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The Gálvez-Carrillo-Kock-Tonks conjecture for locally discrete decomposition spaces. (English) Zbl 07838052
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I. Gálvez-Carrillo et al. [Adv. Math. 331, 952–1015 (2018; Zbl 1403.00023); Adv. Math. 333, 1242–1292 (2018; Zbl 1403.18016); Adv. Math. 371, Article ID 107267, 5 p. (2020; Zbl 1471.18026); Adv. Math. 334, 544–584 (2018; Zbl 1403.18017)] have discovered that the incidence coalgebra construction and Mõbius inversion make sense for objects more general than Mõbius categories, which they call *decompostion spaces*, and which are the same thing as the 2-Segal spaces [*T. Dyckerhoff* and *M. Kapranov*, Higher Segal spaces. Cham: Springer (2019; Zbl 1459.18001)]. It seems likely that all combinatorial coalgebras, bialgebras and Hopf algebras arise from the incidence coalgebra construction of decomposition spaces, which has been shown for most of Schmitt's examples [*W. R. Schmitt*, Can. J. Math. 45, No. 2, 412–428 (1993; Zbl 0781.16026)], restriction species [*I. Gálvez-Carrillo* et al., Int. Math. Res. Not. 2020, No. 21, 7558–7616 (2020; Zbl 1467.16035)] and hereditary species [*L. Carlier*, Int. Math. Res. Not. 2022, No. 8, 5745–5780 (2022; Zbl 1489.18010)]. Gávez-Carrillo et al. work in the fully homotopical setting of simplicial ∞-groupoids, but already the discrete case of the notion is very rich, as exemplified in [*J. E. Bergner* et al., Topology Appl. 235, 445–484 (2018; Zbl 1422.55036); *J. Kock* and *D. I. Spivak*, Proc. Am. Math. Soc. 148, No. 6, 2317–2329 (2020; Zbl 1442.18008)], where the notion is related to constructions in algebraic topology and category theory.

I. Gálvez-Carrillo et al. [Adv. Math. 334, 544–584 (2018; Zbl 1403.18017)] showed that the Lawvere-Menni Hopf algebra [Zbl 1236.18001] is the incidence coalgebra of a decomposition space U. With this discovery the universal property could be stated, showing its nature as a moduli space.

Conjecture. For any decomposition space X the mapping space $\mathrm{map}\,(X, U)$ is contractible.

This paper establishes the first case of the conjecture. The main result, which is the first substantial evidence for the full conjecture, goes as follows.

Theorem 6.6. $\mathrm{map}\,(X, U)$ is a contractible 1-groupoid for every 1-truncated locally discrete decomposition space X.

The level of generality already covers all the classical theory of incidence algebras and Mõbius inversion in combinatorics since locally finite posets, Cartier-Foata monoids, Mõbius categories and Schmitt's examples are all 0-truncated simplicial spaces.

The theorem, namely the contractibility of the 1-groupoids $\mathrm{map}\,(X, U)$, is a 2-categorical statement. The proof given here is based on 2-category theory. The strategy is to build a local strict model, a kind of neighborhood $U_X \subset U$ around the intervals of a given locally discrete decomposition space X. The bulk of the paper is concerned with setting up this local model, showing that it is strict.

The synopsis of the paper goes as follows.

- §1 reviews basic notions and some results on homotopy pullbacks of groupoids.
- §2 introduces some necessary material relating to the notion of slice and coslice of decomposition groupoids, giving the definitiion of interval (Definition 2.12).
- §3 identifies the level of generality, working with strict simplicial groupoids such that all active-inert squares are strict pullbacks and such that d_1 is a discrete isofibration.
- §4 constructs the stretched-culf factorization system in the category of discrete algebraic intervals, coming to the Gávez-Carrillo-Kock-Tonks conjecture.
- §5 defines the decomposition groupoid of all discrete algebraic intervals U [*I. Gálvez-Carrillo* et al., Adv. Math. 334, 544–584 (2018; Zbl 1403.18017)].
- §6 establishes the main theorem (Theorem 6.6).

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MSC:

18N10 2-categories, bicategories, double categories
18N50 Simplicial sets, simplicial objects
06A11 Algebraic aspects of posets
16T15 Coalgebras and comodules; corings

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