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Adjoint functors on the representation category of $\mathcal{OI}$. (English) Zbl 07768216

Algebr. Represent. Theory 26, No. 5, 2015-2038 (2023)

The shift functor $\boldsymbol{S}$, the derivative functor $\boldsymbol{D}$, and the negative shift functor $\boldsymbol{S}^{-1}$ introduced in [*T. Church* et al., Geom. Topol. 18, No. 5, 2951–2984 (2014; Zbl 1344.20016)] as well as the coinduction functor $\boldsymbol{Q}$ introduced in [*W. L. Gan* and *L. Li*, J. Lond. Math. Soc., II. Ser. 92, No. 3, 689–711 (2015; Zbl 1358.18001)] are examples of adjoint functors in the representation theory of the category $\mathcal{FJ}$ of finite sets and injections. It was established in [*W. L. Gan*, J. Pure Appl. Algebra 221, No. 5, 1242–1248 (2017; Zbl 1359.18001)] that $\boldsymbol{S}^{-1}$ is simultaneously the left adjoint of $\boldsymbol{S}$ and the right adjoint of $\boldsymbol{D}$, while it was proved in [*W. L. Gan* and *L. Li*, J. Lond. Math. Soc., II. Ser. 92, No. 3, 689–711 (2015; Zbl 1358.18001)] that $\boldsymbol{Q}$ is the right adjoint of $\boldsymbol{S}$. Analogously, these functors have been constructed for other concrete combinatorial categories occurring in representation stability theory, such as the category $\mathcal{OJ}$ of finite totally ordered sets and order-preserving injections [*W. L. Gan* and *L. Li*, J. Algebra 568, 547–575 (2021; Zbl 1457.18012); *A. Snowden* and *S. Güntürkün*, The representation theory of the increasing monoid. Providence, RI: American Mathematical Society (AMS) (2023; Zbl 1515.20017)] as well as the category $\mathcal{VJ}_q$ of finite dimensional vector spaces over a finite field $\mathbb{F}_q$ and linear injections [*R. Nagpal*, Algebra Number Theory 13, No. 9, 2151–2189 (2019; Zbl 1461.20004); *R. Nagpal*, J. Reine Angew. Math. 781, 187–205 (2021; Zbl 1482.20008)]. *W. L. Gan* et al. [Indiana Univ. Math. J. 69, No. 7, 2325–2338 (2020; Zbl 1461.18006)] introduced the Nakayama functor and its inverse as its right adjoint for representations of $\mathcal{FJ}$ and $\mathcal{VJ}_q$. Furthermore, representation theory of abstract combinatorial categories rigged out in functors sharing similar properties have been investigated [*W. L. Gan* and *L. Li*, Trans. Am. Math. Soc. 371, No. 12, 8513–8534 (2019; Zbl 1468.16029); *W. L. Gan* and *L. Li*, J. Pure Appl. Algebra 223, No. 1, 188–217 (2019; Zbl 1443.13014)].

This paper aims to give a systematic construction of these adjoint functors for the category $\mathcal{OJ}$, describe their adjunction relations, and pursue possible applications in representation theory of $\mathcal{OJ}$, noting that the combinatorial structure of $\mathcal{OJ}$ gives two functorial operations on the morphism set. It is also established that the Nakayama functor induces an equivalence between the Serre quotient of the category of finitely generated modules by the category of finitely generated torsion modules and the category of finite-dimensional modules.

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MSC:

18A22 Special properties of functors (faithful, full, etc.)
16E30 Homological functors on modules (Tor, Ext, etc.) in associative algebras

Keywords:

OI-modules; Nakayama functor; Serre quotient; adjoint functors

Full Text: DOI arXiv

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