-\infty,\infty)$ into ones on $(0,\infty)$, i.e., the market restriction (see Chap. 17(p.117)). However, the market restriction naturally leads to the possibility that the symmetry and analogy which are guaranteed under the integrated theory constructed on $(-\infty,\infty)$ may collapse. In Parts 3(p.131) we demonstrated that the collapse occurs in fact. Thus, it follows that the answer to the question in Motive 1(p.3) is "No!".

P6. **Null-Time-Zone and Deadline-Engulfing**

Our physical recognition in E1(p.285) led, as its inevitable result, to the time concepts of *recognizing time*, *starting time*, *initiating time*, *stopping time*, and *deadline* (see Section 7.1(p.43)), and the concept of the "*initiating time*" inevitably yields the concept of "*optimal initiating time*" (see (7.2.5(p.44))). Then, we found out that there exists the three types of optimal initiating time, the starting time (ⓢ), unregenerate time (⊚), and the deadline (🅓) (see Section 7.2.4.3(p.45)), and that ⊚ and 🅓 led us to the two unexpected phenomena, *null-time-zone* (see Section 7.2.4.5(p.46)) and *deadline-engulfing* (see Section 7.2.4.6(p.46) and Alice 4(p.46)). The two phenomena are the most significant discoveries in this paper in the sense that they strongly press for the comprehensive re-examination of almost all results obtained in conventional researches in which the concept of initiating time has not been introduced. Now, we see from Table 22.1.1(p.234) that ⊚ and 🅓 causing the above two singular properties are not rare; in fact it can occur at the rather high occurrence rates of 22.2% and 33.4% respectively. What is furthermore amazing is that the strictly optimal initiating times, ⊚$_\blacktriangle$ and 🅓$_\blacktriangle$, are possible although at the very small occurrence rates of 2.6% and 3.2% (see *Example* 7.2.1(p.46) and $\overline{\overline{\text{C}}}$5C(p.234)). What is moreover striking is that both ⊚$_\blacktriangle$ and 🅓$_\blacktriangle$ are possible even in the simplest case of $\beta=0$ and $s=0$ (see *Example* 7.2.2(p.46) and $\overline{\overline{\text{C}}}$5D(p.234)).

P7. **Discount Factor**

Presumably, this paper will be the first to define the concepts of *profit* and *cost* through the third concept of *fund*. To be honest, we have always found certain inconsistencies in conventional approaches to *profit* and *cost* where clear definitions are often lacking. Despite the extensive discussions about the discount factor for *profit*, it is surprising that the discount factor for *cost* has been addressed so infrequently. We believe this oversight stems from a misguided assumption that the buying problem is of little importance, as it is merely considered the inverse of the selling problem. This assumption implies that the buying problem can be fully explained by simply reversing the signs of the variables, parameters, constants, etc., defined in the selling problem. However, we emphasize here that this paper demonstrates that the two problems are not inversely related at all.

This study ends with bespeaking the above seven points today, *December 11, 2024*.

Chapter 30

Future Study

30.1 Different Variations

30.1.1 No-Recall-Model

Below are variations of models defined in Section 4.1(p.21).

V1. *Limited search budget* [23,Iku1992] This model involves a limited total budget allocated for search activities. The challenge lies in determining how to distribute this limited budget among search activities at every time point throughout the planning horizon.

V2. *Price mechanism switching* [16,Ee2006] [14,Ee2004] This model allows for the switching of price mechanisms between $\mathbb{R}$-mechanism and $\mathbb{P}$-mechanism at each time point during the planning horizon.

V3. *Several search areas* [24,Iku1995] For instance, consider Tokyo, Kyoto, and Osaka as potential areas where the leading-trader can search for counter-traders. Then, if the leading-trader is in Tokyo today, the decision arises tomorrow whether to stay in Tokyo or to move to which of Kyoto and Osaka.

V4. *Uncertain deadline* [17,Eem2009] In *Example* 1.4.1(p.5), the return home date is not yet definite; it could be imminent or one week later, or the directive itself might be rescinded.

30.1.2 Recall-Model

Below are variations of models defined in Section 4.2(p.23).

V5. *Uncertain recall* [22,Iku1988] This is the model in which the recall of counter-traders once rejected is uncertain.

V6. *Costly recall* [30,Kan1999],[31,Kan2005] This is the model in which some cost must be paid to recall counter-traders once rejected.

V7. *Reserved recall* [42,Sai1998],[43,Sai1999] This is the model in which the availability of recall can be reserved by paying some deposit

30.1.3 Others

In addition to the above variations, in the future we will have other variations. For examples:

V8. *Multiple assets model* This is the model in which multiple assets are traded. In the model, the optimal decision rule depends on the number of assets remaining not yet being traded.

V9. *Lasting effect of search activity* This is the model in which the effect of the search activity that was taken at a certain point in time lasts for a while. The simplest case is that its effect disappears with a given probability p at the next point in time; hence it lasts with the probability $1 - p$.

⋮

30.2 Future Subjects

F1. Further variations

In Section 30.1(p.287) we presented 9 variations of the basic models of asset trading problems. For each variation there exists *one* structured-unit-of models consisting of 24 models (see Section 3.3(p.18)), hence it follows that we have $216 = 9 \times 24$ in all. Furthermore, the following different mixed variations can be considered:

- Model with several search areas and limited search budget
- Model with uncertain deadline and mechanism switching
- Model with limited search budget, uncertain deadline, and mechanism switching
- Model with several search areas, limited search budget, uncertain deadline, and mechanism switching

- ∘ Model with recall, several search areas, limited search budget, uncertain deadline, and mechanism switching
- ∘ Model with uncertain recall, uncertain deadline, and mechanism switching

$\vdots$

Hence, it follows that the number of variations becomes astronomical. In dealing with the vast amount of these variations, the integrated theory will become a powerful tool; analyzing them without this theory would be nearly impossible.

F2. See [F.S] 1(p.237)

In Part 4(p.235) we applied the integrated theory to the recall-model with $\mathbb{R}$-mechanism where it suffices to memorize only the best of once-rejected prices. However, in the recall-model with $\mathbb{P}$-mechanism, we face the difficulty of determining which of the once-rejected prices should be memorized. This problem remains as one of the most perplexing unsolved subjects of study.

F3. See [F.S] 2(p.267), [F.S] 3(p.272), and [F.S] 4(p.272)

In the recall-model, it is remained as an unsolved problems how to analytically examine the conditions on which each of ⓢ, ⊚, and ⓓ occur. This is one of the most challenging study subjects in this paper.

F4. Numerical Experiment

In general, numerical calculation involves computing a given expression by substituting numerical values for constants, parameters, variables, $\cdots$ related to its expression. In this paper, we performed numerical calculations from two distinct perspectives. One is to reconfirm results that have already been proven, the other is to exemplify expectations that are difficult to prove mathematically. We refer to the former as the *numerical example* and the latter as the *numerical experiment*, i.e.,

$$\text{numerical calculation} = \begin{cases} \text{numerical example,} \\ \text{numerical experiment.} \end{cases}$$

Throughout the paper we have:

Numerical Example's 1(p.126), 2(p.126), 3(p.127), 4(p.147), 5(p.187), and 6(p.219)

Numerical Experiment 1(p.316).

When confronting such problems as presented in F1(p.287)-F3(p.288) that are analytically difficult to address, the only methodology available will presumably be numerical experiments.

Appendix

2

A 1 Direct Proof of Underlying Functions of Type $\mathbb{R}$

In this appendix we provide the direct proofs for all lemmas in Section 12.6(p.81) in which they were proven by using Theorem 12.5.1(p.80) (symmetry theorem).

A 1.1 $\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$

For convenience of reference, below let us copy Lemma 12.6.1(p.81).

Lemma A 1.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *strictly increasing on* $(-\infty, b]$.
(f) $\tilde{T}(x) = \mu - x$ *on* $[b, \infty)$ *and* $\tilde{T}(x) < \mu - x$ *on* $(-\infty, b)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $\tilde{T}(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \le \min\{0, \mu - x\}$ *on* $x \in (-\infty, \infty)$.
(i) $\tilde{T}(0) = 0$ *if* $a > 0$ *and* $\tilde{T}(0) = \mu$ *if* $b < 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b > y$, *then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b > \mu$. □

● *Proof* First, for any x and y let us prove the following two inequalities:

$$-(x-y)F(y) \ge \tilde{T}(x) - \tilde{T}(y) \ge -(x-y)F(x) \cdots \mathbf{(1)},$$

$$(x-y)(1-F(y)) \ge \tilde{T}(x) + x - \tilde{T}(y) - y \ge (x-y)(1-F(x)) \cdots \mathbf{(2)}.$$

Then, let $\tilde{T}(x,y) \overset{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)]$ for any x and y.[‡] Since $1 \ge I(\boldsymbol{\xi} < y) \ge 0$ and since $\min\{\boldsymbol{\xi} - x, 0\} \le 0$ and $\min\{\boldsymbol{\xi} - x, 0\} \le \boldsymbol{\xi} - x$, we have $\min\{\boldsymbol{\xi} - x, 0\} \le \min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} < y) \le (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)$, hence from (5.1.11(p.25)) we get $\tilde{T}(x) \le \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)] = \tilde{T}(x,y)$. Accordingly, for any x and y we have

$$\tilde{T}(x) - \tilde{T}(y) \le \tilde{T}(x,y) - \tilde{T}(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} < y)] = -(x-y)\,\mathbf{E}[I(\boldsymbol{\xi} < y)].$$

Since $I(\boldsymbol{\xi} \ge y) + I(\boldsymbol{\xi} < y) = 1$, we have $\tilde{T}(x) - \tilde{T}(y) \le -(x-y)(\mathbf{E}[1 - I(\boldsymbol{\xi} \ge y)]) = -(x-y)(1 - \mathbf{E}[I(\boldsymbol{\xi} \ge y)])$. Then, since

$$\mathbf{E}[I(\boldsymbol{\xi} \ge y)] = \textstyle\int_{-\infty}^{\infty} I(\xi \ge y)f(\xi)d\xi = \int_{y}^{\infty} 1 \times f(\xi)d\xi = \int_{y}^{\infty} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} > y\} = 1 - \Pr\{\boldsymbol{\xi} \le y\} = 1 - F(y),$$

‡If a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

we have $\tilde{T}(x)-\tilde{T}(y)\le -(x-y)F(y)$, hence the far left inequality of (1) holds. Multiplying both sides of the inequality by -1 leads to $-\tilde{T}(x)+\tilde{T}(y)\ge (x-y)F(y)$ or equivalently $\tilde{T}(y)-\tilde{T}(x)\ge -(y-x)F(y)$. Then, interchanging the notations x and y yields $\tilde{T}(x)-\tilde{T}(y)\ge -(x-y)F(x)$, hence the far right inequality of (1) holds. (2) is immediate from adding $x-y$ to (1). Let us note here that $\tilde{T}(x)$ defined by (5.1.11(p.25)) can be rewritten as

$$\begin{aligned}\tilde{T}(x) &= \mathbf{E}[\min\{\boldsymbol{\xi}-x,0\}I(b\ge\boldsymbol{\xi})]+\mathbf{E}[\min\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi}>b)].\cdots\mathbf{(3)}\\ &= \mathbf{E}[\min\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi}\ge a)]+\mathbf{E}[\min\{\boldsymbol{\xi}-x,0\}I(a>\boldsymbol{\xi})].\cdots\mathbf{(4)}.\end{aligned}$$

(a,b) Immediate from the fact that $\min\{\boldsymbol{\xi}-x,0\}$ is continuous and nonincreasing in $x\in(-\infty,\infty)$ for any given $\boldsymbol{\xi}$.

(c) Let $x>y>a$. Then, since $-(x-y)<0$ and $F(y)>0$ due to (2.2.1 (2,3) (p.13)), we have $-(x-y)F(y)<0$, hence $0>\tilde{T}(x)-\tilde{T}(y)$ from (1), i.e., $\tilde{T}(y)>\tilde{T}(x)$, so $\tilde{T}(x)$ is *strictly* decreasing on $(a,\infty)\cdots\mathbf{(5)}$. Suppose $\tilde{T}(a)=\tilde{T}(x)$ for any $x>a$, hence $x-a>0$. Then, for any sufficiently small $\varepsilon>0$ such that $x-a>2\varepsilon>0$ we have $a<a+\varepsilon<x-\varepsilon<x$, hence $\tilde{T}(a)=\tilde{T}(x)<\tilde{T}(a+\varepsilon)\le\tilde{T}(a)$ due to (5) and (b), which is a contradiction. Thus it must be that $\tilde{T}(a)\ne\tilde{T}(x)$ for any $x>a$, i.e., $\tilde{T}(a)>\tilde{T}(x)$ or $\tilde{T}(a)<\tilde{T}(x)$ for any $x>a$. Since the latter is impossible due to (b), it follows that $\tilde{T}(a)>\tilde{T}(x)$ for any $x>a$. From this and (5) it eventually follows that $\tilde{T}(x)$ is strictly decreasing on $[a,\infty)$ instead of (a,∞).

(d) Evident from the fact that $\tilde{T}(x)+x=\mathbf{E}[\min\{\boldsymbol{\xi},x\}]$ from (5.1.11(p.25)) and that $\min\{\boldsymbol{\xi},x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $b>x>y$, hence $F(x)<1$ due to (2.2.1 (1,2) (p.13)). Then, since $(x-y)(1-F(x))>0$, we have $\tilde{T}(x)+x>\tilde{T}(y)+y$ from (2), i.e., $\tilde{T}(x)+x$ is strictly increasing on $(-\infty,b)\cdots\mathbf{(6)}$. Suppose $\tilde{T}(b)+b=\tilde{T}(x)+x$ for any $x<b$. Then, for any sufficiently small $\varepsilon>0$ such that $b-x>\varepsilon$ we have $x<x+\varepsilon<b$, hence $\tilde{T}(b)+b=\tilde{T}(x)+x<\tilde{T}(x+\varepsilon)+x+\varepsilon\le\tilde{T}(b)+b$ due to (6) and (d), which is a contradiction. Thus, $\tilde{T}(x)+x\ne\tilde{T}(b)+b$ for $x<b$, i.e., $\tilde{T}(x)+x>\tilde{T}(b)+b$ or $\tilde{T}(x)+x<\tilde{T}(b)+b$ for $x<b$. Since the former is impossible due to (d), it must be that $\tilde{T}(x)+x<\tilde{T}(b)+b$ for $x<b$. From this and (6) it follows that $\tilde{T}(x)+x$ is strictly increasing on $(-\infty,b]$.

(f) Let $x\ge b$. If $b\ge\boldsymbol{\xi}$, then $x\ge\boldsymbol{\xi}$, hence $\min\{\boldsymbol{\xi}-x,0\}=\boldsymbol{\xi}-x$, and if $\xi>b$, then $f(\xi)=0$ due to (2.2.3 (3) (p.13)). Thus, from (3) we have $\tilde{T}(x)=\mathbf{E}[(\boldsymbol{\xi}-x)I(b\ge\boldsymbol{\xi})]+0=\mathbf{E}[(\boldsymbol{\xi}-x)I(b\ge\boldsymbol{\xi})]+\mathbf{E}[(\boldsymbol{\xi}-x)I(\boldsymbol{\xi}>b)]=\mathbf{E}[(\boldsymbol{\xi}-x)(I(b\ge\boldsymbol{\xi})+I(\boldsymbol{\xi}>b))]=\mathbf{E}[\boldsymbol{\xi}-x]=\mu-x$,[†] hence the former half is true. Then, since $\tilde{T}(b)=\mu-b$ or equivalently $\tilde{T}(b)+b=\mu$, if $b>x$, from (e) we have $\tilde{T}(x)+x<\tilde{T}(b)+b=\mu$, hence $\tilde{T}(x)<\mu-x$, so the latter half is true.

(g) Let $a\ge x$. If $\boldsymbol{\xi}\ge a$, then $\boldsymbol{\xi}\ge x$, hence $\min\{\boldsymbol{\xi}-x,0\}=0$ and if $a>\boldsymbol{\xi}$, then $f(\boldsymbol{\xi})=0$ due to (2.2.3 (1) (p.13)), hence $\mathbf{E}[\min\{\boldsymbol{\xi}-x,0\}I(a>\boldsymbol{\xi})]=0$. Accordingly, we have $\tilde{T}(x)=0$ from (4), hence the latter half is true. Let $x>a$. Then, since $\tilde{T}(x)<\tilde{T}(a)$ from (c) and since $\tilde{T}(a)=0$ from the fact stated just above, we have $\tilde{T}(x)<0$ for $x>a$, hence the former half is true.

(h) From (f) we have $\tilde{T}(x)\le\mu-x$ for any x and from (g) we have $\tilde{T}(x)\le 0$ for any x, thus it follows that $\tilde{T}(x)\le\min\{0,\mu-x\}$ for any x.

(i) From (5.1.11(p.25)) we have $\tilde{T}(0)=\mathbf{E}[\min\{\boldsymbol{\xi},0\}]=\mathbf{E}[\min\{\boldsymbol{\xi},0\}I(a\le\boldsymbol{\xi}\le b)]$. If $a>0$, then $0\le\boldsymbol{\xi}$, hence $\min\{\boldsymbol{\xi},0\}=0$, so $\tilde{T}(0)=\mathbf{E}[0]=0$, and if $b<0$, then $\boldsymbol{\xi}<0$, hence $\min\{\boldsymbol{\xi},0\}=\boldsymbol{\xi}$, so $\tilde{T}(0)=\mathbf{E}[\boldsymbol{\xi}]=\mu$.

(j) If $\beta=1$, then $\beta\tilde{T}(x)+x=\tilde{T}(x)+x$, hence the assertion is true from (d).

(k) Since $\beta\tilde{T}(x)+x=\beta(\tilde{T}(x)+x)+(1-\beta)x$, if $\beta<1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x>y$ and $b>y$. If $x\ge b$, then $\tilde{T}(x)+x\ge\tilde{T}(b)+b>\tilde{T}(y)+y$ due to (d,e), and if $b>x$, then $b>x>y$, hence $\tilde{T}(x)+x>\tilde{T}(y)+y$ due to (e).

(m) From (5.1.11(p.25)) we have

$$\begin{aligned}\lambda\beta\tilde{T}(\lambda\beta\mu+s)+s &= \lambda\beta\,\mathbf{E}[\min\{\boldsymbol{\xi}-\lambda\beta\mu-s,0\}]+s\\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu-\lambda\beta s,0\}]+s\\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu+(1-\lambda\beta)s,s\}],\end{aligned}$$

which is nondecreasing in s and strictly increasing in s if $\lambda\beta<1$.

(n) Evident from (2.2.2(p.13)). ∎

A 1.2 $\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$, and $\tilde{\kappa}_{\mathbb{R}}$

From (5.1.13(p.25)) and (5.1.14(p.25)) and from Lemma A 1.1(p.289) (f) we obtain, noting (10.2.1(p.56)),

$$\tilde{L}(x)\begin{cases} =\lambda\beta\mu+s-\lambda\beta x \text{ on } [b,-\infty) & \cdots(1),\\ <\lambda\beta\mu+s-\lambda\beta x \text{ on } (-\infty,b) & \cdots(2),\end{cases} \tag{A 1.1}$$

$$\tilde{K}(x)\begin{cases} =\lambda\beta\mu+s-\delta x \quad\text{on}\quad [b,\infty) & \cdots(1),\\ <\lambda\beta\mu+s-\delta x \quad\text{on}\quad (-\infty,b) & \cdots(2).\end{cases} \tag{A 1.2}$$

[†] $I(b\ge\boldsymbol{\xi})+I(\boldsymbol{\xi}>b)=1$.

In addition, from (5.1.14(p.25)) and Lemma A 1.1(p.289) (g) we have

$$\tilde{K}(x)\begin{cases} < -(1-\beta)x + s & \text{on} \quad (a,\infty) \quad \cdots(1), \\ = -(1-\beta)x + s & \text{on} \quad (-\infty, a] \quad \cdots(2), \end{cases} \tag{A 1.3}$$

hence we obtain

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty,\infty). \tag{A 1.4}$$

Then, from (A 1.2 (1) (p.290)) and (A 1.3 (2) (p.291)) we get

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta\mu + s + (1-\lambda)\beta x & \text{on} \quad [b,\infty) \quad \cdots(1), \\ \beta x + s & \text{on} \quad (-\infty, a] \quad \cdots(2). \end{cases} \tag{A 1.5}$$

Since $\tilde{K}(x) = \tilde{L}(x) - (1-\beta)x$ from (5.1.14(p.25)) and (5.1.13(p.25)), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\,x_{\tilde{L}} \cdots(1), \qquad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\,x_{\tilde{K}} \cdots(2). \tag{A 1.6}$$

Lemma A 1.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a,\infty)$.*
(d) *Let $s=0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\ge)\ 0$.*
(e) *Let $s>0$.*

1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (=(>))\ x \Leftrightarrow \tilde{L}(x) < (=(>))\ 0$.*
2. *$(\lambda\beta\mu + s)/\lambda\beta \ge (<)\ b \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta\mu+s)/\lambda\beta \ge (<)\ b$.* □

● *Proof* (a-c) Immediate from (5.1.13(p.25)) and Lemma A 1.1(p.289) (a-c).

(d) Let $s=0$. Then, since $\tilde{L}(x) = \lambda\beta\tilde{T}(x)$, from Lemma A 1.1(p.289) (g) we have $\tilde{L}(x) = 0$ for $a \ge x$ and $\tilde{L}(x) < 0$ for $x > a$, hence $x_{\tilde{L}} = a$ by the definition of $x_{\tilde{L}}$ (see Section 5.2(p.27) (b)), so $x_{\tilde{L}} < (\ge)\ x \Rightarrow \tilde{L}(x) < (=)\ 0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x) = 0 \Rightarrow \tilde{L}(x) \ge 0$, we have $\tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\ge)\ 0$.

(e) Let $s>0$.

(e1) From (A 1.1 (1) (p.290)) and from $\lambda>0$ and $\beta>0$ we have $\tilde{L}(x) < 0$ for a sufficiently large $x>0$ such that $x \ge b$. In addition, we have $\tilde{L}(a) = \lambda\beta\tilde{T}(a) + s = s > 0$ from Lemma A 1.1(p.289) (g). Hence, from (c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}} > a$ is immediate from $\tilde{L}(a) > 0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta\mu+s)/\lambda\beta \ge (<)\ b$, from (A 1.1(p.290)) we have $\tilde{L}((\lambda\beta\mu+s)/\lambda\beta) = (<)\ \lambda\beta\mu + s - \lambda\beta(\lambda\beta\mu+s)/\lambda\beta = 0$, hence $x_{\tilde{L}} = (<)\ (\lambda\beta\mu+s)/\lambda\beta \ge (<)\ b$ from (e1). ∎

Corollary A 1.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (\ge)\ 0$.*
(b) *$x_{\tilde{L}} \le (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$.* □

● *Proof* (a) "⇒" is immediate from Lemma A 1.2(p.291) (d,e1). "⇐" is evident by contraposition.

(b) Since $x_{\tilde{L}} < (\ge)\ x \Rightarrow \tilde{L}(x) < (\ge)\ 0$ due to (a) and since $\tilde{L}(x) < (\ge)\ 0 \Rightarrow \tilde{L}(x) \le (\ge)\ 0$, we have $x_{\tilde{L}} < (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$. In addition, if $x_{\tilde{L}} = x$, then $\tilde{L}(x) = \tilde{L}(x_{\tilde{L}}) = 0$ or equivalently $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) = 0$, hence $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) \le 0$. Accordingly, it follows that $x_{\tilde{L}} \le (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$. ∎

Lemma A 1.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$\tilde{K}(x)$ is strictly decreasing on $[a,\infty)$.*
(d) *$\tilde{K}(x)$ is strictly decreasing on $(-\infty,\infty)$ if $\beta<1$.*
(e) *$\tilde{K}(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(f) *$\tilde{K}(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(g) *$\tilde{K}(x)+x$ is strictly increasing on $(-\infty,b]$.*
(h) *If $x>y$ and $b>y$, then $\tilde{K}(x)+x > \tilde{K}(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_{\tilde{K}} = a$ where $x_{\tilde{K}} < (\ge)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\ge)\ 0$.*
(j) *Let $\beta<1$ or $s>0$.*

1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}} < (=(>))\ x \Leftrightarrow \tilde{K}(x) < (=(>))\ 0$.*
2. *$(\lambda\beta\mu+s)/\delta \ge (<)\ b \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta\mu+s)/\delta$.*
3. *Let $\tilde{\kappa} < (=(>))\ 0$. Then $x_{\tilde{K}} < (=(>))\ 0$.* □

• *Proof* (a-c) Immediate from (5.1.14(p.25)) and Lemma A 1.1(p.289) (a-c).

(d) Immediate from (5.1.14(p.25)) and Lemma A 1.1(p.289) (b).

(e) From (5.1.14(p.25)) we have

$$\tilde{K}(x) + x = \lambda\beta\tilde{T}(x) + \beta x + s = \lambda\beta(\tilde{T}(x) + x) + (1-\lambda)\beta x + s \cdots \mathbf{(1)},$$

hence the assertion holds from Lemma A 1.1(p.289) (d).

(f) Obvious from (1) and Lemma A 1.1(p.289) (d).

(g) Clearly from (1) and Lemma A 1.1(p.289) (e).

(h) Let $x > y$ and $b > y$. If $x \geq b$, then $\tilde{K}(x) + x \geq \tilde{K}(b) + b > \tilde{K}(y) + y$ due to (e,g), and if $b > x$, then $b > x > y$, hence $\tilde{K}(x) + x > \tilde{K}(y) + y$ due to (g). Thus, whether $x \geq b$ or $b > x$, we have $\tilde{K}(x) + x > \tilde{K}(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $\tilde{K}(x) = \lambda\tilde{T}(x)$ due to (5.1.14(p.25)), from Lemma A 1.1(p.289) (g) we have $\tilde{K}(x) = 0$ for $a \geq x$ and $\tilde{K}(x) < 0$ for $x > a$, so $x_{\tilde{K}} = a$ by the definition of $x_{\tilde{K}}$ (see Section 5.2(p.27) (b)). Hence $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (=)\ 0$. The inverse holds by contraposition. In addition, since $\tilde{K}(x) = 0 \Rightarrow \tilde{K}(x) \geq 0$, we have $\tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) This proof consists of the following six steps:

- First note (A 1.3 (2) (p.291)). If $\beta < 1$, then $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \geq a$ and if $s > 0$, then, whether $\beta < 1$ or $\beta = 1$, we have $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a$. Hence, whether $\beta < 1$ or $s > 0$, we have $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a$.
- Next note (A 1.2 (1) (p.290)). Then, since $\delta > 0$ from (10.2.2 (1) (p.56)), whether $\beta < 1$ or $s > 0$ we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$.
- Hence, whether $\beta < 1$ or $s > 0$, it follows that there exists the solution x_K.

◦ Let $\beta < 1$. Then, the solution $x_{\tilde{K}}$ is unique from (d).

◦ Let $s > 0$. If $\beta < 1$, the solution $x_{\tilde{K}}$ is unique for the reason just above. If $\beta = 1$, we have $\tilde{K}(a) = s > 0$ from (A 1.3 (2) (p.291)), hence $x_{\tilde{K}} > a$ due to (c), so $\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x = x_{\tilde{K}}$ due to (c), thus the solution $x_{\tilde{K}}$ is unique. Therefore, whether $\beta < 1$ or $\beta = 1$, it follows that the solution $x_{\tilde{K}}$ is unique.

◦ Hence, whether $\beta < 1$ or $s > 0$, it follows that the solution $x_{\tilde{K}}$ is unique.

From all the above, whether $\beta < 1$ or $s > 0$, it eventually follows that the solution $x_{\tilde{K}}$ uniquely exists.

(j2) Let $(\lambda\beta\mu + s)/\delta \geq (<)\ b$. Then, from (A 1.2 (1(2)) (p.290)) we have $\tilde{K}((\lambda\beta\mu + s)/\delta) = (<)\ \lambda\beta\mu + s - \delta(\lambda\beta\mu + s)/\delta = 0$, hence $x_{\tilde{K}} = (<)\ (\lambda\beta\mu + s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.25)), hence $x_{\tilde{K}} < (= (>))\ 0$ from (j1). ∎

Corollary A 1.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.

(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

• *Proof* (a) Clearly $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to Lemma A 1.3(p.291) (i,j1). The inverse holds by contraposition.

(b) Since $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to (a) and since $\tilde{K}(x) < (\geq)\ 0 \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$, we have $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{K}} = x$, then $\tilde{K}(x) = \tilde{K}(x_{\tilde{K}}) = 0$ or equivalently $x_{\tilde{K}} = x \Rightarrow \tilde{K}(x) = 0$, hence $x_{\tilde{K}} = x \Rightarrow \tilde{K}(x) \leq 0$. Accordingly, it follows that $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. ∎

Lemma A 1.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $a < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (=))\ 0$.

(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (>))\ 0$. □

• *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_{\tilde{L}} = a$ from Lemma A 1.2(p.291) (d) and $x_{\tilde{K}} = a$ from Lemma A 1.3(p.291) (i), hence $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) Let $\beta = 1$ and $s > 0$. Then $\tilde{K}(x_{\tilde{L}}) = 0$ from (A 1.6 (1) (p.291)), hence $x_{\tilde{K}} = x_{\tilde{L}}$ from Lemma A 1.3(p.291) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_{\tilde{L}} = a \cdots \mathbf{(1)}$ from Lemma A 1.2(p.291) (d).

◦ If $a < 0$, then $x_{\tilde{L}} < 0$, hence $\tilde{K}(x_{\tilde{L}}) > 0$ from (A 1.6 (1) (p.291)), hence $x_{\tilde{L}} < x_{\tilde{K}}$ from Lemma A 1.3(p.291) (j1), and if $a = (>)\ 0$, then $x_{\tilde{L}} = (>)\ 0$, hence $\tilde{K}(x_L) = (<)\ 0$ from (A 1.6 (1) (p.291)), so $x_{\tilde{L}} = (>)\ x_{\tilde{K}}$ from Lemma A 1.3(p.291) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.

◦ If $a < 0$, from (5.1.17(p.25)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) < 0$ due to Lemma A 1.1(p.289) (g), hence $x_{\tilde{K}} < 0 \cdots \mathbf{(2)}$ from Lemma A 1.3(p.291) (j1), and if $a = (>)\ 0$, from (5.1.17(p.25)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) = 0$ due to Lemma A 1.1(p.289) (g), hence $x_{\tilde{K}} = 0$ from Lemma A 1.3(p.291) (j1) or equivalently $x_{\tilde{K}} = (=)\ 0$. Accordingly, we have the *second relation* "⇒".

(d) Let $\beta < 1$ and $s > 0$. Now, since $\tilde{\kappa} = \tilde{K}(0)$ from (5.1.17(p.25)), if $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$, thus $x_{\tilde{K}} < (= (>))\ 0 \cdots (3)$ from Lemma A 1.3(p.291) (j1). Accordingly $\tilde{L}(x_{\tilde{K}}) < (= (>))\ 0$ from (A 1.6 (2) (p.291)), hence $x_{\tilde{L}} < (= (>)\ x_{\tilde{K}}$ from Lemma A 1.2(p.291) (e1). Thus "⇒" holds and its inverse "⇐" is immediate by contraposition. The last "⇒" is immediate from (3). ∎

Lemma A 1.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) *$\tilde{\mathcal{L}}(s)$ is nondecreasing in s and strictly increasing in s if $\lambda\beta < 1$.*

(b) *Let $\lambda\beta\mu \le a$.*

1. $x_{\tilde{L}} \ge \lambda\beta\mu + s$.
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_{\tilde{L}} > \lambda\beta\mu + s$.*

(c) *Let $\lambda\beta\mu > a$. Then, there exists a $s_{\tilde{\mathcal{L}}} > 0$ such that if $s_{\tilde{\mathcal{L}}} > (\le)\ s$, then $x_{\tilde{L}} < (\ge)\ \lambda\beta\mu + s$.* □

● *Proof* (a) From (5.1.15(p.25)) and (5.1.13(p.25)) we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s) = \lambda\beta\tilde{T}(\lambda\beta\mu + s) + s \cdots (1)$, hence the assertion holds from Lemma A 1.1(p.289) (m).

(b) Let $\lambda\beta\mu \le a$. Then, from (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta\mu) = 0 \cdots (2)$ due to Lemma A 1.1(p.289) (g).

(b1) Since $s \ge 0$, from (a) we have $\tilde{\mathcal{L}}(s) \ge \tilde{\mathcal{L}}(0) = 0$ due to (2) or equivalently $\tilde{L}(\lambda\beta\mu + s) \ge 0$ due to (1), hence $x_{\tilde{L}} \ge \lambda\beta\mu + s$ from Corollary A 1.1(p.291) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0 \cdots (3)$ due to (2) or equivalently $\tilde{L}(\lambda\beta\mu + s) > 0$, hence $x_{\tilde{L}} > \lambda\beta\mu + s$ from Lemma A 1.2(p.291) (e1).

(c) Let $\lambda\beta\mu > a$. From (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta\mu) < 0$ due to Lemma A 1.1(p.289) (g). Noting (A 1.1 (1) (p.290)), for any sufficiently large $s > 0$ such that $\lambda\beta\mu + s \ge b$ and $\lambda\beta\mu + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s) = \lambda\beta\mu + s - \lambda\beta(\lambda\beta\mu + s) = (1 - \lambda\beta)(\lambda\beta\mu + s) \ge 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\tilde{\mathcal{L}}} > 0$ of $\tilde{\mathcal{L}}(s) = 0$. Then $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$ or equivalently $\tilde{L}(\lambda\beta\mu + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta\mu + s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 1.1(p.291) (a) we get $x_{\tilde{L}} < \lambda\beta\mu + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \ge \lambda\beta\mu + s$ for $s \ge s_{\tilde{\mathcal{L}}}$. ∎

Lemma A 1.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have:*

(a) *$\tilde{\kappa} = s$ if $a > 0$ and $\tilde{\kappa} = \lambda\beta\mu + s$ if $b < 0$.*

(b) *Let $\beta < 1$ or $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$.* □

● *Proof* (a) Immediate from (5.1.16(p.25)) and Lemma A 1.1(p.289) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} < (= (>))\ 0$, we have $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.25)), hence $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 1.3(p.291) (j3). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 2 Direct Proof of Underlying Functions of Type $\mathbb{P}$

A 2.1 $\mathscr{A}\{T_{\mathbb{P}}\}$

For convenience of reference, below let us copy Lemma 13.2.1(p.93).

Lemma A 2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) *$T(x)$ is continuous on $(-\infty, \infty)$.*

(b) *$T(x)$ is nonincreasing on $(-\infty, \infty)$.*

(c) *$T(x)$ is strictly decreasing on $(-\infty, b]$.*

(d) *$T(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(e) *$T(x) + x$ is strictly increasing on $[a^\star, \infty)$.*

(f) *$T(x) = a - x$ on $(-\infty, a^\star]$ and $T(x) > a - x$ on $(a^\star, \infty)$.*

(g) *$T(x) > 0$ on $(-\infty, b)$ and $T(x) = 0$ on $[b, \infty)$.*

(h) *$T(x) \ge \max\{0, a - x\}$ on $(-\infty, \infty)$.*

(i) *$T(0) = a$ if $a^\star > 0$ and $T(0) = 0$ if $b < 0$.*

(j) *$\beta T(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*

(k) *$\beta T(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*

(l) *If $x < y$ and $a^\star < y$, then $T(x) + x < T(y) + y$.*

(m) *$\lambda\beta T(\lambda\beta a - s) - s$ is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.*

(n) *$a^\star < a$.* □

A 2.2 $\mathscr{A}\{L_{\mathbb{P}}\}$, $\mathscr{A}\{K_{\mathbb{P}}\}$, $\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$, and $\kappa_{\mathbb{P}}$

Noting Lemma A 2.1(p.293) (f), from (5.1.20(p.26)) and (5.1.21(p.26)) we obtain

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x & \text{on } (-\infty, a^\star] \quad \cdots(1),\\ > \lambda\beta a - s - \lambda\beta x & \text{on } (a^\star, \infty) \quad \cdots(2),\end{cases} \tag{A 2.1}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x & \text{on } (-\infty, a^\star] \quad \cdots(1),\\ > \lambda\beta a - s - \delta x & \text{on } (a^\star, \infty) \quad \cdots(2).\end{cases} \tag{A 2.2}$$

In addition, from (5.1.21(p.26)) and Lemma A 2.1(p.293) (g) we have

$$K(x)\begin{cases} > -(1-\beta)x - s & \text{on } (-\infty, b) \quad \cdots(1),\\ = -(1-\beta)x - s & \text{on } [b, \infty) \quad \cdots(2),\end{cases} \tag{A 2.3}$$

from which we obtain

$$K(x) + x \geq \beta x - s \quad \text{on} \quad (-\infty, \infty). \tag{A 2.4}$$

Then, from (A 2.2 (1) (p.294)) and (A 2.3 (2) (p.294)) we get

$$K(x) + x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x & \text{on } (-\infty, a^\star] \quad \cdots(1),\\ \beta x - s & \text{on } [b, \infty) \quad \cdots(2).\end{cases} \tag{A 2.5}$$

Since $K(x) = L(x) - (1-\beta)x$ from (5.1.21(p.26)) and (5.1.20(p.26)), if x_L and x_K exist, then

$$K(x_L) = -(1-\beta)\, x_L \cdots(1), \quad L(x_K) = (1-\beta)\, x_K \cdots(2). \tag{A 2.6}$$

Lemma A 2.2 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$L(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$L(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *Let $s = 0$. Then $x_L = b$ where $x_L > (\leq)\, x \Leftrightarrow L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.*
(e) *Let $s > 0$.*
  1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\, x \Leftrightarrow L(x) > (= (<))\, 0$.*
  2. *$(\lambda\beta a - s)/\lambda\beta \leq (>)\, a^\star \Leftrightarrow x_L = (>)\, (\lambda\beta a - s)/\lambda\beta > (\leq)\, a^\star$.* □

• *Proof* (a-c) Immediate from (5.1.20(p.26)) and Lemma A 2.1(p.293) (a-c).

(d) Let $s = 0$. Then, since $L(x) = \lambda\beta T(x)$, from Lemma A 2.1(p.293) (g) we have $L(x) > 0$ for $x < b$ and $L(x) = 0$ for $b \leq x$, hence $x_L = b$ by the definition of x_L (see Section 5.2(p.27) (a)), thus $x_L > (\leq)\, x \Rightarrow L(x) > (=)\, 0$. The inverse is true by contraposition. In addition, since $L(x) = 0 \Rightarrow L(x) \leq 0$, we have $L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.

(e) Let $s > 0$.

(e1) From (A 2.1 (1) (p.294)) and from $\lambda > 0$ and $\beta > 0$ we have $L(x) > 0$ for a sufficiently small $x < 0$ such that $x \leq a^\star$. In addition, we have $L(b) = \lambda\beta T(b) - s = -s < 0$ from Lemma A 2.1(p.293) (g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L < b$ is immediate from $L(b) < 0$. The latter half is evident.

(e2) If $(\lambda\beta a - s)/\lambda\beta \leq (>)\, a^\star$, from (A 2.1 (1(2)) (p.294)) we have $L((\lambda\beta a - s)/\lambda\beta) = (>)\, \lambda\beta a - s - \lambda\beta(\lambda\beta a - s)/\lambda\beta = 0$, hence $x_L = (>)\, (\lambda\beta a - s)/\lambda\beta$ from (e1). ▮

Corollary A 2.1 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$x_L > (\leq)\, x \Leftrightarrow L(x) > (\leq)\, 0$.*
(b) *$x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$.* □

• *Proof* (a) "⇒" is immediate from Lemma A 2.2(p.294) (d,e2). "⇐" is evident by contraposition.

(b) Since $x_L > (\leq)\, x \Rightarrow L(x) > (\leq)\, 0$ due to (a) and since $L(x) > (\leq)\, 0 \Rightarrow L(x) \geq (\leq)\, 0$, we have $x_L > (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. In addition, if $x_L = x$, then $L(x) = L(\,x_L\,) = 0$ or equivalently $x_L = x \Rightarrow L(x) = 0$, hence $x_L = x \Rightarrow L(x) \geq 0$. Accordingly, it follows that $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. ▮

Lemma A 2.3 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$K(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(g) *$K(x) + x$ is strictly increasing on $[a^\star, \infty)$.*

(h) *If $x < y$ and $a^\star < y$, then $K(x) + x < K(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*

1. *There uniquely exists x_K where $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.*
2. $(\lambda\beta a - s)/\delta \leq (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.
3. *Let $\kappa > (= (<))\ 0$. Then $x_K > (= (<))\ 0$.* □

● *Proof* (a-c) Immediate from (5.1.21(p.26)) and Lemma A 2.1(p.293) (a-c).

(d) Immediate from (5.1.21(p.26)) and Lemma A 2.1(p.293) (b).

(e) From (5.1.21(p.26)) we have $K(x) + x = \lambda\beta T(x) + \beta x - s = \lambda\beta(T(x) + x) + (1-\lambda)\beta x - s \cdots$ **(1)**, hence the assertion holds from Lemma A 2.1(p.293) (d).

(f) Obvious from (1) and Lemma A 2.1(p.293) (d).

(g) Clearly from (1) and Lemma A 2.1(p.293) (e).

(h) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $K(x) + x \leq K(a^\star) + a^\star < K(y) + y$ due to (e,g). If $a^\star < x$, then $a^\star < x < y$, hence $K(x) + x < K(y) + y$ due to (g). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $K(x) + x < K(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x)$ due to (5.1.21(p.26)), from Lemma A 2.1(p.293) (g) we have $K(x) = 0$ for $b \leq x$ and $K(x) > 0$ for $x < b$, so that $x_K = b$ due to the definition in Section 5.2(p.27) (a). Hence $x_K > (\leq)\ x \Rightarrow K(x) > (=)\ 0$. The inverse holds by contraposition. In addition, since $K(x) = 0 \Rightarrow K(x) \leq 0$, we have $K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) This proof consists of the following six steps:

- First note (A 2.3 (2) (p.294)). If $\beta < 1$, then $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$ and if $s > 0$, then, whether $\beta < 1$ or $\beta = 1$, we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$. Hence, whether $\beta < 1$ or $s > 0$, we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$.
- Next note (A 2.2 (1) (p.294)). Then, since $\delta > 0$ from (10.2.2 (1) (p.56)), whether $\beta < 1$ or $s > 0$ we have $K(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a^\star$.
- Hence, whether $\beta < 1$ or $s > 0$, it follows that there exists the solution x_K.
- ∘ Let $\beta < 1$. Then, the solution x_K is unique from (d).
- ∘ Let $s > 0$. If $\beta < 1$, the solution x_K is unique for the reason just above. If $\beta = 1$, we have $K(b) = -s < 0$ from (A 2.3 (2) (p.294)), hence $x_K < b$ due to (c), so $K(x)$ is strictly decreasing on the neighbourhood of $x = x_K$ due to (c), thus the solution x_K is unique. Therefore, whether $\beta < 1$ or $\beta = 1$, it follows that the solution x_K is unique.
- ∘ Hence, whether $\beta < 1$ or $s > 0$, it follows that the solution x_K is unique.

From all the above, whether $\beta < 1$ or $s > 0$, it eventually follows that the solution x_K uniquely exists.

(j2) Let $(\lambda\beta a - s)/\delta \leq (>)\ a^\star$. Then, from (A 2.2 (1(2)) (p.294)) we have $K((\lambda\beta a - s)/\delta) = (>)\ \lambda\beta a - s - \delta(\lambda\beta a - s)/\delta = 0$, hence $x_K = (>)\ (\lambda\beta a - s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$ from (5.1.24(p.26)), hence $x_K > (= (<))\ 0$ from (j1). ∎

Corollary A 2.2 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) $x_K > (\leq)\ x \Leftrightarrow K(x) > (\leq)\ 0$.
(b) $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. □

● *Proof* (a) Clearly $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to Lemma A 2.3(p.294) (i,j1). The inverse holds by contraposition.

(b) Since $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to (a) and since $K(x) > (\leq)\ 0 \Rightarrow K(x) \geq (\leq)\ 0$, we have $x_K > (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. In addition, if $x_K = x$, then $K(x) = K(x_K) = 0$ or equivalently $x_K = x \Rightarrow K(x) = 0$, hence $x_K = x \Rightarrow K(x) \geq 0$. Accordingly, it follows that $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. ∎

Lemma A 2.4 ($\mathscr{A}\{L_{\mathbb{P}}/K_{\mathbb{P}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_L = x_K = b$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_L = x_K$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $b > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\kappa > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (<))\ 0$.* □

● *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma A 2.2(p.294) (d) and $x_K = b$ from Lemma A 2.3(p.294) (i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (A 2.6 (1) (p.294)), hence $x_K = x_L$ from Lemma A 2.3(p.294) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots$ **(1)** from Lemma A 2.2(p.294) (d).

- ∘ If $b > 0$, then $x_L > 0$, hence $K(x_L) < 0$ from (A 2.6 (1) (p.294)), so $x_L > x_K$ from Lemma A 2.3(p.294) (j1), and if $b = (<)\ 0$, then $x_L = (<)\ 0$, hence $K(x_L) = (>)\ 0$ from (A 2.6 (1) (p.294)), so $x_L = (<)\ x_K$ from Lemma A 2.3(p.294) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.

- If $b > 0$, from (5.1.24(p.26)) we have $K(0) = \lambda\beta T(0) > 0$ due to Lemma A 2.1(p.293) (g), hence $x_K > 0 \cdots$ **(2)** from Lemma A 2.3(p.294) (j1), and if $b = (<)\ 0$, from (5.1.24(p.26)) we have $K(0) = \lambda\beta T(0) = 0$ due to Lemma A 2.1(p.293) (g), hence $x_K = 0$ from Lemma A 2.3(p.294) (j1) or equivalently $x_K = (=)\ 0$. Accordingly, we have the *second relation* "$\Rightarrow$".

(d) Let $\beta < 1$ and $s > 0$. Now, from (5.1.24(p.26)) and (5.1.23(p.26)), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0$ from Lemma A 2.3(p.294) (j1). Accordingly $L(x_K) > (= (<))\ 0$ from (A 2.6 (2) (p.294)), hence $x_L > (= (<))\ x_K$ from Lemma A 2.2(p.294) (e1). ∎

Lemma A 2.5 ($\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$)

(a) *$\mathcal{L}(s)$ is nonincreasing in s and is strictly decreasing in s if $\lambda\beta < 1$.*

(b) *Let $\lambda\beta a \geq b$.*

1. *$x_L \leq \lambda\beta a - s$.*
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta a - s$.*

(c) *Let $\lambda\beta a < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\leq)\ s$, then $x_L > (\leq)\ \lambda\beta a - s$.* □

• *Proof* (a) From (5.1.22(p.26)) and (5.1.20(p.26)) we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta T(\lambda\beta a - s) - s$, hence the assertion holds from Lemma A 2.1(p.293) (m).

(b) Let $\lambda\beta a \geq b$. Then, from (5.1.22(p.26)) and (5.1.20(p.26)) we have $\mathcal{L}(0) = L(\lambda\beta a) = \lambda\beta T(\lambda\beta a) = 0 \cdots$ **(1)** due to Lemma A 2.1(p.293) (g).

(b1) Since $s \geq 0$, from (a) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) \leq 0$, hence $x_L \leq \lambda\beta a - s$ from Corollary A 2.1(p.294) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) < 0$, thus $x_L < \lambda\beta a - s$ from Lemma A 2.2(p.294) (e1).

(c) Let $\lambda\beta a < b$. From (5.1.22(p.26)) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta a) > 0$ due to Lemma A 2.1(p.293) (g). Noting (A 2.1 (1) (p.294)), for any sufficiently large $s > 0$ such that $\lambda\beta a - s \leq a^\star$ and $\lambda\beta a - s < 0$ we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta a - s - \lambda\beta(\lambda\beta a - s) = (1 - \lambda\beta)(\lambda\beta a - s) \leq 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}} > 0$ of $\mathcal{L}(s) = 0$. Then $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta a - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta a - s) \leq 0$ for $s \geq s_{\mathcal{L}}$. Hence, from Corollary A 2.1(p.294) (a) we get $x_L > \lambda\beta a - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta a - s$ for $s \geq s_{\mathcal{L}}$. ∎

Lemma A 2.6 ($\mathscr{A}\{\kappa_{\mathbb{P}}\}$) *We have*:

(a) *$\kappa = \lambda\beta a - s$ if $a^\star > 0$ and $\kappa = -s$ if $b < 0$.*

(b) *Let $\beta < 1$ or $s > 0$, Then $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$.* □

• *Proof* (a) Immediate from (5.1.23(p.26)) and Lemma A 2.1(p.293) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.24(p.26)), hence $x_K > (= (<))\ 0$ from Lemma A 2.3(p.294) (j1). Thus "$\Rightarrow$" was proven. Its inverse "$\Leftarrow$" is immediate by contraposition. ∎

A 3 Direct Proof of Underlying Functions of $\tilde{\text{T}}$ype $\mathbb{P}$

A 3.1 $\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$

Lemma A 3.1

(a) *Let $x \leq a$. Then $z(x) = a$*

(b) *Let $a < x$. Then $a < z(x) < x$.*

(c) *$z(x) \leq b$ for any x.* □

• *Proof* (a) Let $x \leq a$. If $a < z \cdots$ (II), then $x < z$, hence $\tilde{p}(z)(z - x) > 0$ due to (5.1.41 (2) (p.27)), and if $z \leq a \cdots$ (I), then $\tilde{p}(z)(z - x) = 0$ due to (5.1.41 (1) (p.27)) (see Figure A 3.1(p.296) below). Hence $z(x) = a$ due to Def. 5.1.2(p.27).

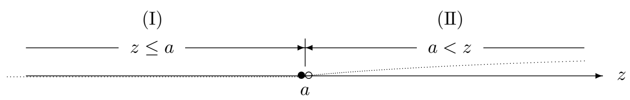


Figure A 3.1: Case $x \leq a$

(b) Let $a < x$. If $x \leq z \cdots$ (III), then $\tilde{p}(z)(z - x) \geq 0$, if $a < z < x \cdots$ (II), then $\tilde{p}(z)(z - x) < 0$ due to (5.1.41 (2) (p.27)), and if $z \leq a \cdots$ (I), then $\tilde{p}(z)(z - x) = 0$ due to (5.1.41 (1) (p.27)) (see Figure A 3.2(p.297) below). Hence, $z(x)$ is given by z on $a < z < x$, i.e., $a < z(x) < x$.

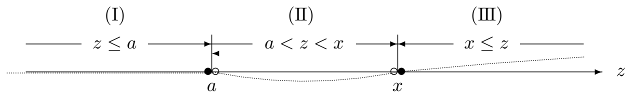


Figure A 3.2: Case $a < x$

(c) Assume that $z(x) > b$ for a certain x. Then, since $\tilde{p}(z(x)) = 1 = \tilde{p}(b)$ due to (5.1.42 (2) (p.27)), from (5.1.38(p.27)) we have $\tilde{T}(x) = z(x) - x > b - x = \tilde{p}(b)(b - x) \geq \tilde{T}(x)$, which is a contradiction. Hence, it must be that $z(x) \leq b$ for any x. ∎

Corollary A 3.1 *$a \leq z(x) \leq b$ for any x.* □

• *Proof* Evident from Lemma A 3.1(p.296). ∎

Lemma A 3.2 *$\tilde{p}(z)$ is nondecreasing on $(-\infty, \infty)$ and strictly increasing in $z \in [a, b]$.* □

• *Proof* The former half is immediate from (5.1.31(p.26)). For $a \leq z' < z \leq b$ we have $\tilde{p}(z) - \tilde{p}(z') = \Pr\{\boldsymbol{\xi} \leq z\} - \Pr\{\boldsymbol{\xi} \leq z'\} = \Pr\{z' < \boldsymbol{\xi} \leq z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.2.3 (2) (p.13))), hence $p(z) > p(z')$, i.e., $p(z)$ is strictly increasing on $[a, b]$. ∎

Lemma A 3.3 *$z(x)$ is nondecreasing on $(-\infty, \infty)$.* □

• *Proof* From (5.1.38(p.27)), for any x and y we have

$$
\begin{aligned}
\tilde{T}(x) &= \tilde{p}(z(x))(z(x) - x) \\
&= \tilde{p}(z(x))(z(x) - y) - (x - y)\tilde{p}(z(x)) \\
&\geq \tilde{T}(y) - (x - y)\tilde{p}(z(x)) \\
&= \tilde{p}(z(y))(z(y) - y) - (x - y)\tilde{p}(z(x)) \\
&= \tilde{p}(z(y))\big(z(y) - x + (x - y)\big) - (x - y)\tilde{p}(z(x)) \\
&= \tilde{p}(z(y))(z(y) - x) + (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x))) \\
&\geq \tilde{T}(x) + (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x))).
\end{aligned}
$$

Hence $0 \geq (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x)))$. Let $x > y$. Then $0 \geq \tilde{p}(z(y)) - \tilde{p}(z(x))$ or equivalently $\tilde{p}(z(x)) \geq \tilde{p}(z(y)) \cdots$ **(1)**. Since $a \leq z(x) \leq b$ and $a \leq z(y) \leq b$ from Corollary A 3.1(p.297), if $z(x) < z(y)$, then $\tilde{p}(z(x)) < \tilde{p}(z(y))$ from Lemma A 3.2(p.297), which contradicts (1). Hence, it must be that $z(x) \geq z(y)$, i.e., $z(x)$ is nondecreasing in $x \in (-\infty, \infty)$. ∎

Lemma A 3.4

(a) *$\tilde{T}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{T}(x) < 0$ on (a, ∞) and $\tilde{T}(x) = 0$ on $(-\infty, a]$.*
(e) *$\tilde{T}(x) \leq b - x$ on $(-\infty, \infty)$.*
(f) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(g) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(h) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(i) *$\tilde{T}(x) \leq \min\{0, b - x\}$ for any $x \in (-\infty, \infty)$.*
(j) *$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing in s and is strictly increasing in s if $\lambda\beta < 1$.* □

• *Proof* (a,b) Immediate from the fact that $\tilde{p}(z)(z - x)$ in (5.1.32(p.26)) is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any z.

(c) Let $x' > x > a$. Then $z(x) > a$ from Lemma A 3.1(p.296) (b). Accordingly, since $\tilde{p}(z(x)) > 0$ due to (5.1.41 (2) (p.27)) and since $z(x) - x > z(x) - x'$, from (5.1.38(p.27)) we have $\tilde{T}(x) = \tilde{p}(z(x))(z(x) - x) > \tilde{p}(z(x))(z(x) - x') \geq \tilde{T}(x')$, i.e., $\tilde{T}(x)$ is *strictly decreasing* on $(a, \infty) \cdots$ **(1)**. Assume $\tilde{T}(a) = \tilde{T}(x)$ for a given $x > a$, so $x - a > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > 2\varepsilon > 0$ we have $a < a + \varepsilon < x - \varepsilon < x$, hence $\tilde{T}(a) = \tilde{T}(x) < \tilde{T}(a + \varepsilon) \leq \tilde{T}(a)$ due to the *strict* unceasingness shown just above and the nonincreasingness in (b), which is a contradiction. Thus, since $\tilde{T}(x) \neq \tilde{T}(a)$ for any $x > a$, we have $\tilde{T}(x) < \tilde{T}(a)$ or $\tilde{T}(x) > \tilde{T}(a)$ for any $x > a$. However, the latter is impossible due to (b), hence only the former holds, i.e., $\tilde{T}(x) < \tilde{T}(a)$ for any $x > a$. From this and (1) it eventually follows that $\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$ instead of on (a, ∞).

(d) Let $x \leq a$. Then, since $z(x) = a$ from Lemma A 3.1(p.296) (a), we have $\tilde{p}(z(x)) = 0$ due to (5.1.41 (1) (p.27)), hence $\tilde{T}(x) = \tilde{p}(z(x))(z(x) - x) = 0$ on $(-\infty, a]$, so $\tilde{T}(a) = 0$. Let $x > a$. Then, from (c) we have $\tilde{T}(x) < \tilde{T}(a) = 0$, i.e., $\tilde{T}(x) < 0$ on (a, ∞).

(e) From (5.1.32(p.26)) and (5.1.42 (2) (p.27)) we see that $\tilde{T}(x) \leq \tilde{p}(b)(b - x) = b - x$ for any x on $(-\infty, \infty)$.

(f) For $x' < x$ we have, from (5.1.38(p.27)),

$$\begin{aligned}\tilde{T}(x)+x &= \tilde{p}(z(x))(z(x)-x)+x\\ &= \tilde{p}(z(x))z(x)+(1-\tilde{p}(z(x)))x\\ &\geq \tilde{p}(z(x))z(x)+(1-\tilde{p}(z(x)))x'\\ &= \tilde{p}(z(x))(z(x)-x')+x' \geq \tilde{T}(x')+x',\end{aligned}$$

hence it follows that $\tilde{T}(x)+x$ is nondecreasing in x on $(-\infty,\infty)$,

(g) If $\beta=1$, then $\beta\tilde{T}(x)+x=T(x)+x$, hence the assertion is true from (f).

(h) Since $\beta\tilde{T}(x)+x=\beta(\tilde{T}(x)+x)+(1-\beta)x$, if $\beta<1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Since $\tilde{T}(x)\leq b-x$ for any x from (e) and $\tilde{T}(x)\leq 0$ for any x from (d), we have $\tilde{T}(x)\leq\min\{0,b-x\}$ for any $x\in(-\infty,\infty)$.

(j) From (5.1.32(p.26)) we have

$$\begin{aligned}\lambda\beta\tilde{T}(\lambda\beta b+s)+s &= \lambda\beta\min_z\tilde{p}(z)(z-\lambda\beta b-s)+s\\ &= \min_z\tilde{p}(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s)+s.\end{aligned}$$

Then, for $s>s'$ we have

$$\begin{aligned}&\lambda\beta\tilde{T}(\lambda\beta b+s)+s-\lambda\beta\tilde{T}(\lambda\beta b+s')-s'\\ &\quad= \min_z p(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s)-\min_z p(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s')+(s-s')\\ &\quad\geq \min_z -p(z)\lambda\beta(s-s')+(s-s')^{\dagger}\\ &\quad\geq \min_z -(s-s')\lambda\beta+(s-s')\quad(\text{due to } p(z)\leq 1 \text{ and } s-s'>0)\\ &\quad= -(s-s')\lambda\beta+(s-s')\\ &\quad= (s-s')(1-\lambda\beta)\geq(>)\ 0 \text{ if } \lambda\beta\leq(<)\ 1.\end{aligned}$$

Hence, since $\lambda\beta\tilde{T}(\lambda\beta b+s)+s\geq(>)\ \lambda\beta\tilde{T}(\lambda\beta b+s')+s'$ if $\lambda\beta\leq(<)\ 1$, it follows that $\lambda\beta\tilde{T}(\lambda\beta b+s)+s$ is nondecreasing in s and strictly increasing in s if $\lambda\beta<1$. ∎

Let us define

$$\begin{aligned}&\tilde{h}(z)=\tilde{p}(z)(z-b)/(1-\tilde{p}(z)),\quad z<b,\\ &\tilde{h}^\star=\inf_{z<b}\tilde{h}(z),\end{aligned}$$

Below, for any x let us define the following successive four assertions:

$$\begin{aligned}&A_1(x)=\langle\!\langle\, z(x)<b\,\rangle\!\rangle,\\ &A_2(x)=\langle\!\langle\,\tilde{T}(b,x)>\tilde{T}(z',x,)\text{ for at least one } z'<b\,\rangle\!\rangle,\\ &A_3(x)=\langle\!\langle\, b-\tilde{h}(z')>x \text{ for at least one } z'<b\,\rangle\!\rangle,\\ &A_4(x)=\langle\!\langle\,\sup_{z<b}\{b-\tilde{h}(z)\}>x\,\rangle\!\rangle.\end{aligned}$$

Proposition A 3.1 For any x we have $A_1(x)\Leftrightarrow A_2(x)\Leftrightarrow A_3(x)\Leftrightarrow A_4(x)$. □

● *Proof* Letting $\tilde{T}(z,x)\overset{\text{def}}{=}\tilde{p}(z)(z-x)$, we can rewrite (5.1.32(p.26)) as $\tilde{T}(x)=\min_z\tilde{T}(z,x)=\tilde{T}(z(x),x)$ (see (5.1.38(p.27))).

1. Let $A_1(x)$ be true for any x. Suppose $\tilde{T}(b,x)\leq\tilde{T}(z',x)$ for all $z'<b$. Then the minimum of $\tilde{T}(z,x)$ is attained at $z=b$ (see Def. 5.1.2(p.27)), i.e., $z(x)=b$, which contradicts $A_1(x)$. Hence it must be that $\tilde{T}(b,x)>\tilde{T}(z',x)$ for at least one $z'<b$, thus $A_2(x)$ becomes true. Accordingly, we have $A_1(x)\Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any x. Then, if $z(x)=b$, we have $\tilde{T}(b,x)>\tilde{T}(z',x)\geq\tilde{T}(x)=\tilde{T}(z(x),x)=\tilde{T}(b,x)$, which is a contradiction, hence it must be that $z(x)<b$ due to Lemma A 3.1(p.296) (c); accordingly, we have $A_2(x)\Rightarrow A_1(x)$. Thus, it follows that we have $A_1(x)\Leftrightarrow A_2(x)$ for any given x.
2. Since $\tilde{p}(b)=1$ from (5.1.42 (2) (p.27)), for $z'<b$ we have

$$\begin{aligned}&\tilde{T}(b,x)-\tilde{T}(z',x)\\ &\quad= \tilde{p}(b)(b-x)-\tilde{p}(z')(z'-x)\\ &\quad= b-x-\tilde{p}(z')(z'-x)\\ &\quad= b-x-\tilde{p}(z')(b-x+z'-b)\\ &\quad= b-x-\tilde{p}(z')(b-x)-\tilde{p}(z')(z'-b)\\ &\quad= (1-\tilde{p}(z'))(b-x)-\tilde{p}(z')(z'-b)\\ &\quad= (1-\tilde{p}(z'))\big(b-x-\tilde{p}(z')(z'-b)/(1-\tilde{p}(z'))\big)\\ &\quad= (1-\tilde{p}(z'))(b-x-\tilde{h}(z'))\\ &\quad= (1-\tilde{p}(z'))(b-\tilde{h}(z')-x).\end{aligned}$$

Accordingly, noting $1>\tilde{p}(z')$ due to (5.1.42 (1) (p.27)), we immediately see that $A_2(x)\Leftrightarrow A_3(x)$ for any given x.

† $\min a(x)-\min b(x)\geq\min\{a(x)-b(x)\}$.

3. Let $A_3(x)$ be true for any x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x) \Rightarrow A_4(x)$. Let $A_4(x)$ be true for any x. Then evidently $b - \tilde{h}(z') > x$ for at least one $z' < b$, hence $A_3(x)$ is true, so we have $A_4(x) \Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x) \Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. ∎

Lemma A 3.5

(a) $-\infty < \tilde{h}^\star < 0$.

(b) $\tilde{x}^\star = b - \tilde{h}^\star > b$.

(c) $\tilde{x}^\star > (\leq)\ x \Leftrightarrow z(x) < (=)\ b$.

(d) $b^\star > b$. □

• *Proof* (a) For any infinitesimal $\varepsilon > 0$ such that $a < a + \varepsilon < b \cdots$ (II) we have $0 < \tilde{p}(a+\varepsilon) < 1$ from (5.1.41 (2) (p.27)) and (5.1.42 (1) (p.27)), hence, $\tilde{h}(a+\varepsilon) = \tilde{p}(a+\varepsilon)(a+\varepsilon-b)/(1-\tilde{p}(a+\varepsilon)) < 0$. If $z \leq a \cdots$ (I), then $\tilde{p}(z) = 0$ due to (5.1.41 (1) (p.27)), hence $\tilde{h}(z) = 0$ for $z \leq a$. From the above we have $\tilde{h}^\star < 0$ (finite) or $\tilde{h}^\star = -\infty$.

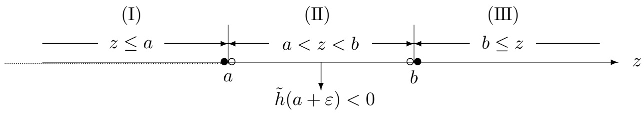


Figure A 3.3: $\tilde{h}(z) = 0$ for $z \leq a$ and $\tilde{h}(a+\varepsilon) < 0$

Assume that $\tilde{h}^\star = -\infty$. Then, there exists at least one z' on $a < z' < b$ such that $\tilde{h}(z') \leq -N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$ (see (2.2.4(p.13))) with any $M > 1$, i.e., $N = M/\underline{f}$, we have $\tilde{h}(z') \leq -M/\underline{f}$, so $\tilde{p}(z')(z'-b)/(1-\tilde{p}(z')) \leq -M/\underline{f}$. Hence, noting (5.1.31(p.26)), we have

$$\tilde{p}(z')(z'-b) \ \leq\ -(1-\tilde{p}(z'))M/\underline{f} = -(1-\Pr\{\boldsymbol{\xi} \leq z'\})M/\underline{f} = -\Pr\{z' < \boldsymbol{\xi}\}M/\underline{f} \cdots (*)$$

where $\Pr\{z' < \boldsymbol{\xi}\} = \int_{z'}^{b} f(w)dw \geq \int_{z'}^{b} \underline{f}dw = (b-z')\underline{f}$. Accordingly, since $\tilde{p}(z')(z'-b) \leq -(b-z')\underline{f}M/\underline{f} = (z'-b)M$, we have $\tilde{p}(z') \geq M > 1$ due to $z' - b < 0$, which is a contradiction. Hence, it must follow that $\tilde{h}^\star > -\infty$.

(b) Since $A_1(x) \Rightarrow A_4(x)$ due to Proposition A 3.1, we can rewrite (5.1.40(p.27)) as

$$\begin{aligned}\tilde{x}^\star &= \sup\{x \mid \sup_{z<b}\{b - \tilde{h}(z)\} > x\} \\ &= \sup_{z<b}\{b - \tilde{h}(z)\} \cdots (1) \\ &= b - \inf_{z<b}\tilde{h}(z) = b - \tilde{h}^\star > b\end{aligned}$$

due to (a), hence (b) holds.

(c) Let $\tilde{x}^\star > x$, hence $\sup_{z<b}\{b - \tilde{h}(z)\} > x$ from (1), so $z(x) < b$ due to $A_4(x) \Rightarrow A_1(x)$. Let $\tilde{x}^\star \leq x$, hence $\sup_{z<b}\{b - \tilde{h}(z)\} \leq x$ from (1). Now, since $\sup_{z<b}\{b - \tilde{h}(z)\} \leq x \Rightarrow z(x) \geq b$ due to the contraposition of $A_4(x) \Leftrightarrow A_1(x)$, we obtain $z(x) = b$ due to Lemma A 3.1(p.296) (c).

(d) First note $\tilde{T}(x) \leq \tilde{p}(z')(z'-x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a < b - \varepsilon$ we have $\tilde{p}(b-\varepsilon) > 0$ from (5.1.41 (2) (p.27)), hence $\tilde{T}(b) \leq \tilde{p}(b-\varepsilon)(b-\varepsilon-b) = -\tilde{p}(b-\varepsilon)\varepsilon < 0$, so adding b to the both sides of this inequality yields $\tilde{T}(b) + b < b$, so $\tilde{T}(x) + x \leq \tilde{T}(b) + b < b$ for $x \leq b$ due to Lemma A 3.4(p.297) (f). Accordingly, if $b^\star \leq b$, we have $\tilde{T}(b^\star) + b^\star \leq \tilde{T}(b) + b < b$, hence from Lemma A 3.4(p.297) (a) we have $\tilde{T}(b^\star+\varepsilon) + b^\star + \varepsilon < b$ for any sufficiently small $\varepsilon > 0$, so $\tilde{T}(b^\star+\varepsilon) < b - (b^\star+\varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.27))). Therefore, it must follow that $b^\star > b$. ∎

Lemma A 3.6

(a) *$\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.*

(b) *$\tilde{T}(x) = b - x$ on $[b^\star, \infty)$ and $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$.*

(c) *$\tilde{T}(0) = b$ if $b^\star < 0$ and $\tilde{T}(0) = 0$ if $a > 0$.*

(d) *If $x > y$ and $b^\star > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$.* □

• *Proof* (a) From (5.1.38(p.27)) we have

$$\tilde{T}(x) + x \ =\ \tilde{p}(z(x))(z(x)-x) + x = \tilde{p}(z(x))z(x) + (1-\tilde{p}(z(x)))x. \cdots (1)$$

∘ Let $\tilde{x}^\star > x$. Then $z(x) < b$ from Lemma A 3.5(p.299) (c), hence $\tilde{p}(z(x)) < 1$ due to (5.1.42 (1) (p.27)), so $1 - \tilde{p}(z(x)) > 0$. If $x > x'$, from (1) we have

$$\tilde{T}(x) + x > \tilde{p}(z(x))z(x) + (1-\tilde{p}(z(x)))x' = \tilde{p}(z(x))(z(x)-x') + x' \geq \tilde{T}(x') + x',$$

i.e., $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$, hence understandably so also on $(-\infty, b^\star]$.

- Let $\tilde{x}^\star \le x$. Then $z(x) = b$ from Lemma A 3.5(p.299) (c), hence $\tilde{p}(z(x)) = 1$ from (5.1.42 (2) (p.27)), so $\tilde{T}(x) = \tilde{p}(z(x))(z(x)-x) = b - x \cdots$ **(2)**. Suppose $b^\star > \tilde{x}^\star$. Then, since $b^\star > b^\star - 2\varepsilon > \tilde{x}^\star$ for an infinitesimal $\varepsilon > 0$, we have $b^\star > b^\star - \varepsilon > \tilde{x}^\star + \varepsilon > \tilde{x}^\star$ or equivalently $\tilde{x}^\star < b^\star - \varepsilon$; accordingly, due to (2) we obtain $\tilde{T}(b^\star - \varepsilon) = b - (b^\star - \varepsilon) \cdots$ **(3)**. Now, due to (5.1.39(p.27)) we have $\tilde{T}(b^\star - \varepsilon) < b - (b^\star - \varepsilon)$, which contradicts (3). Accordingly, it must be that $\tilde{x}^\star \ge b^\star$. Let $x' < x < b^\star$. Then, since $\tilde{x}^\star > x$, we have $z(x) < b$ Lemma A 3.5(p.299) (c), hence $\tilde{p}(z(x)) < 1$ due to (5.1.42 (1) (p.27)) or equivalently $1 - \tilde{p}(z(x)) > 0$. Thus, from (1) we have

$$\tilde{T}(x) + x > \tilde{p}(z(x))z(x) + (1 - \tilde{p}(z(x)))x' = \tilde{p}(z(x))(z(x) - x') + x' \ge \tilde{T}(x') + x',$$

implying that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star) \cdots$ **(4)**. Now let us assume $\tilde{T}(b^\star) + b^\star = \tilde{T}(x) + x$ for any $x < b^\star$. Then, for any sufficiently small $\varepsilon > 0$ such that $b^\star - x > 2\varepsilon > 0$ we have $x < x + \varepsilon < b^\star - \varepsilon < b^\star$, hence $\tilde{T}(b^\star) + b^\star = \tilde{T}(x) + x < \tilde{T}(x+\varepsilon) + x + \varepsilon \le \tilde{T}(b^\star) + b^\star$ due to (4) and Lemma A 3.4(p.297) (f), which is a contradiction. Thus, $\tilde{T}(x) + x \ne \tilde{T}(b^\star) + b^\star$ for $x < b^\star$, i.e., $\tilde{T}(x) + x > \tilde{T}(b^\star) + b^\star$ or $\tilde{T}(x) + x < \tilde{T}(b^\star) + b^\star$ for $x < b^\star$; however, the former is impossible due to the nondecreasing in Lemma A 3.4(p.297) (f), hence it follows that $\tilde{T}(x) + x < \tilde{T}(b^\star) + b^\star$ for $x < b^\star$. From this and (4) it inevitably follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$ instead of $(-\infty, b^\star)$.

Accordingly, whether $\tilde{x}^\star > x$ or $\tilde{x}^\star \le x$, it follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.

(b) Due to (5.1.39(p.27)) we have $\tilde{T}(x) < b - x$ for $x < b^\star$, i.e., $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$, hence the latter half is true. Since $\tilde{T}(x) \le b - x$ on $(-\infty, \infty)$ due to Lemma A 3.4(p.297) (e), we have $\tilde{T}(x) + x \le b \cdots$ **(5)** on $(-\infty, \infty)$. Suppose $\tilde{T}(b^\star) + b^\star < b$. Then, for an infinitesimal $\varepsilon > 0$ we have $\tilde{T}(b^\star + \varepsilon) + b^\star + \varepsilon < b$ due to Lemma A 3.4(p.297) (a), i.e., $\tilde{T}(b^\star + \varepsilon) < b - (b^\star + \varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.27))). Consequently, it must be that $\tilde{T}(b^\star) + b^\star = b \cdots$ **(6)** or equivalently $\tilde{T}(b^\star) = b - b^\star$. Let $x > b^\star$. Then, from Lemma A 3.4(p.297) (f) we have $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star = b$. From this and (5) it must be that $\tilde{T}(x) + x = b$ on $(b^\star, \infty)$, hence $\tilde{T}(x) = b - x$ on $(b^\star, \infty)$. From this and (6) it follows that $\tilde{T}(x) = b - x$ on $[b^\star, \infty)$. Hence the former half is true.

(c) Let $b^\star < 0$. Then, since $0 \in [b^\star, \infty)$, we have $\tilde{T}(0) = b$ from the former half of (b). Now we have $\tilde{T}(0) = \min_z \tilde{p}(z)z \cdots$ **(7)** from (5.1.32(p.26)). Let $a > 0$. Then, if $z \le a$, we have $\tilde{p}(z)z = 0$ from (5.1.41 (1) (p.27)) and if $z > a\,(> 0)$, then $\tilde{p}(z)z > 0$ from (5.1.41 (2) (p.27)). Hence it follows that $\tilde{T}(0) = 0$ due to (7).

(d) Let $x > y$ and $b^\star > y$. If $x \ge b^\star$, then $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star > \tilde{T}(y) + y$ due to Lemma A 3.4(p.297) (f) and (a), and if $b^\star > x$, then $b^\star \ge x > y$, hence $\tilde{T}(x) + x > \tilde{T}(y) + y$ due to (a). Thus, whether $x \ge b^\star$ or $b^\star > x$, we have $\tilde{T}(x) + x > \tilde{T}(y) + y$. ∎

All the results obtained above (see Lemmas A 3.1(p.296) -A 3.6(p.299)) can be complied into Lemma A 3.7(p.300) below.

Lemma A 3.7 **($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$)** *For any $F \in \mathscr{F}$ we have:*

(a)	$\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (a)
(b)	$\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (b)
(c)	$\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (c)
(d)	$\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (f)
(e)	$\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, b^\star]$ $\leftarrow$	$\leftarrow$ Lemma A 3.6(p.299) (a)
(f)	$\tilde{T}(x) = b - x$ *on* $[b^\star, \infty)$ *and* $T(x) < b - x$ *on* $(-\infty, b^\star)$ $\leftarrow$	$\leftarrow$ Lemma A 3.6(p.299) (b)
(g)	$\tilde{T}(x) < 0$ *on* (a, ∞) *and* $T(x) = 0$ *on* $(-\infty, a]$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (d)
(h)	$\tilde{T}(x) \le \min\{0, b - x\}$ *on* $(-\infty, \infty)$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (i)
(i)	$\tilde{T}(0) = b$ *if* $b^\star < 0$ *and* $T(0) = 0$ *if* $a > 0$ $\leftarrow$	$\leftarrow$ Lemma A 3.6(p.299) (c)
(j)	$\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (g)
(k)	$\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (h)
(l)	*If* $x > y$ *and* $b^\star > y$*, then* $T(x) + x > T(y) + y$ $\leftarrow$	$\leftarrow$ Lemma A 3.6(p.299) (d)
(m)	$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$ $\leftarrow$	$\leftarrow$ Lemma A 3.4(p.297) (j)
(n)	$b^\star > b$ $\leftarrow$	$\leftarrow$ Lemma A 3.5(p.299) (d)

A 3.2 $\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$, and $\tilde{\kappa}_{\mathbb{P}}$

From (5.1.33(p.27)) and (5.1.34(p.27)) and from Lemma A 3.7(p.300) (f) we obtain, noting (10.2.1(p.56)),

$$\tilde{L}(x) \begin{cases} = \lambda\beta b + s - \lambda\beta x \text{ on } [b^\star, -\infty) & \cdots (1), \\ < \lambda\beta b + s - \lambda\beta x \text{ on } (-\infty, b^\star) & \cdots (2), \end{cases} \tag{A 3.1}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta b + s - \delta x \text{ on } [b^\star, \infty) & \cdots (1), \\ < \lambda\beta b + s - \delta x \text{ on } (-\infty, b^\star) & \cdots (2). \end{cases} \tag{A 3.2}$$

In addition, from (5.1.34(p.27)) and Lemma A 3.7(p.300) (g) we have

$$\tilde{K}(x) \begin{cases} < -(1-\beta)x + s \text{ on } (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s \text{ on } (-\infty, a] & \cdots (2), \end{cases} \tag{A 3.3}$$

hence we obtain

$$\tilde{K}(x) + x \leq \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{A 3.4}$$

Then, from (A 3.2 (1) (p.300)) and (A 3.3 (2) (p.300)) we get

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta b + s + (1-\lambda)\beta x \text{ on } & [b^\star, \infty) \quad \cdots (1), \\ \beta x + s \quad \text{on} & (-\infty, a] \quad \cdots (2). \end{cases} \tag{A 3.5}$$

Since $\tilde{K}(x) = \tilde{L}(x) - (1-\beta)x$ from (5.1.34(p.27)) and (5.1.33(p.27)), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\, x_{\tilde{L}} \cdots (1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\, x_{\tilde{K}} \cdots (2). \tag{A 3.6}$$

Lemma A 3.8 ($\tilde{L}_{\mathbb{P}}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty, \infty)$.*

(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty, \infty)$.*

(c) *$\tilde{L}(x)$ is strictly decreasing on $[a, \infty)$.*

(d) *Let $s = 0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\geq)\, x \Leftrightarrow \tilde{L}(x) < (=)\, 0 \Rightarrow \tilde{L}(x) < (\geq)\, 0$.*

(e) *Let $s > 0$.*

1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (= (>))\, x \Leftrightarrow \tilde{L}(x) < (= (>))\, 0$.*
2. *$(\lambda\beta b + s)/\lambda\beta \geq (<)\, b^\star \Leftrightarrow x_{\tilde{L}} = (<)\, (\lambda\beta b + s)/\lambda\beta < (\geq)\, b^\star$.* □

• *Proof* (a-c) Immediate from (5.1.33(p.27)) and Lemma A 3.7(p.300) (a-c).

(d) Let $s = 0$. Then, since $\tilde{L}(x) = \lambda\beta\tilde{T}(x)$, from Lemma A 3.7(p.300) (g) we have $\tilde{L}(x) = 0$ for $a \geq x$ and $\tilde{L}(x) < 0$ for $x > a$, hence $x_{\tilde{L}} = a$ by definition (see Section 5.2(p.27) (b)), so $x_{\tilde{L}} < (\geq)\, x \Rightarrow \tilde{L}(x) < (=)\, 0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x) = 0 \Rightarrow \tilde{L}(x) \geq 0$, we have $\tilde{L}(x) < (=)\, 0 \Rightarrow \tilde{L}(x) < (\geq)\, 0$.

(e) Let $s > 0$.

(e1) From (A 3.1 (1) (p.300)) and the assumption of $\lambda > 0$ and $\beta > 0$ we have $\tilde{L}(x) < 0$ for a sufficiently large $x > 0$ such that $x > b^\star$. In addition, we have $\tilde{L}(a) = \lambda\beta\tilde{T}(a) + s = s > 0$ from Lemma A 3.7(p.300) (g). Hence, from (a,c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}} > a$ is immediate from $\tilde{L}(a) > 0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta b + s)/\lambda\beta \geq (<)\, b^\star$, from (A 3.1(p.300)) we have $\tilde{L}((\lambda\beta b + s)/\lambda\beta) = (<)\, \lambda\beta b + s - \lambda\beta(\lambda\beta b + s)/\lambda\beta = 0$, hence $x_{\tilde{L}} = (<)\, (\lambda\beta b + s)/\lambda\beta$ from (e1). ∎

Corollary A 3.2 ($\tilde{L}_{\mathbb{P}}$)

(a) *$x_{\tilde{L}} < (\geq)\, x \Leftrightarrow \tilde{L}(x) < (\geq)\, 0$.*

(b) *$x_{\tilde{L}} \leq (\geq)\, x \Rightarrow \tilde{L}(x) \leq (\geq)\, 0$.* □

• *Proof* (a) Clearly $x_{\tilde{L}} < (\geq)\, x \Rightarrow \tilde{L}(x) < (\geq)\, 0$ from Lemma A 3.8(p.301) (d,e1). The inverse is true by contraposition.

(b) Since $x_{\tilde{L}} < (\geq)\, x \Rightarrow \tilde{L}(x) < (\geq)\, 0$ due to (a) and since $\tilde{L}(x) < (\geq)\, 0 \Rightarrow \tilde{L}(x) \leq (\geq)\, 0$, we have $x_{\tilde{L}} < (\geq)\, x \Rightarrow \tilde{L}(x) \leq (\geq)\, 0$. In addition, if $x_{\tilde{L}} = x$, then $\tilde{L}(x) = \tilde{L}(x_{\tilde{L}}) = 0 \leq 0$ or equivalently $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) \leq 0$, hence it follows that $x_{\tilde{L}} \leq (\geq)\, x \Rightarrow \tilde{L}(x) \leq (\geq)\, 0$. ∎

Lemma A 3.9 ($\tilde{K}_{\mathbb{P}}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty, \infty)$.*

(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty, \infty)$.*

(c) *$\tilde{K}(x)$ is strictly decreasing on $[a, \infty)$.*

(d) *$\tilde{K}(x)$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*

(e) *$\tilde{K}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(f) *$\tilde{K}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*

(g) *$K(x) + x$ is strictly increasing on $(-\infty, b^\star]$.*

(h) *If $x > y$ and $b^\star > y$, then $\tilde{K}(x) + x > \tilde{K}(y) + y$.*

(i) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{K}} = a$ where $x_{\tilde{K}} < (\geq)\, x \Leftrightarrow \tilde{K}(x) < (=)\, 0 \Rightarrow \tilde{K}(x) < (\geq)\, 0$.*

(j) *Let $\beta < 1$ or $s > 0$.*

1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}} < (= (>))\, x \Leftrightarrow \tilde{K}(x) < (= (>))\, 0$.*
2. *$(\lambda\beta b + s)/\delta \geq (<)\, b^\star \Leftrightarrow x_{\tilde{K}} = (<)\, (\lambda\beta b + s)/\delta$.*
3. *Let $\tilde{\kappa} < (= (>))\, 0$. Then $x_{\tilde{K}} < (= (>))\, 0$.* □

● *Proof* (a-c) Evident from (5.1.34(p.27)) and Lemma A 3.7(p.300) (a-c).

(d) Evident from Lemma A 3.7(p.300) (b) and (5.1.34(p.27)).

(e) From (5.1.34(p.27)) we have

$$\tilde{K}(x)+x \;=\; \lambda\beta\tilde{T}(x)+\beta x+s=\lambda\beta(\tilde{T}(x)+x)+(1-\lambda)\beta x+s\cdots\mathbf{(1)},$$

hence the assertion is immediate from Lemma A 3.7(p.300) (d).

(f) Evident from (1) and Lemma A 3.7(p.300) (d).

(g) Evident from (1) and Lemma A 3.7(p.300) (e).

(h) Let $x>y$ and $b^\star>y$. If $x\geq b^\star$, then $\tilde{K}(x)+x\geq\tilde{K}(b^\star)+b^\star>\tilde{K}(y)+y$ due to (e,g), and if $b^\star>x$, then $b^\star>x>y$, hence $\tilde{K}(x)+x>\tilde{K}(y)+y$ due to (g).

(i) Let $\beta=1$ and $s=0$. Then, since $\tilde{K}(x)=\lambda\tilde{T}(x)$ due to (5.1.34(p.27)), from Lemma A 3.7(p.300) (g) we have $\tilde{K}(x)=0$ for $a\geq x$ and $\tilde{K}(x)<0$ for $x>a$, so $x_{\tilde{K}}=a$ by the definition of $x_{\tilde{K}}$ (See Section 5.2(p.27) (b)). Hence $x_{\tilde{K}}<(\geq)\,x\Rightarrow\tilde{K}(x)<(=)\,0$. The inverse is immediate by contraposition. In addition, since $\tilde{K}(x)=0\Rightarrow\tilde{K}(x)\geq 0$, we have $\tilde{K}(x)<(=)\,0\Rightarrow\tilde{K}(x)<(\geq)\,0$.

(j) Let $\beta<1$ or $s>0$.

(j1) First note (A 3.3 (2) (p.300)). Then, if $\beta=1$, then $s>0$, hence $\tilde{K}(x)=s>0$ for any $x\leq a$ and if $\beta<1$, then $\tilde{K}(x)>0$ for any sufficiently small $x<0$ such that $x<a$. Hence, whether $\beta=1$ or $\beta<1$ (for any $0<\beta\leq 1$), we have $\tilde{K}(x)>0$ for any sufficiently small x. Next, for any sufficiently large $x>0$ such that $x\geq b^\star$, from (A 3.2 (1) (p.300)) we have $\tilde{K}(x)<0$ since to $\delta>0$ due to (10.2.2 (1) (p.56)). Hence, it follows that there exists the solution $x_{\tilde{K}}$ for any $0<\beta\leq 1$. Let $\beta<1$. Then, the solution is unique due to (d). Let $\beta=1$, hence $s>0$. Then, since $\tilde{K}(a)=s>0$ from (A 3.3 (2) (p.300)), we have $x_{\tilde{K}}>a$, hence $\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x=x_{\tilde{K}}$ due to (c), implying that the solution $x_{\tilde{K}}$ is unique. Therefore, for any $0<\beta\leq 1$ the solution is unique. Thus, the latter half is immediate.

(j2) Let $(\lambda\beta b+s)/\delta\geq(<)\,b^\star$. Then, from (A 3.2 (1(2)) (p.300)) we have $\tilde{K}((\lambda\beta b+s)/\delta)=(<)\,\lambda\beta b+s-\delta(\lambda\beta b+s)/\delta=0$, hence $x_{\tilde{K}}=(<)\,(\lambda\beta b+s)/\delta$ due to (j1). Its inverse is also true by contraposition.

(j3) If $\tilde{\kappa}<(=(>))\,0$, then $\tilde{K}(0)<(=(>))\,0$ from (5.1.37(p.27)), hence $x_{\tilde{K}}<(=(>))\,0$ from (j1). ∎

The corollary below is used when it is not specified whether $s>0$ or $s=0$.

Corollary A 3.3 ($\tilde{K}_{\mathbb{P}}$)

(a) $x_{\tilde{K}}<(\geq)\,x\Leftrightarrow\tilde{K}(x)<(\geq)\,0$.

(b) $x_{\tilde{K}}\leq(\geq)\,x\Rightarrow\tilde{K}(x)\leq(\geq)\,0$. □

● *Proof* (a) Clearly $x_{\tilde{K}}<(\geq)\,x\Rightarrow\tilde{K}(x)<(\geq)\,0$ due to Lemma A 3.9(p.301) (i,j1). The inverse is immediate by contraposition.

(b) Since $x_{\tilde{K}}<(\geq)\,x\Rightarrow\tilde{K}(x)<(\geq)\,0$ due to (a) and since $\tilde{K}(x)<(\geq)\,0\Rightarrow\tilde{K}(x)\leq(\geq)\,0$, we have $x_{\tilde{K}}<(\geq)\,x\Rightarrow\tilde{K}(x)\leq(\geq)\,0$. In addition, if $x_{\tilde{K}}=x$, then $\tilde{K}(x)=\tilde{K}(x_{\tilde{K}})=0\leq 0$, hence it follows that $x_{\tilde{K}}\leq(\geq)\,x\Rightarrow\tilde{K}(x)\leq(\geq)\,0$. ∎

Lemma A 3.10 ($\tilde{L}_{\mathbb{P}}/\tilde{K}_{\mathbb{P}}$)

(a) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}=a$.

(b) *Let* $\beta=1$ *and* $s>0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}$.

(c) *Let* $\beta<1$ *and* $s=0$. *Then* $a<(=(>))\,0\Rightarrow x_{\tilde{L}}<(=(>))\,x_{\tilde{K}}<(=(=))\,0$.

(d) *Let* $\beta<1$ *and* $s>0$. *Then* $\tilde{\kappa}<(=(>))\,0\Rightarrow x_{\tilde{L}}<(=(>))\,x_{\tilde{K}}<(=(>))\,0$. □

● *Proof* (a) If $\beta=1$ and $s=0$, then $x_{\tilde{L}}=a$ from Lemma A 3.8(p.301) (d) and $x_{\tilde{K}}=a$ from Lemma A 3.9(p.301) (i), hence $x_{\tilde{L}}=x_{\tilde{K}}=a$.

(b) Let $\beta=1$ and $s>0$. Then $\tilde{K}(x_{\tilde{L}})=0$ from (A 3.6 (1) (p.301)), hence $x_{\tilde{K}}=x_{\tilde{L}}$ from Lemma A 3.9(p.301) (j1).

(c) Let $\beta<1$ and $s=0$. Then $x_{\tilde{L}}=a\cdots\mathbf{(1)}$ from Lemma A 3.8(p.301) (d). Suppose $a<0$. Then, since $x_{\tilde{L}}<0$, we have $\tilde{K}(x_{\tilde{L}})>0$ from (A 3.6 (1) (p.301)), hence $x_{\tilde{K}}>x_{\tilde{L}}$ from Lemma A 3.9(p.301) (j1). Furthermore, from (5.1.37(p.27)) and (5.1.36(p.27)) we have $\tilde{K}(0)=\lambda\beta\tilde{T}(0)<0$ due to Lemma A 3.7(p.300) (g), hence $x_{\tilde{K}}<0$ from Lemma A 3.9(p.301) (j1). Suppose $a=(>)\,0$. Then, since $x_{\tilde{L}}=(>)\,0$ from (1), we have $\tilde{K}(x_{\tilde{L}})=(<)\,0$ due to (A 3.6 (1) (p.301)), hence $x_{\tilde{L}}=(>)\,x_{\tilde{K}}$ from Lemma A 3.9(p.301) (j1). Furthermore, from (5.1.37(p.27)) and (5.1.36(p.27)) we have $\tilde{K}(0)=\lambda\beta\tilde{T}(0)=0$ due to Lemma A 3.7(p.300) (g), hence $x_{\tilde{K}}=(=)\,0$ from Lemma A 3.9(p.301) (j1).

(d) Let $\beta<1$ and $s>0$. Since $\tilde{\kappa}=\tilde{K}(0)$ from (5.1.37(p.27)), if $\tilde{\kappa}<(=(>))\,0$, then $\tilde{K}(0)<(=(>))\,0$, hence $x_{\tilde{K}}<(=(>))\,0$ from Lemma A 3.9(p.301) (j1). Accordingly $\tilde{L}(x_{\tilde{K}})<(=(>))\,0$ from (A 3.6 (2) (p.301)), so $x_{\tilde{L}}<(=(>))\,x_{\tilde{K}}$ from Lemma A 3.8(p.301) (e1). ∎

Lemma A 3.11 ($\tilde{\mathcal{L}}_{\mathbb{P}}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s.

(b) *If* $\lambda\beta<1$, *then* $\tilde{\mathcal{L}}(s)$ *is strictly increasing in* s.

(c) *Let* $\lambda\beta b\leq a$.

1. $x_{\tilde{\mathcal{L}}}\geq\lambda\beta b+s$.
2. *Let* $s>0$ *and* $\lambda\beta<1$. *Then* $x_{\tilde{\mathcal{L}}}>\lambda\beta b+s$.

(d) *Let* $\lambda\beta b>a$. *Then, there exists a* $s_{\tilde{\mathcal{L}}}>0$ *such that if* $s_{\tilde{\mathcal{L}}}>(\leq)\,s$, *then* $x_{\tilde{\mathcal{L}}}<(\geq)\,\lambda\beta b+s$. □

● *Proof* (a,b) From (5.1.35(p.27)) and (5.1.33(p.27)) we have $\tilde{\mathcal{L}}(s) = \lambda\beta\tilde{T}(\lambda\beta b + s) + s \cdots$ **(1)**, hence the assertions are true from Lemma A 3.7(p.300) (m).

(c) Let $\lambda\beta\mu \le a$. Then, from (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) = 0 \cdots$ **(2)** due to Lemma A 3.7(p.300) (g).

(c1) Since $s \ge 0$, from (a) we have $\tilde{\mathcal{L}}(s) \ge \tilde{\mathcal{L}}(0) = 0$ due to (2) or equivalently $\tilde{L}(\lambda\beta b + s) \ge 0$, hence $x_{\tilde{L}} \ge \beta b + s$ from Corollary A 3.2(p.301) (a).

(c2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (b) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0$ due (2), hence $\tilde{L}(\lambda\beta b + s) > 0$, so $x_{\tilde{L}} > \lambda\beta b + s$ from Lemma A 3.8(p.301) (e1).

(d) Let $\lambda\beta b > a$. From (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) < 0$ due to Lemma A 3.7(p.300) (g). Noting (A 3.1 (1) (p.300)), for any sufficiently large $s > 0$ such that $\lambda\beta b + s \ge b^\star$ and $\lambda\beta b + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta b + s) = \lambda\beta b + s - \lambda\beta(\lambda\beta b + s) = (1-\lambda\beta)(\lambda\beta b + s) \ge 0$. Accordingly, due to (a) it follows that there exists a $s_{\tilde{\mathcal{L}}} > 0$ where $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$, or equivalently, $\tilde{L}(\lambda\beta b + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta b + s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 3.2(p.301) (a) we have $x_{\tilde{L}} < \beta b + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \ge \beta b + s$ for $s \ge s_{\tilde{\mathcal{L}}}$. ∎

Lemma A 3.12 ($\mathscr{A}\{\tilde{\kappa}_{\mathbb{P}}\}$) *We have*:

(a) $\tilde{\kappa} = \lambda\beta b + s$ *if* $b^\star < 0$ *and* $\tilde{\kappa} = s$ *if* $a > 0$.

(b) *Let* $\beta < 1$ *or* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$. □

● *Proof* (a) Immediate from (5.1.36(p.27)) and Lemma A 3.7(p.300) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} > (= (<))\ 0$, we have $\tilde{K}(0) > (= (<))\ 0$ from (5.1.37(p.27)), hence $x_{\tilde{K}} > (= (<))\ 0$ from Lemma A 3.9(p.301) (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 4 Direct Proof of Assertion Systems

A 4.1 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Since $\tilde{K}(x) + (1-\beta)x = \tilde{L}(x)$ for any x due to (5.1.14(p.25)) and (5.1.13(p.25)), from (6.4.4(p.41)) we have Accordingly:

1. If $\tilde{L}(V_{t-1}) \le 0$, then $V_t - \beta V_{t-1} = \tilde{L}(V_{t-1})$, hence $V_t - \beta V_{t-1} = \min\{\tilde{L}(V_{t-1}), 0\}, \quad t > 1.$ (A 4.1)

$$V_t = \tilde{L}(V_{t-1}) + \beta V_{t-1} = \tilde{K}(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.2}$$

2. If $\tilde{L}(V_{t-1}) \ge 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.3}$$

Now, from (6.4.4(p.41)) with $t = 2$ we have

$$V_2 - V_1 = \min\{\tilde{K}(V_1), -(1-\beta)V_1\}. \tag{A 4.4}$$

Finally, from (A 4.1(p.303)) we see that

$$\tilde{L}(V_{t-1}) < (>)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle})^\dagger. \tag{A 4.5}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.22) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.22})), \qquad \delta = 1 \cdots (2) \quad (\text{see } (10.2.1(\text{p.56}))). \tag{A 4.6}$$

□ **Tom A 4.1** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.14(p.25)) we have $\tilde{K}(x) = \tilde{T}(x) \le 0 \cdots$ **(1)** for any x due to Lemma A 1.1(p.289) (g), hence from (6.4.4(p.41)) and (1) we have $V_t = \min\{\tilde{T}(V_{t-1}) + V_{t-1}, V_{t-1}\} = \min\{\tilde{T}(V_{t-1}), 0\} + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1} \cdots$ **(2)** for $t > 1$.

(a) Since $V_2 = \tilde{T}(V_1) + V_1$, we have $V_2 \le V_1$ due to (1). Suppose $V_{t-1} \ge V_t$. Then, from Lemma A 1.1(p.289) (d) we have $V_t \ge \tilde{T}(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$.

(b) Since $V_1 = \mu$ from (6.4.3(p.41)), we have $V_1 > a$. Suppose $V_{t-1} > a$. Then, noting $b > a$, from (2) we have $V_t > \tilde{T}(a) + a = a$ due to Lemma A 1.1(p.289) (l,g). Accordingly, by induction $V_{t-1} > a$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ due to Lemma A 1.2(p.291) (d), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Lemma A 1.2(p.291) (e1)), so $\tilde{L}(V_{t-1}) < 0 \cdots$ **(3)** for $\tau \ge t > 1$. Hence, from (A 4.1(p.303)) we obtain $V_t - \beta V_{t-1} < 0$ for $\tau \ge t > 1$, i.e., $V_t < \beta V_{t-1}$ for $\tau \ge t > 1$. Accordingly $V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-1}V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$ due to (3) and (A 4.5(p.303)). ∎

Let us define

$$\texttt{S}_{19}\boxed{ⓢ\blacktriangle \mid ◎\ \|} = \left\{ \begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1) \quad \boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \texttt{Conduct}_{\tau\ge t>1\blacktriangle}, \\ (2) \quad \boxed{◎\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \texttt{Conduct}_{\tau\ge t>1\blacktriangle}. \end{array} \right\}$$

†See Section 6.1(p.29).

□ Tom **A 4.2** ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A]$\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
   - ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.*
   - i. *Let $a < 0$ $(\!(\tilde{\kappa} < 0)\!)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
   - ii. *Let $a = 0$ $(\!(\tilde{\kappa} = 0)\!)$.*
     1. *Let $\beta\mu + s \ge b$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
     2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
   - iii. *Let $a > 0$ $(\!(\tilde{\kappa} > 0)\!)$.*
     1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\texttt{S}_{19}$(p.303) $\boxed{\circ\blacktriangle \mid *\|}$ *is true.* □

● *Proof* Let $\beta < 1$ or $s > 0$. Note here (A 4.6 (1,2) (p.303)).

(a) Since $x_{\tilde{K}} \le (\beta\mu + s)/\delta = \beta\mu + s = V_1$ due to Lemma A 1.3(p.291) (j2) and (6.4.3(p.41)), we have $\tilde{K}(V_1) \le 0$ due to Lemma A 1.3(p.291) (j1), hence $V_2 - V_1 \le 0$ from (A 4.4(p.303)), i.e., $V_1 \ge V_2$. Suppose $V_{t-1} \ge V_t$. Then, from (6.4.4(p.41)) and Lemma A 1.3(p.291) (e) we have $V_t \ge \min\{\tilde{K}(V_t)+V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta\mu + s \ge M$ and $a \ge M$, hence $V_1 \ge M$. Suppose $V_{t-1} \ge M$. Then, from Lemma A 1.3(p.291) (e) and (A 1.5 (2) (p.291)) we have $V_t \ge \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \ge \min\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Hence, by induction $V_t \ge M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.4.4(p.41)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \ge 0$, we have $V \le x_{\tilde{K}}$ from Lemma A 1.3(p.291) (j1).

(b) Let $\beta\mu \le a \cdots$ **(1)**. Then $x_{\tilde{L}} \ge \beta\mu + s = V_1$ from Lemma A 1.5(p.293) (b1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 1.1(p.291) (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > 1$. Hence, from (A 4.3(p.303)) we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^1$. Hence $t_\tau^* = 1$ for $\tau > 1$ (see Preference Rule 7.2.1(p.45)), i.e., $\boxed{\bullet\, \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c) Let $\beta\mu > a$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" of the lemma. Then $(\lambda\beta\mu+s)/\delta = \mu+s \cdots$ **(3)** due to (2) and (A 4.6 (1,2) (p.303)). In addition, since $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(4)** from Lemma A 1.4(p.292) (b), we have $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots$ **(5)**.

(c1i) Let $\mu + s \ge b$. Then $x_{\tilde{L}} = x_{\tilde{K}} = \mu + s = V_1$ from (4), Lemma A 1.3(p.291) (j2), (3), and (6.4.3(p.41)). Accordingly, since $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Lemma A 1.2(p.291) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\, \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c1ii) Let $\mu + s < b$. Then $x_{\tilde{L}} = x_{\tilde{K}} < \mu + s = V_1 < b$ from (4), Lemma A 1.3(p.291) (j2), and (6.4.3(p.41)), hence $b > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Lemma A 1.2(p.291) (e1). Then, from (A 4.2(p.303)), Lemma A 1.3(p.291) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Lemma A 1.2(p.291) (e1). Thus, for the same reason as in the proof of Tom A 4.1(p.303) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.

(c2i) Let $a < 0$ $(\!(\tilde{\kappa} < 0)\!)$. Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots$ **(6)** from Lemma A 1.4(p.292) (c $(\!($d$)\!)$). Now, since $x_{\tilde{K}} \le \beta\mu + s$ due to Lemma A 1.3(p.291) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \le V_1$ from (6.4.3(p.41)). Suppose $x_{\tilde{K}} \le V_{t-1}$. Then, from Lemma A 1.3(p.291) (e) we have $V_t \ge \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \ge x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.291) (a). Hence, for the same reason as in the proof of Tom A 4.1(p.303) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{CONDUCT}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii) Let $a = 0$ $(\!(\tilde{\kappa} = 0)\!)$. Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(7)** from Lemma A 1.4(p.292) (c $(\!($d$)\!)$).

(c2ii1) Let $\beta\mu + s \ge b$. Then, $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.291) (j2) and (6.4.3(p.41)). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.2(p.303)) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.3(p.303)), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\, \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2ii2) Let $\beta\mu + s < b$. Then, since $V_1 < b$ from (6.4.3(p.41)), we have $V_{t-1} < b$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta\mu + s = V_1$ from Lemma A 1.3(p.291) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Lemma A 1.2(p.291) (e1), from (A 4.2(p.303)) and Lemma A 1.3(p.291) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.291) (a), for the same reason as in the proof of Tom A 4.1(p.303) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{\tau\blacktriangle}$ for $\tau \ge t > 1$.

(c2iii) Let $a > 0$ $(\!(\tilde{\kappa} > 0)\!)$. Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots$ **(8)** from Lemma A 1.4(p.292) (c $(\!($d$)\!)$).

(c2iii1) Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. First, let $\beta\mu + s \geq b$. Then, since $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.291) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \geq V_1$. Next, let $s_{\tilde{\mathcal{L}}} \leq s$. Then, since $x_{\tilde{L}} \geq \beta\mu + s$ due to Lemma A 1.5(p.293) (c), we have $x_{\tilde{L}} \geq V_1$ from (6.4.3(p.41)). Accordingly, whether $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$, we have $x_{\tilde{L}} \geq V_1$, so $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ from Corollary A 1.1(p.291) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta\mu+s < b \cdots$ **(**9**)** and $s < s_{\tilde{\mathcal{L}}}$. Then, from (8) and Lemma A 1.5(p.293) (c) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta\mu+s = V_1 \cdots$ **(**10**)**, hence $\tilde{K}(V_1) < 0 \cdots$ **(**11**)** from Lemma A 1.3(p.291) (j1). In addition, since $V_1 < b$ due to (9) and (6.4.3(p.41)), we have $V_{t-1} < b$ for $t > 0$ from (a). Now, from (A 4.4(p.303)) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from (6.4.4(p.41)) and Lemma A 1.3(p.291) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10), so $V_1 \geq x_{\tilde{L}}$. Assume that $V_{t-1} \geq x_{\tilde{L}}$ for *all* $t > 1$, hence $V \geq x_{\tilde{L}}$. Now, from (8) and $V \leq x_{\tilde{K}}$ in (a) we have the contradiction of $V \leq x_{\tilde{K}} < x_{\tilde{L}} \leq V$. Hence, it is impossible that $V_{t-1} \geq x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 > V_2 > \cdots > V_{t^\bullet_\tau - 1} > x_{\tilde{L}} \geq V_{t^\bullet_\tau} > V_{t^\bullet_\tau+1} > V_{t^\bullet_\tau+2} > \cdots, \tag{A 4.7}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t^\bullet_\tau \geq t > 1, \qquad x_{\tilde{L}} \geq V_{t-1}, \quad t > t^\bullet_\tau. \tag{A 4.8}$$

Therefore, from Corollary A 1.1(p.291) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots$ **(**12**)** for $t^\bullet_\tau \geq t > 1$ and $\tilde{L}(V_{t-1}) \geq 0 \cdots$ **(**13**)** for $t > t^\bullet_\tau$.

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots$ **(**14**)** for $\tau \geq t > 1$ from (12), for the same reason as in the proof of `Tom` A 4.1(p.303) (b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $t^\bullet_\tau \geq \tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\texttt{S}_{19}$(p.303) (1) is true.
2. Let $\tau > t^\bullet$. First, let $\tau \geq t > t^\bullet_\tau$. Then, since $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (A 4.3(p.303)), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots \textbf{(}15\textbf{)}.$$

Next, let $t^\bullet_\tau \geq t > 1$. Then, from (12) and (A 4.1(p.303)) we have $V_t - \beta V_{t-1} < 0$ for $t^\bullet_\tau \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence

$$V_{t^\bullet_\tau} < \beta V_{t^\bullet_\tau - 1} < \beta^2 V_{t^\bullet_\tau - 2} < \cdots < \beta^{t^\bullet_\tau - 1} V_1 \cdots \textbf{(}16\textbf{)}.$$

From (15) and (16) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} < \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} < \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} < \cdots < \beta^{\tau - 1} V_1,$$

hence we obtain $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$ due to Preference Rule 7.2.1(p.45), i.e., $\boxed{\circledS\ \texttt{ndOIT}_\tau\langle t^\bullet_\tau\rangle}_{\|}$ for $\tau > 1$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (12) and (A 4.5(p.303)). Hence $\texttt{S}_{19}$(p.303) (2) is true. ∎

A 4.2 $\mathscr{A}\{$M:1[ℙ][A]$\}$

Since $K(x) + (1-\beta)x = L(x)$ for any x due to (5.1.21(p.26)) and (5.1.20(p.26)), from (6.4.6(p.41)) we have

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\} \geq 0, \quad t > 1. \tag{A 4.9}$$

Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence
$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.10}$$
2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently
$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.11}$$

Now, from (6.4.6(p.41)) with $t = 2$ we have

$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{A 4.12}$$

Finally, from (A 4.9(p.305)) we see that

$$L(V_{t-1}) > (<)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{A 4.13}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.22) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2\text{(p.22)}), \qquad \delta = 1 \cdots (2) \quad (\text{see } (10.2.1\text{(p.56)})). \tag{A 4.14}$$

□ `Tom` **A 4.3** **(**$\mathscr{A}\{$M:1[ℙ][A]$\}$**)** *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau \geq t > 1\blacktriangle}$. □

● *Proof* Let $\beta=1$ and $s=0$. Then, from (5.1.21(p.26)) we have $K(x)=T(x)\geq 0\cdots(1)$ for any x due to Lemma A 2.1(p.293) (g), hence from (6.4.6(p.41)) and (1) we have
$V_t=\max\{T(V_{t-1})+V_{t-1},V_{t-1}\}=\max\{T(V_{t-1}),0\}+V_{t-1}=T(V_{t-1})+V_{t-1}\cdots(2)$ for $t>1$.

(a) Since $V_2=T(V_1)+V_1$, we have $V_2\geq V_1$ due to (1). Suppose $V_{t-1}\leq V_t$. Then, from Lemma A 2.1(p.293) (d) we have $V_t\leq T(V_t)+V_t=V_{t+1}$. Hence, by induction $V_{t-1}\leq V_t$ for $t>1$, i.e., V_t is nondecreasing in $t>0$.

(b) Since $V_1=a$ from (6.4.5(p.41)), we have $V_1<b$. Suppose $V_{t-1}<b$. Then, noting $a^\star<a<b$ due to Lemma A 2.1(p.293) (n), from (2) we have $V_t<T(b)+b=b$ due to Lemma A 2.1(p.293) (c,g). Accordingly, by induction $V_{t-1}<b$ for $t>1$, hence $L(V_{t-1})>0$ for $t>1$ due to Lemma A 2.2(p.294) (d), so $L(V_{t-1})>0\cdots(3)$ for $\tau\geq t>1$. Hence, from (A 4.9(p.305)) we obtain $V_t-\beta V_{t-1}>0$ for $\tau\geq t>1$, i.e., $V_t>\beta V_{t-1}$ for $\tau\geq t>1$. Accordingly $V_\tau>\beta V_{\tau-1}>\cdots>\beta^{\tau-1}V_1$, hence $t^*_\tau=\tau$ for $\tau>1$, i.e., $\boxed{Ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau>1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau\geq t>1$ due to (3) and (A 4.13(p.305)). ∎

Let us define

$$\texttt{S}_{20}\boxed{Ⓢ\blacktriangle \mid ◎\ \|}=\left\{\begin{array}{l}\text{For any } \tau>1 \text{ there exists } t^\bullet_\tau>1 \text{ such that}\\ (1)\quad \boxed{Ⓢ\ \texttt{dOITs}_{t^\bullet_\tau\geq\tau>1}\langle\tau\rangle}_\blacktriangle \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle},\\ (2)\quad \boxed{◎\ \texttt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}.\end{array}\right\}$$

□ **Tom A 4.4** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta<1$ or $s>0$.*

(a) *V_t is nondecreasing in $t>0$ and converges to a finite $V\geq x_K$ as $t\to\infty$.*

(b) *Let $\beta a\geq b$. Then* $\boxed{●\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.

(c) *Let $\beta a<b$.*

1. *Let $\beta=1$.*
   i. *Let $a-s\leq a^\star$. Then* $\boxed{●\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
   ii. *Let $a-s>a^\star$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta<1$ and $s=0$ ($s>0$).*
   i. *Let $b>0$ ($\kappa>0$). Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $b=0$ ($\kappa=0$).*
      1. *Let $\beta a-s\leq a^\star$. Then* $\boxed{●\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta a-s>a^\star$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $b<0$ ($\kappa<0$).*
      1. *Let $\beta a-s\leq a^\star$ or $s_{\mathcal{L}}\leq s$. Then* $\boxed{●\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta a-s>a^\star$ and $s_{\mathcal{L}}>s$. Then* $\texttt{S}_{20}$(p.306) $\boxed{○\blacktriangle \mid *\|}$ *is true.* □

● *Proof* Let $\beta<1$ or $s>0$. First note (A 4.14(p.305))

(a) Since $x_K\geq(\lambda\beta a-s)/\delta=\beta a-s=V_1$ due to Lemma A 2.3(p.294) (j2) and (6.4.5(p.41)), we have $K(V_1)\geq 0$ due to Lemma A 2.3(p.294) (j1), hence $V_2-V_1\geq 0$ from (A 4.12(p.305)), i.e., $V_1\leq V_2$. Suppose $V_{t-1}\leq V_t$. Then, from (6.4.6(p.41)) and Lemma A 2.3(p.294) (e) we have $V_t\leq\max\{K(V_t)+V_t,\beta V_t\}=V_{t+1}$. Hence, by induction $V_{t-1}\leq V_t$ for $t>1$, i.e., V_t is nondecreasing in $t>0$. Consider a sufficiently large $M>0$ such that $\beta a-s\leq M$ and $b\leq M$, hence $V_1\leq M$. Suppose $V_{t-1}\leq M$. Then, from Lemma A 2.3(p.294) (e) and (A 2.5 (2) (p.294)) we have $V_t\leq\max\{K(M)+M,\beta M\}=\max\{\beta M-s,\beta M\}\leq\max\{M,M\}=M$ due to $\beta\leq 1$ and $s\geq 0$. Hence, by induction $V_t\leq M$ for $t>0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t\to\infty$. Then, from (6.4.6(p.41)) we have $V=\max\{K(V)+V,\beta V\}$, hence $0=\max\{K(V),-(1-\beta)\beta V\}$. Thus, since $K(V)\leq 0$, we have $V\geq x_K$ from Lemma A 2.3(p.294) (j1).

(b) Let $\beta a\geq b\cdots(1)$. Then $x_L\leq\beta a-s=V_1$ from Lemma A 2.5(p.296) (b1) with $\lambda=1$ and $\delta=1$, hence $x_L\leq V_{t-1}$ for $t>1$ from (a). Accordingly, since $L(V_{t-1})\leq 0$ for $t>1$ due to Corollary A 2.1(p.294) (a), we have $L(V_{t-1})\leq 0$ for $\tau\geq t>1$. Hence, from (A 4.11(p.305)) we have $V_t=\beta V_{t-1}$ for $\tau\geq t>1$. Thus $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1$, i.e., $I^\tau_\tau=I^{\tau-1}_\tau=\cdots=I^1_\tau$, hence $t^*_\tau=1$ for $\tau>1$ due to Preference Rule 7.2.1(p.45), i.e., $\boxed{●\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau>1$.

(c) Let $\beta a<b$.

(c1) Let $\beta=1\cdots(2)$, hence $s>0$ due to the assumption "$\beta<1$ or $s>0$" of the lemma. Then $(\lambda\beta a-s)/\delta=a-s\cdots(3)$ due to (2) and (A 4.14 (2) (p.305)). In addition, since $x_L=x_K\cdots(4)$ from Lemma A 2.4(p.295) (b), we have $K(x_L)=K(x_K)=0\cdots(5)$.

(c1i) Let $a-s\leq a^\star$. Then $x_L=x_K=a-s=V_1$ from (4), Lemma A 2.3(p.294) (j2), (3), and (6.4.5(p.41)). Accordingly, since $x_L\leq V_{t-1}$ for $t>1$ from (a), we have $L(V_{t-1})\leq 0$ for $t>1$ due to Lemma A 2.2(p.294) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{●\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau>1$.

(c1ii) Let $a-s>a^\star$. Then $x_L=x_K>a-s=V_1>a^\star$ from (4), Lemma A 2.3(p.294) (j2), and (6.4.5(p.41)), hence $a^\star<V_{t-1}$ for $t>1$ from (a). Suppose $V_{t-1}<x_L$, hence $L(V_{t-1})>0$ from Lemma A 2.2(p.294) (e1). Then, from (A 4.10(p.305)), Lemma A 2.3(p.294) (g), and (4) we have $V_t<K(x_L)+x_L=K(x_K)+x_L=x_L$. Accordingly, by induction $V_{t-1}<x_L$ for $t>1$, hence $L(V_{t-1})>0$ for $t>1$ from Lemma A 2.2(p.294) (e1). Thus, for the same reason as in the proof of Tom A 4.3(p.305) (b) we have $\boxed{Ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau>1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau\geq t>1$.

(c2) Let $\beta < 1$ and $s = 0 \,(\!(s > 0)\!)$.

(c2i) Let $b > 0 \,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots (6)$ from Lemma A 2.4(p.295) (c (d)). Now, since $x_K \geq \beta a - s$ due to Lemma A 2.3(p.294) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_K \geq V_1$ from (6.4.5(p.41)). Suppose $x_K \geq V_{t-1}$. Then, from Lemma A 2.3(p.294) (e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (6), thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.294) (a). Hence, for the same reason as in the proof of Tom A 4.3(p.305) (b) we have ⓢ dOITs$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$ and conduct$_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2ii) Let $b = 0 \,(\!(\kappa = 0)\!)$. Then $x_L = x_K \cdots (7)$ from Lemma A 2.4(p.295) (c (d)).

(c2ii1) Let $\beta a - s \leq a^\star$. Then, $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.294) (j2) and (6.4.5(p.41)). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (7), thus $L(V_{t-1}) = L(x_L) = 0$. Then, from (A 4.10(p.305)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (7). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.11(p.305)), hence, for the same reason as in the proof of (b) we obtain ● dOITd$_\tau\langle 1\rangle_{\|}$ for $\tau > 1$.

(c2ii2) Let $\beta a - s > a^\star$. Then, since $V_1 > a^\star$, we have $V_{t-1} > a^\star$ for $t > 1$ due to (a). In addition, we have $x_K > \beta a - s = V_1$ from Lemma A 2.3(p.294) (j2) and (6.4.5(p.41)). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (7). Then, since $L(V_{t-1}) > 0$ due to Corollary A 2.1(p.294) (a), from (A 4.10(p.305)) and Lemma A 2.3(p.294) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, thus $x_L > V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.294) (a), for the same reason as in the proof of Tom A 4.3(p.305) (b) we have ⓢ dOITs$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$ and Conduct$_{\tau\blacktriangle}$ for $\tau \geq t > 1$.

(c2iii) Let $b < 0 \,(\!(\kappa < 0)\!)$. Then $x_L < x_K \cdots (8)$ from Lemma A 2.4(p.295) (c (d)).

(c2iii1) Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. First, let $\beta a - s \leq a^\star$. Then, since $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.294) (j2), we have $x_L < V_1$ from (8), hence $x_L \leq V_1$. Next, let $s_{\mathcal{L}} \leq s$. Then, since $x_L \leq \beta a - s$ due to Lemma A 2.5(p.296) (c), we have $x_L \leq V_1$ and (6.4.5(p.41)). Accordingly, whether $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, so $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary A 2.1(p.294) (a), for the same reason as in the proof of (b) we obtain ● dOITd$_\tau\langle 1\rangle_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta a - s > a^\star \cdots (9)$ and $s < s_{\mathcal{L}}$. Then, from (8) and Lemma A 2.5(p.296) (c) we have $x_K > x_L > \beta a - s = V_1 \cdots (10)$, hence $K(V_1) > 0 \cdots (11)$ from Lemma A 2.3(p.294) (j1). In addition, since $V_1 > a^\star$ due to (9), we have $V_{t-1} > a^\star$ for $t > 0$ from (a). Now, from (A 4.12(p.305)) and (11) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from (6.4.6(p.41)) and Lemma A 2.3(p.294) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (10). Assume that $V_{t-1} \leq x_L$ for *all* $t > 1$, hence $V \leq x_L$. Now, from (8) and $V \geq x_K$ due to (a) we have the contradiction $V \geq x_K > x_L \geq V$. Hence, it is impossible that $V_{t-1} \leq x_L$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < V_{t^\bullet_\tau + 2} < \cdots, \tag{A 4.15}$$

from which

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t-1}, \quad t > t^\bullet_\tau. \tag{A 4.16}$$

Therefore, from Corollary A 2.1(p.294) (a) we have $L(V_{t-1}) > 0 \cdots (12)$ for $t^\bullet_\tau \geq t > 1$ and $L(V_{t-1}) \leq 0 \cdots (13)$ for $t > t^\bullet_\tau$.

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots (14)$ for $\tau \geq t > 1$ from (12), for the same reason as in the proof of Tom A 4.3(p.305) (b) we have ⓢ dOITs$_\tau\langle\tau\rangle_\blacktriangle$ for $t^\bullet_\tau \geq \tau > 1$ and Conduct$_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence S_{20}(p.306) (1) is true.
2. Let $\tau > t^\bullet_\tau$. Firstly, let $\tau \geq t > t^\bullet_\tau$. Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (A 4.11(p.305)), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (15).$$

Next, let $t^\bullet_\tau \geq t > 1$. Then, from (12) and (A 4.9(p.305)) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet_\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t^\bullet \geq t > 1$, hence

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \beta^2 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (16).$$

From (15) and (16) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau - 1} V_1,$$

hence we obtain $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$ due to Preference Rule 7.2.1(p.45), i.e., ◎ ndOIT$_\tau\langle t^\bullet_\tau\rangle_{\|}$ for $\tau > t^\bullet_\tau$. In addition, we have Conduct$_{t\blacktriangle}$ for $t^\bullet \geq t > 1$ due to (12) and (A 4.13(p.305)). Hence S_{20}(p.306) (2) is true. ▮

A 4.3 $\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]$\}$

Since $\tilde{K}(x) + (1 - \beta)x = \tilde{L}(x)$ due to (5.1.34(p.27)) and (5.1.33(p.27)), from (6.4.8(p.41)) we have

$$V_t - \beta V_{t-1} = \min\{\tilde{L}(V_{t-1}), 0\} \leq 0, \quad t > 1. \tag{A 4.17}$$

Accordingly:

1. If $\tilde{L}(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = \tilde{L}(V_{t-1})$ or equivalently

$$V_t = \tilde{L}(V_{t-1}) + \beta V_{t-1} = \tilde{K}(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.18}$$

2. If $\tilde{L}(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.19}$$

Now, from (6.4.8(p.41)) with $t = 2$ we have

$$V_2 - V_1 = \min\{\tilde{K}(V_1), -(1-\beta)V_1\}. \tag{A 4.20}$$

Finally, from (A 4.17(p.307)) we see that

$$\tilde{L}(V_{t-1}) < (>)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_t). \tag{A 4.21}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.22) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.22})), \qquad \delta = 1 \quad (\text{see } (10.2.1(\text{p.56}))). \tag{A 4.22}$$

□ **Tom A 4.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$. □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.34(p.27)) we have $\tilde{K}(x) = \tilde{T}(x) \leq 0 \cdots$ **(1)** for any x due to Lemma A 3.7(p.300) (g), hence from (6.4.8(p.41)) and (1) we have $V_t = \min\{\tilde{T}(V_{t-1}) + V_{t-1}, V_{t-1}\} = \tilde{T}(V_{t-1}) + V_{t-1} \cdots$ **(2)** for $t > 1$.

(a) Since $V_2 = \tilde{T}(V_1) + V_1$ from (2), we have $V_2 \leq V_1$ due to (1). Suppose $V_{t-1} \geq V_t$. Then, from (2) and Lemma A 3.7(p.300) (d) we have $V_t \geq \tilde{T}(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$.

(b) Since $V_1 = b$ from (6.4.7(p.41)), we have $V_1 > a$. Suppose $V_{t-1} > a$. Then, noting $b^\star > b > a$ due to Lemma A 3.7(p.300) (n), from (2) we have $V_t > \tilde{T}(a) + a = a$ due to Lemma A 3.7(p.300) (l,g). Accordingly, by induction $V_{t-1} > a$ for $t > 1$, hence $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Lemma A 3.8(p.301) (d), thus $\tilde{L}(V_{t-1}) < 0 \cdots$ **(3)** for $\tau \geq t > 1$. Hence, from (A 4.17(p.307)) we obtain $V_t - \beta V_{t-1} < 0$ for $\tau \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly $V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (A 4.21(p.308)). ▮

Let us define

$$\mathtt{S}_{21}\boxed{\circledS\blacktriangle \,|\, \circledcirc\,\|} = \left\{\begin{array}{ll} \multicolumn{2}{l}{\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that}} \\ (1) & \boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}, \\ (2) & \boxed{\circledcirc\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}. \end{array}\right\}$$

□ **Tom A 4.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \geq b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
   - ii. *Let $b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*
     1. *Let $\beta b + s \geq b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
     2. *Let $\beta b + s < b^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau > 1\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
     1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
     2. *Let $\beta b + s < b^\star$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\mathtt{S}_{21}\boxed{\circ\blacktriangle \,|\, *\triangle}$ *is true.* □

● *Proof* Let $\beta < 1$ or $s > 0$. First note (A 4.22 (1,2) (p.308)).

(a) Since $x_{\tilde{K}} \leq (\beta b + s)/\delta = \beta b + s = V_1$ due to Lemma A 3.9(p.301) (j2) and (6.4.7(p.41)), we have $\tilde{K}(V_1) \leq 0$ due to Lemma A 3.9(p.301) (j1), hence $V_2 - V_1 \leq 0$ from (A 4.20(p.308)), i.e., $V_1 \geq V_2$. Suppose $V_{t-1} \geq V_t$. Then, from (6.4.8(p.41)) and Lemma A 3.9(p.301) (e) we have $V_t \geq \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta b + s \geq M$ and $a \geq M$, hence $V_1 \geq M$. Suppose $V_{t-1} \geq M$. Then, from Lemma A 3.9(p.301) (e) and (A 3.5 (2) (p.301)) we have $V_t \geq \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \geq \min\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \geq M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.4.8(p.41)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \geq 0$, we have $V \leq x_{\tilde{K}}$ from Lemma A 3.9(p.301) (j1).

(b) Let $\beta b \leq a \cdots$ **(1)**. Then $x_{\tilde{L}} \geq \beta b + s = V_1$ from Lemma A 3.11(p.302) (c1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ due to Corollary A 3.2(p.301) (a), we have $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > 1$. Hence, from (A 4.19(p.308)) we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$. Thus, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^1$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$ due to Preference Rule 7.2.1(p.45).

(c) Let $\beta b > a$.

(c1) Let $\beta = 1 \cdots (2)$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" of the lemma. Then, we see that $(\lambda\beta b + s)/\delta = b + s \cdots (3)$ due to (2(p.309)) and (A 4.22(p.308)). In addition, since $x_{\tilde{L}} = x_{\tilde{K}} \cdots (4)$ from Lemma A 3.10(p.302) (b), we have $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots (5)$.

(c1i) Let $b + s \ge b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} = b + s = V_1$ from (4), Lemma A 3.9(p.301) (j2, (3), and (6.4.7(p.41)). Accordingly, since $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 3.2(p.301) (a). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c1ii) Let $b + s < b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} < b + s = V_1 < b^\star$ from (4), Lemma A 3.9(p.301) (j2), and (6.4.7(p.41)), hence $b^\star > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Corollary A 3.2(p.301) (a). Then, from (A 4.18(p.307)), Lemma A 3.9(p.301) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Corollary A 3.2(p.301) (a). Thus, for the same reason as in the proof of Tom A 4.5(p.308) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(c2i) Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots (6)$ from Lemma A 3.10(p.302) (c (\!(d)\!)). Now, since $x_{\tilde{K}} \le \beta b + s$ due to Lemma A 3.9(p.301) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \le V_1$ from (6.4.7(p.41)). Suppose $x_{\tilde{K}} \le V_{t-1}$. Then, from Lemma A 3.9(p.301) (e) we have $V_t \ge \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \ge x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.301) (a). Hence, for the same reason as in the proof of Tom A 4.5(p.308) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$, and $\texttt{CONDUCT}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii) Let $a = 0\,(\!(\tilde{\kappa} = 0)\!)$. Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots (7)$ from Lemma A 3.10(p.302) (c (\!(d)\!)).

(c2ii1) Let $\beta b + s \ge b^\star$. Then, $x_{\tilde{K}} = \beta b + s = V_1$ from Lemma A 3.9(p.301) (j2) and (6.4.7(p.41)). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.18(p.307)) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.19(p.308)), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c2ii2) Let $\beta b + s < b^\star$. Then, since $V_1 < b^\star$ from (6.4.7(p.41)), we have $V_{t-1} < b^\star$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta b + s = V_1$ from Lemma A 3.9(p.301) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Corollary A 3.2(p.301) (a), from (A 4.18(p.307)) and Lemma A 3.9(p.301) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.301) (a), for the same reason as in the proof of Tom A 4.5(p.308) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2iii) Let $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots (8)$ from Lemma A 3.10(p.302) (c (\!(d)\!)).

(c2iii1) Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. First let $\beta b + s \ge b^\star$. Then, since $x_{\tilde{K}} = \beta b - s = V_1$ from Lemma A 3.9(p.301) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \ge V_1$. Next let $s_{\tilde{\mathcal{L}}} \le s$. Then, since $x_{\tilde{L}} \ge \beta b + s$ due to Lemma A 3.11(p.302) (d), we have $x_{\tilde{L}} \ge V_1$. Accordingly, whether $\beta b + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$, we have $x_{\tilde{L}} \ge V_1$, thus $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ from Corollary A 3.2(p.301) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c2iii2) Let $\beta b + s < b^\star \cdots (9)$ and $s < s_{\tilde{\mathcal{L}}}$. Then, from (8) and Lemma A 3.11(p.302) (d) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta b + s = V_1 \cdots (10)$, hence $\tilde{K}(V_1) < 0 \cdots (11)$ from Lemma A 3.9(p.301) (j1). In addition, since $V_1 < b^\star$ due to (9), we have $V_{t-1} < b^\star$ for $t > 0$ from (a). Now, from (A 4.20(p.308)) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from Lemma A 3.9(p.301) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10). Assume that $V_{t-1} \ge x_{\tilde{L}}$ for *all* $t > 1$, hence $V \ge x_{\tilde{L}}$ due to (a). Then, from (8) and $V \le x_{\tilde{K}}$ due to (a) we have the contradiction of $V \le x_{\tilde{K}} < x_{\tilde{L}} \le V$. Hence, it is impossible that $V_{t-1} \ge x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t_\tau^\bullet > 1$ such that

$$V_1 > V_2 > \cdots > V_{t_\tau^\bullet - 1} > x_{\tilde{L}} \ge V_{t_\tau^\bullet} > V_{t_\tau^\bullet + 1} > V_{t_\tau^\bullet + 2} > \cdots, \tag{A 4.23}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t_\tau^\bullet \ge t > 1, \qquad x_{\tilde{L}} \ge V_{t-1}, \quad t > t_\tau^\bullet. \tag{A 4.24}$$

Therefore, from Corollary A 3.2(p.301) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots (12)$ for $t_\tau^\bullet \ge t > 1$ and $\tilde{L}(V_{t-1}) \ge 0 \cdots (13)$ for $t > t_\tau^\bullet$.

1. Let $t_\tau^\bullet \ge \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots (14)$ for $\tau \ge t > 1$ from (12), for the same reason as in the proof of Tom A 4.5(p.308) (b) we have $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$. Hence $\texttt{S}_{21}$(p.308) (1) is true.
2. Let $\tau > t_\tau^\bullet$. Firstly, let $\tau \ge t > t_\tau^\bullet$. Then, since $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > t_\tau^\bullet$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \ge t > t_\tau^\bullet$ from (A 4.19(p.308)), thus
$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} \cdots (15).$$
Next, let $t_\tau^\bullet \ge t > 1$. Then, from (12) and (A 4.17(p.307)) we have $V_t - \beta V_{t-1} < 0$ for $t_\tau^\bullet \ge t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t_\tau^\bullet \ge t > 1$, hence
$$V_{t_\tau^\bullet} < \beta V_{t_\tau^\bullet - 1} < \beta^2 V_{t_\tau^\bullet - 2} < \cdots < \beta^{t_\tau^\bullet - 1} V_1 \cdots (16).$$

From (15) and (16) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t_\tau^\bullet} V_{t_\tau^\bullet} < \beta^{\tau-t_\tau^\bullet+1} V_{t_\tau^\bullet-1} < \beta^{\tau-t_\tau^\bullet+2} V_{t_\tau^\bullet-2} < \cdots < \beta^{\tau-1} V_1,$$

hence we obtain $t_\tau^* = t_\tau^\bullet$ for $\tau > t_\tau^\bullet$ due to Preference Rule 7.2.1(p.45), i.e., $\boxed{\circledcirc\ \texttt{ndOIT}_\tau\langle t_\tau^\bullet\rangle}$ for $\tau > t_\tau^\bullet$. In addition, we have $\texttt{Conduct}_{t_\blacktriangle}$ for $t_\tau^\bullet \geq t > 1$ due to (12) and (A 4.21(p.308)). Hence $\texttt{S}_{21}(2)$ is true. ▮

A 5 Optimal Initiating Time of Markovian Decision Processes

This section defines the optimal initiating time (OIT) for Markovian decision processes (MDP) [21,Howard,1960][40,Ross], which can be regarded as the most general model of decision processes.

A 5.1 Standard Definition of Markovian Decision Processes

A 5.1.1 Maximization MDP

Let the process be in a *state* i at a time t (see Figure 2.2.1(p.11)), and if an action x is taken at that time, then a *reward* $r(i,x)$ can be obtained and the present state i changes into j at the next time $t-1$ with a known probability $p(j|i,x)$. By $v_t(i)$ let us denote the maximum of the total expected present discounted *profit* gained over a given planning horizon starting from a time t in a state i. Then we have

$$v_t(i) \ = \ \max_x\bigl\{r(i,x) + \beta \textstyle\sum_j p(j|i,x) v_{t-1}(j)\bigr\}, \quad t > 0, \tag{A 5.1}$$

where $v_0(i)$ is a profit specified for a reason inherent in the process; in many cases, $v_0(i) = \max_x r(i,x)$. Let us call the decision process the *maximization* MDP.

A 5.1.2 Minimization MDP

This is the inverse of the maximization MDP where if an action x is taken at a given time t in a state i, a *cost* $c(i,x)$ must be paid. By $v_t(i)$ let us denote the minimum of the total expected present discounted *cost* over a given planning horizon from starting a time t in a state i. Then we have

$$v_t(i) \ = \ \min_x\bigl\{c(i,x) + \beta \textstyle\sum_j p(j|i,x) v_{t-1}(j)\bigr\}, \quad t > 0, \tag{A 5.2}$$

where $v_0(i)$ is a cost specified for a reason inherent in the process; in many cases, $v_0(i) = \min_x c(i,x)$. Let us call the decision process the *minimization* MDP.

A 5.2 Optimal Initiating Time

A 5.2.1 Initiating State $i_\circ$

Assume that a common state $i_\circ$ is defined for any given initiating time $t \geq 0$, and let us define

$$V_t \overset{\text{def}}{=} v_t(i_\circ), \quad t \leq \tau. \tag{A 5.3}$$

A 5.2.2 Relationship between $V_{[\tau]}$ and $V_{\beta[\tau]}$ (see Section 7.2.4.2(p.45))

In this section, by using some examples, let us demonstrate that the monotonicity of

$$V_{[\tau]} = \{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{t_{qd}}\} \quad \text{(original sequence)}$$

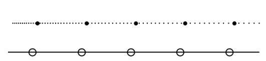
is not always inherited to

$$V_{\beta[\tau]} = \{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^\tau V_{t_{qd}}\} \quad (\beta\text{-adjusted sequence}).$$

Below let

$V_{[\tau]}$ →

$V_{\beta[\tau]}$ →



t_τ^* → • (optimal initiating time)

□ *Example* **A 5.1 (maximization** `MDP`**)** Suppose $V_{[\tau]}$ is strictly increasing in t where

$$V_\tau > V_{\tau-1} > V_{\tau-2} > \cdots > V_0 > 0.$$

In this case, as seen in Figure A 5.1(p.311) below, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0 > 0$, i.e., the monotonicity of $V_{[\tau]}$ is inherited to $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (Ⓢ). □

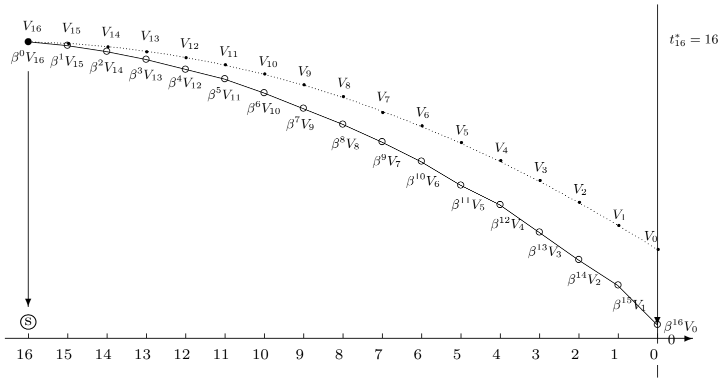


Figure A 5.1: Inheritance of monotonicity

□ *Example* **A 5.2 (maximization** `MDP`**)** Suppose $V_{[\tau]}$ is strictly increasing in t where

$$V_\tau > \beta V_{\tau-1} > V_{\tau-2} > \cdots > V_{\tau-t'} > 0 > V_{\tau-t'-1} > \cdots > V_0.$$

In this case, as seen in Figure A 5.2(p.311) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (Ⓢ). □

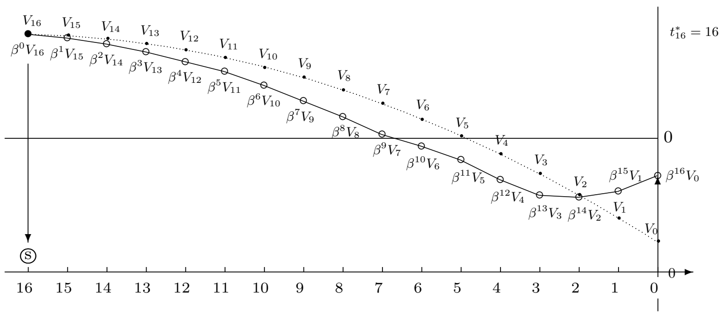


Figure A 5.2: Collapse of monotonicity

□ *Example* **A 5.3 (maximization** `MDP`**)** Suppose $V_{[\tau]}$ is strictly decreasing in t where

$$0 < V_\tau < \beta V_{\tau-1} < V_{\tau-2} < \cdots < V_0.$$

In this case, as seen in Figure A 5.3(p.311) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 6$, i.e., unregenerate (◎). □

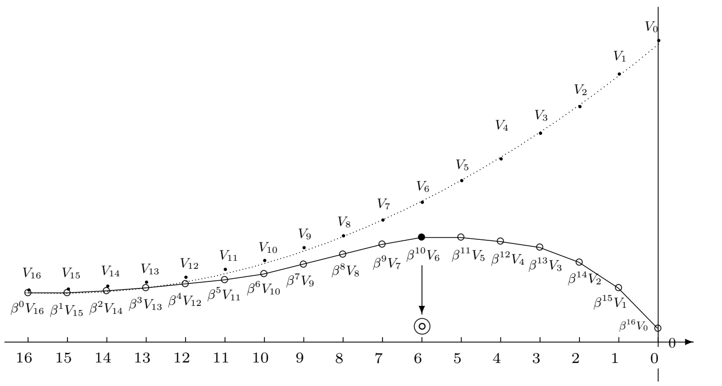


Figure A 5.3: Collapse of monotonicity

□ *Example* **A 5.4 (minimization MDP)** Suppose $V_{[\tau]}$ is strictly decreasing in t where

$$0 < V_\tau < \beta V_{\tau-1} < \cdots < V_{\tau-t'} < 0 < V_{\tau-t'-1} < \cdots < V_0.$$

In this case, as seen in Figure A 5.4(p.312) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (Ⓢ). □

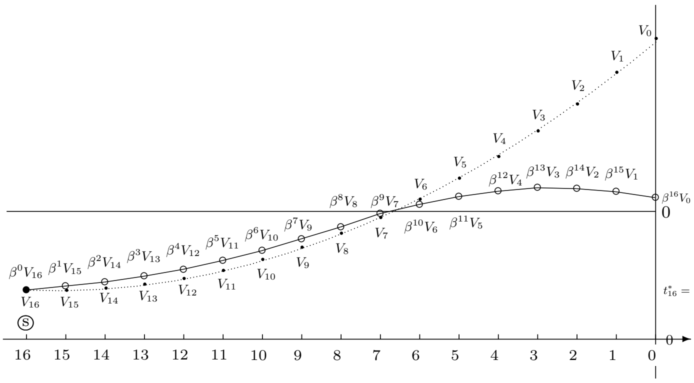


Figure A 5.4: Collapse of monotonicity

A 6 Calculation of Solutions x_K, x_L, and $s_{\mathcal{L}}$

The following lemma is used to numerically calculate the solutions x_K, x_L, and $s_{\mathcal{L}}$ (see Section 5.2(p.27)).

Lemma A 6.1 (x_K , x_L , $s_{\mathcal{L}}$)

(a) $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq x_K \leq \max\{b, 0\}$.
(b) $\min\{a, (\lambda\beta\mu - s)/\lambda\} \leq x_L \leq b$.
(c) $0 \leq s_{\mathcal{L}} \leq \lambda\beta\mu - \min\{a, 0\}$. □

● *Proof* (a)

- Let $x \leq a \cdots$ **(1)**. Now, from (10.2.4 (1) (p.57)) we have $K(x) = \delta\big((\lambda\beta\mu - s)/\delta - x\big)$, hence $K(x) \geq 0$ for $x \leq (\lambda\beta\mu - s)/\delta$. From this and (1) we have $K(x) \geq 0$ for $x \leq \min\{a, (\lambda\beta\mu - s)/\delta\}$, hence $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) \geq 0$.
  1. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\delta\} < x_K \cdots$ **(2)** due to Corollary 10.2.2(p.58) (a).
  2. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) = 0$.
     - If $\beta = 1$ and $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} \geq x_K$ due to Lemma 10.2.2(p.57) (i). Since $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq a < b = x_K$ from Lemma 10.2.2(p.57) (i), we have $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$.
     - If $\beta < 1$ or $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$ due to Lemma 10.2.2(p.57) (j1).

     Accordingly, whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$", we have $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K \cdots$ **(3)**.

  Thus, from (2) and (3) we have $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq x_K \cdots$ **(4)**.
- Let $b \leq x \cdots$ **(5)**. Now, from (10.2.5 (2) (p.57)) we have $K(x) \leq 0$ for $0 \leq x$. From this and (5) we have $K(x) \leq 0$ for $\max\{b, 0\} \leq x$, hence $0 \geq K(\max\{b, 0\})$. Accordingly, we have $x_K \leq \max\{b, 0\} \cdots$ **(6)** due to Corollary 10.2.2(p.58) (a).

From (4) and (6) the assertion becomes true.

(b)

- Let $x \leq a \cdots$ **(7)**. Now, from (10.2.3 (1) (p.57)) we have $L(x) = \lambda\beta\big((\lambda\beta\mu - s)/\lambda\beta - x\big)$, hence $L(x) \geq 0$ for $x \leq (\lambda\beta\mu - s)/\lambda\beta$. From this and (7) we have $L(x) \geq 0$ for $x \leq \min\{a, (\lambda\beta\mu - s)/\lambda\beta\}$, hence $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) \geq 0$.
  1. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} < x_L \cdots$ **(8)** due to Corollary 10.2.1(p.57) (a).
  2. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) = 0$.
     - If $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \geq x_L$ due to Lemma 10.2.1(p.57) (d). Since $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \leq a < b = x_L$ from Lemma 10.2.1(p.57) (d), hence $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$.
     - If $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$ due to Lemma 10.2.1(p.57) (e1).

     Accordingly, whether $s = 0$ or $s > 0$, we have $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L \cdots$ **(9)**.

  Thus, from (8) and (9) we have $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \leq x_L \cdots$ **(10)**.
- Let $b \leq x \cdots$ **(11)**. Then, from (5.1.3(p.25)) and Lemma 10.1.1(p.55) (g) we have $L(x) = -s \leq 0$, hence $0 \geq L(b)$. Accordingly, due to Corollary 10.2.1(p.57) (a) we have $x_L \leq b \cdots$ **(12)**.

From (10) and (12) the assertion becomes true.

(c) From (5.1.5(p.25)) and (5.1.3(p.25)) we have $\mathcal{L}(0) = L(\lambda\beta\mu) = \lambda\beta T(\lambda\beta\mu) \geq 0 \cdots$ **(13)** due to Lemma 10.1.1(p.55) (g). Now, for a sufficiently large $s > 0$ such that $\lambda\beta\mu - s \leq a$ and $\lambda\beta\mu - s \leq 0 \cdots$ **(14)** we have $s \geq \lambda\beta\mu - a$ and $s \geq \lambda\beta\mu$, hence $s \geq \max\{\lambda\beta\mu - a, \lambda\beta\mu\} = \lambda\beta\mu + \max\{-a, 0\} = \lambda\beta\mu - \min\{a, 0\} \cdots$ **(15)**. Then, from (5.1.5(p.25)), (5.1.3(p.25)), and Lemma 10.1.1(p.55) (f) we have

$$\mathcal{L}(s) = \lambda\beta T(\lambda\beta\mu - s) - s = \lambda\beta(\mu - \lambda\beta\mu + s) - s = \lambda\beta\mu - \lambda\beta(\lambda\beta\mu - s) - s = (1 - \lambda\beta)(\lambda\beta\mu - s).$$

Hence, since $1 \geq \lambda\beta$, due to (14) we have $\mathcal{L}(s) \leq 0$ for $s \geq \lambda\beta\mu - \min\{a, 0\}$ due to (15), so $\mathcal{L}(\lambda\beta\mu - \min\{a, 0\}) \leq 0$. From this and (13) we have $\mathcal{L}(0) \geq 0 \geq \mathcal{L}(\lambda\beta\mu - \min\{a, 0\})$, hence due to Lemma 10.2.4(p.59) (a) we have $0 \leq s_{\mathcal{L}} \leq \lambda\beta\mu - \min\{a, 0\}$. ∎

A 6.1 Calculation of Solutions $x_{\tilde{K}}$, $x_{\tilde{L}}$, and $s_{\tilde{\mathcal{L}}}$

Lemma A 6.2 ($x_{\tilde{K}}$, $x_{\tilde{L}}$, $s_{\tilde{\mathcal{L}}}$)

(a) $\max\{b, (\lambda\beta\mu + s)/\delta\} \geq x_{\tilde{K}} \geq \min\{a, 0\}$.

(b) $\max\{b, (\lambda\beta\mu + s)/\lambda\beta\} \geq x_{\tilde{L}} \geq a$.

(c) $0 \leq s_{\tilde{\mathcal{L}}} \leq -\lambda\beta\mu + \max\{b, 0\}$. □

● *Proof* Applying the operation $\mathcal{R}$ to Lemma A 6.1(p.312) leads to

⟨a⟩ $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\delta\} \leq -\hat{x}_K \leq \max\{-\hat{b}, 0\}$.

⟨b⟩ $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\lambda\}\beta \leq -\hat{x}_L \leq -\hat{b}$.

⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} - \min\{-\hat{a}, 0\}$.

The above can be rewritten as below:

⟨a⟩ $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta\} \leq -\hat{x}_K \leq -\min\{\hat{b}, 0\}$.

⟨b⟩ $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \leq -\hat{x}_L \leq -\hat{b}$.

⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$.

The above can be rewritten as below:

⟨a⟩ $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta \geq \hat{x}_K \geq \min\{\hat{b}, 0\}$.

⟨b⟩ $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \geq \hat{x}_L \geq \hat{b}$.

⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$.

Applying the operation $\mathcal{C}_{\mathbb{R}}$ (see Lemma 12.3.1(p.72) (b,g,h,i) to the above yields

⟨a⟩ $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\delta\} \geq x_{\check{\tilde{K}}} \geq \min\{\check{a}, 0\}$.

⟨b⟩ $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\lambda\}\beta \geq x_{\check{\tilde{L}}} \geq \check{a}$.

⟨c⟩ $0 \leq s_{\check{\tilde{\mathcal{L}}}} \leq -\lambda\beta\check{\mu} + \max\{\check{b}, 0\}$.

Finally, applying the operation $\mathcal{I}_{\mathbb{R}}$ (see Lemma 12.3.3(p.73) (b,g,h,i), we obtain (a)-(c) of this lemma. ∎

A 7 Others

A 7.1 Monotonicity of Solution

Proposition A 7.1 In general, for the solution x_t of a given equation $g_t(x) = 0$ we have:

Case A Let $g_t(x)$ is nondecreasing in x for all t.

(I) If $g_t(x)$ is nondecreasing in t for all x, then x_t is nonincreasing in t.

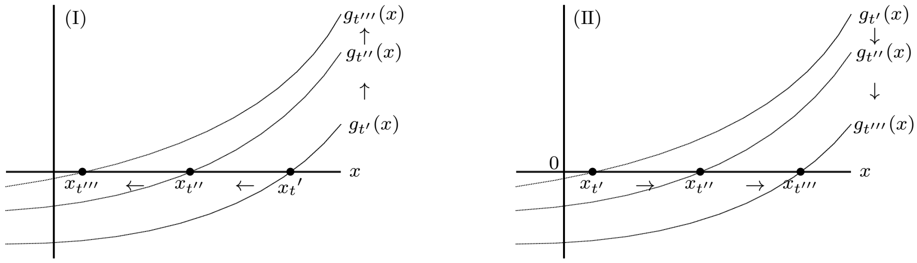
(II) If $g_t(x)$ is nonincreasing in t for all x, then x_t is nondecreasing in t.

Case B Let $g_t(x)$ is nonincreasing in x for all t.

(III) If $g_t(x)$ is nondecreasing in t for all x, then x_t is nondecreasing in t.

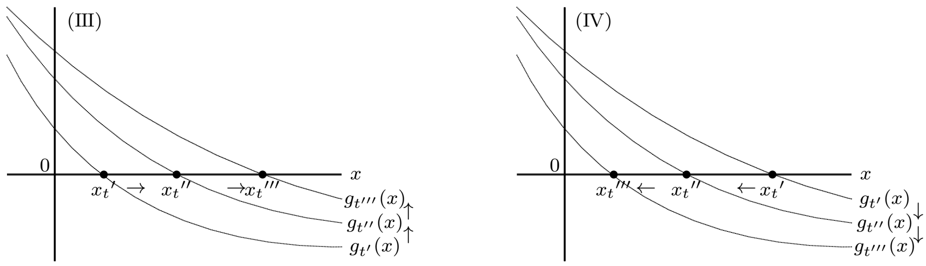
(IV) If $g_t(x)$ is nonincreasing in t for all x, then x_t is nonincreasing in t. □

● *Proof* Evident from Figures A 7.1(p.314) and A 7.2(p.314) below:



$g_t(x)$ is nondecreasing in t (I) $g_t(x)$ is nonincreasing in t (II)

Figure A 7.1: Case A: $g_t(x)$ is nondecreasing in x



$g_t(x)$ is nondecreasing in t (III) $g_t(x)$ is nonincreasing in t (IV)

Figure A 7.2: Case B: $g_t(x)$ is nonincreasing in x

A 7.2 Uniform Probability Density Function

For given a and b such as $-\infty < a < b < \infty$ let consider the uniform probability density function:

$$f(x) = \begin{cases} 0, & x < a, \\ 1/(b-a), & a \le x \le b, \\ 0, & b < x, \end{cases} \tag{A 7.1}$$

where the expectation is $\mu = 0.5(a + b)$. Then we have:

$$T(x) = \begin{cases} 0.5(a+b) - x, & x \le a, & \cdots(1), \\ 0.5(b-x)^2/(b-a), & a \le x \le b, & \cdots(2), \\ 0, & b \le x, & \cdots(3), \end{cases} \tag{A 7.2}$$

where (1) and (3) are immediate from Lemma 10.1.1(p.55) (f,g). Let $a \le x \le b \cdots (2)$. Then, from (5.1.2(p.25)) we have:

$$\begin{aligned} T(x) &= \int_a^b \max\{\xi - x, 0\}(b-a)^{-1} d\xi \\ &= \int_x^b (\xi - x)(b-a)^{-1} d\xi \\ &= (b-a)^{-1} \int_0^{b-x} \eta d\eta \quad (\eta = \xi - x) = 0.5(b-x)^2/(b-a). \end{aligned}$$

A 7.3 Graphs of $T_{\mathbb{R}}(x)$

From Lemma 10.1.1(p.55) (b,f,g) one immediately sees that $T_{\mathbb{R}}(x)$ can be depicted as in Figure A 7.3(p.315) (I) below. Similarly, from Lemma 10.2.2(p.57) (b, (10.2.4 (1) (p.57)), and (10.2.5 (2) (p.57))) we immediately see that $K_{\mathbb{R}}(x)$ can be depicted as in Figure A 7.3(p.315) (II) below.

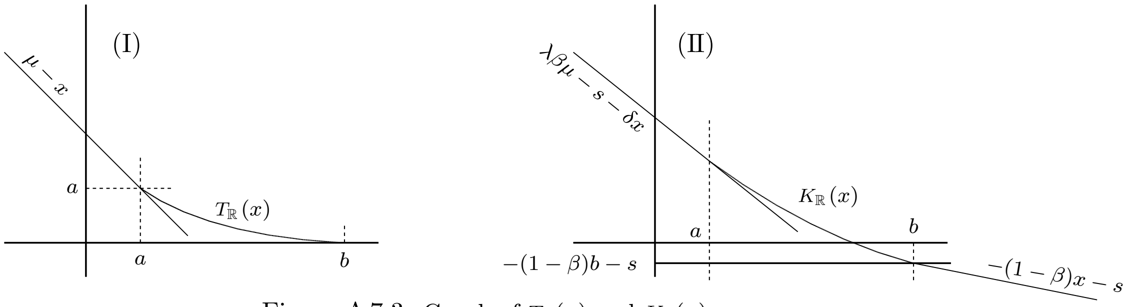


Figure A 7.3: Graph of $T_{\mathbb{R}}(x)$ and $K_{\mathbb{P}}(x)$

A 7.4 Graph of $T_{\mathbb{P}}(x)$

From Lemma 13.2.1(p.93) (b,f,g) we immediately see that $T_{\mathbb{P}}(x)$ can be depicted as in Figure A 7.4 below.

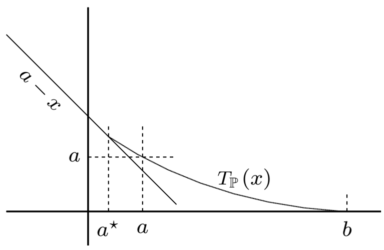


Figure A 7.4: Graph of $T_{\mathbb{P}}(x)$

Here note that $a^\star < a$ (see Lemma 13.2.1(p.93) (n)).

When $f(x)$ is the uniform distribution function (see (A 7.1(p.314))), we can obtain the $a^\star$ as below. Then we have:

$$\begin{aligned}
&p(z) = 1 && \text{for } z \le a && \text{from (5.1.28 (1) (p.26))},\\
&p(z) = \int_z^b f(\xi)d\xi = \int_z^b 1/(b-a)d\xi = (b-z)/(b-a) && \text{for } a \le z \le b && \text{from (5.1.18(p.26))},\\
&p(z) = 0 && \text{for } b \le z && \text{from (5.1.29 (2) (p.26))}.
\end{aligned}$$

Hence we get

$$T(z,x) \overset{\text{def}}{=} p(z)(z-x) = \begin{cases} z - x, & z \le a \quad \cdots \textbf{(1)},\\ (b-z)(z-x)/(b-a), & a \le z \le b \quad \cdots \textbf{(2)},\\ 0, & b \le z \quad \cdots \textbf{(3)}. \end{cases}$$

Then (5.1.19(p.26)) can be expressed as

$$T(x) = \max_z T(z,x) = T(z(x),x) \cdots \textbf{(4)}.$$

Here let us define

$$g^*(z,x) = (b-z)(z-x)/(b-a), \quad z,x \in (-\infty,\infty),$$

which is a quadratic expression of z for any given x. By $z^*(x)$ let us denote z attaining the maximum of $g^*(z,x)$ for a given $x \in (-\infty,\infty)$. Then clearly

$$z^*(x) = (b+x)/2 \cdots \textbf{(5)}.$$

Note that $g^*(z,x)$ can be depicted as the three possible *smooth* curves (dotted curve) in Figure A 7.5(p.316) below, depending on a value that $z^*(x)$ takes on, i.e.,

$$\begin{aligned}
&z^*(x) \le a && \cdots \text{(i)}\\
&a \le z^*(x) \le b && \cdots \text{(ii)}\\
&b \le z^*(x) && \cdots \text{(iii)}
\end{aligned}$$

Accordingly, noting (1) - (3) , we see that $T(z,x)$ can be depicted as the three possible *broken* curves (bold curve), each of which has the line $z - x$ with the angle 45° on $z \le a$ and the horizontal line (z-axis) on $b \le z$.

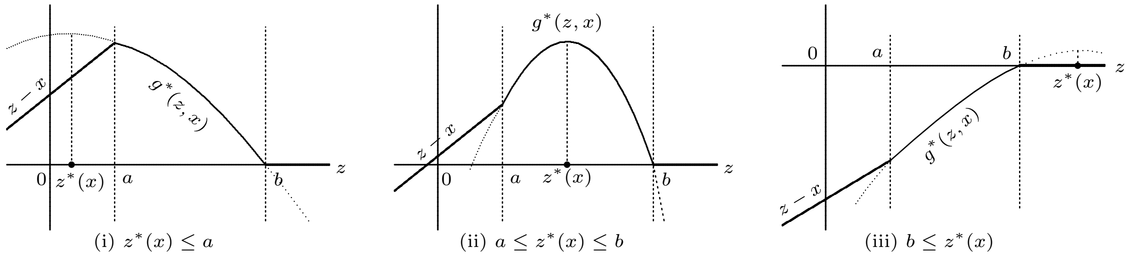


Figure A 7.5: Graph of $g^*(z,x)$ (smooth curve) and $T(z,x)$ (broken curve)

Here note that the $T(z,x)$ is given by the broken curve (see (1) - (3)) and that z maximizing the broken curve is given by $z(x)$ (see (4)). Then, from (5) and Figure A 7.5(p.316) we see that

1. Let $z^*(x) \le a \cdots (1)$, i.e., $(b+x)/2 \le a$, hence $x \le 2a-b$. Then, by definition we have

$$z(x) = a \cdots \mathbf{(}6\mathbf{)}, \quad x \le 2a-b.$$

Hence, from (4) and (1) we have $T(x) = T(a,x) = a - x \cdots \mathbf{(7)}$ on $x \le 2a-b$.

2. Let $a < z^*(x) \le b \cdots (2)$, i.e., $a < (b+x)/2 \le b$, hence $2a-b < x \le b$. Then, by definition we have

$$z(x) = z^*(x) = (b+x)/2 > a \cdots \mathbf{(}8\mathbf{)}, \quad 2a-b < x \le b.$$

Hence, from (4) and (2) we have

$$T(x) = T(z^*(x),x) = (b - z^*(x))(z^*(x)-x)/(b-a) = (b-x)^2/4(b-a), \quad 2a-b < x \le b.$$

Now, since

$$m(x) \stackrel{\text{def}}{=} T(x) - a + x = ((b-x)^2 - 4(b-a)(a-x))/4(b-a),$$

we have

$$m'(x) = (x - 2a + b)/2(b-a) > 0, \quad 2a-b < x \le b,$$

hence $m(x)$ is strictly increasing on $2a-b < x \le b$. In addition to the fact, since it can be easily confirmed that $m(2a-b) = 0$, it follows that $m(x) > 0$ on $2a-b < x \le b$, hence $m(x) = T(x)-a+x > 0$ on $2a-b < x \le b$ or equivalently $T(x) > a-x \cdots \mathbf{(9)}$ on $2a-b < x \le b$.

3. Let $b \le z^*(x) \cdots (3)$, i.e., $b \le (b+x)/2$, hence $b \le x$. Then, by definition we have

$$z(x) = b > a \cdots \mathbf{(}10\mathbf{)}, \quad b \le x.$$

Hence $T(x) = T(b,x) = 0$ from (4) , hence $T(x) = 0 \ge b - x > a - x \cdots \mathbf{(}11\mathbf{)}$ on $b \le x$.

Collecting up (7) , (9) , and (11) , we have

$$T(x) \begin{cases} = a-x, & x \le 2a-b, \\ > a-x, & 2a-b < x \le b, \\ > a-x, & b \le x. \end{cases} \tag{A 7.3}$$

Accordingly, noting (5.1.26(p.26)) and Figure A 7.4(p.315) , from (A 7.3(p.316)) we immediately see that

$$a^\star = 2a - b \cdots (1). \tag{A 7.4}$$

Similarly, collecting up (6) , (8) , and (10) , we have

$$z(x) \begin{cases} = a, & x \le 2a-b, \\ > a, & 2a-b < x \le b, \\ > a, & b \le x. \end{cases} \tag{A 7.5}$$

Accordingly, noting (5.1.27(p.26)), we immediately see that

$$x^\star = 2a - b \cdots (2). \tag{A 7.6}$$

Numerical Experiment **1** (Discontinuity of $z(x)$ (Dr. Mong Shan Ee)) $z(x)$ is not always continuous in $x = x^\star$; in fact we can demonstrate a numerical example in which $z(x)$ is not continuous in $x = x^\star$. For example let us consider $F(w)$ with $f(w)$ such that $f(w) \approx 0.05701$ on $[0.1, 0.599]$, $f(w)$ is a triangle on $[0.599, 0.7]$ with its maximum at $w = 0.6$, and $f(w) \approx 0.06982$ on $[0.7, 3.0]$. Then we have $z(x) \approx 0.599$ for $x \le 0.48568$ and $z(x) \approx 1.7$ for $x < 0.48568$, i.e., $z(x)$ is discontinuous at $x = 0.48568$. □

A 7.5 Economic Implications of Market Partition

The three restricted markets defined in Section 17.2(p.117) implies the following:

- Positive market $\mathscr{F}^+$ In an asset trading problem in the real world, the price is usually positive, i.e., the problem is defined on the positive market $\mathscr{F}^+$, called the *input market* in the sense that all goods are first input in the market.
- Mixed market $\mathscr{F}^\pm$ For example, suppose you must waste a piece of well-worn furniture, say a book cabinet, sofa bed and so on. For such a good, normally the two kinds of receiving-sides (buyers) may appear: One who pays some money on the ulterior motive that some profit might be obtained by reselling it and the other who requires some money for the reason that some cost may be incurred for its disposal. This market can be regarded as a market in which the positive market and the negative market are mixed; let us call the market the *secondhand market*.
- Negative market $\mathscr{F}^-$ The trading problem in A3.5(p.18) is defined on this market; let us call the market the *junk market*. □

Remark A 7.1 (life of durable goods) A new durable good (automobile, house furnishings, TV and so on) is first placed on the positive market $\mathscr{F}^+$ (input market), deteriorates year by year, a while later is drove to the mixed market $\mathscr{F}^\pm$ (*secondhand market*), before long moves into the negative market $\mathscr{F}^-$ (*junk market*), and then finally is recycled or dumped. This deterioration flow implies that the probability density functions of price transfers from right to left as seen in Figure A 7.6(p.317) below. □

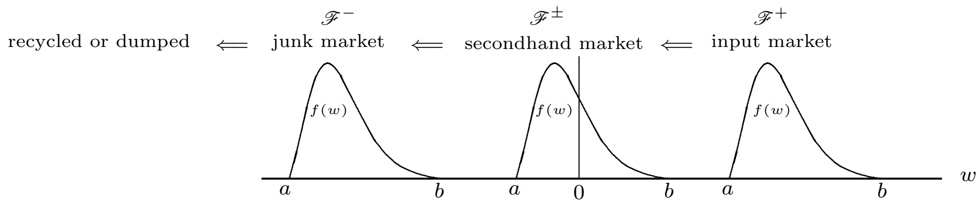


Figure A 7.6: Deterioration transition of goods (life of goods)

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♡

Many decision theories discussed by researchers have traditionally been framed as mathematical theories. In contrast, this paper approaches "decision" as a subject of study within the natural sciences (see Section 1.3(p.4)). It is important to note that some researchers may have objections to this viewpoint. However, one should recognize that the truth of mathematics resides within mathematics itself, and the truth of physics resides within physics; there is no direct relationship between these two types of truth. To illustrate, physicists sometimes refer to the term "mathematics" as "arithmetic", using it merely as a tool, akin to how carpenters use hammers. While a good hammer is necessary for building a good structure, it would be a mistake to think that a good structure cannot be built without a good hammer. As Albert Einstein famously stated:

As far as the laws of mathematics refer to reality, they are not certain,
and as far as they are certain, they do not refer to reality.

— Albert Einstein —

♢

This paper, which began with a proposition by Dr. Professor Shizuo Senju on March 31, 1966
concludes with this apothegm on December 11, 2024.

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