Coquand, Thierry; Ruch, Fabian; Sattler, Christian
Constructive sheaf models of type theory. (English) Zbl 07547339
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The principal objective in this paper is to provide a constructive version of the notion of models of univalent type theory [*V. Voevodsky*, Math. Struct. Comput. Sci. 25, No. 5, 1278–1294 (2015; Zbl 1361.68192)], using in a crucial way the fact that we have a constructive interpretation of univalence [*C. Cohen* et al., LIPIcs – Leibniz Int. Proc. Inform. 69, Article 5, 34 p. (2018; Zbl 1434.03036); *I. Orton* and *A. M. Pitts*, LIPIcs – Leibniz Int. Proc. Inform. 62, Article 24, 19 p. (2016; Zbl 1370.03016); *I. Orton* and *A. M. Pitts*, Log. Methods Comput. Sci. 14, No. 4, Paper No. 23, 33 p. (2018; Zbl 1509.03054)], which can be relativized to any presheaf model. The main point is that the operation sending an object to its object of descent data (a compatible collection of elements of its restrictions) defines a left-exact modality [*K. Quirin* and *N. Tabareau*, J. Formaliz. Reason. 9, No. 2, 131–161 (2016; Zbl 1454.03021); *E. Rijke* et al., Log. Methods Comput. Sci. 16, No. 1, Paper No. 2, 79 p. (2020; Zbl 1489.03005); *The Univalent Foundations Program*, Homotopy type theory. Univalent foundations of mathematics. Princeton, NJ: Institute for Advanced Study; Raleigh, NC: Lulu Press (2013; Zbl 1298.03002)], which can be used to build internally models of univalent type theory.

This work opens the possibility of generalizing the rich collection of results about sheaf models of intuitionistic logic as in [*A. S. Troelstra* and *D. van Dalen*, Constructivism in mathematics. An introduction. Volume II. Amsterdam etc.: North-Holland (1988; Zbl 0661.03047)] to sheaf models of univalent type theory, extending the previous work in [*T. Coquand* et al., in: Proceedings of the 2017 32nd annual ACM/IEEE symposium on logic in computer science, LICS 2017, Reykjavík University, Reykjavík, Iceland, June 20–23, 2017. Piscataway, NJ: IEEE Press. Article No. 70, 11 p. (2017; Zbl 1452.03037)] to a complete model of univalence and having no restrictions for representing (higher) data types. The present semantics has already been used in [*M. Z. Weaver* and *D. R. Licata*, in: Proceedings of the 2020 35th annual ACM/IEEE symposium on logic in computer science, LICS 2020, virtual event, July 8–11, 2020. New York, NY: Association for Computing Machinery (ACM). 915–928 (2020; Zbl 1498.03038)] for building a constructive model of directed univalence.

The synopsis of the paper goes as follows. The authors first introduce the notion of lex operations as an operation acting on [loc. cit.] and families of types. A descent data operation is then a lex operation defining a left-exact modality. These two notions are formulated genuinely syntactically in the framework of type theory. It is then shown how to instantiate these operations for cubical presheaves, where one can understand the notion of being modal for a descent data operation as a generalization of the sheaf condition, while the compatibility requirements are expressed up to path equality instead of being expressed as strict equalities. The authors next provide some examples and the applications to the unprovability of countable choice and to the model of the algebraic closure of a field. In an appendix, it is explicated how some of the results about descent data operations can be generalized to accessible left-exact modalities.

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