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The reductive Borel-Serre compactification as a model for unstable algebraic K-theory.
(English) Zbl 07782626
Sel. Math., New Ser. 30, No. 1, Paper No. 10, 93 p. (2024)

Let A be an associative ring and M a finitely generated projective A-module. This paper introduces a category $\mathrm{RBS}(M)$, proving several theorems which show that its geometric realization functions as well-behaved unstable algebraic K-theory space. Unstable algebraic K-theory can naturally be viewed not as a bare anima or homotopy type, but rather as a stratified homotopy type, with structure very much akin to those stratified homotopy type arising from familiar compactifications of locally symmetric spaces [*S. Zucker*, Invent. Math. 70, 169–218 (1982; Zbl 0508.20020); *M. Ørsnes Jansen*, Int. Math. Res. Not. 2023, No. 19, 16394–16452 (2023; Zbl 07794926); *A. Borel* and *J.-P. Serre*, Comment. Math. Helv. 48, 436–491 (1973; Zbl 0274.22011)].

The principal results of the paper are the following four theorems.

Theorem. Let A be a ring and M a split noetherian finitely generated projective A-module. The map

$$GL(M) = \pi_1 BGL(M) \to \pi_1 |\mathrm{RBS}(M)|$$

is surjective with kernel $E(M)$, so we have

$$\pi_1 |\mathrm{RBS}(M)| = GL(M)/E(M)$$

Theorem. Let A be a ring with many units in the sense of [Zbl 0684.18001] and let M a split noetherian finitely generated projective A-module. Then the comparison map

$$c : BGL(M) \to |\mathrm{RBS}(M)|$$

is a $\mathbb{Z}$-homology isomorphism.

Suppose furthermore that every summand of M is free. Then c is an isomorphism on homology with all local coefficient systems.

Theorem. Let k be a finite field of characteristic p and V a finite-dimensional k-vector space. Then we have

(1) $|\mathrm{RBS}(V)|$ is a simple space;

(2) The map
$$|\mathrm{RBS}(V)| \to *$$
is an $\mathbb{F}_p$-homology isomorphism;

(3) The map
$$BGL(V) \to |\mathrm{RBS}(V)|$$
is a $\mathbb{Z}[1/p]$-homology isomorphism.

Theorem. Let A be a ring. Let $\mathcal{M}$ denote a set of representatives for the isomorphism classes of finitely generated projective A-modules. Then there is a natural structure of a monoidal category on $\bigsqcup_{M\in\mathcal{M}} \mathrm{RBS}(M)$ and an identification

$$K(A) \simeq \left| \bigsqcup_{M\in\mathcal{M}} \mathrm{RBS}(M) \right|^{gp}$$

of $K(A)$ with the group completion of the realization of this monoidal category.

There is the plus construction definition in unstable algebraic K-theory. If $n \geq 3$, the subgroup $E_n(A) \subset GL_n(A)$ generated by elementary matrices is perfect [*C. A. Weibel*, The K-book. An introduction to algebraic K-theory. Providence, RI: American Mathematical Society (AMS) (2013; Zbl 1273.19001), Lemma

1.3.2], so one can form the plus construction on $BGL_n(A)$ which kills the normal subgroup generated by $E_n(A)$. By the second theorem above, this agrees with $|\mathrm{RBS}(A^n)|$ provided that A is commutative and local with infinite residue field. On the other hand, the third theorem above shows that for finite fields, the two definitions differ, and the one in this paper yields an unstable algebraic K-theory which is much simpler and closer in nature to the stable K-theory.

Allen Yuan's partial K-theory [*A. Yuan*, J. Am. Math. Soc. 36, No. 1, 107–175 (2023; Zbl 1506.55009)] is very similar to the proposed model in this paper. Partial K-theory is defined essentially so as to make the analogue of the fourth theorem above a tautology, whereas the proof of the theorem in this paper takes many pages of simplicial manipulations. That is to say, Yuan takes Waldhausen's S-dot construction [Zbl 1506.55009], and instead of freely making a group-like E_1-anima out of it, which produces $K(A)$, he freely makes an E_1-anima without group-like construction, and this is the definition of $K^\partial(A)$. Clearly, partial K-theory should be similar to the E_1-anima $|\coprod \mathrm{RBS}(M)|$, because the S-dot construction exactly encodes fibrations and their associated gradeds with all compatibilities, and this is the essence of the author's RBS categories as well. However, when Yuan unravels K^∂ into something concrete, it ends up being slightly more combinatorially intricate, in that the basic objects are not lists of finitely generated projective modules, but lists of lists of finitely generated projective modules. The two models for unstable K-theory unwind to the same thing when all flags on have length ≤ 2, but in other cases they are a priori different and it is not clear whether or not the anima are nontheless equivalent.

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MSC:

18F25 Algebraic K-theory and L-theory (category-theoretic aspects)
19D06 Q- and plus-constructions
32S60 Stratifications; constructible sheaves; intersection cohomology (complex-analytic aspects)

Full Text: DOI arXiv

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