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Epistemic Logic of Reciprocal Empathy:

Surfaces to Deeper Layers and Latent Infinity

by

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Epistemic Logic of Reciprocal Empathy: Surfaces to Deeper Layers and Latent Infinity*

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Abstract

We present epistemic logic REL_β of reciprocal empathy with two persons, where reciprocal empathization, "to put oneself into the other's shoes", from one person to the other and *vice versa,* is formalized. Logic REL_β is an fixed-point extension of epistemic logic KD adding perception operators $\mathbf{Pc}_i(\cdot,\cdot)$ for persons $i = 1, 2$; the point of REL_β is that the interpersonal epistemic depths are bounded by β, which can be as shallow as $\beta = 3$. In fact, logic REL_β captures the infinity entailed by the fixed-point argument, which we call "latent infinity". Our main objective is to study the syntactical logic REL_β, but Kripke semantics is needed for a full study; we prove the soundness/completeness of REL_β. Using it, we provide various properties of REL_β; using them, we show what the latent infinity is and how it is hidden. We apply REL_β to an example of communication and coordination due to David Lewis. It is formulated in terms of non-logical axioms within REL_β. In this application, we see different degrees of sharing common thoughts dependent upon on issues. This application shows that our approach has great potential for studies of social problems.

Key words: Reciprocal empathy, Fixed-point logics, Boundedness of interpersonal epistemic depths, Hilbert-style proof theory, Kripke semantics, Latent infinity.

1 Introduction

An individual empathizes with another through the mental act "to put oneself into the other's shoes" in order to acquire what the other believes and/or thinks. In addition, he may think that the other engages in the symmetric act. We call the first as *mere empathization* and to the second as *reciprocal empathization.* We focus on the latter and formalize this reciprocity by introducing new epistemic operators for empathized beliefs. The resulting logic, REL_β, is a fixed point extension of a KD type 2-person epistemic logic, where the extension is made by adding individual inference abilities. Moreover, we introduce a bound β on depths of interpersonal (and intrapersonal) reasoning. Our result reveals that the reciprocity of empathization implies a hidden infinity for any $\beta \geq 3$, and REL_β still captures the central part of REL_ω (with no bound) even for shallow β's such as 3. We call this the "*latent infinity*".

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Our target object is a social situation where interpersonal logical reasoning of an individual person plays a crucial role. We follow the basic attitude that the syntactics and semantics are complementary to each other in having a complete picture of a logical system. Logic REL_β is used to describe such a situation, and the semantics is used to suport the analysis in REL_β. In particular, to describe a social situation in details, we follow the traditional separation between *logic* and *(formal) theory*; we formalize a social situation as a theory. We explain theoretical developments of logic REL_β in Section 1.1, and discuss our motivational attitude and required methodological changes in Section 1.2.

1.1 Theoretical developments

We adopt the syntactic approach to formalize reciprocal empathization. It is called *reciprocal empathy logic,* REL_β, in which we introduce a new operator, called *perception (-empathization) operator* $\mathbf{Pc}_i(\cdot,\cdot)$ for each person $i = 1, 2$, and a depth restriction on interpersonal reasoning parameterized by $\beta \leq \omega := \{0, 1, ...\}$. The depth β is an upper bound for the nested depths of *belief operators* $\mathbf{B}_i(\cdot)$ as well as $\mathbf{Pc}_i(\cdot,\cdot)$ $(i = 1, 2)$, which represents a bound on the ability to conduct interpersonal (and/or intrapersonal) inferences for person i.

The operator $\mathbf{Pc}_i(\cdot,\cdot)$ connects the external world to person i's mind, as the name suggests, and conducts interpersonal reasonings. This interpretation is substantiated by one axiom and one inference rule on each $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$. Thus, the operator plays two roles;

(i): person i's perception of the state-of-affairs, including the other's perception;

(ii): his logical reasoning, including his thinking of the other's reasoning.

Empathization corresponds to (i) and is a source of i's interpersonal beliefs. Then, (ii) may develop such beliefs into more complex ones.[1] These roles will be well observed in Lewis's example in Section 5.

We provide a series of logics REL_β with $0 \leq \beta \leq \omega$, depicted in Table 1.1, where REL_β is a sub-system of $\mathrm{REL}_{\beta'}$ for $\beta < \beta' \leq \omega$. In particular, it is classical logic CL for $\beta = 0$, and it is epistemic logic KD_β for $\beta = 1, 2$. These logics do not allow for reciprocal empathization; mere empathization appears for $\beta = 2$. Reciprocal empathization becomes possible in logic REL_β for β with $3 \leq \beta \leq \omega$. When $\beta = \omega$, no interpersonal bounds are imposed.

Table 1.1; the series of REL_β $(0 \leq \beta \leq \omega)$

REL_0	$\rightarrow$	REL_1	$\rightarrow$	REL_2	$\rightarrow$	REL_3	$\rightarrow$	$\cdots$	$\rightarrow$	$\cdots$	REL_ω
$\shortparallel$		$\shortparallel$		$\shortparallel$		$\shortparallel$					
CL		KD_1		KD_2		Re.Em.					Re.Em.

We adopt the standard (serial) Kripke semantics, except for the restriction β on the set of formulae for semantical valuation. Thus, the semantical models are uniform over REL_β except for the set of formulae. In Section 3.1, we provide the (*soundness/*) *completeness theorem* for logic REL_β $(0 \leq \beta \leq \omega)$ with respect to Kripke semantics; a proof of completeness will be given Sections 7 and 8. For $\beta \leq 2$, the operators $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$ do not appear; and the

[1] Dynamic epistemic logic takes these properties except for the reciprocity property, i.e. the last part of (i) (cf., van Ditmarsch, *et al.* [13], and Benthem-Smets [6]). Here, dynamics is included in the semantics and then, the axiomatization is considered. We separate dynamic as a formal theory from our logic, which will be discussed in Section 5.

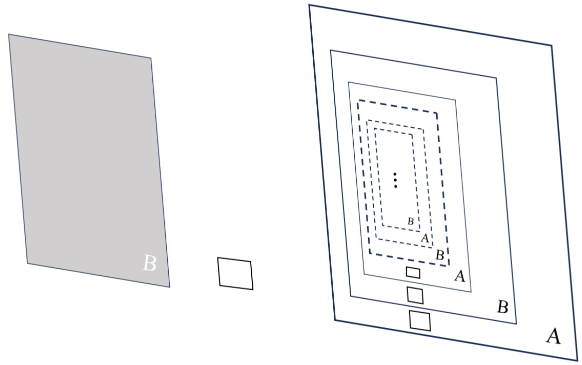

Figure 1: Two mirrors and infinite images

completeness theorem holds in simpler forms. For $\beta \geq 3$, the semantical structure includes an infinite structure with respect to interpersonal reasoning, but logic REL_β with a finite β includes no explicit infinity. This was mentioned above as latent infinity.

The latent infinity could be found in an analogy of an unbounded number of images created between two mirrors with a person in the box in the middle as in Fig.1.1, who looks at Mirror A. He finds that the box is in the image of Mirror A, this image of Mirror A reflects onto Mirror B, this again reflects onto Mirror A, and so on. The word "so on" means that the third image becomes the same as the first image; after the 3rd step of reflection in Mirror A, the same argument could be repeated.

We interpret this analogy from the viewpoints of the reciprocal empathy logic REL_β and the corresponding Kripke semantics. From the syntactical perspective, reciprocity means returning to the third image with the reflection of the original mirror image that contains the whole situation; this reciprocity is captured by the axioms and inference rules for $\mathbf{Pc}_i(\cdot,\cdot)$ in logic REL_β. From the semantical viewpoint, on the other hand, the unbounded number of reflected images is captured in a semantical model in that an unbounded number of truth valuations are required across accessible possible worlds, but the formulae to be valuated are still restricted. Thus, the latent infinity is revealed from the semantical structure, but it is hidden in the series of logics in Table 1.1 for REL_β with $\beta \geq 3$, which will be discussed in Section 4.

To elucidate the concept of latent infinity in logic REL_β, we need a few meta-theorems, which will be given in Sections 2, 3, and 4. First, logic $\mathrm{REL}_{\beta'}$ is a *conservative extension* of REL_β for $\beta < \beta' \leq \omega$; the behavior of REL_β is the same as in $\mathrm{REL}_{\beta'}$ as long as the formulae for REL_β are concerned. This is further conservatively extended into the infinitary logic in Hu, *et al.* [16]. These imply that REL_β includes some infinity, and its latency is discussed in Section 4; when $\beta \leq \omega$, $\mathbf{Pc}_i(C_1, C_2)$ is *implicitly definable*, i.e., its operational meaning is determined by the axiom and inference rule for $\mathbf{Pc}_i(\cdot,\cdot)$, but it is not *explicitly definable,* i.e., it is not expressed in terms of the persons' belief operators, but it is possible in the infinitary logic of [16].

Lewis [23], Section I.3, talked about how deep the layers of interpersonal beliefs would be required in a social situation. However, there is no definitive answer given in his analysis; in p.32 and p.52, it is argued that people could not go to deep layers but "And the more orders (degrees), the better" in p.33. Our study shows that the required depth β is 3 to have reciprocal empathization.

Reciprocal empathization has some similarity to the concept of *common belief* (/knowledge) in that both are related to a situation shared by the two people in which interpersonal inference/beliefs are described. The difference is: reciprocal empathization is individual inference conducted by a person, but common belief describes the situation where interpersonal beliefs are shared by the two persons. In this sense, our treatment is based on methodological individualism (von Mises [27], Chap.I, Section 4). In Section 6, we consider how they differ mathematically. It is shown that common belief is not expressible in logic REL_β, but the extension of REL_β having self-consciousness of empathization can describe common belief.

These sections have focused on logical properties described in REL_β, which correspond to (ii). The perceptions in (i) appear in the application of logic REL_β in Section 5. Because we need an additional methodological treatment from logic REL_β for this application, we postpone a explanation of it to Section 1.2, and will discuss methodological changes we adopt in the present paper.

For the completeness proof of REL_β given in Sections 7 and 8, we are indebted to the proof of the common belief logic due to Fagin, *et al.* [15][2], but we need new arguments to treat the bound β for the individualistic fixed-point arguments. In the unbounded case $\beta = \omega$, the standard argument to construct a countermodel requires some modifications, but in the bounded case $\beta < \omega$, more drastic modifications are required. We will take two steps of constructing a full countermodel in Sections 7 and 8.

Finally, a small number of papers treating epistemic logics with bounded interpersonal reasoning. Kaneko-Suzuki [20] and [21] developed epistemic logics KD with bounds so that interpersonal reasoning abilities are parallelly bounded in syntax and semantics. Their concern was to give various meta-theorems for studies of game theoretical decision making; some will be used for the study of latent infinity in Section 4. Nevertheless, they touch neither reciprocal empathization nor latent infinity. Arthaud-Rinard [1], Larotonda-Primiero [22] studied computer scientific complexity. Larotonda-Primiero [22] treated three-valued epistemic logic with bounded semantic reasoning. The focusses of these papers remain in semantics.

1.2 An application of REL_β and methodological remarks

As indicated above, our theoretical developments have deviated from the standard epistemic logic in various manners. The deviations are motivated from the methodological perspective of social science. We postpone a discussion on these issues to Section 9.1 after a full development of logic REL_β. Still, some comments on Sections 5 and 6 help understand the development.

As stated above, we take an example due to Lewis [23], p.52 in Sections 5:

(*): you and I have met, we have been talking together, you must leave before our business is done; so you say you will return to the same place tomorrow. Imagine the case. Clearly, I will expect you to return. You will expect me to expect to return. I will expect you to expect me to expect you to return. Perhaps, there will be one or two orders more.

This situation is observed in our everyday lives. We formalize this situation as a theory within the language of REL_β, following the distinction between a *logic* and a *(formal) theory* in the tradition of first-order predicate logic (cf., Barwise [5], Mendelson [26]), though REL_β is a propositional logic.

[2] Some papers use different methods, cf., Meyer-van der Hoek [28].

We describe (*) as a theory with "non-logical axioms" proper to the situation (*), which we call postulates. As stated in Section 1.1, $\mathbf{Pc}_i(\cdot,\cdot)$ serves as an interface between the external world and the individual's mind; the postulates are the properties describing the interface and reasonings in the mind of person i. Some postulates are about the physical source of empathized beliefs and others are about the social contexts where the empathized beliefs emerge; these are classified in (i). Also, $\mathbf{Pc}_i(\cdot,\cdot)$ has an ability of conducting logical reasoning, which is in (*ii*). One postulate, called the *trustworthiness of words,* is classified in (*ii*), which formalizes the reciprocal trust of the contents of the words being exchanged. Although (*) is simple and observed in our everyday lives, it contains quite different bases for reciprocal empathization.

We have taken (*) for an object of our application through a theory within the language of logic REL_β; our motivational perspective is that social situations are objects of study. The bound β on interpersonal reasoning is naturally introduced from this perspective. Our methodological perspective differs from most of the epistemic-logic literature (Fagin, *et al.* [15], Meyer-van der Hoek [28], Benthem [7]) where semantic models and/or variants are the target objects. These methodological principles will be discussed more in Section 9.1. These considerations address suitable directions of further extensions of our approach, to be mentioned in Section 9.2.

The paper is organized as follows: Section 2 gives a language with a bound β on interpersonal interactions, and develops a Hilbert-style proof theory. Section 3 formulates the Kripke model of KD type, where the bound β enters only the definition of valuation of a formula. Then, the soundness/completeness theorem is presented. Section 4 discusses latent infinity. Section 5 formulates Lewis's (*). Section 6 compares the concept of common belief with that of reciprocal empathization. Sections 7 and 8 prove the completeness of logic REL_β with respect to Kripke semantics. Section 9 addresses some methodological reflections and suitable directions of further extensions of our approach.

2 Reciprocal Empathy Logic REL_β

In this section, we prepare the set of formulae with a bound β for epistemic depths, and formulate logic REL_β. We give two theorems describing the properties of reciprocal empathy.

2.1 Language

We use the following primitive symbols:

propositional variables: $\mathbf{p}_0, \mathbf{p}_1, ...$;
logical connectives: $\neg$ (not), $\supset$ (imply), $\wedge$ (and);
unary belief operators: $\mathbf{B}_1(\cdot), \mathbf{B}_2(\cdot)$;
binary reciprocal empathization operators: $\mathbf{Pc}_1(\cdot,\cdot)$ and $\mathbf{Pc}_2(\cdot,\cdot)$;
parentheses: (,).

We denote the set of propositional variables by PV. The conjunction symbol $\wedge$ is applied to any finite nonempty set of formulae Φ. For $i = 1, 2$, the operator symbol $\mathbf{Pc}_i(\cdot,\cdot)$ describes person i's perception of "a state of affairs at a particular time", which is binary operator with two object formulae in the parentheses $(\cdot,\cdot)$. We call the formula $\mathbf{Pc}_i(C_1, C_2)$ *reciprocal empathized beliefs*; the reason for this term will be explained after the axiom and inference rule for it. We stipulate that when person i is given, the other person is denoted as j.

The formulae are defined by the following induction on the lengths of formulae:

(F-o): any propositional variable p is a formula;

(F-i): if A, B are formulae, so are $(\neg A), (A \supset B)$;

(F-ii): if Φ is a finite nonempty set of formulae, $(\wedge\Phi)$ is a formula;

(F-iii): if A is a formula, so is $\mathbf{B}_i(A)$ for $i = 1, 2$;

(F-iv): if C_1, C_2 are formulae, so is $\mathbf{Pc}_i(C_1, C_2)$ for $i = 1, 2$.

We denote the *set of all formulae* by $\mathcal{P}$. We abbreviate the parentheses, e.g., $(A \supset B)$ and $(\wedge\Phi)$ as $A \supset B$ and $\wedge\Phi$ and may use different parenthesis such as $[,]$, when no confusions are expected. Also, $\wedge\{A, B\}$ is often denoted as $A \wedge B$, and $(A \supset B) \wedge (B \supset A)$ is denoted as $A \equiv B$. We also write $\boldsymbol{C} = (C_1, C_2)$ and $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2)$. We say that a formula A is an *epistemic* formula iff it is expressed as $\mathbf{B}_i(C)$ or $\mathbf{Pc}_i(C_1, C_2)$ for $i = 1, 2$, which is abbreviated as an *ep-formula.* We call $\mathbf{B}_i(C)$ and $\mathbf{Pc}_i(\boldsymbol{C})$, respectively, a *b-formula* and an *pc-formula*.

An pc-formula $\mathbf{Pc}_i(C_1, C_2)$ means that when person i is in a state of affairs with the other person j, he observes the piece of information relevant for him expressed by formula C_i and the piece C_j for player j. Thus, the pc-operator $\mathbf{Pc}_i(\cdot, \cdot)$ describes an interfaces between the external world and the internal mind of i. Also, $\mathbf{Pc}_i(\cdot, \cdot)$ will serve as the source that player i uses to start reciprocal empathization in an interaction with the other person. The two operators $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$ will have intimate interactions with personal beliefs operators $\mathbf{B}_i(\cdot)$, $i = 1, 2$. Belief operator $\mathbf{B}_i(\cdot)$ describes the internal beliefs and logical reasoning within himself. The precise meaning of empathization will be given by the axiom and inference rule for $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$ in Section 2.2.

A crucial part of our theory is to investigate the depths of interpersonal reasoning needed to achieve reciprocal empathization and the implications of cognitive bounds for the process of reciprocal empathization. To this end, we introduce the depth measure δ to count the nested depths of $\mathbf{B}_i(\cdot)$ and $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$. Formally, we define δ by the following induction;

(δ-o): $\delta(p) = 0$ for all propositional variables p;

(δ-i): $\delta(\neg A) = \delta(A)$ and $\delta(A \supset B) = \max(\delta(A), \delta(B))$;

(δ-ii): $\delta(\wedge\Phi) = \max\{\delta(A) : A \in \Phi\}$, where Φ is a finite nonempty set of formulae;

(δ-iii): $\delta(\mathbf{B}_i(A)) = \delta(A) + 1$ for $i = 1, 2$;

(δ-iv): for $\delta(\mathbf{Pc}_i(C_1, C_2)) = \max(\delta(C_1), \delta(C_2)) + 1$ for $i = 1, 2$.

For example, $\delta(\mathbf{Pc}_1[\mathbf{B}_2(p_1), \mathbf{B}_1\mathbf{Pc}_2(p_1, p_2)]) = \max(\delta(\mathbf{B}_2(p_1)), \delta(\mathbf{B}_1\mathbf{Pc}_2(p_1, p_2)) + 1 = \max(1, 2) + 1 = 3$. Note that we do not differentiate between $\mathbf{B}_i(\cdot)$ and $\mathbf{Pc}_i(\cdot, \cdot)$ $(i = 1, 2)$ in the above definition of δ. We note that $\delta(\mathbf{Pc}_1(\boldsymbol{C})) = \delta(\mathbf{Pc}_2(\boldsymbol{C}))$ for any $\boldsymbol{C} = (C_1, C_2)$ in $\mathcal{P} \times \mathcal{P}$. This fact will be used without referring.

Let β is a natural number or the ordinal $\omega = \{0, 1, ...\}$, i.e., β is from $\omega \cup \{\omega\}$. This β is intended to be the upper bound for the depths of formulae $\delta(C)$ up to which a person can perform interpersonal reasoning. Since β is an ordinal number, we should write a constraint as, say, $2 + \delta(A) \leq \beta$. However, since our main concern is the case $\beta < \omega$, we may write $\delta(A) \leq \beta - 2$ for simplicity. When $\beta = \omega$, this means $\delta(A) \leq \beta - 2 = \omega$.

Let a bound β be given. Whenever an pc-formula $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2)$ is allowed, we require that person i can have two more layers of interpersonal reasoning relative to the pc-formula. This requirement expressed by the following:

$$\delta(\mathbf{Pc}_i(\boldsymbol{C})) \leq \beta - 2 \text{ for any subformula } \mathbf{Pc}_i(\boldsymbol{C}) \text{ of } A. \qquad (1)$$

The set of formulae permissible by bound β is given as follows:

$$\mathcal{P}_\beta = \{A \in \mathcal{P} : \delta(A) \leq \beta \text{ and (1) holds for } A\}. \tag{2}$$

That is, $A \in \mathcal{P}_\beta$ has at most depth β and if it contains $\mathbf{Pc}_i(\boldsymbol{C})$, it's depth is at least than $\beta - 2$. Condition (1) plays the crucial role in the present paper.

In Table 1.1, $\mathbf{Pc}_i(\cdot,\cdot)$ is allowed in $\mathcal{P}_\beta$ with $\beta \geq 3$. When $\beta = \omega$, the set $\mathcal{P}_\omega$ coincides with the entire set of formulae $\mathcal{P}$, since the restriction (1) is void. When $\beta < \omega$, (2) allows $\delta(A) = \beta$ but requires A to satisfy (1). When $\beta \leq 2$, the set $\mathcal{P}_\beta$ has no occurrences of becomes $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$. These cases have no instances of reciprocal empathization. When $\beta = 0$, $\mathcal{P}_\beta$ has no ep-formulae.

The set $\mathcal{P}_\beta$ is *subformula-closed*, i.e., if $C \in \mathcal{P}_\beta$, then any subformula C' of C is also in $\mathcal{P}_\beta$. This fact is a key to our analysis, but since it is simple, we may not refer to it. However, when we mention different sets of formulae, we should be conscious about subformula-closedness.

2.2 Formulation of logic $\mathbf{REL}_\beta$

The axioms and inference rules for REL_β are stated as follows: For any $A, B, C \in \mathcal{P}_\beta$ and a finite nonempty set of formulae Φ in $\mathcal{P}_\beta$,

L1: $A \supset (B \supset A)$;
L2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;
L3: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;
L4: $\wedge\Phi \supset A$, where $A \in \Phi$;

MP: $\dfrac{A \supset B \quad A}{B}$ and $\wedge$**-Rule**: $\dfrac{\{A \supset B : B \in \Phi\}}{A \supset \wedge\Phi}$.

These axioms (schemata) and inference rules are drawn form classical propositional logic.[3,4] We adopt the following axiom schemata and inference rule for $\mathbf{B}_i(\cdot)$: for all $\mathbf{B}_i(A), \mathbf{B}_i(C) \in \mathcal{P}_\beta$ and $i = 1, 2$,

K: $\mathbf{B}_i(A \supset C) \supset (\mathbf{B}_i(A) \supset \mathbf{B}_i(C))$;
D: $\neg\mathbf{B}_i(\neg A \wedge A)$;
Nec: $\dfrac{A}{\mathbf{B}_i(A)}$.

These K, D, and Nec, in addition to L1 - L4, MP, and $\wedge$-Rule compose the epistemic logic KD_β with the language $\mathcal{P}_\beta$. When $\beta \geq 1$, Axioms K, D, and Inference rule Nec are not vacuous in that each has some instances in $\mathcal{P}_\beta$. Here, person i has the classical logical ability.

Finally, the *reciprocal empathy logic* REL_β is defined as the system by adding *Axiom* (schema) *AEM* and (Inference) *Rule IEM* to KD_β. They express a fixed-point property of the perception operators $\mathbf{Pc}_i(\boldsymbol{C}), i = 1, 2$.

AEM (Axiom for Empathization): For any $\mathbf{Pc}_j(\boldsymbol{C}) \in \mathcal{P}_\beta$ and $i = 1, 2$,

[3] In the literature of epistemic logic, it is typical to start with classical valid formulae instead of Axioms L1 to L4 and the inference rules. Our direct concern is shallow depth of interpersonal reasoning, but this is closed related to shallow depths of intrapersonal reasoning, too. From this point of view, we start explicitly with Axioms L1 to L4 and the inference rules.

[4] Here, $\mathbf{B}_i(A)$ and $\mathbf{Pc}_i(A_1, A_2)$ $(i = 1, 2)$ in $\mathcal{P}_\beta$ are treated as if propositional variables. In KD_β below, $\mathbf{Pc}_i(\boldsymbol{C}), i = 1, 2$ are similarly treated as if propositinal variables.

$\mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$.

IEM (Inference Rule for Empathization): For any $\mathbf{Pc}_i(\boldsymbol{C}), \mathbf{B}_j(D_i) \in \mathcal{P}_\beta$, $i = 1, 2$,

$$\frac{D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j) \qquad D_j \supset \mathbf{B}_j(C_j) \wedge \mathbf{B}_j(D_i)}{D_i \supset \mathbf{Pc}_i(\boldsymbol{C})}.$$

As mentioned in Section 1.1, the operator $\mathbf{Pc}_i(\cdot, \cdot)$ has two roles: (i) the interface from the external world to person i's internal mind and (ii) one's logical reasoning including the other's reasoning. An example illustrating these roles will be discussed in Section 5. Axiom AEM and Rule IEM describe interpersonal reasoning. Indeed, Axiom AEM directly includes interpersonal reasoning. Rule IEM literally states that if other formulae D_i and D_j satisfy the same property as postulated by AEM, D_i implies $\mathbf{Pc}_i(\boldsymbol{C})$, i.e., it is the deductively weakest in the formulae having the property described by AEM. Incidentally, AEM and IEM form the fixed-point, which will be shown in Theorem 2.2.

Now, a *proof* in logic REL_β is a triple $\langle X, >; \varphi\rangle$ with the following conditions:

P1: $(X, >)$ is a finite tree with the root x_0;

P2: $\varphi : X \to \mathcal{P}_\beta$ such that

(i): if x is a leaf of the tree $(X, >)$, then $\varphi(x)$ is an instance of L1-L4, K, D, or AEM in $\mathcal{P}_\beta$;

(ii): for any non-leaf $x \in X$,

$$\frac{\{\varphi(y) : y \text{ is an immediate predecessor of } x\}}{\varphi(x)}$$

is an instance of MP, $\wedge$-rule, Nec, or IEM in $\mathcal{P}_\beta$.

When $\varphi(x_0) = A$, we call $\langle X, >; \varphi\rangle$ a *proof of* A in REL_β. We say that A is provable in REL_β, denoted by $\vdash_\beta A$, iff there is a proof of A in REL_β.

When $\vdash_\beta A$, all formulae in the proof of A are required to be in $\mathcal{P}_\beta$. However, the negative assertion $\nvdash_\beta A$ can arise in two different cases:

(*P-a*): $A \in \mathcal{P}_\beta$, but there is no proof of A in REL_β; and (*P-b*): $A \notin \mathcal{P}_\beta$.

When $A \in \mathcal{P}_\beta$, it is enough to think about candidate proofs of A in REL_β, which is unprovability in the standard sense. In the alternative case $A \notin \mathcal{P}_\beta$, we have $\nvdash_{\beta'} A$ directly by the definition of a proof of A.

An application of logic REL_β to a social problem takes the form of a (formal) theory described within logic REL_β. For this, we introduce *non-logical axioms* in REL_β. Let Γ be a subset of $\mathcal{P}_\beta$. We define $\Gamma \vdash_\beta A$ iff $\vdash_\beta A$ or $\vdash_\beta \wedge\Phi \supset A$ for some nonempty finite subset Φ of Γ. If $\Gamma = \emptyset$, this is $\vdash_\beta A$. We can regard Γ as a set of non-logical axioms, and we will use this for an application in Section 5.

Lemma 2.1 states some basic facts on the provability relation $\vdash_\beta$. These should be proved since we adopt a particular axiomatization of classical logic and that of epistemic logic KD^2 with the set of formulae $\mathcal{P}_\beta$. Proofs of $\langle 0\rangle$ and $\langle 1\rangle$ are found in Mendelson [26], p.31, and a proof of $\langle 2\rangle$ is in Kaneko [17], Lemma 11.1. A proof of $\langle 3\rangle$ is given in [17], Lemma 4.1.(3), p.25. We use them without referring to Lemma 2.1.

Lemma 2.1 For any $A, B, C, \wedge\mathbf{B}_i(\Phi) \in \mathcal{P}_\beta$, and $i = 1, 2$,

$\langle \mathbf{0}\rangle$: $\vdash_\beta A \supset A$; $\langle \mathbf{1}\rangle$: $\vdash_\beta A \supset B$ and $\vdash_\beta B \supset C$ imply $\vdash_\beta A \supset C$;

$\langle \mathbf{2}\rangle$: $\vdash_\beta [A \wedge B \supset C] \equiv [A \supset (B \supset C)]$; $\langle \mathbf{3}\rangle$: $\vdash_\beta \mathbf{B}_i(\wedge\Phi) \supset \wedge\mathbf{B}_i(\Phi)$ and $\vdash_\beta \wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$.

Here, we describe person i's inference ability through the reciprocal empathization operator $\mathbf{Pc}_i(\cdot,\cdot)$. Theorem 2.1 states that the KD-properties of the belief operator $\mathbf{B}_i(\cdot)$ are preserved by $\mathbf{Pc}_i(\cdot,\cdot)$ as an extension of the unary operator $\mathbf{B}_i(\cdot)$ to the binary operator $\mathbf{Pc}_i(\cdot,\cdot)$. Proofs of the assertions are given in Section 2.3; we follow the convention that the results given in each section will be proved in the last subsection, except for the completeness part of Theorem 3.1.

Theorem 2.1 (Logical properties of $\mathbf{Pc}_i(\cdot,\cdot)$) Let β be in $N \cup \{\omega\}$ with $\beta \geq 3$ and $i = 1, 2$. Assume that all the formulae in the assertions are in $\mathcal{P}_\beta$.

$\langle \mathbf{1} \rangle$(**Necessitation**) If $\vdash_\beta A_1 \wedge A_2$, then $\vdash_\beta \mathbf{Pc}_i(A_1, A_2)$.

$\langle \mathbf{2} \rangle$(**Unilateral K**) $\vdash_\beta \mathbf{Pc}_i(A_1 \supset B_1, A_2) \wedge \mathbf{Pc}_i(A_1, A_2) \supset \mathbf{Pc}_i(B_1, A_2)$ and

$$\vdash_\beta \mathbf{Pc}_i(A_1, A_2 \supset B_2) \wedge \mathbf{Pc}_i(A_1, A_2) \supset \mathbf{Pc}_i(A_1, B_2).$$

$\langle \mathbf{3} \rangle$(**Bilateral K**) $\vdash_\beta \mathbf{Pc}_i(A_1 \supset B_1, A_2 \supset B_2) \wedge \mathbf{Pc}_i(A_1, A_2) \supset \mathbf{Pc}_i(B_1, B_2)$.

$\langle \mathbf{4} \rangle$($\mathbf{Pc}_i$-$\wedge$) $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi, B_2) \equiv \wedge\{\mathbf{Pc}_i(A_1, B_2) : A_1 \in \Phi\}$ and

$$\vdash_\beta \mathbf{Pc}_i(A_1, \wedge\Phi) \equiv \wedge\{\mathbf{Pc}_i(A_1, B_2) : B_2 \in \Phi\}.$$

$\langle \mathbf{5} \rangle$(**D**) $\vdash_\beta \mathbf{Pc}_i(\bot, B) \supset \bot$ and $\vdash_\beta \mathbf{Pc}_i(A, \bot) \supset \bot$.

Thus, the inference properties owned by $\mathbf{Pc}_i(\cdot,\cdot)$ are parallel to the corresponding properties of $\mathbf{B}_i(\cdot)$. In addition to these, Theorem 2.2 shows the reciprocity of empathized beliefs. The contrapositive of the only-if assertion of $\langle 2 \rangle$ is obtained in the sense of $(P\text{-}b)$ above.

Theorem 2.2 (Fixed-point properties) Let $\mathbf{Pc}_i(\boldsymbol{C}) \in \mathcal{P}$ and $i = 1, 2$.

$\langle \mathbf{1} \rangle$**(Interpersonal fixed-point):** If $\mathbf{Pc}_i(\boldsymbol{C}) \in \mathcal{P}_\beta$, then

$$\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{B}_i[C_i \wedge \mathbf{Pc}_j(\boldsymbol{C})]. \tag{3}$$

$\langle \mathbf{2} \rangle$**(Depths for reciprocal empathized beliefs):** $\delta(\mathbf{Pc}_i(\boldsymbol{C})) \leq \beta - 2$ if and only if

$$\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{B}_i[C_i \wedge \mathbf{B}_j[C_j \wedge \mathbf{Pc}_i(\boldsymbol{C})]]. \tag{4}$$

The right-hand of (3) expresses the idea that person i perceives the current state of affair and enters into the mind of person j to acknowledge j's perception of the current state of affair. This is one step of reciprocal empathization. Then, $\langle 2 \rangle$ asserts that the process returns to $\mathbf{Pc}_i(\boldsymbol{C})$ in person i's own perception through simulating j's thinking. Thus, (4) is a fixed-point statement with respect to $\mathbf{Pc}_i(\boldsymbol{C})$.

Calculating the depth of the formula in (4), we find $\delta(\mathbf{Pc}_i(\boldsymbol{C})) \leq \beta - 2$, which implies $\mathbf{Pc}_i(\boldsymbol{C}) \in \mathcal{P}_\beta$ by (2). This is coherent with our purpose to study reciprocal empathization. In other words, when $\beta \leq 2$, logic REL_β has no capacity to describe reciprocal empathization. When $\beta = 3$, only non-epistemic formulae, i.e., $\delta(C_1) = \delta(C_2) = 0$, are the objects for reciprocal empathization. This case will be used in an application in Section 5.

Theorem 2.2.$\langle 2 \rangle$ means that in the two mirrors example in Section 1.1, full reciprocity is revealed in the second inner mirror image of A. As stated, it is enough to notice this reciprocity for practical understanding of AEM and IEM. Latency and infinity will be studied in more detailed manner, which will be given in Section 4.

It may be better to put subscript i to the Axiom and Rule for $\mathbf{Pc}_i(\cdot,\cdot)$ as AEM_i and IEM_i to emphasize that they describe person i's logical inferences. In particular, the upper formulae of Rule IEM are symmetric for i and j. To emphasize person i's inference, it may be helpful to change it to the assertion for $i = 1, 2$ that for any $\mathbf{Pc}_i(\boldsymbol{C}), \mathbf{B}_j(D_i) \in \mathcal{P}_\beta$, *there is* a $\mathbf{B}_i(D_j) \in \mathcal{P}_\beta$ such that the inference described above is assumed. Nevertheless, this formulation is equivalent to the above for IEM.

2.3 Proofs

Proof of Theorem 2.1.⟨1⟩ Suppose $\vdash_\beta A_i$ for $i = 1, 2$. Let $D_i = A_i$ for $i = 1, 2$. Then, by Nec, we have $\vdash_\beta \mathbf{B}_i(D_k)$ for $i, k = 1, 2$. Thus, $\vdash_\beta \mathbf{B}_i(D_1) \wedge \mathbf{B}_i(D_2)$. Since $\vdash_\beta \mathbf{B}_i(D_1) \wedge \mathbf{B}_i(D_2) \supset (D_i \supset \mathbf{B}_i(D_1) \wedge \mathbf{B}_i(D_2))$ by Axiom L1, we have $\vdash_\beta D_i \supset \mathbf{B}_i(D_1) \wedge \mathbf{B}_i(D_2)$ for $i = 1, 2$, which are expressed as $\vdash_\beta D_i \supset \mathbf{B}_i(A_i) \wedge \mathbf{B}_i(D_j)$ for $i = 1, 2$. Since thy are the upper formulae of inference IEM, we have $\vdash_\beta D_i \supset \mathbf{Pc}_i(A_1, A_2)$ for $i = 1, 2$. Since $D_i = A_i$ for $i = 1, 2$, we have $\vdash_\beta \mathbf{Pc}_i(A_1, A_2)$ for $i = 1, 2$.

⟨**2**⟩ We show only $\vdash_\beta \mathbf{Pc}_i(A_1, A_2 \supset B_2) \wedge \mathbf{Pc}_i(A_1; A_2) \supset \mathbf{Pc}_i(A_1, B_2)$ for $i = 1, 2$. The other is shown similarly. Denote $F_i = \mathbf{Pc}_i(A_1, A_2 \supset B_2) \wedge \mathbf{Pc}_i(A_1, A_2)$ for $i = 1, 2$. By Axiom AEM, $\vdash_\beta F_1 \supset \mathbf{B}_1(A_1)$. Similarly, we have $\vdash_\beta F_2 \supset \mathbf{B}_2(A_2)$ and $\vdash_\beta F_2 \supset \mathbf{B}_2(A_2 \supset B_2)$. Thus, $\vdash_\beta F_2 \supset \mathbf{B}_2(B_2)$. By Axiom AEM, we have $\vdash_\beta F_i \supset \mathbf{B}_i(F_j)$ for $i = 1, 2$. Thus, $\vdash_\beta F_1 \supset \mathbf{B}_1(A_1) \wedge \mathbf{B}_1(F_2)$ and $\vdash_\beta F_2 \supset \mathbf{B}_2(B_2) \wedge \mathbf{B}_2(F_1)$. By Rule IEM, we have $\vdash_\beta F_i \supset \mathbf{Pc}_i(A_1, B_2)$ for $i = 1, 2$.

⟨**3**⟩ Denote $F_i = \mathbf{Pc}_i(A_1 \supset B_1, A_2 \supset B_2) \wedge \mathbf{Pc}_i(A_1, A_2)$ for $i = 1, 2$. By Axiom AEM and Lemma 2.1.⟨3⟩, $\vdash_\beta F_i \supset \mathbf{B}_i(A_i \supset B_i) \wedge \mathbf{B}_i(A_i)$. Hence, $\vdash_\beta F_i \supset \mathbf{B}_i(B_i)$ for $i = 1, 2$. Similarly, we have $\vdash_\beta F_i \supset \mathbf{B}_i(F_j)$ for $i = 1, 2$. Thus, $\vdash_\beta F_i \supset \mathbf{B}_i(B_i) \wedge \mathbf{B}_i(F_j)$ for $i = 1, 2$. By Rule IEM, we have $\vdash_\beta F_i \supset \mathbf{Pc}_i(B_1, B_2)$.

⟨**4**⟩ First, we show $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi, A_2) \supset \wedge\{\mathbf{Pc}_i(A, A_2) : A \in \Phi\}$. Let A be arbitrary in Φ. Then, $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi \supset A, A_2 \supset A_2) \supset (\mathbf{Pc}_i(\wedge\Phi, A_2) \supset \mathbf{Pc}_i(A, A_2))$ by Lemma 2.1.⟨2⟩ as well as ⟨3⟩ above, and also $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi \supset A, A_2 \supset A_2)$ by ⟨1⟩. Hence, $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi, A_2) \supset \mathbf{Pc}_i(A, A_2)$. Since A is arbitrary in Φ, we have $\vdash_\beta \mathbf{Pc}_i(\wedge\Phi, A_2) \supset \wedge\{\mathbf{Pc}_i(A, A_2) : A \in \Phi\}$ by $\wedge$-rule.

Consider the converse. Let $D_i = \wedge\{\mathbf{Pc}_i(A, A_2) : A \in \Phi\}$ for $i = 1, 2$. Since $\vdash_\beta D_1 \supset \mathbf{B}_1(A)$ for all $A \in \Phi$ and $\vdash_\beta D_1 \supset \wedge\mathbf{B}_1(\Phi)$, we have, by Lemma 2.1.⟨3⟩, $\vdash_\beta D_1 \supset \mathbf{B}_1(\wedge\Phi)$. Similarly, we have $\vdash_\beta D_1 \supset \mathbf{B}_1(\wedge\{\mathbf{Pc}_2(A, A_2) : A \in \Phi\})$. Thus, $\vdash_\beta D_1 \supset \mathbf{B}_1(\wedge\Phi) \wedge \mathbf{B}_1(\wedge\{\mathbf{Pc}_2(A, A_2) : A \in \Phi\})$. Similarly, we have $\vdash_\beta D_2 \supset \mathbf{B}_2(A_2) \wedge \mathbf{B}_2(\wedge\{\mathbf{Pc}_1(A, A_2) : A \in \Phi\}$. These are written as $\vdash_\beta D_1 \supset \mathbf{B}_1(\wedge\Phi) \wedge \mathbf{B}_1(D_2)$ and $\vdash_\beta D_2 \supset \mathbf{B}_2(A_2) \wedge \mathbf{B}_2(D_1)$. Regarding these as the upper formulae of Rule IEM, we have $\vdash_\beta D_i \supset \mathbf{Pc}_i(\wedge\Phi, A_2)$ for $i = 1, 2$.

⟨**5**⟩ We prove $\vdash_\beta \mathbf{Pc}_i(\perp, B) \supset \perp$ for $i = 1, 2$. Let $D_i = \mathbf{Pc}_i(\perp, B), i = 1, 2$. By AEM, we have $\vdash_\beta D_1 \supset \mathbf{B}_1(\perp) \wedge \mathbf{B}_1(D_2)$ and $\vdash_\beta D_2 \supset \mathbf{B}_2(B) \wedge \mathbf{B}_2(D_1)$. By Axiom D for $\mathbf{B}_1(\cdot)$, we have $\vdash_\beta D_1 \supset \perp$. By the second, we have $\vdash_\beta D_2 \supset \mathbf{B}_2(D_1)$. Hence, $\vdash_\beta D_2 \supset \mathbf{B}_2(\perp)$. By Axiom D for $\mathbf{B}_2(\cdot)$ again, we have $\vdash_\beta D_2 \supset \perp$.$\blacksquare$

Proof of Theorem 2.2.⟨1⟩: It suffices to prove $\vdash_\beta \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C}) \supset \mathbf{Pc}_i(\boldsymbol{C})$. Let $D_k = \mathbf{B}_k(C_k) \wedge \mathbf{B}_k\mathbf{Pc}_{k'}(\boldsymbol{C}), k, k' = 1, 2$ $(k \neq k')$. We should prove that the upper formulae of Rule IEM are provable in REL_β, that is,

$$\vdash_\beta D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j) \text{ and } \vdash_\beta D_j \supset \mathbf{B}_j(C_j) \wedge \mathbf{B}_j(D_i). \tag{5}$$

Once this is proved, we have $\vdash_\beta D_i \supset \mathbf{Pc}_i(\boldsymbol{C})$ by Rule IEM. First, we observe $\delta(D_k) = \delta(\mathbf{Pc}_k(\boldsymbol{C})) + 1$ for $k = 1, 2$. Indeed, since $\delta(\mathbf{Pc}_j(\boldsymbol{C})) + 2 \leq \beta$ by (1), we have $\delta(\mathbf{B}_k(D_{k'})) = \delta(D_{k'}) + 1 = \delta(\mathbf{B}_{k'}\mathbf{Pc}_k(\boldsymbol{C})) + 1 = (\delta(\mathbf{Pc}_k(\boldsymbol{C})) + 1) + 1 = \delta(\mathbf{Pc}_k(\boldsymbol{C})) + 2 \leq \beta$. Thus, both formulae of (5) are permissible in $\mathcal{P}_\beta$.

By Lemma 2.1.$\langle 1\rangle$, $\vdash_\beta D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$. Noting $\delta(\mathbf{B}_i[\mathbf{B}_j(C_j) \wedge \mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})]) \leq \beta$ by (1), by Nec, $\vdash_\beta \mathbf{B}_i[\mathbf{Pc}_j(\boldsymbol{C}) \supset \mathbf{B}_j(C_j) \wedge \mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})]$, which implies, by Axiom K, $\vdash_\beta \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C}) \supset \mathbf{B}_i[\mathbf{B}_j(C_j) \wedge \mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})]$. Thus, $\vdash_\beta D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i[\mathbf{B}_j(C_j) \wedge \mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})]$, i.e., $\vdash_\beta D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j)$. This holds for $i = 1, 2$. Hence, we have both statements of (5).

$\langle\mathbf{2}\rangle$:**(Only-if)**: By $\langle 1\rangle$, we have $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{B}_i[C_i \wedge \mathbf{Pc}_j(\boldsymbol{C})]$ for $i = 1, 2$. By Nec and K, we have $\vdash_\beta \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C}) \equiv \mathbf{B}_i[\mathbf{B}_j[C_j \wedge \mathbf{Pc}_i(\boldsymbol{C})]]$. We have, abbreviating $\vdash_\beta$, the following calculation in REL_β; $\mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{B}_i(C_i) \wedge \mathbf{B}_i[\mathbf{Pc}_j(\boldsymbol{C})] \equiv \mathbf{B}_i(C_i) \wedge \mathbf{B}_i[\mathbf{B}_j[C_j \wedge \mathbf{Pc}_i(\boldsymbol{C})]] \equiv \mathbf{B}_i[C_i \wedge \mathbf{B}_j[C_j \wedge \mathbf{Pc}_i(\boldsymbol{C})]]$. By (1), the last equivalent formulae are in $\mathcal{P}_\beta$.

(if): By definition of provability, the formula of (4) is in $\mathcal{P}_\beta$, i.e., $\mathbf{Pc}_i(\boldsymbol{C}) \in \mathcal{P}_\beta$, which implies $\delta(\mathbf{Pc}_i(\boldsymbol{C})) \leq \beta - 2$ by (1).■

3 Kripke Semantics for REL_β

Logic REL_β ($0 \leq \beta \leq \omega$) is concise for presenting provable formulae, but the corresponding Kripke semantics is complementary to have evaluations of negative and/or structural results in REL_β. The main result is the soundness/completeness theorem for REL_β ($0 \leq \beta \leq \omega$) with the restricted set of formulae $\mathcal{P}_\beta$ given by (2). It reveals the hierarchal structure of our logics indexed by depth β in Table 1.1 more clearly. Indeed, as an immediate corollary, we obtain a conservative extension result over REL_β and $\mathrm{REL}_{\beta'}$ with $\beta < \beta' \leq \omega$.

3.1 Kripke models and valuations

We define the Kripke frame for REL_β, which is uniform over different depths β ($0 \leq \beta \leq \omega$). A *Kripke frame* $K = (W, R_1, R_2)$ is given as a triple of a nonempty set of *possible worlds* and *accessibility relations* R_i, $i = 1, 2$. Each R_i is a binary relation over W satisfying the *seriality* condition: for each $w \in W$, wR_iu for some $u \in W$. A *Kripke model* M is a pair of a Kripke frame $K = (W; R_1, R_2)$ and a truth assignment $\sigma : W \times PV \to \{\top, \bot\}$. We focus on serial Kripke models $M = (K, \sigma)$, which will be simply called a *Kripke model*.

We need one more concept; we say that a finite sequence $\langle(w_0, i_0), (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu\rangle$ is an *al-chain* (*alternating chain*) from (w, i) iff $\nu \geq 1$, $w_0, w_1, ..., w_\nu$ are possible worlds in W and $i_0, i_1, ..., i_{\nu-1}$ are players in $\{1, 2\}$ with $(w_0, i_0) = (w, i)$ such that

$$w_t R_{i_t} w_{t+1} \text{ for } t \leq \nu - 1 \text{ and } i_t \neq i_{t+1} \text{ for } t \leq \nu - 2.$$

By seriality, for any $w \in W$ and $i = 1, 2$, there is always an al-chain of any length $\langle(w_0, i_0), (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu\rangle$ from (w, i). The set of al-chains is infinite, even if W is a finite set.

Now, we define the *valuation (unary) relation* $(M, w) \models C$ for any $w \in W$ inductively with respect to the lengths of formulae by

V0: for any $p \in PV$, $(M, w) \models p$ if and only if $\sigma(w, p) = \top$;

V1: $(M, w) \models \neg C$ if and only if $(M, w) \not\models C$;

V2: $(M, w) \models A \supset C$ if and only if $(M, w) \not\models A$ or $(M, w) \models C$;

V3: $(M, w) \models \wedge\Phi$ if and only if $(M, w) \models C$ for all $C \in \Phi$;
for any $\mathbf{B}_i(C) \in \mathcal{P}_\beta$ and $i = 1, 2$,

V4: $(M, w) \models \mathbf{B}_i(C)$ if and only if $(M, u) \models C$ for any $u \in W$ with wR_iu;

for any $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2) \in \mathcal{P}_\beta$ and $i = 1, 2$,

V5: $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$ if and only if $(M, w_\nu) \models C_{i_{\nu-1}}$ for any al-chain $\langle (w_0, i_0), (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu \rangle$ from $(w_0, i_0) = (w, i)$.

V0 to V4 are standard, but V5 is new and needs some comments. The right-hand side has no constraints on the lengths of al-chains, i.e., $(M, w_\nu) \models C_{i_{\nu-1}}$ is required even for any w_ν with distance from w further than the distance β from w. Only in the both sides, the valuated formulae are required to be in $\mathcal{P}_\beta$. This uniformity provides conservativeness, which will be discussed after Theorem 3.1.

We say that a formula $C \in \mathcal{P}_\beta$ is *valid* iff $(M, w) \models C$ for all models $M = ((W, R_1, R_2), \sigma)$ and all $w \in W$. The validity of C is denoted by $\models C$.

We have the soundness-completeness theorem for logic REL_β with respect to the above semantics. A proof of the soundness (only-if) part is given in Section 3.2. The completeness (if) part will be proved in Sections 7 and 8.

Theorem 3.1 (Soundness-completeness) Let β be any ordinal with $0 \leq \beta \leq \omega$ and $A \in \mathcal{P}_\beta$. Then, $\vdash_\beta A$ if and only if $\models A$.

This theorem covers the series of logics REL_β's with $0 \leq \beta \leq \omega$ in Table 1.1. It starts with REL_0 (classical logic), $\mathrm{REL}_1 = \mathrm{KD}_1$,$\mathrm{REL}_2 = \mathrm{KD}_2$, and then from REL_3 to REL_ω. The operators $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$ appear from REL_3 to REL_ω, where reciprocal empathization is possible. In REL_3, no nested occurrences of $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$ are allowed, REL_β with $\beta \geq 4$ has some freedom, and REL_ω has no constraints on interpersonal reasoning. We are interested in reciprocal empathization included from REL_3 to REL_ω. From the viewpoint of applications, the logics REL_β with $\beta = 3$ or slightly higher are more important than those with large β. $\mathrm{REL}_2 = \mathrm{KD}_2$ is regarded as a logic of mere empathization, which will be discussed in Sections 5 and 9.[5]

In the proof of completeness, we construct a finite countermodel. This implies the finite model property.

Theorem 3.2 (Finite model property) Let β be any ordinal with $0 \leq \beta \leq \omega$. For any $A \in \mathcal{P}_\beta$, if $\nvdash_\beta A$, there is a finite model $M = (K, \sigma)$ such that $(M, w) \not\models A$ for some $w \in W$.

Thus, we can restrict the semantics to the set of finite models to have equivalence between provability and validity.

The idea of a conservative extension is basic to comparisons between two logics (/theories) in the literature of mathematical logic.[6] In this paper, we give two results on conservative extensions, which will play crucial roles in Section 4 for the consideration of latent infinity in terms of explicit/implicit definability of $\mathbf{Pc}_i(C_1, C_2)$. First, it is directly applied to comparisons between logics REL_β and $\mathrm{REL}_{\beta'}$ when $\beta' > \beta$.

Theorem 3.3 (Conservative extension 1) Let $0 \leq \beta \leq \beta' \leq \omega$. Then, for any $C \in \mathcal{P}_\beta$,

$$\vdash_\beta C \text{ if and only if } \vdash_{\beta'} C. \tag{6}$$

[5] KD_1 is the KD system with no nested occurrences of $\mathbf{B}_i(\cdot), i = 1, 2$, and KD_2 is the KD system allowing only nest occurrences of depth 2. In general, KD_β allows nest occurrences of $\mathbf{B}_i(\cdot), i = 1, 2$ up to β. These are special cases of Kaneko-Suzuki [21]'s KD logics with shallow depths. In that approach, more individualistic versions of nested structures are discussed with respect to $\mathbf{B}_i(\cdot), i = 1, 2$.

[6] For example, the axiomatic set theory NBG is an conservative extension of ZF (cf., Mendelson [26], Chap.4, p.224).

A proof is straightforward from Theorem 3.1. Specifically, for any $C \in \mathcal{P}_\beta$ with $\beta \leq \beta' \leq \omega$, the semantical valuation relation $(M, w) \models C$ is uniform on $\beta' \leq \omega$. Using this fact, when $\beta \leq \beta' \leq \omega$ and $C \in \mathcal{P}_\beta \subseteq \mathcal{P}_{\beta'}$, it holds that $\vdash_\beta C$ if and only if $\models C$ if and only if $\vdash_{\beta'} C$. Thus, we have (6).

As mentioned in Section 1, our logical approach to social situations has emphasis on small β. The left part, with small β, of Table 1 are more important than the right part with large β. As long as the target formulae are restricted within the bound β, this already captures the provability of $\mathrm{REL}_{\beta'}$. We will see the power of conservativeness in the study of the "latent infinity" in Section 4.

Kaneko-Suzuki [21] gave various meta-theorems in epistemic logic KD_ω and its subsystems with shallow epistemic depths. The following corollary is useful particularly for applications of their meta-theorems. For β ($0 \leq \beta \leq \omega$), logic KD_β is obtained from REL_β by restricting $\mathcal{P}_\beta$ to $\{A \in \mathcal{P}_\beta : A$ contains no $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2\}$. By Theorem 3.1, the soundness-completeness theorem holds for KD_β. The following corollary enables us to convert the meta-theorems given in [21] to KD_ω.

Corollary 3.4 (Conservative extension 2) REL_ω is a conservative extension of KD_ω.

3.2 Proof of soundness

Soundness is proved by the induction over a proof $\langle (X, >), \varphi \rangle$ of a formula A in REL_β from its leaves. For a leaf $x \in X$, $\varphi(x)$ is an instance of the axiom schemata L1-L4, K, D, and AEM. We check the validity of each instance of these axioms. Then, the induction step is to show that validity is preserved by the inference rules, MP, $\wedge$-Rule, Nec, and IEM. By the principle of mathematical induction, we have the validity of A.

Let A be an instance of L1 to L4 or K, D. It is standard to show that $(M, w) \models A$ for any model $M = (K, \sigma)$ and any $w \in W$. This does not depend upon depth β. Also, it is proved in the standard manner that inference rules MP, $\wedge$-rule, and Nec preserve validity. This step does not depend upon β, either.

Now, we prove that Axiom AEM is valid; for any $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2) \in \mathcal{P}_\beta$, $\models \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$ for $i = 1, 2$. It suffices to show for any Kripke model $M = ((W, R_1, R_2), \sigma)$ and any world $w \in W$, $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$. Suppose that $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$. By V5, it holds that $(M, w_\nu) \models C_{i_{\nu-1}}$for any al-chain $\langle (w_0, i_0), (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu \rangle$ with $(w_0, i_0) = (w, i)$. When $\nu = 1$, this implies $(M, w_1) \models C_{i_0}$ $(= C_i)$. Hence, $(M, w) \models \mathbf{B}_i(C_i)$. Returning an arbitrary ν, we have $(M, w_\nu) \models C_{i_{\nu-1}}$ since $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$. We focus on al-chain $\langle (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu \rangle$ with $(w_1, i_1) = (w_1, j)$ from (w_1, i_1). Since this al-chain is arbitrary, we have $(M, w_1) \models \mathbf{Pc}_j(\boldsymbol{C})$. Since w_1 is arbitrary and wR_iw_1, we have $(M, w) \models \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{A})$. Thus, $(M, w) \models \mathbf{Pc}_i(\boldsymbol{A}) \supset \mathbf{B}_i(A_i) \wedge \mathbf{B}_i\mathbf{Pc}_j(A)$.

Finally, we prove that Rule IEM preserves validity $\models$. Suppose that for some $\mathbf{Pc}_i(\boldsymbol{C}), D_i \in \mathcal{P}_\beta$ $(i = 1, 2)$,

$$\models D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j) \text{ for } i, j = 1, 2 \text{ and } j \neq i. \tag{7}$$

Let $M = ((W; R_1, R_2), \sigma)$ be any model and w any world in W. Suppose $(M, w) \models D_i$. Let $\langle (w_0, i_0), (w_1, i_1), ..., (w_{\nu-1}, i_{\nu-1}), (w_\nu, i_\nu), \cdots \rangle$ be an arbitrary infinite sequence from $(w_0, i_0) = (w, i)$ so that for any ν, its initial segment $\langle (w_0, i_0), ..., (w_{\nu-1}, i_{\nu-1}), w_\nu \rangle$ is an al-chain from

$(w_0, i_0) = (w, i)$. It suffices to show by induction over $\nu \geq 1$ that

$$(M, w_\nu) \models C_{i_{\nu-1}} \text{ for all } \nu \geq 1. \tag{8}$$

Once this is proved, by V5, we have $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$. Thus, $(M, w) \models D_i \supset \mathbf{Pc}_i(\boldsymbol{C})$ by V2.

Let us see (8). We focus on odd ν's, and prove by induction that for any odd $\nu \geq 1$,

$$(M, w_\nu) \models C_{i_{\nu-1}},\ (M, w_{\nu+1}) \models C_{i_\nu},\ \text{and } (M, w_{\nu+1}) \models D_{i_{\nu+1}}. \tag{9}$$

This is enough for (8) including even ν's. The induction base and induction step are uniformly proved, and we show it for any odd ν. Since $i_{\nu-1} = i$ and $w_{\nu-1} R_{i_{\nu-1}} w_\nu$, we have $(M, w_\nu) \models C_i$ and $(M, w_\nu) \models D_j$, by the first statement of (7) and the induction hypothesis $(M, w_{\nu-1}) \models D_{i_{\nu-1}}$ (this is the supposition of the entire proof when $\nu = 1$). Thus, we have shown $(M, w_\nu) \models C_{i_{\nu-1}}$ and $(M, w_\nu) \models D_{i_\nu}$. Now, we go to the case $\nu + 1$. By the second of (7), we have $(M, w_\nu) \models D_{i_\nu} \supset \mathbf{B}_{i_\nu}(C_{i_\nu}) \wedge \mathbf{B}_{i_\nu}(D_{i_{\nu+1}})$. Hence, $(M, w_\nu) \models \mathbf{B}_{i_\nu}(C_{i_\nu}) \wedge \mathbf{B}_{i_\nu}(D_{i_{\nu+1}})$. Since $w_\nu R_{i_\nu} w_{\nu+1}$, we have $(M, w_{\nu+1}) \models C_{i_\nu}$ and $(M, w_{\nu+1}) \models D_{i_{\nu+1}}$. We complete the proof of (9).■

4 Latent Infinity in Logic REL_β

We elucidate the "latent infinity" in logic REL_β, by studying the two concepts, *explicit definability* and *implicit definability* (cf., Troelstra-Schwichtenberg [35], Maksimova [24]), for pc-formulae $\mathbf{Pc}_i(\cdot, \cdot), i = 1, 2$.[7] In an infinitary extension of REL_β given in Hu, *et al.* [16], the formula $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2)$ is expressed as an infinitary conjunction of b-formulae. We show the explicit indefinability in REL_β for any β ($3 \leq \beta \leq \omega$) that there is no (finitary) formula to express $\mathbf{Pc}_i(\boldsymbol{C})$. On the other hand, it is shown that $\mathbf{Pc}_i(\cdot, \cdot)$ is operationally and uniquely determined in logic REL_β up to the formulae in $\mathcal{P}_\beta$. These mean that the latent infinity in REL_β is hidden in the operational property for any β ($3 \leq \beta \leq \omega$), and become explicitly expressed in the infinitary logic of [16].

4.1 Explicit indefinability of $\mathbf{Pc}_i(\cdot, \cdot)$ in logic REL_β

The *explicit definability* of $\mathbf{Pc}_i(\cdot, \cdot)$ in REL_β means that $\mathbf{Pc}_i(\boldsymbol{C})$ is expressed as a formula in REL_β including no occurrences of $\mathbf{Pc}_1(\cdot, \cdot)$ and $\mathbf{Pc}_2(\cdot, \cdot)$. First, Theorem 4.1 gives a necessary and sufficient bound m for $\mathbf{Pc}_i(\boldsymbol{C})$ to entail $\mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(C_{i_m})$. We use the same term for a sequence $(i_1, ..., i_m)$ of $1, 2$ as an al-chain in Section 3; it is an *al-chain* iff $i_t \neq i_{t+1}$ for $t = 1, ..., m-1$.

Theorem 4.1 (Surface to deeper layers) Let $\beta \leq \omega$, $\mathbf{Pc}_i(\boldsymbol{C}) \in \mathcal{P}_\beta$, and $(i_1, ..., i_m)$ an al-chain with $i_1 = i$ and $m \geq 1$. Then, $\delta(\mathbf{Pc}_i(\boldsymbol{C})) + m - 1 \leq \beta$ if and only if

$$\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(C_{i_m}). \tag{10}$$

Let $\beta < \omega$. When $\delta(C_1) = \delta(C_2) = 0$, the maximum length m is β, but if $\boldsymbol{C}$ contains some pc-subformulae, m is smaller than β. When $\beta = \omega$, the former statement of the assertion is vacuous, and the latter holds for all al-chains $(i_1, ..., i_m)$ with $i_1 = i$ and of any length $m < \omega$. Nevertheless, in REL_ω, it is not allowed to take the conjunction of the set of these beliefs.

[7] Explicit definability and implicit definability are equivalent in the first-order theory, known as Beth's theorem (see Troelstra-Schwichtenberg [35]).

On the other hand, we take the infinitary logic, denoted by HKS, of Hu, *et al.* [16] with a suitable choice of a language including the operator symbols $\mathbf{Pc}_i(\cdot,\cdot), i = 1,2$ with Axiom AEM and Rule IEM, where its provability relation by $HKS \vdash$. Then, it holds:

$$HKS \vdash \mathbf{Pc}_i(\boldsymbol{C}) \equiv \wedge\{\mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(C_{i_m}) : (i_1,...,i_m) \text{ is an al-chain with } i_1 = i\}. \tag{11}$$

The proof is straightforward in logic HKS. In this sense, $\mathbf{Pc}_i(\boldsymbol{C})$ is explicitly definable in logic HKS. In fact, this logic is a conservative extension of REL_β $(\beta \leq \omega)$; the provability $HKS \vdash$ coincides with $\vdash_\beta$ within the set of formulae $\mathcal{P}_\beta$.[8] Of course, since the above infinitary formula is not in $\mathcal{P}_\beta$, it could be conjectured that there is no formula in $\mathcal{P}_\beta$ such that it expresses $\mathbf{Pc}_i(\boldsymbol{C})$ in logic REL_β.

The following theorem gives an affirmative answer to this conjecture.

Theorem 4.2 (Explicit indefinability $\mathbf{Pc}_i(\cdot,\cdot)$) Let $3 \leq \beta \leq \omega$. Let $p_1, p_2 \in PV$. Then, there is no $C \in \mathcal{P}_\beta$ such that no pc-formula $\mathbf{Pc}_k(\cdot,\cdot), k = 1,2$ occurs as subformulae of C and $\vdash_\beta \mathbf{Pc}_i(p_1,p_2) \equiv C$.

Theorem 4.2 asserts that $\mathbf{Pc}_i(p_1,p_2)$ cannot be expressed by any formula without including $\mathbf{Pc}_k(\cdot,\cdot), k = 1,2$ in logic REL_β for $\beta \leq \omega$. This result is perfect contrast to (11); explicit definability never holds for REL_β for any $\beta \leq \omega$, but explicit definability appears in infinitary logic HKS.

The rough idea of a proof is as follows: suppose that such a C exists in $\mathcal{P}_\beta$. Note $\delta(C) < \omega$. Then $\vdash_\omega \mathbf{Pc}_i(p_1,p_2) \equiv C$. This and Theorem 4.1 for $\beta = \omega$ imply $\vdash_\omega C \supset \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(p_{i_m})$ for any al-chain $(i_1,...,i_m)$ with $i_1 = i$ and any $m < \omega$. We take an $m > \delta(C)$. Then, a formula $\mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(p_{i_m})$ of depth m is derived from C with the smaller depth $\delta(C)$ than m. Corollary 3.4 implies that Lemma 4.1 (the depth lemma of Kaneko-Suzuki [21]) can be applied and we have $\vdash_\omega \neg C$. By Theorem 3.3 (conservativity), $\vdash_\beta \neg C$. This will be shown to be impossible. A full proof will be given in Section 4.3.

4.2 Implicit definability of $\mathbf{Pc}_i(\cdot,\cdot)$ in REL_β

Here, we show that $\mathbf{Pc}_i(C_1,C_2)$ is still *implicitly* definable in the sense that $\mathbf{Pc}_i(C_1,C_2)$ is operationally determined uniquely in logic REL_β. To formulate this concept, we add binary operators $\mathbf{Pc}'_i(\cdot,\cdot), i = 1,2$ to the list of basic symbols in Section 2.1. We define the set of formulae $\mathcal{P}'_\beta$ in the same manner in Section 2.1, except for one additional step, (F$'$-v), for $\mathbf{Pc}'_i(C'_1,C'_2)$, $i = 1,2$, where C'_1, C'_2 are already defined by (F$'$-o) to (F$'$-iv). The depth $\delta(C')$ is defined so that it is the maximum nested depth counting all occurrences of $\mathbf{B}_i(\cdot)$, $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{Pc}'_i(\cdot,\cdot)$ in C' for $i = 1,2$. We assume the same axioms L1 to L4, K, D, Nec, $\wedge$-rule, Nec in $\mathcal{P}'_\beta$, while in addition to AEM and IEM for $\mathbf{Pc}_i(\cdot,\cdot)$, we add the corresponding AEM$'$ and IEM$'$ for $\mathbf{Pc}'_i(\cdot,\cdot)$ in $\mathcal{P}'_\beta$. This extended logic is denoted by $(\text{REL}'_\beta, \mathcal{P}'_\beta)$, and its provability relation by $\vdash'_\beta$. In fact, Theorem 4.1, where neither $\mathbf{Pc}_k(\cdot,\cdot)$ nor $\mathbf{Pc}'_k(\cdot,\cdot)$, $k = 1,2$ occurs in C, holds for this extension $(\text{REL}'_\beta, \mathcal{P}'_\beta)$, which is obtained by repeating the proof of Theorem 4.2. Hence, explicit definability does not hold.

Let us focus on the operational meanings of $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{Pc}'_i(\cdot,\cdot)$ in $(\text{REL}'_\beta, \mathcal{P}'_\beta)$. The following lemma states that $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{Pc}'_i(\cdot,\cdot)$ are deductively equivalent in REL'_β. That

[8] In fact, the soundness/completeness theorem for the above HKS needs small modifications of Kripke semantics and the proof given in Hu, *et. al* [16]. But since (11) is proved within HKS, the modifications are insubstantive.

is, although $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{Pc}'_i(\cdot,\cdot)$ are not expressed by a formula without including $\mathbf{Pc}_k(\cdot,\cdot)$ and $\mathbf{Pc}'_k(\cdot,\cdot)$, they are operationally equivalent.

Lemma 4.1 (Implicit definability 1): Let $\mathbf{Pc}_i(\boldsymbol{C}), \mathbf{Pc}'_i(\boldsymbol{C}') \in \mathcal{P}'_\beta$. If $\vdash'_\beta C_k \equiv C'_k$ for $k = 1, 2$, then, $\vdash'_\beta \mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{Pc}'_i(\boldsymbol{C}')$ for $i = 1, 2$.

This states that $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{Pc}'_i(\cdot,\cdot)$ are operationally equivalent within $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$. However, $\mathbf{Pc}_i(\cdot,\cdot)$ is originally in $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$, and $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$ is larger than $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$. To have a full answer to the question of implicit definability for $\mathbf{Pc}_i(\cdot,\cdot)$, we return to $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ from $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$.

Before this, let us see how larger than $\mathcal{P}_\beta$ the set $\mathcal{P}'_\beta$ is. Let $C \in \mathcal{P}_\beta$. Let λ be the number of occurrences, $\mathbf{Pc}_i(D_1, D_2), i = 1, 2$ in C. The cardinality of the set of formulae obtained from C by substituting, or not, $\mathbf{Pc}'_k(D_1, D_2), k = 1, 2$ for $\mathbf{Pc}_k(\cdot,\cdot)$ is 2^λ. These formulae are all in $\mathcal{P}'_\beta$. Thus, the set $\mathcal{P}'_\beta$ is much larger than $\mathcal{P}_\beta$. Nevertheless, we can embed $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$ into $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$.

We have the translation ϕ from $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$ to $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$; for any $C \in \mathcal{P}'_\beta$, $\phi(C)$ is the formula obtained from C by eliminating all occurrences "$'$" in C. Then, ϕ is an onto map from $\mathcal{P}'_\beta$ to $\mathcal{P}_\beta$. We have the following theorem, which means that $\mathbf{Pc}_i(\boldsymbol{C})$ is operationally defined without ambiguity. Recall the assumption $3 \leq \beta \leq \omega$.

Theorem 4.3 (Implicit definability 2): For any $C' \in \mathcal{P}'_\beta$, $\vdash'_\beta C'$ if and only if $\vdash_\beta \phi(C')$.

Thus, the extended logic $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$ is the same as $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ only with symbolic differences. In other words, the operational property of $\mathbf{Pc}_i(\boldsymbol{C})$ is fully determined even in $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$. This operational property is latent in the sense that it is not explicitly represented in $\mathcal{P}_\beta$.

Finally, let us see the soundness/completeness of $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ is converted to the soundness/completeness of $(\mathrm{REL}'_\beta, \mathcal{P}'_\beta)$. First, we adopt the set of formulae $\mathcal{P}'_\beta$; the valuations of V1 to V4 are the same forms, and V5 is applied to formulae $\mathbf{Pc}'_i(\boldsymbol{C})$. Then, by induction on the lengths of formulae, we have

$$\models C' \text{ if and only if } \models \phi(C'). \tag{12}$$

Let $C' \in \mathcal{P}'_\beta$ and $\vdash'_\beta C$. This provability is translated into $\vdash_\beta \phi(C')$ by Theorem 4.3. By Theorem 3.1, this is equivalent to $\models \phi(C')$, and by (12), we have $\models C$.

4.3 Proofs

Proof of Theorem 4.1.(If): Let $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_m}(C_{i_m})$ $(i = i_1 = 1, 2)$. These formulae are in $\mathcal{P}_\beta$ by the definition of a proof in REL_β. Hence, $\delta(C_1 \wedge C_2) + m \leq \beta$. This is written as $\delta(\mathbf{Pc}_i(\boldsymbol{C})) - 1 + m \leq \beta$, i.e., $\delta(\mathbf{Pc}_i(\boldsymbol{C})) + m - 1 \leq \beta$.

(Only-If): We prove (10) by induction over k $(1 \leq k \leq m)$). By Axiom AEM, we have $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i)$ for $i = 1, 2$. These are the claims of (10) for $k = 1$. Suppose the induction hypothesis that (10) holds for k $(1 \leq k \leq m - 1)$. Consider player $j \neq i$. Then, since $\vdash_\beta \mathbf{Pc}_j(\boldsymbol{C}) \supset \mathbf{B}_{j_1} \cdots \mathbf{B}_{j_k}(C_{j_k})$, we have $\vdash_\beta \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C}) \supset \mathbf{B}_i\mathbf{B}_{j_1} \cdots \mathbf{B}_{j_k}(C_{j_k})$ by Nec and K. Since $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$ by AEM, we have $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i\mathbf{B}_{j_1} \cdots \mathbf{B}_{j_k}(C_{j_k})$ for $i = 1, 2$. By the induction principle, we have (10).■

To prove Theorem 4.2, we refer to a theorem for KD_ω (cf., Theorem 5.5, p.185 in Kaneko-Suzuki [21]), which is regarded as a theorem in REL_ω by Corollary 3.4. The following lemma is a simplified form needed for the proof of Theorem 4.2.

Lemma 4.2 (Depth lemma) Let C, D be non-pc-formulae in $\mathcal{P}_\omega$ with $\delta(C) < k$. For a sequence $(i_1, ..., i_k)$ in $\{1, 2\}$, if $\mathrm{KD}_\omega \vdash C \supset \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_k}(D)$, then $\mathrm{KD}_\omega \vdash \neg C$ or $\mathrm{KD}_\omega \vdash D$.

We say that a formula C is consistent in REL_β iff $\nvdash_\beta \neg C$.

Proof of Theorem 4.2: Suppose, on the contrary, that there is a non-pc-formula $C \in \mathcal{P}_\beta$ such that $\vdash_\beta C \equiv \mathbf{Pc}_i(p_1, p_2)$. First, we see that C is consistent in REL_β. On the contrary, let C be inconsistent, i.e., $\vdash_\beta \neg C$, which implies $\vdash_\beta \neg \mathbf{Pc}_i(p_1, p_2)$. Consider the model $((W, R_1, R_2), \sigma)$ so that $W = \{w\}$ and R_i is reflexive for $i = 1, 2$, and $\sigma(w, p) = \top$ for all $p \in PV$. Then, $((W, R_1, R_2), \sigma) \models \mathbf{Pc}_i(p_1, p_2)$ by V5. By Theorem 3.1 (soundness), $\nvdash_\beta \neg \mathbf{Pc}_i(p_1, p_2)$, which is impossible. Thus, C is consistent in REL_β.

Since $C \in \mathcal{P}_\beta$, $\delta(C) := \ell < \omega$ even if $\beta = \omega$. Since $\vdash_\beta C \equiv \mathbf{Pc}_i(p_1, p_2)$, we have $\vdash_\omega C \equiv \mathbf{Pc}_i(p_1, p_2)$. Thus, we have by Theorem 4.1,

$$\vdash_\omega C \supset \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_{\ell+1}}(p_{i_{\ell+1}}), \tag{13}$$

where $(i_1, ..., i_{\ell+1})$ is an al-chain with $i_1 = i$. Since REL_ω is a conservative extension of KD_ω as stated in Corollary 3.4, the provability relation $\vdash_\omega$ in (13) can be replaced by the provability relation $\mathrm{KD}_\omega \vdash$.

Now, we apply Lemma 4.2 to the formula in (13) with $\mathrm{KD}_\omega \vdash$; thus, $\mathrm{KD}_\omega \vdash \neg C$ or $\mathrm{KD}_\omega \vdash p_{i_{\ell+1}}$. The latter is simply impossible. Hence, $\mathrm{KD}_\omega \vdash \neg C$; since REL_ω is an extension of KD_ω, we have $\vdash_\omega \neg C$. By Theorem 3.2 again, we have $\vdash_\beta \neg C$. This is impossible since C is consistent in REL_β, i.e., $\nvdash_\beta \neg C$. In sum, the supposition of the existence of a non-pc-formula C such that $\vdash_\beta C \equiv \mathbf{Pc}_i(p_1, p_2)$ is wrong, we have the assertion of Theorem 4.2.■

Proof of Lemma 4.1. Since the two operators are symmetric, it suffices to show that $\vdash'_\beta \mathbf{Pc}'_i(\boldsymbol{C}') \supset \mathbf{Pc}_i(\boldsymbol{C})$. Let $D_i = \mathbf{Pc}'_i(\boldsymbol{C}')$, $i = 1, 2$. By AEM for $\mathcal{P}^*_\beta$, we have $\vdash'_\beta \mathbf{Pc}'_i(\boldsymbol{C}') \supset \mathbf{B}_i(C'_i) \wedge \mathbf{B}_i\mathbf{Pc}'_j(\boldsymbol{C}')$ for $i = 1, 2$. These are expressed as $\vdash'_\beta D_i \supset \mathbf{B}_i(C'_i) \wedge \mathbf{B}_i(D_j)$ for $i = 1, 2$. Since $\vdash'_\beta C_k \equiv C'_k$ for $k = 1, 2$, we have $\vdash'_\beta D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j)$ for $i = 1, 2$. These are regarded as the upper formulae of IEM for $\mathcal{P}^*_\beta$, we have $\vdash'_\beta D_i \supset \mathbf{Pc}_i(\boldsymbol{C})$, i.e., $\vdash'_\beta \mathbf{Pc}'_i(\boldsymbol{C}) \supset \mathbf{Pc}_i(\boldsymbol{C})$ for $i = 1, 2$.■

Proof of Theorem 4.3. The *if* part means that REL'_β is an extension of REL_β. The *only if* part is essential. Let C be a given formula in $\mathcal{P}'_\beta$. Suppose that $P = (X, >, \varphi)$ is a proof A in REL'_β. We prove by induction of the proof structure from its leaves that $\vdash_\beta \phi \cdot \varphi(x)$ for all $x \in X$. Let x be a leaf in P. Then, $\varphi(x) = C$ is an instance of L1 to L4, K, D, and AEM. The formula $\phi \cdot \varphi(x) = \phi(C)$ preserves the structure of being the same axiom. Also, since $\phi(C) \in \mathcal{P}_\beta$, we have $\vdash_\beta \phi(C)$. Now, consider a non-leaf node x. We make an induction hypothesis that $\vdash_\beta \phi \cdot \varphi(y)$ for any immediate predecessors y of x. We need to consider the four cases: MP, $\wedge$-Rule, Nec, and IEM. The structures of MP, $\wedge$-Rule, and Nec are preserved by ϕ, and these inferences can be applied. Thus, $\vdash_\beta \phi \cdot \varphi(x)$. Finally, consider Rule IEM, where there two possibilities:

$$\frac{D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i(D_j) \quad D_j \supset \mathbf{B}_j(C_j) \wedge \mathbf{B}_j(D_j)}{D_i \supset \mathbf{Pc}'_i(C_1, C_2)},$$

and the other is the last formula in the lower case is $\mathbf{Pc}_i(C_1, C_2)$. The induction hypothesis is: $\vdash_\beta \phi(D_i) \supset \mathbf{B}_i(\phi(C_i)) \wedge \mathbf{B}_i(\phi(D_j))$ for $i = 1, 2$. Since these are upper formulae of IEM, we have in $\vdash_\beta \phi(D_i) \supset \mathbf{Pc}_i(\phi(C_1), \phi(C_2))$ for $i = 1, 2$. In the other case, we have the same conclusion.

Thus, by the induction principle, we have $\vdash_\beta \phi \cdot \varphi(x)$ for all $x \in X$.■

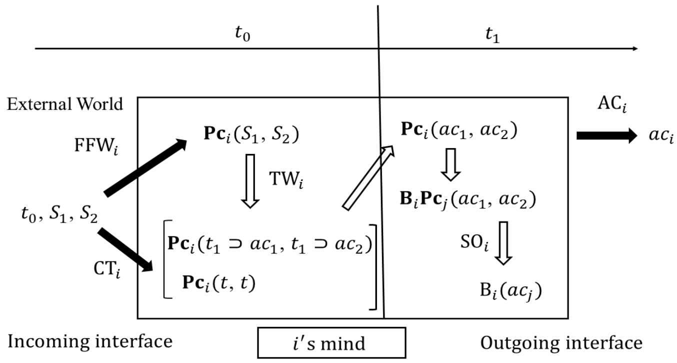

Figure 2: Interface betweem the external world and i's mind

5 An Application of Logic REL$_\beta$

Logic REL$_\beta$ provides a logical frame to describe a social situation where individuals think about action taken with reciprocal empathization operators $\mathbf{Pc}_1(\cdot,\cdot)$ and $\mathbf{Pc}_2(\cdot,\cdot)$. However, a social situation has its own structure not included in REL$_\beta$. In Section 5.1, we take the quotation (*) from Lewis [23] given in Section 1.2 and describe it as a theory within REL$_\beta$, specifying suitable atomic formulae. We keep the theory simple and tractable to highlight the importance of reciprocal empathization for each person's thought and action taking. In Section 5.2, we give a remark on the necessity of reciprocity, comparing with mere empathization.

5.1 A theory based on Lewis's example

We redescribe the quotation (*) in Section 1.2 by adding some supplemental words, in order to eliminate ambiguities, without changing the essential part of the story.

(**): Two persons i and j met today. At the end, both said, "I will come tomorrow." As they go home, each, reflecting on what was said, forms beliefs of his and the other's plans tomorrow. These beliefs then lead to the actual meeting tomorrow.

We use the conceptual scheme to understand (**) from person i's viewpoint:

(o): events in the external world;

(i): perception of an event in the *incoming interface* from the external world to i's mind;[9]

(ii): processing the perceived information through reciprocal empathization within i's mind;

(iii): bringing out an action in the *outgoing interface* from i's mind to the external world.

In Fig.2, the rectangular represents the mind of person i. (o) happens in the external world, and (i) and (iii) are the interface between the external world and i's mind, yet in the opposite directions. In (ii), his thought process works in his mind.

Based on the above scheme, we translate (**) into the language of REL$_\beta$ with $\beta \geq 3$. We use the following atomic symbols, instead of pure propositional variables:

[9] In dynamic epistemic logic, the incoming interface is formulated inside a logic from the viewpoint of semantics (and then from the syntactical viewpoint). See Ditmarsch, *et al.* [13] and Benthem-Smets [6] .

t_0 : day t_0 has arrived; and t_1 : day t_1 has arrived;

ac_i : person i comes; and $sy_i(\cdot)$: person i says "$\cdot$" ($i = 1, 2$).

Symbols t_0 and t_1 are interpreted as "today" and "tomorrow". Symbol ac_i means "person i comes", and $sy_i(\cdot)$ means that person i says "$\cdot$". In (**), when they leave, each i ($i = 1, 2$) says "I will come tomorrow", which is formally expressed as

$$sy_i(t_1 \supset ac_i). \tag{14}$$

This is the verbal utterance, and an operational meaning is given by a non-logical axiom. We abbreviate $sy_i(t_1 \supset ac_i)$ as S_i for $i = 1, 2$, and $\boldsymbol{S} = (S_1, S_2)$. In this example, the set of (pseudo) atomic formulae is given as $PV = \{t_0, t_1, S_1, S_2, ac_1, ac_2\}$. Here, $S_i, i = 1, 2$ have internal structures $sy_i(t_1 \supset ac_i)$.

We adopt the following non-logical axioms:

BSA (Beginning of the state of affairs): t_0, S_1, S_2.

ESA (End of the state of affairs): t_1.

These belong to (o), indicating the beginning and the ending of state of affairs in (**). Time t_0 plays only the role of starting the situation; t_0 occurs nowhere else in the following analysis. On the other hand, t_1 occurs as a subformula of $S_i = sy_i(t_1 \supset ac_i)$, $i = 1.2$, and it plays a specific role in the following.

The other non-logical axioms describe the two persons' observations, thoughts, and final actions to be taken, which we call *postulates*. We note that each of the following postulates is specific to person i, meaning that $\mathbf{Pc}_j(\cdot, \cdot)$ occurs in the mind of i. Keeping this note, we explain each postulate, say CT_i, as CT_1 and CT_2 together. Later, we will return to the note on individuality.

Persons 1 and 2 observe each of two points of time t_0, t_1, and also observe S_1, S_2 at time t_0.

Postulate 0 (CT$_i$)(Common perception of time): $t \supset \mathbf{Pc}_i(t, t)$, where $t = t_0, t_1$.

Postulate 1 (FFW$_i$)(Face to face exchange of words): $S_1 \wedge S_2 \supset \mathbf{Pc}_i(S_1, S_2)$.

These belong to category (i), stating that each of $1, 2$ receives the symbolic pieces of information and the other receiving them.

We postulate that the two persons trust each other with the contents of utterances; this belongs to (ii).

Postulate 2 (TW$_i$)(Trustworthiness of words): $\mathbf{Pc}_i[S_1 \supset (t_1 \supset ac_1), S_2 \supset (t_1 \supset ac_2)]$.

TW_i substantiates the symbolic utterances with the intended meanings, based on "*inductive standards and background information*" (Lewis [23], p.53), which are shared between i and j. This may be better understood by the equivalent formula, due to Theorem 2.2.$\langle 1 \rangle$,

$$\mathbf{B}_i((S_i \supset (t_i \supset ac_i)) \wedge \mathbf{Pc}_j[S_1 \supset (t_1 \supset ac_1), S_2 \supset (t_1 \supset ac_2)]). \tag{15}$$

Person i extracts the content $(t_i \supset ac_i)$ from S_i and believes that person j does the same, and also believe that j empathizes with person i as well.

The next postulate connects person i's internal thought with his action choice and anticipation of j's action choice.

Postulate 3 $(\mathbf{AC}_i)$**(Action in the external world)**: $\mathbf{Pc}_i(ac_1, ac_2) \supset ac_i$;

$(\mathbf{SO}_i)$**(Simulating the other's reasoning)**: $\mathbf{B}_i[\mathbf{Pc}_j(ac_1, ac_2) \supset ac_j]$.

AC_i means that person i induces ac_i to be acted in the external world from i's $\mathbf{Pc}_i(ac_1, ac_2)$. For SO_i, person i simulates person j's reasoning by projecting AC_i to person j, which forms i's anticipation of j's action to be taken. The latter is better understood by recalling that the premise of AC_i is equivalent to $\mathbf{B}_i(ac_i \wedge \mathbf{Pc}_j(ac_1, ac_2))$ by Theorem 2.2.$\langle 1 \rangle$; thus, it contains $\mathbf{Pc}_j(ac_1, ac_2)$ in the mind of person i. In this sense, AC_i and SO_i are regarded as two sides of one postulate. Nevertheless, AC_i is in (iii), but SO_i remains in (ii).

Let i be either 1 and 2. Theorem 5.1 describes the thought process along the state-of-affairs with/within i's mind.

Theorem 5.1 (Thoughts along the state-of-affairs) Let $\beta \geq 3$. We have the following thought process.

$\langle \mathbf{1} \rangle$**(Receipts of words)** $\{t_0, S_1, S_2\} \cup \{\mathrm{FFW}_i\} \vdash_\beta \mathbf{Pc}_i(\mathbf{S})$.

$\langle \mathbf{2a} \rangle$**(Trustworthiness of words)** $\{t_0, S_1, S_2\} \cup \{\mathrm{FFW}_i, \mathrm{TW}_i\} \vdash_\beta \mathbf{Pc}_i(t_1 \supset ac_1, t_1 \supset ac_2)$.

$\langle \mathbf{2b} \rangle$**(Action to be taken at** t_1**)** $\{t_0, S_1, S_2\} \cup \{\mathrm{FFW}_i, \mathrm{TW}_i\} \cup \{t_1\} \cup \{\mathrm{CT}_i\} \vdash_\beta \mathbf{Pc}_i(ac_1, ac_2)$.

$\langle \mathbf{3a} \rangle$**(Action taking)** $\{t_0, S_1, S_2\} \cup \{\mathrm{FFW}_i, \mathrm{TW}_i\} \cup \{t_1\} \cup \{\mathrm{CT}_i, \mathrm{AC}_i\} \vdash_\beta ac_i$.

$\langle \mathbf{3b} \rangle$**(Anticipation)** $\{t_0, S_1, S_2\} \cup \{\mathrm{FFW}_i, \mathrm{TW}_i\} \cup \{t_1\} \cup \{\mathrm{CT}_i, \mathrm{SO}_i\} \vdash_\beta \mathbf{B}_i(ac_j)$.

The process is described in Fig.3 from the objective events t_0, S_1, S_2 in the external world to the information processing within the mind of person i, and, after he fully conducts relevant (interpersonal) inferences, the process goes to the external word once more, except for his anticipation in $\langle 3b \rangle$ staying in his mind. Fig.3 can be regarded as simplified proofs of $\langle 1 \rangle$ to $\langle 3a \rangle$ and $\langle 3b \rangle$. It is from the non-logical axioms BSA, ESA, and postulates CT_i, FFW_i, TW_i, AC_i are connected by MP, $(\bullet)$, and $(\blacktriangledown)$:

$$\frac{\vdash_\beta \mathbf{Pc}_i(A_1 \supset C_1, A_2 \supset C_2) \quad \vdash_\beta \mathbf{Pc}_i(A_1, A_2)}{\vdash_\beta \mathbf{Pc}_i(C_1, C_2)} \ (\bullet) \quad \frac{\vdash_\beta \mathbf{B}_i(A \supset C) \quad \vdash_\beta \mathbf{B}_i(A)}{\vdash_\beta \mathbf{B}_i(C)} \ (\blacktriangledown) \qquad (16)$$

The left $(\bullet)$ is a permissible inference, derived from Theorem 2.1.$\langle 3 \rangle$. The part above ac_i in Fig.3 is regarded as a formal proof in the sense of Section 2. The part to $\langle 3b \rangle$ needs $(\blacktriangledown)$; the right bottom part after $\mathbf{Pc}_i(ac_1, ac_2)$ is the formal derivation, except for the use of Theorem 2.2.$\langle 1 \rangle$. As stated above, the premise $\mathbf{B}_i(\mathbf{Pc}_j(ac_1, ac_2))$ of SO_i is derived from $\mathbf{Pc}_i(ac_1, ac_2)$. Based on this, person i simulates person j's reasoning to action taking. The whole diagram in Fig.3 is the evolution of thoughts in the state of affairs (**).

The assertions in Theorem 5.1 are all about person i's observations and thoughts in that $\mathbf{Pc}_j(\cdot, \cdot)$ occurs only in i's mind. Thus, the postulates are purely individualistic in the methodological sense.

When all the postulates are assumed for both persons, each person comes and meets the other at t_1, fulfilling each person's anticipation. Indeed, let $\Phi_i = \{\mathrm{CT}_i, \mathrm{FFW}_i, \mathrm{TW}_i, \mathrm{AC}_i\}$ and $\Phi_i^* = \Phi_i \cup \{\mathrm{SO}_i\}$. The successful coordination of actions and anticipations is expressed as;

$$\{t_0, S_1, S_2\} \cup (\Phi_1 \cup \Phi_2) \cup \{t_1\} \vdash_\beta ac_1 \wedge ac_2; \qquad (17)$$

$$\{t_0, S_1, S_2\} \cup (\Phi_1^* \cup \Phi_2^*) \cup \{t_1\} \vdash_\beta \mathbf{B}_1(ac_2) \wedge \mathbf{B}_2(ac_1). \qquad (18)$$

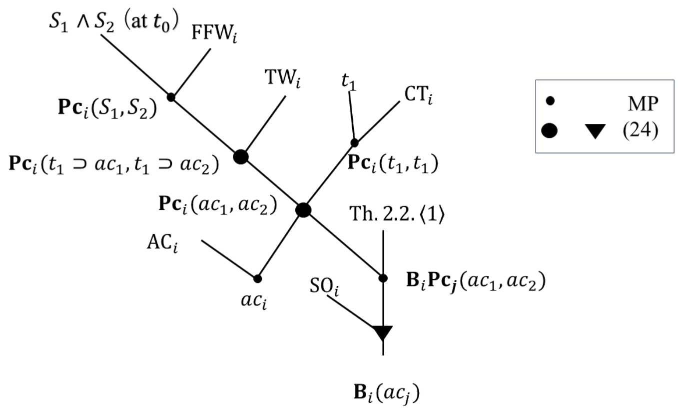

Figure 3: Thought dynamics of person i

If the postulates hold only for i but not for j, person i goes to see j but j does not show up. Person i would then realize his anticipation was wrong; this would involve dynamic revisions on his beliefs. We leave the analysis of such dynamic revision for future research. Instead, we evaluate each postulate from the viewpoint of reciprocal empathization.

All of the postulates 0 to 3 include reciprocal empathization, but their bases for reciprocity differ. First, CT$_i$ and FFW$_i$ are about person i's perception of objective information; CT$_i$ is about global time, and reciprocity is reasonable for people with basic education. FFW$_i$ is about their face-to-face communication in a specific situation where vision plays a crucial role and education/culture are not particularly important. The situation is similar to the two mirrors example depicted in Fig.1 in the physical sense.[10] In these postulates, reciprocal empathization is objectively reasonable, yet upon different bases.

Postulate TW$_i$ differs from the above in terms of its justification. Trustworthiness can be achieved only through convention of a society, and it has to be formed together with the society through experiences and repeated interactions. Lewis emphasized the role of convention to achieve coordination. Our formulation of how convention influences individuals' mental process, however, is closer to Mead's [25] concept of "generalized others", whereby an individual may perceive "common expectations" of physical or mental activities and project such expectations into others' mental activities. Our reciprocal empathization capture this process. Both Lewis' and Mead's conceptions emphasize community dependence, and in this sense, TW$_i$ is community-dependent.[11]

Reciprocal empathized beliefs occur in AC$_i$ and SO$_i$ as premises, that is, in the postulates, the action and anticipation occur only when the reciprocal empathization is achieved. These presume the coordination nature of the situation - - each individual would not have an incentive to come unless there is a common expectation to do so. These beliefs could be deduced from the reciprocal empathized beliefs in the other postulates. In this sense, these are of very different types from CT$_i$, FFW$_i$, and TW$_i$.

[10] This is reminiscent of Plato's [32], book IV, 507, "the analogy of the sun" in that face-to-face communication between the two persons leads sharing commonly what they communicate by the help of light.

[11] How "trustworthiness" comes between people is discussed in fields of psychology, philosophy etc, where various types of trustworthiness are discussed (cf., Moran [29], Räikkä [33]).

Among the postulates except postulate 3, TW_i is crucial in consideration of social behavior of people, since the others hold almost independent from a society. The consideration of a failure of coordination after (17) and (18) can be applied to the case where we remove the underlying assumption for TW_i, e.g., person 1 is from a community with a virtue of trustworthiness, but 2 is from a community without it. Person 1 may notice that the failure is caused by the different cultural backgrounds between 1 and 2, and he may revise some attitude by learning 2's background.

5.2 Mere empathization

Up to now, we assumed that $\beta \geq 3$ and empathization is reciprocal. Consider now $\beta = 2$ and hence each person is only capable of mere empathization, which is formulated as

$$\mathbf{Pc}_i^o(C_1, C_2) := \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{B}_j(C_j). \tag{19}$$

This is the right-hand side of (4) of Theorem 2.2.$\langle 2 \rangle$ with the elimination of $\mathbf{Pc}_i(C_1, C_2)$. That is, person i perceives his own situation, C_i, and is aware of j's situation, C_j, but i does not go further to think about j's reasoning or expectations. Thus, person i' empathization is not reciprocal.

How do we consider the state of affair (**) with mere empathization? Since each person is still capable of receiving information, we can still formulate Postulates CT_i and FFW_i by replacing $\mathbf{Pc}_i(\cdot,\cdot)$ by $\mathbf{Pc}_i^o(\cdot,\cdot)$. For the postulates TW_i and AC_i, we can mechanically proceed on the same substitution; this is not for SO_i since $\beta = 2$. With these substitutions, Theorem 5.1 still holds for $\mathbf{Pc}_i^o(\cdot,\cdot)$ up to $\langle 3a \rangle$, that is, as long as person i uses mere empathization operator $\mathbf{Pc}_i^o(\cdot,\cdot)$ in Postulates 0 to 3 except for SO_i, he deduces action ac_i. However, the operators $\mathbf{Pc}_i^o(\cdot,\cdot)$ do not satisfy Axiom AEM, since $\mathbf{Pc}_i^o(C_1, C_2) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i\mathbf{Pc}_j^o(C_1, C_2)$ is not in $\mathcal{P}_2$. This implies that SO_i is not available and $\langle 3b \rangle$ does not hold; thus, person i has no anticipation $\mathbf{B}_i(ac_j)$.

This lack of anticipation can be a serious threat to coordination. In case of reciprocal empathization, both individuals arrive at the conclusion that he should go and anticipation of the other's coming based on AC_i and SO_i, and this anticipation would reinforces his motive to come as he understands the reason for the other person's coming In case of mere empathization, however, the modified AC_i require person i to come based on his mere belief on j's intention, without thinking about j's underlying reasoning. The same difficulty is found for TW_i with the mechanical substitution of $\mathbf{Pc}_i^o(\cdot,\cdot)$ for $\mathbf{Pc}_i(\cdot,\cdot)$; in the modified TW_i, i can only trust j's words without the reciprocal trust, and this is the sole basis of his own action.

6 Comparisons with the Concept of Common Belief

We studied reciprocal empathization based on individualistic reasoning. Common belief is an interpersonal concept and is regarded as basic in game theory/economics.[12] These interpersonal

[12] Aumann [2] defined the concept of common knowledge in the information partition model in game theory, equivalently, the S5 Kripke model. He gave the definitions of common knowledge in terms of accessibility relations and in terms of the minimal common coarsening of the information partitions. The latter may appear to have a finite structure, but in fact, the minimal common coarsening is the partition of the possible worlds so that each cell includes all accessible worlds. In this sense, it is a super structure of the information partitions, which is very opposite to our consideration.

concepts are related; technically speaking, both are characterized in terms of the fixed-point arguments (cf., Fagin, *et al.* [15], Meyer-van der Hoek [28]).[13] The difference is that reciprocal empathization is conducted by an individual person, while common belief is an attribute of a situation where beliefs of people are shared; since who conduct logical inference is not described in the latter, it deviates from methodological individualism. Nevertheless, we can ask the question of whether common belief is attained in terms of reciprocal empathization in REL_β. We show a general negative answer as well as some positive answer.

6.1 Definition of common belief

We formulate common belief logic with the KD type base logic and a bound β on interpersonal reasoning, which is denoted as CBL_β. Comparisons are made in an extension of REL_β and CBL_β. First, let us define the extension of REL_β.

In addition to the list of primitive symbols given in Section 2.1, we add the *common belief operator* $\mathbf{C}_B(\cdot)$, which is a unary operator symbol. Then, the set of all formulae is denoted by $\mathcal{P}^*$ without a bound on interpersonal reasoning. Then, we define the set of formulae $\mathcal{P}^*_\beta$ as follows;

$$\delta(\mathbf{C}_B(C)) \leq \beta - 2 \text{ for any subformula } \mathbf{C}_B(C) \text{ of } A; \tag{20}$$

$$\mathcal{P}^*_\beta = \{A \in \mathcal{P}^* : \delta(A) \leq \beta \text{ and (1) \& (20) hold for } A\}. \tag{21}$$

This $\mathcal{P}^*_\beta$ is an extension of $\mathcal{P}_\beta$. For a study of the common belief concept $\mathbf{C}_B(C)$ itself, the condition $\delta(\mathbf{C}_B(C)) \leq \beta - 1$ is enough instead of (20), but since we compare it with the reciprocal empathized belief $\mathbf{Pc}_i(\cdot\cdot)$, we assume it.

In addition to the logical axioms and Inference rules listed for REL_β in Section 2, we adopt the following logical axiom and inference rule for $\mathbf{C}_B(\cdot)$, where we denote $\mathbf{B}_1(A) \wedge \mathbf{B}_2(A)$ by $\mathbf{B}_N(A)$; for $\mathbf{C}_B(A) \in \mathcal{P}^*_\beta$,

Axiom ACB: $\mathbf{C}_B(A) \supset \mathbf{B}_N(A) \wedge \mathbf{B}_N\mathbf{C}_B(A)$;

Inference Rule ICB: for any $D \in \mathcal{P}^*_\beta$,

$$\frac{D \supset \mathbf{B}_N(A) \wedge \mathbf{B}_N(D)}{D \supset \mathbf{C}_B(A)}.$$

The logic defined is denoted as REL^*_β, and its provability relation is by $\vdash^*_\beta$. The *common belief logic* CBL_β is defined by restricting REL^*_β to the set $\mathcal{P}^{CB}_\beta := \{A \in \mathcal{P}^*_\beta : A$ includes no occurrences of $\mathbf{Pc}_i(\mathbf{C}), i = 1, 2\}$, and its provability relation is denoted by $\vdash^{CB}_\beta$. Now, we have the three logics $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$, $(\mathrm{REL}^*_\beta, \mathcal{P}^*_\beta)$, and $(\mathrm{CBL}_\beta, \mathcal{P}^{CB}_\beta)$ for each β with $0 \leq \beta \leq \omega$. Both $\mathcal{P}_\beta$ and $\mathcal{P}^{CB}_\beta$ are subsets of $\mathcal{P}^*_\beta$, i.e., $\mathcal{P}_\beta \subseteq \mathcal{P}^*_\beta \supseteq \mathcal{P}^{CB}_\beta$, and $(\mathrm{REL}^*_\beta, \mathcal{P}^*_\beta)$ is an extension of both $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ and $(\mathrm{CBL}_\beta, \mathcal{P}^{CB}_\beta)$.

The Kripke semantics is defined for $\mathcal{P}^*_\beta$ by extending the valuation relation $\models$ by adding the following $\mathrm{V5}^{CB}$ to V0 to V5 in Section 3.1;

$\mathbf{V5}^{CB}$: $(M, w) \models \mathbf{C}_B(A)$ if and only if $(M, w_{m+1}) \models A$ for any chain $[w_0, ..., w_m, w_{m+1}]$

[13] Baltag *et al.* [3] considered the problem of common knowledge from the viewpoint of public announcement logic, introducing an infinitary knowledge operator.

of any length $m+1$ with $w_0 = w$,
where a *chain* is simply a sequence $[w_0, ..., w_m, w_{m+1}]$ with $(w_t, w_{t+1}) \in R_1 \cup R_2$ for $t = 0, ..., m$.

For the present purpose, we need only the soundness of $(\mathrm{REL}^*_\beta, \mathcal{P}^*_\beta)$, which can be proved similarly to the proof of soundness for $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ in Section 3.2.

Lemma 6.1 (Soundness for REL^*_β**)**: For any $C \in \mathcal{P}^*_\beta$, if $\vdash^*_\beta C$, then $\models C$.

The fixed-point property holds for $\mathbf{C}_B(A)$ in $(\mathrm{REL}^*_\beta, \mathcal{P}^*_\beta)$ (and in $(\mathrm{CBL}_\beta, \mathcal{P}^{cb}_\beta)$) in a simpler form than in Theorem 2.1.$\langle 1\rangle$ and $\langle 2\rangle$. Indeed, for any $\mathbf{B}_N\mathbf{C}_B(C) \in \mathcal{P}^*$, it holds that $\mathbf{B}_N\mathbf{C}_B(C) \in \mathcal{P}^*_\beta$ if and only if

$$\vdash^*_\beta \mathbf{C}_B(C) \equiv \mathbf{B}_N(C) \wedge \mathbf{B}_N\mathbf{C}_B(C). \tag{22}$$

This is proved in a similar manner to the proofs of Theorem 2.1.$\langle 1\rangle$ and $\langle 2\rangle$.

Logics $(\mathrm{REL}_\beta, \mathcal{P}_\beta)$ and $(\mathrm{CBL}_\beta, \mathcal{P}^{CB}_\beta)$ differ in the subjects taking logical reasoning. Perception/reciprocal empathization represented by $\mathbf{Pc}_i(\cdot,\cdot)$ is individualistic in that the subject operating logical inference is an individual person. On the other hand, Axiom ACB and Rule ICB determine the properties of distributed beliefs in a situation; who reasons in ACB and ICB are unclear. We raise the question: Is there any way to represent $\mathbf{C}_B(\cdot)$ in terms of $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$? If this is answered in an affirmative way, the common belief $\mathbf{C}_B(\cdot)$ could be regarded as representing a situation where the individual persons interact and reach common belief. But the answers we give are not straightforward.

First, we observe the difference between $\mathbf{Pc}_i(\cdot,\cdot)$ and $\mathbf{C}_B(\cdot)$ at two levels;

(C-i): $\mathbf{Pc}_i(\cdot,\cdot)$ allows two different target formulae C_1 and C_2, while $\mathbf{C}_B(\cdot)$ targets a single C;

(C-ii): even when $C_1 = C_2 = C$, operator $\mathbf{Pc}_i(C_1, C_2)$ may depend upon $i = 1, 2$,
but $\mathbf{C}_B(C_1) = \mathbf{C}_B(C_2)$ are simply identical.

(C-i) means that each of C_1, C_2 in $\mathbf{Pc}_i(C_1, C_2)$ in Lewis's example indicates the word uttered by the corresponding person, and (C-ii) that even when the persons meet the same formula, they may still take it differently, possibly because the persons come from different communities. In general, the same words are sometimes differently understood by people, indicated in the end of Section 5. In order to express $\mathbf{C}_B(\cdot)$ by $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$, we need to eliminate these two dependences.

First, we drop the dependence of C_k upon $k = 1, 2$;

$$\mathbf{Pc}_i(C) := \mathbf{Pc}_i(C, C) \text{ for } i = 1, 2. \tag{23}$$

Then, we eliminate the dependence upon the subscript i of $\mathbf{Pc}_i(C)$ by

$$\mathbf{C}^*_B(C) := \mathbf{Pc}_1(C) \wedge \mathbf{Pc}_2(C). \tag{24}$$

The question is whether this formula $\mathbf{C}^*_B(C)$ expresses $\mathbf{C}_B(C)$ for any $C \in \mathcal{P}^*_\beta$. One way is positively answered; a proof is given in Section 6.1. The result (25) relies only upon (23).

Lemma 6.2 Let $\beta \geq 3$ and $C \in \mathcal{P}^*_\beta$ with $\mathbf{C}_B(C) \in \mathcal{P}^*_\beta$. Then, $\vdash^*_\beta \mathbf{C}_B(C) \supset \mathbf{Pc}_i(C)$ for $i = 1, 2$; thus,

$$\vdash^*_\beta \mathbf{C}_B(C) \supset \mathbf{C}^*_B(C). \tag{25}$$

However, the converse of (25) does not necessarily hold. Consider the model in Diagram 6.1. There, $(M, w) \models \mathbf{Pc}_1(p) \wedge \mathbf{Pc}_2(p)$ but $(M, w) \not\models \mathbf{C}_B(p)$ since $(M, v_1) \not\models p$; we have $(M, w) \not\models$

$\mathbf{C}_B^*(p) \supset \mathbf{C}_B(p)$. By Lemma 6.1 (soundness for REL_β^*), we have $\nvdash_\beta^* \mathbf{C}_B^*(p) \supset \mathbf{C}_B(p)$.

Diagram 6.1; $M = ((W, R_1, R_2), \sigma)$

$$\begin{array}{|lclcl|} \hline & \longrightarrow_1 & & \longrightarrow_1 & v_1\colon \neg\, p \circlearrowleft_{1,2} \\ w: & & u_1\colon p & & \\ & \searrow_2 & & \searrow_2 & \\ & & u_2\colon p \circlearrowleft_{12} & & v_2\colon p \circlearrowleft_{1,2} \\ \hline \end{array}$$

Thus, the formula $\mathbf{C}_B(C) \supset \mathbf{C}_B^*(C)$ holds in REL_β^* but these have a gap. We consider two possible adjustments to close the gap; each is to strengthen

(a): the formula $\mathbf{C}_B^*(C)$ in ($\mathcal{P}_\beta$,REL_β); or (b): the requirement for $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$.

It would be nice if (a) is possible, because logic REL_β would be enough for REL_β^* to consider where common belief is achieved by individual reciprocal empathization $\mathbf{Pc}_i(\cdot,\cdot)$. However, we have the following general negative result on representation of $\mathbf{C}_B(C)$ in terms of formulae in $\mathcal{P}_\beta$. A proof will be given in Section 6.2.

Theorem 6.1 (Impossibility of representation of $\mathbf{C}_B(\cdot)$ in REL_β^*) For any $\beta \geq 0$, there is no formula $A^*(p) \in \mathcal{P}_\beta$ with $p \in PV$ such that $\vdash_\beta^* A^*(p) \equiv \mathbf{C}_B(p)$.

This theorem covers even the possibility that $A^*(p)$ may include $\mathbf{Pc}_i(C_1, C_2), i = 1, 2$ and $C_1 \neq C_2$. Thus, this theorem is more general than the answer required for the question about the comparison between $\mathbf{C}_B(C)$ and $\mathbf{C}_B^*(C)$.

In fact, for the equivalence between $\mathbf{C}_B(C)$ and $\mathbf{C}_B^*(C)$, it is enough to strengthen Axiom AEM and Rule IEM. Now, we consider the general formulation of the extension of the logic ($\mathcal{P}_\beta$,REL_β) and define the strengthened version ($\mathcal{P}_\beta$,REL_β^S) as follows: for $\mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2) \in \mathcal{P}_\beta, i = 1, 2$ and $\mathbf{B}_i[D_i \wedge D_j] \in \mathcal{P}_\beta$,

Axiom AEM^S: $\mathbf{Pc}_i(C_1, C_2) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i[\mathbf{Pc}_i(C_1, C_2) \wedge \mathbf{Pc}_j(C_1, C_2)]$.

Rule IEM^S:

$$\frac{D_i \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i[D_i \wedge D_j] \quad D_j \supset \mathbf{B}_j(C_i) \wedge \mathbf{B}_j[D_i \wedge D_j]}{D_i \supset \mathbf{Pc}_i(\boldsymbol{C})}.$$

The new logic obtained is denoted by REL_β^S. This strengthening is within the perspective of individualistic inference.[14]

Now, we return to the problem of the representation of $\mathbf{C}_B(C)$ by $\mathbf{C}_B^*(C)$. We consider the extended logic ($\mathcal{P}^*$,REL_β^{S*}) including the common belief formulae, where we allow Axiom AEM^S and Rule IEM^S in REL_β^{S*}. We denote the provability relation in REL_β^{S*} by $\vdash_\beta^{S*}$. Recall $\mathbf{C}_B^*(C) = \mathbf{Pc}_1(C) \wedge \mathbf{Pc}_2(C)$ has each person's interpersonal empathization as well as his reflection on himself. In fact, this captures the common belief; the equivalence between $\mathbf{C}_B(C)$ and $\mathbf{C}_B^*(C)$ holds in REL_β^{*SR}. Now, we return to the problem of the representation of $\mathbf{C}_B(C)$ by $\mathbf{C}_B^*(C)$.

Theorem 6.2 (Equivalence) $\vdash_\beta^{S*} \mathbf{C}_B(C) \equiv \mathbf{C}_B^*(C)$ for any $\mathbf{C}_B(C) \in \mathcal{P}_\beta^*$.

The gap between the common belief $\mathbf{C}_B(C)$ and $\mathbf{C}_B^*(C) = \mathbf{Pc}_1(C) \wedge \mathbf{Pc}_2(C)$ is fulfilled. It is interpreted as meaning that common belief $\mathbf{C}_B(C)$ is captured by $\mathbf{C}_B^*(C)$ in REL_β^S in the scope of individualism. Nevertheless, this is not regarded simply as a resolution of the question of

[14] A study of REL_β^S together with the role of *Axiom PI*: $\mathbf{B}_i(C) \supset \mathbf{B}_i\mathbf{B}_i(C)$ in REL_β remains open.

whether the concept of common belief is achieved by efforts of individual persons. One reason is that the above success is obtained only for the two-person case. For the case with $|N| \geq 3$, it is standard to define the common belief $\mathbf{C}_B(C)$ in CBL_β in terms of the same axiom and rule as ACB and ICB. However, the binary version of Axiom AEM and Rule IEM are still basic for reciprocal empathization. When $|N| \geq 3$, the common belief in CBL_β is drastically different from the reciprocal empathization in REL_β.

Reciprocal empathization is interpersonal thinking, but the extension REL_β^S is obtained by applying the same argument into one person's internal reflection upon himself. This order of extension is interpreted as meaning that REL_β is more foundational than REL_β^S. This view is compatible with Mead's [25] symbolic interactionism, which will be discussed in Section 9.1.

6.2 Proofs

6.2.1 Proofs of Lemmas 6.1 and Theorem 6.2

We prove of the lemmas and Theorem 6.2 first before Theorem 6.1, because its proof is longer and complicated.

Proof of Lemma 6.1 It suffices to add the following two steps to the proof of soundness for REL_β given in Section 3.2; $\models \mathbf{C}_B(C) \supset \mathbf{B}_N(C) \wedge \mathbf{B}_N\mathbf{C}_B(C)$, and if $\models D \supset \mathbf{B}_N(C) \wedge \mathbf{B}_N(D)$, then $\models D \supset \mathbf{C}_B(C)$. Either case can be proved in the same manners as the corresponding proofs of Axiom AEM and Rule IEM in the soundness part of Theorem 3.1.■

Proof of Lemma 6.2 By Axiom ACB, we have $\vdash_\beta^* \mathbf{C}_B(C) \supset \mathbf{B}_i(C) \wedge \mathbf{B}_i\mathbf{C}_B(C)$ for $i = 1, 2$. Let $D_i = \mathbf{C}_B(C)$, $i = 1, 2$. The previous statement is $\vdash_\beta^* D_i \supset \mathbf{B}_i(C) \wedge \mathbf{B}_i(D_j)$ for $i = 1, 2$. Regarding these as the upper formulae of Rule IEM, we have $\vdash_\beta^* D_i \supset \mathbf{Pc}_i(C)$ for $i = 1, 2$. This implies $\vdash_\beta^* \mathbf{C}_B(C) \supset \mathbf{Pc}_1(C) \wedge \mathbf{Pc}_2(C)$, i.e., $\vdash_\beta^* \mathbf{C}_B(C) \supset \mathbf{C}_B^*(C)$.■

Proof of Theorem 6.2. Since $\vdash_\beta^* \mathbf{C}_B(C) \supset \mathbf{C}_B^*(C)$ by Lemma 6.2, it suffices to show $\vdash_\beta^{S*} \mathbf{C}_B^*(C) \supset \mathbf{C}_B(C)$. By Axiom AEM, it holds that $\vdash_\beta^* \mathbf{C}_B^*(C) \supset \mathbf{B}_N(C)$. First, let us see $\vdash_\beta^{S*} \mathbf{C}_B^*(C) \supset \mathbf{B}_N\mathbf{C}_B^*(C)$. By Axiom AEM^S, it holds that $\vdash_\beta^{S*} \mathbf{Pc}_i(C) \supset \mathbf{B}_i[\mathbf{Pc}_i(C) \wedge \mathbf{Pc}_j(C)]$ for $i = 1, 2$, which implies $\vdash_\beta^{S*} \mathbf{C}_B^*(C) \supset \mathbf{B}_N(\mathbf{C}_B^*(C))$. Then, $\vdash_\beta^{S*} \mathbf{C}_B^*(C) \supset \mathbf{B}_N(C) \wedge \mathbf{B}_N\mathbf{C}_B^*(C)$. Letting $D = \mathbf{C}_B^*(C)$ and regarding $\vdash_\beta^{S*} D \supset \mathbf{B}_N(C) \wedge \mathbf{B}_N(D)$ as the upper formula of Rule IEM^S, we have $\vdash_\beta^{S*} D \supset \mathbf{C}_B(C)$, i.e., $\vdash_\beta^{S*} \mathbf{C}_B^*(C) \supset \mathbf{C}_B(C)$.■

6.2.2 Proof of Theorem 6.1

The theorem holds for $\beta \leq 2$ in the vacuous sense that for any $A^*(p) \in \mathcal{P}_\beta$, the formula $A^*(p) \equiv \mathbf{C}_B(p)$ is not in $\mathcal{P}_\beta^*$ because of (20) and (21). In the following, we assume $\beta \geq 3$.

Throughout the following, we suppose that there is a formula $A^*(p) \in \mathcal{P}_\beta$ such that

$$\vdash_\beta^* A^*(p) \equiv \mathbf{C}_B(p). \tag{26}$$

We abbreviate $A^*(p)$ as A^* in the following.We will derive a contradiction from (26). First, it follows from (26) that $\nvdash_\beta^* \neg A^*$. Indeed, if $\vdash_\beta^* \neg A^*$, then, $\vdash_\beta^* \neg\mathbf{C}_B(p)$; eliminating all ep-operators from the proof of $\neg\mathbf{C}_B(p)$ in REL_β^*, we obtain a proof of $\neg p$ in classical logic CL, which is a contradiction; thus $\nvdash_\beta^* \neg A^*$. Since REL_β^* is an extension of REL_β, $\vdash_\beta \neg A^*$ implies

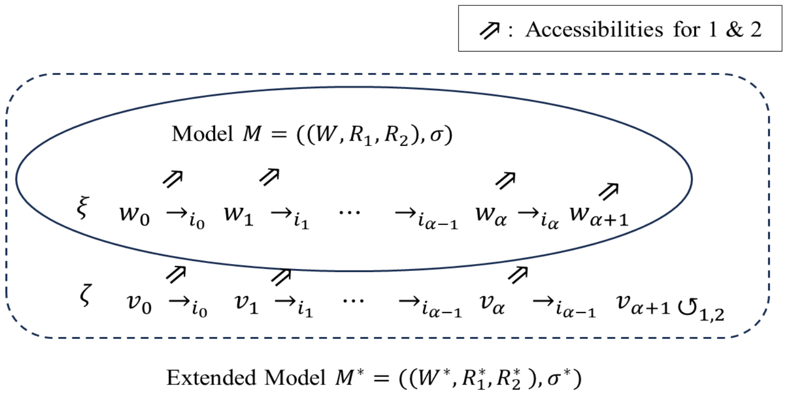

Figure 4: Extended Model

$\vdash_\beta^* \neg A^*$. The contrapositive is that $\nvdash_\beta^* \neg A^*$ implies $\nvdash_\beta \neg A^*$. Thus, $\nvdash_\beta \neg A^*$. Applying Theorem 3.1 (completeness for REL$_\beta$), there is a Kripke model $M = ((W, R_1, R_2), \sigma)$ and a world $w_0 \in W$ such that

$$(M, w_0) \nvDash \neg A^*, \text{ equivalently, } (M, w_0) \models A^*. \tag{27}$$

We will extend the model $M = ((W, R_1, R_2), \sigma)$ to $M^* = ((W^*, R_1^*, R_2^*), \sigma^*)$ so that $(M^*, w_0) \models A^*$ but $(M^*, w_0) \nvDash \mathbf{C}_B(p)$; thus, $(M^*, w_0) \nvDash A^* \supset \mathbf{C}_B(p)$. By Lemma 6.1 (soundness for REL$_\beta^*$), we get $\nvdash_\beta^* A^* \supset \mathbf{C}_B(p)$, a contradiction to (26). This completes the proof; now, we construct a model $M^* = ((W^*, R_1^*, R_2^*), \sigma^*)$.

Strategy of construction of a counter model: We denote $\delta(A^*) = \alpha$. We consider an al-chain $\xi = \langle (w_0, i_0), ..., (w_\alpha, i_\alpha), w_{\alpha+1} \rangle$ in the model $M = ((W, R_1, R_2), \sigma)$, which is fixed in the following. When these worlds are all distinct, the required $M^* = ((W^*, R_1^*, R_2^*), \sigma^*)$ is simple, but some worlds need to be identical when $\delta(A^*) = \alpha$ is larger than the cardinality of W.

Specifically, let $v_0, v_1, \cdots, v_{\alpha+1}$ be new symbols not in W. Consider the *new chain* $\zeta = \langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_\alpha, i_{\alpha-1}), v_{\alpha+1} \rangle$, where ζ is not an al-chain since the persons assigned to $v_{\alpha-1}$ and v_α is $i_{\alpha-1}$. We define the new frame (W^*, R_1^*, R_2^*); $W^* = W \cup \{v_0, v_1, \cdots, v_{\alpha+1}\}$ and the extended accessibility relations $R_i^*, i = 1, 2$ are as follows:

- **AC1**: for any $t = 0, ..., \alpha - 1$, $v_t R_i^* u$ iff $w_t R_i u$ or $[u = v_{t+1}$ & $i = i_t]$;
- **AC2**: for v_α, $v_\alpha R_i^* u$ iff $[w_\alpha R_i u$ & $i = i_\alpha]$ or $[u = v_{\alpha+1}$ & $i = i_{\alpha-1}]$;
- **AC3**: for $v_{\alpha+1}$, $v_{\alpha+1} R_i^* u$ iff $u = v_{\alpha+1}$.

The three cases are exclusive and exhaustive; each R_i^* is uniquely defined. AC1 says that for $t = 0, ..., \alpha - 1$, R_i^* at v_t mimics R_i at w_t, i.e., $v_t R_i^* u$ if and only if $w_t R_i^* u$ for $i = 1, 2$, but additionally, person i_t connects v_t to v_{t+1} except for $t = \alpha$. AC2 states two scenarios from v_α; if $i = i_\alpha$, then $\langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_\alpha, i_\alpha), u \rangle$ is an al-chain going to $u = w_{\alpha+1}$, and if $i = i_{\alpha-1}$, the chain becomes $\zeta = \langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_\alpha, i_{\alpha-1}), v_{\alpha+1} \rangle$, which is the unique connection to $v_{\alpha+1}$. AC3 means that $v_{\alpha+1}$ is an end world with reflexivity for both persons.

Recall that $R_i, i = 1, 2$ are serial in W. By AC1 to AC3, each $R_i^*, i = 1, 2$ has has an accessible possible world for each $v_0, ..., v_\alpha, v_{\alpha+1}$. Thus, we have the next lemma.

Lemma 6.3 R_i^* is a serial relation on W^* for $i = 1, 2$.

The following lemma, immediate from AC1 to AC3, plays a crucial role in the following.

Lemma 6.4 In (W^*, R_1^*, R_2^*), v_0 has no predecessor, and for $t = 0, ..., \alpha$, v_{t+1} has the unique predecessor v_t for $R_{i_t}^*$ but no predecessors for $R_{j_t}^*$ $(j_t \neq i_t)$.

We extend the assignment σ to $\sigma^* : W^* \longrightarrow \{\top, \bot\}$ as follows: for any $(w, q) \in W^* \times PV$,

$$\sigma^*(w,q) = \begin{cases} \sigma(w,q) & \text{if } w \in W \\ \sigma(w_t,q) & \text{if } w = v_t, t = 1, ..., \alpha \\ \bot & \text{if } w = v_{\alpha+1}. \end{cases} \tag{28}$$

Thus, σ^* is identical to σ over W, it mimics σ over $w_1, \cdots, w_\alpha$ for $v_1, ..., v_\alpha$, and it takes $\bot$ at $v_{\alpha+1}$ for any $q \in PV$. Now, we have the extended Kripke model $M^* = ((W^*, R_1^*, R_2^*), \sigma^*)$.

The following is a simple observation.

Lemma 6.5 $(M^*, v_0) \nvDash \mathbf{C}_B(p)$.

Proof. Since there is a chain from v_0 to $v_{\alpha+1}$ and $(M^*, v_{\alpha+1}) \nvDash p$ by (28), we have $(M^*, v_0) \nvDash \mathbf{C}_B(p)$ by V5^{CB}.■

Finally, we show $(M^*, v_0) \models A^*$. Once this is shown, we have $(M^*, v_0) \nvDash A^* \supset \mathbf{C}_B(p)$. By Lemma 6.1 (soundness for REL$_\beta^*$), we have the final target $\nvdash_\beta^* A^* \supset \mathbf{C}_B(p)$.

Verification of $(M^*, v_0) \models A^*$: Recall $(M, w_0) \models A^*$ in (27). We will prove that this is preserved in the extended model M^*. The last connection from v_α to $v_{\alpha+1}$ with $i_{\alpha-1}$ prevents a connection from v_0 to $v_{\alpha+1}$ with an al-chain; so, $\mathbf{Pc}_i(\cdot)$ in A^* is not valuated at $v_{\alpha+1}$; A^* is not either since $A^* \in \mathcal{P}_\beta$. But $\mathbf{C}_B(p)$ depends upon the valuation at $v_{\alpha+1}$ by V5^{CB}. After all, the extension M^* is obtained from M in order to have these aims.

Now, we show that the valuation $\models$ in M is preserved in M^*.

Lemma 6.6 For any $C \in \mathcal{P}_\beta$,

$$\text{for any } w \in W,\ (M^*, w) \models C \text{ if and only if } (M, w) \models C. \tag{29}$$

Proof. The difference between M^* and M is that M^* has $\{v_0, ..., v_{\alpha+1}\}$ additional to W. However, Lemma 6.4 implies that at any $w \in W$, either person has no references to $\{v_0, ..., v_{\alpha+1}\}$. Hence, the valuation in (M^*, w) is determined by the valuations in M. The exact proof is given by induction on the lengths of formulae.■

Now, we prove $(M^*, v_0) \models A^*$. For this, we decompose A^* along the chain $\zeta = \langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_{\alpha-1}, i_{\alpha-1}), v_{\alpha+1} \rangle$ from (v_0, i_0). For the decomposition, we use some concepts. Let $C \in \mathcal{P}_\alpha$. We say that D is a *maximal* i*-subformula* of C iff its outmost symbol is $\mathbf{B}_i(\cdot)$ or $\mathbf{Pc}_i(\cdot, \cdot)$ and is not in the scope of an ep-subformula of C. We denote the set of maximal i-subformula of C by $Max(C, i)$ and define $Max(C) := \cup_{i=1,2} Max(C, i)$. Also, let $PV_d(C)$ be the set of (direct) propositional variables occurring in C but not in the scope of an ep-subformula of C. To follow the chain ζ, $C \in Max(C', i)$ is decomposed. For $C = \mathbf{B}_i(C^0)$ or $C = \mathbf{Pc}_i(C_1^0, C_2^0) \in Max(C', i)$, C^0 or C_i^0 is the *content formula* of C.

Now, we define two types of sets $C(t)$ and $E(t)$ along $\zeta = \langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_\alpha, i_{\alpha-1}), v_{\alpha+1} \rangle$. The following induction definition is opposite to the process of generating formulae. In step 0, we define the two set;

$$C(0) = \{A^*\} \text{ and } E(0) = Max(A^*) \cup PV_d(A^*).$$

First, $C(0) = \{A^*\}$ is given, and then, $E(0)$ is the set of the maximal ep-formulae and direct propositional formulae. To step 1, we do not keep propositional variables in $E(0)$. The set $E(0)$ may have ep-formulae $\mathbf{B}_i(C^0)$ and/or $C = \mathbf{Pc}_i(C_1^0, C_2^0)$ and $i \neq i_0$. These formulae are valuated by referring to W but not to v_1. Hence, we consider the set $C(1) = \{D^0 : D \in \cup_{C \in E(0)} Max(C, i_0)\}$ collecting the content formulae D^0 of $D \in \cup_{C \in E(0)} Max(C, i_0)$. In step 1, each D^0 is decomposed in the same way in Step 0 from A^*.

The induction step is based on the same idea. Suppose that the sets $C(t)$ and $E(t)$ are given for v_t $(t \leq \alpha - 1)$. $E(t)$ is the set of ep-formulae and propositional variables. We define the next sets for v_{t+1}:

$$\begin{aligned} C(t+1) &= \{D^0 : D \in \cup_{C \in E(t)} Max(C, i_t)\}; \\ E(t+1) &= \cup_{D^0 \in C(t+1)} [Max(D^0) \cup PV_d(D^0)]. \end{aligned}$$

When $E(t)$ includes $\mathbf{B}_i(\cdot)$ or $\mathbf{Pc}_i(\cdot, \cdot)$ $(i \neq i_t, t \leq \alpha)$, its valuation is done through referring to some worlds in W. Thus, the pair $C(t)$ and $E(t)$ are defined up to step α, i.e., v_α.

The above decomposition of A^* is made along $\zeta = \langle (v_0, i_0), \cdots, (v_{\alpha-1}, i_{\alpha-1}), (v_\alpha, i_{\alpha-1}), v_{\alpha+1} \rangle$ up to $(v_\alpha, i_{\alpha-1})$. Incidentally, the valuation at v_α referring to $v_{\alpha+1}$ is only through $R_{i^*_{\alpha-1}}$; this is necessary for Lemma 6.5.

The decomposition process is opposite to the construction of formula A^*. Lemma 6.7 follows the above decomposition.

Lemma 6.7 Let $t = 0, ..., \alpha$. Any formula $D^0 \in C(t)$ is obtained from $E(t)$ by some finitely repeated applications of logical connectives $\neg, \supset$, and $\wedge$.

We have the following.

Lemma 6.8.$\langle 1 \rangle$ For $t = 0, ..., \alpha - 1$, if $Max(C^0, i_t) \neq \emptyset$ for some $C \in E(t)$, then $0 \leq \delta(E(t+1) < \delta(E(t))$; and $\langle 2 \rangle$ $\delta(E(\alpha)) = 0$.

Proof. $\langle 2 \rangle$ follows $\langle 1 \rangle$. Consider $\langle 1 \rangle$. Let $Max(C^0, i_{t+1}) \neq \emptyset$ for some $C \in E(t)$. Then, $E(t)$ has an i_t-formula, and the outermost ep-operator of such a formula is eliminated. In this case, the depths are decreased, i.e., $\delta(E(t+1)) < \delta(E(t))$.■

Now, we have the final lemma.

Lemma 6.9 For any $C \in C(t) \cup E(t)$ and $t = 0, ..., \alpha$,

$$(M^*, v_t) \models C \text{ if and only if } (M, w_t) \models C. \tag{30}$$

Proof. Let us prove (30) by double induction on the decomposition process backwardly and on the lengths of formulae for each $t = \alpha, ..., 0$. Let $t = \alpha$. Then, by Lemma 6.8, $C \in E(\alpha)$ is a propositional variable $p \in PV$. This is the very base of our induction. By (28), we have (30) for C. Now, let $C \in C(\alpha)$. This C is constructed from $p \in E(\alpha)$ by the connectives $\neg, \supset$, and $\wedge$. The induction hypothesis is that (30) holds for any immediate subformulae of C. Then, (30) holds for C. Thus, (30) holds for all $C \in C(\alpha) \cup E(\alpha)$ by the induction on the lengths of formulae.

Diagram 6.2

$$\begin{array}{ccccccccc} \cdots & & C(t) & & C(t+1) & & \cdots\ C(\alpha-1) & & C(\alpha) \\ \cdots & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \cdots\quad \uparrow & \swarrow & \uparrow \\ \cdots & & E(t) & & E(t+1) & & \cdots\ E(\alpha-1) & & E(\alpha) \\ & & v_t & & v_{t+1} & & \cdots\quad v_{\alpha-1} & & v_\alpha \end{array}$$

Suppose the induction hypothesis IDH$(t+1)$ that (30) holds for any formula $D \in C(t+1) \cup E(t+1)$. We show that (30) holds for all $C \in E(t)$. IDH$(t+1)$ will be used only twice in the following; Lemma 6.6 plays the corresponding role in various places.

Suppose that $C \in E(t)$. There are three cases: (i) C is a propositional variable, (ii) $[C = \mathbf{B}_i(C'), i = 1, 2]$, and (iii) $[C = \mathbf{Pc}_i(C'_1, C'_2), i = 1, 2]$. In (i), we have (30) by (28). In (ii) and (iii), we need to cover both cases $i = i_t$ and $i \neq i_t$.

Consider (ii). Suppose $(M, w_t) \nvDash \mathbf{B}_i(C')$. Then, $(M, u) \nvDash C'$ for some $u \in W$ with $w_t R_i u$. By Lemma 6.6, we have $(M^*, u) \nvDash C'$. Also, since $w_t R_i u$ implies $v_t R_i^* u$ by AC1, we have $(M^*, u) \nvDash C'$. Thus, $(M^*, v_t) \nvDash \mathbf{B}_i(C')$. This case covers both $i = i_t$ and $i \neq i_t$.

Conversely, suppose $(M^*, v_t) \nvDash \mathbf{B}_i(C')$. Then, $(M^*, u) \nvDash C'$ for some $u \in W^*$ with $v_t R_i^* u$. Let $u \in W$. Then $(M, u) \nvDash C'$ by Lemma 6.6 and $w_t R_i u$ by AC1; thus, $(M, w_t) \nvDash \mathbf{B}_i(C')$. This covers both $i = i_t$ and $i \neq i_t$. Now, let $u \notin W$. Then $v_t R_i^* u$ implies $u = v_{t+1}$ and $i = i_t$ by Lemma 6.4; moreover, $C' \in C(t+1)$. By IDH$(t+1)$, $(M^*, v_{t+1}) \nvDash C'$ implies $(M, w_{t+1}) \nvDash C'$. Since $w_t R_{i_t} w_{t+1}$ and $(M, w_{t+1}) \nvDash C'$. Thus, $(M, w_t) \nvDash \mathbf{B}_i(C')$.

Consider (iii), i.e., $\mathbf{Pc}_i(\boldsymbol{C'}) = \mathbf{Pc}_i(C'_1, C'_2)$. Suppose that $(M, w_t) \nvDash \mathbf{Pc}_i(\boldsymbol{C'})$. Then, $(M, u_{\nu+1}) \nvDash C'_{\ell_\nu}$ for some al-chain $\langle (u_0, \ell_0), ..., (u_\nu, \ell_\nu), u_{\nu+1} \rangle$ in W with $(u_0, \ell_0) = (w_t, i)$ in W. We change this al-chain to $\langle (v_t, \ell_0), (u_1, \ell_1), ..., (u_\nu, \ell_\nu), u_{\nu+1} \rangle$, i.e., only u_0 is replaced by v_t, which is an al-chain in (W^*, R_1^*, R_2^*) by AC1. By Lemma 6.6, $(M^*, u_{\nu+1}) \nvDash C'_{\ell_\nu}$. Thus, $(M^*, v_t) \nvDash \mathbf{Pc}_i(\boldsymbol{C'})$. This step covers both cases $i = i_t$ and $i \neq i_t$.

Next, suppose $(M^*, v_t) \nvDash \mathbf{Pc}_i(\boldsymbol{C'})$. Then, $(M^*, u_{\nu+1}) \nvDash C'$ for some al-chain $\langle (u_0, \ell_0), ..., (u_\nu, \ell_\nu), u_{\nu+1} \rangle$ with $(u_0, \ell_0) = (v_t, i)$. In this case, there are two cases (A) $u_{\nu+1} \in W$ and (B) $\{u_0, ..., u_\nu, u_{\nu+1}\}$ is included in $\{v_t, ..., v_\alpha\}$. In (B), the case $i \neq i_t$ is excluded by Lemma 6.4..

Consider case (A): First, $u_{\nu+1} \in W$ and $(M^*, u_{\nu+1}) \nvDash C'_{\ell_\nu}$, so $(M, u_{\nu+1}) \nvDash C'_{\ell_\nu}$ by Lemma 6.6. Suppose that the sequence $u_0, ..., u_\nu, u_{\nu+1}$ consists of two part $u_0 = v_t, ..., u_k = v_k$ and $u_{k+1}, ..., u_{\nu+1}$ in W. We consider the new al-chain $\langle (w_t, \ell_0), ..., (w_k, \ell_{k-t}), (u_{k+1}, \ell_{k-t+1}), ..., (u_\nu, \ell_\nu), u_{\nu+1} \rangle$ in (W, R_1, R_2). Thus, $(M, w_t) \nvDash \mathbf{Pc}_i(\boldsymbol{C'})$. This step covers both cases $i = i_t$ and $i \neq i_t$.

Consider case (B): $\{u_0, ..., u_\nu, u_{\nu+1}\}$ is simply replaced by $\{w_t, ..., w_{t+\nu}, w_{t+(\nu+1)}\}$. By IDH$(t+1)$, $(M^*, u_{\nu+1}) \nvDash C'_\nu$ if and only if $(M, w_{t+(\nu+1)}) \nvDash C'_{\ell_\nu}$. Hence, for the al-chain $\langle (w_t, i_t), ..., (w_{t+\nu}, i_{t+\nu}), w_{t+(\nu+1)} \rangle$ from (w_t, i_t), we have $(M, w_{t+(\nu+1)}) \nvDash C'_{\ell_\nu}$. Thus, $(M, w_t) \nvDash \mathbf{Pc}_i(\boldsymbol{C'})$.

It remains that the induction step t is to extend the equivalence of (30) for $C \in C(t)$. We have the extension by the induction on the length of formulae from $C \in E(t)$ taking the cases of $\neg, \wedge$, and $\supset$. Thus, we have (30) for all $t = \alpha, ..., 1, 0$.■

By Lemma 6.9, we have $(M, w_0) \models A^*$ if and only if $(M^*, v_0) \models A^*$. Since $(M, w_0) \models A^*$ by (27), we have $(M^*, v_0) \models A^*$. By Lemma 6.5, we have $(M^*, v_0) \nvDash A^* \supset C_B(p)$, and by Lemma 6.1, $\nvdash_\beta^* A^* \supset C_B(p)$. This is a contradiction to the existence of a formula A^* in $\mathcal{P}_\beta$ with $\vdash_\beta^* A^* \equiv C_B(p)$. We complete the proof of Theorem 6.1.

7 Completeness Proof of $\mathbf{REL}_\beta$: Step 1 for A with $\delta(A) = \beta - 1$

In Sections 7 and 8, we prove completeness of logic REL_β; that is, for any β with $0 \leq \beta \leq \omega$ and for any $A \in \mathcal{P}_\beta$, $\models A$ implies $\vdash_\beta A$. As usual, we show its contrapositive, i.e., if $\nvdash_\beta A$, there is a counter Kripke model for A. Nevertheless, the construction requires significant modifications from the standard construction because of the restriction on the formulae by bound β. Moreover, for a finite β, the proof takes two steps; the first step is based on the standard construction, but various modifications and new arguments are needed to accommodate Axiom AEM and Rule IEM, because of the bound β. The two steps are;

Step 1: for any $A \in \mathcal{P}_\beta$ with $\nvdash_\beta A$ and $\delta(A) \leq \beta - 1$, we construct a countermodel;

Step 2: for any $A \in \mathcal{P}_\beta$ with $\nvdash_\beta A$ and $\delta(A) = \beta$, the countermodel of Step 1 is extended to A.

Steps 1 and 2 are given in Sections 7 and 8. Case $\delta(A) = \beta$ is essential for our research program because this gives the boundary of reciprocal empathization. When $\beta = \omega$, Step 1 is enough.

We give brief explanations of Steps 1 and 2, and the difficulties we encounter. First, we give three remarks for Step 1.

(1a): Construction of a counter Kripke model is based on the standard idea of the epistemic logic with common knowledge/belief (cf., Fagin, *et al.* [15]).

(1b): The fact that the operator $\mathbf{Pc}_i(\cdot, \cdot)$ is binary creates complications not present in the common knowledge logic, and we prepare several basic lemmas to take care of them.

(1c): We still need the condition $\delta(A) \leq \beta - 1$ in this construction. The restriction (1) for the set $\mathcal{P}_\beta$ is not enough, because formulae like $\mathbf{B}_i\mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})$ with $\delta(\mathbf{Pc}_i(\boldsymbol{C})) = \beta - 2$ is allowed in $\mathcal{P}_\beta$ but in the construction of a countermodel, we need to apply Rule IEM to a maximal consistent set containing $\mathbf{B}_i\mathbf{B}_j\mathbf{Pc}_i(\boldsymbol{C})$, which requires one layer deeper than what is allowed in $\mathcal{P}_\beta$. Thus, Step 1 assumes that the target formula A satisfies $\delta(A) \leq \beta - 1$.

In Step 2, we take a new method to extend a countermodel for $\delta(A) \leq \beta - 1$ to the case $\delta(A) = \beta$. The new method is as follows. First, we divide the set of subformulae of A into two parts: the first set consists of subformulae C for which $\delta(A) \leq \beta - 1$ holds and the second set consists of the remaining subformulae. Using this division, Step 1 is applied to the first set and we have a Kripke model for them. Then, we extend this Kripke model to a model containing the remaining subformulae, and show that the extended model is a counter model for A. This extension is specific to the second set of formulae; we can avoid Rule IEM in the new part of the extension.

This section has three subsections: Section 7.1 prepares various basic lemmas. Section 7.2 defines a Kripke model, and Section 7.3 proves that it is a countermodel of $A \in \mathcal{P}_\beta$ with $\delta(A) \leq \beta - 1$ and $\nvdash_\beta A$. This section finishes Step 1, and we have Step 2 in Section 8.

7.1 Preparations

We stipulate that $\omega - k = \omega$ for any $k < \omega$; so when $\beta = \omega$, both (1) and $\delta(A) \leq \beta - 1$ hold automatically. In Sections 7.1.1 and 7.1.2, some preparations are given.

7.1.1 Preparation 1

We prepare some basic facts: Let $\mathcal{A}_m^o = \{A_0, ..., A_m\}$ be a finite set of formulae, where $A_0, ..., A_m$ are distinct, and $\mathcal{A}_m^* = \mathcal{A}_m^o \cup \{\neg A : A \in \mathcal{A}_m^o\}$. By stipulating $\wedge\emptyset$ to be $(\neg p) \vee p$, the consistency of a finite, possibly empty, set of formulae Γ can be generally defined by $\nvdash_\beta \wedge\Gamma \supset (\neg p) \wedge p$. The empty set $\emptyset$ is consistent because of the soundness theorem for REL_β. We say that a subset w of $\mathcal{A}_m^*$ is a *maximal consistent subset* iff $\wedge w$ is consistent and for any $A \in \mathcal{A}_m^o$, $A \in w$ or $\neg A \in w$. Let $\mathbb{W}(\mathcal{A}_m^*)$ be the set of maximal consistent subsets of $\mathcal{A}_m^*$. We can construct a maximal consistent set in the standard manner (cf., Chellas [9], Sec.2.6); thus, $\mathbb{W}(\mathcal{A}_m^*)$ is nonempty. We write $\varphi_w = \wedge w$ for $w \in \mathbb{W}(\mathcal{A}_m^*)$.

We denote the set of all subformulae of A by $\mathrm{Sub}(A)$, and define $\mathrm{Sub}(\Phi) = \cup_{A\in\Phi}\mathrm{Sub}(A)$ for a set of formulae Φ. The following lemma is basic and will be used in the main proof of completeness. Note that by $\langle \mathbf{1} \rangle$ of this lemma, each $w \in \mathbb{W}(\mathcal{A}_m^*)$ has cardinality $m+1$.

Lemma 7.1.$\langle \mathbf{0} \rangle$ For any consistent subset v of $\mathcal{A}_m^*$, there is a maximal consistent subset $u \in \mathbb{W}(\mathcal{A}_m^*)$ with $v \subseteq u$;

$\langle \mathbf{1} \rangle$ if $w \in \mathbb{W}(\mathcal{A}_m^*)$, then either $A_t \in w$ or $\neg A_t \in w$ for each $t \leq m$.

$\langle \mathbf{2} \rangle$ if $w \in \mathbb{W}(\mathcal{A}_m^*)$, then $w \cap \mathcal{A}_{m-1} \in \mathbb{W}(\mathcal{A}_{m-1}^*)$; and if $w \in \mathbb{W}(\mathcal{A}_{m-1}^*)$, then $w \cup \{A_m\} \in \mathbb{W}(\mathcal{A}_m^*)$ or $w \cup \{\neg A_m\} \in \mathbb{W}(\mathcal{A}_m^*)$;

$\langle \mathbf{3} \rangle$ for any consistent subset $v \subseteq \mathcal{A}_m^*$, $\vdash_\beta \wedge v \equiv \vee_{v\subseteq w \in \mathbb{W}(\mathcal{A}_m^*)}\varphi_w$;

$\langle \mathbf{4} \rangle$ for any nonempty subset V of $\mathbb{W}(\mathcal{A}_m^*)$, $\vdash_\beta \neg(\vee_{w\in(\mathbb{W}(\mathcal{A}_m^*)-V)}\varphi_w) \supset \vee_{w\in V}\varphi_w$.

Proof The standard construction of a maximal consistent set guarantees $\langle 0 \rangle$. We prove $\langle 1 \rangle$, $\langle 2 \rangle$, and $\langle 3 \rangle$. Taking $v = \emptyset$ in $\langle 3 \rangle$, it follows that $\vdash_\beta \vee_{w\in\mathbb{W}(\mathcal{A}_m^*)}\varphi_w$. This is equivalent to $\vdash_\beta (\vee_{w\in(\mathbb{W}(\mathcal{A}_m^*)-V)}\varphi_w) \vee (\vee_{w\in V}\varphi_w)$, for any nonempty subset V of $\mathbb{W}(\mathcal{A}_m)$, which is equivalent to $\langle 4 \rangle$.

$\langle 0 \rangle$: Suppose that neither belongs to w. Since w is maximal consistent, both $w \cup \{A_t\}$ and $w \cup \{\neg A_t\}$ are inconsistent. This implies $\vdash_\beta \wedge w \supset \neg A_t$ and $\vdash_\beta \wedge w \supset \neg\neg A_t$, which means that w is inconsistent, a contradiction. Hence, $A_t \in w$ or $\neg A_t \in w$. If both hold, then $\vdash_\beta \wedge w \supset A_t$ and $\vdash_\beta \wedge w \supset \neg A_t$, a contradiction. Thus, either $A_t \in w$ or $\neg A_t \in w$.

$\langle 2 \rangle$: Consider the former: Let $w \in \mathbb{W}(\mathcal{A}_m^*)$. Then, $w \cap \mathcal{A}_{m-1}^*$ is consistent. By $\langle 1 \rangle$, $w \cap \mathcal{A}_{m-1}^*$ is maximal in $\mathcal{A}_{m-1}^*$. Now consider the latter assertion. Let $w \in \mathbb{W}(\mathcal{A}_{m-1}^*)$. Then $w \cup \{A_m\}$ or $w \cup \{\neg A_m\}$ is consistent; in either case, it is maximally consistent in $\mathcal{A}_m^*$ by $\langle 1 \rangle$.

$\langle 3 \rangle$: Let $\mathcal{V}$ be the set of consistent subsets of $\mathcal{A}_m^*$. We prove the assertion by the induction over $\mathcal{V}$ with respect to the cardinality of $v \in \mathcal{V}$ from the base case where v is a maximally consistent set in $\mathcal{V}$ in the descending order. Now, consider the induction base where ν is maximally consistent in $\mathcal{A}_m^*$. In this case, the assertion of $\langle 3 \rangle$ is written as $\vdash_\beta \wedge v \equiv \vee\{\varphi_v\}$ and the disjunction $\vee\{\varphi_v\}$ is equivalent to $\varphi_v = \wedge v$ and the result follows directly.

Now, let ℓ be a number with $1 \leq \ell \leq m$; we note $|\mathcal{A}_m| = m+1$. Let $\mathcal{V}_\ell = \{v \in \mathcal{V} : |v| = \ell\}$. The induction hypothesis is that for each $v' \in \mathcal{V}_{\ell+1}$,

$$\vdash_\beta \wedge v' \equiv \vee_{v'\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)}\varphi_u. \tag{31}$$

Now, we choose any $v \in \mathcal{V}_\ell$, and let $\mathcal{V}(v) = \{v \cup \{C\} : v \cup \{C\} \in \mathcal{V}\}$. First, we prove $\vdash_\beta \varphi_v \equiv \vee_{v'\in\mathcal{V}(v)}\varphi_{v'}$. Let $\mathcal{A}_m^o(v) = \{C \in \mathcal{A}_m^o : C \in v \text{ or } \neg C \in v\}$. Then, we have

$$\vdash_\beta \vee_{v'\in\mathcal{V}(v)}\varphi_{v'} \equiv \vee_{C\in\mathcal{A}_m^o-\mathcal{A}_m^o(v)}[(\varphi_v \wedge C) \vee (\varphi_v \wedge \neg C)].$$

Abbreviating $\vdash_\beta$ and connecting the formulae directly by $\equiv$, we have

$$\begin{aligned} \vee_{v'\in\mathcal{V}(v)}\varphi_{v'} &\equiv \vee_{C\in A_m^o-A_m^o(v)}[(\varphi_v \wedge C) \vee (\varphi_v \wedge \neg C)] \\ &\equiv \vee_{C\in A_m^o-A_m^o(v)}\varphi_v \wedge (C \vee \neg C) \equiv \vee_{C\in A_m^o-A_m^o(v)}\varphi_v \equiv \varphi_v. \end{aligned}$$

Thus, $\vdash_\beta \varphi_v \equiv \vee_{v'\in\mathcal{V}(v)}\varphi_{v'}$. Now, we have;

$$\begin{aligned}\vee_{v'\in\mathcal{V}(v)}\varphi_{v'} &\equiv \vee_{v'\in\mathcal{V}(v)} \vee_{v'\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)} \varphi_u \ (\because (31)) \\ &\equiv [\vee_{v\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)}\varphi_u] \vee [\vee_{v'\in\mathcal{V}(v),|v'|=\ell+1} \vee_{v'\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)} \varphi_u] \\ &\equiv [\vee_{v\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)}\varphi_u].\end{aligned}$$

Thus, $\vdash_\beta \varphi_v \equiv \vee_{v\subseteq u\in\mathbb{W}(\mathcal{A}_m^*)}\varphi_u$. We have completed the induction step. ■

7.1.2 Preparation 2

We construct a model based on a given formula $A \in \mathcal{P}_\beta$. Such a model is typically constructed directly from the set of all subformulae of A. Here, we extend the set of all subformulae of A to accommodate the fixed-point argument in Axiom AEM and Rule IEM. To have it, we focus on the pc-subformulae of A. We define

$$\Theta(A) = \{D : D \text{ is an pc-subformula of } A\}. \tag{32}$$

In the following, we assume that

$$\Theta(A) \text{ is nonempty.} \tag{33}$$

The case $\Theta(A) = \emptyset$ is covered by the following proof by ignoring the cases of an pc-formula.

Let $D \in \Theta(A)$, i.e., $D = \mathbf{Pc}_l(\boldsymbol{C})$ for $l = 1$ or 2 and some $\boldsymbol{C} = (C_1, C_2)$. We define

$$\mathrm{SI}(D) = \{\mathbf{Pc}_i(\boldsymbol{C}), \mathbf{B}_i(C_i), \mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C}), \mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C})) : i = 1, 2\}. \tag{34}$$

That is, we associate the set $\mathrm{SI}(D)$ with D. Note that $\mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C}))$ looks redundant but it will be used in Lemma 8.5.

Then, we take all subformulae of A and all subformulae (of C) of $\mathrm{SI}(D),\ D \in \Theta(A)$, that is,

$$\begin{aligned}\mathrm{Sub}^o(A) &= \{C : C \text{ is a subformula of } A\} \\ \cup\{C &: C \text{ is a sub-formula of } \mathrm{SI}(D), D \in \Theta(A)\}.\end{aligned} \tag{35}$$

The addition of the second set will be used when we handle pc-formulue in our induction steps in Section 7.2. Then, we extend this set adding the negations of all formulae in $\mathrm{Sub}^o(A)$, that is,

$$\mathrm{Sub}^*(A) = \{\neg C,\ C : C \in \mathrm{Sub}^o(A)\}. \tag{36}$$

This is used to construct the set of possible worlds by taking all maximal consistent sets in it.

In the following, we denote $\delta(\mathrm{Sub}^*(A)) = \max\{\delta(C) : C \in \mathrm{Sub}^*(A)\}$.

Lemma 7.2 $\mathrm{Sub}^*(A)$ is subformula-closed, and if $\delta(A) \leq \beta - 1$, then $\delta(\mathrm{Sub}^*(A)) \leq \beta - 1$.

Proof Subformula-closedness follows from definitions (35) and (36). For epistemic depths, consider two cases. First, let $D = \mathbf{Pc}_j(\boldsymbol{C}) \in \Theta(A)$. Then, $\delta(D) \leq \beta - 2$ by (1). We should consider any formulae $\mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})$ added in (34); $\delta(\mathbf{B}_i\mathbf{Pc}_j(\boldsymbol{C})) = 1 + \delta(\mathbf{Pc}_j(\boldsymbol{C})) \leq 1 + (\beta - 2) = \beta - 1$. Let $C \in \mathrm{Sub}^*(A)$ but let C be not added in (34). Then, $\delta(C) \leq \delta(A)$. Since $\delta(A) \leq \beta - 1$, we have $\delta(C) \leq \delta(A) \leq \beta - 1$. Hence, $\delta(\mathrm{Sub}^*(A)) \leq \beta - 1$.■

7.2 Construction of a countermodel for $A \in \mathcal{P}_\beta$ with $\delta(A) \leq \beta - 1$

Now, we focus on a formula A and assume

$$\delta(A) \leq \beta - 1 \text{ and } \nvdash_\beta A. \tag{37}$$

Our goal is to construct a model $M = (F, \sigma) = ((W, R_1, R_2), \sigma)$ so that $(M, w) \nvDash A$ for some $w \in W$.

We can apply Lemma 7.1 by taking $\mathrm{Sub}^o(A)$,$\mathrm{Sub}^*(A)$ as $\mathcal{A}^o_m, \mathcal{A}^*_m$. We denote the set of maximal consistent subsets of $\mathrm{Sub}^*(A)$, by

$$\mathrm{Con}^*(A) := \mathbb{W}(\mathrm{Sub}^*(A)). \tag{38}$$

Since $\nvdash_\beta A$, we have the consistency of $\neg A$. By Lemma 7.1.$\langle 0 \rangle$ and $\langle 1 \rangle$, there is a $w \in \mathrm{Con}^*(A)$ such that

$$A \notin w. \tag{39}$$

Then, we construct a model $M = ((W, R_1, R_2), \sigma)$ with $W = \mathrm{Con}^*(A)$ so that $(M, w) \nvDash A$, which is the final goal.

The following observations are simple observations.

Lemma 7.3 For each $w \in \mathrm{Con}^*(A)$, we have

$\langle 1 \rangle$: for any $\neg C \in \mathrm{Sub}^*(A)$, either $C \in w$ or $\neg C \in w$;

$\langle 2 \rangle$: for any $(D \supset D') \in \mathrm{Sub}^*(A)$, $(D \supset D') \in w$ if and only if $D \notin w$ or $D' \in w$;

$\langle 3 \rangle$: for any $\wedge\Phi \in \mathrm{Sub}^*(A)$, $\wedge\Phi \in w$ if and only if $C \in w$ for any $C \in \Phi$.

Proof We look only at $\langle \mathbf{1} \rangle$. Let $\neg C \in \mathrm{Sub}^*(A)$. Since w is maximal and consistent, it holds that $C \in w$ or $\neg C \in w$.■

We denote, by $u^{-\mathbf{B}_i}$, the set $\{C : \mathbf{B}_i(C) \in u\}$ for any set of formulae u. We define a model $M = (F, \sigma) = ((W; R_1, R_2), \sigma)$ as follows:

M1: $W = \mathrm{Con}^*(A) = \mathbb{W}(\mathrm{Sub}^*(A))$;

M2: $R_i = \{(u, v) \in W^2 : u^{-\mathbf{B}_i} \subseteq v\}$ for $i = 1, 2$;

M3: for any $(w, p) \in W \times PV$, $\sigma(w, p) = \top$ if and only if $p \in w$.

The above $M = (F, \sigma)$ is a model for logic REL_β. It suffices to see the seriality for R_i, $i = 1, 2$.

Lemma 7.4 (Seriality) The relation R_i is serial for $i = 1, 2$.

Proof Let $u \in W$. It suffices to show that $u^{-\mathbf{B}_i}$ is consistent; then, there exists some $v \in \mathrm{Con}^*(A)$ such that $u^{-\mathbf{B}_i} \subseteq v$, i.e., $(u, v) \in R_i$. Indeed, if $\vdash_\beta \wedge u^{-\mathbf{B}_i} \supset (\neg p \wedge p)$ for some $p \in PV$, by Nec and Axiom K, $\vdash_\beta \wedge u \supset \mathbf{B}_i(\neg p \wedge p)$; so by Axiom D, u is inconsistent, a contradiction to $u \in W$.■

We write $\delta(w) = \max\{\delta(C) : C \in w\}$ for $w \in W$. Then, for any $w \in W$, we have $\delta(w) = \delta(\mathrm{Sub}^*(A))$ by Lemma 7.3.$\langle 1 \rangle$. When $\delta(A) \leq \beta - 1$, by Lemma 7.2, it holds that $\delta(w) = \delta(\mathrm{Sub}^*(A)) = \delta(A) \leq \beta - 1$. We write this fact as a lemma, because it is crucial in the proof of the last induction step, specifically, in the if-part of the case $C = \mathbf{Pc}_i(C) = \mathbf{Pc}_i(C_1, C_2)$.

Lemma 7.5 (Depth $\beta - 1$) Let $\delta(A) \leq \beta - 1$. Then, $\delta(w) \leq \beta - 1$ for any $w \in W$.

7.3 Main part of the completeness proof of Step 1

Now, we show that for any $C \in \text{Sub}^*(A)$ and any $u \in W$,

$$C \in u \text{ if and only if } (M, u) \models C. \tag{40}$$

Once (40) is shown, we have $(M, w) \not\models A$ because $A \notin w$ by (39). Thus, $M = ((W, R_1, R_2), \sigma)$ is a countermodel for A. Nevertheless, for the connection to Step 2, i.e., $\delta(A) = \beta$, (40) will play the role of the induction base of the proof to be taken in Section 8.

We prove (40) by induction on the subformula structure of A. Specifically, we consider the partial ordering $\succ$ by representing the immediate subformulae relation over $\text{Sub}^*(A)$. The pair $(\text{Sub}^*(A), \succ)$ is regarded as a tree, though some paths may go through the same formula. Its root is A, the leaves are propositional variables, and connections of nodes are made by inference rules.

The induction base is for a propositional variable $C = p \in \text{Sub}^*(A)$. In this case, by M3 and V1,

$$p \in u \text{ if and only if } \sigma(w, p) = \top \text{ if and only if } (M, u) \models p.$$

Now, we go to the induction step. Let C be a non-propositional formula. The possible cases of the outmost connective of C are $\neg$, $\supset$, $\wedge$, $\mathbf{B}_i(\cdot)$, and $\mathbf{Pc}_i(\cdot, \cdot)$. Now, we make the induction hypothesis that for any immediate subformula C' of C,

$$C' \in u \text{ if and only if } (M, u) \models C'. \tag{41}$$

The *if* part of case $\mathbf{Pc}_i(\cdot, \cdot)$ includes another induction argument along an al-sequence in an entangled manner with the induction along (41).

Case (a) $C = \neg C'$: Since $C' \in \text{Sub}^*(A)$, we have $\neg C' \in \text{Sub}^*(A)$. It holds by the hypothesis (41) that $C' \in u$ if and only if $(M, u) \models C'$, equivalently, $C' \notin u$ if and only if $(M, u) \not\models C'$. The former is equivalent to $\neg C' \in u$ by Lemma 7.3.$\langle 1 \rangle$, and the latter is equivalent to $(M, u) \models \neg C'$. Thus, we have $\neg C' \in u$ if and only if $(M, u) \models \neg C'$.

Case (b) $C = F \supset G$: Since $F, G \in \text{Sub}^*(A)$, we have $F \supset G \in u$ if and only if $F \notin u$ or $G \in u$ by Lemma 7.3.$\langle 2 \rangle$; by the induction hypothesis, the latter is equivalent to $(M, u) \not\models F$ or $(M, u) \models G$, which is equivalent to $(M, u) \models F \supset G$.

The case of $C = \wedge \Phi$ is similar.

Case (c) $C = \mathbf{B}_i(C')$: The induction hypothesis (41) is assumed for C'. First, we show that $(M, w) \models \mathbf{B}_i(C')$ implies $\mathbf{B}_i(C') \in w$. Suppose $(M, w) \models \mathbf{B}_i(C')$. We claim that $w^{-\mathbf{B}_i} \cup \{\neg C'\}$ is inconsistent. Suppose, on the contrary, it is consistent. Using Lemma 7.1.$\langle 0 \rangle$, there exists some $u \in W$ such that $w^{-\mathbf{B}_i} \cup \{\neg C'\} \subseteq u$, which implies $C' \notin u$. By the induction hypothesis, $(M, u) \not\models C'$. Since $w^{-\mathbf{B}_i} \subseteq u$, we have wR_iu by M2. Hence, $(M, w) \not\models \mathbf{B}_i(C')$, a contradiction to the starting supposition. Thus, $w^{-\mathbf{B}_i} \cup \{\neg C'\}$ is inconsistent; so $\vdash_\beta \wedge w^{-\mathbf{B}_i} \supset C'$. This implies $\vdash_\beta \wedge w \supset \mathbf{B}_i(C')$. Thus, $\mathbf{B}_i(C') \in w$, since w is a maximal consistent set. Note that $\delta(\wedge w) \leq \beta - 1$.

Conversely, suppose that $\mathbf{B}_i(C') \in w$. Then, $C' \in w^{-\mathbf{B}_i}$. Take any u with wR_iu, i.e., $w^{-\mathbf{B}_i} \subseteq u$. Hence, $C' \in u$. By the induction hypothesis, $(M, u) \models C'$. This holds for any u with wR_iu. Thus, $(M, w) \models \mathbf{B}_i(C')$.

Now, we go to the last case.

Case (d) $C = \mathbf{Pc}_i(\boldsymbol{C}) = \mathbf{Pc}_i(C_1, C_2)$. The *if* part is crucial.

(*Only-if*): Suppose $\mathbf{Pc}_i(\boldsymbol{C}) \in w$. Let $\langle (w_0, i_0), ..., (w_\nu, i_\nu), w_{\nu+1} \rangle$ be an arbitrary al-chain with $(w_0, i_0) = (w, i)$. We show, by induction along this al-chain, that C_{i_k} and $\mathbf{Pc}_{i_{k+1}}(\boldsymbol{C})$ are in w_{k+1} for all k $(0 \leq k \leq \nu)$. Let $k = 0$. Then, $w_0 = w, i_0 = i$. Since $\mathbf{Pc}_i(\boldsymbol{C}) \in w$ and $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i \mathbf{Pc}_j(\boldsymbol{C})$ by AEM, we have $\mathbf{B}_i(C_i), \mathbf{B}_i \mathbf{Pc}_j(\boldsymbol{C}) \in w$. Because wR_iw_1, we have $w^{-\mathbf{B}_i} \subseteq w_1$; so $C_i \in w_1$ and $\mathbf{Pc}_{i_1}(\boldsymbol{C}) \in w_1$. This is the assertion of the induction base. Now, suppose the induction hypothesis that $C_{i_k} \in w_{k+1}$ and $\mathbf{Pc}_{i_{k+1}}(\boldsymbol{C}) \in w_{k+1}$. By Axiom AEM, we have $\mathbf{B}_{i_{k+1}}(C_{i_{k+1}}) \in w_{k+1}$ and $\mathbf{B}_{i_{k+1}} \mathbf{Pc}_{i_{k+2}}(\boldsymbol{C}) \in w_{k+1}$. Again, since $w_{k+1} R_{i_{k+1}} w_{k+2}$, i.e., $w_{k+1}^{-\mathbf{B}_{i_{k+1}}} \subseteq w_{k+2}$, we have $C_{i_{k+1}} \in w_{k+2}$ and $\mathbf{Pc}_{i_{k+2}}(\boldsymbol{C}) \in w_{k+2}$. This concludes the induction argument for the al-chain. Thus, $C_{i_{k-1}} \in w_k$ for $k = 1, ..., \nu + 1$. Focusing on $k = \nu + 1$, we have $C_{i_\nu} \in w_{\nu+1}$. By the induction hypothesis for (41), we have $(M, w_{\nu+1}) \models C_{i_\nu}$. Since $\langle (w, i_0), , ..., (w_\nu, i_\nu), w_{\nu+1} \rangle$ is an arbitrary al-chain, we have $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$.

(*If*): Suppose $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$. Since it was shown in Section 3.2 that Axiom AEM, i.e., $\mathbf{Pc}_i(\boldsymbol{C}) \supset \mathbf{B}_i(C_i) \wedge \mathbf{B}_i \mathbf{Pc}_j(\boldsymbol{C})$, is valid with respect to $\models$, we have $(M, w') \models \mathbf{Pc}_j(\boldsymbol{C})$ for all $w' \in W$ with wR_iw'. The existence of such a w' is ensured by the Lemma 7.4 (Seriality). Now, we define $W^l_{\boldsymbol{C}} = \{u \in W : (M, u) \models \mathbf{Pc}_l(\boldsymbol{C})\}$ for $l = 1, 2$. Because of $(M, w) \models \mathbf{Pc}_i(\boldsymbol{C})$ and $(M, w') \models \mathbf{Pc}_j(\boldsymbol{C})$, it is guaranteed that

$$w \in W^i_{\boldsymbol{C}} \text{ and } w' \in W^j_{\boldsymbol{C}}. \tag{42}$$

By Lemma 7.5, $\delta(u) \leq \beta - 1$ for all $u \in W^l_{\boldsymbol{C}}$ for $l = 1, 2$. Since $\varphi_u := \wedge u$ has the depth $\delta(\varphi_u) = \delta(u) \leq \beta - 1$, φ_u belongs to $\mathcal{P}_\beta$, which holds for any $u \in W^l_{\boldsymbol{C}}$. Let $\chi_{W^l_{\boldsymbol{C}}} := \vee\{\varphi_u : u \in W^l_{\boldsymbol{C}}\}$ for $l = 1, 2$. This has the depth $\delta(\chi_{W^l_{\boldsymbol{C}}}) = \max\{\delta(u) : u \in W^l_{\boldsymbol{C}}\} \leq \beta - 1$, and $\delta(\mathbf{B}_l(\chi_{W^{l'}_{\boldsymbol{C}}})) = \delta(\chi_{W^{l'}_{\boldsymbol{C}}}) + 1 \leq \beta$. We will show that for $l, l' = 1, 2$ $(l \neq l')$,

$$\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_l(C_l) \wedge \mathbf{B}_l(\chi_{W^{l'}_{\boldsymbol{C}}}). \tag{43}$$

These formulae are in $\mathcal{P}_\beta$ and are regarded as the upper formulae of Rule IEM. Once (43) is proved, we have, by Rule IEM, $\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{Pc}_l(\boldsymbol{C})$ for $l = 1, 2$. Since $\vdash_\beta \varphi_w \supset \chi_{W^i_{\boldsymbol{C}}}$ by (42), we have $\vdash_\beta \varphi_w \supset \mathbf{Pc}_i(\boldsymbol{C})$. By Lemma 7.1.$\langle 1 \rangle$, we have $\mathbf{Pc}_i(\boldsymbol{C}) \in w$. Now, we have the *if*-part of (41) in the case $C = \mathbf{Pc}_i(\boldsymbol{C})$, under the assumption that (43) is proved.

Now, we show (43); the proof is up to the end of this subsection. First, we show

$$\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_l(C_l) \text{ for } l = 1, 2. \tag{44}$$

Let u be an arbitrary world in $W^l_{\boldsymbol{C}}$. By the definition of $\chi_{W^l_{\boldsymbol{C}}}$, it holds that $(M, u) \models \mathbf{Pc}_l(\boldsymbol{C})$. By this and V4,

$$(M, v) \models C_l \textit{ for any } v \text{ with } uR_lv. \tag{45}$$

Now, we claim that $u^{-\mathbf{B}_l} \cup \{\neg C_l\}$ is inconsistent. Suppose, on the contrary, it is consistent. By Lemma 7.1.$\langle 0 \rangle$, there exists some $v \in W$ such that $u^{-\mathbf{B}_l} \cup \{\neg C_l\} \subseteq v$, which implies $C_l \notin v$ and uR_lv. By the induction hypothesis for (41), $(M, v) \not\models C_l$, a contradiction to (45). Thus, $u^{-\mathbf{B}_l} \cup \{\neg C_l\}$ is inconsistent; so $\vdash_\beta \wedge u^{-\mathbf{B}_l} \supset C_l$. This implies $\vdash_\beta \wedge u \supset \mathbf{B}_l(C_l)$, and furthermore $\mathbf{B}_l(C_l) \in u$. Since this holds for an arbitrary $u \in W^l_{\boldsymbol{C}}$, we have $\vdash_\beta \vee\{\wedge u : u \in W^l_{\boldsymbol{C}}\} \supset \mathbf{B}_l(C_l)$, that is, $\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_l(C_l)$.

Finally, we show

$$\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_l(\chi_{W^{l'}_{\boldsymbol{C}}}) \text{ for } l, l' \text{ with } l \neq l'. \tag{46}$$

We first derive an equivalent formula to (46).

Lemma 7.6 $\langle \mathbf{1} \rangle \vdash_\beta \chi_{W^{l'}_{\boldsymbol{C}}} \equiv \neg(\vee_{v \in W - W^{l'}_{\boldsymbol{C}}} \varphi_v)$ and $\langle \mathbf{2} \rangle \vdash_\beta \chi_{W^{l'}_{\boldsymbol{C}}} \equiv \wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \neg\varphi_v$.

Proof Since $\vdash_\beta \neg(\vee_{v \in W - W^{l'}_{\boldsymbol{C}}} \varphi_v) \equiv \wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \neg\varphi_v$ by the classical de Morgan's low, the above two are equivalent. The implication $\vdash_\beta \neg(\vee_{v \in W - W^{l'}_{\boldsymbol{C}}} \varphi_v) \supset \chi_{W^{l'}_{\boldsymbol{C}}}$ is by Lemma 7.1.$\langle 4 \rangle$. Consider the converse. By Lemma 7.1.$\langle 1 \rangle$, if u, v are distinct in W, then there is a $C \in \mathrm{Sub}^*(A)$ such that $\vdash_\beta \varphi_u \supset C$ and $\vdash_\beta \varphi_v \supset \neg C$; the latter is equivalent to $\vdash_\beta C \supset \neg\varphi_v$. Thus, $\vdash_\beta \varphi_u \supset \neg\varphi_v$. This holds for all distinct $u, v \in W$. Hence, for any $u \in W^{l'}_{\boldsymbol{C}}$, we have $\vdash_\beta \varphi_u \supset \wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \neg\varphi_v$, which implies $\vdash_\beta \vee_{u \in W^{l'}_{\boldsymbol{C}}} \varphi_u \supset \wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \neg\varphi_v$. This is further equivalent to $\vdash_\beta \chi_{W^{l'}_{\boldsymbol{C}}} \supset \neg(\vee_{v \in W - W^{l'}_{\boldsymbol{C}}} \varphi_v)$.■

Using Lemma 7.6.$\langle 2 \rangle$, (46) is changed into $\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_i(\wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \neg\varphi_v)$, which is written, by Lemma 2.1.$\langle 3 \rangle$, as

$$\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \wedge_{v \in W - W^{l'}_{\boldsymbol{C}}} \mathbf{B}_l(\neg\varphi_v). \tag{47}$$

We consider the following assertion; since $\chi_{W^l_{\boldsymbol{C}}} = \vee\{\varphi_u : u \in W^l_{\boldsymbol{C}}\}$, (47) is derived from (48) but not necessarily the converse;

$$\vdash_\beta \varphi_u \supset \mathbf{B}_l(\neg\varphi_v) \text{ for any } u \in W^l_{\boldsymbol{C}} \text{ and any } v \in W - W^{l'}_{\boldsymbol{C}}. \tag{48}$$

Now, (48) implies (47), and (47) is equivalent to (46). Thus, it remains to show (48).

Suppose, on the contrary, that (48) does not hold for some $u \in W^l_{\boldsymbol{C}}$ and some $v \in W - W^{l'}_{\boldsymbol{C}}$; that is, $\nvdash_\beta \varphi_u \supset \mathbf{B}_l(\neg\varphi_v)$, i.e., $\nvdash_\beta \neg(\varphi_u \wedge \neg\mathbf{B}_l(\neg\varphi_v))$; thus, φ_u and $\neg\mathbf{B}_i(\neg\varphi_v)$ are consistent. We have the following lemma.

Lemma 7.7 If φ_u and $\neg\mathbf{B}_i(\neg\varphi_v)$ are consistent, then $u^{-B_l} \subseteq v$.

Proof We prove the contrapositive; suppose $u^{-B_l} \nsubseteq v$. That is, $C \in u^{-B_l}$ but $C \notin v$ for some C. Then, $\neg C \in v$; so, $\vdash_\beta \varphi_\nu \supset \neg C$, equivalently, $\vdash_\beta C \supset \neg\varphi_v$. Since $\delta(\varphi_v) \leq \beta - 1$ by Lemma 7.5, we can apply NEC and K, and we have $\vdash_\beta \mathbf{B}_i(C) \supset \mathbf{B}_i(\neg\varphi_v)$. Since $\mathbf{B}_i(C) \in u$, we have $\vdash_\beta \varphi_u \supset \mathbf{B}_i(C)$; so, $\vdash_\beta \varphi_u \supset \mathbf{B}_i(\neg\varphi_v)$. Thus, φ_u and $\neg\mathbf{B}_i(\neg\varphi_v)$ are inconsistent.■

Let us return to the proof of (48); we are still assuming that (48) does not hold. We have Lemma 7.7; $u^{-B_l} \subseteq v$, that is, uR_lv. We show that this implies $(M, v) \models \mathbf{Pc}_{l'}(\boldsymbol{C})$. Let $\langle (w_1, i_1), ..., (w_\nu, i_\nu), w_{\nu+1} \rangle$ be any al-chain with $(w_1, i_1) = (v, l')$. Since uR_iv, $\langle (u, i), (w_1, i_1), ..., (w_\nu, i_\nu), w_{\nu+1} \rangle$ is an al-chain from (u, i). Since $u \in W^l_{\boldsymbol{C}}$, it holds that $(M, u) \models \mathbf{Pc}_l(\boldsymbol{C})$. This implies $(M, w_{\nu+1}) \models C_{i_\nu}$. Because $\langle (w_1, i_1), ..., (w_\nu, i_\nu), w_{\nu+1}) \rangle$ is any al-chain with $(w_1, i_1) = (v, l')$, we have $(M, v) \models \mathbf{Pc}_{l'}(\boldsymbol{C})$. This is a contradiction to the choice of $v \in W - W^{l'}_{\boldsymbol{C}}$. We derived a contradiction from the assumption that (48) does not hold. Thus, we have (48).

Now, we can return to the main target (43). We have (48), which implies (47), which is equivalent to (46). Combining (46) with (44), we have $\vdash_\beta \chi_{W^l_{\boldsymbol{C}}} \supset \mathbf{B}_l(C_l) \wedge \mathbf{B}_l(\chi_{W^{l'}_{\boldsymbol{C}}})$ for $l = 1, 2$. This is (43).■

8 Completeness Proof of $\mathbf{REL}_\beta$: Step 2 with $\delta(A) = \beta$

Here, we eliminate the condition $\delta(A) \leq \beta - 1$ for completeness, and start simply with the condition

$$\delta(A) = \beta \text{ and } \nvdash_\beta A. \tag{49}$$

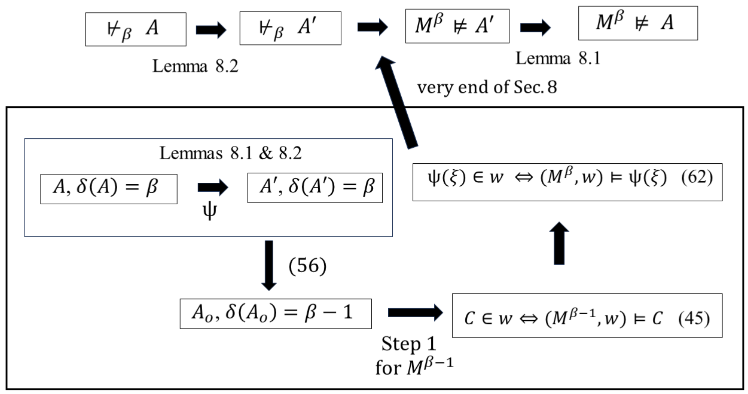

Figure 5: Process to the countermodel M^β

The proof takes various steps describe in Fig.5. The large rectangle is the main part without assuming $\nvdash_\beta A$, and then, its result is applied to $\nvdash_\beta A$ in the top line. In the top left of the rectangle, we start with a given formula A with $\delta(A) = \beta$, and transform the formula A to an equivalent formula A' by the function ψ so that $\delta(A') = \beta - 1$. Then, Section 8.2 extracts the part A_o of A', satisfying the $\delta(A_o) = \beta - 1$; we construct a model for A_0 as the model M given of Section 7.2 is denoted by $M^{\beta-1}$. For this model, (41) of Section 7.3 holds. We will extend the model $M^{\beta-1}$ to M^β by adding a new part. Then, (41) is extended to (57) with ψ. This can be applied to the top line. Then, we have a countermodel for A with (49).

8.1 New part of subformulae of A with $\delta(A) = \beta$

Let $A \in \mathcal{P}_\beta$ and $\delta(A) = \beta \geq 3$. We focus on the trunk part of the decomposition tree of formula A up to the first occurrences of a propositional variable, a b-formula, or a pc-formula. The obstacle for the method of Section 7 is the case where decomposition stops at a pc-formula $\mathbf{Pc}_i(C_1, C_2)$; the valuations are difficult to be controlled, since it is valuated by al-chains of all lengths. We avoid this obstacle by substituting $\mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(C_1, C_2))$ for $\mathbf{Pc}_i(C_1, C_2)$, since $\mathbf{Pc}_i(C_1, C_2)$ is equivalent to $\mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(C_1, C_2))$ by Theorem 2.1. Then, we get the new formula A'. An example is shown in Fig.5. It is a simple idea, but because the new decomposition tree will be used in an induction proof, we need a rigorous definition.

We denote the binary relation $\succ$ over the subformulae of A; for $C, C' \in \mathrm{Sub}(A)$, we define $C \succ C'$ iff C' is an immediate subformula C. We say that $\xi = [\xi_0, \xi_1, ..., \xi_t]$ is a *trunk path* from A iff

(o): $\xi_0 = A$ and $\xi_k \succ \xi_{k+1}$ for $k = 0, ..., t-1$;

(i): if $t \geq 1$, then, the outmost connectives of $\xi_0, \xi_1, ..., \xi_{t-1}$ are $\neg$, $\supset$, or $\wedge$.

A trunk path starts with the root $\xi_0 = A$ with decompositions with logical connectives $\neg$, $\supset$, or $\wedge$. We denote the *set* of all trunk paths from $\xi_0 = A$ by Ξ_{trk}. We say that $\xi = [\xi_0, \xi_1, ..., \xi_m] \in \Xi_{trk}$ is a *maximal trunk path* iff it is a trunk path and

(ii)(o-p): $\xi_m \in PV$; (o-B): ξ_m is a b-formula; or (o-PC): ξ_m is a pc-formula..

That is, ξ is an outcome of the decomposition process until (ii) holds. In the case $m = 0$,

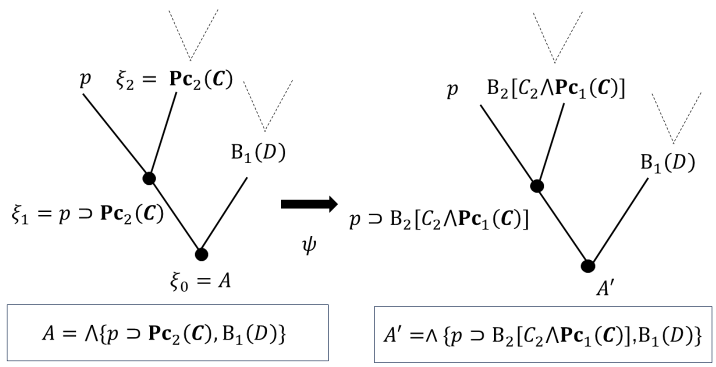

Figure 6: A trunk with no naked pc-formulae

$A = \mathbf{B}_i(C)$ for some C and $i = 1, 2$ and A is not an pc-formula since $\delta(A) = \beta \geq 3$. The set Ξ_{trk} expresses the outcomes of the decomposition process from A. Fig.6 depicts two examples of the trunks $\Xi_{trk}(A)$ and $\Xi_{trk}(A')$.

As stated above, (o-PC) is an obstacle for constructing countermodel for A. Nevertheless, if we find another formula A' so that A' is equivalent to A in REL$_\beta$ and only (o-p) or (o-B) of (ii) holds for A'. Then, if M is a countermodel for A', so is M for A.

Lemma 8.1 Let $C, C' \in \mathcal{P}_\beta$, and $M = ((W, R_1, R_2), \sigma)$ a model. Let $\vdash_\beta C \equiv C'$. Then, for any $w \in W$,

$$(M, w) \models C \text{ if and only if } (M, w) \models C'.$$

Proof Let $\vdash_\beta C \equiv C'$. By soundness for REL$_\beta$, it holds that $(M, w) \models C \equiv C'$ for all $w \in W$. For any $w \in W$, $(M, w) \models C$ implies $(M, w) \models C'$, and *vice versa*.■

Now, we construct the function $\psi : \Xi_{trk} \longrightarrow \mathcal{P}_\beta$ by induction from its maximal trunk paths:

(IB): For any maximal path $\xi = [\xi_0, \xi_1, ..., \xi_m] \in \Xi_{trk}$,

$$\psi(\xi) = \begin{cases} \xi_m & \text{if } \xi_m \in PV \\ \mathbf{B}_i(C) & \text{if } \xi_m = \mathbf{B}_i(C) \\ \mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C})) & \text{if } \xi_m = \mathbf{Pc}_i(\boldsymbol{C}). \end{cases}$$

The case $\xi_m = \mathbf{Pc}_i(\boldsymbol{C})$ is changed. Let $\xi = [\xi_0, \xi_1, ..., \xi_t] \in \Xi_{trk}$ be non-maximal in Ξ_{trk}. Suppose the induction hypothesis that $\psi(\xi \cdot \xi_{t+1})$ is already defined for all $\xi \cdot \xi_{t+1} \in \Xi_{trk}$. Then,

($\neg$): if $\xi_t = \neg\xi_{t+1}$, then $\psi(\xi) = \neg\psi(\xi \cdot \xi_{t+1})$;

($\supset$): if $\xi_t = \xi_{t+1} \supset \xi'_{t+1}$, then $\psi(\xi) = \psi(\xi \cdot \xi_{t+1}) \supset \psi(\xi \cdot \xi'_{t+1})$;

($\wedge$): if $\xi_t = \wedge\Phi$, then $\psi(\xi) = \wedge\{\psi(\xi \cdot \xi_{t+1}) : \xi_{t+1} \in \Phi\}$.

Conditions (IB), ($\neg$), ($\supset$), and ($\wedge$) guarantee that this inductive definition gives a unique value $\psi(\xi)$ to each trunk path $\xi \in \Xi_{trk}$. Since $\delta(\mathbf{Pc}_i(C)) \leq \beta - 2$ if $\xi_m = \mathbf{Pc}_i(C)$, it holds that $\delta(\mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C}))) \leq \beta - 1$. This implies that $\delta(\xi) \leq \beta$ for all $\xi \in \Xi_{trk}$. In Fig.5, the trunk derived from $A = \wedge\{p \supset \mathbf{Pc}_2(\boldsymbol{C}), \mathbf{B}_1(D)\}$ is depicted as the left tree, where $\xi(D) = D$ is assumed, and the trunk obtained by ψ is depicted as the right tree with $\psi([\xi_0]) = A' = \wedge\{p \supset \mathbf{B}_2[C_2 \wedge \mathbf{Pc}_1(\boldsymbol{C})], \mathbf{B}_1(D)\}$.

If Ξ_{trk} has no naked pc-formulas, then $\psi(\xi) = \xi_t$ for all $\xi = [\xi_0, \xi_1, ..., \xi_t] \in \Xi_{trk}$. But our concern is the case where Ξ_{trk} has some naked ps-formula. In this case, the equality $\psi(\xi) = \xi_t$ becomes an equivalence in REL_β.

Lemma 8.2 $\vdash_\beta \psi(\xi) \equiv \xi_t$ for all $\xi = [\xi_0, \xi_1, ..., \xi_t] \in \Xi_{trk}$.

Proof We prove the assertion by induction on $(\Xi_{trk}, \succ)$ from its maximal $\xi \in \Xi_{trk}$. The induction base is $\vdash_\beta \psi(\xi) \equiv \xi_m$ for maximal $\xi = [\xi_0, \xi_1, ..., \xi_m] \in \Xi_{trk}$. If ξ_m is a propositional variable or a b-formula, then, $\psi(\xi) = \xi_m$ by IB; thus, $\vdash_\beta \psi(\xi) \equiv \xi_m$. If $\xi_m = \mathbf{Pc}_i(\boldsymbol{C})$, it holds by Theorem 2 1.$\langle 1 \rangle$ that $\vdash_\beta \mathbf{Pc}_i(\boldsymbol{C}) \equiv \mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C}))$. By IB, $\vdash_\beta \psi(\xi) \equiv \xi_m$.

Now, suppose that $\xi = [\xi_0, \xi_1, ..., \xi_t] \in \Xi_{trk}$ is non-maximal. Then, there are three cases to be considered: $(\neg)$ $\xi_t = \neg C$; $(\supset)$ $\xi_t = C \supset D$, and $(\wedge)$ ξ_t is $\wedge\Phi$. The induction hypothesis is that $\vdash_\beta \psi(\xi \cdot \xi_{t+1}) \equiv \xi_{t+1}$ for all $\xi \cdot \xi_{t+1}$ with $\xi_t \succ \xi_{t+1}$. We consider only $(\wedge)$; $(\neg)$ and $(\supset)$ are similar. Then, $\xi_t = \wedge\Phi$. The induction hypothesis is expressed as $\vdash_\beta \psi(\xi \cdot C) \equiv C$ for all $C \in \Phi$. By Axiom L4 and Lemma 2.1.$\langle 1 \rangle$, this implies $\vdash_\beta \wedge\psi(\xi \cdot C) \supset C$ for all $C \in \Phi$; thus, $\vdash_\beta \wedge\{\psi(\xi \cdot C) : C \in \Phi\} \supset \wedge\Phi$ by $\wedge$-Rule. The converse can be proved in the same manner. Thus, $\vdash_\beta \wedge\{\psi(\xi \cdot C) : C \in \Phi\} \equiv \wedge\Phi$; so, $\vdash_\beta \psi(\xi) \equiv \xi_t$ by $(\wedge)$ and $\xi_t = \wedge\Phi$.■

8.2 Main part of the completeness proof

Recall $A \in \mathcal{P}_\beta$ with $\delta(A) = \beta$. Then, let $A' = \psi([\xi_0])$ where $\xi_0 = A$, and define

$$\Phi_o = \{C : \psi(\xi) = \mathbf{B}_i(C) \text{ for some } i \text{ and } \xi \text{ is maximal in } \Xi_{trk}\}. \tag{50}$$

In Fig.5, since the transformed $A' = \psi([\xi_0])$ is $\wedge\{p \supset \mathbf{B}_2[C_2 \wedge \mathbf{Pc}_1(\boldsymbol{C})], \mathbf{B}_1(D)\}$, Φ_o is the set $\{C_2 \wedge \mathbf{Pc}_1(\boldsymbol{C}), D\}$.

For a maximal $\xi = [\xi_0, \xi_1, ..., \xi_m] \in \Xi_{trk}$, $\psi(\xi)$ is in PV or is a b-formula by (IB). In (50), Φ_o collects the contents of such b-formulae. Now, let $A_o = \wedge\Phi_o$. Since $\delta(A) = \beta$, we have

$$\delta(A_o) = \beta - 1. \tag{51}$$

Section 7 gives a model for A_o, denoted by $M^{\beta-1} = ((W^{\beta-1}; R_1^{\beta-1}, R_2^{\beta-1}), \sigma^{\beta-1})$.

We extend $M^{\beta-1}$ by adding the parts of A' that are not in A_o. For this purpose, we write $\mathrm{Sub}(A'; \beta-1)$ and $\mathrm{Sub}(\{A', A_o\})$ as:

$$\mathrm{Sub}(A'; \beta-1) = \mathrm{Sub}(A_o); \text{ and } \mathrm{Sub}(A'; \beta) = \mathrm{Sub}(\{A', A_o\}).$$

If $\psi(\xi) = p$ for a maximal $\xi \in \Xi_{trk}$ and $p \in PV$, then $p \in \mathrm{Sub}(A'; \beta)$. For $\kappa = \beta - 1, \beta$, we define

$$\mathrm{Sub}^o(A'; \kappa) = \mathrm{Sub}(A'; \kappa) \cup \{C : C \text{ is a sub-formula of } \mathrm{SI}(D) \text{ for some } D \in \Theta(A')\}; \tag{52}$$

$$\mathrm{Sub}^*(A'; \kappa) = \{\neg C, C : C \in \mathrm{Sub}^o(A'; \kappa)\}.$$

The additional part of $\mathrm{Sub}^o(A'; \kappa)$ in (52) is the same for $\kappa = \beta - 1, \beta$, because of (34). The set $\mathrm{Sub}^*(A; \beta - 1)$ corresponds to $\mathrm{Sub}^*(A)$ of Section 7.1.2.

For $\kappa = \beta - 1, \beta$, the set of maximal consistent subsets of $\mathrm{Sub}^*(A'; \kappa)$ in REL_β is denoted by

$$\mathrm{Con}^*(A'; \kappa) := \mathbb{W}(\mathrm{Sub}^*(A'; \kappa)). \tag{53}$$

It holds that Con$^*(A';\beta-1)\cap$ Con$^*(A';\beta)=\emptyset$, i.e., they are mutually exclusive. Indeed, since $\delta(w)=\kappa$ for $w\in$ Con$^*(A';\kappa)$ and $\kappa=\beta-1,\beta$, any $w\in$ Con$^*(A';\beta-1)$ differs from any $w'\in$Con$^*(A';\beta)$; we have mutual exclusiveness. In the new model M^β, we adopt the union Con$^*(A';\beta-1)\cup$ Con$^*(A';\beta)$ as the set of possible worlds.

Now, we extend the model $M^{\beta-1}=((W^{\beta-1};R_1^{\beta-1},R_2^{\beta-1}),\sigma^{\beta-1})$ for A_o by adding the remaining structure for A'. It would be convenient to write down the definition of $M^{\beta-1}=((W^{\beta-1};R_1^{\beta-1},R_2^{\beta-1}),\sigma^{\beta-1})$:

M1$^{\beta-1}$: $W^{\beta-1}=$ Con$^*(A';\beta-1)$;

M2$^{\beta-1}$: $R_i^{\beta-1}=\{(u,v)\in[W^{\beta-1}]^2: u^{-\mathbf{B}_i}\subseteq v\}$;

M3$^{\beta-1}$: for any $(w,p)\in W^{\beta-1}\times PV$, $\sigma^{\beta-1}(w,p)=\top$ if and only if $p\in w$.

We extend $M^{\beta-1}$ to a model $M^\beta=((W^\beta;R_1^\beta,R_2^\beta),\sigma^\beta)$ as follows:

M1$^\beta$: $W^\beta=W^{\beta-1}\cup$ Con$^*(A';\beta)$;

M2$^\beta$: for $i=1,2$, $R_i^\beta=\{(u,v)\in[W^{\beta-1}]^2: u^{-\mathbf{B}_i}\subseteq v]\}$
$\cup\{(u,v)\in \text{Con}^*(A';\beta)\times W^{\beta-1}: u^{-\mathbf{B}_i}\subseteq v]\}$;

M3$^\beta$: for any $(w,p)\in W^\beta\times PV$, $\sigma^\beta(w,p)=\top$ if and only if $p\in w$.

Since Con$^*(A';\beta-1)\cap$Con$^*(A';\beta)=\emptyset$, the set of possible worlds W^β is obtained by adding the new part Con$^*(A';\beta)$ to $W^{\beta-1}$. The accessibility R_i^β has the two parts:

$$\{(u,v)\in[W^{\beta-1}]^2: u^{-\mathbf{B}_i}\subseteq v\}\text{ and }\{(u,v)\in\text{Con}^*(A';\beta)\times W^{\beta-1}: u^{-\mathbf{B}_i}\subseteq v\}. \quad (54)$$

The first part of (54) keeps the accessibility $R_i^{\beta-1}$ of **M2**$^{\beta-1}$, and the second part represents the connection from Con$^*(A';\beta)$ to $W^{\beta-1}$, which implies that a formula $\psi(\xi)$, $\xi\in\Xi_{trk}$ is valuated by referring to worlds in $W^{\beta-1}$. Since $W^{\beta-1}$ and Con$^*(A';\beta)$ are mutually exclusive, so are $[W^{\beta-1}]^2=[W^{\beta-1}]\times[W^{\beta-1}]$ and Con$^*(A';\beta)\times W^{\beta-1}$. This allows us to separate applications of the two accessibilities in (54). Also, $\sigma^\beta(\cdot,\cdot)$ is an extension of $\sigma^{\beta-1}(\cdot,\cdot)$.

Lemma 8.3 (Seriality) R_i^β is serial over W^β for $i=1,2$.

Proof When $w\in W^{\beta-1}=$ Con$^*(A';\beta-1)$, Lemma 7.4 can be regarded as this assertion. Now, let $w\in$ Con$^*(A';\beta)$. Then, $w^{-\mathbf{B}_i}$ is consistent and is a subset of Sub$^*(A';\beta-1)$. Thus, there is some $v\in$ Con$^*(A';\beta-1)$ with $w^{-\mathbf{B}_i}\subseteq v$. This means that R_i^β is serial.$\blacksquare$

The new model M^β preserves the previous model $M^{\beta-1}$ with respect to $\models$.

Lemma 8.4 (Preserving the previous valuation) For any $u\in W^{\beta-1}$ and $C\in$ Sub$^*(A';\beta-1)$,

$$(M^\beta,u)\models C\text{ if and only if }(M^{\beta-1},u)\models C. \quad (55)$$

$$C\in u\text{ if and only if }(M^\beta,u)\models C. \quad (56)$$

Proof First, we see that (56) follows (55). Indeed, let $u\in W^{\beta-1}$ and $C\in$Sub$^*(A';\beta-1)$ be arbitrarily chosen. We can apply the proof of Section 7.3 to obtain that $C\in u$ if and only if $(M^{\beta-1},u)\models C$. This and (55) imply (56). We prove (55) by induction on the length of a formula in Sub$^*(A';\beta-1)$.

The induction base is the case: $C=p\in PV$. Since $u\in W^{\beta-1}$, it holds that $(M^\beta,u)\models p\Longleftrightarrow \sigma^\beta(u,p)=\top\Longleftrightarrow p\in u\Longleftrightarrow\sigma^{\beta-1}(u,p)=\top\Longleftrightarrow(M^{\beta-1},u)\models p$.

Consider a non-propositional formula in $C\in$ Sub$^*(A';\beta-1)$. The induction hypothesis is that for any $u\in W^{\beta-1}$, (55) holds for any immediate subformula C' of C. We have the five cases according to the outmost connectives of C; $\neg$, $\supset$, $\wedge$, $\mathbf{B}_i(\cdot)$, and $\mathbf{Pc}_i(\cdot,\cdot)$ $(i = 1, 2)$. We consider the cases $\supset$, $\mathbf{B}_i(\cdot)$, and $\mathbf{Pc}_i(\cdot,\cdot)$.

Let $C = C' \supset C''$. Then, $(M^\beta, u) \models C' \supset C''$ if and only if $(M^\beta, u) \not\models C'$ or $(M^\beta, u) \models C''$ if and only if $(M^{\beta-1}, u) \not\models C'$ or $(M^{\beta-1}, u) \models C''$ if and only if $(M^{\beta-1}, u) \models C' \supset C''$, where the second equivalence is by the induction hypothesis.

Now, let $\mathbf{B}_i(C') \in \mathrm{Sub}^*(A'; \beta - 1)$. Then, $(M^\beta, u) \models \mathbf{B}_i(C')$ if and only if $(M^\beta, v) \models C'$ for all v with $uR^\beta v$. Since $u \in W^{\beta-1}$, we have, by M1$^\beta$ and M2$^\beta$, $uR^\beta v \iff [(u, v) \in [w^{\beta-1}]^2$ & $u^{-B_i} \subseteq v] \iff uR^{\beta-1}v$, we can apply the induction hypothesis to the last, it is equivalent to $(M^{\beta-1}, v) \models C'$ for all v with $uR^{\beta-1}v$, and further equivalent to $(M^{\beta-1}, u) \models \mathbf{B}_i(C')$.

Let $\mathbf{Pc}_i(C) \in \mathrm{Sub}^*(A'; \beta-1)$. By V5, $(M^\beta, u) \models \mathbf{Pc}_i(C)$ is equivalent to that $(M^\beta, w_{\nu+1}) \models C_{i_\nu}$ for any al-chain $\langle (w_0, i_0), ..., (w_\nu, i_\nu), w_{\nu+1} \rangle$ with $(w_0, i_0) = (w, i)$. Since $u \in W^{\beta-1}$, all accessibilities in this al-chain are in $(W^{\beta-1}, R_1^{\beta-1}, R_2^{\beta-1})$ by M2$^\beta$. Hence, the above is equivalent to $(M^{\beta-1}, u) \models \mathbf{Pc}_i(C)$.■

Finally, we show that the assertion of Lemma 8.4 is extended to $\mathrm{Con}^*(A'; \beta)$ and M^β with the subformula tree $(\Xi_{trk}, \succ)$ and the function ψ.

Lemma 8.5 For any $\xi = [\xi_0, ..., \xi_t] \in \Xi_{trk}$ and $w \in \mathrm{Con}^*(A'; \beta)$,

$$\psi(\xi) \in w \text{ if and only if } (M^\beta, w) \models \psi(\xi). \tag{57}$$

Proof. We prove the assertion by induction over $(\Xi_{trk}, \succ)$ from its maximal trunk paths. The inductive base is essential. Let $\xi = [\xi_0, ..., \xi_m]$ be a maximal path in Ξ_{trk}. By (IB), $\psi(\xi) = p$ or $\psi(\xi) = \mathbf{B}_i(C)$.

Let $\xi = [\xi_0, ..., \xi_m] \in \Xi_{trk}$ with $\xi_m = p \in PV$. Then, $\psi(\xi) = p$ by (IB). Let $w \in \mathrm{Con}^*(A'; \beta)$. Then, by M3$^\beta$, $p \in w$ if and only if $\sigma^\beta(w, p) = \top$. By V0, this is equivalent to $(M^\beta, w) \models p$.

Next, let ξ_m be a b-formula $\mathbf{B}_i(C)$. Then, $\psi(\xi) = \mathbf{B}_i(C)$ and C belongs to $\mathrm{Sub}^*(A'; \beta - 1)$. If ξ_m is an pc-formula $\mathbf{Pc}_i(\boldsymbol{C})$, then $\psi(\xi) = \mathbf{B}_i(C_i \wedge \mathbf{Pc}_j(\boldsymbol{C}))$ and $C = C_i \wedge \mathbf{Pc}_j(\boldsymbol{C})$ belongs to $\mathrm{Sub}^*(A'; \beta - 1)$. Thus, in either case, C belongs to $\mathrm{Sub}^*(A'; \beta - 1)$. We prove (57) for $\psi(\xi) = \mathbf{B}_i(C)$.

Suppose $\mathbf{B}_i(C) \in w \in \mathrm{Con}^*(A'; \beta)$. Take any $u \in W^\beta$ with $wR_i^\beta u$, i.e.,$w^{-\mathbf{B}_i} \subseteq u$. By M2$^\beta$, $u \in W^{\beta-1}$. Since $C \in w^{-\mathbf{B}_i} \subseteq u$, we have $C \in u$. By (56) of Lemma 8.4, we have $(M^\beta, u) \models C$. Since u is arbitrary, we can write $(M^\beta, u) \models C$ for all u with $wR_i^\beta u$. That is, $(M^\beta, w) \models \mathbf{B}_i(C)$.

Conversely, suppose $(M^\beta, w) \models \mathbf{B}_i(C)$. We claim that $w^{-\mathbf{B}_i} \cup \{\neg C\}$ is inconsistent. Suppose, on the contrary, that $w^{-\mathbf{B}_i} \cup \{\neg C\}$ is consistent. Since $w^{-\mathbf{B}_i} \cup \{\neg C\}$ is a subset of $\mathrm{Sub}^*(A; \beta-1)$, using Lemma 7.1.(0), there exists some $u \in \mathrm{Con}^*(A'; \beta - 1)$ such that $w^{-\mathbf{B}_i} \cup \{\neg C\} \subseteq u$, which implies $w^{-\mathbf{B}_i} \subseteq u$ and $\{\neg C\} \subseteq u$. Hence, $C \notin u$. Since $\delta(C) \leq \beta - 1$ and $u \in \mathrm{Con}^*(A'; \beta - 1)$, it holds that $(M^\beta, u) \not\models C$. Since $w^{-\mathbf{B}_i} \subseteq u$, we have $wR_i^\beta u$ by M2$^\beta$. Hence, $(M^\beta, w) \not\models \mathbf{B}_i(C)$, a contradiction to the starting supposition. Thus, $w^{-\mathbf{B}_i} \cup \{\neg C\}$ is inconsistent; so $\vdash_\beta \wedge w^{-\mathbf{B}_i} \supset C$. This implies $\vdash_\beta \wedge w \supset \mathbf{B}_i(C)$. Thus, $\mathbf{B}_i(C) \in w$, since w is a maximal consistent subset of $\mathrm{Sub}^*(A'; \beta)$.

Thus, we have shown the inductive base. The next step is to go along $\succ$. We should consider only the three cases where the outmost connective of ξ_m is $\neg$, $\supset$, and $\wedge$. Here, we consider the case $\neg$, i.e., $\xi_k = \neg C$ for some C. The other two cases are similar. By $(\neg)$, $\psi(\xi) = \neg\psi(\xi \cdot C)$. The inductive hypothesis is (57) for $\psi(\xi \cdot C)$. First, suppose $\psi(\xi) \in w \in \mathrm{Con}^*(A'; \beta)$. That is,

$\neg\psi(\xi \cdot C) \in w$. Hence, $\psi(\xi \cdot C) \notin w$. By the inductive hypothesis, we have $(M^\beta, w) \nvDash \psi(\xi \cdot C)$. This implies $(M^\beta, w) \models \neg\psi(\xi \cdot C)$. The converse is obtained by tracing this argument back.■

Since $\nvdash_\beta A$ by the starting assumption (49), it holds that $\neg A$ is consistent. Recall the root of Ξ_{trk} is $\xi = [A]$. Since $\psi([A]) = A'$, by Lemma 8.2, $\vdash_\beta A' \equiv A$. Hence, $\neg A'$ is consistent. This implies that there is a maximally consistent w in $\mathrm{Con}^*(A'; \beta)$ such that $\neg A' \in w$. Thus, $A' \notin w$. By Lemma 8.5, we have $(M^\beta, w) \nvDash A'$. Thus, M^β is a countermodel for A'. By Lemma 8.2, M^β is a countermodel for A, too.

9 Discussions from the Social Scientific Perspective

In the development of logic REL_β, we have made quite a few deviations from the traditional thoughts in epistemic logic. They are necessitated because we target social situations where bounded interpersonal reasoning is prominent. We give some discussions on these deviations from the social scientific perspective, which will be given in Section 9.1. Then, we mention possible extensions of our approach in Section 9.2.

9.1 Methodological reflections

First, we emphasize the *principle of methodological individualism* (MD) in the ontological sense (von Mises [27], Chap.II, Section 4, and Kaneko [18], Chap.6) that an individual person is the unit of mental/physical action taking in society.[15] This may be regarded as a methodological principle in epistemic logic, but the formulation of common belief CBL_β is incompatible with MD, i.e., Axiom ACB and Rule ICB are not individualistic as pointed out in Section 6. Axiom ARE and Rule IRE for REL_β are purely individualistic in the ontological sense.

From the social scientific perspective, MD has a more serious implications. Since social science targets social situations with human beings, we should take boundedness of each individual's abilities of memory/thought seriously. This aspect is captured by a bound β for interpersonal (/intrapersonal) reasoning for a person. From the viewpoint of bounded rationality, an individual person is limited in his cognitive/epistemic activity in various ways, cf., Simon [34]. Theorem 3.1 for logic REL_β captures one aspect of individual limitations in interpersonal interactions. In this context, how reciprocal empathization is developed is not straightforward. For this, we go to the notion of "*the generalized other*" due to Mead [25] (see also Collins [10], Chap.7). It is developed in one's societal background, which Mead calls the *genesis*.

By the genesis, he means that an individual person has gradually developed his intelligence and his sense of self from his baby age to adulthood age in society such as home, neighborhood, schools, and so on, emphasizing interactions with other people. After these, person i can practice reciprocal empathization with another person, j who shares with the same community background with i. The pc-operators $\mathbf{Pc}_i(\cdot,\cdot), i = 1, 2$ together with Axiom AER and Rule IER can be interpreted as expressing Mead's concept of the generalized other. As we argued in Section 5, the generalized other depends highly upon issues as well as the communities where they come from.

In epistemic logic, self-introspection is typically assumed and is regarded as a basic axiom.

[15] This differs from the reductionistic MD in that any cause for a social event can be reduced to some inner elements of an individual person, without considering the level of a social structure.

However, Mead [25], Supplementary Essay III, pp.354-378 argues that intrapersonal thinking is rooted in interpersonal interactions. Consistent with his argument, we take the perspective that interpersonal empathization is more basic and one's introspection is a product from interpersonal beliefs by projecting them on himself. This has the implication that logic REL_β^S given in Section 6.1 is a derivative of logic REL_β.

9.2 Further extensions

***(a)*: Possible extensions of logic $\mathbf{REL}_\beta$**: Although we have focused on logic REL_β with two persons, it would be important to extend it to the n-person case with $n < \omega$. A basic extension of REL_β is logic $\mathrm{REL}_\beta(n)$, where the number of persons is n but reciprocal empathization is kept in a bilateral manner as in REL_β. By the nature of reciprocal empathy, the bilateral assumption is crucial. Still, we need to be careful with some notations; the pc-operator for i against j needs the subscript such as $\mathbf{Pc}_{i,j}(C_i, C_j)$, Axiom AEM as $\mathrm{AEM}_{i,j}$, and Rule IEM as $\mathrm{IEM}_{i,j}$. We need to think about whether or not interactions between different pairs of persons should be taken into account seriously. In such a case, the bound β may need to be redefined.

Mathematically speaking, there are many possible extensions where reciprocal empathization takes more complex forms than the bilateral reciprocal one. Nevertheless, we should work on an extension, keeping the conceptual bases discussed in Section 9.1.

***(b)*: Extension of Lewis's example**: Lewis's example in Section 5 demonstrated several aspects crucial to our understanding of convention, empathization, and communication in social situations. In general, the situation may involve more people, and in addition to (a), an extension of Lewis's example has its own specific problems. For example, one can study the situation where people from various different communities who may or may not understand each other depending upon issues. Then, we can discuss how people could communicate to have successful coordination. These are indicative to have a new development of the concept of "convention" due to Lewis [23] depending upon communities. These may include the present world issue "selective empathy", meaning that empathization is applied to some group of people but not to some others (cf., Wang, *et al.* [36]).

***(c)*: Applications to game theory**: (a) and (b) indicate new potential directions for game theory, when we take reciprocal empathy seriously. The central theme in game theory is to study strategic behavior, and the standard solution concept is Nash equilibrium. The common practice is to consider Nash equilibrium in the following manner: each player maximizes his own payoff, taking his anticipation of the other's equilibrium behavior as given. However, the process in which this anticipation is formed is rarely discussed in the literature. Such anticipation formation is essentially a problem of empathization, in which each player attempts to simulate the other's thinking and may reach reciprocity as well. Our framework can be used to study this problem directly and explore its various implications to strategic reasoning.

In this regard, our framework also uncovers a connection between this anticipation formation and Mead's [25] concept of "generalized others", as our analysis of the Lewis example in Section 5 shows. Thus, our framework may allow for a more serious introduction of social backgrounds and contexts into the current game theory (in which the game is analyzed as an isolated phenomenon). A related problem has been treated by Kaneko-Matsui [19] in the context of discrimination/prejudice, in which past experiences play a crucial role in players' expectations of others' behavior, but the epistemic elements are not discussed there. This may lead to another direction.

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