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One-dimensional topological theories with defects: the linear case. (English) [Zbl 07807660](#)

Im, Mee Seong (ed.) et al., Algebraic and topological aspects of representation theory. Virtual AMS special session on geometric and algebraic aspects of quantum groups and related topics, virtual, November 20–21, 2021. Providence, RI: American Mathematical Society (AMS). *Contemp. Math.* 791, 105–146 (2024)

Universal construction [*C. Blanchet* et al., *Topology* 34, No. 4, 883–927 (1995; [Zbl 0887.57009](#)); *M. Khovanov*, “Decorated one-dimensional cobordisms and tensor envelopes of noncommutative recognizable power series”, Preprint, [arXiv:2010.05730](#)] starts with an evaluation function for closed n -manifolds to produce state space for closed $(n - 1)$ -manifolds and maps between these spaces associated to n -cobordisms, which results in a functor from the category of n -dimensional cobordisms to the category of vector spaces usually failing to be a TQFT with the tensor product of states for two $(n - 1)$ -manifolds N_1, N_2 properly embedded into the state space for their union

$$A(N_1) \otimes A(N_2) \hookrightarrow A(N_1 \sqcup N_2)$$

The universal construction turns out to be riveting already in low-dimensions, including in dimensions two [*M. Khovanov*, “Universal construction of topological theories in two dimensions”, Preprint, [arXiv:2007.03361](#); *M. Khovanov* et al., *Sel. Math., New Ser.* 28, No. 4, Paper No. 71, 68 p. (2022; [Zbl 1496.18018](#)); *M. Khovanov* et al., *Commun. Math. Phys.* 385, No. 3, 1835–1870 (2021; [Zbl 1490.57039](#)); *M. Khovanov* and *R. Sazdanovic*, *J. Pure Appl. Algebra* 225, No. 6, Article ID 106592, 24 p. (2021; [Zbl 1480.16037](#))] and one [*P. Gustafson* et al., *Lett. Math. Phys.* 113, No. 5, Paper No. 93, 38 p. (2023; [Zbl 07743387](#)); *M. S. Im* and *M. Khovanov*, “Topological theories and automata”, Preprint, [arXiv:2202.13398](#); *M. S. Im* et al., “Universal construction in monoidal and non-monoidal settings, the Brauer envelope, and pseudocharacters”, Preprint, [arXiv:2303.02696](#); *M. S. Im* and *P. Zimmer*, *Involve* 15, No. 2, 319–331 (2022; [Zbl 1499.18039](#))]. In the latter case, one needs to add zero-dimensional defects with labels in a set Σ . An oriented interval with a collection of Σ -labelled defects encodes a word ω , that is an element of the free monoid Σ^* on the set Σ . An oriented circle with labels in Σ encodes a word up to cyclic eigenvalues. Given an evaluation of each word and a separate evaluation of words up to cyclic equivalence, there is an associated rigid linear monoidal category [*M. Khovanov*, “Decorated one-dimensional cobordisms and tensor envelopes of noncommutative recognizable power series”, Preprint, [arXiv:2010.05730](#)].

This paper studies this category for a rational evaluation α . The Karoubi closure of the resulting category can be reduced to the Karoubi closure of a category built from a symmetric Frobenius algebra \mathcal{K} that can be extracted from α (§2.4). §§2.1–2.3 deal with the setup, basic theory and various examples. §3 reviews thin flat surface 2D TQFTs associated to symmetric Frobenius algebras, explaining how to enhance these TQFTs by 0-dimensional defects floating along the boundary that carry elements of the algebra. Comparisons between one-dimensional theories with defects and two-dimensional theories without defects are addressed throughout the paper. The Boolean analogues of these categories and their relation to automata and regular languages were investigated in [*M. S. Im* and *M. Khovanov*, “Topological theories and automata”, Preprint, [arXiv:2202.13398](#)].

For the entire collection see [[Zbl 07807550](#)].

Reviewer: [Hirokazu Nishimura \(Tsukuba\)](#)

MSC:

- [18M05](#) Monoidal categories, symmetric monoidal categories
- [18M30](#) String diagrams and graphical calculi
- [57K16](#) Finite-type and quantum invariants, topological quantum field theories (TQFT)
- [16W60](#) Valuations, completions, formal power series and related constructions (associative rings and algebras)
- [15A63](#) Quadratic and bilinear forms, inner products

Keywords:

rational noncommutative power series; topological theory; universal construction; topological quantum field theory (TQFT); defects in TQFT; Brauer categories; symmetric Frobenius algebras

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References:

- [1] Blanchet, C., Topological quantum field theories derived from the Kauffman bracket, *Topology*, 883-927 (1995) · [Zbl 0887.57009](#) · [doi:10.1016/0040-9383\(94\)00051-4](#)
- [2] Bar-Natan, Dror, Khovanov's homology for tangles and cobordisms, *Geom. Topol.*, 1443-1499 (2005) · [Zbl 1084.57011](#) · [doi:10.2140/gt.2005.9.1443](#)
- [3] Berstel, Jean, Zeta functions of formal languages, *Trans. Amer. Math. Soc.*, 533-546 (1990) · [Zbl 0797.68092](#) · [doi:10.2307/2001573](#)
- [4] Brundan, Jonathan, Foundations of Frobenius Heisenberg categories, *J. Algebra*, 115-185 (2021) · [Zbl 1471.18019](#) · [doi:10.1016/j.jalgebra.2021.02.020](#)
- [5] Brundan, Jonathan, Quantum Frobenius Heisenberg categorification, *J. Pure Appl. Algebra*, Paper No. 106792, 50 pp. pp. (2022) · [Zbl 1490.18020](#) · [doi:10.1016/j.jpaa.2021.106792](#)
- [6] Caprau, Carmen, Twin TQFTs with Frobenius algebras, *J. Math.*, Art. ID 407068, 25 pp. pp. (2013) · [Zbl 1486.81146](#) · [doi:10.1155/2013/407068](#)
- [7] Paul Gustafson, Mee-Seong Im, Remy Kaldawy, Mikhail Khovanov, and Zachary Lihn, Automata and one-dimensional TQFTs with defects, *arXiv preprint <https://arxiv.org/abs/2301.00700>* (2023), 1-36. · [Zbl 07743387](#)
- [8] Mee-Seong Im and Mikhail Khovanov, Topological theories and automata, *arXiv preprint <https://arxiv.org/abs/2202.13398>* (2022), 1-70.
- [9] Mee-Seong Im, Mikhail Khovanov, and Victor Ostrik, Universal construction in monoidal and non-monoidal settings, the Brauer envelope, and pseudocharacters, *arXiv preprint <https://arxiv.org/abs/2303.02696>* (2023), 1-59.
- [10] Im, Mee Seong, One-dimensional topological theories with defects and linear generating functions, *Involve*, 319-331 (2022) · [Zbl 1499.18039](#) · [doi:10.2140/involve.2022.15.319](#)
- [11] Khovanov, Mikhail, A functor-valued invariant of tangles, *Algebr. Geom. Topol.*, 665-741 (2002) · [Zbl 1002.57006](#) · [doi:10.2140/agt.2002.2.665](#)
- [12] Khovanov, Mikhail, $sl(3)$ link homology, *Algebr. Geom. Topol.*, 1045-1081 (2004) · [Zbl 1159.57300](#) · [doi:10.2140/agt.2004.4.1045](#)
- [13] Khovanov, Mikhail, An invariant of tangle cobordisms, *Trans. Amer. Math. Soc.*, 315-327 (2006) · [Zbl 1084.57021](#) · [doi:10.1090/S0002-9947-05-03665-2](#)
- [14] Khovanov, Mikhail, Heisenberg algebra and a graphical calculus, *Fund. Math.*, 169-210 (2014) · [Zbl 1304.18019](#) · [doi:10.4064/fm225-1-8](#)
- [15] \bysame, Decorated one-dimensional cobordisms and tensor envelopes of noncommutative recognizable power series, *arXiv preprint <https://arxiv.org/abs/2010.05730>* (2020), 1-33.
- [16] \bysame, Universal construction of topological theories in two dimensions, *arXiv preprint <https://arxiv.org/abs/2007.03361>* (2020), 1-56.
- [17] Mikhail Khovanov and Nitu Kitchloo, A deformation of Robert-Wagner foam evaluation and link homology, *arXiv preprint <https://arxiv.org/abs/2004.14197>* (2020), 1-58.
- [18] Khovanov, Mikhail, Two-dimensional topological theories, rational functions and their tensor envelopes, *Selecta Math. (N.S.)*, Paper No. 71, 68 pp. pp. (2022) · [Zbl 1496.18018](#) · [doi:10.1007/s00029-022-00785-z](#)
- [19] Mikhail Khovanov and You Qi, Introduction to categorification: Section 2, Topological Quantum Field Theories, Lecture notes: https://www.math.columbia.edu/~khovanov/cat2020/notes/2_TQFT.pdf and n-d TQFT (2020), 1-22.
- [20] Khovanov, Mikhail, Evaluating thin flat surfaces, *Comm. Math. Phys.*, 1835-1870 (2021) · [Zbl 1490.57039](#) · [doi:10.1007/s00220-021-04011-3](#)
- [21] Mikhail Khovanov and Radmila Sazdanovic, Bilinear pairings on two-dimensional cobordisms and generalizations of the Deligne category, *arXiv preprint <https://arxiv.org/abs/2007.11640>* (2020), 1-15.
- [22] \bysame, Diagrammatic categorification of the Chebyshev polynomials of the second kind, *J. Pure Appl. Algebra* 225 (2021), no. 6, Paper No. 106592, 23 pp., <https://doi.org/10.1016/j.jpaa.2020.106592> <https://doi.org/10.1016/j.jpaa.2020.106592> · [Zbl 1480.16037](#)
- [23] Aaron Lauda, Frobenius algebras and planar open string topological field theories, *arXiv preprint <https://arxiv.org/abs/math/0508349>* (2005), 1-66.
- [24] \bysame, Open-Closed Topological Quantum Field Theory and Tangle Homology, Ph.D. thesis, University of Cambridge, 2006, pp. 1-169.
- [25] Lazaroiu, C. I., On the structure of open-closed topological field theory in two dimensions, *Nuclear Phys. B*, 497-530 (2001) · [Zbl 0983.81090](#) · [doi:10.1016/S0550-3213\(01\)00135-3](#)
- [26] Lauda, Aaron D., State sum construction of two-dimensional open-closed topological quantum field theories, *J. Knot Theory Ramifications*, 1121-1163 (2007) · [Zbl 1148.57039](#) · [doi:10.1142/S0218216507005725](#)
- [27] Lauda, Aaron D., Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras, *Topology Appl.*, 623-666 (2008) · [Zbl 1158.57038](#) · [doi:10.1016/j.topol.2007.11.005](#)

- [28] Lauda, Aaron D., Open-closed TQFTS extend Khovanov homology from links to tangles, *J. Knot Theory Ramifications*, 87-150 (2009) · [Zbl 1161.57018](#) · [doi:10.1142/S0218216509006793](#)
- [29] Licata, Anthony, Hecke algebras, finite general linear groups, and Heisenberg categorification, *Quantum Topol.*, 125-185 (2013) · [Zbl 1279.20006](#) · [doi:10.4171/QT/37](#)
- [30] Ehud Meir, Interpolations of monoidal categories and algebraic structures by invariant theory, arXiv preprint <https://arxiv.org/abs/2105.04622>arXiv:2105.04622 (2021), 1-31. · [Zbl 1509.16040](#)
- [31] Gregory~W. Moore and Graeme Segal, D-branes and K-theory in 2D topological field theory, arXiv preprint <https://arxiv.org/abs/hep-th/0609042>arXiv:hep-th/0609042 (2006), 1-88.
- [32] McSween, Alexandra, Affine oriented Frobenius Brauer categories, *Comm. Algebra*, 742-756 (2023) · [Zbl 1506.18020](#) · [doi:10.1080/00927872.2022.2113401](#)
- [33] Mackaay, Marco, The universal \mathfrak{sl}_3 -link homology, *Algebr. Geom. Topol.*, 1135-1169 (2007) · [Zbl 1170.57011](#) · [doi:10.2140/agt.2007.7.1135](#)
- [34] Retakh, Vladimir, Differential topology, infinite-dimensional Lie algebras, and applications. Noncommutative rational functions and Farber's invariants of boundary links, *Amer. Math. Soc. Transl. Ser. 2*, 237-246 (1999), Amer. Math. Soc., Providence, RI · [Zbl 0971.68103](#) · [doi:10.1090/trans2/194/11](#)
- [35] Rosso, Daniele, A general approach to Heisenberg categorification via wreath product algebras, *Math. Z.*, 603-655 (2017) · [Zbl 1366.18006](#) · [doi:10.1007/s00209-016-1776-9](#)
- [36] Robert, Louis-Hadrien, A closed formula for the evaluation of foams, *Quantum Topol.*, 411-487 (2020) · [Zbl 1476.57054](#) · [doi:10.4171/qt/139](#)
- [37] Savage, Alistair, Frobenius Heisenberg categorification, *Algebr. Comb.*, 937-967 (2019) · [Zbl 1459.18008](#) · [doi:10.5802/alco.73](#)
- [38] \bysame , Affine oriented Frobenius Brauer categories, <https://alistairsavage.ca/talks/2021-Savage-AOFBC.pdf>Presentation Slides (2021), 1-33.
- [39] Saima Samchuck-Schnarch, Frobenius Brauer Categories, <http://dx.doi.org/10.20381/ruor-28137>uOttawa Research (2022), 1-107, PhD Thesis, University of Ottawa.

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