On group rings over a modular field which possess radicals expressible as principal ideals.

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§ 1. Introduction.

For an arbitrary field K a group $\mathfrak{G}$ of finite order determines an associative algebra $\Gamma(\mathfrak{G}, K)$, the group ring over K. As is well known, $\Gamma(\mathfrak{G}, K)$ is not semisimple if the characteristic of K is a prime number p dividing the order of $\mathfrak{G}$. In the present paper we shall study the structure of $\Gamma(\mathfrak{G}, K)$ for the case where the radical of $\Gamma(\mathfrak{G}, K)$ is expressible as a principal ideal.

Let A be a ring with a unit element which satisfies the minimum condition for left and right ideals. A is called to be *quasi-primary*, if A is indecomposable as a two-sided ideal and A is a direct sum of indecomposable left ideals which have the same multiplicity. If A is a direct sum of two-sided ideals which themselves are quasi-primary rings, we shall say that A is *quasi-primary-decomposable*. In case A is an algebra over an algebraically closed field K, the second condition in the definition of quasi-primary rings amounts to saying that all irreducible representations of A in K have the same degree. On the basis of these definitions it will be shown that the radical of A is a principal left ideal as well as a principal right ideal if and only if A is quasi-primary-decomposable and generalized uni-serial in the sense of T. NAKAYAMA[1].

Let p be a fixed prime number and $\mathfrak{G}$ a group of finite order $g = p^a g'$ with $(p, g') = 1$. We denote by $\mathfrak{P}$ a p-Sylow-subgroup of $\mathfrak{G}$ and by $\mathfrak{H}$ the largest normal subgroup of $\mathfrak{G}$ which has an order prime to p. These designations shall be retained throughout the present paper.

Let K be an algebraically closed field of characteristic p. Then

1) T. NAKAYAMA: On Frobeniusean algebras II, Ann, of Math., **42** (1941)-pp. 1-21.

it will be shown in § 4 that $\Gamma(\mathfrak{G}, K)$ is quasi-primary-decomposable if and only if $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ and the factor group $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is commutative. In § 5 we prove that the radical of the group ring $\Gamma(\mathfrak{G}, K)$ is expressible as a principal left ideal and as a principal right ideal if and only if $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{P}$ is a cyclic group.

In conclusion we shall mention a theorem that for a group $\mathfrak{G}$ with a cyclic p-Sylow-subgroup the normalizer of any subgroup of an order $p^h (h > 0)$ has the group ring whose radical is a principal ideal.

§ 2. Rings whose radicals are principal ideals.

Let A be a ring which has a unit element and satisfies the minimum condition (whence also the maximum condition) for left and right ideals.

Theorem 1. *In order that the radical N of a ring A be a principal left ideal and a principal right ideal: $N = Ac = dA$, it is necessary and sufficient that A be quasi-primary-decomposable and generalized uni-serial.*

Proof. 1) *Necessity.* From the assumption that $N = Ac = dA$ it follows that $N = cA = Ad$ and A is a generalized uni-serial ring[2]. Let A be a direct sum of indecomposable left ideals $Ae_{\kappa i}$:

$$A = Ae_{11} + \cdots + Ae_{1, f(1)} + \cdots + Ae_{n1} + \cdots + Ae_{n, f(n)},$$

where $e_{\kappa i}$ are mutually orthogonal primitive idempotent elements, $Ae_{\kappa i}$ is isomorphic to $Ae_{\kappa 1}$ for $i = 1, 2, \ldots, f(\kappa)$ and $Ae_{\kappa i}$ is not isomorphic to $Ae_{\tau j}$ if $\kappa \neq \tau$.

Then it is easily seen that N^ν $(= Ac^\nu = c^\nu A)$ is expressible as a direct sum of $Ac^\nu e_{\kappa i}$:

$$N^\nu = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} Ac^\nu e_{\kappa i}, \qquad \nu = 1, 2, \ldots, \tag{1}$$

and similarly as a direct sum of $Ae_{\tau j} c^\nu$:

$$N^\nu = \sum_{\tau=1}^{n} \sum_{j=1}^{f(\tau)} Ae_{\tau j} c^\nu, \qquad \nu = 1, 2, \ldots. \tag{2}$$

2) T. NAKAYAMA: Note on uni-serial and generalized uni-serial rings, Proc. Imp. Acad. Tokyo, **16** (1940), pp. 285–289. G. AZUMAYA and T. NAKAYAMA, On absolutely uniserial algebras, Jap. Jour. of Math., **19** (1948), pp, 263–273, Theorem 4.

Now we shall prove that

(3) $Ac^{\nu}e_{\kappa i}$ is isomorphic to $Ac^{\nu}e_{\kappa 1}$ for $i = 1, 2, \ldots, f(\kappa)$,

(4) $Ae_{\tau j}c^{\nu}$ is isomorphic to $Ae_{\tau 1}c^{\nu}$ for $j = 1, 2, \ldots, f(\tau)$,

(5) $Ac^{\nu}e_{\kappa i}$ is not isomorphic to $Ac^{\nu}e_{\tau j}$ if $\kappa \neq \tau$,

(6) $Ae_{\tau j}c^{\nu}$ is not isomorphic to $Ae_{\kappa i}c^{\nu}$ if $\tau \neq \kappa$,

(7) $Ac^{\nu}e_{\kappa i}$ and $Ae_{\tau j}c^{\nu}$ are indecomposable left ideals,

where we exclude $Ac^{\nu}e_{\kappa i}$ and $Ae_{\tau j}c^{\nu}$ such that $Ac^{\nu}e_{\kappa i} = 0$, $Ae_{\tau j}c^{\nu} = 0$.

Proof of (3) and (4) is obvious.

Proof of (5). Suppose that $Ac^{\nu}e_{\kappa i} \cong Ac^{\nu}e_{\tau j}$. Then we have $N^{\nu}e_{\kappa i}/N^{\nu+1}e_{\kappa i} \cong Ae_{\mu k}/Ne_{\mu k}$, $N^{\nu}e_{\tau j}/N^{\nu+1}e_{\tau j} \cong Ae_{\mu k}/Ne_{\mu k}$ for some $e_{\mu k}$, since A is generalized uni-serial. If we take elements a, b such that $a \in e_{\mu k}N^{\nu}e_{\kappa i}$, $b \in e_{\mu k}N^{\nu}e_{\tau j}$, $a \bar{\in} N^{\nu+1}$, $b \bar{\in} N^{\nu+1}$, then it holds that $e_{\mu k}N^{\nu} = aA = bA$. Hence $e_{\mu k}N^{\nu}$ is homomorphic to $e_{\kappa i}A$ as well as to $e_{\tau j}A$ and so we have $\kappa = \tau$.

Proof of (6) is obvious from the fact that $Ae_{\tau j}c^{\nu}$ is homomorphic to $Ae_{\tau j}$.

Proof of (7). $Ac^{\nu}e_{\kappa i}$ is homomorphic to some $Ae_{\mu k}$ and $Ae_{\tau j}c^{\nu}$ is homomorphic to $Ae_{\tau j}$. Since each $Ae_{\kappa i}$ has only one composition series, $Ac^{\nu}e_{\kappa i}$ and $Ae_{\tau j}c^{\nu}$ are indecomposable left ideals.

Thus (3), (4), (5), (6) and (7) are proved. Hence, by the Krull-Schmidt theorem, there exists a one-to-one correspondence φ between the subsets of $\{1, 2, \ldots, n\}$ such that

(8) $$Ac^{\nu}e_{\kappa 1} \cong Ae_{\varphi(\kappa),1}c^{\nu},$$

(9) $$f(\kappa) = f(\varphi(\kappa)),$$

where we exclude those $Ac^{\nu}e_{\kappa i}$ and $Ae_{\tau j}c^{\nu}$ which reduce to zero. From (8) it follows further that

(10) $$N^{\nu}e_{\kappa 1}/N^{\nu+1}e_{\kappa 1} \cong Ae_{\varphi(\kappa),1}/Ne_{\varphi(\kappa),1} \quad \text{if} \quad Ac^{\nu}e_{\kappa 1} \neq 0.$$

Therefore the number $f(\kappa)$ is the same for any $Ae_{\kappa i}$ contained in a fixed indecomposable two-sided ideal. This shows that A is quasi-primary-decomposable.

2) *Sufficiency.* Let us put

$$Ne_{\kappa 1}/N^{2}e_{\kappa 1} \cong Ae_{\varphi(\kappa),1}/Ne_{\varphi(\kappa),1}.$$

Then we find, as in the proof of (5), that $\kappa \neq \tau$ implies $\varphi(\kappa) \neq \varphi(\tau)$. Since $Ae_{\kappa 1}$ and $Ae_{\varphi(\kappa), 1}$ are contained in the same indecomposable two-sided ideal, we have $f(\kappa) = f(\varphi(\kappa))$, according to the assumption that A is quasi-primary-decomposable. For an element $c_{\kappa i}$ such that $c_{\kappa i} \in e_{\varphi(\kappa), i} N e_{\kappa i}$, $c_{\kappa i} \bar{\in} N^2$, we have

$$(11) \qquad Ac_{\kappa i} = Ne_{\kappa i}, \qquad c_{\kappa i}A = e_{\varphi(\kappa), i}N.$$

Hence, if we put $c = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} c_{\kappa i}$, we obtain a direct sum decomposition: $Ac = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} Ac_{\kappa i}$. By (11) we conclude that $N = Ac$. Similarly cA is a direct sum of $c_{\kappa i}A = e_{\varphi(\kappa), i}N$, and $N = cA$. Thus the proof of Theorem 1 is completed.

Remark. As is easily shown, a generalized uni-serial ring is not quasi-primary-decomposable in general[2a].

Lemma 1. *If A is generalized uni-serial and quasi-Frobeniusean*[3], *then the length of a composition series of $Ae_{\kappa i}$ is the same for any $Ae_{\kappa i}$ contained in a fixed indecomposable two-sided ideal.*

Proof. Suppose that $Ne_{\kappa 1}/N^2 e_{\kappa 1} \cong Ae_{\tau 1}/Ne_{\tau 1}$. Since $Ne_{\kappa 1} = Ab$, $e_{\tau 1}N = bA$ for an element b such that $b \in e_{\tau 1}Ne_{\kappa 1}$, $b \bar{\in} N^2$, the mapping $ae_{\tau 1} \rightarrow ae_{\tau 1}b$ induces a homomorphism of $Ae_{\tau 1}$ onto $Ab = Ne_{\kappa 1}$, and hence we have an isomorphism $Ae_{\tau 1}/Ae_{\tau 1} \frown l(N) \cong Ne_{\kappa 1}$. Since A is quasi-Frobeniusean, we have $l(N) = r(N)$ and $Ae_{\tau 1} \frown r(N)$ is a simple left ideal. Thus our lemma is proved.

Remark. From Lemma 1 it follows that for a symmetric algebra A which is generalized uni-serial the factor groups appearing in the composition series of indecomposable left ideals belonging to the same block have the following types:

$$\begin{array}{llllllll}
Ae_{\kappa 1}: \alpha_1 & \alpha_2 \; \ldots\ldots \; \alpha_m & \alpha_1 \; \ldots\ldots \; \alpha_m & \ldots\ldots \; \alpha_m & \alpha_1 \\
Ae_{\tau 1}: \alpha_2 & \alpha_3 \; \ldots\ldots \; \alpha_1 & \alpha_2 \; \ldots\ldots \; \alpha_1 & \ldots\ldots \; \alpha_1 & \alpha_2 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & & \\
Ae_{\lambda 1}: \alpha_m & \alpha_1 \; \ldots\ldots \; \alpha_{m-1} & \alpha_m \; \ldots\ldots \; \alpha_{m-1} & \ldots\ldots \; \alpha_{m-1} & \alpha_m ,
\end{array}$$

where α_i denotes a type of groups. Hence the Cartan matrix corresponding to this block is of the form

2a) The set of those matrices of order 3 which are of the form $\begin{pmatrix} a & o & o \\ b & c & d \\ e & f & g \end{pmatrix}$ with coefficients in a given field is a generalized uni-serial ring, but not quasi-primary-decomposable.

3) T. NAKAYAMA: loc. cit., 1), p. 8.

$$\begin{pmatrix} s+1 & s & \cdots\cdots & s \\ s & s+1 & \cdots\cdots & s \\ & \cdots\cdots\cdots & & \\ s & s & & s+1 \end{pmatrix}.$$

§ 3. The group ring $\Gamma(\mathfrak{G}, K)$ in the case where $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$.

1. We assume throughout this section that $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$[4] and that the ground field K is an algebraically closed field of characteristic p.

The group ring $\Gamma(\mathfrak{H}, K)$, which we shall denote also by $\Gamma(\mathfrak{H})$ for the sake of simplicity, is clearly semisimple, and hence it is expressed as a direct sum of simple two-sided ideals. Let

$$(12) \qquad \Gamma(\mathfrak{H}) = \sum_{\kappa=1}^{m} \sum_{i=1}^{t_\kappa} \Gamma(\mathfrak{H}) e_{\kappa i}$$

be such a decomposition of $\Gamma(\mathfrak{H})$, where $e_{\kappa i}$ are mutually orthogonal idempotent elements, and if we denote by $\mathfrak{G}_\kappa$ the set of elements G of $\mathfrak{G}$ for which $G^{-1}e_{\kappa 1}G = e_{\kappa 1}$, the following relations are assumed to be valid:

$$(13) \qquad \mathfrak{G}_\kappa = \{G\,;\, G^{-1}e_{\kappa 1}G = e_{\kappa 1}\}, \quad \mathfrak{G} = \mathfrak{G}_\kappa Q_{\kappa 1} + \cdots + \mathfrak{G}_\kappa Q_{\kappa t_\kappa},$$
$$Q_{\kappa 1} = E, \ (\mathfrak{G} : \mathfrak{G}_\kappa) = t_\kappa,$$

$$(14) \qquad Q_{\kappa i}^{-1} e_{\kappa 1} Q_{\kappa i} = e_{\kappa i}, \qquad i = 1, 2, \ldots, t_\kappa,$$

$$(15) \qquad e_{11} = \frac{1}{h}\sum_{H\in\mathfrak{H}} H, \quad h = (\mathfrak{H} : E), \quad t_1 = 1.$$

For any element G of $\mathfrak{G}_\kappa$ the mapping $a \to G^{-1}aG$ $(a \in \Gamma(\mathfrak{H})e_{\kappa 1})$ defines an automorphism of the simple ring $\Gamma(\mathfrak{H})e_{\kappa 1}$ and hence there exists an element $M(G)$ of $\Gamma(\mathfrak{H})e_{\kappa 1}$ such that

$G^{-1}aG = M(G)^{-1}aM(G)$, for every element a of $\Gamma(\mathfrak{H})e_{\kappa 1}$.

The element $M(G)$ is determined by G uniquely apart from a factor belonging to the field K. Here we can put

$M(H) = He_{\kappa 1}$, for any element H of $\mathfrak{H}$.

We choose representatives S_σ, S_τ, ... from each coset $S_\sigma\mathfrak{H}$, $S_\tau\mathfrak{H}$, ... of $\mathfrak{G}_\kappa$ mod $\mathfrak{H}$ (the representative of the coset $\mathfrak{H}$ shall be a unit

4) For the significance of the subgroup $\mathfrak{H}$ cf. the introduction.

element E of $\mathfrak{G}$), and determine $M(S_\sigma)$, $M(S_\tau)$, ... in any way, and define

$$M(HS_\sigma) = M(H)M(S_\sigma)\,, \qquad \text{for} \quad H \in \mathfrak{H}\,.$$

Then the factor set $\{c(G_1, G_2);\ G_1, G_2 \in \mathfrak{G}_\kappa\}$ arising from the relations

$$M(G_1)M(G_2) = c(G_1, G_2)M(G_1 G_2) \tag{16}$$

is essentially a factor set of $\mathfrak{G}_\kappa/\mathfrak{H}$, that is,

$$c(S_\sigma H, S_\tau H') = c(HS_\sigma, H'S_\tau) = c(S_\sigma, S_\tau)$$

for any $H, H' \in \mathfrak{H}$. Hence we may denote $c(S_\sigma, S_\tau)$ by $c(\sigma, \tau)$ $(\sigma, \tau \in \mathfrak{G}_\kappa/\mathfrak{H})$.

If we put

$$u_0(\sigma) = S_\sigma \cdot M(S_\sigma)^{-1}, \qquad \sigma \in \mathfrak{G}_\kappa/\mathfrak{H}\,,$$

then we have

$$u_0(\sigma)u_0(\tau) = c(\sigma, \tau)^{-1}u_0(\sigma\tau)\,. \tag{17}$$

Hence all the elements of the form

$$\sum_{\sigma \in \mathfrak{G}_\kappa/\mathfrak{H}} x_\sigma u_0(\sigma)\,. \qquad x_\sigma \in K \tag{18}$$

constitute an algebra $\mathfrak{D}_\kappa^0$ over K and we have

$$\Gamma(\mathfrak{G}_\kappa, K)e_{\kappa 1} = \Gamma(\mathfrak{H})e_{\kappa 1} \times \mathfrak{D}_\kappa^0\,. \tag{19}$$

Moreover, if we put

$$e_\kappa = e_{\kappa 1} + e_{\kappa 2} + \cdots + e_{\kappa t_\kappa} \tag{20}$$

$$d_{ij} = Q_{\kappa i}^{-1} e_{\kappa 1} Q_{\kappa j}\,, \quad i, j = 1, \ldots, t_\kappa\,, \tag{21}$$

$$u(\sigma) = \sum_{i=1}^{t_\kappa} Q_{\kappa i}^{-1} u_0(\sigma) Q_{\kappa i} = \sum_{i=1}^{t_\kappa} d_{i1} u_0(\sigma) d_{1i}\,, \tag{22}$$

$$\tilde{c}_{\lambda\mu} = \sum_{i=1}^{t_\kappa} Q_{\kappa i}^{-1} c_{\lambda\mu} Q_{\kappa i} = \sum_{i=1}^{t_\kappa} d_{i1} c_{\lambda\mu} d_{1i}\,, \tag{23}$$

where $c_{\lambda\mu}$ are matrix-units in $\Gamma(\mathfrak{H})e_{\kappa 1}$: $\Gamma(\mathfrak{H})e_{\kappa 1} = \sum K c_{\lambda\mu}$ $(1 \leqq \lambda, \mu \leqq f_\kappa')$, then we can easily prove that

$$\Gamma(\mathfrak{G}, K)e_\kappa = \mathfrak{A}_\kappa \times \mathfrak{B}_\kappa \times \mathfrak{D}_\kappa \,, \tag{24}$$

where

$$\mathfrak{A}_\kappa = \sum K\tilde{c}_{\lambda\mu} \cong \Gamma(\mathfrak{H})e_{\kappa 1} \cong K_{f'_\kappa} \,, \tag{25}$$

$$\mathfrak{B}_\kappa = \sum Kd_{ij} \cong K_{t_\kappa} \,, \tag{26}$$

$$\mathfrak{D}_\kappa = \sum_{\sigma \in \mathfrak{G}_\kappa / \mathfrak{H}} u(\sigma)K \cong \mathfrak{D}^0_\kappa \,. \tag{27}$$

Therefore we obtain

$$\Gamma(\mathfrak{G}, K)e_\kappa \cong \Gamma(\mathfrak{H}, K)e_{\kappa 1} \times \mathfrak{D}_\kappa \times K_{t_\kappa} \,. \tag{28}$$

and in particular

$$\Gamma(\mathfrak{G}, K)e_1 \cong \Gamma(\mathfrak{G}/\mathfrak{H}, K), \quad e_1 = e_{11} = \frac{1}{h}\sum_{H \in \mathfrak{H}} H \,. \tag{29}$$

If we denote the centrum of $\Gamma(\mathfrak{H})$ by $\mathfrak{Z}$ and define $G \circ x = GxG^{-1}$ for any element x of $\mathfrak{Z}$, $\mathfrak{Z}$ may be considered as a left $\mathfrak{G}/\mathfrak{H}$-module. By means of the basis $e_{11}, \ldots, e_{mt_m}$ of $\mathfrak{Z}$ we have a representation of $\mathfrak{G}$ by permutations of the elements $e_{11}, \ldots, e_{mt_m}$. On the other hand, the totality of the sums of conjugate elements of $\mathfrak{H}$ forms a basis of $\mathfrak{Z}$, by which we obtain another representation of $\mathfrak{G}$ by permutations of the classes of conjugate elements of $\mathfrak{H}$. The number m is equal to the multiplicity of the 1-representation in the former representation. Hence the number m is equal to the number of classes of conjugate elements of $\mathfrak{G}$ contained in $\mathfrak{H}$.

We summarize these results as follows[5].

Theorem 2. *Let $\mathfrak{H}$ be the largest normal subgroup of $\mathfrak{G}$ which has an order prime to p, and let*

$$\Gamma(\mathfrak{H}, K) = \sum_{\kappa=1}^{m} \sum_{i=1}^{t_\kappa} \Gamma(\mathfrak{H}, K)e_{\kappa i} \tag{12}$$

be a decomposition of $\Gamma(\mathfrak{H}, K)$ into a direct sum of simple two-sided ideals $\Gamma(\mathfrak{H}, K)e_{\kappa i}$ and let us further assume that the relations (13), (14), (15) *hold.*

5) Cf. A. H. CLIFFORD: Representations induced in an invariant subgroup, Ann. of Math., **38** (1937), pp. 533–550; T. NAKAYAMA and K. SHODA, Über die Darstellung einer endlichen Gruppe durch halblineare Transformationen, Jap. Jour. of Math., **12** (1936), pp. 109-122.

Then if we put

$$e_\kappa = e_{\kappa 1} + e_{\kappa 2} + \cdots + e_{\kappa t_\kappa}, \tag{20}$$

$\Gamma(\mathfrak{G}, K)$ is a direct sum of two-sided ideals $\Gamma(\mathfrak{G}, K)e_\kappa$, and

$$\Gamma(\mathfrak{G}, K)e_\kappa \cong \Gamma(\mathfrak{H}, K)e_{\kappa 1} \times \mathfrak{D}_\kappa \times K_{t_\kappa}, \tag{28}$$

$$\Gamma(\mathfrak{G}, K)e_1 \cong \Gamma(\mathfrak{G}/\mathfrak{H}, K), \tag{29}$$

where $\mathfrak{D}_\kappa$ is isomorphic to the (generalized) group ring of $\mathfrak{G}_\kappa/\mathfrak{H}$ over K with a factor set $\{c(\sigma, \tau)^{-1}\}$ (cf. (17), (18), (27)). The number m of the direct summands $\Gamma(\mathfrak{G}, K)e_\kappa$ is equal to the number of the classes of conjugate elements of $\mathfrak{G}$ which are contained in $\mathfrak{H}$.

2. Now we shall state a lemma which is easily obtained from a theorem of R. Brauer and C. Nesbitt[6].

Lemma 2. *The number of blocks of lowest kind of $\mathfrak{G}$ is equal to the number of classes of conjugate elements of $\mathfrak{N}$ contained in $\mathfrak{M}$, where $\mathfrak{N}$ is the normalizer of a p-Sylow-subgroup $\mathfrak{P}$ and $\mathfrak{M}$ is the largest normal subgroup of $\mathfrak{N}$ which has an order prime to p.* (It is to be noted that the centralizer $\mathfrak{C}$ of $\mathfrak{P}$ is a direct product of $\mathfrak{M}$ and $\mathfrak{C} \frown \mathfrak{P}$).

According to Lemma 2 we obtain

Corollary to Theorem 2. *In case $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$, $\Gamma(\mathfrak{G}, K)e_1$ is the indecomposable two-sided ideal corresponding to the first block of $\mathfrak{G}$.*

If $\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$, then $\mathfrak{H}\mathfrak{P}$ is also a normal subgroup of $\mathfrak{G}$, and we have $\mathfrak{N} = \mathfrak{G}$, $\mathfrak{M} = \mathfrak{H}$ in the notations of Lemma 2. Since the blocks of $\mathfrak{G}$ are all of lowest kind then, we obtain the following theorem.

Theorem 3. *In case a p-Sylow-subgroup $\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ the decomposition of $\Gamma(\mathfrak{G}, K)$ described in Theorem 2 is a direct sum decomposition into indecomposable two-sided ideals, that is, $\Gamma(\mathfrak{G}, K)e_\kappa$ is indecomposable as a two-sided ideal for every κ, $\kappa = 1, 2, \ldots, m$.*

3. We shall now study the structure of $\mathfrak{D}_\kappa$. Since $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ it is readily seen that $(\mathfrak{G}_\kappa \frown \mathfrak{H}\mathfrak{P})/\mathfrak{H}$ is a p-Sylow-subgroup of $\mathfrak{G}_\kappa/\mathfrak{H}$ and is normal in $\mathfrak{G}_\kappa/\mathfrak{H}$. Let us put

6) R. Brauer and C. Nesbitt: On the modular characters of groups, Ann. of Math., **42** (1941), pp. 556-590, Theorem 2.

$$\mathfrak{H}\mathfrak{P}_{\kappa} = \mathfrak{G}_{\kappa} \frown \mathfrak{H}\mathfrak{P}, \quad \mathfrak{S}_{\kappa} = \mathfrak{G}_{\kappa}/\mathfrak{H}$$

and denote $\mathfrak{H}\mathfrak{P}_{\kappa}/\mathfrak{H}$ by $\mathfrak{P}_{\kappa}$ simply.

Then there exists, by a theorem of I. Schur, a subgroup $\mathfrak{L}_{\kappa}$ such that $\mathfrak{S}_{\kappa} = \mathfrak{L}_{\kappa}\mathfrak{P}_{\kappa}$, $\mathfrak{S}_{\kappa}/\mathfrak{P}_{\kappa} \cong \mathfrak{L}_{\kappa}$. Since K is a field of characteristic p, any factor set belonging to a p-group is associate to a factor set 1 (such that $c(G, G') = 1$ for any elements G, G' of the group) in K, and hence it is easily shown that we can determine the elements $v(G) = k_G u(G)$, $k_G \in K$ so that they satisfy the following relations:

$$v(P)v(Q) = v(PQ)\,, \tag{30}$$

$$v(PL) = v(P)v(L)\,, \tag{31}$$

$$v(L)^{-1}v(P)v(L) = v(L^{-1}PL)\,, \tag{32}$$

where P, $Q \in \mathfrak{P}_{\kappa}$ and $L \in \mathfrak{L}_{\kappa}$. Correspondiug to these $v(G)$ we obtain a factor set which shall be denoted by $\{c(G, G')\}$. According to the relations (30), (31), (32) it is seen that this factor set is essentially a factor set of the group $\mathfrak{L}_{\kappa}$. Now let us put

$$\mathfrak{R}_{\kappa} = \sum_{P \in \mathfrak{P}_{\kappa}} \mathfrak{D}_{\kappa}(v(P) - v(E)) = \sum_{L \in \mathfrak{L}_{\kappa}} \sum_{P \in \mathfrak{P}_{\kappa}} v(L)(v(P) - v(E))K\,.$$

We shall prove that $\mathfrak{R}_{\kappa}$ is the radical of $\mathfrak{D}_{\kappa}$. From the relations (30), (31), (32) it follows that $\mathfrak{R}_{\kappa}$ is a two-sided ideal. As is well known[7], $\sum_{P \in \mathfrak{P}_{\kappa}} (v(P) - v(E))K$ constitutes the radical of the group ring $\sum_{P \in \mathfrak{P}_{\kappa}} v(P)K$, and hence it is easily verified that $\mathfrak{R}_{\kappa}$ is a nilpotent ideal, if we make use of the relations (30)–(32).

On the other hand, if we construct the (generalized) group ring $\mathfrak{D}_{\kappa}^{*}$ of $\mathfrak{L}_{\kappa}$ over K with a factor set $\{c(L, L')\,;\, L, L' \in \mathfrak{L}_{\kappa}\}$:

$$\mathfrak{D}_{\kappa}^{*} = \sum_{L \in \mathfrak{L}_{\kappa}} v(L)K\,,$$

then $\mathfrak{D}_{\kappa}^{*}$ is a semisimple algebra since the order of $\mathfrak{L}_{\kappa}$ is prime to p, and $\mathfrak{D}_{\kappa}/\mathfrak{R}_{\kappa} \cong \mathfrak{D}_{\kappa}^{*}$. Therefore $\mathfrak{R}_{\kappa}$ is the radical of $\mathfrak{D}_{\kappa}$ and

7) S. A. JENNINGS: Trans. Amer. Math. Soc., **50** (1941), pp. 175-185. Since $v(P) - v(E)$ is a nilpotent element, it follows readily from a theorem of Wedderburn that $\sum(v(P) - v(E))K$ is the radical of the group ring $\sum v(P)K$. J. H. M. WEDDERBURN, Ann. of Math., **38** (1937), p. 854.

$$\mathfrak{D}_\kappa = \mathfrak{D}_\kappa^* + \mathfrak{N}_\kappa\,, \qquad \mathfrak{D}_\kappa^* \frown \mathfrak{N}_\kappa = 0\,.$$

4. If we decompose the semisimple algebra $\mathfrak{D}_\kappa^*$ into a direct sum of simple left ideals: $\mathfrak{D}_\kappa^* = \mathfrak{D}_\kappa^* e_1^* + \cdots + \mathfrak{D}_\kappa^* e_s^*$, then e_j^* is a primitive idempotent of $\mathfrak{D}_\kappa$ and $\mathfrak{D}_\kappa e_j^*$ is an indecomposable left ideal of $\mathfrak{D}_\kappa$. The representation obtained by $\mathfrak{D}_\kappa e_j^* / \mathfrak{N}_\kappa e_j^*$ is an irreducible representation of $\mathfrak{D}_\kappa$.

Let $U_1, \ldots, U_k$ be the distinct indecomposable constituents of the regular representation of $\mathfrak{G}$ in K and $F_1, \ldots, F_k$ the distinct irreducible representations of $\mathfrak{G}$ corresponding to $U_1, \ldots, U_k$. We denote the degree of F_λ by f_λ and that of U_λ by u_λ. Then, if U_λ is obtained by an indecomposable left ideal of $\Gamma(\mathfrak{G}, K)e_\kappa$, we have $u_\lambda = p^{d_\kappa} f_\lambda$, where $(\mathfrak{P}_\kappa : E) = p^{d_\kappa}$, and we see that f_λ is divisible exactly by p^{a-d_κ}. Since the degree of $\mathfrak{D}_\kappa^* e_j^*$ over K divides the order of $\mathfrak{L}_\kappa$, $f_\lambda / p^{a-d_\kappa}$ divides g'.

Theorem 4. *Suppose that $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$. Then, if U_λ belongs to a block of defect d we have $u_\lambda = p^d f_\lambda$ and f_λ / p^{a-d} divides g', where $g = p^a g'$, $(g', p) = 1$. $a-d$ is the exact exponent to which p divides the degree of any irreducible representation of $\mathfrak{G}$ belonging to a block of defect d.*

5. Suppose that $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is commutative. Then $\mathfrak{L}_\kappa \cong \mathfrak{S}_\kappa / \mathfrak{P}_\kappa$ is also commutative. Let us denote by $\mathfrak{L}_\kappa^o$ the set of elements L of $\mathfrak{L}_\kappa$ such that $c(X, L) = c(L, X)$ for every elements X of $\mathfrak{L}_\kappa$. Then it is easily seen that $\mathfrak{L}_\kappa^o$ is a subgroup of $\mathfrak{L}_\kappa$ and

$$\mathfrak{Z} = \sum_{L \in \mathfrak{L}_\kappa^o} v(L) K$$

is the centrum of $\mathfrak{D}_\kappa^*$. $\mathfrak{Z}$ is isomorphic to the group ring $\Gamma(\mathfrak{L}_\kappa^o, K)$ of $\mathfrak{L}_\kappa^o$ over K, and hence it is a direct sum of r simple ideals: $\mathfrak{Z} = \mathfrak{Z}\tilde{e}_1 + \cdots + \mathfrak{Z}\tilde{e}_r$, $r = (\mathfrak{L}_\kappa^o : E)$. Therefore

$$\mathfrak{D}_\kappa^* = \mathfrak{D}_\kappa^* \tilde{e}_1 + \cdots + \mathfrak{D}_\kappa^* \tilde{e}_r$$

is a direct decomposition of $\mathfrak{D}_\kappa^*$ into simple two-sided ideals. Suppose that the irreducible representation of $\mathfrak{D}_\kappa^* : v(L) \to D(L)$ is obtained by regarding $\mathfrak{D}_\kappa^* e_1^*$ as a representation module of $\mathfrak{D}_\kappa^*$. Then we can easily prove that, for characters χ and χ' of $\mathfrak{L}_\kappa$, the irreducible representations of $\mathfrak{D}_\kappa^*$

$$D_\chi : v(L) \rightarrow \chi(L)D(L)\,, \quad D_{\chi'} : v(L) \rightarrow \chi'(L)D(L)$$

are distinct if and only if χ and χ' are distinct when considered as characters of $\mathfrak{L}^\circ_\kappa$. Thus the representation obtained by $\mathfrak{D}^*_\kappa e^*_j$ is equivalent to some D_χ and hence we have $(\mathfrak{L}_\kappa : E) = (\mathfrak{L}^\circ_\kappa : E)f^2$, where $f^2 = (\mathfrak{D}^*_\kappa \tilde{e}_j : K)$, $j = 1.\ 2, \ldots, r$.

Theorem 5. *In case $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is a commutative group the degree of F_λ is the same as that of F_μ if F_λ and F_λ belong to the same block. Namely, in this case the group ring $\Gamma(\mathfrak{G}, K)$ is quasi-primary-decomposable.*

6. We conclude this section with the following remark. Suppose that $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$. Then the blocks of $\mathfrak{G}$ are all of lowest kind if and only if $\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$. Let us assume that the blocks of $\mathfrak{G}$ are all of lowest kind. Then $\mathfrak{G}_\kappa$ must contain $\mathfrak{H}\mathfrak{P}$, and hence every element of $\mathfrak{P}$ commutes with every element of the centrum of $\Gamma(\mathfrak{H}, K)$. If we denote by $\mathfrak{P}^*$ the intersection of the centralizer of $\mathfrak{H}$ with $\mathfrak{P}$, $\mathfrak{P}/\mathfrak{P}^*$ may be considered as a group of automorphisms of $\mathfrak{H}$ which leave the classes of conjugate elements of $\mathfrak{H}$ invariant. Hence we have $\mathfrak{P} = \mathfrak{P}^*$[8]. Thus $\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$. As is shown by R. Brauer, the above statement does not hold in general without the assumption that $\mathfrak{H}\mathfrak{P}$ is normal.

§ 4. Quasi-primary-decomposable group rings.

1. Let $\Gamma = \Gamma(\mathfrak{G}, K)$ be a quasi-primary-decomposable group ring over an algebraically closed field K of characteristic p. If Γe_1 is the indecomposable two-sided ideal corresponding to the first block of $\mathfrak{G}$ (to wh'ch the 1-representation belongs) and e_1 is an idempotent lying in the centrum of Γ, then the set of elements G of $\mathfrak{G}$ such that $Ge_1 = e_1$ constitutes a normal subgroup of $\mathfrak{G}$. As is pointed out by K. Iwasawa[9], this group coincides with $\mathfrak{H}$. Let

$$\mathfrak{D} : G \rightarrow D(G)$$

be a representation obtained by considering Γe_1 as a representation module. This is a faithful representation of $\mathfrak{G}/\mathfrak{H}$. By the assump-

8) Cf. A. Speiser: Theorie der Gruppen von endlicher Ordnung, 1927, Satz 108.

9) K. Iwasawa: Shijô Sugaku Danwa-kai, No. 246 (1942), p. 1589.

tion that Γ is quasi-primary-decomposable the irreducible constituents of $\mathfrak{D}$ are all of degree one, and so we can assume that the coefficients $d_{ij}(G)$ of the matrix $D(G)$ are zero if $i<j$. Then it is easily verified that the set of elements G such that $d_{jj}(G)=1$ for every j forms a p-Sylow-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}/\mathfrak{H}$. Moreover $\tilde{\mathfrak{P}}$ is normal in $\mathfrak{G}/\mathfrak{H}$ and $(\mathfrak{G}/\mathfrak{H})/\tilde{\mathfrak{P}}$ is commutative. This shows that for a p-Sylow-subgroup $\mathfrak{P}$ the group $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is commutative. Conversely, if $\mathfrak{H}\mathfrak{P}$ is normal in $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is commutative then $\Gamma(\mathfrak{G}, K)$ is quasi-primary-decomposable, as is already shown by Theorem 5. Hence the following theorem is established.

Theorem 6. *Let $\mathfrak{H}$ be the largest normal subgroup of $\mathfrak{G}$ which has an order prime to p and $\mathfrak{P}$ a p-Sylow-subgroup of $\mathfrak{G}$. Then the group ring $\Gamma(\mathfrak{G}, K)$ of $\mathfrak{G}$ over an algebraically closed field K of characteristic p is quasi-primary-decomposable if and only if* 1) *$\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ and* 2) *$\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is a commutative group.*

Remark. Another proof of the "if" part of Theorem 6 is obtained as follows. We shall first prove that the indecomposable two-sided ideal corresponding to the first block is isomorphic to $\Gamma(\mathfrak{G}/\mathfrak{H}, K)$ (cf. Theorem 2). Then we can proceed in the same way as in the proof of Lemma 2 in the paper of M. Osima[10].

Corollary. *In order that every irreducible m·dular representation of $\mathfrak{G}$ is of degree one it is necessary and sufficient that a p-Sylow-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is normal and $\mathfrak{G}/\mathfrak{P}$ is commutative.*

2. As a special case we have the following theorem due to M. Osima[11].

Theorem 7. *$\Gamma(\mathfrak{G}, K)$ is primary-decomposable if and only if it holds that $\mathfrak{G}=\mathfrak{H}\mathfrak{P}$. In this case the decomposition of $\Gamma(\mathfrak{G}, K)$ described in Theorem 2 is a direct sum decomposition into indecomposable two-sided ideals.*

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10) M. OSIMA: On primary decomposable group rings, Proc. Phys. -math. Soc. Japan **24** (1942), pp. 1-9.

11) M. OSIMA: loc. cit. Let K be an algebraic number field such that every absolutely irreducible representation can be written with coefficients in K. Our Theorem 2 is valid for such a field K. Hence all the ordinary irreducible representations of $\mathfrak{G}$ remain irreducible as modular representations if (and only if) $\mathfrak{G}=\mathfrak{H}\mathfrak{P}$ and $\mathfrak{D}_\kappa$ is commutative for such a field K and for every κ. This gives an explanation to Osima's theorem 8 in his paper cited above.

Proof. We have only to prove the "if" part and the second part of the theorem. Since $\mathfrak{G} = \mathfrak{H}\mathfrak{P}$ the group $\mathfrak{G}_\kappa/\mathfrak{H}$ in Theorem 2 may be regarded as a subgroup $\mathfrak{P}_\kappa$ of $\mathfrak{P}$. Hence $\mathfrak{D}_\kappa$ is isomorphic to $\Gamma(\mathfrak{P}_\kappa, K)$ which is completely primary, and therefore $\Gamma(\mathfrak{G}, K)e_\kappa$ is a primary ring. This completes the proof of the theorem.

§ 5. Group rings with radicals expressible as principal ideals.

1. We shall now prove the following theorem.

Theorem 8. *In order that the radical of the group ring $\Gamma(\mathfrak{G}, K)$ be a principal left ideal as well as a principal right ideal it is necessary and sufficient that* 1) $\mathfrak{H}\mathfrak{P}$ *is a normal subgroup of* $\mathfrak{G}$ *and* 2) $\mathfrak{P}$ *is a cyclic group.*

Proof. 1) *Necessity.* Suppose that the radical of $\Gamma(\mathfrak{G}, K)$ is a principal left ideal and a principal right ideal. Then by Theorem 1 $\Gamma(\mathfrak{G}, K)$ is quasi-primary-decomposable. According to Theorem 6 $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is commutative. The indecomposable two-sided ideal of $\Gamma(\mathfrak{G}, K)$ corresponding to the first block is equivalent to $\Gamma(\mathfrak{G}/\mathfrak{H}, K)$ as an algebra. Therefore $\Gamma(\mathfrak{G}/\mathfrak{H}, K)$ is quasi-primary and generalized uni-serial. We put $\mathfrak{S} = \mathfrak{G}/\mathfrak{H}$ and denote a p-Sylow-subgroup of $\mathfrak{S}$ by $\mathfrak{P}$. Then $\mathfrak{P}$ is a normal subgroup and $\mathfrak{S}/\mathfrak{P}$ is commutative, and further $\Gamma(\mathfrak{S}, K)$ is generalized uni-serial. As is well known, there exists a subgroup $\mathfrak{L}$ of $\mathfrak{S}$ such thot $\mathfrak{S} = \mathfrak{L}\mathfrak{P}$, $\mathfrak{S}/\mathfrak{P} \cong \mathfrak{L}$. Now we put $e = \dfrac{1}{l}\sum_{L\in\mathfrak{L}} L$, $l = (\mathfrak{L} : E)$. Then e is a primitive idempotent in $\Gamma(\mathfrak{S}, K)$. If we denote the radical of $\Gamma(\mathfrak{S}, K)$ by N, then $\Gamma(\mathfrak{S}, K)e$, Ne, N^2e, ... is a composition series of $\Gamma(\mathfrak{S}, K)e$, and the degree of $N^ie/N^{i+1}e$ is one. On denoting the radical of $\Gamma(\mathfrak{P}, K)$ by N_0, we have $\Gamma(\mathfrak{S}, K)e = \Gamma(\mathfrak{P}, K)e$, $N^ie = N_0^ie$. Since N_0^ie is isomorphic to N_0^i as a left $\Gamma(\mathfrak{P}, K)$-module, $\Gamma(\mathfrak{P}, K)$, N_0, N_0^2, ... is a composition series of $\Gamma(\mathfrak{P}, K)$. This shows that the group ring $\Gamma(\mathfrak{P}, K)$ is uni-serial. Therefore $\mathfrak{P}$ is a cyclic group, as is well known[10].

2) *Sufficiency.* Let us assume that $\mathfrak{H}\mathfrak{P}$ is normal in $\mathfrak{G}$ and $\mathfrak{P}$ is a cyclic group. Then the factor group $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is isomorphic to a subgroup of the group of all automorphisms of $\mathfrak{P}$. This group of automorphisms is a cyclic group of order $p^{a-1}(p-1)$, when $p > 2$ and $(\mathfrak{P} : E) = p^a$. Therefore $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ is a cyclic group

of an order dividing $p-1$. In case $p=2$ we have $\mathfrak{G}=\mathfrak{H}\mathfrak{P}$. Therefore $\Gamma(\mathfrak{G}, K)$ is quasi-primary-decomposable, and the results of § 2 are applicable to our case. In this case the radical $\mathfrak{R}_\kappa$ is expressed in the form

$$\mathfrak{R}_\kappa = \mathfrak{D}_\kappa c_\kappa = c_\kappa \mathfrak{D}_\kappa , \quad c_\kappa = v(P_\kappa) - v(E) ,$$

where P_κ is a generator of $\mathfrak{P}_\kappa$. This shows that the radical of $\Gamma(\mathfrak{G}, K)e_\kappa$ is expressible as a principal left ideal and as a principal right ideal. Thus the theorem is completely proved.

2. Let the radical of $\Gamma(\mathfrak{G}, K)$ be a principal ideal. By the notations in § 2 we have $\mathfrak{G}_\kappa \frown \mathfrak{H}\mathfrak{P} = \mathfrak{H}\mathfrak{P}_\kappa$, $\mathfrak{P}_\kappa \leqq \mathfrak{P}$. Here $\mathfrak{G}_\kappa/\mathfrak{H}\mathfrak{P}_\kappa$ is isomorphic to a subgroup $\mathfrak{G}_\kappa\mathfrak{P}/\mathfrak{H}\mathfrak{P}$ of $\mathfrak{G}/\mathfrak{H}\mathfrak{P}$ and the latter is a cyclic group of an order prime to p. Hence any factor set of the group $\mathfrak{G}_\kappa/\mathfrak{H}\mathfrak{P}_\kappa \cong \mathfrak{L}_\kappa$ is associate to 1 and so we have

$$\mathfrak{D}_\kappa \cong \Gamma(\mathfrak{G}_\kappa/\mathfrak{H}, K) .$$

The above result holds if K is an algebraically closed field of characteristic zero ; for such a field K we have

$$\Gamma(\mathfrak{G}, K)e_\kappa \cong \Gamma(\mathfrak{H}, K)e_{\kappa 1} \times \Gamma(\mathfrak{G}_\kappa/\mathfrak{H}, K) \times K_{t_\kappa}$$

since every q-Sylow-subgroup of $\mathfrak{G}_\kappa/\mathfrak{H}$ is cyclic for every prime factor q of the order of $\mathfrak{G}_\kappa/\mathfrak{H}$ and consequently any factor set of $\mathfrak{G}_\kappa/\mathfrak{H}$ is associate to 1 by a theorem of I. Schur[12].

Assume that $(\mathfrak{P}_\kappa : E) > 0$. If $\mathfrak{C}_\kappa$ is the centralizer of $\mathfrak{P}_\kappa$, the centralizer of $\mathfrak{H}\mathfrak{P}_\kappa/\mathfrak{H}$ in $\mathfrak{G}/\mathfrak{H}$ is $\mathfrak{H}\mathfrak{C}_\kappa/\mathfrak{H}$, as is shown easily. By Lemma 8 which will be proved in the next section, we see that $\mathfrak{H}\mathfrak{C}_\kappa = \mathfrak{H}\mathfrak{P}$. Hence the centralizer of $\mathfrak{H}\mathfrak{P}_\kappa/\mathfrak{H}$ in $\mathfrak{G}_\kappa/\mathfrak{H}$ is identical with $(\mathfrak{G}_\kappa/\mathfrak{H}) \frown (\mathfrak{H}\mathfrak{P}/\mathfrak{H}) = \mathfrak{H}\mathfrak{P}_\kappa/\mathfrak{H}$. Thus the centralizer of the p-Sylow-subgroup of $\mathfrak{G}_\kappa/\mathfrak{P}$ coincides with itself, and the structure of $\mathfrak{G}_\kappa/\mathfrak{H}$ is completely known.

For an algebraically closed field K of characteristic p the group ring $\Gamma(\mathfrak{G}_\kappa/\mathfrak{H}, K)$ is indecomposable as a two-sided ideal by virtue of Lemma 2, and it is quasi-primary as well as generalized uniserial.

We can easily determine the ordinary and modular irreducible representations of $\mathfrak{G}_\kappa/\mathfrak{H}$. The matrix of decomposition numbers

12) I. SCHUR : Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, Crelle **127** (1904), pp. 20-50.

of $\mathfrak{G}_\kappa/\mathfrak{H}$ and hence of the block of $\mathfrak{G}$ corresponding to $\Gamma(\mathfrak{G}, K)e_\kappa$ takes the following form :

$$D_\kappa = \begin{array}{c} \left(\begin{array}{c} 1\ 0\ldots.0 \\ 0\ 1\ldots.0 \\ \ddots \\ 0\ 0\ldots.1 \\ 1\ 1\ldots.1 \\ \cdots \\ 1\ 1\ldots.1 \end{array}\right) \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m_\kappa \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} s_\kappa \end{array}, \qquad s_\kappa = \frac{p^{d_\kappa}-1}{m_\kappa},\ m_\kappa \mid p-1\ ,$$

where $(\mathfrak{P}_\kappa : E) = p^{d_\kappa}$, $m_\kappa = (\mathfrak{G}_\kappa : \mathfrak{H}\mathfrak{P}_\kappa)$. Hence the matrix of Cartan invariants is

$$\begin{pmatrix} s_\kappa+1 & s_\kappa & \ldots s_\kappa \\ s_\kappa & s_\kappa+1 & s_\kappa \\ \ldots & \ldots & \ldots \\ s_\kappa & s_\kappa \ldots & s_\kappa+1 \end{pmatrix}, \text{ which is equivalent to } \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ & & & p^{d_\kappa} \end{pmatrix}.$$

The form of the Cartan matrix is also known from the remark at the end of § 1.

Thus we obtain the following theorem by applying a theorem of R. Brauer and C. Nesbitt[13].

Theorem 9. *Let the radical of the group ring $\Gamma(\mathfrak{G}, K)$ be a principal ideal: $\Gamma(\mathfrak{G}, K)c = c\Gamma(\mathfrak{G}, K)$. Then the summand $\Gamma(\mathfrak{G}, K)e_\kappa$ in the decomposition of Theorem 2 is indecomposable as a two-sided ideal if $(\mathfrak{P}_\kappa : E) = p^{d_\kappa} > 1$. In case $(\mathfrak{P}_\kappa : E) = 1$, that is, $\mathfrak{G}_\kappa/\mathfrak{H}$ has an order m_κ prime to p, $\Gamma(\mathfrak{G}, K)e_\kappa$ is expressed as a direct sum of m_κ simple two-sided ideals. The number of blocks of positive defect d is equal to the number of the classes $\mathfrak{K}_\nu$ of p-regular elements such that the order of the normalizer of any element of $\mathfrak{K}_\nu$ is divisible by p^d but not by p^{d+1}.*

§ 6. Some remarks concerning a group with a cyclic p-Sylow-subgroup.

In this section we assume that the p-Sylow-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a cyclic group. Let $\mathfrak{P}_d$ be the subgroup of $\mathfrak{P}$ with an order

13) R. Brauer and C. Nensbitt: loc. cit., p. 569.

p^d ($d \geqq 1$), and let us denote the centralizer and normalizer of $\mathfrak{P}_d$ by $\mathfrak{C}_d$ and $\mathfrak{N}_d$ respectively. $\mathfrak{C}_d$ is a normal subgroup of $\mathfrak{N}_d$ and we have

$$\mathfrak{P} = \mathfrak{P}_a \supset \mathfrak{P}_{a-1} \supset \cdots \supset \mathfrak{P}_1 , \quad \mathfrak{C} = \mathfrak{C}_a \leqq \mathfrak{C}_{a-1} \leqq \cdots \leqq \mathfrak{C}_1 ,$$
$$\mathfrak{N} = \mathfrak{N}_a \leqq \mathfrak{N}_{a-1} \leqq \cdots \leqq \mathfrak{N}_1 .$$

$\mathfrak{N}/\mathfrak{C}$ is a cyclic group of an order dividing $p-1$.

Lemma 3. *Any two elements of $\mathfrak{C}_d$ which are conjugate in $\mathfrak{G}$ are also conjugate in $\mathfrak{N}_d$.*

Proof is easy; we have only to notice that for $\mathfrak{P}$ there exists only one subgroup with a given order since $\mathfrak{P}$ is cyclic.

Lemma 4. $\mathfrak{N}_d = \mathfrak{N}\mathfrak{C}_d = \mathfrak{C}_d\mathfrak{N} , \quad \mathfrak{N} = \mathfrak{N}_a .$

Proof. If $G \in \mathfrak{N}_d$, then $G^{-1}\mathfrak{P}_d G = \mathfrak{P}_d$. Let us put $Q = G^{-1}PG$, where P is a generator of $\mathfrak{P}_d$. Then we have $P,\ Q \in \mathfrak{P}_d \leqq \mathfrak{C}$. Hence by Lemma 3 P and Q are conjugate in $\mathfrak{N}$, and there exists an element N of $\mathfrak{N}$ such that $P = N^{-1}QN$. This shows that $GN \in \mathfrak{C}_d$. Therefore $G \in \mathfrak{C}_d\mathfrak{N}$. Thus we have $\mathfrak{N}_d \leqq \mathfrak{C}_d\mathfrak{N} = \mathfrak{N}\mathfrak{C}_d$. Since it is clear that $\mathfrak{C}_d\mathfrak{N} \leqq \mathfrak{N}_d$, Lemma 4 is proved.

Lemma 5. $\mathfrak{N} \frown \mathfrak{C}_d = \mathfrak{C} , \quad \mathfrak{C} = \mathfrak{C}_a .$

Proof. It is sufficient to prove that $\mathfrak{N} \frown \mathfrak{C}_d \leqq \mathfrak{C}$ for $p > 2$, since in case $p = 2$ we have $\mathfrak{N} = \mathfrak{C}$ (and $\Gamma(\mathfrak{G}, K)$ is primary-decomposable). Let G be an element of $\mathfrak{N} \frown \mathfrak{C}_d$. If the order of G is divisible by p, G is expressed as a product of M and Q, where M, Q are both powers of G and the order of M is prime to p and that of Q is a power of p. The elements M and Q belong to $\mathfrak{N} \frown \mathfrak{C}_d$. Let P be a generator of $\mathfrak{P}$ and put

$$M^{-1}PM = P^r , \quad (r, p) = 1 .$$

If γ is a primitive root mod p^a and n is the smallest exponent for which M^n commutes with P, then $n \mid p-1$, since $M^{p-1} \in \mathfrak{C}$, and there exists an integer k such that

$$r \equiv \gamma^{k\frac{p^{a-1}(p-1)}{n}} \pmod{p^a} , \quad 0 < k < n , \quad (k, n) = 1 .$$

On the other hand, M belongs to $\mathfrak{C}_d$ and hence $M^{-1}P^{p^{a-d}}M = P^{p^{a-d}}$, that is, we have $p^{a-d} \equiv rp^{a-d} \pmod{p^a}$ and consequently $r \equiv 1 \pmod{p^d}$. Therefore

$$\gamma^{k\frac{p^{a-1}(p-1)}{n}} \equiv 1 \pmod{p^d} .$$

Since γ is also a primitive root mod $p^d (d \geqq 1)$ and $(k, n) = 1$, $n \mid p-1$, we have $n = 1$. This shows that M is an element of $\mathfrak{C}$.

The element Q belongs to $\mathfrak{P}$, since $\mathfrak{P}$ is a p-Sylow-subgroup of $\mathfrak{N}$ and is normal in $\mathfrak{N}$. Thus we have $G \varepsilon \mathfrak{C}$, which implies that $\mathfrak{N} \frown \mathfrak{C}_d \leqq \mathfrak{C}$.

Lemma 6. $\mathfrak{N}_d/\mathfrak{C}_d \cong \mathfrak{N}/\mathfrak{C}$.

Proof is obvious from Lemmas 4 and 5.

Lemma 7. *If $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$*[14]*, then we have*

$$\mathfrak{H}\mathfrak{C}_d \frown \mathfrak{N}_d = \mathfrak{C}_d , \quad \mathfrak{H}\mathfrak{C}_d \frown \mathfrak{N} = \mathfrak{C} .$$

Proof. Since it is easy to see that if $G = HN \in \mathfrak{H}\mathfrak{C}_d \frown \mathfrak{N}_d$, $H \in \mathfrak{H}$, $N \in \mathfrak{C}_d$ and $P \in \mathfrak{P}_d$, we have $GPG^{-1}P^{-1} \in \mathfrak{H} \frown \mathfrak{P}_d = E$, it follows that $\mathfrak{H}\mathfrak{C}_d \frown \mathfrak{N}_d = \mathfrak{C}_d$. The second part of the lemma now follows directly from Lemma 5.

Lemma 8. *Under the same assumption as in Lemma 7 we have $\mathfrak{H}\mathfrak{C}_d = \mathfrak{H}\mathfrak{P}$.*

Proof. Since $\mathfrak{H}\mathfrak{C}_d \geqq \mathfrak{H}\mathfrak{P}$, we have only to prove that $\mathfrak{H}\mathfrak{C}_d \leqq \mathfrak{H}\mathfrak{P}$. First we remark that $\mathfrak{H}\mathfrak{C} = \mathfrak{H}\mathfrak{P}$ and $\mathfrak{G} = \mathfrak{H}\mathfrak{N}$, since the centralizer or normalizer of $\mathfrak{H}\mathfrak{P}/\mathfrak{H}$ in $\mathfrak{G}/\mathfrak{H}$ is $\mathfrak{H}\mathfrak{C}/\mathfrak{H}$ or $\mathfrak{H}\mathfrak{N}/\mathfrak{H}$. Hence we see that $\mathfrak{G}/\mathfrak{H}\mathfrak{P} = \mathfrak{H}\mathfrak{N}/\mathfrak{H}\mathfrak{C} = (\mathfrak{H}\mathfrak{C})\mathfrak{N}/\mathfrak{H}\mathfrak{C} \cong \mathfrak{N}/\mathfrak{N} \frown \mathfrak{H}\mathfrak{C} = \mathfrak{N}/\mathfrak{C}$ by Lemma 7. On the other hand we have $\mathfrak{H}\mathfrak{N}_d \geqq \mathfrak{H}\mathfrak{N}$ and so $\mathfrak{H}\mathfrak{N}_d = \mathfrak{G}$. Furthermore by Lemmas 4 and 7 we see that $\mathfrak{G}/\mathfrak{H}\mathfrak{C}_d = (\mathfrak{H}\mathfrak{C}_d)\mathfrak{N}/\mathfrak{H}\mathfrak{C}_d \cong \mathfrak{N}/\mathfrak{C}$. Since $\mathfrak{H}\mathfrak{C}_d \geqq \mathfrak{H}\mathfrak{P}$, we have therefore $\mathfrak{H}\mathfrak{C}_d = \mathfrak{H}\mathfrak{P}$.

Returning to the general case we obtain

Lemma 9. *The totality of all the p-regular elements of $\mathfrak{C}_d$ forms a normal subgroup $\mathfrak{M}_d$ of $\mathfrak{N}_d$ and $\mathfrak{C}_d = \mathfrak{M}_d\mathfrak{P}$. Here we have $\mathfrak{M}_a \leqq \mathfrak{M}_{a-1} \leqq \cdots \leqq \mathfrak{M}_1$. In case $\mathfrak{H}\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$ we have further $\mathfrak{M}_1 \leqq \mathfrak{H}$.*

Proof is obvious by a theorem of Burnside[15].

For a group $\mathfrak{G}$ such that $\mathfrak{H}\mathfrak{P}$ is normal in $\mathfrak{G}$ we can put $\mathfrak{M}_0 = \mathfrak{H}$, $\mathfrak{C}_0 = \mathfrak{M}_0\mathfrak{P}$, $\mathfrak{N}_0 = \mathfrak{G}$ by Lemma 9. This shows the peculiarity of such a group among the groups with cyclic p-Sylow-subgroups.

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14) For the significance of the subgroug $\mathfrak{H}$ cf. the introduction.

15) Cf. A. SPEISER : loc. cit., Satz 120.

According to Lemma 9 and Theorem 8 we obtain

Theorem 10. *If a p-Sylow-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a cyclic group, then the normalizer of any subgroup of an order $p^d (d \geqq 1)$ satisfies the conditions* 1) *and* 2) *in Theorem 8. Its group ring possesses the radical expressible as a principal ideal.*

Finally we mention a theorem which follows readily from Theorem 9, Lemma 3 end a theorem of R. Brauer[16].

Theorem 11. *Let $\mathfrak{G}$ be a group of finite order which has a cyclic p-Sylow-subgroup. Then the number of p-blocks of a positive defect d is equal to the number of the classes $\mathfrak{R}_\nu$ of p-regular elements such that the order of the normalizer of any element of $\mathfrak{R}_\nu$ is divisible by p^d but not by p^{d+1}. The Cartan matrix corresponding to a block of defect d has a determinant p^d.*

16) R. Brauer: On the arithmetic in a group ring, Proc. Nat. Acad. Scad. Sci. U.S.A., **30** (1944), pp. 109-114.