A generalization of a theorem of C. Kuratowski concerning functional spaces.

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Let R be a separable metric space and I_{2n+1} a $(2n+1)$-dimensional cube in a $(2n+1)$-dimensional Euclidean space. Then the set of all continuous mappings of R into I_{2n+1} turns out to be a complete metric space I_{2n+1}^R, if we define a metric as usually: $\rho(f, g) = \sup_{x \in R} \rho(f(x), g(x))$, where we denote by ρ the metric in I_{2n+1}.

The purpose of the present note is to establish the following theorem.

Theorem. *Let $A_1, A_2, \ldots, A_m$ be a finite number of closed sets in a separable metric space R of dimension at most n. Then the set Φ of all continuous mappings f of R into I_{2n+1} such that*

$$\overline{f(A_1)}.\overline{f(A_2)}.\ \ldots.\ \overline{f(A_m)} = \overline{f(A_1 A_2 \ldots . A_m)}$$

is a dense G_δ-set in I_{2n+1}^R. In particular, in case $A_1 A_2 \ldots . A_m = 0$, Φ is open.

In the case where $m = 2$ and $A_1 A_2$ is a compact set, this theorem has already been proved by C. Kuratowski and applied to the problem of compactifications of regularly one-dimensional spaces[1]. It is to be noted that for a finite open covering $\mathfrak{U} = \{U_1, \ldots, U_m\}$ of R a continuous mapping f of R into I_{2n+1} is a $\mathfrak{U}$-mapping in the sense of W. Hurewicz and H. Wallman[2] if and only if $\overline{f(R-U_1)} \ldots . \overline{f(R-U_m)} = 0$, and that a mapping which is a $\mathfrak{U}_i$-mapping for every i is a homeomorphism if $\mathfrak{U}_1, \mathfrak{U}_2 \ldots$ form a basic sequence of coverings of R. Thus the theorem stated above seems indispensable for discussing the problem of compactifications of topological spaces by the method of functional spaces.

Now we proceed to the proof of the theorem. For this purpose it is sufficient to prove the theorem for the case $m = 2$. In this

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1) C. Kuratowski, Quelques théorèmes sur le plongement topologique des espaces, Fund. Math., 30 (1938), pp. 8-13.

2) W. Hurewicz and H. Wallman, Dimension theory, 1941, pp. 60–61.

case we write, for the sake of simplicity, A, B instead of A_1, A_2, and assume that $A\cdot B\neq 0$; for the case that $A\cdot B=0$ the proof may be carried out easily.

For any positive number ε let us denote by $\Phi(\varepsilon)$ the set of all continuous mappings f of R into I_{2n+1} such that

$$\overline{f(A)}\cdot\overline{f(B)}\subset S(f(A\cdot B),\ \varepsilon),$$

where $S(X,\varepsilon)$ means the set of all points x for which $\rho(x, X)<\varepsilon$. Then the theorem will be established by Baire's theorem if the following three lemmas are proved.

Lemma 1. $$\Phi=\prod_{i=1}^{\infty}\Phi(1/i).$$

Lemma 2. *For any positive number ε the set $\Phi(\varepsilon)$ is an open set in I_{2n+1}.*

Lemma 3. *For $\varepsilon>0$ the set $\Phi(\varepsilon)$ is dense in I_{2n+1}^R.*

Lemma 1 is proved easily; we have only to recall that $\overline{f(A\cdot B)}=\prod_{i=1}^{\infty}S(f(A\cdot B),\ 1/i)$.

Proof of Lemma 2. Let us assume that f belongs to $\Phi(\varepsilon)$. Then for any point x of $\overline{f(A)}\cdot\overline{f(B)}$ we have $\rho(x, f(A\cdot B))<\varepsilon$. Since $\overline{f(A)}\cdot\overline{f(B)}$ is compact, we have

$$0\leqq\alpha<\varepsilon,\quad \text{where}\quad \alpha=\sup_{x\in\overline{f(A)}\cdot\overline{f(B)}}\rho(x, f(AB)).$$

Take a number β such that $\alpha<\beta<\varepsilon$. Then we have for some positive integer m

$$\overline{S(f(A),\ 1/m)}\cdot\overline{S(f(B),\ 1/m)}\subset S(f(AB),\ \beta),$$

since I_{2n+1} is compact and

$$\overline{f(A)}\cdot\overline{f(B)}=\prod_{i=1}^{\infty}[\overline{S(f(A),\ 1/i)}\cdot\overline{S(f(B),\ 1/i)}]\subset S(f(A\cdot B),\ \beta).$$

Let δ be a positive number such that $\delta<\dfrac{1}{m}$, $\delta<\varepsilon-\beta$.

If $\rho(f, g)<\delta$ for $g\in I_{2n+1}^R$, we have

$$\overline{g(A)}\cdot\overline{g(B)}\subset\overline{S(f(A),\ \delta)}\cdot\overline{S(f(B)),\ \delta)}\subset\overline{S(f(A),\ 1/m)}\cdot\overline{S(f(B),\ 1/m)}$$
$$\subset S(f(A\cdot B),\ \beta)\subset S(g(A\cdot B),\ \beta+\delta)\subset S(g(A\cdot B),\ \varepsilon),$$

and hence $g \in \Phi(\varepsilon)$. This proves that $\Phi(\varepsilon)$ is open.

Proof of Lemma 3. Let f be an arbitrary continuous mapping of R into I_{2n+1} and let us construct open sets $H_1, \ldots, H_m$ such that

$$I_{2n+1} = H_1 + H_2 + \cdots + H_m; \quad \delta(H_i) < \varepsilon/2, \ i = 1, \ldots, m.$$

By the hypothesis that $dim\ R \leq n$ there is a closed covering $\{F_1, \ldots, F_m\}$ of R such that $F_i \subset f^{-1}(H_i)$, $i = 1, \ldots, m$ and the order of $\{F_1, \ldots, F_m\}$ does not exceed $n+1$. Moreover we can construct open sets $L_1, \ldots, L_m$ such that

(1) $$F_i \subset L_i, \ \bar{L}_i \subset f^{-1}(H_i), \ i = 1, \ldots, m;$$

(2) $\{\bar{L}_1, \ldots, \bar{L}_m, A, B\}$ is similar to $\{F_1, \ldots, F_m, A, B\}$.

For a set of indices $\{i_1, \ldots, i_k\}$ satisfying the condition

(3) $$L_{i_1} \ldots L_{i_k} \cdot A \neq 0, \ L_{i_1} \ldots L_{i_k} \cdot B \neq 0, \ L_{i_1} \ldots L_{i_k} \cdot A \cdot B = 0,$$

we can construct an open set $L(i_1, \ldots, i_k)$ such that

(4) $$A \cdot B \subset L(i_1, \ldots, i_k),$$

(5) $$L(i_1, \ldots, i_k) \cdot [\bar{L}_{i_1} \ldots \bar{L}_{i_k}] = 0.$$

This is possible, since the condition (3) implies $F_{i_1} \ldots F_{i_k} A \cdot B = 0$ and hence we have $\bar{L}_{i_1} \ldots \bar{L}_{i_k} A \cdot B = 0$ by (2).

Denote by L_0 the intersection of all $L(i_1, \ldots, i_k)$ for which $\{i_1, \ldots, i_k\}$ satisfies the condition (3), and put

$$G_i = L_i \cdot L_0, \quad i = 1, 2, \ldots, m.$$

Then the relations

(6) $$\bar{G}_{i_1} \ldots \bar{G}_{i_k} \cdot A \neq 0, \quad \bar{G}_{i_1} \ldots \bar{G}_{i_k} \cdot B \neq 0$$

imply the relation

(7) $$G_{i_1} \ldots G_{i_k} \cdot A \cdot B \neq 0.$$

Because, if $G_{i_1} \ldots G_{i_k} A \cdot B = 0$, we have $L_{i_1} \ldots L_{i_k} A \cdot B = 0$, since $A \cdot B \subset L_0$, and hence $\{i_1, \ldots, i_k\}$ satisfies the condition (3), and consequently we have $\bar{G}_{i_1} \ldots \bar{G}_{i_k} \subset \bar{L}_0 \cdot \bar{L}_{i_1} \ldots \bar{L}_{i_k} = 0$ by (5), since $L_0 \subset L(i_1, \ldots, i_k)$; this contradicts (6).

Now let us assume that

$A \cdot B \cdot G_i \neq 0, \ i = 1, 2, \ldots, r; \ A \cdot B \cdot G_i = 0, \ i = r+1, \ldots, m.$

Then there exist open sets $U_1, \ldots, U_r$ such that

(8) $\bar{G}_i \subset U_i, \quad \bar{U}_i \subset f^{-1}(H_i), \quad i = 1, \ldots, r,$

(9) $\{\bar{U}_1, \ldots, \bar{U}_r, A, B\}$ is similar to $\{\bar{G}_1, \ldots, \bar{G}_r, A, B\}$.

Since the order of $\{\bar{G}_1, \ldots, \bar{G}_r\}$ is not greater than $n+1$ and $dim\ R \leqq n$, we can construct an open covering $\{X_i, Y_j, Z_j, i = 1, 2, \ldots, r; j = 1, 2, \ldots, m\}$ such that

(10) $G_i \subset X_i \subset U_i, \quad i = 1, 2, \ldots, r,$

(11) $Y_j \subset (R-B) f^{-1}(H_j), \ Z_j \subset (R-A) f^{-1}(H_j), \ j = 1, 2, \ldots, m,$

(12) the order of $\{X_1, \ldots, X_r, Y_1, \ldots, Y_m, Z_1, \ldots, Z_m\} \leqq n+1,$

according to an addition theorem in dimension theory.[3)]

Then the relations

(13) $X_{i_1} \ldots X_{i_k} \cdot A \neq 0, \ X_{i_1} \ldots X_{i_k} \cdot B \neq 0$

imply the relation

(14) $X_{i_1} \ldots X_{i_k} \cdot A \cdot B \neq 0.$

Because $\bar{X}_{i_1} \ldots \bar{X}_{i_k}$ intersects both A and B, and hence $\bar{U}_{i_1} \ldots \bar{U}_{i_k}$ intersects A and B, and consequently $\bar{G}_{i_1} \ldots \bar{G}_{i_k}$ intersects A and B (by (9)), that is, $\bar{G}_{i_1} \ldots \bar{G}_{i_k}$ satisfies the condition (6); this proves (14) by (7) and (10).

Since the diameters of the sets $f(X_i), f(Y_j), f(Z_j)$ are all less than $\varepsilon/2$, we can find points x_i, y_j, z_j of I_{2n+1} such that x_i, y_j, z_j ($i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, m$) lie in general position and the diameters of the sets $x_i + f(X_i)$, $y_j + f(Y_j)$, $z_j + f(Z_j)$ are all less than $\varepsilon/2$. We shall define a barycentric mapping g of R into I_{2n+1} as follows: Assign to each of the points x_i the weight $\rho(p, R-X_i)$ and to each of the points y_j the weight $\rho(p, R-Y_j)$ and to each of the points z_j the weight $\rho(p, R-Z_j)$, and denote by $g(p)$ the center of gravity of the system of points x_i, y_j, z_j ($i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, m$) with

3) K. Morita, On the dimension of normal spaces, forthcoming in Jap. Jour. Math. Cf. also. On the sum theorem in dimension theory, Sugaku, Vol. 1 No. 3 (1948) (in Japanese).

these weights. Here ρ means a metric in R. Then it is easy to see that

(15) $$g \in I^R_{2n+1}, \quad \rho(f, g) < \varepsilon .$$

We shall show that

(16) $$g \in \Phi(\varepsilon).$$

Denote by $P_{A \cdot B}$ the sum of simplexes $[x_{i_1}, \ldots, x_{i_k}]$ such that $X_{i_1} \ldots X_{i_k} \cdot A \cdot B \neq 0$, and denote by P_A the sum of simplexes $[x_{i_1}, \ldots, x_{i_k}, y_{i_{k+1}}, \ldots, y_{i_l}]$ such that $X_{i_1} \ldots X_{i_k} \cdot Y_{i_{k+1}} \ldots Y_{i_l} \cdot A \neq 0$, and by P_B the sum of simplexes $[x_{i_1}, \ldots, x_{i_k}, z_{i_{k+1}}, \ldots, z_{i_l}]$ such that $X_{i_1} \ldots X_{i_k} \cdot Z_{i_{k+1}} \ldots Z_{i_l} \cdot B \neq 0$. Then these $P_A, P_B, P_{A \cdot B}$ are all polytopes and it is seen that

(17) $$g(A) \subset P_A, \quad g(B) \subset P_B, \quad g(A \cdot B) \subset P_{A \cdot B} .$$

We shall now prove that

(18) $$P_A P_B = P_{A \cdot B}.$$

It is evident that $P_{A \cdot B} \subset P_A \cdot P_B$, and so we have only to prove that $P_A \cdot P_B \subset P_{A \cdot B}$. From (11) we see that a simplex belonging to $P_A \cdot P_B$ must be of the form $[x_{i_1}, \ldots, x_{i_k}]$. Then we have $X_{i_1} \ldots X_{i_k} \cdot A \neq 0$, $X_{i_1} \ldots X_{i_k} \cdot B \neq 0$ and hence by (14) $X_{i_1} \ldots X_{i_k} A \cdot B \neq 0$. This proves that $[x_{i_1}, \ldots, x_{i_k}] \subset P_{A \cdot B}$. Therefore $P_A \cdot P_B \subset P_{A \cdot B}$. Thus the equality (18) is verified.

Let $[x_{i_1}, \ldots, x_{i_k}]$ be a "Grundsimplex" of (the complex associated with) P_{AB}. For any point p of $X_{i_1} \ldots X_{i_k} \cdot A \cdot B$ we have $\rho(f(p), x_{i_\nu}) < \varepsilon/2$, $\nu = 1, 2, \ldots, k$ and hence the diameter of the simplex $[x_{i_1}, \ldots, x_{i_k}]$ is less than ε. On the other hand, $g(p)$ belongs to $[x_{i_1}, \ldots, x_{i_k}]$. Hence we have $[x_{i_1}, \ldots, x_{i_k}] \subset S(g(p), \varepsilon)$ and therefore

(19) $$P_{A \cdot B} \subset S(g(A \cdot B), \varepsilon).$$

From (17), (18), (19) it follows that

$$\overline{g(A)} \cdot \overline{g(B)} \subset P_A \cdot P_B = P_{A \cdot B} \subset S(g(A \cdot B), \varepsilon),$$

that is, $g \in \Phi(\varepsilon)$.

From (15), (16) it is seen that $\Phi(\varepsilon)$ is dense in I^R_{2n+1}. This completes the proof of Lemma 3.