Jun, Jaiung; **Szczesny, Matt**; **Tolliver, Jeffrey**

Proto-exact categories of modules over semirings and hyperrings. (English) Zbl 07710314

J. Algebra 631, 517-557 (2023)

Proto-exact categories were introduced by *T. Dyckerhoff* and *M. Kapranov* [Higher Segal spaces. Cham: Springer (2019; Zbl 1459.18001)] as a generalization of Quillen exact categories [*D. Quillen*, Lect. Notes Math. None, 85–147 (1973; Zbl 0292.18004)], providing a flexible framework for exact sequencces in non-additive categories. Several interesting combinatorial categories, such as the category of matroids [*C. Eppolito* et al., Math. Z. 296, No. 1–2, 147–167 (2020; Zbl 1442.18017)] and the category of representations over a quiver over $\mathbb{F}_1$ [*M. Szczesny*, Int. Math. Res. Not. 2012, No. 10, 2377–2404 (2012; Zbl 1288.14012); *J. Jun* and *A. Sistko*, "Coefficient quivers, $\mathbb{F}_1$-representations, and Euler characteristics of quiver Grassmannians", Preprint, arXiv:2112.06291; Algebr. Represent. Theory 26, No. 1, 207–240 (2023; Zbl 07659756)], are equipped with a proto-exact structure.

The principal objective in this paper is to enlarge the catalogue of non-additive proto-exact categories, showing that the categories of modules over semirings as well as hyperrings are so. Modules over an idempotent semiring are closely related to matroid theory [*J. Giansiracusa* and *N. Giansiracusa*, Manuscr. Math. 156, No. 1–2, 187–213 (2018; Zbl 1384.05063)] and modules over a hyperring have an interesting connection to finite incidence geometries [*A. Connes* and *C. Consani*, in: Casimir force, Casimir operators and Riemann hypothesis. Mathematics for innovation in industry and science. Proceedings of the conference, Fukuoka, Japan, November 9–13, 2009. Berlin: de Gruyter. 147–198 (2010; Zbl 1234.14002); *J. Jun*, Commun. Algebra 46, No. 3, 942–960 (2018; Zbl 1419.20008)] and matroids [*M. Baker* and *N. Bowler*, Adv. Math. 343, 821–863 (2019; Zbl 1404.05022)]. The paper provides the following results.

- The category Mod_H of modules over a semiring H is a proto-exact category (Theorem 3.14).
- The category $\mathcal{L}$ of algebraic lattices is a proto-exact category (Theorem 4.15).
- The category Mod_R of modules over a hyperring R is a proto-exact category (Theorem 5.11).

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18E13 Protomodular categories, semi-abelian categories, Mal'tsev categories
05B35 Combinatorial aspects of matroids and geometric lattices
06B05 Structure theory of lattices
16Y20 Hyperrings
16Y60 Semirings

Keywords:

proto-exact category; semiring; hyperring; lattice; saturated module over a semiring; algebraic lattice; geometric lattice; incidence geometry

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