

On Bicompatifications of Semibicompact Spaces

By

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A Hausdorff space R is called semibicompact, after L. Zippin¹⁾, if for any point p of R and any neighbourhood U of p there exists an open set V such that $p \in V \subset U$ and $B(V)$ is bicompat²⁾. For such a space H. Freudenthal has established a theory of "ends"³⁾. His theory is concerned with the uniquely determined bicompatification $\gamma(R)$ of a semibicompact space R , but he has dealt primarily with the perfectly separable case. In the present paper we shall discuss the same problem from the standpoint of the theory of completions of spaces with respect to uniformities⁴⁾ and supplement the theory by the removal of the second axiom of countability for R and by the determination of the character of the bicompatification $\gamma(R)$.

§ 1. The bicompatification $\gamma(R)$

Let R be a semibicompact Hausdorff space. Then R is easily shown to be a regular space. For convenience an open or closed set A of R will be called γ -open or γ -closed if its boundary $B(A)$ is bicompat, and a finite covering composed of γ -open sets will be called a γ -covering.

Lemma 1. *For any γ -covering $\{G_1, \dots, G_m\}$ there exists a γ -covering $\{H_1, \dots, H_m\}$ such that $\bar{H}_i \subset G_i$, $i=1, 2, \dots, m$.*

1) L. Zippin: On semicompact spaces, Amer. Jour. of Math., **57** (1935), 327-341.

2) Here as well as in the sequel we denote by $B(X)$ the boundary of a subset X of R : $B(X) = \bar{X} \cdot \bar{R} - \bar{X}$.

3) H. Freudenthal: Neuaufbau der Endentheorie, Ann. of Math., **43** (1942), 261-279.

4) A. Weil: Sur les espaces à structure uniforme et sur la topologie générale, Act. Sci. Ind. 551, 1937; J.W. Tukey: convergence and uniformity in topology, 1940; K. Morita: On the simple extension of a space with respect to a uniformity, I. II. III. IV, Proc. Acad. Japan **27** (1951), 65-72, 130-137, 166-171, . These notes of ours will be cited with S. I, S. II, S. III, S. IV respectively.

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Proof. From the bicompaictness of $B(G_i)$ and the regularity of R it follows that there exist a finite number of γ -open sets V_1, \dots, V_n such that

$$(\sum_{i=2}^m B(G_i))(R - \sum_{i=2}^m G_i) \subset \sum_{i=1}^n V_i, \quad \bar{V}_i \subset G_1.$$

If we put $H_1 = (R - \sum_{i=2}^m \bar{G}_i) + \sum_{i=1}^n V_i$, then we see that $\bar{H}_1 \subset G_1$ and $\{H_1, G_2, G_3, \dots, G_m\}$ is a γ -covering. Repeated application of such a process proves Lemma 1.

As an immediate consequence of Lemma 1 we have

Lemma 2. *For any γ -covering there exists a star refinement which is a γ -covering.*

Theorem 1. *Let R be a semibicompact Hausdorff space. Then R is a completely regular space and there exists a bicompa Hausdorff space S with the following properties:*

- (a) *S contains R as a dense subspace.*
- (b) *For any point p of S and any neighbourhood U of p there exists an open set V of S such that $p \in V \subset U$ and $B(V) \subset R$.*
- (c) *For any bicompa Hausdorff space S' with the properties (a), (b) there exists a continuous mapping of S onto S' which leaves each point of R invariant.*

Such a space S is essentially unique and will be denoted by $\gamma(R)$.

Remark. In case S has a countable basis, the condition (b) is equivalent to the condition

$$(b)' \quad \dim(S - R) \leq 0.$$

H. Freudenthal uses (b') instead of (b); he deals primarily with the separable case.

In case R is topologically complete in the sense of E. Čech, the condition (b) is likewise equivalent to (b'). Indeed in this case $S - R$ is an F_σ -set and hence if we express $S - R$ as a sum of bicompa sets $A_n (n=1, 2, \dots)$ then (b') implies $\dim A_n \leq 0$ and consequently by a theorem on dimension theory⁵⁾ we have (b).

Proof of Theorem 1. Let $\{\mathfrak{M}_\alpha; \alpha \in \Omega\}$ be the family of all γ -coverings of R . Then $\{\mathfrak{M}_\alpha\}$ is a completely regular T-uniformity

5) K. Morita, On the dimension of normal spaces I, Jap. Jour. of Math. vol. 20 (1950) pp. 5-36, Theorem 3.-4.

agreeing with the topology (cf. S. I.), that is, R is a uniform space with a uniformity $\{\mathfrak{M}_\alpha\}$ in the sense of A. Weil and J. W. Tukey. By S. II, §4, the simple extension (=completion) R^* of R with respect to $\{\mathfrak{M}_\alpha\}$ can be considered as the bicomcompact extension of R with respect to a closed basis composed of all γ -closed sets. Hence if G is a γ -open set of R , then \bar{G} and $R-G$ are also γ -closed and

$$\widetilde{\bar{G}} \cdot (\widetilde{R-G}) = \widetilde{\bar{G}} \cdot (\widetilde{R-G}) = \bar{G} \cdot (R-G),$$

since $\bar{G} \cdot (R-G)$ is bicomcompact, where \sim denotes the closure operation in R^* . Since $\widetilde{\bar{G}} = \widetilde{\bar{G}} = \bar{G}^*$ and $\widetilde{R-G} = R^* - G^*$, we have

$$\bar{G}^* \cdot (R^* - G^*) = \bar{G} \cdot (R-G).$$

Hence R^* has the properties (a) and (b).

Let S' be any bicomcompact Hausdorff space with the properties (a), (b). If we denote by $\{\mathfrak{B}_\lambda\}$ the family of all finite open coverings \mathfrak{B}_λ of S' such that the boundary of any set of \mathfrak{B}_λ is contained in R , then $\{\mathfrak{B}_\lambda\}$ is a completely regular T-uniformity of S' agreeing with the topology and S' is complete with respect to $\{\mathfrak{B}_\lambda\}$. If we put $\mathfrak{N}_\lambda = \{V \cdot R; V \in \mathfrak{B}_\lambda\}$, then \mathfrak{N}_λ is a γ -covering of R and, by S. IV, Theorem 2, S' can be considered as the simple extension of R with respect to $\{\mathfrak{N}_\lambda\}$. Hence by S. IV, Theorem 3 there exists a continuous mapping of R^* onto S' which leaves each point of R invariant. Thus R^* has the property (c).

If S' has further the property (c), $\{\mathfrak{N}_\lambda\}$ is equivalent to $\{\mathfrak{M}_\alpha\}$ by virtue of S. IV, Theorem 3, and hence there exists a homeomorphism of R^* onto S' which leaves each point of R invariant. This completes our proof^{5a)}.

§2. A characterization of $\gamma(R)$

Theorem 2. $\gamma(R)$ is characterized as a bicomcompact Hausdorff space S with the properties (a), (b), and (c)':

(c)' Any two disjoint γ -closed sets A, B of R have disjoint closures in S .

Proof. As is shown in the proof of Theorem 1, any bicomcompact Hausdorff space S satisfying (a), (b) may be considered as the simple

5a) The space constructed by Freudenthal is essentially coincident with $\gamma(R)$. H. Nakano communicated to me a different proof for the existence of a bicomcompact Hausdorff space S satisfying (a), (b).

extension of R with respect to a completely regular T -uniformity $\{\mathfrak{N}_\lambda\}$ consisting of γ -coverings. By the remark at the end of §2 in S. II, the condition (c') is equivalent to the condition that for any binary γ -covering there exists a refinement $\mathfrak{N}_\lambda \in \{\mathfrak{N}_\lambda\}$. The latter condition means that $\{\mathfrak{N}_\lambda\}$ is equivalent to $\{\mathfrak{M}_\omega\}$. This proves our theorem.

Theorem 3. *Any γ -closed set F of R is semibicompact as a space and $\gamma(F)$ is homeomorphic to the closure of F in the space $\gamma(R)$.*

Proof. The first part of the theorem is obvious. Any γ -closed subset of the space F is also a γ -closed subset in the whole space R . Hence the second part of the theorem follows readily from Theorem 2.

§3. The character of $\gamma(R)$

By the *character* of a space we mean the least cardinal number m such that there exists an open basis of cardinal number m .

Theorem 4. *Let m be the character of a semibicompact space R and n the cardinal number of the family of all open-closed sets of R , and let m^* and n^* be the corresponding cardinal numbers for the space $\gamma(R)$. Then we have $m^* = mn$, that is, m^* is equal to m or n according as $m \geq n$ or $m < n$. Here we assume that m is infinite.*

Proof. It is sufficient to prove

$$\begin{array}{ll} (1) & n = n^*, \\ (2) & m^* \geq n^* \\ (3) & m^* \geq m, \\ (4) & m^* \leq mn. \end{array}$$

If A is an open-closed set of R , then its closure \tilde{A} in $\gamma(R)$ is open-closed, and conversely if C is open-closed in $\gamma(R)$, so is $C \cdot R$ in R and $C = \widetilde{C \cdot R}$. This proves (1).

If \mathfrak{G}^* is an open basis of cardinal number m^* for $\gamma(R)$, then any open-closed set A of $\gamma(R)$ is expressible as a finite sum of sets of \mathfrak{G}^* because of the bicompatness of A and hence we have (2). The relation (3) is obvious.

We shall now prove (4). Let \mathfrak{G} be an open basis of cardinal number m for R which consists of γ -open sets⁶⁾, and let \mathfrak{H} be the family of all finite sums of sets of \mathfrak{G} . We denote by \mathfrak{K} the family of all the sets K of the form: $K = H + P$, where $H \in \mathfrak{H}$ and P is

6) If $\{G_\alpha\}$ is an open basis of cardinal number m , then each G_α is expressible as a sum of γ -open sets $G_{\alpha\gamma}$, m in number. We may put $\mathfrak{G} = \{G_{\alpha\gamma}\}$.

an open-closed set of the subspace $R-H$. As is easily shown, K is open and $B(K) \subset B(H)$, and hence any set of \mathfrak{R} is γ -open.

We shall first show that the cardinal number of \mathfrak{R} is at most m_n . For this purpose it is sufficient to prove that the family $\mathfrak{L}(H)$ of all open-closed sets of the subspace $R-H$ has cardinal number at most m_n .

$\alpha)$ If $P \in \mathfrak{L}(H)$ and $P \cdot B(H) = 0$, then P is an open-closed set of R . Hence the cardinal number of the family of such P is at most n .

$\beta)$ If $P \in \mathfrak{L}(H)$ and $P \cdot B(H) \neq 0$, then $X = P \cdot B(H)$ is an open-closed set of the subspace $B(H)$. By the same argument as in the proof of (2) we see that the cardinal number of the family of all open-closed sets of the space $B(H)$ is at most m . If we have $X = Q \cdot B(H)$ for another $Q \in \mathfrak{L}(H)$, then $P-Q$ and $Q-P$ are sets of the type discussed in $\alpha)$ and Q is obtained from P by $Q = \{P - (P-Q)\} + (Q-P)$. Hence the cardinal number of the family $\{P; P \cdot B(H) \neq 0, P \in \mathfrak{L}(H)\}$ is also at most m_n .

Thus the cardinal number of \mathfrak{R} is at most m_n . We shall next prove that for γ -open sets L, M such that $\bar{M} \subset L$ there exists a set K of \mathfrak{R} such that $M \subset K \subset L$. Since $B(M)$ is bicomact there exists a set H of \mathfrak{S} such that $B(M) \subset H \subset L$. As is easily shown, $(R-H) \cdot M$ is an open-closed set of the subspace $R-H$. Hence if we put $K = H + (R-H)M$ we have $M \subset K \subset L$ and $K \in \mathfrak{R}$.

Therefore, in view of Lemma 1, the family $\{\mathfrak{N}_\lambda\}$ of all the coverings which consist of a finite number of sets of \mathfrak{R} is equivalent to the uniformity $\{\mathfrak{M}_\alpha\}$ consisting of all γ -coverings. Hence $\gamma(R)$ may be considered as the simple extension of R with respect to $\{\mathfrak{N}_\lambda\}$ and consequently we have (4). This completes our proof.

As an immediate consequence we obtain

Theorem 5. *Let R be a semibicomact space with a countable basis. Then $\gamma(R)$ has a countable basis if and only if there exist at most a countable number of open-closed sets in R .*

The above proof of Theorem 4 shows the validity of the following theorems.

Theorem 6. *If a Hausdorff space R is zero-dimensional in the sense of Menger-Urysohn, so is $\gamma(R)$.*

Theorem 7. *If a Hausdorff space R is regularly one-dimensional (that is, each point has an arbitrarily small neighbourhood with a boundary consisting of a finite number of points), so also is $\gamma(R)$.*

§4 The quasi-component space.

Let $\{\mathfrak{Q}_\lambda\}$ be the family of all finite coverings of R which consist of mutually disjoint open-closed sets. If we identify two points x, y of R such that $x \in S(y, \mathfrak{Q}_\lambda)$ for every \mathfrak{Q}_λ and introduce a new topology in R by taking the family of all open-closed sets of R as a basis of open sets for this new topology, then the identification space of R with this new topology is called the quasi-component space of R and it will be denoted by $L(R)$. The mapping f of R onto $L(R)$ induced by the above identification is a uniformly continuous mapping of R onto $L(R)$, where we consider R as a space with a uniformity $\{\mathfrak{W}_\alpha\}$ (cf. the proof of Theorem 1) and $L(R)$ as a space with a uniformity $\{f(\mathfrak{Q}_\lambda)\}$ consisting of finite open coverings which are obtained as the images of \mathfrak{Q}_λ by f . Then by S. II. Theorem 3 f can be extended to a continuous mapping f^* from $\gamma(R)$ into $\gamma(L(R))$. It is easy to see that f^* induces a homeomorphism of $L(\gamma(R))$ onto $\gamma(L(R))$. Hence we have

Theorem 8. *The quasi-component space $L(\gamma(R))$ of $\gamma(R)$ is homeomorphic to $\gamma(L(R))$.*

The character of $L(R)$ is not greater than \mathfrak{n} , where the meaning of \mathfrak{m} and \mathfrak{n} are the same as in §3, and hence the character of $\gamma(L(R))$ is equal to \mathfrak{n} by Theorem 4 in case \mathfrak{n} is infinite. The latter fact is also seen from (1) in §3 and Theorem 8. If $\mathfrak{n} \leq \mathfrak{s}_0$, any decreasing sequence of non-empty open-closed sets of R has a non-empty intersection, since otherwise there would exist a countable number of non-empty open-closed sets V_n , $n=1, 2, \dots$ such that $R = \bigcup_{n=1}^{\infty} V_n$, $V_i \cdot V_j = 0$ for $i < j$, and hence we would have $\mathfrak{n} > \mathfrak{s}_0$. Hence we have $L(R) = \gamma(L(R))$. Conversely if $L(R)$ is a compactum then we have $\gamma(L(R)) = L(R)$ and $\mathfrak{n} \leq \mathfrak{s}_0$. Therefore we have by Theorem 5

Theorem 9. *For a semibicompact Hausdorff space R with a countable basis, $\gamma(R)$ has a countable basis if and only if the quasi-component space $L(R)$ is a compactum.*

The original formulation of Freudenthal will then be obtained from Theorem 9 if $L(R)$ is replaced by $\gamma(L(R))$ ⁷⁾.

§5 The case where R has a countable basis.

Theorem 10. *Let R be a semibicompact Hausdorff space with a*

7) H. Freudenthal, loc. cit. Satz VI.

countable basis. Then there exists a compactum S with the properties (a), (b) in Theorem 1.

Proof. By the assumption there is a uniformity $\{\mathfrak{U}_n; n=1, 2, \dots\}$ which consists of a countable number of γ -coverings and agrees with the topology. For a covering \mathfrak{G} of R and a subset X of R we denote by $(\mathfrak{G}|X)$ the family $\{G; G \cdot X \neq 0, G \in \mathfrak{G}\}$. We can then construct a countable number of γ -coverings $\mathfrak{B}_n, n=1, 2, \dots$ with the following properties:

α_n) \mathfrak{B}_n is a star refinement of \mathfrak{B}_{n-1} .

β_n) \mathfrak{B}_n is a refinement of \mathfrak{U}_n .

γ_n) \mathfrak{B}_n is a refinement of the coverings

$$\{V_j^{(i)}, R - \overline{V_j^{(i)}}, (\mathfrak{B}_{n-1}|B(V_j^{(i)}))\} \text{ for } i=1, 2, \dots, n-1; \\ j=1, \dots, r_i, \text{ where } \mathfrak{B}_i = \{V_j^{(i)}; j=1, 2, \dots, r_i\}.$$

This construction is possible by induction with the aid of Lemma 2. Then $\{\mathfrak{B}_n\}$ is a completely regular T-uniformity agreeing with the topology. Let R^* be the simple extension of R with respect to $\{\mathfrak{B}_n\}$. By S. II. Theorem 2 the boundary of $(V_j^{(i)})^* = R^* - \overline{(R - V_j^{(i)})}$ in R^* coincides with the boundary of $V_j^{(i)}$ in R (cf. the proof of Theorem 1). Hence R^* is a compactum with the properties (a), (b).

As an application of Theorem 10 we can now prove the following theorem of L. Zippin^{7a)}.

Theorem 11. *If R is a semibicompact, separable metric space and topologically complete in the sense of M. Fréchet, then there exists a compactum S such that S contains R as a dense subset and $S - R$ consists of a countable number of points.*

Proof. By the assumption R is a G_δ -set in the compact metric space R^* constructed in Theorem 10, so that there exists a countable number of compact sets $A_n, n=1, 2, \dots$ such that $R^* - R = A_1 + A_2 + \dots$. The subspace $(A_1 + \dots + A_i) - (A_1 + \dots + A_{i-1})$ is a locally compact separable metric space of dimension at most zero, and hence it is expressible as a sum of a countable number of disjoint compact sets.

Therefore there exists a countable number of disjoint compact sets $X_n, n=1, 2, \dots$ such that

7a) Cf. L. Zippin, loc. cit.

$$R^* - R = \sum_{n=1}^{\infty} X_n; \quad X_i \cdot X_j = 0 \quad \text{for } i \neq j.$$

Since $\dim X_i \leq 0$, there exist a finite number of disjoint compact sets $X_{i_j} (j=1, 2, \dots, s_i)$ of diameter less than $1/i$ such that $X_i = X_{i_1} + X_{i_2} + \dots + X_{i_{s_i}}$. On changing the notations we see that there can be found a countable number of closed sets $C_n, n=1, 2, \dots$ such that

$$(5) \quad R^* - R = \sum_{n=1}^{\infty} C_n; \quad C_i \cdot C_j = 0 \quad \text{for } i \neq j,$$

(6) the diameter of C_i converges to zero as $i \rightarrow \infty$.

It is easy to see that the collection of the sets $C_n, n=1, 2, \dots$ and the points p of R defines an upper semi-continuous decomposition of R^* . By constructing this decomposition space we have a desired space S .

Remark. The above proof gives likewise a proof of a theorem of G. Nöbeling to the effect that any regularly one-dimensional separable metrizable space can be imbedded in a regularly one-dimensional compactum⁸⁾.

8) G. Nöbeling, Math. Ann. **104** (1931), 81-91; C. Kuratowski, Fund. Math. **30** (1938), 8-13; for the theorem used by Kuratowski cf. also K. Morita, Sci. Rep. of Tokyo Bunrika Daigaku, Sec. A, **4** (1949), No. 84, 151-155.