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Lagrangian cobordism functor in microlocal sheaf theory. I. (English) Zbl 07738257

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This paper constructs a Lagrangian cobordism functor between microlocal sheaf categories of compact objects, and its right adjoint functor between microlocal sheaf categories of proper objects, using the result of *D. Nadler* and *V. Shende* ["Sheaf quantization in Weinstein symplectic manifolds", Preprint, arXiv:2007.10154], which is independent of Floer theory and symplectic field theory. The main result goes as follows.

Theorem. Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds and $L \subset \partial_\infty X \times \mathbb{R}$ an exact Lagrangian cobordism from Λ_- to Λ_+. There is a cobordism functor between the microlocal sheaf categories of compact objects

$$\Phi_L^* : \mu\mathrm{Sh}^c_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}) \to \mu\mathrm{Sh}^c_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \otimes_{\mathrm{Loc}^c(\Lambda_-)} \mathrm{Loc}^c(L)$$

and a fully faithful adjoint functor between microlocal sheaf categories of proper objects

$$\Phi_L : \mu\mathrm{Sh}^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{\mathrm{Loc}^b(\Lambda_-)} \mathrm{Loc}^b(L) \to \mu\mathrm{Sh}^b_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R})$$

such that concatenations of cobordisms give rise to compositions of cobordism functors.

In particular, when $X = T^*M$, there is a cobordism functor between compact sheaves

$$\Phi_L^* : \mathrm{Sh}^c_{\Lambda_+}(M) \to \mathrm{Sh}^c_{\Lambda_-,}(M) \otimes_{\mathrm{Loc}^c(\Lambda_-)} \mathrm{Loc}^c(L)$$

and a fully faithful adjoint functor between proper sheaves

$$\Phi_L : \mathrm{Sh}^b_{\Lambda_-}(M) \times_{\mathrm{Loc}^b(\Lambda_-)} \mathrm{Loc}^b(L) \to \mathrm{Sh}^b_{\Lambda_+}(M)$$

The author compares his approach with *S. Guillermou* et al. [Duke Math. J. 161, No. 2, 201–245 (2012; Zbl 1242.53108)] and *X. Jin* and *D. Treumann* ["Brane structures in microlocal sheaf theory", Preprint, arXiv:1704.04291].

In the past few years, *D. Treumann* and *E. Zaslow* [Adv. Theor. Math. Phys. 22, No. 5, 1289–1345 (2018; Zbl 07430949)] and *R. Casals* and *E. Zaslow* [Geom. Topol. 26, No. 8, 3589–3745 (2022; Zbl 07678845)] have developed systematic approaches to compute the number of microlocal rank 1 sheaves over $\mathbb{F}_q$ for certain Legendrian surfaces using flag moduli. Combining with the author's fully faithful cobordism functor on proper sheaves, he gets new obstructions to Lagrangian cobordisms for these Legendrian surfaces.

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MSC:

53D37 Symplectic aspects of mirror symmetry, homological mirror symmetry, and Fukaya category
14J33 Mirror symmetry (algebro-geometric aspects)
35A27 Microlocal methods and methods of sheaf theory and homological algebra applied to PDEs

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