Berezin-Toeplitz quantization of vector bundles over Kähler manifolds

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1 Introduction

One of the approaches of the quantum theory of gravity or the theory of everything is string theory. String theory is a theory in which the fundamental constituents of the universe are one-dimensional strings rather than zero-dimensional point particles. This simple modification makes a huge difference from theories of fundamental point particles. The spectrum of a single first-quantized string contains various different types of states with particular masses and spins. More surprisingly, there is a particular state corresponding to the graviton, which is a quantum of gravitational interaction, and the string theory is naturally incorporate the quantum theory of gravity.

Various studies of superstring theory and M-theory [1–3] suggests that the noncommutative geometry may play an essential role in the description of spacetime in Planck scale. In the Einstein's classical theory of gravity, we assume that the spacetime is a smooth manifold, which means that the spacetime coordinates are a set of real numbers. On the contrary, in the noncommutative geometry, we assume that the spacetime coordinates are noncommutative operators on a suitable Hilbert space. When the Hilbert space is finite-dimensional, the spacetime coordinates are finite dimensional square matrices. We call this kind of noncommutative geometry as fuzzy geometry or matrix geometry and it is deeply related to matrix models of superstring theory and M-theory.

In the study of the fuzzy geometry, a theory called the matrix regularization [4] plays an essential role to uncover the relationship between the commutative geometry and the fuzzy geometry. The matrix regularization is a map from functions on a manifold to corresponding matrices on a fuzzy geometry. Using this map, one can construct a matrix model of superstring theory (or M-theory) from a world-sheet theory of a single string (or world-volume theory of a single membrane). Therefore, this theory is important to study the relationship between string (or a membrane) states to the corresponding matrix states. Let us briefly discuss a mathematical aspect of matrix regularization. Let us consider a symplectic manifold (M, ω), which is an even-dimensional manifold equipped with a nondegenerate closed two-form ω. Let $2n$ be the dimension of M. From the symplectic structure ω, one can naturally define a volume form $\mu_\omega := \omega^{\wedge n}/n!$ and the Poisson bracket $\{f, g\} = X_f g$. Here, f, g are smooth functions on M and X_f is the Hamiltonian vector field. In this sense, a symplectic manifold is a mathematical generalization of the phase space. Now, let us also assume M is closed, i.e. M is a compact manifold without boundary. Then, the matrix regularization is defined as a sequence of linear maps $T_p : \mathcal{C}^\infty(M, \mathbb{C}) \to M_{N_p}(\mathbb{C})$ satisfying the following axioms [5]:

$$\lim_{p\to\infty} |T_p(f)T_p(g) - T_p(fg)| = 0, \tag{1.1}$$

$$\lim_{p\to\infty} |(\mathrm{i}\hbar_p)^{-1}[T_p(f), T_p(g)] - T_p(\{f,g\})| = 0, \tag{1.2}$$

$$\lim_{p\to\infty} (2\pi\hbar_p)^n \mathrm{Tr}\, T_p(f) = \int_M \mu_\omega f. \tag{1.3}$$

Here, p is an integer, $\mathcal{C}^\infty(M,\mathbb{C})$ is the set of smooth complex functions, $\{N_p\}$ is a sequence of strictly increasing integers, $\hbar_p = (2\pi p)^{-1}$, and $|\cdot|$ is a matrix norm. In this sense, the matrix regularization is an analog of the quantization of a classical phase space. From these relations, one can derive the matrix models from the worldvolume actions of a membrane or a string [4].

For a Kähler manifold M, the existence of the map T_p satisfying (1.1), (1.3) and (1.3) is known and one of the construction of such maps is known as a Berezin-Toeplitz quantization [6]. A Kähler manifold is a special type of symplectic manifold equipped with an integrable complex structure and Riemannian structures with some compatibility condition. From the integral complex structure, one can define a finite dimensional Hilbert space $\mathcal{H}$ as a space of square integrable holomorphic sections of some line bundle $L^{\otimes p}$. Let Π be the orthogonal projection from the space of all sections of $L^{\otimes p}$ to $\mathcal{H}$. Then, the Toeplitz operator of f defined by $T_p(f) := \Pi f \Pi$ is shown to satisfy all the properties of the matrix regularization. For a general symplectic manifold, where the complex structure is not necessarily integrable, there is a method called spin^c Berezin-Toeplitz quantization [7,8]. From an almost complex structure, one can construct suitable spinor fields and corresponding Dirac operator with finite dimensional kernel. Then, the spin^c Toeplitz operator defined by $T_p(f) := \Pi f \Pi$, where Π is the projection from a space of suitable spinor fields to the kernel of the Dirac operator, is shown to satisfy all the properties of the matrix regularization.

In this dissertation, we summarize a series of studies of the spin^c Berezin-Toeplitz quantization of vector bundles over a general closed Kähler manifold [9,10] (similar problems are also considered in [11, 12]). We define the spin^c Toeplitz operator of a section of a Hermitian vector bundle and derive a large-p asymptotic expansion of the product of two Toeplitz operators $T_p(s)T_p(t)$ for any sections s, t up to the first order in $\hbar_p$. From this expansion, we obtain some of the important relations of the Toeplitz operators, including a generalization of (1.1), (1.3) and (1.3). In our framework, we give explicit two examples of monopole bundles and tensor bundles. As a first example, we study the matrix regularization of monopole bundles over a complex projective space $\mathbb{CP}^n$ [10] and one-dimensional complex torus $\mathbb{T}^2_\tau$ [13]. This study is important to describe membranes (or strings) with monopole charges. As a second example, we study the matrix regularization of tensor bundles [14]. This study is important to describe various tensor fields over a fuzzy manifold, which are essential for gauge theories and gravitational theories.

The organization of this dissertation is as follows. In section 2, we review a few essential mathematical notions which are necessary to define the spin^c Berezin-Toeplitz quantization of vector bundles. In section 3, we study the Berezin-Toeplitz quantization for general vector bundles and derive important properties from the asymptotic expansion. In section 4 and 5, we consider Berezin-Toeplitz quantization of monopole bundles over $\mathbb{CP}^n$ and $\mathbb{T}^2_\tau$, respectively. In section 6, we consider Berezin-Toeplitz quantization of tensor bundles over a Kähler manifold and give a simple example of the Berezin-Toeplitz quantization of vector fields on the square torus $\mathbb{T}^2_{\tau=\mathrm{i}}$. In section 7, we give a summary of this dissertation and discuss the future problems. In Appendix A, we summarize some of the notations and definitions of basic mathematical terms, which are extensively used in this dissertation. In Appendix B, we review the M-theory [15] and its connection to the BFSS matrix model [1, 16]. In Appendix C, we derive the BFSS matrix model using the matrix regularization of a single M2-brane [4].

2 Mathematical Preliminaries

In this section, we summarize basic mathematical techniques used for the studies of spin^c Berezin-Toeplitz quantization.

2.1 Symplectic Geometry

In this subsection, we review the basic properties of the symplectic geometry. A comprehensive reference of this topic can be found for example in [17].

It is well-known that the classical theory in Hamiltonian formalism is naturally described in the language of symplectic geometry. As we will see below, the symplectic manifold is thought of as a generalization of the phase space of the classical system. Thus, we can expect that the notion of quantization is defined for a general symplectic manifold. As we will see later, the Berezin-Toeplitz quantization, which is the main topic of this dissertation, is a quantization of a compact symplectic manifold.

Let us first give a definition of a symplectic manifold.

Definition 1. Let M be an even-dimensional differentiable manifold. A symplectic structure or a symplectic form ω on M is defined as a closed nondegenerate differential two-form:

$$\mathrm{d}\omega = 0, \qquad {}^\forall u \in TM_x : \ \omega_x(u,v) = 0 \quad \Rightarrow \quad v = 0 \in TM_x,$$

at each point $x \in M$. Here, d is the exterior derivative, TM is the tangent bundle of M and TM_x is the tangent vector space at point $x \in M$. The pair (M, ω) is then called a symplectic manifold.

The symplectic structure can be defined only for even-dimensional manifolds because of the following logic. Let A be a skew-symmetric matrix with size d. Then, we have $\det A = \det A^{\mathrm{T}} = \det(-A) = (-1)^d \det A$. If d is odd, then $\det A$ is automatically zero and thus A should be degenerate. Since ω is closed, one can locally write it as $\omega = \mathrm{d}\theta$ for some one-form θ called the symplectic potential. Now, let us give several important examples of the symplectic manifolds.

Example 2.1. Let us consider the even-dimensional Euclidean space $\mathbb{R}^{2n}$ with the standard coordinate system $\{q^1, q^2, \cdots q^n, p^1, p^2, \cdots, p^n\}$. One can prove that the two-form

$$\omega := \sum_{i=1}^{n} \mathrm{d}p^i \wedge \mathrm{d}q^i$$

defines a symplectic structure on $\mathbb{R}^{2n}$. Here, $\wedge$ is the exterior product of forms. In classical mechanics, a system of a particle moving in an n-dimensional Euclidean space $\mathbb{R}^n$ is described by the phase space $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, whose coordinates $\{q^1, q^2, \cdots q^n, p^1, p^2, \cdots, p^n\}$ specify the particle's position and its conjugate momentum.

Example 2.2. Let X be an n-dimensional differentiable manifold. The cotangent bundle T^*X is a $2n$-dimensional noncompact differentiable manifold. Let $\{q^1, q^2, \cdots, q^n\}$ be a local coordinate system of a patch $U \subset X$. Any element $\phi \in T^*M_x$ can be uniquely written by $\phi = \sum_{i=1}^n p_i \mathrm{d}q^i$ for some real numbers $\{p^1, p^2, \cdots, p^n\}$. Then, one can define a coordinate system of $\pi^{-1}(U) \subset T^*X$ by $\{q^1, q^2, \cdots q^n, p^1, p^2, \cdots, p^n\}$. This coordinate system is called the standard coordinate system of T^*X. On the cotangent bundle T^*X, there exists a one-form θ, locally written by

$$\theta = \sum_{i=1}^n p^i \mathrm{d}q^i,$$

in the standard coordinates. Without using the local coordinates, θ is defined as $\theta_\alpha(V) := \alpha(\pi_*(V))$, where $\alpha \in T^*X$, $V \in T(T^*X)_\alpha$ and $\pi_* : T(T^*X) \to TX$ is a projection derived from the bundle projection $\pi : T^*X \to X$. The one-form θ is called the canonical one-form on the cotangent bundle T^*X. Then, one can define the canonical two-form

$$\omega := \mathrm{d}\theta = \sum_{i=1}^n \mathrm{d}p^i \wedge \mathrm{d}q^i,$$

which induces a symplectic structure on T^*X. In classical mechanics, a system of a particle moving in X is described by a point in the phase space T^*X. This means that the phase space in general has a natural symplectic structure.

In these two examples, the manifolds we considered are noncompact. In the theory of matrix regularization, we are mainly interested in compact symplectic manifolds. Some examples of the compact symplectic manifolds are the complex projective space $\mathbb{CP}^n$ and the one-dimensional torus $\mathbb{T}^2_\tau$. These manifolds will be considered in section 4 and 5 and we will not deal these manifolds in this section.

For a general $2n$-dimensional symplectic manifold (M, ω), there exists a useful local coordinate system called the canonical coordinate system $\{q^i, p^i\}_{i=1}^n$ such that the symplectic form takes the following form:

$$\omega = \sum_{i=1}^n \mathrm{d}p^i \wedge \mathrm{d}q^i.$$

This is a consequence of the Darboux theorem and the canonical coordinate system is sometimes called the Darboux coordinate system.

Now, let us consider some of the important properties of the general symplectic manifolds. From the nondegeneracy of ω, there is a linear isomorphism from the tangent vector space TM_x to the cotangent vector space T^*M_x by $v \mapsto \iota_v\omega$. Here, ι_u is the interior product with a tangent vector $u \in TM_x$ defined by

$$^\forall v_i \in TM_x : \ (\iota_u\alpha)(v_1, v_2, \cdots, v_{p-1}) = \alpha(u, v_1, v_2, \cdots, v_{p-1}), \tag{2.1}$$

for a p-form α. In this sense, the symplectic structure can be thought of as a skew-symmetric analog of the Riemannian structure. Using this isomorphism, one can define a class of special tangent vector fields which play an essential role in the symplectic geometry.

Definition 2. Let (M, ω) be a symplectic manifold and $\mathcal{C}^\infty(M, \mathbb{R})$ be a set of smooth real functions on M. For $f \in \mathcal{C}^\infty(M, \mathbb{R})$, the Hamiltonian vector field of f denoted by X_f is uniquely defined by the relation

$$\iota_{X_f}\omega = \mathrm{d}f. \tag{2.2}$$

Let us discuss the basic properties of the Hamiltonian vector fields. In the canonical coordinate system $\{q^i, p^i\}$, one can locally write

$$X_f = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial}{\partial q^i} \right).$$

By acting X_f on a smooth function g, we have a local expression

$$X_f g = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \right),$$

which is the familiar Poisson bracket appearing in the classical mechanics. Thus, it is natural to consider the following structure in the coordinate free manner.

Definition 3. Let (M, ω) be a symplectic manifold. We define the Poisson bracket $\{\cdot, \cdot\}$: $\mathcal{C}^\infty(M, \mathbb{R}) \times \mathcal{C}^\infty(M, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})$ by

$$\{f, g\} := X_f g, \tag{2.3}$$

for $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$.

Proposition 2.3. *The Poisson bracket defined above satisfies the following set of properties:*

Skew-symmetry

$$\{f, g\} = -\{g, f\},$$

Bilinear

$$\{f, ag + bh\} = a\{f, g\} + b\{f, h\},$$

Leibniz's rule

$$\{f, gh\} = g\{f, h\} + \{f, g\}h,$$

Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

for any $f, g, h \in \mathcal{C}^\infty(M, \mathbb{R})$ *and* $a, b \in \mathbb{C}$.

Proof. The first property immediately follows from $\{f, g\} = X_f g = \iota_{X_f} \mathrm{d}g = \iota_{X_f}\iota_{X_g}\omega = -\omega(X_f, X_g)$. The second and the third properties are also easily derived from the linarity and Leibniz rule of vector fields. The Jacobi identity is actually equivalent to the condition $\mathrm{d}\omega = 0$. Let us use the identity

$$\mathrm{d}\alpha(u, v, w) = u\alpha(v, w) + v\alpha(w, u) + w\alpha(u, v) - \alpha([u, v], w) - \alpha([v, w], u) - \alpha([w, u], v), \quad (2.4)$$

which holds for any vectors u, v, w and any two-form α. This can be obtained from the Cartan homotopy formula,

$$\mathcal{L}_u = \mathrm{d}\iota_u + \iota_u \mathrm{d}, \quad (2.5)$$

and other identities

$$\iota_u\iota_v + \iota_v\iota_u = 0, \quad \iota_{[u,v]} = \mathcal{L}_u\iota_v - \iota_v\mathcal{L}_u.$$

Here, $\mathcal{L}_u$ is the Lie derivative. Then, we have

$$\begin{aligned} \mathrm{d}\omega(X_f, X_g, X_h) &= X_f\omega(X_g, X_h) + X_g\omega(X_h, X_f) + X_h\omega(X_f, X_g) \\ &\quad - \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g) \\ &= -\{f, \{g, h\}\} - \{g, \{h, f\}\} - \{h, \{f, g\}\} + [X_f, X_g]h + [X_g, X_h]f + [X_h, X_f]g. \end{aligned}$$

Here,

$$\begin{aligned} &[X_f, X_g]h + [X_g, X_h]f + [X_h, X_f]g \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{g, \{h, f\}\} - \{h, \{g, f\}\} + \{h, \{f, g\}\} - \{f, \{h, g\}\} \\ &= 2(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}). \end{aligned}$$

Hence, we obtain

$$\mathrm{d}\omega(X_f, X_g, X_h) = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\},$$

for any f, g, h and the Jacobi identity is satisfied if and only if ω is closed. □

This shows that $(\mathcal{C}^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ form a Lie algebra called the Poisson algebra. Note that the Jacobi identity implies

$$[X_f, X_g] = X_{\{f,g\}}. \quad (2.6)$$

This shows that the map $f \mapsto X_f$ induces a Lie algebra homomorphism.

Let us comment two other important properties of the Hamiltonian vector fields.

First, from (2.5), it is easy to see that the Hamiltonian vector fields preserve the symplectic structure:

$$\mathcal{L}_{X_f}\omega = \mathrm{d}\iota_{X_f}\omega + \iota_{X_f}\mathrm{d}\omega = \mathrm{d}^2 f = 0.$$

This implies that the Hamiltonian vector fields generates a symplectomorphism, which is a diffeomorphism preserving the symplectic structure. In general, the vector fields preserving ω are locally written by the Hamiltonian vector fields. To see this, let u be a vector field and impose the condition $\mathcal{L}_u\omega = 0$. Then, from (2.5), this condition is equivalent to $\iota_u\omega$ being closed. Since any closed form is locally exact, there exist a local function f such that $u = X_f$. In particular, in the case where the manifold M is simply connected, any closed form is globally exact, and therefore any vector fields preserving ω are given by the Hamiltonian vector fields.

Secondly, let us consider the Hamilton formalism of classical mechanics. A particle's state at a specific time is specified by its position and momentum. In general, a particle's state changes as time elapses. The motion of a particle is thus specified by a curve on a symplectic manifold M, where time play as a parameter of the curve. Let us consider the simplest case of a phase space $(\mathrm{R}^{2n}, \omega = \sum_{i=1}^n \mathrm{d}p^i \wedge \mathrm{d}q^i)$. The Hamiltonian mechanics, there exists a special function H on R^{2n} called the Hamiltonian (We here assume that the Hamiltonian is time independent.). Then, the particle's trajectory in the phase space $\{q^i(t), p^i(t)\}$ is characterized by the Hamilton's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}q^i(t) = \frac{\partial H}{\partial p^i}(q^i(t), p^i(t)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p^i(t) = -\frac{\partial H}{\partial q^i}(q^i(t), p^i(t)).$$

More elegantly, for a smooth function $f(q^i, p^i)$, which corresponds to some physical observables, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(q^i(t), p^i(t)) = -\{H, f\}(q^i(t), p^i(t)) = -X_H f(q^i(t), p^i(t)).$$

This shows that the particle's trajectory is the flow generated Hamiltonian vector field of the Hamiltonian.

Finally, there is a natural volume form defined on a symplectic manifold.

Definition 4. Let (M, ω) be a $2n$-dimensional symplectic manifold. The Liouville volume form μ_ω is defined by

$$\mu_\omega = \frac{\omega^{\wedge n}}{n!}, \tag{2.7}$$

where $\omega^{\wedge n}$ denotes n-fold exterior product.

The Liouville form is a volume form on M, meaning that μ_ω is a nowhere vanishing $2n$-form, due to the nondegeneracy of ω. In the canonical coordinate system $\{q^i, p^i\}_{i=1}^n$, the Liouville form is written as

$$\mu_\omega = \mathrm{d}q^1 \wedge \mathrm{d}p^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}p^2 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p^n,$$

which is evidently nowhere vanishing. From the definition of the Liouville form, we can see that the Hamiltonian vector field X_f preserves the Liouville volume form $\mathcal{L}_{X_f}\mu_\omega = 0$ known as the Liouville theorem. The volume form induces a measure on the space of functions on M, which allows the integration of functions.

2.2 Prequantum line bundle

In this subsection, we introduce a line bundle called the prequantum line bundle.

The prequantum line bundle plays an important role in various schemes of quantization of symplectic manifold, such as the geometric quantization [18], and the Berezin-Toeplitz quantization [6, 7].

Definition 5. Let M be a differentiable manifold and L be a complex Hermitian line bundle over M. Let ∇^L be a Hermitian connection with respect to the Hermitian inner product. L is called the prequantum line bundle if the first Chern class of L denoted as $c_1(L)$ is a symplectic form on M. Here, the first Chern class of L is defined by

$$c_1(L) = \frac{\mathrm{i}}{2\pi} R^L,$$

where $R^L := (\nabla^L)^2$ is the curvature of L and i is the imaginary unit.

Note that the first Chern class is in the second integer cohomology $[\alpha] \in H^2(M, \mathbb{Z})$, i.e. for any two-dimensional submanifold $\Sigma \subset M$,

$$\int_\Sigma c_1(L) \in \mathbb{Z}.$$

Thus, the existence of such a line bundle can be paraphrased by the existence of a symplectic form ω which belong to the integer cohomology class $H^2(M, \mathbb{Z})$. A manifold allowing such a structure is called quantizable.

Now, let us discuss how the notion of prequantum line bundle arises in the theory of quantization. Let (M, ω) be a symplectic manifold. In the Dirac quantization axiom, the quantization map $Q : \mathcal{C}^\infty(M, \mathbb{C}) \to \mathrm{End}(\mathcal{H})$ is a map satisfying the following set of axioms:

1. $Q(af + bg) = aQ(f) + bQ(g)$,
2. $Q(1) = \mathbf{1}_\mathcal{H}$,
3. $Q(f)^* = Q(\bar{f})$,
4. $Q(\{f, g\}) = (\mathrm{i}\hbar)^{-1}[Q(f), Q(g)]$,
5. If $\{f_i\}$ is complete, $\{Q(f_i)\}$ is complete,

for any $f, g \in \mathcal{C}^\infty(M, \mathbb{C})$ and $a, b \in \mathbb{C}$. Here, $(\cdot)^*$ denotes the adjoint with respect to the Hermitian inner product on $\mathcal{H}$ and $\bar{f}$ denotes the complex conjugate of f. Also, a set of functions $\{f_i\}$ is called complete if ${}^\forall i : \ \{f_i, f\} = 0 \ \Rightarrow f = \text{const.}$ and a set of operators $\{F_i\}$ is called complete if ${}^\forall i : [F_i, F] = 0 \ \Rightarrow F = \text{const.} \times \mathbf{1}_\mathcal{H}$. As shown by Groenewold and Van Hove, there is no such map

Q which satisfy all of the conditions given above. However, we can construct a map satisfying all the conditions except for the last requirement 5. This is called the prequantization. As we will see below, we can construct the prequantization map using the prequantum line bundle L.

As a first guess, the Lie algebra homomorphism of the Hamiltonian vector fields (2.6) motivates us to consider a map

$$Q_1(f) = \mathrm{i}\hbar X_f,$$

which acts on the space of smooth functions. This map obviously satisfies the axiom 1,3 and 4, but the condition 2 is not satisfied. Then, let us consider a map

$$Q_2(f) = \mathrm{i}\hbar X_f + f$$

as a second guess. This map satisfies the axiom 1,2 and 3. Let us check whether the condition 4 is satisfied. Unfortunately, this map fails to satisfy the condition 4:

$$[Q_2(f), Q_2(g)] = [\mathrm{i}\hbar X_f + f, \mathrm{i}\hbar X_g + g] = (\mathrm{i}\hbar)^2 X_{\{f,g\}} + \mathrm{i}\hbar 2\{f, g\} = \mathrm{i}\hbar Q_2(\{f, g\}) + \mathrm{i}\hbar\{f, g\}.$$

Let us assume that the prequantization map is of the form

$$Q_3(f) = Q_2(f) + R(f),$$

where $R(f)$ will be determined below. First, R should be linear and hence we expect that $R(f)$ contains terms which are first order in f and X_f. Secondly, due to the constraint $R(1) = 0$, we expect $R(f)$ should be first order in X_f. Thus, we expect that $R(f)$ is of the form

$$R(f) = gX_f + \alpha(X_f),$$

for some function g and a one-form α. For a general symplectic manifold (M, ω), one cannot introduce specific functions or one-forms constructed solely by the symplectic form ω. This implies that g is a constant and $\alpha = 0$. As one can readily check, the map $Q_3(f) = Q_2(f) + aX_f$ does not satisfy the condition 4 for any complex constant a. However, there is a symplectic potential θ, which is a locally defined one-form satisfying $\omega = \mathrm{d}\theta$. Thus, one can locally consider

$$Q(f) = \mathrm{i}a(X_f + \mathrm{i}b\theta(X_f)) + f.$$

Here, a and b are some constant, which are real due to the axiom 3. Let us introduce an operator $\nabla_u = u + \mathrm{i}b\theta(u)$ for a tangent vector u. Then, $Q(f)$ is written by

$$Q(f) = \mathrm{i}a\nabla_{X_f} + f.$$

Using the identity $[\nabla_u, \nabla_v] - \nabla_{[u,v]} = R(u, v)$, where $R = \mathrm{i}b\mathrm{d}\theta = \mathrm{i}b\omega$, one finds

$$\begin{aligned}
[Q(f), Q(g)] &= (\mathrm{i}a)^2[\nabla_{X_f}, \nabla_{X_g}] + \mathrm{i}2a\{f, g\} \\
&= (\mathrm{i}a)^2\mathrm{i}b\omega(X_f, X_g) + (\mathrm{i}a)^2\nabla_{[X_f,X_g]} + \mathrm{i}2a\{f, g\} \\
&= -(\mathrm{i}a)^2\mathrm{i}b\{f, g\} + (\mathrm{i}a)^2\nabla_{\{f,g\}} + \mathrm{i}2a\{f, g\} \\
&= \mathrm{i}aQ(\{f, g\}) + \mathrm{i}a(ab + 1)\{f, g\}.
\end{aligned}$$

Thus, by taking $a = \hbar,\ b = -\hbar^{-1}$, the prequantization map Q satisfies all the desired properties. Note that the symplectic potential θ is only defined for some local patch and there is an arbitrariness $\theta \mapsto \theta + \mathrm{d}h$, for any smooth function h. This motivates us to consider a line bundle L whose connection is locally written as ∇ and correspondingly the curvature is $R^L = -\mathrm{i}\hbar^{-1}\omega$. In this sense, Q should be considered as an operator on sections of L rather than functions.

2.3 Canonical spin^c structure

In this subsection, we define the canonical spin^c bundle from an almost complex structure and its compatible metric on a manifold. The sections of this bundle are complex spinor fields with an particular U(1) connection. References of this topic can be found in [8, 19].

First, let us define an almost complex structure.

Definition 6. Let M be an even-dimensional differentiable manifold. An almost complex structure J on M is defined as a linear map $J_x : TM_x \to TM_x$ at every point $x \in M$ such that

$$(J_x)^2 = -\mathbf{1}_{TM_x}.$$

Here, $\mathbf{1}_{TM_x}$ is the identity map on TM_x. The pair (M, J) is then called an almost complex manifold.

Note that the almost complex structure is defined only for even dimensional manifolds because

$$(\det J_x)^2 = \det(J_x^2) = \det(-\mathbf{1}_{TM_x}) = (-1)^{\dim M} \geq 0.$$

Now, let us discuss the splitting of the complexified tangent vector space $TM_x \otimes \mathbb{C}$ using the almost complex structure J_x. From $(J_x)^2 = -\mathbf{1}_{TM_x}$, the eigenvalues of J_x acting on the complex vector space $TM_x \otimes \mathbb{C}$ are $\pm\mathrm{i}$, where i is the imaginary unit. Thus, we can define a splitting $TM_x \otimes \mathbb{C} = T^{(1,0)}M_x \oplus T^{(0,1)}M_x$ according to the eigenvalues $+\mathrm{i}, -\mathrm{i}$, respectively. Note that the complex conjugation $\bar{\cdot} : TM_x \otimes \mathbb{C} \to TM_x \otimes \mathbb{C}$ maps $v \in T^{(1,0)}M_x$ to $\bar{v} \in T^{(0,1)}M_x$ and vice versa. This implies that the each vector space $T^{(1,0)}M_x, T^{(0,1)}M_x$ is isomorphic to each other under the complex conjugation. Consequently, we can consider the splitting of the complexified tangent vector bundle

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M,$$

and the corresponding dual bundle decomposition

$$T^*M \otimes \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M.$$

Let us consider the exterior algebra bundle $\Lambda(T^{*(0,1)}M) = \bigoplus_{k=0}^{\dim M/2} \Lambda^k(T^{*(0,1)}M)$. We introduce a notation

$$\Lambda^{0,\bullet} := \Lambda(T^{*(0,1)}M), \quad \Lambda^{0,k} := \Lambda^k(T^{*(0,1)}M), \quad \Lambda^{0,\mathrm{even}} := \bigoplus_{k:\mathrm{even}} \Lambda^{0,k}, \quad \Lambda^{0,\mathrm{odd}} := \bigoplus_{k:\mathrm{odd}} \Lambda^{0,k}. \tag{2.8}$$

The bundle $\Lambda^{0,\bullet} = \Lambda^{0,\text{even}} \oplus \Lambda^{0,\text{odd}}$ is the fundamental $\mathbb{Z}_2$ bundle, which induces the spin^c structure on M. Let us define a Clifford action on $\Lambda^{0,\bullet}$. Before doing so, we first introduce a notion of compatible metric.

Definition 7. Let (M, J) be an almost complex manifold. A Riemannian metric g on M, which is a positive-definite inner product of tangent vectors $g_x : TM_x \times TM_x \to \mathbb{R}$ is called compatible with J if

$$^\forall u, v \in TM_x : \quad g_x(J_x u, J_x v) = g_x(u, v),$$

at each point $x \in M$.

Then, we can introduce a Clifford action on $\Lambda^{0,\bullet}$ as follows.

Definition 8. The Clifford action of $u \in TM_x \otimes \mathbb{C}$ on $\Lambda_x^{0,\bullet}$ is defined by

$$c(u) := \sqrt{2} \left(\bar{u}^{(1,0)*} \wedge - \iota_{u^{(0,1)}} \right). \tag{2.9}$$

Here, the complexified tangent vector u is decomposed as $u = u^{(1,0)} + u^{(0,1)} \in T^{(1,0)} M_x \oplus T^{(0,1)} M_x$ and $\bar{u}^{(1,0)*}$ is defined by $^\forall v \in TM_x :\ g_x(u^{(1,0)}, v) = \bar{u}^{(1,0)*}(v)$, using the compatible metric g. The interior product with a complexified vector $u^{(1,0)}$ is defined similarly as in (2.1).

Note that the map $c(u)$ interchanges the $\mathbb{Z}_2$-grading, that is, $c(u)$ maps elements of $\Lambda_x^{0,\text{even}}$ to $\Lambda_x^{0,\text{odd}}$ and vice versa. Now, let us prove the following proposition.

Proposition 2.4. *The Clifford action defined in* (2.9) *satisfies*

$$c(u)c(v) + c(v)c(u) = -2\langle u, v \rangle, \tag{2.10}$$

for any $u, v \in TM_x \otimes \mathbb{C}$. Here, we defined $\langle u, v \rangle := g_x(u, v)$.

Proof. First, let us evaluate the left-hand side. By expanding $c(u)c(v)$, we obtain

$$c(u)c(v) = 2\bar{u}^{(1,0)*} \wedge \bar{v}^{(1,0)*} \wedge -2\bar{u}^{(1,0)*} \wedge \iota_{v^{(1,0)}} - 2\iota_{u^{(0,1)}} \bar{v}^{(1,0)*} \wedge +2\iota_{u^{(1,0)}} \iota_{v^{(1,0)}}.$$

The first and the forth term is skew-symmetric under the exchange of u and v. Hence, we have

$$c(u)c(v) + c(v)c(u) = -2\bar{u}^{(1,0)*} \wedge \iota_{v^{(0,1)}} - 2\iota_{u^{(0,1)}} \bar{v}^{(1,0)*} \wedge -2\bar{v}^{(1,0)*} \wedge \iota_{u^{(0,1)}} - 2\iota_{v^{(0,1)}} \bar{u}^{(1,0)*} \wedge .$$

Let us use the identity

$$\iota_w(\alpha \wedge \beta) = (\iota_w \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_w \beta),$$

for any tangent vector w, p-form α and q-form β. Then, we have

$$c(u)c(v) + c(v)c(u) = -2\bar{u}^{(1,0)*}(v^{(0,1)}) - 2\bar{v}^{(1,0)*}(u^{(0,1)}) = -2(g_x(u^{(1,0)}, v^{(0,1)}) + g_x(v^{(1,0)}, u^{(0,1)})).$$

For the right-hand side, by using the assumption that g is compatible with J, one can show

$$\begin{aligned}\langle u, v\rangle &= g_x(u,v) = \frac{1}{2}(g_x(u,v) + g_x(J_x u, J_x v)) \\ &= \frac{1}{2}(g_x(u^{(1,0)} + u^{(0,1)}, v^{(1,0)} + v^{(0,1)}) - g_x(u^{(1,0)} - u^{(0,1)}, v^{(1,0)} - v^{(0,1)})) \\ &= g_x(u^{(1,0)}, v^{(0,1)}) + g_x(v^{(1,0)}, u^{(0,1)}).\end{aligned}$$

Thus, we have (2.10). □

Let $\dim M = 2n$. Let $\{w_i\}_{i=1}^n$ be an orthonormal frame of $T^{(1,0)}M$ and $\{w^i\}_{i=1}^n$ be its dual frame. Then, one can take an orthonormal frame of TM by

$$e_{2i-1} = \frac{1}{\sqrt{2}}(w_i + \bar{w}_i), \quad e_{2i} = \frac{\mathrm{i}}{\sqrt{2}}(w_i - \bar{w}_i).$$

For w_i and its complex conjugate $\bar{w}_i$, the Clifford action is given by

$$c(w_i) = \sqrt{2}\bar{w}^i \wedge, \quad c(\bar{w}_i) = -\sqrt{2}\iota_{\bar{w}_i}. \tag{2.11}$$

This shows that the combination $c(w_{i_1})c(w_{i_2})\cdots c(w_{i_j})1$ for $1 \le i_1 < i_1 < \cdots < i_j \le n$ form a frame of $\Lambda^{0,\bullet}$. Here, $1 \in \Lambda^{0,\bullet}$ and j runs from 0 to n. Also, by introducing the Hermitian inner product on $\Lambda^{\mathrm{odd}}(T^{*(0,1)}M)$ induced from the compatible metric g, one can show

$$c(w_i)^* = -c(\bar{w}_i), \quad c(\bar{w}_i)^* = -c(w_i),$$

where $(\cdot)^*$ denotes the adjoint with respect to the Hermitian inner product. Correspondingly, we have

$$c(e_i)^* = -c(e_i). \tag{2.12}$$

Let us define the Clifford connection on $\Lambda^{0,\bullet}$.

Definition 9. Let (M, J) be a $2n$-dimensional almost complex manifold with compatible metric g. Let ∇^{TM} be the Levi-Civita connection on TM, i.e. the unique torsion-free connection satisfying $\nabla^{T^*M\otimes T^*M}g = 0$. Then, the Clifford connection $\nabla^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}$ is defined by

$$\nabla^{\Lambda^{0,\bullet}} := \mathrm{d} + \frac{1}{4}\sum_{i,j=1}^{2n}\langle\Gamma^{TM}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det}. \tag{2.13}$$

Here, Γ^{TM} is the connection one-form $\nabla^{TM}e_i = \Gamma^{TM}e_i$ and $\Gamma^{\det}$ is the connection one-form of the determinant bundle $\det(T^{(1,0)}M) := \Lambda^n(T^{(1,0)}M)$, that is,

$$\nabla^{\det}(w_1 \wedge w_2 \wedge \cdots \wedge w_n) = \Gamma^{\det}(w_1 \wedge w_2 \wedge \cdots \wedge w_n) = \sum_{i=1}^n\langle\Gamma^{TM}w_i, \bar{w}_i\rangle(w_1 \wedge w_2 \wedge \cdots \wedge w_n).$$

One can also write

$$\nabla^{\Lambda^{0,\bullet}} = \mathrm{d} - \sum_{i,j=1}^{n} \langle \Gamma^{TM} \bar{w}_i, w_j \rangle \bar{w}^i \wedge \iota_{\bar{w}_j} + \frac{1}{2} \sum_{i,j=1}^{n} \langle \Gamma^{TM} w_i, w_j \rangle \iota_{\bar{w}_i} \iota_{\bar{w}_j} + \frac{1}{2} \sum_{i,j=1}^{n} \langle \Gamma^{TM} \bar{w}_i, \bar{w}_j \rangle \bar{w}^i \wedge \bar{w}^j \wedge. \quad (2.14)$$

Note that the Clifford connection preserves the $\mathbb{Z}_2$-grading of $\Lambda^{0,\bullet}$. The Clifford connection has the following property.

Proposition 2.5. *For any $u, v \in TM_x \otimes \mathbb{C}$,*

$$[\nabla_u^{\Lambda^{0,\bullet}}, c(v)] = c(\nabla_u^{TM} v). \quad (2.15)$$

Proof. The only noncommuting part is

$$[\nabla_u^{\Lambda^{0,\bullet}}, c(v)] = \frac{1}{4} \sum_{i,j} \langle \Gamma^{TM}(u) e_i, e_j \rangle [c(e_i) c(e_j), c(v)].$$

Using the commutator and the skew-commutator relation $[AB, C] = A\{B, C\} - \{A, C\}B$, which holds for any linear operators A, B, C, we have

$$\begin{aligned} [\nabla_u^{\Lambda^{0,\bullet}}, c(v)] &= \frac{1}{4} \sum_{i,j} \langle \Gamma^{TM}(u) e_i, e_j \rangle (c(e_i)\{c(e_j), c(v)\} - \{c(e_i), c(v)\} c(e_j)) \\ &= -\frac{1}{2} \sum_{i,j} \langle \nabla_u^{TM} e_i, e_j \rangle (c(e_i) \langle e_j, v \rangle - \langle e_i, v \rangle c(e_j)) \\ &= c(\nabla_u^{TM} v). \end{aligned}$$

Here, we used $\langle \nabla_u^{TM} e_i, e_j \rangle = u \langle e_i, e_j \rangle - \langle e_i, \nabla_u^{TM} e_j \rangle = -\langle e_i, \nabla_u^{TM} e_j \rangle$ and $\sum_j \langle v, e_j \rangle e_j = v$. □

Finally, let us close this subsection by calculating the curvature of the connection $\nabla^{\Lambda^{0,\bullet}}$.

Proposition 2.6. *The curvature $R^{\Lambda^{0,\bullet}} := (\nabla^{\Lambda^{0,\bullet}})^2$ is given by*

$$R^{\Lambda^{0,\bullet}} = \frac{1}{4} \sum_{i,j} \langle R^{TM} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} R^{\det}. \quad (2.16)$$

Here, $R^{\det} = (\nabla^{\det})^2 = \mathrm{d}\Gamma^{\det}$ is the curvature of $\det(T^{(1,0)}M)$

Proof. From (2.13), we have

$$\begin{aligned} R^{\Lambda^{0,\bullet}} &= \left(\mathrm{d} + \frac{1}{4} \sum_{i,j} \langle \nabla^{TM} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \Gamma^{\det} \right)^2 \\ &= \frac{1}{4} \sum_{i,j} \mathrm{d} \langle \nabla^{TM} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{16} \sum_{i,j,k,l} \langle \nabla^{TM} e_i, e_j \rangle \langle \nabla^{TM} e_k, e_l \rangle c(e_i) c(e_j) c(e_k) c(e_l) + \frac{1}{2} R^{\det}. \end{aligned}$$

For the first term, we have

$$\frac{1}{4}\sum_{i,j}\mathrm{d}\langle\nabla^{TM}e_i,e_j\rangle c(e_i)c(e_j)=\frac{1}{4}\sum_{i,j}\langle(\nabla^{TM})^2e_i,e_j\rangle c(e_i)c(e_j)-\frac{1}{4}\sum_{i,j}\langle\nabla^{TM}e_i,\nabla^{TM}e_j\rangle c(e_i)c(e_j)$$
$$=\frac{1}{4}\sum_{i,j}\langle R^{TM}e_i,e_j\rangle c(e_i)c(e_j)-\frac{1}{4}\sum_{i,j}\langle\nabla^{TM}e_i,\nabla^{TM}e_j\rangle c(e_i)c(e_j).$$

Using the commutator and the skew-commutator relations $[AB,C]=A[B,C]+[A,C]B=A\{B,C\}-\{A,C\}B$ for any linear operators A,B,C, we have

$$\begin{aligned}[c(e_i)c(e_j),c(e_k)c(e_l)]&=c(e_i)[c(e_j),c(e_k)c(e_l)]+[c(e_i),c(e_k)c(e_l)]c(e_j)\\&=-c(e_i)c(e_k)\{c(e_j),c(e_l)\}+c(e_i)\{c(e_j),c(e_k)\}c(e_l)\\&\quad-c(e_k)\{c(e_i),c(e_l)\}c(e_j)+\{c(e_i),c(e_k)\}c(e_l)c(e_j)\\&=2\delta_{jl}c(e_i)c(e_k)-2\delta_{jk}c(e_i)c(e_l)+2\delta_{il}c(e_k)c(e_j)-2\delta_{ik}c(e_l)c(e_j).\end{aligned}$$

Thus, the second term can be written as

$$\begin{aligned}&\frac{1}{16}\sum_{i,j,k,l}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_k,e_l\rangle c(e_i)c(e_j)c(e_k)c(e_l)\\&=\frac{1}{32}\sum_{i,j,k,l}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_k,e_l\rangle[c(e_i)c(e_j),c(e_k)c(e_l)]\\&=\frac{1}{16}\sum_{i,j,k}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_k,e_j\rangle c(e_i)c(e_k)-\frac{1}{16}\sum_{i,j,l}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_j,e_l\rangle c(e_i)c(e_l)\\&\quad+\frac{1}{16}\sum_{i,j,k}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_k,e_i\rangle c(e_k)c(e_j)-\frac{1}{16}\sum_{i,j,l}\langle\nabla^{TM}e_i,e_j\rangle\langle\nabla^{TM}e_i,e_l\rangle c(e_l)c(e_j)\\&=\frac{1}{4}\sum_{i,j}\langle\nabla^{TM}e_i,\nabla^{TM}e_j\rangle c(e_i)c(e_j).\end{aligned}$$

Therefore, we obtain (2.16).

□

2.4 Spin^c Dirac operator

Now, let us define a spin^c Dirac operator. Most of the propositions and theorems given in this subsection can be found in [8].

Definition 10. Let (M,J) be an almost complex manifold. Let g be a compatible metric and (E,h^E) be a Hermitian vector bundle over M with a Hermitian connection ∇^E. Let $\mathcal{C}^\infty(M,F)$ denote a space of smooth sections of a vector bundle F over M. We define a spin^c Dirac operator acting on $\mathcal{C}^\infty(M,\Lambda^{0,\bullet}\otimes E)$ by

$$D^E=\sum_i c(e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i},\tag{2.17}$$

where $\nabla^{\Lambda^{0,\bullet}\otimes E} = \nabla^{\Lambda^{0,\bullet}} \otimes \mathbf{1}_E + \mathbf{1}_{\Lambda^{0,\bullet}} \otimes \nabla^E$.

To show the self-adjointness of D^E, let us consider the following. Let (M, g) be a d-dimensional Riemannian manifold and let μ_g be the Riemannian volume form locally given by

$$\mu_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d, \tag{2.18}$$

using the local coordinates $\{x^i\}$. Let (F, h^F) be a Hermitian vector bundle over M with a Hermitian connection ∇^F. We define a Hermitian inner product on $\mathcal{C}^\infty(M, F)$ by

$$(s, t) := \int_M \mu_g(x)\, h^F(s(x), t(x)), \tag{2.19}$$

and the divergence of a vector field $u \in \mathcal{C}^\infty(M, TM)$ by

$$(\operatorname{div} u)(x) := \sum_i \langle \nabla^{TM}_{e_i} u, e_i \rangle_x.$$

Then, we have the following proposition.

Proposition 2.7. *The adjoint of the connection* $(\nabla^F_u)^*$ *is given by*

$$(\nabla^F_u)^* = -\nabla^F_u - \operatorname{div} u. \tag{2.20}$$

Proof. From the definition, we have

$$\begin{aligned}(\nabla^F_u s, t) &= \int_M \mu_g\, h^F(\nabla^F_u s, t) = \int_M \mu_g\, u(h^F(s,t)) - \int_M \mu_g\, h^F(s, \nabla^F_u t) \\ &= \int_M \mathcal{L}_u(\mu_g\, h^F(s,t)) - \int_M (\mathcal{L}_u \mu_g)\, h^F(s,t) - \int_M \mu_g\, h^F(s, \nabla^F_u t).\end{aligned}$$

Using (2.5), we can show that the first term vanishes because of $\mathrm{d}(\mu_g\, h^F(s,t)) = 0$, the Stokes' theorem $\int_M \mathrm{d}(\cdots) = \int_{\partial M}(\cdots)$ and the assumption $\partial M = \emptyset$. The second term can be evaluated as follows. Let $\{e^i\}$ be the dual orthonormal frame of T^*M. Then, one obtains

$$\mathcal{L}_u \mu_g = \mathcal{L}_u(e^1 \wedge e^2 \wedge \cdots \wedge e^d) = \sum_i ((\mathcal{L}_u e^i)(e_i))\mu_g = -\sum_i \langle e_i, \mathcal{L}_u e_i \rangle \mu_g.$$

Here, we used $\mathcal{L}_u(\alpha(v)) = (\mathcal{L}_u \alpha)(v) + \alpha(\mathcal{L}_u v)$ for any one-form α and any vector v. Using $\mathcal{L}_u v = [u, v] = \nabla^{TM}_u v - \nabla^{TM}_v u$ and $\langle \nabla^{TM}_u e_i, e_i \rangle = -\langle e_i, \nabla^{TM}_u e_i \rangle = 0$, one finds

$$\mathcal{L}_u \mu_g = \sum_i \langle \nabla^{TM}_{e_i} u, e_i \rangle \mu_g = \operatorname{div} u.$$

Therefore, we obtain (2.20). □

Then, we can show the following property.

Proposition 2.8. *The spin^c Dirac operator is formally self-adjoint with respect to the Hermitian inner product*

$$(s,t) := \int_M \mu_g(x)\langle s(x), t(x)\rangle_{\Lambda^{0,\bullet}\otimes E}.$$

Here, $s,t \in \mathcal{C}^\infty(M, \Lambda^{0,\bullet}\otimes E)$ and $\langle\cdot,\cdot\rangle_{\Lambda^{0,\bullet}\otimes E}$ is the Hermitian inner product induced from g and h^E.

Proof. From (2.12) and (2.15), we find

$$\begin{aligned}
(s, D^E t) &= \sum_i (s, c(e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i} t) = -\sum_i (c(e_i)s, \nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i} t) \\
&= \sum_i (\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i} c(e_i)s, t) + \sum_i (\operatorname{div} e_i\, c(e_i)s, t) \\
&= (D^E s, t) + \sum_i (c(\nabla^{TM}_{e_i} e_i)s, t) + \sum_{i,j} (\langle \nabla^{TM}_{e_j} e_i, e_j\rangle\, c(e_i)s, t) \\
&= (D^E s, t) + \sum_i (c(\nabla^{TM}_{e_i} e_i)s, t) - \sum_{i,j} (\langle e_i, \nabla^{TM}_{e_j} e_j\rangle\, c(e_i)s, t) \\
&= (D^E s, t).
\end{aligned}$$

□

The spin^c Dirac operator D^E is a first order differential operator and its square $(D^E)^2$ is related to the Laplace operator which is a second order differential operator. Before calculating $(D^E)^2$, let us define the Bochner Laplacian.

Definition 11. Let (F, h^F) be a Hermitian vector bundle over M with a Hermitian connection ∇^F. The Bochner Laplacian Δ^F is defined by

$$\Delta^F := -\sum_i \left[(\nabla^F_{e_i})^2 - \nabla^F_{\nabla^{TM}_{e_i} e_i} \right]. \tag{2.21}$$

Proposition 2.9. *The Bochner Laplacian Δ^F can be written as*

$$\Delta^F = \sum_i (\nabla^F_{e_i})^* \nabla^F_{e_i}. \tag{2.22}$$

Proof. Using (2.20), we have

$$\sum_i (\nabla^F_{e_i})^* \nabla^F_{e_i} = -\sum_i (\nabla^F_{e_i})^2 - \sum_{i,j} \langle \nabla^{TM}_{e_j} e_i, e_j\rangle \nabla^F_{e_i} = -\sum_i \left[(\nabla^F_{e_i})^2 - \nabla^F_{\nabla^{TM}_{e_i} e_i} \right].$$

□

This implies that Δ^F is a positive semidefinite self-adjoint operator.

Now, let us show the following theorem called the Lichnerowicz formula.

Theorem 2.10. *The spin^c Dirac operator defined in* (2.17) *satisfies*

$$(D^E)^2 = \Delta^{\Lambda^{0,\bullet}\otimes E} + \frac{1}{4}K + \frac{1}{2}\sum_{i,j}\left(R^E + \frac{1}{2}R^{\det}\right)(e_i,e_j)c(e_i)c(e_j). \tag{2.23}$$

Here, K is the scalar curvature defined by $K := -\sum_{i,j}\langle R^{TM}(e_i,e_j)e_i,e_j\rangle$.

Proof. From (2.10) and (2.15), we find

$$\begin{aligned}
(D^E)^2 &= \frac{1}{2}\sum_{i,j}\left(c(e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}c(e_j)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} + c(e_j)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j}c(e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\right)\\
&= \frac{1}{2}\sum_{i,j}\left(c(e_i)c(e_j)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} + c(e_j)c(e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j}\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\right)\\
&\quad + \frac{1}{2}\sum_{i,j}\left(c(e_i)c(\nabla^{TM}_{e_i}e_j)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} + c(e_j)c(\nabla^{TM}_{e_j}e_i)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\right)\\
&= \frac{1}{2}\sum_{i,j}\left(c(e_i)c(e_j)+c(e_j)c(e_i)\right)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} + \frac{1}{2}\sum_{i,j}c(e_j)c(e_i)\left[\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j},\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\right]\\
&\quad + \sum_{i,j,k}\langle\nabla^{TM}_{e_i}e_j,e_k\rangle c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j}\\
&= -\sum_i\left(\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_i}\right)^2 + \frac{1}{2}\sum_{i,j}c(e_j)c(e_i)\left[(R^{\Lambda^{0,\bullet}}+R^E)(e_j,e_i) + \nabla^{\Lambda^{0,\bullet}\otimes E}_{[e_j,e_i]}\right]\\
&\quad + \sum_{i,j,k}\langle\nabla^{TM}_{e_i}e_j,e_k\rangle c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j}.
\end{aligned}$$

By using $\langle\nabla^{TM}_{e_i}e_j,e_k\rangle = -\langle e_j,\nabla^{TM}_{e_i}e_k\rangle$, the third term can be written as

$$\begin{aligned}
\sum_{i,j,k}\langle\nabla^{TM}_{e_i}e_j,e_k\rangle c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} &= -\sum_{i,k}c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{\nabla^{TM}_{e_i}e_k}\\
&= -\frac{1}{2}\sum_{i,k}(\{c(e_i),c(e_k)\}+[c(e_i),c(e_k)])\nabla^{\Lambda^{0,\bullet}\otimes E}_{\nabla^{TM}_{e_i}e_k}\\
&= \sum_i\nabla^{\Lambda^{0,\bullet}\otimes E}_{\nabla^{TM}_{e_i}e_i} - \frac{1}{2}\sum_{i,k}c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{\nabla^{TM}_{e_i}e_k-\nabla^{TM}_{e_k}e_i}\\
&= \sum_i\nabla^{\Lambda^{0,\bullet}\otimes E}_{\nabla^{TM}_{e_i}e_i} - \frac{1}{2}\sum_{i,k}c(e_i)c(e_k)\nabla^{\Lambda^{0,\bullet}\otimes E}_{[e_i,e_k]}.
\end{aligned}$$

Thus, one obtains

$$(D^E)^2 = \Delta^{\Lambda^{0,\bullet}\otimes E} + \frac{1}{2}\sum_{i,j}(R^{\Lambda^{0,\bullet}}+R^E)(e_i,e_j)c(e_i)c(e_j).$$

From (2.16), we have

$$\frac{1}{2}\sum_{i,j}R^{\Lambda^{0,\bullet}}(e_i,e_j)c(e_i)c(e_j) = \frac{1}{8}\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle c(e_k)c(e_l)c(e_i)c(e_j) + \frac{1}{4}\sum_{i,j}R^{\det}(e_i,e_j)c(e_i)c(e_j).$$

Using $\langle R^{TM}(u,v)w,y\rangle = \langle R^{TM}(w,y)u,v\rangle$ for any vectors u,v,w,y, we have

$$\frac{1}{2}\sum_{i,j} R^{\Lambda^{0,\bullet}}(e_i,e_j)c(e_i)c(e_j) = \frac{1}{8}\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle c(e_i)c(e_j)c(e_k)c(e_l)+\frac{1}{4}\sum_{i,j} R^{\det}(e_i,e_j)c(e_i)c(e_j).$$

From $R^{TM}(u,v)w + R^{TM}(v,w)u + R^{TM}(w,u)v = 0$, one has

$$\begin{aligned}&\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle c(e_i)c(e_j)c(e_k)c(e_l)\\ &= -\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle (c(e_j)c(e_k)c(e_i) + c(e_k)c(e_i)c(e_j))c(e_l).\end{aligned}$$

To evaluate this, let us calculate the following:

$$\begin{aligned}c(e_j)c(e_k)c(e_i) + c(e_k)c(e_i)c(e_j) &= -c(e_j)c(e_i)c(e_k) - c(e_i)c(e_k)c(e_j) - 4\delta_{ik}c(e_j)\\ &= 2c(e_i)c(e_j)c(e_k) + 2\delta_{ij}c(e_k) + 2\delta_{jk}c(e_i) - 4\delta_{ik}c(e_j).\end{aligned}$$

Thus, one obtains

$$\begin{aligned}&3\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle c(e_i)c(e_j)c(e_k)c(e_l)\\ &= -\sum_{i,j,k,l}\langle R^{TM}(e_i,e_j)e_k,e_l\rangle (2\delta_{ij}c(e_k) + 2\delta_{jk}c(e_i) - 4\delta_{ik}c(e_j))c(e_l)\\ &= 6\sum_{i,j,l}\langle R^{TM}(e_i,e_j)e_i,e_l\rangle c(e_j)c(e_l) = 3\sum_{i,j,l}\langle R^{TM}(e_i,e_j)e_i,e_l\rangle \{c(e_j),c(e_l)\}\\ &= 6K.\end{aligned}$$

Hence, we have

$$\frac{1}{2}\sum_{i,j} R^{\Lambda^{0,\bullet}}(e_i,e_j)c(e_i)c(e_j) = \frac{1}{4}K + \frac{1}{4}\sum_{i,j} R^{\det}(e_i,e_j)c(e_i)c(e_j),$$

which gives (2.23). □

For the Berezin-Toeplitz quantization given in the next section, let us consider the following set up. Let (M,g) be a $2n$-dimensional Riemannian manifold. Let us assume that there exists a prequantum line bundle (L,h^L) with a Hermitian connection ∇^L over M. We set a symplectic form by $\omega = c_1(L) = \frac{\mathrm{i}}{2\pi}R^L$. Then, there exists an almost complex structure J satisfying

$$g(Ju,Jv) = g(u,v), \quad \omega(Ju,Jv) = \omega(u,v), \quad \omega(u,Ju) > 0,$$

for any vector fields u,v. Using an orthonormal frame of $T^{(1,0)}M$ denoted by $\{w_i\}_{i=1}^n$, we define a $n\times n$ matrix r by $r_{ij} = R^L(w_i,\bar{w}_j)$. Then, we can see that r is self-adjoint and positive definite. Thus,

$$\tau = \sum_{i=1}^n R^L(w_i,\bar{w}_i), \quad \rho = -\sum_{i,j=1}^n R^L(w_i,\bar{w}_j)c(w_j)c(\bar{w}_i), \tag{2.24}$$

are a positive definite function and a positive definite operator, respectively.

Let (E, h^E) be a Hermitian vector bundle with a Hermitian connection ∇^E. We consider a sequence of spin^c Dirac operators $\{D^{p,E}\}_{p\in\mathbb{N}}$ on $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E)$. Here, $D^{p,E} := D^{L^{\otimes p}\otimes E}$. Then, the following theorem holds for $D^{p,E}$.

Theorem 2.11.

$$(D^{p,E})^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} - p\tau + p\rho + \tilde{R}^E. \tag{2.25}$$

Here, $\tilde{R}^E$ is defined by

$$\tilde{R}^E = \frac{1}{4}K + \frac{1}{2}\sum_{i,j}\left(R^E + \frac{1}{2}R^{\det}\right)(e_i, e_j)c(e_i)c(e_j). \tag{2.26}$$

Proof. Let us calculate the following:

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^{2n} R^L(e_i, e_j)c(e_i)c(e_j) &= \frac{1}{2}\sum_{i,j=1}^{n}\left(R^L(w_i, \bar{w}_j)c(\bar{w}_i)c(w_j) + R^L(\bar{w}_i, w_j)c(w_i)c(\bar{w}_j)\right)\\ &= -\sum_{i,j=1}^{n} R^L(w_i, \bar{w}_j)c(w_j)c(\bar{w}_i) - \sum_{i=1}^{n} R^L(w_i, \bar{w}_i)\\ &= \rho - \tau.\end{aligned}$$

Then, by (2.11), (2.23) and (2.24), we obtain (2.25). □

Using this expansion, we can show the lower bound of the Laplacian $\Delta^{L^{\otimes p}\otimes E}$.

Proposition 2.12. *Let (F, h^F) be a Hermitian vector bundle with a Hermitian connection ∇^F. Them, there exists a positive number $C > 0$, which does not depend on p, such that*

$$\Delta^{L^{\otimes p}\otimes F} - p\tau \geq -C. \tag{2.27}$$

Proof. For $s \in \mathcal{C}^\infty(M, L^{\otimes p} \otimes F) \subset \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes F)$, (2.22) and (2.25) implies that

$$|D_p^F s|^2 = \sum_{i=1}^{2n} |\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes F} s|^2 - p(s, \tau s) + (s, \tilde{R}^F s).$$

Here, the norm is defined by the section inner product $|s|^2 = (s, s)$. From (2.14), we have

$$\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes F} s = \nabla^{L^{\otimes p}\otimes F} s + \frac{1}{2}\sum_{i,j=1}^{n}\langle \Gamma^{TM}\bar{w}_i, \bar{w}_j\rangle \bar{w}^i \wedge \bar{w}^j \wedge s, \tag{2.28}$$

and this implies

$$0 \leq |D_p^F s|^2 \leq \sum_{i=1}^{2n} |\nabla_{e_i}^{L^{\otimes p}\otimes F} s|^2 + \frac{1}{4}\sum_{i=1}^{2n}\left|\sum_{j,k=1}^{n}\langle \Gamma^{TM}(e_i)\bar{w}_j, \bar{w}_k\rangle \bar{w}^j \wedge \bar{w}^k \wedge s\right|^2 - p(s, \tau s) + (s, \tilde{R}^F s),$$

using the triangle inequality of the norm. Hence, (2.27) holds. □

Using this proposition, one can obtain some important property of $D^{p,E}$.

Theorem 2.13. *There exists a positive number $C > 0$, which does not depend on p, such that*

$$|D^{p,E}\psi|^2 \geq (ap - C)|\psi|^2, \tag{2.29}$$

for any $\psi \in \mathcal{C}^\infty(M, \Lambda^{0,>0} \otimes L^{\otimes p} \otimes E)$. Here, $a > 0$ is the maximum number satisfying $\rho - a\mathbf{1} \geq 0$ and $\Lambda^{0,>0} := \bigoplus_{i=1}^{\dim M} \Lambda^{0,i}$.

Proof. By putting $F = \Lambda^{0,\bullet} \otimes E$ in (2.27), we find

$$\begin{aligned} |D^{p,E}\psi|^2 &= \sum_i |\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}\psi|^2 - p(\psi, \tau\psi) + p(\psi, \rho\psi) + (\psi, \tilde{R}^E\psi) \\ &\geq -C|\psi|^2 + p(\psi, \rho\psi) + (\psi, \tilde{R}^E\psi). \end{aligned}$$

Since we have a bound $(\psi, \rho\psi) \geq a|\psi|^2$ for $\psi \in \mathcal{C}^\infty(M, \Lambda^{0,>0} \otimes L^{\otimes p} \otimes E)$. Hence, we obtain

$$|D^{p,E}\psi|^2 \geq -C|\psi|^2 + ap|\psi|^2 + (\psi, \tilde{R}^E\psi),$$

which proves (2.29). □

Using this bound, one can prove the spectral gap of $D^{p,E}$ from (2.29).

Theorem 2.14.

$$\operatorname{Spec}\left((D^{p,E})^2\right) \subset \{0\} \cup (ap - C, +\infty). \tag{2.30}$$

Proof. Let $\psi = \psi_+ + \psi_- \in \mathcal{C}^\infty(M, \Lambda^{0,\mathrm{even}} \otimes L^{\otimes p} \otimes E) \oplus \mathcal{C}^\infty(M, \Lambda^{0,\mathrm{odd}} \otimes L^{\otimes p} \otimes E)$ be an eigensection of $(D^{p,E})^2$ with an eigenvalue $\lambda > 0$. If $\psi_- \neq 0$, (2.29) implies $\lambda \geq ap - C$. If $\psi_- = 0$, we then have $(D^{p,E})^2 D^{p,E}\psi^+ = \lambda D^{p,E}\psi^+$. Then, $D^{p,E}\psi^+ \in \mathcal{C}^\infty(M, \Lambda^{0,\mathrm{odd}} \otimes L^{\otimes p} \otimes E)$, which is nonzero by the assumption $\lambda > 0$, is also an eigensection of $(D^{p,E})^2$ with eigenvalue λ. Thus, (2.29) again implies $\lambda \geq ap - C$. □

One can also show the following theorem.

Theorem 2.15. *Let us consider restrictions of the Dirac operator*

$$D_+^{p,E} := D^{p,E}|_{\mathcal{C}^\infty(M,\Lambda^{0,\mathrm{even}}\otimes L^{\otimes p}\otimes E)}, \quad D_-^{p,E} := D^{p,E}|_{\mathcal{C}^\infty(M,\Lambda^{0,\mathrm{odd}}\otimes L^{\otimes p}\otimes E)}.$$

For $p \geq \frac{C}{a}$, we have

$$\ker D^{p,E} \subset \mathcal{C}^\infty(M, \Lambda^{0,0} \otimes L^{\otimes p} \otimes E) = \mathcal{C}^\infty(M, L^{\otimes p} \otimes E). \tag{2.31}$$

In particular, we have

$$\ker D_-^{p,E} = \{0\}. \tag{2.32}$$

Finally, let us evaluate the dimension of $\ker D^{p,E}$.

Theorem 2.16. *Let us assume that M is compact. For $p \geq \frac{C}{a}$, we have*

$$\dim\ker D^{p,E} = \int_M \mathrm{Td}(T^{(1,0)}M)\mathrm{ch}(L^{\otimes p}\otimes E_i), \tag{2.33}$$

where $\mathrm{Td}(F)$ *is the Todd class of a vector bundle F defined as*

$$\mathrm{Td}(F) := \det\left(\frac{-\mathrm{i}R^F/2\pi}{\exp(-\mathrm{i}R^F/2\pi)-1}\right),$$

and $\mathrm{ch}(F)$ *is the Chern character of F defined as*

$$\mathrm{ch}(F) := \mathrm{Tr}\exp(\frac{\mathrm{i}R^F}{2\pi}).$$

Proof. From (2.32), we have $\dim\ker D^{p,E}_{-} = 0$ for $p \geq \frac{C}{a}$ and hence

$$\dim\ker D^{p,E} = \dim\ker D^{p,E}_{+} + \dim\ker D^{p,E}_{-} = \dim\ker D^{p,E}_{+} - \dim\ker D^{p,E}_{-}.$$

For the case of compact manifold M, one can show that $D^{p,E}_{+}$ is a Fredholm operator and we can define the analytical index $\mathrm{Ind}\, D^{p,E}_{+} := \dim\ker D^{p,E}_{+} - \dim\ker D^{p,E}_{-}$. By the Atiyah-Singer index theorem, $\mathrm{Ind}\, D^{p,E}_{+}$ is equal to the topological index and thus we obtain (2.33). □

We can also compute the leading large-p expansion

$$\dim\ker D^{p,E} = \mathrm{rank}(E)\int_M \mathrm{e}^{\frac{\mathrm{i}p}{2\pi}R^L} = \mathrm{rank}(E_i)\, p^n\int_M \mu_\omega + \mathcal{O}(p^{n-1}),$$

where μ_ω is the Liouville volume form defined in (2.7).

2.5 Identities for Kähler manifolds

In this subsection, we define a Kähler manifold, which is a special type of symplectic manifold. In some proof of the asymptotic properties of the Toeplitz operator, we assume the Kähler structure of the base manifold M. For a more comprehensive reference of the Kähler manifolds, we refer to [20].

First, let us define the integrability of an almost complex structure.

Definition 12. Let (M, J) be an almost complex manifold. Then, the almost complex structure J is called integrable if the torsion

$$N_J(u,v) := [u,v] + J([Ju,v]+[u,Jv]) - [Ju,Jv]$$

vanishes for all $u, v \in \mathcal{C}^\infty(M, TM)$.

Then, we define a Kähler manifold as follows.

Definition 13. Let g, ω and J be a Riemannian metric, a symplectic structure and an integrable complex structure on M, respectively. A manifold (M, g, ω, J) is called Kähler if the triple (g, ω, J) satisfying the compatibility condition

$$\omega(u, v) = g(Ju, v), \tag{2.34}$$

for any $u, v \in \mathcal{C}^\infty(M, TM)$.

A particularly important property of the Kähler structure is that J commutes with the Levi-Civita connection ∇^{TM}, which is essential in some of the proofs in the next section. This property is obtained from the following proposition proposition.

Definition 14. Let (M, J) be an almost complex manifold and let g be a compatible metric. Let us define a nondegenerate two-form $\tilde{\omega}(u, v) := g(Ju, v)$. Then, we have

$$[\nabla^{TM}, J] = 0 \quad \Leftrightarrow \quad J\text{: integrable and } \mathrm{d}\tilde{\omega} = 0.$$

Here, ∇^{TM} is the Levi-Civita connection on TM.

Proof. First, let us show $[\nabla^{TM}, J] = 0 \Rightarrow J$: integrable and $\mathrm{d}\tilde{\omega} = 0$. Using $J^2 = -\mathbf{1}$ and $\nabla^{TM}_u v - \nabla^{TM}_v u = [u, v]$ for any $u, v \in \mathcal{C}^\infty(M, TM)$, one finds

$$\begin{aligned} N_J(u, v) &= [u, v] + J\nabla^{TM}_{Ju} v - J\nabla^{TM}_v Ju + J\nabla^{TM}_u Jv - J\nabla^{TM}_{Jv} u - [Ju, Jv] \\ &= [J, \nabla^{TM}_{Ju}]v - [J, \nabla^{TM}_v]Ju + [J, \nabla^{TM}_u]Jv - [J, \nabla^{TM}_{Jv}]u \\ &= 0, \end{aligned} \tag{2.35}$$

that is, J is integrable. For $\mathrm{d}\tilde{\omega}$, let us use (2.4):

$$\begin{aligned} \mathrm{d}\tilde{\omega}(u, v, w) &= u\tilde{\omega}(v, w) + v\tilde{\omega}(w, u) + w\tilde{\omega}(u, v) - \tilde{\omega}([u, v], w) - \tilde{\omega}([v, w], u) - \tilde{\omega}([w, u], v) \\ &= ug(Jv, w) + vg(Jw, u) + wg(Ju, v) - g(J[u, v], w) - g(J[v, w], u) - g(J[w, u], v) \\ &= g(\nabla^{TM}_u Jv, w) + g(Jv, \nabla^{TM}_u w) + g(\nabla^{TM}_v Jw, u) + g(Jw, \nabla^{TM}_v u) \\ &\quad + g(\nabla^{TM}_w Ju, v) + g(Ju, \nabla^{TM}_w v) \\ &\quad - g(J\nabla^{TM}_u v, w) + g(J\nabla^{TM}_v u, w) - g(J\nabla^{TM}_v w, u) + g(J\nabla^{TM}_w v, u) \\ &\quad - g(J\nabla^{TM}_w u, v) + g(J\nabla^{TM}_u w, v) \\ &= g([\nabla^{TM}_u, J]v, w) + g([\nabla^{TM}_v, J]w, u) + g([\nabla^{TM}_w, J]u, v) \\ &= 0. \end{aligned} \tag{2.36}$$

Secondly, let us show J: integrable and $\mathrm{d}\tilde{\omega} = 0 \Rightarrow [\nabla^{TM}, J] = 0$. From (2.35) and (2.36), we have

$$\begin{aligned}
0 &= g(N_J(u,v),w) + \mathrm{d}\tilde{\omega}(Ju,v,w) + \mathrm{d}\tilde{\omega}(u,Jv,w) \\
&= g([J,\nabla^{TM}_{Ju}]v - [J,\nabla^{TM}_{v}]Ju + [J,\nabla^{TM}_{u}]Jv - [J,\nabla^{TM}_{Jv}]u, w) \\
&\quad + g([\nabla^{TM}_{Ju}, J]v, w) + g([\nabla^{TM}_{v}, J]w, Ju) + g([\nabla^{TM}_{w}, J]Ju, v) \\
&\quad + g([\nabla^{TM}_{u}, J]Jv, w) + g([\nabla^{TM}_{Jv}, J]w, u) + g([\nabla^{TM}_{w}, J]u, Jv) \\
&= -g([J,\nabla^{TM}_{v}]Ju, w) - g([J,\nabla^{TM}_{Jv}]u, w) + g([\nabla^{TM}_{v}, J]w, Ju) + g([\nabla^{TM}_{w}, J]Ju, v) \\
&\quad + g([\nabla^{TM}_{Jv}, J]w, u) + g([\nabla^{TM}_{w}, J]u, Jv) \\
&= -g(J\nabla^{TM}_{v}Ju, w) - g(\nabla^{TM}_{v}u, w) - g(J\nabla^{TM}_{Jv}u, w) + g(\nabla^{TM}_{Jv}Ju, w) \\
&\quad + g(\nabla^{TM}_{v}Jw, Ju) - g(J\nabla^{TM}_{v}w, Ju) - g(\nabla^{TM}_{w}u, v) - g(J\nabla^{TM}_{w}Ju, v) \\
&\quad + g(\nabla^{TM}_{Jv}Jw, u) - g(J\nabla^{TM}_{Jv}w, u) + g(\nabla^{TM}_{w}Ju, Jv) - g(J\nabla^{TM}_{w}u, Jv) \\
&= -2g((J\nabla^{TM}_{w}J + \nabla^{TM}_{w})u, v).
\end{aligned}$$

Thus, using the assumptions $N_J = \mathrm{d}\tilde{\omega} = 0$, we have

$$J\nabla^{TM}_{w}J + \nabla^{TM}_{w} = 0 \quad \Rightarrow \quad [\nabla^{TM}, J] = 0.$$

□

From this proposition, the commutativity $[\nabla^{TM}, J] = 0$ is achieved if and only if the triple $(g, \tilde{\omega}, J)$ satisfies the Kähler condition (2.34).

Now, we state some important properties, which only hold for Kähler manifolds.

Proposition 2.17. *Let (M, g, ω, J) be a Kähler manifold. The Bochner Laplacian Δ^F defined in* (2.21) *can also be written as*

$$\Delta^F = \sum_i \left(2(\nabla^F_{\bar{w}_i})^* \nabla^F_{\bar{w}_i} + R^F(w_i, \bar{w}_i)\right). \tag{2.37}$$

Here, $\{w_i\}$ is an orthonormal frame of $T^{(1,0)}M$.

Proof. From the definition (2.21), we have

$$\begin{aligned}
\Delta^F &= -\sum_i \left(\nabla^F_{w_i}\nabla^F_{\bar{w}_i} + \nabla^F_{\bar{w}_i}\nabla^F_{w_i} - \nabla^F_{\nabla^{TM}_{w_i}\bar{w}_i} - \nabla^F_{\nabla^{TM}_{\bar{w}_i}w_i}\right) \\
&= -\sum_i \left(2\nabla^F_{w_i}\nabla^F_{\bar{w}_i} - R^F(w_i, \bar{w}_i) - \nabla^F_{[w_i,\bar{w}_i]} - \nabla^F_{\nabla^{TM}_{w_i}\bar{w}_i} - \nabla^F_{\nabla^{TM}_{\bar{w}_i}w_i}\right) \\
&= -\sum_i \left(2\nabla^F_{w_i}\nabla^F_{\bar{w}_i} - R^F(w_i, \bar{w}_i) - 2\nabla^F_{\nabla^{TM}_{w_i}\bar{w}_i}\right) \\
&= \sum_i \left(2(\nabla^F_{\bar{w}_i})^*\nabla^F_{\bar{w}_i} + 2(\operatorname{div} w_i)\nabla^F_{\bar{w}_i} + R^F(w_i, \bar{w}_i) + 2\nabla^F_{\nabla^{TM}_{w_i}\bar{w}_i}\right).
\end{aligned}$$

For the second term, we have

$$\sum_i (\operatorname{div} w_i)\nabla^F_{\bar{w}_i} = \sum_{i,j}\langle \nabla^{TM}_{e_j} w_i, e_j\rangle \nabla^F_{\bar{w}_i} = \sum_{i,j}\langle \nabla^{TM}_{w_j} w_i, \bar{w}_j\rangle \nabla^F_{\bar{w}_i} + \sum_{i,j}\langle \nabla^{TM}_{\bar{w}_j} w_i, w_j\rangle \nabla^F_{\bar{w}_i}.$$

From $[\nabla^{TM}, J] = 0$, ∇^{TM} preserves the splitting $TM = T^{(1,0)}M \oplus T^{(0,1)}M$ and consequently $\nabla^{TM}_{\bar{w}_j} w_i \in \mathcal{C}^\infty(M, T^{(1,0)}M)$. Thus, $\langle \nabla^{TM}_{\bar{w}_j} w_i, w_j\rangle = 0$ since the inner product $\langle u, v\rangle = g(u,v)$ is compatible with J. Again, by using this property, we have

$$\begin{aligned}\sum_i (\operatorname{div} w_i)\nabla^F_{\bar{w}_i} &= \sum_{i,j}\langle \nabla^{TM}_{w_j} w_i, \bar{w}_j\rangle \nabla^F_{\bar{w}_i} = \sum_{i,j}\langle \nabla^{TM}_{w_j} e_i, \bar{w}_j\rangle \nabla^F_{e_i} = -\sum_{i,j}\langle e_i, \nabla^{TM}_{w_j} \bar{w}_j\rangle \nabla^F_{e_i} \\ &= -\sum_j \nabla^F_{\nabla^{TM}_{w_j}\bar{w}_j}.\end{aligned}$$

Therefore, we obtain (2.37). □

Theorem 2.18. *Let (M, g, ω, J) be a Kähler manifold. Then, we have*

$$(D^{p,E})^2 = 2\sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} + p\rho + \hat{R}^E, \tag{2.38}$$

and

$$\rho = -2\pi \sum_i c(w_i)c(\bar{w}_i). \tag{2.39}$$

Here, $\hat{R}^E := \tilde{R}^E + \sum_i R^{\Lambda^{0,\bullet}\otimes E}(w_i, \bar{w}_i)$ where $\tilde{R}^E$ is defined in (2.26).

Proof. From (2.25) and (2.37), we have

$$\begin{aligned}(D^{p,E})^2 &= 2\sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} + \sum_i R^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(w_i, \bar{w}_i) - p\tau + p\rho + \tilde{R}^E \\ &= 2\sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} + p\rho + \hat{R}^E.\end{aligned}$$

Here, we used $\tau = \sum_i R^L(w_i, \bar{w}_i)$ in the last line. For the Kähler case (2.34), we have

$$\frac{\mathrm{i}}{2\pi} R^L(w_i, \bar{w}_j) = \omega(w_i, \bar{w}_j) = \mathrm{i} g(w_i, \bar{w}_j) = \mathrm{i}\delta_{ij}$$

and therefore we have (2.39). □

The following proposition is used for the asymptotic expansion of the Berezin-Toeplitz quantization in the following section.

Proposition 2.19. *Let (M, g, ω, J) be a Kähler manifold. On $\mathcal{C}^\infty(M, \Lambda^{0,1} \otimes L^{\otimes p} \otimes E)$, the inverse of $(D^{p,E})^2$ can be written as*

$$\begin{aligned}(D^{p,E})^{-2} = \frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E - \frac{\hbar_p^2}{4}\sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ + \frac{\hbar_p^2}{4}(D^{p,E})^{-2}\hat{R}^E \sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ + \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\sum_{i,j} (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* (\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ + \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\sum_{i,j} (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* (\nabla_{\nabla_{w_j}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ij}^E) \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}.\end{aligned} \tag{2.40}$$

Here, we defined $K_{ij}^E := R^{\Lambda^{0,\bullet}\otimes E}(w_j, \bar{w}_i) + \operatorname{div}(\nabla_{\bar{w}_i}^{TM} w_j) - \bar{w}_i(\operatorname{div} w_j)$ and

$$\hbar_p := (2\pi p)^{-1}. \tag{2.41}$$

Proof. For $\psi \in \mathcal{C}^\infty(M, \Lambda^{0,0} \otimes L^{\otimes p} \otimes E)$, we have

$$\rho c(w_j)\psi = -2\pi \sum_i c(w_i)c(\bar{w}_i)c(w_j)\psi = -2\pi \sum_i c(w_i)\{c(\bar{w}_i), c(w_j)\}\psi = 4\pi c(w_j)\psi.$$

Then, (2.38) becomes

$$(D^{p,E})^2 = 2\sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + 2\hbar_p^{-1} + \hat{R}^E.$$

on $\mathcal{C}^\infty(M, \Lambda^{0,1} \otimes L^{\otimes p} \otimes E)$. Thus, its inverse can be written as

$$(D^{p,E})^{-2} = \frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\left[2\sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + \hat{R}^E\right] \tag{2.42}$$

on $\mathcal{C}^\infty(M, \Lambda^{0,1} \otimes L^{\otimes p} \otimes E)$. By using (2.42) to itself, we obtain

$$\begin{aligned}(D^{p,E})^{-2} = \frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E - \frac{\hbar_p^2}{2}\sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ + \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\left[2\sum_j (\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + \hat{R}^E\right] \sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}\end{aligned}$$

Let us calculate

$$\begin{aligned}[\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}, (\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*] &= -[\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}, \nabla_{w_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + \operatorname{div} w_j] \\ &= R^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(w_j, \bar{w}_i) + \nabla_{[w_j, \bar{w}_i]}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} - \bar{w}_i(\operatorname{div} w_j) \\ &= \hbar_p^{-1}\delta_{ij} + \nabla_{\nabla_{w_j}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ij}^E.\end{aligned} \tag{2.43}$$

Then, we have

$$
\begin{aligned}
(D^{p,E})^{-2} = &\frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E - \frac{\hbar_p^2}{2}\sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \\
&+ \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\hat{R}^E \sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \\
&+ \hbar_p^2 (D^{p,E})^{-2} \sum_{i,j} (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j} \\
&+ \hbar_p (D^{p,E})^{-2} \sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \\
&+ \hbar_p^2 (D^{p,E})^{-2} \sum_{i,j} (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\nabla^{TM}_{w_j}\bar{w}_i} + (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\nabla^{TM}_{w_i}\bar{w}_j})^* + K^E_{ij}) \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j} \\
= &\frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E - \frac{\hbar_p^2}{2}\sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \\
&+ \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\hat{R}^E \sum_i (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \\
&+ \hbar_p^2 (D^{p,E})^{-2} \sum_{i,j} (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j})^* \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i} \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j} \\
&- (D^{p,E})^{-2} + \frac{\hbar_p}{2} - \frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E \\
&+ \hbar_p^2 (D^{p,E})^{-2} \sum_{i,j} (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^* (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\nabla^{TM}_{w_j}\bar{w}_i} + (\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\nabla^{TM}_{w_i}\bar{w}_j})^* + K^E_{ij}) \nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j}.
\end{aligned}
$$

Here, we used (2.42) in the second equality. Thus, we obtain (2.40). □

Proposition 2.20. *Let (M, g, ω, J) be a Kähler manifold and we assume that (2.31) holds. For $\psi \in \ker D^{p,E} \subset \mathcal{C}^\infty(M, L^{\otimes p} \otimes E)$, we have*

$$\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{u}}\psi = \nabla^{L^{\otimes p}\otimes E}_{\bar{u}}\psi = 0, \tag{2.44}$$

for any $u \in \mathcal{C}^\infty(M, T^{(1,0)}M)$.

Proof. From (2.28), we have $\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}\psi = \nabla^{L^{\otimes p}\otimes E}\psi$. Since $c(\bar{w}_i)\phi = 0$ for $\phi \in \mathcal{C}^\infty(M, L^{\otimes p} \otimes E)$, we have

$$D^{p,E}\psi = \sum_i c(w_i)\nabla^{L^{\otimes p}\otimes E}_{\bar{w}_i}\psi = 0 \quad \Rightarrow \quad \nabla^{L^{\otimes p}\otimes E}_{\bar{w}_i}\psi = 0.$$

□

Proposition 2.21. *Let (M, g, ω, J) be a Kähler manifold. For $\psi \in \ker D^{p,E}$ with $|\psi| = \mathcal{O}(\hbar_p^0)$, we have an estimation*

$$
\begin{aligned}
&|(\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^*\psi| = \mathcal{O}(\hbar_p^{-\frac{1}{2}}), \\
&|(\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_i})^*(\nabla^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}_{\bar{w}_j})^*\psi| = \mathcal{O}(\hbar_p^{-1}).
\end{aligned}
\tag{2.45}
$$

Proof. From (2.43) and (2.44), we have

$$|(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi|^2 = (\psi, \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) = (\psi, (\hbar_p^{-1} + K_{ii}^E)\psi) = \mathcal{O}(\hbar_p^{-1}).$$

Similarly, we have

$$\begin{aligned}
&|(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi|^2 \\
&= (\psi, \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) \\
&= (\psi, \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) \\
&\quad + (\psi, \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\hbar_p^{-1} + \nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ii}^E)(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) \\
&= (\psi, (\hbar_p^{-1}\delta_{ij} + \nabla_{\nabla_{w_j}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + K_{ij}^E)(\hbar_p^{-1}\delta_{ij} + (\nabla_{\nabla_{w_j}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ji}^E)\psi) \\
&\quad + (\psi, \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\hbar_p^{-1} + \nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ii}^E)(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) \\
&= \mathcal{O}(\hbar_p^{-2}) + (\psi, \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}(\nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*)(\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^*\psi) \\
&= \mathcal{O}(\hbar_p^{-2}).
\end{aligned}$$

□

3 Berezin-Toeplitz quantization for a vector bundle

In this section, we consider the Berezin-Toeplitz quantization for vector bundles and derive various properties of the quantization map [10].

3.1 Toeplitz operators for a vector bundle

Let (M, g) be a $2n$-dimensional Riemannian manifold and assume that there exists a prequantum line bundle (L, h^L) with a Hermitian connection ∇^L over M. We set a symplectic form by $\omega = c_1(L) = \frac{\mathrm{i}}{2\pi} R^L$. Let J be an almost complex structure satisfying

$$g(Ju, Jv) = g(u, v), \quad \omega(Ju, Jv) = \omega(u, v), \quad \omega(u, Ju) > 0, \tag{3.1}$$

for any vector fields u, v. Such an almost complex structure can be introduced by $J = \tilde{J}(-\tilde{J}^2)^{-\frac{1}{2}}$ where $\tilde{J}$ is a skew-adjoint linear map uniquely determined by

$$\omega(u, v) = g(\tilde{J}u, v), \tag{3.2}$$

for any vector fields u, v. Using (g, J), we introduce the fundamental $\mathbb{Z}_2$-graded spin^c bundle by $\Lambda^{0,\bullet} = \Lambda^{0,\mathrm{even}} \oplus \Lambda^{0,\mathrm{odd}}$, which is defined in (2.8).

Let (E_1, h^{E_1}) and (E_2, h^{E_2}) be finite-rank Hermitian vector bundles with Hermitian connections ∇^{E_1} and ∇^{E_2}, respectively. As we will see below, we define a Toeplitz operator of a section of a homomorphism bundle $\mathrm{Hom}(E_2, E_1)$. Here, the homomorphism bundle $\mathrm{Hom}(E_2, E_1)$ is defined as a vector bundle whose fiber $\mathrm{Hom}(E_2, E_1)_x$ is a vector space of linear maps from $(E_2)_x$ to $(E_1)_x$ at every point $x \in M$. Note that any finite-rank Hermitian vector bundle is isomorphic to some homomorphism bundles, that is, we can treat any vector bundle as a homomorphism bundle. Let $\mathcal{C}^\infty(M, F)$ be a space of smooth sections of a vector bundle F over M. From the homomorphism structure, a section $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$ can be thought of as a linear map $\mathcal{C}^\infty(M, E_2) \to \mathcal{C}^\infty(M, E_1)$, using the pointwise multiplication $(ss_2)(x) = s(x)s_2(x)$ for $s_2 \in \mathcal{C}^\infty(M, E_2)$ and $ss_2 \in \mathcal{C}^\infty(M, E_1)$. The connection of $\mathrm{Hom}(E_2, E_1)$ can be introduced using the compatibility condition

$$\nabla^{E_1}(ss_2) = (\nabla^{\mathrm{Hom}(E_2,E_1)}s)s_2 + s(\nabla^{E_2}s_2). \tag{3.3}$$

Let us consider a tensor product bundle $\Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a$ for $a = 1, 2$. On $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a)$, we equip an inner product

$$(\psi, \psi') = \int_M \mu_g(x) \langle \psi(x), \psi'(x) \rangle, \tag{3.4}$$

where μ_g is the Riemannian volume form defined as (2.18) and $\langle \cdot, \cdot \rangle$ is the Hermitian inner products on the fiber induced from the Hermitian metric of $\Lambda^{0,\bullet}$, L and E_a. The norm on this space is defined by $|\psi|^2 = (\psi, \psi)$. We denote the corresponding by $L^2(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a)$ the L^2 completion of

$\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a)$. Let D^{p,E_a} be a spin^c Dirac operator on $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a)$ defined as (2.17) and the Bergman projection is defined as the orthogonal projection

$$\Pi^{p,E_a} : L^2(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a) \to \ker D^{p,E_a}.$$

Now, let us define the Toeplitz operator of $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$. Let us consider $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$ to be a linear map on a broader space

$$s : \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_2) \to \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_1).$$

Definition 15. The Toeplitz operator of $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$ is defined as

$$T_p^{(E_1,E_2)}(s) = \Pi^{p,E_1}\, s\, \Pi^{p,E_2}, \tag{3.5}$$

which is a linear map $L^2(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_2) \to L^2(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_1)$.

Note that this operator is essentially a nontrivial map from $\ker D^{p,E_2}$ to $\ker D^{p,E_1}$ and therefore it is represented by a $(\dim \ker D^{p,E_1}) \times (\dim \ker D^{p,E_2})$ matrix. From the definition, we have

$$T_p^{(E_1,E_2)}(s)^* = \Pi^{p,E_2}\, s^*\, \Pi^{p,E_1}. \tag{3.6}$$

Here, $T_p^{(E_1,E_2)}(s)^*$ is the Hermitian adjoint of $T_p^{(E_1,E_2)}(s)$ with respect to the inner product (3.4) and $s^* \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_1, E_2))$ is the adjoint of $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$ determined by $h^{E_1}(u_1, s(x)u_2) = h^{E_2}(s^*(x)u_1, u_2)$ for $u_a \in (E_a)_x$ at each point $x \in M$.

We can also observe that the map $T_p^{(E,E)}$ is unital. Let E be a vector bundle and let us consider the endomorphism bundle $\mathrm{End}(E) := \mathrm{Hom}(E, E)$. Then, there exists the identity section $\mathbf{1}_E \in \mathcal{C}^\infty(M, \mathrm{End}(E))$. Then, we can see that the

$$T_p^{(E,E)}(\mathbf{1}_E) = \mathbf{1}_{\ker D^{p,E}},$$

where $\mathbf{1}_{\ker D^{p,E}}$ is the identity operator on $\ker D^{p,E}$. This shows that the map $T_p^{(E,E)}$ preserves the identity element.

3.2 Asymptotic expansion of Toeplitz operators

We can also consider a section $t \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_3, E_2))$ and its Toeplitz operator $T_p^{(E_2,E_3)}(t) = \Pi^{p,E_2}\, t\, \Pi^{p,E_3}$. As in the case of Berezin-Toeplitz quantization of functions, we expect that there exists an asymptotic expansion

$$\left| T_p^{(E_1,E_2)}(s) T_p^{(E_2,E_3)}(t) - \sum_{l=0}^{k} \hbar_p^l T_p^{(E_1,E_3)}(C_l(s,t)) \right| = \mathcal{O}(\hbar_p^{k+1}),$$

for any k. Here, $\hbar_p$ is defined in (2.41) and $\{C_l\}_{l=0}^\infty$ is a sequence of bilinear maps

$$C_l : \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1)) \times \mathcal{C}^\infty(M, \mathrm{Hom}(E_3, E_2)) \to \mathcal{C}^\infty(M, \mathrm{Hom}(E_3, E_1)).$$

First, let us obtain the leading coefficient $C_0(s,t)$.

Theorem 3.1. *Let us assume that p is large enough to satisfy* (2.32)*. Then, we have*

$$\left|T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) - T_p^{(E_1,E_3)}(st)\right| = \mathcal{O}(\hbar_p).$$

Proof. First, we have

$$T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) = \Pi^{p,E_1}\, s\,\Pi^{p,E_2}\, t\,\Pi^{p,E_3} = T_p^{(E_1,E_3)}(st) - \Pi^{p,E_1}\, s(\mathbf{1} - \Pi^{p,E_2})t\,\Pi^{p,E_3}.$$

The operator $\mathbf{1} - \Pi^{p,E_2}$ is the orthogonal projection to the orthogonal complement $(\ker D^{p,E_2})^\perp$. We can directly check

$$\mathbf{1} - \Pi^{p,E_2} = D^{p,E_2}P^{p,E_2}D^{p,E_2}.$$

where

$$P^{p,E_2}\psi := D_-^{p,E_2}(D_+^{p,E_2}D_-^{p,E_2})^{-2}D_+^{p,E_2}\psi_+ + (D_+^{p,E_2}D_-^{p,E_2})^{-1}\psi_-,$$

for $\psi = \psi_+ + \psi_- \in \mathcal{C}^\infty(M, \Lambda^{0,\mathrm{even}}\otimes L^{\otimes p}\otimes E_2)\oplus\mathcal{C}^\infty(M, \Lambda^{0,\mathrm{odd}}\otimes L^{\otimes p}\otimes E_2)$. Here, $D_+^{p,E_2}D_-^{p,E_2}$ is strictly positive from (2.32) and therefore the inverse $(D_+^{p,E_2}D_-^{p,E_2})^{-1}$ exists on $\mathcal{C}^\infty(M, \Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E)$. Hence, we reach

$$\begin{aligned} T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) &= T_p^{(E_1,E_3)}(st) - \Pi^{p,E_1}\, sD^{p,E_2}P^{p,E_2}D^{p,E_2}t\,\Pi^{p,E_3}. \\ &= T_p^{(E_1,E_3)}(st) + \sum_{i,j}\Pi^{p,E_1}c(\bar{w}_i)(\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)P^{p,E_2}c(w_j)(\nabla_{\bar{w}_j}^{\mathrm{Hom}(E_3,E_2)}t)\Pi^{p,E_3}, \end{aligned}$$

using $D^{p,E_3}\Pi^{p,E_3} = \Pi^{p,E_1}D^{p,E_1}$ and (3.3). On $(\ker D^{p,E_2})^\perp$, we have

$$P^{p,E_2} = (D^{p,E_2})^{-2},$$

since $(D^{p,E_2})^2$ is strictly positive on $(\ker D^{p,E_2})^\perp$. For $\psi \in \ker D^{p,E_3}$, $\sum_j c(w_j)(\nabla_{\bar{w}_j}^{\mathrm{Hom}(E_3,E_2)}t)\psi \in \mathcal{C}^\infty(M, \Lambda^{0,1}\otimes L^{\otimes p}\otimes E_3) \subset (\ker D^{p,E_2})^\perp$. Thus, we have

$$T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) = T_p^{(E_1,E_3)}(st) + \sum_{i,j}\Pi^{p,E_1}c(\bar{w}_i)(\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)(D^{p,E_2})^{-2}c(w_j)(\nabla_{\bar{w}_j}^{\mathrm{Hom}(E_3,E_2)}t)\Pi^{p,E_3}. \tag{3.7}$$

Using (2.30), we have $(D^{p,E_2})^{-2} = \mathcal{O}(\hbar_p)$ and therefore

$$|T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) - T_p^{(E_1,E_3)}(st)| = \mathcal{O}(\hbar_p). \tag{3.8}$$

□

This means that the product of sections is approximated by the operator products under the Berezin-Toeplitz quantization.

Let us now assume that (M, g, ω, J) is Kähler. Then, we can obtain the coefficient $C_1(s,t)$.

Theorem 3.2. *Let (M, g, ω, J) be a Kähler manifold and let us assume that p is large enough to satisfy* (2.32)*. Then, we have*

$$\left| T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) - T_p^{(E_1,E_3)}(st) + \hbar_p \sum_i T_p^{(E_1,E_3)}((\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)(\nabla_{\bar{w}_i}^{\mathrm{Hom}(E_3,E_2)}t)) \right| = \mathcal{O}(\hbar_p^2). \tag{3.9}$$

Proof. This proof is based on the technique of [21]. From (2.40), (3.7) becomes

$$\begin{aligned} T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t) &= T_p^{(E_1,E_3)}(st) + \frac{\hbar_p}{2}\sum_{i,j} \Pi^{p,E_1} c(\bar{w}_i)(\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)c(w_j)(\nabla_{\bar{w}_j}^{\mathrm{Hom}(E_3,E_2)}t)\Pi^{p,E_3} + \epsilon \\ &= T_p^{(E_1,E_3)}(st) - \hbar_p \sum_i T_p^{(E_1,E_3)}((\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)(\nabla_{\bar{w}_i}^{\mathrm{Hom}(E_3,E_2)}t)) + \epsilon. \end{aligned} \tag{3.10}$$

Here, we used $c(\bar{w}_i)c(w_j) = -2\delta_{ij}$ on $\mathcal{C}^\infty(M, L^{\otimes p} \otimes E)$ and ϵ is given by

$$\epsilon = \sum_{i,j} \Pi^{p,E_1} c(\bar{w}_i)(\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)}s)K^{E_2}c(w_j)(\nabla_{\bar{w}_j}^{\mathrm{Hom}(E_3,E_2)}t)\Pi^{p,E_3},$$

where

$$\begin{aligned} K^E = &-\frac{\hbar_p}{2}(D^{p,E})^{-2}\hat{R}^E - \frac{\hbar_p^2}{4}\sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ &+ \frac{\hbar_p^2}{4}(D^{p,E})^{-2}\hat{R}^E \sum_i (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ &+ \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\sum_{i,j} (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* (\nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* \nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} \\ &+ \frac{\hbar_p^2}{2}(D^{p,E})^{-2}\sum_{i,j} (\nabla_{\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* (\nabla_{\nabla_{w_j}^{TM}\bar{w}_i}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E} + (\nabla_{\nabla_{w_i}^{TM}\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E})^* + K_{ij}^E) \nabla_{\bar{w}_j}^{\Lambda^{0,\bullet}\otimes L^{\otimes p}\otimes E}. \end{aligned}$$

Using (2.45), (2.44) and $(D^{p,E_2})^{-2} = \mathcal{O}(\hbar_p)$, we can estimate that

$$|\epsilon| = \mathcal{O}(\hbar_p^2). \tag{3.11}$$

Therefore, we obtain (3.9). □

The consequence of the asymptotic (3.9) is as follows. First, we define generalizations of the Poisson bracket and the commutator.

Definition 16. Let E be a vector bundle over a symplectic manifold (M, ω). For $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $s \in \mathcal{C}^\infty(M, E)$, we define a generalized Poisson bracket as

$$\{f, s\} := \nabla_{X_f}^E s, \tag{3.12}$$

where X_f is the Hamiltonian vector field defined in (2.2).

Definition 17. Let us consider the setup of section 3.1. For $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$, we define a generalized commutator as

$$[T_p(f\mathbf{1}), T_p^{(E_1,E_2)}(s)] := T_p^{(E_1,E_1)}(f\mathbf{1}_{E_1})T_p^{(E_1,E_2)}(s) - T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_2)}(f\mathbf{1}_{E_2}), \tag{3.13}$$

where $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ are the identity elements of $\mathcal{C}^\infty(M, \mathrm{End}(E_1))$ and $\mathcal{C}^\infty(M, \mathrm{End}(E_2))$, respectively.

Theorem 3.3. *Let (M, g, ω, J) be a Kähler manifold and let us assume that p is large enough to satisfy* (2.32)*. Then, we have*

$$\left|(\mathrm{i}\hbar_p)^{-1}[T_p(f\mathbf{1}), T_p^{(E_1,E_2)}(s)] - T_p^{(E_1,E_2)}(\{f, s\})\right| = \mathcal{O}(\hbar_p), \tag{3.14}$$

for $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$.

Proof. From (3.10) and (3.11), we have

$$\left|(\mathrm{i}\hbar_p)^{-1}[T_p(\mathbf{1}), T_p^{(E_1,E_2)}(s)] - \mathrm{i}\sum_i T_p^{(E_1,E_3)}((w_i f)(\nabla_{\bar{w}_i}^{\mathrm{Hom}(E_2,E_1)} s) - (\bar{w}_i f)(\nabla_{w_i}^{\mathrm{Hom}(E_2,E_1)} s))\right| = \mathcal{O}(\hbar_p).$$

From (2.34), the symplectic form can be written as $\omega = \mathrm{i}\sum_i w^i \wedge \bar{w}^i$, where $\{w^i\}$ is the dual frame of $\{w_i\}$. Then, (2.2) implies

$$X_f = \mathrm{i}\sum_i [(w_i f)\bar{w}_i - (\bar{w}_i f)w_i]. \tag{3.15}$$

Thus, we obtain (3.14). □

For the trivial bundle $E_1 = E_2 = M \times \mathbb{C}$, (3.8) and (3.14) reduce to (1.1) and (1.2), respectively.

3.3 Trace of the Toeplitz operator

In this subsection, we consider a general $2n$-dimensional symplectic manifold (M, g, ω, J) satisfying (3.1). Let us consider an endomorphism bundle $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ for a vector bundle E over M. Then, the Toeplitz operator of $s \in \mathcal{C}^\infty(M, \mathrm{End}(E))$ is given by

$$T_p^{(E,E)}(s) = \Pi^{p,E} s\, \Pi^{p,E}.$$

In this case, we can consider a trace of the Toeplitz operator.

Theorem 3.4. *Let M be a $2n$-dimensional manifold.*

$$\lim_{p\to\infty} (2\pi\hbar_p)^n \,\mathrm{Tr}\, T_p^{(E,E)}(s) = \int_M \mu_\omega \,\mathrm{tr}_E\, s. \tag{3.16}$$

Here, tr_E is the trace of fiber space E induced by the fiber inner product and μ_ω is the Liouville volume form defined in (2.7).

Proof. Using the Schwartz kernel, the trace of $T_p^{(E,E)}(s)$ can be represented as

$$\operatorname{Tr} T_p^{(E,E)}(s) = \int_M \mu_g(x) \operatorname{tr}_{\Lambda^{0,\bullet}\otimes E} \left(B_p(x,x)s(x)\right),$$

where $B_p(x,y)$ is the Bergman kernel defined by

$$(\Pi^{p,E}\psi)(x) = \int_M \mu_g(y) B_p(x,y)\psi(y),$$

for any $\psi \in \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E)$. Note that we consider a kernel with respect to the Riemannian volume form μ_g. In [22], the diagonal of the Bergman kernel $B_p(x,x)$ has the following large-p asymptotic form,

$$B_p(x,x) = (2\pi\hbar_p)^{-n}(\det \tilde{J})^{\frac{1}{2}} P_x \mathbf{1}_{E_x} + \mathcal{O}(\hbar_p^{-n+1}),$$

where $\tilde{J}$ is determined by (3.2) and P_x is the projection $\Lambda_x^{0,\bullet} \to \Lambda_x^{0,0}$ and $\mathbf{1}_{E_x} \in \operatorname{End}(E_x)$ is the identity operator. From (3.2), we have $(\det \tilde{J})^{\frac{1}{2}}\mu_g = \mu_\omega$ and therefore we obtain (3.16). □

We can see that the relation (3.16) is a generalization of 1.3.

From this correspondence, we can show the correspondence of inner products.

Theorem 3.5. *For $s, t \in \mathcal{C}^\infty(M, \operatorname{Hom}(E_2, E_1))$, we define a Hermitian inner product of Toeplitz operators*

$$(T_p^{(E_1,E_2)}(s), T_p^{(E_1,E_2)}(t)) := (2\pi\hbar_p)^n \operatorname{Tr}(T_p^{(E_1,E_2)}(s)^* T_p^{(E_1,E_2)}(t)). \tag{3.17}$$

Then, we have the following correspondence for a Kähler manifold (M, g, ω, J):

$$\lim_{p\to\infty} (T_p^{(E_1,E_2)}(s), T_p^{(E_1,E_2)}(t)) = (s,t). \tag{3.18}$$

Here, the inner product of sections is given by (2.19) *with $h^{\operatorname{Hom}(E_2,E_1)}(s(x), t(x)) = \operatorname{tr}_{E_2}(s^*(x)t(x))$.*

Proof. From (3.6), (3.8) and (3.16), we have

$$\lim_{p\to\infty} (T_p^{(E_1,E_2)}(s), T_p^{(E_1,E_2)}(t)) = \int_M \mu_\omega h^{\operatorname{Hom}(E_2,E_1)}(s,t).$$

For a Kähler manifold (M, g, ω, J), we have $\mu_\omega = \mu_g$ and therefore obtain (3.18). □

3.4 Bochner Laplacian and its matrix regularization

Let (M, g) be a Riemannian manifold. From the Nash embedding theorem, there exists an smooth isometric embedding $X = (X^1, X^2, \cdots, X^d) : M \to \mathbb{R}^d$ satisfying

$$g = \sum_{a=1}^d \mathrm{d}X^a \otimes \mathrm{d}X^a, \tag{3.19}$$

for a sufficiently large d.

Using the following theorem, the Bochner Laplacian (2.21) can be expressed in terms of the isometric functions and the generalized commutator (3.13) in the case of Kähler manifold.

Theorem 3.6. *Let (M, g, ω, J) be a Kähler manifold and E be a vector bundle over M. Then, we have*

$$\Delta^E s = -\sum_a \{X^a, \{X^a, s\}\}, \tag{3.20}$$

for any $s \in \mathcal{C}^\infty(M, E)$.

Proof. From (3.12) and (3.15), we have

$$-\sum_a \{X^a, \{X^a, s\}\} = \sum_{a,i,j} ((w_i X^a)\nabla^E_{\bar{w}_i} - (\bar{w}_i X^a)\nabla^E_{w_i})((w_j X^a)\nabla^E_{\bar{w}_j} - (\bar{w}_j X^a)\nabla^E_{w_j})s.$$

Note that (3.19) can be written as $\sum_a (uX^a)(vX^a) = g(u, v) = \langle u, v\rangle$. Hence, we obtain

$$\begin{aligned}
-\sum_a \{X^a, \{X^a, s\}\} &= -\sum_i \left(\nabla^E_{w_i}\nabla^E_{\bar{w}_i} + \nabla^E_{\bar{w}_i}\nabla^E_{w_i}\right) s - \sum_{i,j} (\langle w_i, \bar{w}_i\bar{w}_j\rangle - \langle \bar{w}_i, w_i\bar{w}_j\rangle)\nabla^E_{w_j} s \\
&\quad + \sum_{i,j} (\langle w_i, \bar{w}_i w_j\rangle - \langle \bar{w}_i, w_i w_j\rangle)\nabla^E_{\bar{w}_j} s \\
&= -\sum_i \nabla^E_{e_i}\nabla^E_{e_i} s - \sum_{i,j} \langle [w_i, \bar{w}_i], \bar{w}_j\rangle \nabla^E_{w_j} s + \sum_{i,j} \langle [w_i, \bar{w}_i], w_j\rangle \nabla^E_{\bar{w}_j} s \\
&= -\sum_i \nabla^E_{e_i}\nabla^E_{e_i} s + \sum_i \nabla^E_{\nabla^{TM}_{e_i} e_i} s.
\end{aligned}$$

□

Now, we consider the matrix Laplacian, which corresponds to the Bochner Laplacian (2.21) in large-p limit. From (3.20), it is natural to define the matrix Laplacian $\hat{\Delta}$ as follows.

Definition 18.

$$\hat{\Delta}^{\mathrm{Hom}(E_1,E_2)} T_p^{(E_1,E_2)}(s) := \hbar_p^{-2} \sum_a [T_p(X^a \mathbf{1}), [T_p(X^a \mathbf{1}), T_p^{(E_1,E_2)}(s)]], \tag{3.21}$$

for $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$.

It is easy to show that $\hat{\Delta}$ is a formally self-adjoint positive semidefinite operator with respect to the inner product (3.17). We have the following large-p correspondence of Laplacians.

Theorem 3.7. *Let (M, g, ω, J) be a Kähler manifold and let us assume that p is large enough to satisfy* (2.32). *Then, we have*

$$|\hat{\Delta}^{\mathrm{Hom}(E_1,E_2)} T_p^{(E_1,E_2)}(s) - T_p^{(E_1,E_2)}(\Delta^{\mathrm{Hom}(E_2,E_1)} s)| = \mathcal{O}(\hbar_p),$$

for any $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$.

In case of $M = \mathbb{CP}^n$, the Laplacian on $\mathcal{C}^\infty(M, L^{\otimes q})$ is related to the quadratic Casimir and we can use the techniques of the representation theory. For a more general setup, let us consider the following.

Proposition 3.8. *Let (M,ω) be a symplectic manifold and let L be a prequantum line bundle over M. We define a differential operator on $\mathcal{C}^\infty(M, L^{\otimes q})$ by*

$$Q(f) := \nabla^{L^{\otimes q}}_{X_f} - \mathrm{i}2\pi q f,$$

for $f \in \mathcal{C}^\infty(M,\mathbb{C})$. Then, we have

$$[Q(f), Q(g)] = Q(\{f,g\}),$$

for any $f, g \in \mathcal{C}^\infty(M,\mathbb{C})$.

Proof. From (2.6) and $R^{L^{\otimes q}}(X_f, X_g) = -\mathrm{i}2\pi q\omega(X_f, X_g) = \mathrm{i}2\pi q\{f,g\}$, we obtain

$$[Q(f), Q(g)] = [\nabla^{L^{\otimes q}}_{X_f}, \nabla^{L^{\otimes q}}_{X_g}] - \mathrm{i}4\pi q\{f,g\} = \nabla^{L^{\otimes q}}_{X_{\{f,g\}}} + R^{L^{\otimes q}}(X_f, X_g) - \mathrm{i}4\pi q\{f,g\} = Q(\{f,g\}).$$

□

Proposition 3.9. *Let us assume that the smooth isometric embedding $X : M \to \mathbb{R}^d$ satisfy*

$$\{X^a, X^b\} = C \sum_c f_{abc} X^c. \tag{3.22}$$

Here, f_{abc} is a skew-symmetric structure constant of some Lie algebra $\mathfrak{g}$ and C is a real constant number. Then, we can consider a self-adjoint operator on $\mathcal{C}^\infty(M, L^{\otimes q})$ by

$$\mathcal{L}^a := \mathrm{i}C^{-1} Q(X^a). \tag{3.23}$$

Then, $\{\mathcal{L}^a_{(q)}\}$ satisfies

$$[\mathcal{L}^a, \mathcal{L}^b] = \mathrm{i} \sum_c f_{abc} \mathcal{L}^c.$$

Theorem 3.10. *Let (M, g, ω, J) be a Kähler manifold and assume that the smooth isometric embedding $X : M \to \mathbb{R}^d$ satisfy* (3.22) *and*

$$|X|^2 := \sum_a (X^a)^2 = \mathrm{const.}$$

Then, the Bochner Laplacian $\Delta^{L^{\otimes q}}$ can be written as

$$\Delta^{L^{\otimes q}} = C^2 \sum_a (\mathcal{L}^a)^2 - 4\pi^2 q^2 |X|^2. \tag{3.24}$$

Proof. From (3.23) and (3.20), we have

$$C^2 \sum_a (\mathcal{L}^a)^2 = \sum_a (\mathrm{i}\nabla^{L^{\otimes q}}_{X_{X^a}} + 2\pi q X^a)^2 = \Delta^{L^{\otimes q}} + \mathrm{i}4\pi q \sum_a X^a \nabla^{L^{\otimes q}}_{X_{X^a}} + 4\pi^2 q^2 |X|^2.$$

Using $X_{fg} = fX_g + gX_f$, we have $\nabla^{L^{\otimes q}}_{X_{|X|^2}} = 2\sum_a X^a \nabla^{L^{\otimes q}}_{X_{X^a}} = 0$. Therefore, we obtain (3.24). □

3.5 Rectangular matrices as off-diagonals of a block diagonal matrix

Let E_1 and E_2 be vector bundles over M and let $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$. Then, the Toeplitz operator of s defined in (3.5) is a map $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_2) \to \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_1)$. Thus, we can naturally consider an extended linear map

$$T_p^{(E_1 \oplus E_2, E_1 \oplus E_2)} \begin{pmatrix} f_1 & s \\ t & f_2 \end{pmatrix} := \begin{pmatrix} T_p^{(E_1,E_1)}(f_1) & T_p^{(E_1,E_2)}(s) \\ T_p^{(E_2,E_1)}(t) & T_p^{(E_2,E_2)}(f_2) \end{pmatrix},$$

for $f_1 \in \mathcal{C}^\infty(M, \mathrm{End}(E_1))$, $f_2 \in \mathcal{C}^\infty(M, \mathrm{End}(E_2))$ and $t \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_1, E_2))$. Here, $E_1 \oplus E_2$ is the Whitney sum bundle and it corresponds to the Toeplitz operator of the following section:

$$S := \begin{pmatrix} f_1 & s \\ t & f_2 \end{pmatrix} \in \mathcal{C}^\infty(M, \mathrm{End}(E_1 \oplus E_2)).$$

Using this formulation, one can see that the matrix Laplacian (3.21) is naturally appears in the off-diagonal of the following operation

$$\hat{\Delta}^{\mathrm{End}(E_1 \oplus E_2)}(S) := \hbar_p^{-2} \sum_a \left[\hat{X}^a, \left[\hat{X}^a, T_p^{(E_1 \oplus E_2, E_1 \oplus E_2)}(S)\right]\right],$$

where

$$\hat{X}^a := \begin{pmatrix} T_p^{(E_1,E_1)}(X^a \mathbf{1}_{E_1}) & \\ & T_p^{(E_2,E_2)}(X^a \mathbf{1}_{E_2}) \end{pmatrix}.$$

In the context of matrix models of string theories and M-theories, such a block diagonal matrix configuration corresponds to the two-body problem with objects with the same geometry. Therefore, one may use the matrix regularization of homomorphism bundle to uncover the relations of off-diagonal matrices and the two-body interactions.

4 Monopole bundle over fuzzy $\mathbb{CP}^n$

In this subsection, we consider the Berezin-Toeplitz quantization smooth sections of $L^{\otimes q}$ over the complex projective space $\mathbb{CP}^n$ [10] (similar studies in other construction are given for example in [23, 24]).

4.1 Geometry of $\mathbb{CP}^n$

Let us first define the complex projective space $\mathbb{CP}^n$ of complex dimension n.

Definition 19. Let Z, Z' be elements of $\mathbb{C}^{n+1} \setminus \{0\}$. We define an equivalence relation $\sim$ by

$$Z \sim Z' \quad :\Leftrightarrow \quad {}^{\exists} c \in \mathbb{C} \setminus \{0\} : \; Z = cZ'.$$

Then, we define $\mathbb{CP}^n$ as a quotient space

$$\mathbb{CP}^n = \mathbb{C}^{n+1} / \sim .$$

Let $[Z] = [Z^0, Z^1, \cdots, Z^n]$ be a representative of the equivalence class of Z. Then, $\mathbb{CP}^n$ can be covered by $n+1$ patches $\{U_\alpha\}_{\alpha=0}^n$ where $U_\alpha := \{[Z] \in \mathbb{CP}^n \mid Z^\alpha \neq 0\}$. We define the inhomogeneous coordinates $(z^1_{(\alpha)}, z^2_{(\alpha)}, \cdots, z^n_{(\alpha)})$ of U_α by

$$z^i_{(\alpha)} = \begin{cases} Z^{i-1}/Z^\alpha & (i = 1, 2, \cdots, \alpha - 1) \\ Z^i/Z^\alpha & (i = \alpha, \alpha + 1, \cdots, n) \end{cases}.$$

Now, let us introduce a Kähler structure of $\mathbb{CP}^n$ as follows. Let K_α be a local function on U_α defined by

$$K_\alpha(x) := \log(1 + \sum_{i=1}^n |z^i_{(\alpha)}(x)|^2) = \log(\sum_{i=0}^n |Z^i/Z^\alpha|^2).$$

For $x \in U_\alpha \cap U_\beta$, we have $K_\alpha(x) = K_\beta(x) + \log(Z^\beta/Z^\alpha) + \log(\overline{Z^\beta/Z^\alpha})$ and consequently we obtain $\partial\bar{\partial}K_\alpha = \partial\bar{\partial}K_\beta$. Here, $\partial, \bar{\partial}$ are the Dolbeault differentials. Thus, we can introduce a closed two-form ω locally by

$$\omega = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}K. \tag{4.1}$$

We omit the subscripts of the patch unless it is necessary. In terms of the local complex coordinates $\{z^i\}$, ω is expressed as

$$\omega = \frac{\mathrm{i}}{2\pi}\sum_{i,j=1}^n \frac{(1 + |z|^2)\delta_{ij} - \bar{z}^i z^j}{(1 + |z|^2)^2} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j. \tag{4.2}$$

Here, $|z|^2 := \sum_{i=1}^n |z^i|^2$. The normalization of this symplectic form makes $[\omega] \in H^2(\mathbb{CP}^n, \mathbb{Z})$. This can be shown by the following argument. Since the second homology of $\mathbb{CP}^n$ is generated by

$\mathbb{CP}^1 \subset \mathbb{CP}^n$, we only need to show $\int_{\mathbb{CP}^1} \omega \in \mathbb{Z}$. The symplectic form on $\mathbb{CP}^1$ is $\omega = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}z\wedge \mathrm{d}\bar{z}}{1+|z|^2}$, where z is the local inhomogeneous complex coordinates of $\mathbb{CP}^1$. Then, we can show

$$\int_{\mathbb{CP}^1} \omega = 1.$$

Let J be an almost complex structure defined by $J(\partial/\partial z^i) = \mathrm{i}\partial/\partial z^i,\ J(\partial/\partial \bar{z}^i) = -\mathrm{i}\partial/\partial \bar{z}^i$. Then, we define g by $g(u,v) := \omega(u, Jv)$ for any $u, v \in \mathcal{C}^\infty(\mathbb{CP}^n, T\mathbb{CP}^n)$, which is locally written as

$$g = \frac{1}{2\pi}\sum_{i,j=1}^{n} \frac{(1+|z|^2)\delta_{ij} - \bar{z}^i z^j}{(1+|z|^2)^2}(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i). \tag{4.3}$$

The triple (g, ω, J) defines a Kähler structure of $\mathbb{CP}^n$.

Let us consider the isometric embedding of $\mathbb{CP}^n$ into $\mathbb{R}^{n^2+2n}$. We can choose a particular representative of homogeneous coordinate $\zeta = (\zeta^0, \zeta^1, \cdots, \zeta^n)$ such that $|\zeta|^2 = 1$ up to an overall $\mathrm{U}(1)$ phase factor. For example, on the patch U_0, ζ is related to the inhomogeneous coordinate $\{z^i\}$ by

$$\zeta = u\frac{(1, z^1, z^2, \cdots, z^n)}{\sqrt{1+|z|^2}} \in \mathbb{C}^{n+1},$$

for $u \in \mathrm{U}(1)$. Then, we can define a rank 1 hermitian projection $P_\zeta := \zeta\zeta^*$, which is a $(n+1)\times(n+1)$ matrix-valued function on $\mathbb{CP}^n$. Let $\{T^a\}_{a=1}^{n^2+2n}$ be a basis of $\mathfrak{su}(n+1)$ satisfying

$$T^a T^b = \frac{1}{2(n+1)}\delta_{ab}\mathbf{1}_{\mathbb{C}^{n+1}} + \frac{1}{2}\sum_{c=1}^{n^2+2n}(d_{abc} + \mathrm{i}f_{abc})T^c.$$

Here, T^a are traceless Hermitian matrices and d_{abc} and f_{abc} are the symmetric and skew-symmetric structure constants, respectively. Then, we can expand P_ζ as

$$P_\zeta = \frac{\mathbf{1}}{n+1} + 2\pi^{\frac{1}{2}}\sum_{a=1}^{n^2+2n} X^a T^a. \tag{4.4}$$

Here, $\{X^a\}_{a=1}^{n^2+2n}$ is a set of n^2+2n smooth real functions of $\mathbb{CP}^n$, which are given by

$$X^a = \pi^{-\frac{1}{2}}\zeta^* T^a \zeta. \tag{4.5}$$

From $P_\zeta^2 = P_\zeta$, we have

$$\sum_{a=1}^{n^2+2n} X^a X^a = \frac{n}{2\pi(n+1)}, \quad \sum_{a,b=1}^{n^2+2n} d_{abc}X^a X^b - \frac{n-1}{\pi^{\frac{1}{2}}(n+1)}X^c = 0.$$

By the tedious calculation, we also have [1]

$$\{X^a, X^b\} = -2\pi^{\frac{1}{2}}\sum_{c=1}^{n^2+2n} f_{abc}X^c,$$

[1]This calculation can be easily derived from (4.16) and (4.20), which will be shown later.

where we used the following local form Hamiltonian vector field (2.2) induced from (4.2):

$$X_f = \mathrm{i}2\pi(1+|z|^2)\sum_{i,j=1}^{n}(\delta_{ij}+z^i\bar{z}^j)\left(\frac{\partial f}{\partial z^i}\frac{\partial}{\partial \bar{z}^j}-\frac{\partial f}{\partial \bar{z}^j}\frac{\partial}{\partial z^i}\right). \tag{4.6}$$

Note that the metric (4.3) can be written as

$$g = \frac{1}{2\pi}\sum_{i,j=0}^{n}(\delta_{ij}-\bar{\zeta}^i\zeta^j)(\mathrm{d}\zeta^i\otimes \mathrm{d}\bar{\zeta}^j+\mathrm{d}\bar{\zeta}^j\otimes\mathrm{d}\zeta^i) = \frac{1}{2\pi}\operatorname{Tr}(\mathrm{d}P_\zeta\otimes \mathrm{d}P_\zeta),$$

and thus (4.4) implies

$$g = \sum_{a=1}^{n^2+2n}\mathrm{d}X^a\otimes\mathrm{d}X^a.$$

Therefore, $X = (X^1, X^2, \cdots, X^{n^2+2n}) : \mathbb{CP}^n \to \mathbb{R}^{n^2+2n}$ is a smooth isometric embedding.

4.2 Zero modes of the Dirac operator on $\mathbb{CP}^n$

In order to evaluate the matrix element of the Toeplitz operators, we need to construct a complete orthonormal basis of the kernel of the Dirac operator on $\mathcal{C}^\infty(\mathbb{CP}^n, \Lambda^{0,\bullet}\otimes L^{\otimes p})$.

Let D^p be the spin^c Dirac operator on $\mathcal{C}^\infty(\mathbb{CP}^n, \Lambda^{0,\bullet}\otimes L^{\otimes p})$, which is defined in (2.17). As shown in (2.44), any $f^{(p)} \in \ker D^p \subset \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p})$ is simplified to

$$\nabla^{L^{\otimes p}}_{\partial/\partial\bar{z}^i} f^{(p)} = 0, \tag{4.7}$$

for large enough p. From (4.1) and $\omega = \frac{\mathrm{i}}{2\pi}R^L = \frac{\mathrm{i}}{2\pi}\mathrm{d}A^L$, one can take $A^L = -\frac{1}{2}(\partial-\bar{\partial})K$ and therefore (4.7) becomes

$$\left(\frac{\partial}{\partial \bar{z}^i}+\frac{pz^i}{2(1+|z|^2)}\right)f^{(p)} = 0.$$

The general solutions to this equation are

$$f^{(p)} = (1+|z|^2)^{-p/2}\phi(z), \tag{4.8}$$

where $\phi(z)$ is an arbitrary holomorphic function.

Now, let us consider how $f^{(p)}(z)$ transforms under the coordinate change. By considering this, we identify the expression of $f^{(p)} \in \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p})$ in terms of the normalized homogeneous coordinate $\zeta = (\zeta^0, \zeta^1, \cdots, \zeta^n)$. First, we introduce a notation as follows. For $r \in \mathbb{N}$, we define a set

$$\Sigma_r := \{0, 1, \cdots, n\}^r/\text{permutation}.$$

For $\boldsymbol{\alpha}_r = (\alpha_1, \alpha_2, \cdots, \alpha_r), \boldsymbol{\beta}_r = (\alpha_1, \alpha_2, \cdots, \alpha_r) \in \Sigma_r$, we define

$$\zeta^{\boldsymbol{\alpha}_r} := \zeta^{\alpha_1}\zeta^{\alpha_2}\cdots\zeta^{\alpha_r}, \quad \bar{\zeta}^{\boldsymbol{\beta}_r} := \bar{\zeta}^{\beta_1}\bar{\zeta}^{\beta_2}\cdots\bar{\zeta}^{\beta_p}.$$

Let us also define $\mathrm{Pol}_{k+q,k}(\zeta,\bar\zeta)$ as a set of all polynomials of $\zeta^i, \bar\zeta^j$ of degree $(k+q,k)$, i.e.

$$\mathrm{Pol}_{k+q,k}(\zeta,\bar\zeta) := \mathrm{Span}_{\mathbb{C}}\left(\{\zeta^{\boldsymbol{\alpha}_{k+p}}\bar\zeta^{\boldsymbol{\beta}_k}\}_{\boldsymbol{\alpha}_{k+p}\in\Sigma_{k+p},\boldsymbol{\beta}_k\in\Sigma_k}\right).$$

Then, we have the following proposition.

Proposition 4.1.

$$\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p}) = \bigoplus_{k=0}^{\infty}\mathrm{Pol}_{k+p,k}(\zeta,\bar\zeta).$$

Proof. On the overlapping patch $U_\alpha \cap U_\beta$, A^L transforms as $A^L(z_{(\alpha)}) = A^L(z_{(\beta)}) - \mathrm{d}\lambda(z_{(\beta)})$ where

$$\lambda(z_{(\beta)}) = -\frac{1}{2}\left[\log\left(\frac{Z^\alpha}{Z^\beta}\right) - \log\left(\frac{\bar Z^\alpha}{\bar Z^\beta}\right)\right],$$

in terms of the homogeneous coordinate $[Z]$. Correspondingly, any element $f^{(p)} \in \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p})$ transforms as

$$f^{(p)}(z_{(\alpha)}) = \mathrm{e}^{p\lambda(z_{(\beta)})}f^{(p)}(z_{(\beta)}) = \left(\frac{Z^\alpha}{Z^\beta}\right)^{-\frac{p}{2}}\left(\frac{\bar Z^\alpha}{\bar Z^\beta}\right)^{\frac{p}{2}} f^{(p)}(z_{(\beta)}).$$

Here, $f^{(p)}(z_{(\alpha)})$ means that it is a function of $\frac{Z^0}{Z^\alpha}, \frac{Z^1}{Z^\alpha}, \cdots, \frac{Z^n}{Z^\alpha}$ and their complex conjugates $\frac{\bar Z^0}{\bar Z^\alpha}, \frac{\bar Z^1}{\bar Z^\alpha}, \cdots, \frac{\bar Z^n}{\bar Z^\alpha}$. Thus, $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p})$ is spanned by the elements of the following form:

$$\left(\sum_{i=0}^{n}|Z^i|^2\right)^{-k-\frac{p}{2}} Z^{\sigma_1}Z^{\sigma_2}\cdots Z^{\sigma_{k+p}}\bar Z^{\tau_1}\bar Z^{\tau_2}\cdots\bar Z^{\tau_k},$$

where $k \in \mathbb{N}$. □

From (4.8) and Proposition 4.1, we then have the following theorem.

Theorem 4.2. *For large enough p, we have*

$$\ker D^p = \mathrm{Pol}_{p,0}(\zeta,\bar\zeta). \tag{4.9}$$

In the following theorem, we find the complete orthonormal basis of $\ker D^p$ with respect to the inner product

$$(f^{(p)}, g^{(p)}) := \int_{\mathbb{CP}^n}\mu\,\overline{f^{(p)}}g^{(p)},$$

for $f^{(p)}, g^{(p)} \in \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes p})$. Here, $\mu := \mu_g = \mu_\omega$ is the volume form of $\mathbb{CP}^n$.

Theorem 4.3. *Let us define*

$$f^{(p)}_{\boldsymbol{\alpha}_p} := \sqrt{\frac{(p+n)!}{[\boldsymbol{\alpha}_p]!}}\zeta^{\boldsymbol{\alpha}_p}, \tag{4.10}$$

for $\boldsymbol{\alpha}_p \in \Sigma_p$. Here, $[\boldsymbol{\alpha}_p]! := \prod_{i=0}^{n}([\boldsymbol{\alpha}_p]_i!)$ where $[\boldsymbol{\alpha}_p]_i$ is the number of components of $\boldsymbol{\alpha}_p$ equal to $i \in \{0,1,\cdots,n\}$. Then, we have

$$(f^{(p)}_{\boldsymbol{\alpha}_p}, f^{(p)}_{\boldsymbol{\beta}_p}) = \delta_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = \begin{cases}1 & (\boldsymbol{\alpha}_p = \boldsymbol{\beta}_p)\\ 0 & (\boldsymbol{\alpha}_p \neq \boldsymbol{\beta}_p)\end{cases}.$$

Proof. First, let us express $\zeta^{\boldsymbol{\alpha}_p}$ and $\zeta^{\boldsymbol{\beta}_p}$ in terms of the local complex coordinate $\{z^i\}$:

$$\zeta^{\boldsymbol{\alpha}_p} = u^p \frac{(z^1)^{a_1}(z^2)^{a_2}\cdots(z^n)^{a_n}}{(1+|z|^2)^{p/2}}, \quad \zeta^{\boldsymbol{\beta}_p} = u^p \frac{(z^1)^{b_1}(z^2)^{b_2}\cdots(z^n)^{b_n}}{(1+|z|^2)^{p/2}}.$$

Here, u is a U(1) factor coming from the arbitrariness of the choice of ζ and $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are sets of nonnegative integers satisfying $\sum_{i=1}^n a_i \le p$ and $\sum_{i=1}^n b_i \le p$, respectively. Then, we have to show

$$(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) = \frac{(p - \sum_{i=1}^n a_i)! \prod_{i=1}^n (a_i!)}{(p+n)!} \delta_{ab}, \tag{4.11}$$

where $\delta_{ab} := \prod_{i=1}^n \delta_{a_i b_i}$.

Below, we give a proof of (4.11). Since the measure on $\mathbb{CP}^n \setminus U_\alpha$ for any α is zero, we only need to integrate over a single patch:

$$\begin{aligned}(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) &= \int_{\mathbb{C}^n} \mu \frac{(z^1)^{a_1}(z^2)^{a_2}\cdots(z^n)^{a_n}(\bar{z}^1)^{b_1}(\bar{z}^2)^{b_2}\cdots(\bar{z}^n)^{b_n}}{(1+|z|^2)^p} \\ &= \pi^{-n} \int_{\mathbb{R}^{2n}} \frac{\prod_{i=1}^n (x^{2i-1} - \mathrm{i}x^{2i})^{a_i}(x^{2i-1} + \mathrm{i}x^{2i})^{b_i}}{(1+|x|^2)^{p+n+1}} \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^{2n}.\end{aligned}$$

Here, we used the real coordinates defined by $z^i = x^{2i-1} + \mathrm{i}x^{2i}$ and used $\sqrt{\det g} = \pi^{-n}(1+|x|^2)^{-n-1}$. Using the angular coordinates $x^{2i-1} = \rho_i \cos\theta_i$, $x^{2i} = \rho_i \sin\theta_i$, we then obtain

$$(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) = \pi^{-n} \prod_{i=1}^n \left(\int_0^\infty \rho_i \mathrm{d}\rho_i \int_0^{2\pi} \mathrm{d}\theta_i \right) \frac{\prod_{i=1}^n \left(\rho_i \mathrm{e}^{\mathrm{i}\theta_i}\right)^{a_i} \left(\rho_i \mathrm{e}^{-\mathrm{i}\theta_i}\right)^{b_i}}{(1+\sum_{i=1}^n \rho_i^2)^{p+n+1}}.$$

The integral over θ_i gives a factor $\delta_{a_i b_i}$ and we then have

$$(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) = 2^n \delta_{a,b} \int_{[0,\infty)^n} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2 \cdots \mathrm{d}\rho_n}{(1+\sum_{i=1}^n \rho_i^2)^{p+n+1}} \prod_{i=1}^n \rho_i^{2a_i+1}.$$

Now, we employ the spherical coordinates $(\rho, \phi_1, \phi_2, \cdots, \phi_{n-1}) \in [0,\infty) \times [0, \pi/2]^{n-1}$ defined by

$$\rho_1 = \rho\cos\phi_1, \quad \rho_2 = \rho\sin\phi_1\cos\phi_2, \quad \cdots, \quad \rho_{n-1} = \rho\left(\prod_{i=1}^{n-2}\sin\phi_i\right)\cos\phi_{n-1}, \quad \rho_n = \rho\prod_{i=1}^{n-1}\sin\phi_i,$$

and we obtain

$$(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) = 2^n \delta_{a,b} \int_0^\infty \mathrm{d}\rho \frac{\rho^{2\sum_{i=1}^n (a_i+1)-1}}{(1+\rho^2)^{p+n+1}} \prod_{i=1}^{n-1} \left(\int_0^{\pi/2} \mathrm{d}\phi_i \sin^{2\sum_{j=i+1}^n (a_j+1)-1}(\phi_i) \cos^{2a_i+1}(\phi_i) \right).$$

Let us use the Beta function

$$B(x,y) = 2\int_0^{\pi/2} \mathrm{d}\phi \sin^{2x-1}\phi \, \cos^{2y-1}\phi = 2\int_0^\infty \mathrm{d}\rho \frac{\rho^{2x-1}}{(1+\rho^2)^{x+y}},$$

which is defined for complex variables x, y with $\Re x, \Re y > 0$. Then, we find

$$(\zeta^{\boldsymbol{\alpha}_p}, \zeta^{\boldsymbol{\beta}_p}) = \delta_{ab} B\left(\sum_{i=1}^{n}(a_i+1), p+1-\sum_{i=1}^{n} a_i\right)\prod_{i=1}^{n-1} B\left(\sum_{j=i+1}^{n}(a_j+1), a_i+1\right).$$

Using $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\Gamma(x+1) = x!$, we therefore obtain (4.11). □

By counting the number of independent symmetric polynomials of degree p with $n+1$ variables, we have

$$\dim\ker D^p = \frac{(p+n)!}{p!n!}. \tag{4.12}$$

This is consistent with the index theorem. From (2.33), we have

$$\dim\ker D^p = \int_{\mathbb{CP}^n}\left(\frac{\omega}{1-\mathrm{e}^{-\omega}}\right)^{n+1}\mathrm{e}^{p\omega} = C_{p,n}\int_{\mathbb{CP}^n}\omega^n.$$

where we used the residue theorem

$$C_{p,n} := \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z^{n+1}}\left(\frac{z}{1-\mathrm{e}^{-z}}\right)^{n+1}\mathrm{e}^{pz} = \frac{1}{2\pi\mathrm{i}}\oint\mathrm{d}z\frac{\mathrm{e}^{pz}}{(1-\mathrm{e}^{-z})^{n+1}}.$$

The integration contour is a counterclockwise circle around $z = 0$. By simple calculations, we find

$$C_{p,n} = \frac{(p+n)!}{p!n!}, \qquad \int_{\mathbb{CP}^n}\omega^n = 1,$$

which reproduces (4.12).

4.3 Matrix regularization of embedding functions

Let $\Pi^p : \mathcal{C}^\infty(\mathbb{CP}^n, \Lambda^{0,\bullet}\otimes L^{\otimes p})$ be the orthogonal projection. Then, we define the Toeplitz operator of a function $f \in \mathcal{C}^\infty(\mathbb{CP}^n, \mathbb{C})$ by

$$T_p(f) = \Pi^p f\,\Pi^p.$$

Here, we consider the Toeplitz operator of components isometric embedding $X : \mathbb{CP}^n \to \mathbb{R}^{n^2+2n}$ defined in (4.5). The result is summarized in the following theorem.

Theorem 4.4.

$$T_p(X^a) = -\frac{\pi^{-\frac{1}{2}}}{p+n+1}L^a_{(p,0,\cdots,0)}.$$

Here, $\{L^a_{(p,0,\cdots,0)}\}_{a=1}^{n^2+2n}$ *are irreducible representation of* $\{T^a\}_{a=1}^{n^2+2n}$ *with Dynkin index* $(p,0,\cdots,0)$,

$$[L^a_{(p,0,\cdots,0)}, L^b_{(p,0,\cdots,0)}] = \mathrm{i}\sum_{c=1}^{n^2+2n} f_{abc}L^a_{(p,0,\cdots,0)}, \qquad \sum_{a=1}^{n^2+2n}(L^a_{(p,0,\cdots,0)})^2 = \frac{np(p+n+1)}{2(n+1)}\mathbf{1}.$$

Proof. Let us consider the matrix element

$$T_p(X^a)_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} := (f^{(p)}_{\boldsymbol{\alpha}_p}, T_p(X^a) f^{(p)}_{\boldsymbol{\beta}_p}) = (f^{(p)}_{\boldsymbol{\alpha}_p}, X^a f^{(p)}_{\boldsymbol{\beta}_p}).$$

From (4.5) and (4.10), we have

$$\begin{aligned} T_p(X^a)_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} &= \pi^{-\frac{1}{2}} \sum_{i,j=0}^{n} T^a_{ij} (\zeta^i f^{(p)}_{\boldsymbol{\alpha}_p}, \zeta^j f^{(p)}_{\boldsymbol{\beta}_p}) \\ &= \pi^{-\frac{1}{2}} \frac{(p+n)!}{\sqrt{[\boldsymbol{\alpha}_p]![\boldsymbol{\beta}_p]!}} \sum_{i,j=0}^{n} T^a_{ij} (\zeta^{\boldsymbol{\alpha}_p \oplus i}, \zeta^{\boldsymbol{\beta}_p \oplus j}). \end{aligned}$$

Here, we defined a map $\oplus : \Sigma_r \times \Sigma_s \to \Sigma_{r+s}$ such that $\boldsymbol{\alpha}_r \oplus \boldsymbol{\gamma}_s = (\alpha_1, \alpha_2, \cdots, \alpha_r, \gamma_1, \gamma_2, \cdots, \gamma_s) \in \Sigma_{r+s}$. Then, we have

$$(L^a_{(p,0,\cdots,0)})_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = -\sqrt{\frac{[\boldsymbol{\alpha}_p]!}{[\boldsymbol{\beta}_p]!}} \sum_{i,j=0}^{n} T^a_{ij} ([\boldsymbol{\alpha}_p]_i + 1) \delta_{\boldsymbol{\alpha}_p \oplus i, \boldsymbol{\beta}_p \oplus j}. \tag{4.13}$$

This gives

$$(L^a_{(p,0,\cdots,0)} L^b_{(p,0,\cdots,0)})_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = \sqrt{\frac{[\boldsymbol{\alpha}_p]!}{[\boldsymbol{\beta}_p]!}} \sum_{i,j,k,l=0}^{n} T^a_{ij} T^b_{kl} ([\boldsymbol{\alpha}_p]_i + 1)([\boldsymbol{\alpha}_p]_k + \delta_{ik} - \delta_{jk} + 1) \delta_{\boldsymbol{\alpha}_p \oplus i \oplus k, \boldsymbol{\beta}_p \oplus j \oplus l}. \tag{4.14}$$

Therefore, we find

$$\begin{aligned} [L^a_{(p,0,\cdots,0)}, L^b_{(p,0,\cdots,0)}]_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} &= -\sqrt{\frac{[\boldsymbol{\alpha}_p]!}{[\boldsymbol{\beta}_p]!}} \sum_{i,l=0}^{n} [T^a, T^b]_{il} ([\boldsymbol{\alpha}_p]_i + 1) \delta_{\boldsymbol{\alpha}_p \oplus i, \boldsymbol{\beta}_p \oplus l} \\ &= \mathrm{i} \sum_{c=1}^{n^2+2n} (L^c_{(p,0,\cdots,0)})_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}. \end{aligned}$$

Secondly, let us calculate the quadratic Casimir. From (4.14)

$$\sum_{a=1}^{n^2+2n} (L^a_{(p,0,\cdots,0)})^2_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = \sqrt{\frac{[\boldsymbol{\alpha}_p]!}{[\boldsymbol{\beta}_p]!}} \sum_{i,j,k,l=0}^{n} \sum_{a=1}^{n^2+2n} T^a_{ij} T^a_{kl} ([\boldsymbol{\alpha}_p]_i + 1)([\boldsymbol{\alpha}_p]_k + \delta_{ik} - \delta_{jk} + 1) \delta_{\boldsymbol{\alpha}_p \oplus i \oplus k, \boldsymbol{\beta}_p \oplus j \oplus l}.$$

Using the Fierz identity

$$\sum_{a=1}^{n^2+2n} T^a_{ij} T^a_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{n+1} \delta_{ij} \delta_{lk} \right), \tag{4.15}$$

we obtain

$$\sum_{a=1}^{n^2+2n} (L^a_{(p,0,\cdots,0)})^2_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p} = \frac{np(p+n+1)}{2(n+1)} \delta_{\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p}.$$

□

4.4 Bochner Laplacian on $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ and its spectrum

Here, we consider the spectral analysis of the Bochner Laplacian on $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$.

Let us employ the technique of Proposition 3.9. Let us define differential operators $\{\mathcal{L}_a\}_{a=1}^{n^2+2n}$ on $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ by

$$\mathcal{L}^a := -\frac{\mathrm{i}}{2\pi^{\frac{1}{2}}}\left(\nabla^{L^{\otimes q}}_{X_{X^a}} - \mathrm{i}2\pi q X^a\right). \tag{4.16}$$

Then, we have

$$[\mathcal{L}^a, \mathcal{L}^b] = \mathrm{i}\sum_{c=1}^{n^2+2n} f_{abc}\mathcal{L}^c.$$

By Theorem 3.10, we then have

$$\Delta^{L^{\otimes q}} = 2\pi\left(2\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2 - \frac{q^2 n}{n+1}\right).$$

Thus, the eigenvalue of $\Delta^{L^{\otimes q}}$ is $2\pi\left(2E - \frac{q^2n}{n+1}\right)$, where E is an eigenvalue of $\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2$.

Now, we study the spectral analysis of $\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2$.

Theorem 4.5. *The eigenvalues and their associated eigenvectors of $\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2$ are given by*

$$E_k = \frac{1}{2}\left((k+q)(k+n) + k(k+q+n) + \frac{q^2 n}{n+1}\right), \tag{4.17}$$

and

$$f^{(q)}_{k,w} = \sum_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k} c_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k,w}\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k}, \tag{4.18}$$

for $k \in \mathbb{N}$. Here, $c_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k,w} := c_{\sigma_1\cdots\sigma_{k+q},\tau_1\cdots\tau_k,w}$ is completely symmetric in σ_a and τ_b, respectively, and traceless under any contraction between σ_a and τ_b. The index w labels the degeneracy of the eigenvectors with eigenvalues E_k, that is, w labels linearly independent the completely symmetric traceless tensor of $(k+q,k)$ type.

Proof. Let us show this theorem in two different approaches.

First approach is to use the representation theory of $\mathfrak{su}(n+1)$. Let $V_{(d_1,d_2,\cdots,d_n)}$ be the irreducible representation space of $\mathfrak{su}(n+1)$ with Dynkin index $(d_1,d_2,\cdots,d_n)$ and let $V^*_{(d_1,d_2,\cdots,d_n)}$ be the representation space of the complex conjugate representation of $(d_1,d_2,\cdots,d_n)$. Then, (4.9) and Theorem 4.4 implies that

$$\ker D^p = \mathrm{Pol}_{p,0}(\zeta,\bar{\zeta}) = V_{(p,0,\cdots,0)}.$$

Using Proposition 4.1 implies

$$\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q}) = \bigoplus_{k=0}^{\infty} V_{(q+k,0,\cdots,0)} \otimes V^*_{(k,0,\cdots,0)}.$$

Using the decomposition into irreducible representation $V_{(k+q,0,\cdots,0)} \otimes V^*_{(k,0,\cdots,0)} = \bigoplus_{i=0}^k V_{(i+q,0,\cdots,0,i)}$, we have

$$\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q}) = \bigoplus_{k=0}^\infty V_{(k+q,0,\cdots,0,k)}. \tag{4.19}$$

Therefore, the eigenvalues of $\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2$ are the quadratic Casimir for the representations $(k+q,0,\cdots,0,k)$, which are given by (4.17). The corresponding eigenvectors belong to $V_{(k+q,0,\cdots,0,k)}$ and therefore they are of the form (4.18), where the index w labels different weights of $V_{(k+q,0,\cdots,0,k)}$.

Second approach is a purely analytic computation. After a long calculation using (4.5), (4.6) and (4.16), we find

$$\mathcal{L}^a(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k}) = \sum_{i,j=0}^n T^a_{ij}\left(-[\boldsymbol{\sigma}_{k+q}]_i\zeta^{\boldsymbol{\sigma}_{k+q}\ominus i\oplus j}\bar{\zeta}^{\boldsymbol{\tau}_k} + [\boldsymbol{\tau}_k]_j\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k\ominus j\oplus i}\right). \tag{4.20}$$

Here, we defined $\ominus$ such that for $\boldsymbol{\alpha}_{r+s}\in\Sigma_{r+s}, \boldsymbol{\beta}_s\in\Sigma_s$,

$$\boldsymbol{\alpha}_{r+s}\ominus\boldsymbol{\beta}_s := \begin{cases}\boldsymbol{\gamma}_r\in\Sigma_r, & (\text{if } {}^\exists\boldsymbol{\gamma}_r:\ \boldsymbol{\gamma}_r\oplus\boldsymbol{\beta}_s = \boldsymbol{\alpha}_{r+s})\\ \mathbf{0}\in\Sigma_0, & (\text{otherwise})\end{cases}$$

and we set $\zeta^{\mathbf{0}} = \bar{\zeta}^{\mathbf{0}} = 0$. Also, we are using the local complex coordinate on patch U_0 for simplicity. From (4.15), we obtain

$$\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k}) = E_k\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k} - \sum_{i,j=0}^n[\boldsymbol{\sigma}_{k+q}]_i[\boldsymbol{\tau}_k]_i\zeta^{\boldsymbol{\sigma}_{k+q}\ominus i\oplus j}\bar{\zeta}^{\boldsymbol{\tau}_k\ominus i\oplus j}.$$

From $[\boldsymbol{\sigma}_{k+q}]_i[\boldsymbol{\tau}_k]_i = \sum_{a=1}^{k+p}\sum_{b=1}^p\delta_{i,\sigma_a}\delta_{\sigma_a,\tau_b}$, the traceless property $\sum_{\sigma_a=0}^n\delta_{\sigma_a,\tau_b}c_{\boldsymbol{\sigma}_{k+q},\boldsymbol{\tau}_k,w} = 0$ implies

$$\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2 f^{(q)}_{k,w} = E_k f^{(q)}_{k,w}.$$

□

4.5 Matrix regularization of $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ and the matrix Laplacian

Here, we explicitly evaluate the Toeplitz operators of $\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ and discuss spectral analysis of the matrix Laplacian $\hat{\Delta}^{L^{\otimes q}}$.

Let us define the Toeplitz operator of $f^{(q)}\in\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ by

$$T_p(f^{(q)}) = \Pi^{p+q}f^{(q)}\Pi^p.$$

The matrix elements of the Toeplitz operator of $\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k}\in\mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$ is given by

$$\begin{aligned} T_p(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k})_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} :=& (f^{(p+q)}_{\boldsymbol{\alpha}_{p+q}}, T_p(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k})f^{(p)}_{\boldsymbol{\beta}_p}) \\ =& \frac{\sqrt{(p+q+n)!(p+n)!}}{(p+q+k+n)!}\sqrt{\frac{[\boldsymbol{\alpha}_{p+q}]!}{[\boldsymbol{\beta}_p]!}}\frac{[\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k]!}{[\boldsymbol{\alpha}_{p+q}]!}\delta_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k,\boldsymbol{\beta}_p\oplus\boldsymbol{\sigma}_{k+q}}. \end{aligned} \tag{4.21}$$

Now, let us consider the matrix Laplacian (3.21),

$$\hat{\Delta}^{L^{\otimes q}}(T_p(f^{(q)})) = (2\pi p)^2 \sum_{a=1}^{n^2+2n} [T_p(X^a \mathbf{1}), [T_p(X^a \mathbf{1}), T_p(f^{(q)})]].$$

We introduce the following operation on Toeplitz operators

$$\hat{\mathcal{L}}^a T_p(f^{(q)}) := L^a_{(p+q,0,\cdots,0)} T_p(f^{(q)}) - T_p(f^{(q)}) L^a_{(p,0,\cdots,0)}. \tag{4.22}$$

This operator satisfies

$$[\hat{\mathcal{L}}^a, \hat{\mathcal{L}}^b] = \mathrm{i} \sum_{c=1}^{n^2+2n} f_{abc} \hat{\mathcal{L}}^c,$$

and hence $\{\hat{\mathcal{L}}^a\}_{a=1}^{n^2+2n}$ they are representations of $\{T^a\}_{a=1}^{n^2+2n}$. Using Theorem 4.4, we can show

$$\hat{\Delta}^{L^{\otimes q}}(T_p(f^{(q)})) = \frac{2\pi p^2}{(p+q+n+1)(p+n+1)} \left(2 \sum_{a=1}^{n^2+2n} (\hat{\mathcal{L}}^a)^2 - \frac{q^2 n}{n+1} \right) T_p(f^{(q)}).$$

Thus, the eigenvalue of $\hat{\Delta}^{L^{\otimes q}}$ is $\frac{2\pi p^2}{(p+q+n+1)(p+n+1)} \left(2E - \frac{q^2 n}{n+1} \right)$, where E is an eigenvalue of $\sum_{a=1}^{n^2+2n} (\hat{\mathcal{L}}^a)^2$.

Before going to the spectral analysis of $\sum_{a=1}^{n^2+2n} (\hat{\mathcal{L}}^a)^2$, let us show the following correspondence.

Theorem 4.6.

$$T_p(\mathcal{L}^a f^{(q)}) = \hat{\mathcal{L}}^a T_p(f^{(q)}), \tag{4.23}$$

for any $f^{(q)} \in \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$.

Proof. From (4.20) and (4.21), we have

$$\begin{aligned} T_p(\mathcal{L}^a(\zeta^{\boldsymbol{\sigma}_{k+q}} \bar{\zeta}^{\boldsymbol{\tau}_k}))_{\boldsymbol{\alpha}_{p+q}, \boldsymbol{\beta}_p} &= \frac{\sqrt{(p+q+n)!(p+n)!}}{(p+q+k+n)!} \sqrt{\frac{[\boldsymbol{\alpha}_{p+q}]!}{[\boldsymbol{\beta}_p]!}} \frac{[\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k]!}{[\boldsymbol{\alpha}_{p+q}]!} \\ &\quad \times \sum_{i,j=0}^n T^a_{ij} \left(-[\boldsymbol{\sigma}_{k+q}]_i + [\boldsymbol{\tau}_k]_j \frac{[\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k \ominus j \oplus i]!}{[\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k]!} \right) \delta_{\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k \oplus i, \boldsymbol{\beta}_p \oplus \boldsymbol{\sigma}_{k+q} \oplus j}. \end{aligned}$$

On the other hand, the matrix element of $\mathcal{L}^a T_p(\zeta^{\boldsymbol{\sigma}_{k+q}} \bar{\zeta}^{\boldsymbol{\tau}_k})$ is given by

$$\begin{aligned} (\mathcal{L}^a T_p(\zeta^{\boldsymbol{\sigma}_{k+q}} \bar{\zeta}^{\boldsymbol{\tau}_k}))_{\boldsymbol{\alpha}_{p+q}, \boldsymbol{\beta}_p} &= \frac{\sqrt{(p+q+n)!(p+n)!}}{(p+q+k+n)!} \sqrt{\frac{[\boldsymbol{\alpha}_{p+q}]!}{[\boldsymbol{\beta}_p]!}} \frac{[\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k]!}{[\boldsymbol{\alpha}_{p+q}]!} \\ &\quad \times \sum_{i,j=0}^n T^a_{ij} \left(-[\boldsymbol{\alpha}_{p+q} \oplus i]_j \frac{[\boldsymbol{\alpha}_{p+q} \oplus i \ominus j \oplus \boldsymbol{\tau}_k]!}{[\boldsymbol{\alpha}_{p+q} \oplus \boldsymbol{\tau}_k]!} + [\boldsymbol{\beta}_p \oplus j]_i \right) \delta_{\boldsymbol{\alpha}_p \oplus i \oplus \boldsymbol{\tau}_k, \boldsymbol{\beta}_p \oplus \boldsymbol{\sigma}_{k+q} \oplus j}, \end{aligned}$$

using (4.21) and (4.13). Therefore, we have

$$
\begin{aligned}
&\left(T_p(\mathcal{L}^a(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k})) - \mathcal{L}^a T_p(\zeta^{\boldsymbol{\sigma}_{k+q}}\bar{\zeta}^{\boldsymbol{\tau}_k})\right)_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} \\
&= \frac{\sqrt{(p+q+n)!(p+n)!}}{(p+q+k+n)!}\sqrt{\frac{[\boldsymbol{\alpha}_{p+q}]!}{[\boldsymbol{\beta}_p]!}\frac{[\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k]!}{[\boldsymbol{\alpha}_{p+q}]!}} \\
&\quad\times\sum_{i,j=0}^{n} T^a_{ij}\left(-[\boldsymbol{\sigma}_{k+q}\oplus\boldsymbol{\beta}_p\oplus j]_i + [\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k\oplus i]_j\frac{[\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k\ominus j\oplus i]!}{[\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k]!}\right)\delta_{\boldsymbol{\alpha}_{p+q}\oplus\boldsymbol{\tau}_k\oplus i,\boldsymbol{\beta}_p\oplus\boldsymbol{\sigma}_{k+q}\oplus j} \\
&= 0.
\end{aligned}
$$

This holds for any $\boldsymbol{\sigma}_{k+q}\in\Sigma_{k+q}, \boldsymbol{\tau}_k\in\Sigma_k$ and any $k\in\mathbb{N}$ and thus we show (4.23). □

Using this theorem, we can easily obtain the eigenvalues and eigenvectors of $\sum_{a=1}^{n^2+2n}(\hat{\mathcal{L}}^a)^2$ as follows.

Theorem 4.7. *The eigenvalues of $\sum_{a=1}^{n^2+2n}(\hat{\mathcal{L}}^a)^2$ are E_k for $k=0,1,\cdots,p$ and the corresponding eigenvectors are $T_p(f^{(q)}_{k,w})$.*

Proof. The eigenvalues of $\sum_{a=1}^{n^2+2n}(\hat{\mathcal{L}}^a)^2$ can be obtained using the representation theory of $\mathfrak{su}(n+1)$. From the definition (4.22), we can see that $\mathcal{L}^a$ acts on the representation space

$$
V_{(p+q,0,\cdots,0)}\otimes V^*_{(p,0,\cdots,0)} = \bigoplus_{k=0}^{p} V_{(k+q,0,\cdots,0,k)}.
$$

This is a similar to the irreducible decomposition (4.19) except for the cut-off $k\le p$. Thus, the eigenvalues are E_k for $k=0,1,\cdots,p$.

More explicitly, we can use (4.23) to identify the correspondence of eigenvalues or eigenvectors of $\sum_{a=1}^{n^2+2n}(\mathcal{L}^a)^2$ and $\sum_{a=1}^{n^2+2n}(\hat{\mathcal{L}}^a)^2$. Note that the matrix element of $T_p(f^{(q)}_{k,w})$ is given as

$$
T_p(f^{(q)}_{k,w})_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p}\propto(\zeta^{\boldsymbol{\alpha}_{p+q}}\bar{\zeta}^{\boldsymbol{\beta}_q}, f^{(q)}_{k,w}).
$$

Since $\zeta^{\boldsymbol{\alpha}_{p+q}}\bar{\zeta}^{\boldsymbol{\beta}_q}$ can be expanded by the basis $\{f^{(q)}_{k',w'}\}_{k'\le p}$, we find

$$
{}^{\forall}k>p:\quad T_p(f^{(q)}_{k,w})=0.
$$

For $k\le p$, $T_p(f^{(q)}_{k,w})\neq 0$ and (4.23) implies

$$
\sum_{a=1}^{n^2+2n}(\hat{\mathcal{L}}^a)^2 T_p(f^{(q)}_{k,w}) = \sum_{a=1}^{n^2+2n} T_p((\mathcal{L}^a)^2 f^{(q)}_{k,w}) = E_k T_p(f^{(q)}_{k,w}).
$$

□

From this theorem, we see that the spectrum of $\hat{\Delta}^{L^{\otimes q}}$ is the truncated version of the spectrum of $\Delta^{L^{\otimes q}}$ up to a difference of order $\mathcal{O}(\hbar_p)$:

$$\frac{2\pi p^2}{(p+q+n+1)(p+n+1)}\left(2E_k - \frac{q^2 n}{n+1}\right) = 2\pi\left(2E_k - \frac{q^2 n}{n+1}\right) + \mathcal{O}(\hbar_p).$$

Finally, let us prove the following correspondence.

Theorem 4.8. *Let us normalize* $\{f^{(q)}_{k,w}\}$ *such that*

$$\left(f^{(q)}_{k,w}, f^{(q)}_{k',w'}\right) = \delta_{k,k'}\delta_{w,w'}.$$

Then, the inner products of $\{T_p(f^{(q)}_{k,w})\}$ *with respect to* (3.17) *are*

$$\left(T_p(f^{(q)}_{k,w}), T_p(f^{(q)}_{k',w'})\right) = \frac{(p+q+n)!(p+n)!}{p^n(p-k)!(p+q+k+n)!}\delta_{k,k'}\delta_{w,w'},$$

for $k, k' \leq p$.

Proof. First, let us show

$$\left(T_p(f^{(q)}_{k,w}), T_p(f^{(q)}_{k',w'})\right) \propto \delta_{k,k'}\delta_{w,w'}.$$

To show this, let us consider the Cartan-Weyl basis $\{H_a, E_\alpha\}$ of $\mathfrak{su}(n+1)$ satisfying

$$[H_a, H_b] = 0, \quad [H_a, E_{\pm\alpha}] = \pm\alpha_a E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}] = \sum_a \alpha_a H_a, \quad E^*_\alpha = E_{-\alpha}. \tag{4.24}$$

Here, $\{H_a\}_{a=1}^n$ be the basis of Cartan subalgebra and α denotes a root vector. The standard choice of $\{H_a\}_{a=1}^n$ is $H_a := T^{a^2+2a}$ where T^{a^2+2a} is defined as a diagonal matrix with the following entries:

$$T^{a^2+2a}_{ii} := \begin{cases} \left(\frac{2}{a^2+a}\right)^{\frac{1}{2}}, & (0 \leq i < a) \\ -\left(\frac{2}{a^2+a}\right)^{\frac{1}{2}} a, & (i = a) \\ 0. & (a < i \leq n) \end{cases} \tag{4.25}$$

Now, we define the irreducible representation maps $\rho : \mathfrak{su}(n+1) \to \mathrm{End}(V)$ and $\hat{\rho} : \mathfrak{su}(n+1) \to \mathrm{End}(\hat{V})$ such that

$$\begin{aligned} V &= \mathrm{Span}_{\mathbb{C}}(\{f_{k,w}\}), & \rho(T^a) &= \mathcal{L}^a, \\ \hat{V} &= \mathrm{Span}_{\mathbb{C}}(\{T_p(f_{k,w})\}), & \hat{\rho}(T^a) &= \hat{\mathcal{L}}^a. \end{aligned}$$

Then, the correspondence (4.23), which can be neatly expressed as

$$\hat{\rho}(v)T_p(f^{(q)}) = T_p(\rho(v)f^{(q)}), \tag{4.26}$$

for any $v \in \mathfrak{su}(n+1)$ and $f^{(q)} \in \mathcal{C}^\infty(\mathbb{CP}^n, L^{\otimes q})$. From now on, we take w as the weight vector $w = (w_1, w_2, \cdots, w_n)$ such that

$$\rho(H_a) f^{(q)}_{k,w} = w_a f^{(q)}_{k,w}.$$

Then, by the self-adjointness of $\sum_a (\hat{\mathcal{L}}^a)^2$ and $\hat{\rho}(H_a)$, we find

$$\left(T_p(f^{(q)}_{k,w}), T_p(f^{(q)}_{k',w'})\right) = C_{n,p,q,k,w}\delta_{k,k'}\delta_{w,w'},$$

for a constant $C_{n,p,q,k,w}$. Moreover, using (4.24) and (4.26), one can show

$$(T_p(f^{(q)}_{k,w+\alpha}), T_p(f^{(q)}_{k,w+\alpha})) = (T_p(f^{(q)}_{k,w}), T_p(f^{(q)}_{k,w})),$$

for any w and α. This shows that $C_{n,p,q,k,w}$ does not depend on w.

From the above argument, we then only need to calculate $|T_p(f_{k,w})|^2 := (T_p(f_{k,w}), T_p(f_{k,w}))$ for a fixed weight w for each k. For example, let us consider the highest weight $w_{\max}$. In our convention (4.25), we find

$$f^{(q)}_{k,w_{\max}} = \sqrt{\frac{(2k+q+n)!}{k!(k+q)!}} (\zeta^1)^{k+q} (\bar{\zeta}^0)^k,$$

using (4.20). From (4.21), we find

$$T_p(f^{(q)}_{k,w_{\max}})_{\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_p} = \sqrt{\frac{(2k+q+n)!(p+q+n)!(p+n)!}{k!(k+q)!((p+q+k+n)!)^2}} \frac{[\boldsymbol{\alpha}_{p+q} \oplus \mathbf{0}_k]!}{\sqrt{[\boldsymbol{\alpha}_{p+q}]![\boldsymbol{\beta}_p]!}} \delta_{\boldsymbol{\alpha}_{p+q}\oplus\mathbf{0}_k, \boldsymbol{\beta}_p \oplus \mathbf{1}_{k+q}},$$

where we defined $\mathbf{0}_k = (0, 0, \cdots, 0) \in \Sigma_k$ and $\mathbf{1}_{k+q} = (1, 1, \cdots, 1) \in \Sigma_{k+q}$. The only non-vanishing matrix elements of $T_p(f^{(q)}_{k,w_{\max}})$ are

$$T_p(f^{(q)}_{k,w_{\max}})_{\boldsymbol{\gamma}_{p-k}\oplus\mathbf{1}_{k+q}, \boldsymbol{\gamma}_{p-k}\oplus\mathbf{0}_k} = \sqrt{\frac{(2k+q+n)!(p+q+n)!(p+n)!}{k!(k+q)!((p+q+k+n)!)^2}} \frac{[\boldsymbol{\gamma}_{p-k} \oplus \mathbf{0}_k \oplus \mathbf{1}_{k+q}]!}{\sqrt{[\boldsymbol{\gamma}_{p-k} \oplus \mathbf{1}_{k+q}]![\boldsymbol{\gamma}_{p-k} \oplus \mathbf{0}_k]!}},$$

for any $\boldsymbol{\gamma}_{p-k} \in \Sigma_{p-k}$. Hence, we obtain

$$\begin{aligned} |T_p(f^{(q)}_{k,w_{\max}})|^2 &= \frac{(2k+q+n)!(p+q+n)!(p+n)!}{p^n k!(k+q)!((p+q+k+n)!)^2} \sum_{\boldsymbol{\gamma}_{p-k}\in\Sigma_{p-k}} \frac{([\boldsymbol{\gamma}_{p-k} \oplus \mathbf{0}_k \oplus \mathbf{1}_{k+q}]!)^2}{[\boldsymbol{\gamma}_{p-k} \oplus \mathbf{1}_{k+q}]![\boldsymbol{\gamma}_{p-k} \oplus \mathbf{0}_k]!} \\ &= \frac{(2k+q+n)!(p+q+n)!(p+n)!}{p^n((p+q+k+n)!)^2} \sum_{i_1=0}^{p-k} \sum_{i_2=0}^{p-k-i_1} \cdots \sum_{i_n=0}^{p-k-\sum_{j=1}^{n-1} i_j} \binom{p - \sum_{j=1}^n i_j}{k} \binom{i_1+k+q}{k+q}, \end{aligned}$$

where we rewrite the summation as

$$\sum_{\boldsymbol{\gamma}_{p-k}\in\Sigma_{p-k}} F([\boldsymbol{\gamma}_{p-k}]_0, [\boldsymbol{\gamma}_{p-k}]_1, \cdots, [\boldsymbol{\gamma}_{p-k}]_n) = \sum_{i_1=0}^{p-k} \sum_{i_2=0}^{p-k-i_1} \cdots \sum_{i_n=0}^{p-k-\sum_{j=1}^{n-1} i_j} F(p-k-\sum_{j=1}^n i_j, i_1, \cdots, i_n).$$

Let us use the identities for the binomial coefficients,

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}, \qquad \sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1},$$

where they are called the hockey-stick identity and the Chu-Vandermonde identity, respectively. Then, we find

$$\begin{aligned} |T_p(f^{(q)}_{k,w_{\max}})|^2 &= \frac{(2k+q+n)!(p+q+n)!(p+n)!}{p^n((p+q+k+n)!)^2} \sum_{i_1=0}^{p-k} \binom{p+n-1-i_1}{k+n-1} \binom{i_1+k+q}{k+q} \\ &= \frac{(p+q+n)!(p+n)!}{p^n(p-k)!(p+q+k+n)!}. \end{aligned}$$

□

From this theorem, we can see that the large-p correspondence of the inner products (3.18).

5 Monopole bundle over fuzzy $\mathbb{T}^2_\tau$

In this subsection, we consider the Berezin-Toeplitz quantization of smooth sections of $L^{\otimes q}$ over the one-dimensional complex torus $\mathbb{T}^2_\tau$ [13].

5.1 Geometry of $\mathbb{T}^2_\tau$

First, let us define the one-dimensional complex torus $\mathbb{T}^2$ as follows.

Definition 20. Let τ be an element of the complex upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} | \Im\tau > 0\}$. For $z, z' \in \mathbb{C}$, we introduce an equivalence relation $\sim$ such that

$$z \sim z' \quad :\Leftrightarrow \quad {}^\exists k, l \in \mathbb{Z} : \; z - z' = k + \tau l. \tag{5.1}$$

Then, we define $\mathbb{T}^2_\tau$ as

$$\mathbb{T}^2_\tau := \mathbb{C}/\sim .$$

We also use the real coordinates $\{x, y\}$ such that

$$z = x + \tau y, \tag{5.2}$$

which are identified by $x \sim x + 1$ and $y \sim y + 1$, respectively.

Now, let us introduce a Kähler structure of $\mathbb{T}^2_\tau$ as follows. We first define a symplectic form on $\mathbb{C}$ by

$$\omega = \frac{\mathrm{i}}{2\Im\tau} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Since ω is not depends on the choice of the representative under the identification (5.1), ω is also a symplectic form on $\mathbb{T}^2_\tau$. Then, we have $\int_{\mathbb{T}^2_\tau} \omega = 1$, which implies $[\omega] \in H^2(\mathbb{T}^2_\tau, \mathbb{Z})$. Let J be an almost complex structure defined by $J(\partial/\partial z) = \mathrm{i}\partial/\partial z$, $J(\partial/\partial \bar{z}) = -\mathrm{i}\partial/\partial \bar{z}$. Then, the Kähler metric g is given by

$$g = \frac{1}{2\Im\tau}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z).$$

Then, the triple (g, ω, J) defines a Kähler structure of $\mathbb{T}^2_\tau$.

5.2 Zero modes of the Dirac operator on $\mathbb{T}^2_\tau$

In this subsection, we explicitly construct a complete orthonormal basis of the kernel of the Dirac operator on $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \Lambda^{0,\bullet} \otimes L^{\otimes p})$ [25].

Let D^p be the spin^c Dirac operator on $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \Lambda^{0,\bullet} \otimes L^{\otimes p})$ defined in (2.17). As shown in (2.44), any $f^{(p)} \in \ker D^p \subset \mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes p})$ satisfy

$$\nabla^{L^{\otimes p}}_{\partial/\partial\bar{z}} f^{(p)} = 0,$$

for large enough p. From $\omega = \frac{\mathrm{i}}{2\pi} R^L = \frac{\mathrm{i}}{2\pi}\mathrm{d}A^L$, one can take

$$A^L = -\frac{\pi}{2\Im\tau}(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z}),$$

and therefore

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{p\pi}{2\Im\tau}z\right) f^{(p)} = 0. \tag{5.3}$$

Sections of the nontrivial bundle $L^{\otimes p}$ should properly transforms under the coordinate changes. For $\mathbb{T}^2_\tau$, this requirement imposes the following boundary conditions.

Proposition 5.1. *Elements of $\mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes p})$ can be written as*

$$f^{(p)} = \mathrm{e}^{\mathrm{i}\frac{p\pi}{2}\frac{\Im(z^2)}{\Im\tau}}\theta(z), \tag{5.4}$$

where θ satisfies the boundary conditions

$$\begin{cases} \theta(z+1) = \theta(z), \\ \theta(z+\tau) = \mathrm{e}^{-\mathrm{i}p\pi\Re\tau}\mathrm{e}^{-\mathrm{i}2p\pi\Re z}\theta(z). \end{cases} \tag{5.5}$$

Proof. Under the coordinate shifts, the connection one-form A^L and $f^{(p)}$ transforms as

$$\begin{cases} A^L(z+1) = A^L(z) - \mathrm{i}\frac{\pi}{\Im\tau}\mathrm{d}(\Im z), \\ A^L(z+\tau) = A^L(z) - \mathrm{i}\frac{\pi}{\Im\tau}\mathrm{d}(\Im(\bar{\tau}z)), \end{cases} \quad \text{and} \quad \begin{cases} f^{(p)}(z+1) = \mathrm{e}^{\mathrm{i}p\pi\frac{\Im z}{\Im\tau}} f^{(p)}(z), \\ f^{(p)}(z+\tau) = \mathrm{e}^{\mathrm{i}p\pi\frac{\Im(\bar{\tau}z)}{\Im\tau}} f^{(p)}(z). \end{cases} \tag{5.6}$$

By putting (5.4) into (5.6), we obtain (5.5). □

Let us solve the zero mode equation (5.3). From the first condition of (5.5), we have

$$\theta(z) = \sum_{n\in\mathbb{Z}} c_n(\Im z)\mathrm{e}^{\mathrm{i}2\pi n\Re z},$$

for some sequence of complex functions $\{c_n\}_{n\in\mathbb{Z}}$. Then, (5.3) implies

$$\frac{\mathrm{d}c_n}{\mathrm{d}\Im z}(\Im z) = -2\pi\left(p\frac{\Im z}{\Im\tau} + n\right)c_n(\Im z) \quad \Rightarrow \quad c_n(\Im z) = c_n(0)\mathrm{e}^{-p\pi\frac{(\Im z)^2}{\Im\tau}}\mathrm{e}^{-2\pi n\Im z}.$$

Furthermore, the second condition of (5.5) imposes

$$c_{n+p}(0) = \mathrm{e}^{\mathrm{i}\pi(2n+p)\tau}c_n(0) \quad \Rightarrow \quad c_n(0) = c_{\bar{n}_p}\mathrm{e}^{\mathrm{i}\tau\frac{\pi}{p}n^2}.$$

Here, $c_{\bar{n}_p}$ is the complex number which only depends on the congruence class of modulo p,

$$\bar{n}_p = \{pl + n|\ l \in \mathbb{Z}\}.$$

Thus, we obtain

$$\theta(z) = \mathrm{e}^{-p\pi\frac{(\Im z)^2}{\Im\tau}} \sum_{n\in\mathbb{Z}} c_{\bar{n}_p} \mathrm{e}^{\mathrm{i}\tau\frac{\pi}{p}n^2} \mathrm{e}^{\mathrm{i}2\pi nz},$$

which has p linearly independent modes labeled by $\bar{a}_p \in \mathbb{Z}_p$,

$$\theta_{\bar{a}_p}(z) = c_{\bar{a}_p} \mathrm{e}^{-p\pi\frac{(\Im z)^2}{\Im\tau}} \sum_{n\in\bar{a}_p} \mathrm{e}^{\mathrm{i}\tau\frac{\pi}{p}n^2} \mathrm{e}^{\mathrm{i}2\pi nz}.$$

Using this result, we can construct a complete orthonormal basis of $\ker D^p$ with respect to the inner product

$$(f^{(p)}, g^{(p)}) := \int_{\mathbb{T}^2_\tau} \mu\, \overline{f^{(p)}} g^{(p)},$$

for $f^{(p)}, g^{(p)} \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes p})$. Here, $\mu := \omega$ is the volume form of $\mathbb{T}^2_\tau$.

Theorem 5.2. *For $\bar{a}_p \in \mathbb{Z}_p$, we define*

$$f^{(p)}_{\bar{a}_p}(z) := (2p\Im\tau)^{1/4} \mathrm{e}^{\mathrm{i}p\pi\frac{z\Im z}{\Im\tau}} \sum_{n\in\bar{a}_p} \mathrm{e}^{\mathrm{i}\tau\frac{\pi}{p}n^2} \mathrm{e}^{\mathrm{i}2\pi nz}. \tag{5.7}$$

Then, $\{f^{(p)}_{\bar{a}_p}\}_{\bar{a}_p\in\mathbb{Z}_p}$ is an orthonormal basis of $\mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes p})$,

$$(f^{(p)}_{\bar{a}_p}, f^{(p)}_{\bar{b}_p}) = \delta_{\bar{a}_p,\bar{b}_p}.$$

Proof. Using (5.2), we have

$$(f^{(p)}_{\bar{a}_p}, f^{(p)}_{\bar{b}_p}) = (2p\Im\tau)^{\frac{1}{2}} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \sum_{n\in\bar{a}_p}\sum_{m\in\bar{b}_p} \mathrm{e}^{\mathrm{i}\frac{\pi}{p}(-\bar{\tau}n^2+\tau m^2)} \mathrm{e}^{-\mathrm{i}2\pi(n\bar{\tau}-m\tau)y} \mathrm{e}^{-\mathrm{i}2\pi(n-m)x}.$$

The integration over x gives a factor of δ_{nm} and thus we obtain

$$\begin{aligned}
(f^{(p)}_{\bar{a}_p}, f^{(p)}_{\bar{b}_p}) &= (2p\Im\tau)^{\frac{1}{2}} \delta_{\bar{a}_p,\bar{b}_p} \sum_{n\in\bar{a}_p} \mathrm{e}^{-\frac{2\pi}{p}(\Im\tau)n^2} \int_0^1 \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \mathrm{e}^{-4\pi n(\Im\tau)y} \\
&= (2p\Im\tau)^{\frac{1}{2}} \delta_{\bar{a}_p,\bar{b}_p} \sum_{n\in\bar{a}_p} \int_0^1 \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)(y+\frac{n}{p})^2} \\
&= (2p\Im\tau)^{\frac{1}{2}} \delta_{\bar{a}_p,\bar{b}_p} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \\
&= \delta_{\bar{a}_p,\bar{b}_p}.
\end{aligned}$$

□

Since the cardinality of $\mathbb{Z}_p$ is p, we have

$$\dim\ker D^p = p,$$

which is consistent with (2.33),

$$\dim\ker D^p = \int_{\mathbb{T}^2} \mathrm{e}^{p\omega} = p.$$

5.3 The spectrum of the Bochner Laplacian $\Delta^{L^{\otimes q}}$

In this subsection, we obtain the eigenvalues and the corresponding eigenvectors of $\Delta^{L^{\otimes q}}$, which will be quantized in the next subsection.

First, let us consider the case for $q = 0$. In this case, the Bochner Laplacian on $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathbb{C})$ is given as

$$\Delta^{\mathbb{C}} = -\frac{1}{\Im\tau}\left(|\tau|^2\frac{\partial^2}{\partial x^2} - 2\Re\tau\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\right),$$

Since any elements of $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathbb{C})$ are periodic in $z \mapsto z+1, z+\tau$, the eigenvalues and their corresponding eigenvectors of $\Delta^{\mathbb{C}}$ are

$$E_{k,l} = \frac{4\pi^2|\tau k - l|^2}{\Im\tau}, \quad f_{k,l} := \mathrm{e}^{\mathrm{i}2\pi kx}\mathrm{e}^{\mathrm{i}2\pi ly}, \tag{5.8}$$

for $k, l \in \mathbb{Z}$. The eigenvectors are orthonormal

$$(f_{k,l}, f_{k',l'}) = \delta_{k,k'}\delta_{l,l'}.$$

Let us consider the case for $q \neq 0$. The Bochner Laplacian on $\mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes q})$ is

$$\Delta^{L^{\otimes q}} = -2\Im\tau\left(\nabla^{L^{\otimes q}}_{\partial/\partial z}\nabla^{L^{\otimes q}}_{\partial/\partial \bar{z}} + \nabla^{L^{\otimes q}}_{\partial/\partial \bar{z}}\nabla^{L^{\otimes q}}_{\partial/\partial z}\right).$$

From $[\nabla^{L^{\otimes q}}_{\partial/\partial z}, \nabla^{L^{\otimes q}}_{\partial/\partial \bar{z}}] = qR^L(\partial/\partial z, \partial/\partial\bar{z}) = \frac{\pi q}{\Im\tau}$, we can introduce the ladder operators

$$a_q := -\mathrm{i}\sqrt{\frac{\Im\tau}{\pi q}}\nabla^{L^{\otimes q}}_{\partial/\partial \bar{z}}, \quad a_q^* := -\mathrm{i}\sqrt{\frac{\Im\tau}{\pi q}}\nabla^{L^{\otimes q}}_{\partial/\partial z}, \tag{5.9}$$

satisfying $[a_q, a_q^*] = 1$ and $[a_q, a_q] = [a_q^*, a_q^*] = 0$. Then, $\Delta^{L^{\otimes q}}$ can be written as

$$\Delta^{L^{\otimes q}} = 4\pi q\left(N_q + \frac{1}{2}\right),$$

where $N_q := a_q^* a_q$ is the number operator. Thus, the eigenvalues of $\Delta^{L^{\otimes q}}$ are

$$E_m = 4\pi q\left(m + \frac{1}{2}\right),$$

for $m \in \mathbb{N}$. Note that the eigenvectors with $m = 0$ are vanished by the action of $a_q^* \propto \nabla^{L^{\otimes q}}_{\partial/\partial\bar{z}}$ and thus the eigenspace with $m = 0$ is $\ker D^p$. Thus, the eigenvector $f^{(q)}_{m,\bar{a}_q}$ with eigenvalue E_m is given by

$$f^{(q)}_{m,\bar{a}_q} = \frac{(a_q^*)^m}{\sqrt{m!}}f^{(q)}_{\bar{a}_q} = \frac{(2q\Im\tau)^{1/4}}{\sqrt{2^m m!}}\mathrm{e}^{\mathrm{i}\pi q\frac{z\Im z}{\Im\tau}}\sum_{n\in\bar{a}_q} H_m\left(\sqrt{2\pi q\Im\tau}\left(\frac{\Im z}{\Im\tau} + \frac{n}{q}\right)\right)\mathrm{e}^{\mathrm{i}\tau\frac{\pi}{q}n^2}\mathrm{e}^{\mathrm{i}2\pi nz}, \tag{5.10}$$

where $\bar{a}_q \in \mathbb{Z}_q$. Here, H_m is the physicist's Hermite polynomial:

$$H_m(x) = (-1)^m\mathrm{e}^{x^2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{e}^{-x^2}.$$

Using the algebra of the ladder operators, we can show

$$(f^{(q)}_{m,\bar{a}_q}, f^{(q)}_{n,\bar{b}_q}) = \delta_{mn}\delta_{\bar{a}_q,\bar{b}_q}. \tag{5.11}$$

5.4 Toeplitz operators of eigenvectors of $\Delta^{L^{\otimes q}}$

Let $\Pi^p : \mathcal{C}^\infty(\mathbb{T}^2_\tau, \Lambda^{0,\bullet} \otimes L^{\otimes p})$ be the orthogonal projection. We define the Toeplitz operator of $f^{(q)} \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, L^{\otimes q})$ by

$$T_p(f^{(q)}) = \Pi^{p+q} f^{(q)} \Pi^p.$$

In this subsection, we explicitly evaluate the matrix elements

$$T_p(f^{(q)})_{\bar{a}_{p+q},\bar{b}_p} := (f^{(p+q)}_{\bar{a}_{p+q}}, T_p(f^{(q)}) f^{(p)}_{\bar{b}_p})$$

for the eigenvectors of the Bochner Laplacian $\Delta^{L^{\otimes q}}$.

Let us obtain the Toeplitz operators of the eigenvectors of $\Delta^{L^{\otimes q}}$. The result for $q = 0$ is the following.

Theorem 5.3. *Let us introduce a operator $F^{k,l}_p$ with the following matrix elements*

$$(F^{k,l}_p)_{\bar{a}_p,\bar{b}_p} := \mathrm{e}^{-\mathrm{i}\frac{2\pi l}{p} a} \delta_{\bar{a}_p,\overline{b+k}_p}.$$

Then, we find

$$T_p(f_{k,l}) = \mathrm{e}^{-\frac{\pi|\tau k - l|^2}{2p\Im\tau}} \mathrm{e}^{\frac{kl}{p}\pi} F^{k,l}_p. \tag{5.12}$$

Proof.

$$\begin{aligned}
&T_p(\mathrm{e}^{\mathrm{i}2\pi kx}\mathrm{e}^{\mathrm{i}2\pi ly})_{\bar{a}_p,\bar{b}_p} \\
&= (2p\Im\tau)^{\frac{1}{2}} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \sum_{n\in\bar{a}_p}\sum_{m\in\bar{b}_p} \mathrm{e}^{\mathrm{i}\frac{\pi}{p}(-\bar{\tau}n^2+\tau m^2)} \mathrm{e}^{-\mathrm{i}2\pi(n\bar{\tau}-m\tau-l)y} \mathrm{e}^{-\mathrm{i}2\pi(n-m-k)x} \\
&= (2p\Im\tau)^{\frac{1}{2}} \delta_{\bar{a}_p,\overline{b+k}_p} \sum_{n\in\bar{a}_p} \mathrm{e}^{\mathrm{i}\frac{\pi}{p}(-\bar{\tau}n^2+\tau(n-k)^2)} \int_0^1 \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \mathrm{e}^{-2\pi(2n\Im\tau+\mathrm{i}(k\tau-l))y} \\
&= (2p\Im\tau)^{\frac{1}{2}} (F^{k,l}_p)_{\bar{a}_p,\overline{b+k}_p} \mathrm{e}^{-\frac{\pi(|\tau|^2k^2-2kl\tau+l^2)}{2p\Im\tau}} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-2p\pi(\Im\tau)y^2} \\
&= \mathrm{e}^{-\frac{\pi(|\tau|^2k^2-2kl\tau+l^2)}{2p\Im\tau}} (F^{k,l}_p)_{\bar{a}_p,\overline{b+k}_p}.
\end{aligned}$$

□

The operator $F^{k,l}_p$ has the following properties.

Proposition 5.4.

$$\begin{aligned}
&{}^\forall m,n \in \mathbb{Z}: \quad F^{k+mp,l+np}_p = F^{k,l}_p, \\
&F^{k,l}_p F^{k',l'}_p = \mathrm{e}^{\mathrm{i}\frac{2\pi k l'}{p}} F^{k+k',l+l'}_p, \quad (F^{k,l}_p)^* = \mathrm{e}^{\mathrm{i}\frac{2\pi kl}{p}} F^{-k,-l}_p.
\end{aligned} \tag{5.13}$$

In particular, the last two equations and $F^{0,0}_p = \mathbf{1}$ imply that $F^{k,l}_p$ is unitary. The last equation also implies

$$F^{k,l}_p F^{k',l'}_p = \mathrm{e}^{\mathrm{i}\frac{2\pi(kl'-k'l)}{p}} F^{k',l'}_p F^{k,l}_p.$$

We also have the following orthogonality with respect to (3.17).

Theorem 5.5.

$$(T_p(f_{k,l}),T_p(f_{k',l'}))=\mathrm{e}^{-\frac{\pi(|\tau k-l|^2+|\tau k'-l'|^2)}{2p\Im\tau}}\mathrm{e}^{\frac{kl+k'l'}{p}\pi}\delta_{\bar{k}_p,\bar{k}'_p}\delta_{\bar{l}_p,\bar{l}'_p}.$$

Proof.

$$\begin{aligned}(T_p(f_{k,l}),T_p(f_{k',l'}))&=p^{-1}\sum_{a,b=0}^{p-1}\mathrm{e}^{-\frac{\pi(|\tau k-l|^2+|\tau k'-l'|^2)}{2p\Im\tau}}\mathrm{e}^{\frac{kl+k'l'}{p}\pi}\mathrm{e}^{-\mathrm{i}\frac{2\pi(l-l')}{p}a}\delta_{\bar{a}_p,\overline{b+k}_p}\delta_{\bar{a}_p,\overline{b+k'}_p}\\&=p^{-1}\sum_{a=0}^{p-1}\mathrm{e}^{-\frac{\pi(|\tau k-l|^2+|\tau k'-l'|^2)}{2p\Im\tau}}\mathrm{e}^{\frac{kl+k'l'}{p}\pi}\mathrm{e}^{-\mathrm{i}\frac{2\pi(l-l')}{p}a}\delta_{\bar{k}_p,\bar{k}'_p}\\&=\mathrm{e}^{-\frac{\pi(|\tau k-l|^2+|\tau k'-l'|^2)}{2p\Im\tau}}\mathrm{e}^{\frac{kl+k'l'}{p}\pi}\delta_{\bar{k}_p,\bar{k}'_p}\delta_{\bar{l}_p,\bar{l}'_p}.\end{aligned}$$

□

The Toeplitz operators of the eigenvectors of $\Delta^{L^{\otimes q}}$ for $q\neq 0$ is the following.

Theorem 5.6.

$$T_p(f^{(q)}_{m,\bar{c}_q})_{\bar{a}_{p+q},\bar{b}_p}=\sqrt{\frac{p^m}{(p+q)^{m+1}}}\sum_{t=1}^{p+q}f^{(pq(p+q))}_{m,\overline{pc-qb+pqt}_{pq(p+q)}}(0)\delta_{\bar{a}_{p+q},\overline{b+c+qt}_{p+q}}.\tag{5.14}$$

Proof. We first show

$$f^{(q)}_{m,\bar{c}_q}(z)f^{(p)}_{\bar{b}_p}(z)=\sum_{t=1}^{p+q}\sum_{m'=0}^{m}\sqrt{\frac{m!}{(m-m')!m'!}\frac{q^{m'}p^{m-m'}}{(p+q)^{m+1}}}f^{(p+q)}_{m',\overline{b+c+qt}_{p+q}}(z)f^{(pq(p+q))}_{m-m',\overline{pc-qb+pqt}_{pq(p+q)}}(0).\tag{5.15}$$

To show this, let us consider the theta function with characteristics,

$$\vartheta_{a,b}(\nu,\tau)=\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\pi(n+a)^2\tau}\mathrm{e}^{\mathrm{i}2\pi(n+a)(\nu+b)}.$$

Then, we can write (5.7) as

$$f^{(p)}_{\bar{a}_p}(z)=(2p\Im\tau)^{1/4}\mathrm{e}^{\mathrm{i}p\pi\frac{z\Im z}{\Im\tau}}\vartheta_{\frac{a}{p},0}(pz,p\tau).$$

There is a following identity [26],

$$\vartheta_{\frac{c}{q},0}(qz,q\tau)\,\vartheta_{\frac{b}{p},0}(pw,p\tau)=\sum_{t=1}^{p+q}\vartheta_{\frac{b+c+qt}{p+q},0}((p+q)\tilde{z},(p+q)\tau)\vartheta_{\frac{pc-qb+pqt}{pq(p+q)},0}(pq(p+q)\tilde{w},pq(p+q)\tau),$$

where

$$\tilde{z}:=\frac{qz+pw}{p+q},\quad\tilde{w}:=\frac{z-w}{p+q}.$$

This implies

$$f^{(q)}_{\bar{c}_q}(z)f^{(p)}_{\bar{b}_p}(w) = (p+q)^{-\frac{1}{2}}\sum_{t=1}^{p+q} f^{(p+q)}_{\overline{b+c+qt}_{p+q}}(\tilde{z})f^{(pq(p+q))}_{\overline{pc-qb+pqt}_{pq(p+q)}}(\tilde{w}).$$

From (5.9), we have a relation

$$a^*_q(z) = \sqrt{\frac{q}{p+q}}a^*_{p+q}(\tilde{z}) + \sqrt{\frac{p}{p+q}}a^*_{pq(p+q)}(\tilde{w}).$$

Then, (5.10) implies

$$f^{(q)}_{m,\bar{c}_q}(z)f^{(p)}_{\bar{b}_p}(w) = \sum_{t=1}^{p+q}\sum_{m'=0}^{m}\sqrt{\frac{m!}{(m-m')!m'!}\frac{q^{m'}p^{m-m'}}{(p+q)^{m+1}}}f^{(p+q)}_{m',\overline{b+c+qt}_{p+q}}(\tilde{z})f^{(pq(p+q))}_{m-m',\overline{pc-qb+pqt}_{pq(p+q)}}(\tilde{w}).$$

By putting $z = w$, we obtain (5.15).

Using (5.15) and (5.11), we have

$$\begin{aligned}
&T_p(f^{(q)}_{m,\bar{c}_q})_{\bar{a}_{p+q},\bar{b}_p} \\
&= \sum_{t=1}^{p+q}\sum_{m'=0}^{m}\sqrt{\frac{m!}{(m-m')!m'!}\frac{q^{m'}p^{m-m'}}{(p+q)^{m+1}}}f^{(pq(p+q))}_{m-m',\overline{pc-qb+pqt}_{pq(p+q)}}(0)(f^{(p+q)}_{0,\bar{a}_{p+q}}, f^{(p+q)}_{m',\overline{b+c+qt}_{p+q}}) \\
&= \sqrt{\frac{p^m}{(p+q)^{m+1}}}\sum_{t=1}^{p+q} f^{(pq(p+q))}_{m,\overline{pc-qb+pqt}_{pq(p+q)}}(0)\delta_{\bar{a}_{p+q},\overline{b+c+qt}_{p+q}}.
\end{aligned}$$

□

5.5 The isometric embedding of $\mathbb{T}^2_\tau$ and the Matrix Laplacian

Let us consider an element

$$A = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Then, we define a set of functions

$$X^1 = \frac{\rho^{-\frac{1}{2}}}{2\pi}\Re f_{\alpha,\beta}, \quad X^2 = \frac{\rho^{-\frac{1}{2}}}{2\pi}\Im f_{\alpha,\beta}, \quad X^3 = \frac{\rho^{\frac{1}{2}}}{2\pi}\Re f_{\gamma,\delta}, \quad X^4 = \frac{\rho^{\frac{1}{2}}}{2\pi}\Im f_{\gamma,\delta}, \tag{5.16}$$

where $f_{\alpha,\beta}(x,y) = \mathrm{e}^{\mathrm{i}2\pi(\alpha x+\beta y)}$ and $\rho \in (0,\infty)$. Using the direct calculation, one can easily find that $X = (X^1, X^2, X^3, X^4)$ gives the isometric embedding $\mathbb{T}^2_\tau \to \mathbb{R}^4$ with moduli parameter

$$\tau(\rho, A) = \frac{\rho^{-1}\alpha\beta + \rho\gamma\delta + \mathrm{i}}{\rho^{-1}\alpha^2 + \rho\gamma^2}. \tag{5.17}$$

For $A \in \mathrm{SL}(2,\mathbb{Z})$, we define $\Gamma_A : \mathbb{H} \to \mathbb{H}$ such that $\Gamma_A z = \frac{\alpha z+\beta}{\gamma z+\delta}$. Using this action, we have $\tau(\rho, A) = -\overline{\Gamma_{A^{-1}}(\mathrm{i}\rho)}$. Thus, this isometric embedding is applicable only for subset of the moduli parameters, which are essentially purely imaginary parameters up to the $\mathrm{SL}(2,\mathbb{Z})$.

Below, we set the moduli parameter as (5.17) and the isometric embedding as (5.16). Using (5.12), their Toeplitz operators are given by

$$
\begin{aligned}
T_p(X^1) &= \frac{c_p^{\alpha,\beta}\rho^{-\frac{1}{2}}}{4\pi}\left\{F_p^{\alpha,\beta}+(F_p^{\alpha,\beta})^*\right\}, \quad T_p(X^2) = \frac{c_p^{\alpha,\beta}\rho^{-\frac{1}{2}}}{4\pi\mathrm{i}}\left\{F_p^{\alpha,\beta}-(F_p^{\alpha,\beta})^*\right\}, \\
T_p(X^3) &= \frac{c_p^{\gamma,\delta}\rho^{\frac{1}{2}}}{4\pi}\left\{F_p^{\gamma,\delta}+(F_p^{\gamma,\delta})^*\right\}, \quad T_p(X^4) = \frac{c_p^{\gamma,\delta}\rho^{\frac{1}{2}}}{4\pi\mathrm{i}}\left\{F_p^{\gamma,\delta}-(F_p^{\gamma,\delta})^*\right\},
\end{aligned}
\tag{5.18}
$$

where

$$
c_p^{\alpha,\beta} := \mathrm{e}^{-\frac{\pi|\tau\alpha-\beta|^2}{2p\Im\tau}}\mathrm{e}^{\frac{\alpha\beta}{p}}.
$$

Then, the matrix Laplacian (3.21) is given as

$$
\hat{\Delta}^{L^{\otimes q}}T_p(f^{(q)}) = (2\pi p)^2\sum_{a=1}^{4}[T_p(X^a\mathbf{1}),[T_p(X^a\mathbf{1}),T_p(f^{(q)})]].
$$

By a simple calculation, we find

$$
\begin{aligned}
\hat{\Delta}^{L^{\otimes q}}T_p(f^{(q)}) = p^2\Big(&\left[((c_{p+q}^{\alpha,\beta})^2+(c_p^{\alpha,\beta})^2)\rho^{-1}+((c_{p+q}^{\gamma,\delta})^2+(c_p^{\gamma,\delta})^2)\rho\right]T_p(f^{(q)}) \\
&- c_{p+q}^{\alpha,\beta}c_p^{\alpha,\beta}\rho^{-1}\left[F_{p+q}^{\alpha,\beta}T_p(f^{(q)})(F_p^{\alpha,\beta})^*+(F_{p+q}^{\alpha,\beta})^*T_p(f^{(q)})F_p^{\alpha,\beta}\right] \\
&- c_{p+q}^{\gamma,\delta}c_p^{\gamma,\delta}\rho\left[F_{p+q}^{\gamma,\delta}T_p(f^{(q)})(F_p^{\gamma,\delta})^*+(F_{p+q}^{\gamma,\delta})^*T_p(f^{(q)})F_p^{\gamma,\delta}\right]\Big).
\end{aligned}
\tag{5.19}
$$

For the case of $q=0$, we can obtain the eigenvectors and the eigenvalues of $\hat{\Delta}^{\mathbb{C}}$ as follows. Using (5.13), we have

$$
\begin{aligned}
\hat{\Delta}^{\mathbb{C}}T_p(f_{k,l}) &= 4p^2\left[(c_p^{\alpha,\beta})^2\rho^{-1}\sin^2\left(\frac{\pi(\alpha l-\beta k)}{p}\right)+(c_p^{\gamma,\delta})^2\rho\sin^2\left(\frac{\pi(\gamma l-\delta k)}{p}\right)\right]T_p(f_{k,l}) \\
&= (2\pi)^2\left(\rho^{-1}(\alpha l-\beta k)^2+\rho(\gamma l-\delta k)^2\right)T_p(f_{k,l})+\mathcal{O}(\hbar_p) \\
&= \frac{(2\pi)^2|\tau k-l|^2}{\Im\tau}T_p(f_{k,l})+\mathcal{O}(\hbar_p).
\end{aligned}
$$

This shows that $T_p(f_{k,l})$ are eigenvectors of $\hat{\Delta}^{\mathbb{C}}$ and their eigenvalues approach (5.8) as p increases.

For $q\neq 0$, (5.19) is related to the Harper's equation as discussed in [13]. To see this, let us use the different orthonormal bases for $\ker D^{p+q}$ and $\ker D^p$ such that

$$
(F_p^{\alpha,\beta})_{\bar{a}_p,\bar{b}_p} = \delta_{\bar{a}_p,\overline{b+1}_p}, \quad (F_p^{\gamma,\delta})_{\bar{a}_p,\bar{b}_p} = \mathrm{e}^{-\mathrm{i}\frac{2\pi}{p}a}\delta_{\bar{a}_p,\bar{b}_p},
$$

and express the matrix Laplacian (5.19) in terms of this basis:

$$
\begin{aligned}
&\left(\hat{\Delta}^{L^{\otimes q}}T_p(f^{(q)})\right)_{\bar{a}_{p+q},\bar{b}_p} = p^2CT_p(f^{(q)})_{\bar{a}_{p+q},\bar{b}_p} \\
&- p^2D\left(T_p(f^{(q)})_{\overline{a-1}_{p+q},\overline{b-1}_p}+2\lambda\cos\left(\frac{2\pi a}{p+q}-\frac{2\pi b}{p}\right)T_p(f^{(q)})_{\bar{a}_{p+q},\bar{b}_p}+T_p(f^{(q)})_{\overline{a+1}_{p+q},\overline{b+1}_p}\right),
\end{aligned}
$$

where

$$C := ((c_{p+q}^{\alpha,\beta})^2 + (c_p^{\alpha,\beta})^2)\rho^{-1} + ((c_{p+q}^{\gamma,\delta})^2 + (c_p^{\gamma,\delta})^2)\rho, \quad D := c_{p+q}^{\alpha,\beta} c_p^{\alpha,\beta} \rho^{-1}, \quad \lambda := \frac{c_{p+q}^{\gamma,\delta} c_p^{\gamma,\delta}}{c_{p+q}^{\alpha,\beta} c_p^{\alpha,\beta}} \rho^2.$$

Now, let us express the Toeplitz operator $T_p(f^{(q)})$ in terms of a $p(p+q)$-dimensional vector

$$v = \bigoplus_{r=1}^{\gcd(p,q)} v_r, \quad v_r = \left(T_p(f^{(q)})_{\bar{1}_{p+q}, \overline{r+1}_p}, T_p(f^{(q)})_{\bar{2}_{p+q}, \overline{r+2}_p}, \cdots, T_p(f^{(q)})_{\bar{\nu}_{p+q}, \overline{r+\nu}_p} \right),$$

where $\nu := \frac{p(p+q)}{\gcd(p,q)}$. The corresponding representation of $\hat{\Delta}^{L^{\otimes q}}$ acting on v is given as

$$\hat{\Delta}^{L^{\otimes q}} = \bigoplus_{r=1}^{\gcd(p,q)} \hat{\Delta}_r^{L^{\otimes q}}, \quad \hat{\Delta}_r^{L^{\otimes q}} = p^2 C - p^2 D H_{\frac{r}{p}}^{\lambda, \frac{q}{p(p+q)}},$$

where $H_\omega^{\lambda,\alpha}$ is the almost Mathieu operator defined by

$$(H_\omega^{\lambda,\alpha} u)_{\bar{a}_\nu} = u_{\overline{a+1}_\nu} + u_{\overline{a-1}_\nu} + 2\lambda \cos(2\pi(\alpha a + \omega)) u_{\bar{a}_\nu}.$$

Thus, the eigenvalue problem of $\hat{\Delta}^{L^{\otimes q}}$ is the eigenvalue problem of the almost Mathieu operator. This kind of problem naturally arises in the study of the quantum hall effects. Since $f_{m,\bar{a}_q}^{(q)}$ given in (5.10) is the exact eigenvector of $\Delta^{L^{\otimes q}}$, the corresponding Toeplitz operator $T_p(f_{m,\bar{a}_q}^{(q)})$ given in (5.14) should be the approximate eigenvector of $\hat{\Delta}^{L^{\otimes q}}$. From this statement, we can construct the approximate eigenvector of $H_{\frac{r}{p}}^{\lambda, \frac{q}{p(p+q)}}$.

6 Application to tensor fields

In this section, we construct the matrix regularization of tensor fields [14] as an application of the theory given in section 3.

6.1 Toeplitz operators for tensor fields

Let M be a compact Kähler manifold and let $\dim_{\mathbb{C}} M = n$. For $k, l \in \mathbb{N}$, we define a tensor bundle of type (k, l) over M by $T^l_k M := T^*M^{\otimes k} \otimes TM^{\otimes l}$. Then, a smooth tensor field of type (k, l) is defined as an element of $\mathcal{C}^\infty(M, T^l_k)$, which can be locally expressed as

$$f^l_k = (f^l_k)_{i_1 \cdots i_k}{}^{j_1 \cdots j_l} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_l}},$$

using the real coordinates $\{x^i\}$. Throughout this section, we use the Einstein summation convection for repeated indices of the local coordinates. Then, we can define a linear map $\mathcal{C}^\infty(M, T^l_k M) \times \mathcal{C}^\infty(M, T^0_l M) \to \mathcal{C}^\infty(M, T^0_k M)$ by

$$f^l_k\, g_l := (f^l_k)_{i_1 \cdots i_k}{}^{j_1 \cdots j_l} (g_l)_{j_1 \cdots j_l} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k}. \tag{6.1}$$

Thus, the tensor bundle $T^l_k M$ can be considered as a homomorphism bundle $\mathrm{Hom}(T^0_l M, T^0_k M)$. Note that there are infinitely many other choices of defining such homomorphism structures. Let $\Pi^{p,k}$ be the orthogonal projection from $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes T^0_k M)$ to the kernel of the spin^c Dirac operator $D^{p,k} := D^{p, T^0_k M}$. Then, we define the Toeplitz operator of $f^l_k \in \mathcal{C}^\infty(M, T^l_k M)$ as

$$T^{(k,l)}_p(f^l_k) := \Pi^{p,k} f^l_k \Pi^{p,l}. \tag{6.2}$$

For $f^l_k \in \mathcal{C}^\infty(M, T^l_k M)$ and $g^m_l \in \mathcal{C}^\infty(M, T^m_l M)$, we can consider the product of the Toeplitz operators. The relation (3.8) implies

$$T^{(k,l)}_p(f^l_k) T^{(l,m)}_p(g^m_l) = T^{(k,m)}_p(f^l_k\, g^m_l) + \mathcal{O}(\hbar_p).$$

Here, the corresponding product of tensor fields induced from the homomorphism structure (6.1) is

$$f^l_k\, g^m_l = (f^l_k)_{i_1 \cdots i_k}{}^{h_1 \cdots h_l} (g^m_l)_{h_1 \cdots h_l}{}^{j_1 \cdots j_m} \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_l}}.$$

Now, consider the identity element $\mathbf{1}_k \in \mathcal{C}^\infty(M, T^k_k M)$ such that

$$\mathbf{1}_k = \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}.$$

Then, (3.14) implies that

$$\lim_{p \to \infty} \left| (\mathrm{i}\hbar_p)^{-1} (T^{(k,k)}_p(f \mathbf{1}_k) T^{(k,l)}_p(f^l_k) - T^{(k,l)}_p(f^l_k) T^{(l,l)}_p(f \mathbf{1}_l)) - T^{(k,l)}_p(\{f, f^l_k\}) \right| = 0.$$

Let us define a Hermitian fiber inner product on $T^0_k M$ by

$$h^{T^0_k M}(s_k, t_k) := g^{i_1 j_1} \cdots g^{i_k j_k} (s_k)_{i_1 \cdots i_k} \cdot (t_k)_{j_1 \cdots j_k}, \tag{6.3}$$

where (g^{ij}) is the inverse of the metric (g_{ij}). Then, (3.16) implies

$$\lim_{p\to\infty} (2\pi\hbar_p)^n \operatorname{Tr} T^{(k,k)}_p (f^k_k) = \int_M \mu_\omega (f^k_k)_{i_1\cdots i_k}{}^{i_1\cdots i_k}. \tag{6.4}$$

In addition, (6.3) implies that the adjoint of the tensor field $f^l_k \in \mathcal{C}^\infty(M, T^l_k)$ is

$$(f^l_k)^* = g^{i_1 i'_1} \cdots g^{i_k i'_k} g_{j_1 j'_1} \cdots g_{j_l j'_l} \overline{(f^l_k)_{i_1\cdots i_k}{}^{j_1\cdots j_l}}\, \mathrm{d}x^{j'_1} \otimes \cdots \otimes \mathrm{d}x^{j'_l} \otimes \frac{\partial}{\partial x^{i'_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i'_k}}. \tag{6.5}$$

Then, (3.6) implies

$$T^{(l,k)}_p((f^l_k)^*) = T^{(k,l)}_p(f^l_k)^*. \tag{6.6}$$

Before closing this subsection, we give another formulation of tensor fields as follows. Let $\{e_a\}_{=1,\cdots,2n}$ be an orthonormal frame of TM and $\{e^a\}_{a=1,\cdots,2n}$ be the dual frame. From these fields, we define

$$\begin{aligned} E_{a_1 a_2 \cdots a_k} &:= e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_k} \in \mathcal{C}^\infty(M, T^k_0 M), \\ E^{b_1 b_2 \cdots b_l} &:= e^{b_1} \otimes e^{b_2} \otimes \cdots \otimes e^{a_l} \in \mathcal{C}^\infty(M, T^0_l M). \end{aligned}$$

Then, one can define a set of $(2n)^{k+l}$ functions $\{f_{a_1 a_2\cdots a_k}{}^{b_1 b_2 \cdots b_l}\}$ from f^l_k such that

$$f_{a_1 a_2\cdots a_k}{}^{b_1 b_2 \cdots b_l} := (E_{a_1 a_2 \cdots a_k})^{i_1 i_2 \cdots i_k} (E^{b_1 b_2 \cdots b_l})_{j_1 j_2 \cdots j_l} (f^l_k)_{i_1 i_2 \cdots i_k}{}^{j_1 j_2 \cdots j_l}.$$

The Toeplitz operator of $f_{a_1 a_2\cdots a_k}{}^{b_1 b_2 \cdots b_l}$ then satisfies

$$T^{(0,0)}_p (f_{a_1 a_2\cdots a_k}{}^{b_1 b_2 \cdots b_l}) = T^{(0,k)}_p (E_{a_1 a_2 \cdots a_k}) T^{(k,l)}_p (f^l_k) T^{(l,0)}_p (E^{a_1 a_2 \cdots a_l}) + \mathcal{O}(\hbar_p).$$

From this viewpoint, one can represent a rectangular matrix corresponding to a tensor field by a set of square matrices corresponding to functions with orthonormal indices.

6.2 Symplectomorphism on tensor fields

A diffeomorphism $\phi : M \to M$ is called the symplectomorphism if it preserves the symplectic form:

$$\phi^* \omega = \omega.$$

Here, $\phi^* X$ is the pullback of a tensor field X defined by

$$(\phi^* X)(x) = X(\phi(x)).$$

In this subsection, we prove the following theorem.

Theorem 6.1. *Let ϕ be a symplectomorphism generated by Hamiltonian vector fields* (2.2)*. Then, we have*

$$T_p^{(k,l)}(\phi^* f_k^l) = G_k T_p^{(k,l)}(f_k^l) G_l^{-1} + \mathcal{O}(\hbar_p), \tag{6.7}$$

for any $f_k^l \in \mathcal{C}^\infty(M, T_k^l M)$. Here, $G_k \in \mathrm{GL}(\dim\ker D^{p,k}, \mathbb{C})$ for $k > 0$ and $G_0 \in \mathrm{U}(\dim\ker D^{p,0})$.

This is consistent with a well-known $\mathrm{U}(N)$ symmetry of the matrix theories, which corresponds to the area preserving diffeomorphism symmetry of the scalar field theory on commutative manifold [1,2]. Before showing (6.7), let us discuss some of the properties of the symplectomorphism. First property is that the pullback satisfies

$$\begin{aligned} \phi^*(X \otimes Y) &= (\phi^* X) \otimes (\phi^* Y), \\ \int_M Z &= \int_M \phi^* Z, \end{aligned} \tag{6.8}$$

for any tensor fields X, Y of arbitrary types and for any two-form fields Z. Then, the pullback of (k,k)-type tensor field f_k^k using a symplectomorphism ϕ satisfies

$$\int_M \mu_\omega \operatorname{tr}(f_k^k) = \int_M \mu_\omega \operatorname{tr}(\phi^* f_k^k).$$

Here, we used the Liouville theorem $\phi^*\mu_\omega = \mu_\omega$ and the fact that the pullback operation commutes with the contraction operation tr of tensor indices. Let $\mathbb{C}^{m_1\times m_2}$ be a set of all $m_1 \times m_2$ matrices with complex entries. Then, let us consider a map $\phi_{m_1\times m_2} : \mathbb{C}^{m_1\times m_2} \to \mathbb{C}^{m_1\times m_2}$, which corresponds to the pullback operation ϕ^* in large-p limit. From the linearity of pullback operation and (6.8), $\phi_{m_1\times m_2}$ should be a linear map satisfying

$$\phi_{m_1\times m_3}(AB) = \phi_{m_1\times m_2}(A)\phi_{m_2\times m_3}(B) + \mathcal{O}(\hbar_p),$$

for any $A \in \mathbb{C}^{m_1\times m_2}$ and $B \in \mathbb{C}^{m_2\times m_3}$. In addition, the trace correspondence (6.4) implies

$$\operatorname{Tr} A = \operatorname{Tr}[\phi_{m_1\times m_1}(A)] + \mathcal{O}(\hbar_p),$$

for any $A \in \mathbb{C}^{m_1\times m_2}$. Then, we can expect that $\phi_{m_1\times m_2}$ is of the following form

$$\phi_{m_1\times m_2}(A) = M_{m_1} A M_{m_2}^{-1} + \mathcal{O}(\hbar_p),$$

for $M_{m_1} \in \mathrm{GL}(m_1, \mathbb{C})$ and $M_{m_2} \in \mathrm{GL}(m_2, \mathbb{C})$, which satisfy all the desired properties. By this naive argument, we expect that the matrix transformation corresponding to the pullback of the symplectomorphism (including the one which is not generated by the Hamiltonian vector fields) takes the form (6.7).

Now, let us derive (6.7) from the following argument.

Proof. First, we consider the infinitesimal form of the symplectomorphism $\phi(x) = x + \epsilon V$, where ϵ is the infinitesimal real parameter. Then, the infinitesimal form of the pullback of $f_k^l \in \mathcal{C}^\infty(M, T_k^l M)$ can be written a

$$\phi^* f_k^l = f_k^l + \epsilon \mathcal{L}_V f_k^l + \mathcal{O}(\epsilon^2),$$

where $\mathcal{L}_V$ is a Lie derivative along V defined by

$$\begin{aligned}(\mathcal{L}_V f_k^l)_{i_1\cdots i_k}{}^{j_1\cdots j_l} &= (\nabla_V f_k^l)_{i_1\cdots i_k}{}^{j_1\cdots j_l} \\ &\quad - (\nabla_j V^{j_1})(f_k^l)_{i_1\cdots i_k}{}^{j j_2\cdots j_l} \cdots - (\nabla_j V^{j_l})(f_k^l)_{i_1\cdots i_k}{}^{j_1 j_2\cdots j_{l-1} j} \\ &\quad + (\nabla_{i_1} V^i)(f_k^l)_{i i_2\cdots i_k}{}^{j_1\cdots j_l} + \cdots + (\nabla_{i_k} V^i)(f_k^l)_{i_1 i_2\cdots i_{k-1} i}{}^{j_1\cdots j_l}.\end{aligned}$$

Here, $\nabla_i := \nabla_{\partial/\partial x^i}$. Let us consider the Hamiltonian vector field $V = X_f$. Using $\nabla X_f \in \mathcal{C}^\infty(M, T_1^1 M)$, we define a section of T_k^k by

$$t_k = \sum_{i=0}^{k-1} \mathbf{1}_i \otimes \nabla X_f \otimes \mathbf{1}_{k-1-i},$$

for $k > 0$ and for $k = 0$, we set $t_0 = 0$. Then, we can show

$$\mathcal{L}_{X_f} f_k^l = \{f, f_k^l\} + t_k f_k^l - f_k^l t_l.$$

Using (3.8) and (3.14), we find

$$\begin{aligned}T_p^{(k,l)}(\mathcal{L}_{X_f} f_k^l) &= \left((\mathrm{i}\hbar_p)^{-1} T_p^{(k,k)}(f\mathbf{1}_k) + T_p^{(k,k)}(t_k)\right) T_p^{(k,l)}(f_k^l) \\ &\quad - T_p^{(k,l)}(f_k^l)\left((\mathrm{i}\hbar_p)^{-1} T_p^{(l,l)}(f\mathbf{1}_l) + T_p^{(l,l)}(t_l)\right) + \mathcal{O}(\hbar_p).\end{aligned}$$

Therefore, we obtain

$$T_p^{(k,l)}(\phi^* f_k^l) = G_k T_p^{(k,l)}(f_k^l) G_l^{-1} + \mathcal{O}(\hbar_p),$$

where

$$G_k = \exp((\mathrm{i}\hbar_p)^{-1} \epsilon T_p^{(k,k)}(f\mathbf{1}_k) + \epsilon T_p^{(k,k)}(t_k)). \tag{6.9}$$

Note that $T_p^{(k,k)}(f\mathbf{1}_k)$ is self-adjoint but $T_p^{(k,k)}(t_k)$ for $k > 0$ is not for general $f \in \mathcal{C}^\infty(M, \mathbb{R})$. In addition, we have $T_p^{(0,0)}(t_0) = 0$. This implies that $G_k \in \mathrm{GL}(\dim\ker D^{p,k}, \mathbb{C})$ for $k > 0$ and $G_0 \in \mathrm{U}(\dim\ker D^{p,0})$. Therefore, for any finite transformations which are generated by Hamiltonian vector fields, we obtain (6.7). □

In particular subset of the symplectomorphism, let us consider a symplectomorphism ϕ, which also preserves the metric $\phi^* g = g$. Such diffeomorphisms is called the isometries. Then, we have the following theorem.

Theorem 6.2. *Let ϕ be a symplectomorphism generated by Hamiltonian vector fields, which also preserves the metric $\mathcal{L}_{X_f} g = 0$. Then, we have*

$$T_p^{(k,l)}(\phi^* f_k^l) = U_k T_p^{(k,l)}(f_k^l) U_l^* + \mathcal{O}(\hbar_p),$$

for any $f_k^l \in \mathcal{C}^\infty(M, T_k^l M)$. Here, $U_k \in \mathrm{U}(\dim\ker D^{p,k})$ for any $k \in \mathbb{N}$.

Proof. The vector fields preserving the metric are called Killing vector fields. If $u = u^i \frac{\partial}{\partial x^i} \in \mathcal{C}^\infty(M, TM)$ is a Killing vector field, it satisfies the Killing equation $\nabla_i u^j + \nabla^j u_i = 0$, where the tensor indices are raised by the inverse of the metric g^{ij} and lowered by the metric g_{ij}. Now, let us assume that the Hamiltonian vector field X_f is a Killing vector. Then, we

$$\nabla_i (X_f)^j = \frac{1}{2}(\nabla_i (X_f)^j - \nabla^j (X_f)_i),$$

which means that ∇X_f is skew-adjoint in the sense of (6.5). Thus, from (6.6), $T_p^{(k,k)}(t_k)$ is also skew-adjoint, which implies that the transformation matrix G_k obtained in (6.9) is unitary. □

The arguments given above are consistent with the following. Let $f_k^k \in \mathcal{C}^\infty(M, T_k^k M)$ be self-adjoint in the sense of (6.5). Then, (6.6) implies that $T_p^{(k,k)}(f_k^k)$ is also self-adjoint. Let us consider the symplectomorphism ϕ. Since the adjoint of tensor fields (6.5) depends on the metric except for the $(0,0)$-tensor fields (i.e. functions), ϕf_k^k is not self-adjoint in general. This is consistent with Theorem 6.1, that is, $G_k T_p^{(k,k)}(f_k^k) G_k^{-1}$ for $G_k \in \mathrm{GL}(\dim\ker D^{p,k}, \mathbb{C})$ is not self-adjoint in general, except for the case $k = 0$ where $G_0 \in \mathrm{U}(\dim\ker D^{p,0})$. In the case where ϕ also preserves the metric g, ϕf_k^k is self-adjoint. This is consistent with Theorem 6.2, that is, $U_k T_p^{(k,k)}(f_k^k) U_k^{-1}$ for $U_k \in \mathrm{U}(\dim\ker D^{p,k})$ is also self-adjoint.

6.3 Application to one-form field theory

In this subsection, we consider the matrix regularization of one-form field $A = A_i \mathrm{d}x^i \in \mathcal{C}^\infty(M, T^*M)$ over a two-dimensional manifold M by using the Berezin-Toeplitz quantization of the vector bundle T^*M. As an example, we explicitly write the matrix action for fuzzy $\mathbb{T}^2_{\mathrm{i}}$ and we showed that the matrix action for massless one-form field has a matrix gauge symmetry which corresponds to the U(1) gauge symmetry in the large-p limit.

First, let us consider the action

$$S = \frac{1}{2}\int_M \omega F^{ij} F_{ij} + m^2 \int_M \omega A^i A_i,$$

where m is a mass parameter and $F_{ij} := \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}$. For the massless case $m = 0$, there is a U(1) gauge symmetry $A \mapsto A' = A + \mathrm{d}\lambda$ for $\lambda \in \mathcal{C}^\infty(M, \mathbb{R})$. Since M is two-dimensional, we have

$$F^{ij} F_{ij} = 2\left(\sum_a (\mathrm{d}X^a)^* \{X^a, A\}\right)^2.$$

Here, $\{X^a\}_{a=1,\cdots,d}$ is the isometric embedding. Thus, the action can be written as

$$S = \int_M \omega \left(\sum_a (\mathrm{d}X^a)^* \{X^a, A\}\right)^2 + m^2 \int_M \omega A^* A. \tag{6.10}$$

Now, let us apply the matrix regularization to this action. Using the notation of (6.2), we define the Toeplitz operators

$$\widehat{A} = T_p^{(1,0)}(A), \quad \widehat{\mathrm{d}X^a} = T_p^{(1,0)}(\mathrm{d}X^a).$$

Then, we define the matrix action which approximates (6.10) in large-p limit,

$$S_{\mathrm{MM}} = p \operatorname{Tr} \widehat{F}^2 + pm^2 \operatorname{Tr}(\widehat{A}^* \widehat{A}). \tag{6.11}$$

Here,

$$\widehat{F} = (\mathrm{i}\hbar_p)^{-1} \sum_a \widehat{\mathrm{d}X^a}^* [\widehat{X^a}, \widehat{A}],$$

where $[\widehat{X^a}, \widehat{A}] := T_p^{(1,1)}(X^a \mathbf{1}_1)\widehat{A} - \widehat{A} T_p^{(0,0)}(X^a)$.

6.4 Matrix action on $\mathbb{T}^2_{\mathrm{i}}$ and its $\mathrm{U}(1)$ gauge symmetry

In the case of the one-dimensional complex torus $\mathbb{T}^2_\tau$, we can explicitly calculate the Toeplitz operators of X^a and $\mathrm{d}X^a$, which appear in the matrix action (6.11). As already discussed in section 5.5, the isometric embedding of $\mathbb{T}^2_\tau$ is given by (5.16). For simplicity, we assume $\tau = \mathrm{i}$, which is the case for $\rho = 1$ and $A = I \in \mathrm{SL}(2, \mathbb{Z})$. In this case, the isometric embedding functions are given by

$$X^1 = (2\pi)^{-1} \cos(2\pi x), \quad X^1 = (2\pi)^{-1} \sin(2\pi x),$$
$$X^3 = (2\pi)^{-1} \cos(2\pi y), \quad X^4 = (2\pi)^{-1} \sin(2\pi y).$$

Their Toeplitz operators $T_p^{(0,0)}(X^a)$ are given by

$$T_p(X^1) = \frac{\mathrm{e}^{-\frac{\pi}{2p}}}{4\pi}(V_p + V_p^*), \quad T_p(X^2) = \frac{\mathrm{e}^{-\frac{\pi}{2p}}}{4\pi\mathrm{i}}(V_p - V_p^*),$$
$$T_p(X^3) = \frac{\mathrm{e}^{-\frac{\pi}{2p}}}{4\pi}(U_p + U_p^*), \quad T_p(X^4) = \frac{\mathrm{e}^{-\frac{\pi}{2p}}}{4\pi\mathrm{i}}(U_p - U_p^*),$$

where $V_p := F_p^{1,0}$ and $U_p := F_p^{0,1}$ are clock-shift matrices. Thus, we need to compute $T_p^{(1,0)}(\mathrm{d}X^a)$ and $T_p^{(1,1)}(X^a \mathbf{1}_1)$ as follows.

Since $\mathbb{T}^2_{\mathrm{i}}$ is flat, we have a decomposition $\ker D^{p,1} = \ker D^{p,0} \oplus \ker D^{p,0}$ such that

$$\Psi = \psi \mathrm{d}x + \phi \mathrm{d}y,$$

for $\Psi \in \ker D^{p,1}$ and $\psi, \phi \in \ker D^{p,0}$. Here, (x, y) is the real coordinates. Hence, we have a corresponding block matrix decomposition,

$$\hat{A} = \begin{pmatrix} \hat{A}_x \\ \hat{A}_y \end{pmatrix} = \begin{pmatrix} T_p^{(0,0)}(A_x) \\ T_p^{(0,0)}(A_y) \end{pmatrix},$$
$$\widehat{\mathrm{d}X^a} = \begin{pmatrix} T_p^{(0,0)}(\partial_x X^a) \\ T_p^{(0,0)}(\partial_y X^a) \end{pmatrix}, \quad T_p^{(1,1)}(X^a \mathbf{1}_1) = \begin{pmatrix} T_p^{(0,0)}(X^a) & 0 \\ 0 & T_p^{(0,0)}(X^a) \end{pmatrix}.$$

Here, $A = A_x \mathrm{d}x + A_y \mathrm{d}y$ and we introduced $\partial_x := \partial/\partial x$ and $\partial_y := \partial/\partial y$. From (5.16), we have

$$\partial_x X^1 = -2\pi X^2, \quad \partial_x X^2 = 2\pi X^1, \quad \partial_y X^1 = \partial_y X^2 = 0,$$
$$\partial_y X^3 = -2\pi X^4, \quad \partial_y X^4 = 2\pi X^3, \quad \partial_x X^3 = \partial_x X^4 = 0.$$

Thus, we obtain

$$\begin{aligned}\hat{F} &= (\mathrm{i}\hbar_p)^{-1} \sum_a \Big(T_p^{(0,0)}(\partial_x X^a)[T_p^{(0,0)}(X^a), \hat{A}_x] + T_p^{(0,0)}(\partial_y X^a)[T_p^{(0,0)}(X^a), \hat{A}_y] \Big) \\ &= 2\pi(\mathrm{i}\hbar_p)^{-1} \Big(T_p^{(0,0)}(X^1)[T_p^{(0,0)}(X^2), \hat{A}_x] - T_p^{(0,0)}(X^2)[T_p^{(0,0)}(X^1), \hat{A}_x] \\ &\qquad + T_p^{(0,0)}(X^3)[T_p^{(0,0)}(X^4), \hat{A}_y] - T_p^{(0,0)}(X^4)[T_p^{(0,0)}(X^3), \hat{A}_y] \Big).\end{aligned}$$

Using (5.18), we have

$$\hat{F} = \hat{\partial}_x \hat{A}_y - \hat{\partial}_y \hat{A}_x,$$

where

$$\hat{\partial}_x M := \frac{p\mathrm{e}^{-\frac{\pi}{p}}}{2} \left(V_p[V_p^*, M] - V_p^*[V_p, M] \right), \quad \hat{\partial}_y M := -\frac{p\mathrm{e}^{-\frac{\pi}{p}}}{2} \left(U_p[U_p^*, M] - U_p^*[U_p, M] \right),$$

for any $M \in \mathbb{C}^{p\times p}$. These operators correspond to ∂_x and ∂_y in the large-p limit. As we can directly check, we have

$$[\hat{\partial}_x, \hat{\partial}_y]M = 0, \tag{6.12}$$

for any $M \in \mathbb{C}^{p\times p}$. Thus, the matrix $\hat{F}$ corresponds to $F_{xy} = \partial_x A_y - \partial_y A_x$ in the large-p limit. Therefore, the matrix action can be written as

$$S_{\mathrm{MM}} = p \operatorname{Tr}(\hat{\partial}_x \hat{A}_y - \hat{\partial}_y \hat{A}_x)^2 + pm^2 \operatorname{Tr}(\hat{A}_x^2 + \hat{A}_y^2).$$

For the massless case $m = 0$, there exists a symmetry

$$\begin{pmatrix} \hat{A}_x \\ \hat{A}_y \end{pmatrix} \mapsto \begin{pmatrix} \hat{A}_x \\ \hat{A}_y \end{pmatrix} + \begin{pmatrix} \hat{\partial}_x \widehat{\lambda} \\ \hat{\partial}_y \widehat{\lambda} \end{pmatrix},$$

for any $\widehat{\lambda} \in \mathbb{C}^{p\times p}$. This follows from the fact that $\hat{\partial}_x$ and $\hat{\partial}_y$ are linear and (6.12). This transformation corresponds to the U(1) gauge symmetry $A \mapsto A + \mathrm{d}\lambda$ in large-p limit.

7 Conclusion and future problems

In this dissertation, we studied the Berezin-Toeplitz quantization of a vector bundle over a general Kähler manifold M, which is developed in [9, 10, 14]. In this formalism, we treated the vector bundle as a homomorphism bundle $\mathrm{Hom}(E_2, E_1)$. Then, its section $s \in \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1))$ can be considered as linear operators between vector spaces of suitable spinor fields, $s : \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_2) \to \mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_1)$. By restricting the vector spaces $\mathcal{C}^\infty(M, \Lambda^{0,\bullet} \otimes L^{\otimes p} \otimes E_a)$ to the finite-dimensional kernels of spin^c Dirac operators $\ker D^{p,E_a}$, we defined a quantization map $T_p^{(E_1,E_2)} : \mathcal{C}^\infty(M, \mathrm{Hom}(E_2, E_1)) \to \mathrm{Hom}(\ker D^{p,E_2}, \ker D^{p,E_1})$, where $T_p^{(E_1,E_2)}(s)$ can be represented as a matrix with finite size. We obtained a large-p asymptotic expansion of the product $T_p^{(E_1,E_2)}(s)T_p^{(E_2,E_3)}(t)$ for any $s \in \mathrm{Hom}(E_2, E_1)$ and $t \in \mathrm{Hom}(E_3, E_2)$, up to the first order in $\hbar_p$. In the zeroth order of this asymptotic expansion, we derived the correspondence of the product of Toeplitz operators and the product of sections in large-p limit. In the first order of this asymptotic expansion, we derived the correspondence of the generalized commutator of Toeplitz operators and the generalized Poisson bracket of sections in large-p limit. These correspondences are natural generalizations of the well-known relation of matrix regularization of functions. A particular usefulness of the correspondence of the generalized commutator and the generalized Poisson bracket concerns with the matrix Laplacian. For a Kähler manifold, the Bochner Laplacian on sections can be written in terms of the generalized Poisson bracket and the isometric embedding. Thus, we can define the corresponding matrix Laplacian acting on Toeplitz operators by the generalized commutator and the Toeplitz operators of the isometric embedding. As an application of this formalism, we considered the matrix regularization of monopole bundles over $\mathbb{CP}^n$ and $\mathbb{T}^2_\tau$. In particular, our formulation correctly reproduces the results of the monopole bundles over the fuzzy $\mathbb{CP}^n$ given in [23, 24]. Another application of this formalism is the matrix regularization of tensor fields over a Kähler manifold. We treated the tensor field of type (k, l) as a section of the homomorphism bundle $\mathrm{Hom}(T^*M^{\otimes l}, T^*M^{\otimes k})$ and we explicitly analyzed the properties of the Toeplitz operators of sections of this bundle, such as the matrix counterpart of the transformation induced from the pullback of the symplectomorphism. We also explicitly consider the matrix regularization of one-form field on $\mathbb{T}^2_{\mathrm{i}}$ and construct the matrix action. We showed that the matrix action has a matrix gauge symmetry which corresponds to the U(1) gauge symmetry in the large-p limit, if the one-form field is massless.

The Berezin-Toeplitz quantization of vector bundle is applicable to the matrix regularization of a wide class of fields. The possible applications of this work are as follows. First, using the technique of matrix regularization of tensor fields, one may construct a fuzzy version of the higher spin theories [27, 28]. Second, since the spin^c bundle $\Lambda^{0,\bullet}$ itself is a vector bundle, it is possible to consider a matrix regularization of spinor fields. The theories of spinor fields on lattice have the issues of doublers and chiral anomaly. Then, it is possible consider similar problems on fuzzy spaces [29–34]. Using our method, it is interesting to consider the similar problems on a general Kähler manifold. It is also possible to construct a fuzzy field theory on manifolds with various background fields such as instanton configurations. It is important to uncover how the various background field configurations are incorporated in the framework of matrix configurations.

Finally, let us discuss some possible generalizations of this study. In section 3, we defined the Berezin-Toeplitz quantization for vector bundles over a general symplectic manifold and most of the properties of the Toeplitz operators are considered for a general symplectic manifold. However, we failed to show the general asymptotic expansion of the Toeplitz operators for general symplectic case, except for the leading term $C_0(s,t)$. In our technique, we had to assume that M is Kähler in order to obtain the subleading coefficient $C_1(s,t)$, which is important to derive the correspondence of the generalized commutator and the generalized Poisson bracket. However, in [7], it is possible to obtain the asymptotic properties of the Toeplitz operators of functions on a general symplectic manifold using the asymptotic expansion of the Bergman kernel. Thus, we expect that the correspondence of the generalized commutator and the generalized Poisson bracket can be shown for a general symplectic manifold. In [7], the Berezin-Toeplitz quantization of functions over non-compact manifolds or orbifolds are also considered and therefore it may be possible to consider the Berezin-Toeplitz quantization of vector bundles over such general spaces.

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A Notation and conventions

- The sets of all natural numbers, integers, real numbers and complex numbers are respectively denoted as $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$. Here, the natural number is defined as a nonnegative integers, i.e. $\mathbb{N} = \{0, 1, 2, \cdots\}$. We also denote by $\mathbb{Z}_p$ the ring of integers modulo p, where the congruence class is denoted as $\bar{a}_p := \{n \in \mathbb{Z} | n - a \in p\mathbb{Z}\}$.

- For a ring R, the set of all $n \times m$ matrices with entries in R is denoted by $R^{n\times m}$. For particular subsets with group structures, $\mathrm{GL}(n, R) \subset R^{n\times n}$ is the set of all invertible square matrices, $\mathrm{SL}(n, R) \subset R^{n\times n}$ is the set of all square matrices with unit determinant. We also denote by $\mathrm{U}(n) \subset \mathbb{C}^{n\times n}$ the set of all unitary matrices and by $\mathrm{O}(n) \subset \mathbb{R}^{n\times n}$ the set of all orthogonal matrices.

- The imaginary unit is denoted as i. The complex conjugation of $c \in \mathbb{C}$ is denoted as $\bar{c}$. The real part and imaginary part of $c \in \mathbb{C}$ are respectively denoted as $\Re c$ and $\Im c$.

- A manifold M refers to a manifold without boundary.

- For a manifold M, we denote by TM the tangent vector bundle, and denote by T^*M the cotangent vector bundle.

- For vector bundles E, F over a manifold M, E^* denotes a dual bundle of E, $E \otimes F$ denotes the tensor product bundle, $E^{\otimes n}$ denotes the n-fold tensor product of E, $E \oplus F$ denotes the Whitney sum bundle, $\mathrm{Hom}(E, F)$ denotes a homomorphism bundle and $\mathrm{End}(E) := \mathrm{Hom}(E, E)$.

- The exterior algebra bundle of a vector bundle E over M is denoted as $\Lambda(E) = \bigoplus_{i=0}^{\mathrm{rank} E} \Lambda^i(E)$.

- For a vector bundle E over M, we denote by E_x the fiber (a vector space) at $x \in M$.

- For a field $\mathbb{K}$, we denote by $\mathcal{C}^\infty(M, \mathbb{K})$ the set of all smooth $\mathbb{K}$-valued functions over a manifold M. For a vector bundle E over a manifold M, we denote by $\mathcal{C}^\infty(M, E)$ the set of all smooth sections of E.

- A Hermitian vector bundle (E, h^E) over a manifold M is a vector bundle equipped with a Hermitian inner product h^E on each fiber E_x, which is smoothly varying in $x \in M$.

- A Hermitian connection ∇^E of a Hermitian vector bundle (E, h^E) is a connection of E satisfying the compatibility condition $\mathrm{d}h^E(s, t) = h^E(\nabla^E s, t) + h^E(s, \nabla^E t)$

B M-theory and BFSS matrix model

In this appendix, we review the conjectures of the M-theory [15]. Then, we review the BFSS matrix model, which is postulated as as a consistent formulation of the M-theory in the infinite momentum [1] or DLCQ limit [16].

B.1 Conjectures of the M-Theory

M-theory is a hypothetical theory which unites the 5 different types of the superstring theories, type I, IIA, IIB, heterotic SO(32) and heterotic $E_8 \times E_8$ [15]. There are numerous evidences that there might be such a theory, but the formulation of M-theory is yet unknown. In this subsection, we summarize some of the conjectures or postulates which M-theory should respect.

1. *M-theory is a supersymmetric 11-dimensional theory*
   The M-theory is postulated to be a theory with 11-dimensional target spaces with supersymmetry. It does not contradict with the fact that the superstring theory is defined on a 10-dimensional target space since the analysis of the superstring theory is given in perturbative method. It is postulated that the extra 11th dimension opens up in the limit of nonperturbative superstring theory.

2. *The low energy effective theory of the M-theory is the 11-dimensional supergravity theory*
   As the low energy effective actions of the superstring theories are the 10-dimensional supergravity theories (plus some Yang-Mills theories in the case of type I and heterotic theories), the low energy effective theory of the M-theory is postulated as a maximally supersymmetric 11-dimensional supergravity theory.

3. *M-theory contains M2-branes and M5 branes*
   From the fact that the 11-dimensional supergravity theory contains 3-form gauge field, the fundamental objects of the M-theory are expected to couple to the 3-form gauge fields. Thus, the electrically coupled objects are extended in 2+1-dimension and the magnetically coupled objects are extended in 5+1-dimension. They are called M2-branes and M5-branes, respectively.

4. *Codimension* 1 *compactification of the M-theory gives superstring theories*
   It is known that the type IIA supergravity theory (the low energy effective theory of type IIA superstring theory) can be obtained as a S^1 compactification of the 11-dimensional supergravity theory. This suggests that the S^1 compactification of the M-theory would give the type IIA superstring theory, not only in the low energy level. Similarly, it is conjectured that the compactification of the M-theory onto the orbifold $S^1/\mathbb{Z}_2$ would give the heterotic

$E_8 \times E_8$ superstring theory. Moreover, one finds that the resulting string coupling is related to the size of the compactified space.

B.2 Type IIA superstring theory and M-Theory

In this subsection, we discuss the connections of the type IIA superstring theory and the M-theory. In the previous subsection, we mentioned the relation of the type IIA superstring theory and the M-theory only in the low energy and classical level. However, there are some evidences that this connection holds in the quantum stringy level.

Let us suppose that the type IIA is indeed a 11-dimensional theory compactified on a circle with radius R. Then, the type IIA theory is a Kaluza-Klein theory. Let be $\{x^\mu\}_{\mu=0}^{9}$ the coordinates of uncompactified 10-dimensional space $\mathbb{R}^{1,9}$ and y be the compactified coordinate of S^1 such that $y \sim y + 2\pi R$. Then, any complex function $\phi \in \mathcal{C}^\infty(\mathbb{R}^{1,9} \times S^1, \mathbb{C})$ can be expended as [2]

$$\phi(x,y) = \sum_{n\in\mathbb{Z}} \phi_n(x) \mathrm{e}^{iny/R}.$$

The kinetic term of ϕ is proportional to the Laplacian Δ' on $\mathcal{C}^\infty(\mathbb{R}^{1,9} \times S^1, \mathbb{C})$ is then given as

$$\Delta'\phi(x,y) = \sum_{n\in\mathbb{Z}} \left(\Delta\phi_n(x) - \frac{n^2}{R^2}\phi_n(x)\right) \mathrm{e}^{iny/R},$$

where Δ is the Laplacian on $\mathcal{C}^\infty(\mathbb{R}^{1,9}, \mathbb{C})$. Thus, as a 10-dimensional theory, we have infinite number of fields $\{\phi_n\}_{n\in\mathbb{Z}}$ called Kaluza-Klein modes (KK modes) which have infinite tower of masses $\frac{|n|}{R}$. In the limit of $R \to 0$ (which corresponds to the perturbative limit in the superstring theory), the higher modes ($|n| > 0$) become infinitely heavy and therefore decouple. However, for $R > 0$ (which corresponds to the nonperturbative superstring theory), such modes should exist in the 10-dimensional theory. The type IIA theory contains nonperturbative object called D-branes. In particular, the D0-brane has a definite mass and the D0-brane is a BPS state, which do not interact with each other. Thus, the total mass of the $|n|$ D0-branes is simply $|n|$ times the mass of a single D0-brane. We see that ϕ_n state corresponds to a system of $|n|$ D0-branes or anti-D0-branes.

Let us examine the relation of string coupling and the compactification radius. Let $(M^{1,9}, g)$ be a pseudo-Riemannian manifold with signature $(1,9)$. Then, the bosonic part of the type IIA supergravity action S_{IIA} on M is given by

$$\begin{aligned} S_{\mathrm{IIA}}[g,\phi,B_2,A_1,A_3] :=& \frac{1}{2\kappa_{10}^2}\int \mu_g \mathrm{e}^{-2\phi}\left(K - 4\phi\Delta\phi - \frac{1}{2}|H_3|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int \left(\mu_g(|F_2|^2 + |\tilde{F}_4|^2) + B_2 \wedge F_4 \wedge F_4\right) \end{aligned}.$$

[2]For more general fields (sections of a vector bundle), we can do the similar expansion with some modifications. For example, if the vector bundle has a nontrivial holonomy on S^1, we have to impose the twisted boundary condition on S^1 and it gives the different Fourier expansion.

where μ_g is the Riemannian volume form, K is the scalar curvature, ϕ is the scalar dilaton field, Δ is the Hodge Laplacian. For the field content of this action, B_2 is a 2-form field, A_1 is a R-R 1-form field, and A_3 is a R-R 3-form field and we introduced H_3, F_2, F_4 and $\tilde{F}_4$ as

$$H_3 := \mathrm{d}B_2,\ F_2 := \mathrm{d}A_1,\ F_4 := \mathrm{d}A_3,\ \tilde{F}_4 := F_4 - A_1 \wedge A_3,$$

respectively. The first term of the action comes from the NS-NS sector and the second term comes from the R-R sector. The proportionality factor κ_{10} is a 10-dimensional gravitational constant, which is related to the string constant α' by

$$\kappa_{10}^2 = \frac{1}{2}(2\pi)^7 \alpha'^4. \tag{B.1}$$

We also use the notation

$$\int \mu_g |F_p|^2 := \int F_p \wedge \star F_p,$$

where $\star$ denotes the Hodge star. On the other hand, the bosonic part of the 11-dimensional supergravity action S_{11} on a pseudo-Riemannian manifold $(M^{1,10}, G)$ with signature $(1,10)$ is given by

$$S_{11}[G, C_3] = \frac{1}{2\kappa_{11}^2} \int \mu_G \left(K - \frac{1}{2}|F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int C_3 \wedge F_4 \wedge F_4$$

where C_3 is a 3-form field and $F_4 := \mathrm{d}C_3$. Let us assume $M^{1,10} = M^{1,9} \times S^1$. Then, the type IIA supergravity action S_{IIA} can be obtained from the 11-dimensional supergravity action S_{11} by the following rules:

$$G(u,v) = \mathrm{e}^{-\frac{2}{3}\phi} g(u,v) + \mathrm{e}^{\frac{4}{3}\phi} A(u)A(v), \quad G(u, \partial/\partial y) = \mathrm{e}^{\frac{4}{3}\phi} A(u), \quad G(\partial/\partial y, \partial/\partial y) = \mathrm{e}^{\frac{4}{3}\phi}$$
$$C(u,v,w) = A(u,v,w), \quad C(u,v,\partial/\partial y) = \frac{2}{3} B(u,v)$$

for any tangent vector fields u, v, w on $M^{1,9}$. Here, y is the coordinate of S^1. Let us identify $y \sim y + 2\pi\sqrt{\alpha'}$. Then, the radius of compactification R_{11} is

$$R_{11} = \left\langle \sqrt{G(\partial/\partial y, \partial/\partial y)\alpha'} \right\rangle = \mathrm{e}^{\frac{2}{3}\langle\phi\rangle}\sqrt{\alpha'} = g_{\mathrm{s}}^{\frac{2}{3}}\sqrt{\alpha'}$$

where $\langle\cdot\rangle$ is the the expectation value with respect to the partition function of the superstring theory and $g_{\mathrm{s}} := \mathrm{e}^{\langle\phi\rangle}$ is the string coupling. This shows that the size of extra dimension is related to string coupling (i.e. dilaton) and the extra dimension opens up in the nonperturbative case.

Let us revisit the discussion of Kaluza-Klein modes. We discussed that the n-th mode of Kaluza-Klein mode has a mass $|n|/R$. This should be slightly modified in our case. With our identification of y, a 11-dimensional scalar field $\Phi(x,y)$ can be expanded as

$$\Phi(x,y) = \sum_{n\in\mathbb{Z}} \Phi_n(x) \mathrm{e}^{iny/\sqrt{\alpha'}}.$$

Then, the Hodge Laplacian Δ' on $(M^{1,10}, G)$ acting on Φ becomes

$$\Delta'\Phi(x,y) = \sum_{n\in\mathbb{Z}} \left(\mathrm{e}^{-\frac{2}{3}\phi}\Delta\Phi_n(x) - \frac{n^2}{R_{11}^2}\Phi_n(x) \right) \mathrm{e}^{\mathrm{i}ny/\sqrt{\alpha'}}.$$

where Δ is the Hodge Laplacian on $(M^{1,9}, g)$. Thus, the mass of the Kaluza-Klein mode $\Phi_n(x)$ will be

$$m_n = \frac{|n|}{R_{11}}\mathrm{e}^{\frac{1}{3}\phi} = \frac{|n|}{g_{\mathrm{s}}\sqrt{\alpha'}}. \tag{B.2}$$

Note that the tension of the Dp-brane is given by

$$T_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2} g_{\mathrm{s}}^{-1}.$$

Therefore, we conclude that the n-th Kaluza-Klein mode indeed corresponds to $|n|$ D0-branes of type IIA superstring theory.

Finally, let us define the Planck mass in 11-dimension. (B.2) shows that the compactification radius measured by the 10-dimensional metric g is

$$\tilde{R}_{11} := g_{\mathrm{s}}\sqrt{\alpha'}. \tag{B.3}$$

By comparing the coefficients of the Einstein-Hilbert actions of type IIA supergravity theory and 11-dimensional supergravity theory, we have

$$\frac{2\pi\tilde{R}_{11}}{\kappa_{11}^2} = \frac{1}{\kappa_{10}^2}.$$

Since κ_{10} is given by (B.1), we obtain

$$2\kappa_{11}^2 = (2\pi)^8 g_{\mathrm{s}}^3 (\alpha')^{9/2}.$$

Thus, it is natural to define the 11-dimensional Planck mass M_{11} by

$$M_{11} := g_{\mathrm{s}}^{-1/3}(\alpha')^{-\frac{1}{2}},$$

which satisfies

$$2\kappa_{11}^2 = (2\pi)^8 M_{11}^{-9}. \tag{B.4}$$

Similarly, the 11-dimensional Planck length is given by

$$l_{11} = 1/M_{11}.$$

Combining (B.3) and (B.4), we can express g_{s} and α' by the 11-dimensional parameter $\tilde{R}_{11}$ and M_{11}:

$$g_{\mathrm{s}} = (\tilde{R}_{11}M_{11})^{3/2}, \quad \alpha' = \tilde{R}_{11}^{-1}M_{11}^{-3}.$$

B.3 M-Theory and the BFSS conjecture

Let us consider the following action:

$$
\begin{aligned}
S[X^1, X^2, \cdots, X^9, \psi] &= \frac{1}{2g_{\mathrm{s}}\sqrt{\alpha'}} \int_{\mathbb{R}} \mathrm{d}t \, \mathrm{Tr}\, L(t), \\
L(t) &:= \sum_a \dot{X}^a(t)\dot{X}^a(t) + \frac{1}{2}\sum_{\alpha,\beta}[X^a(t), X^b(t)]^2 + \psi^{\mathrm{T}}(t)\left(\mathrm{i}\dot{\psi}(t) - \sum_a \Gamma_a[X^a(t), \psi(t)]\right).
\end{aligned}
\tag{B.5}
$$

Here, $\{X^a\}_{a=1}^9$ is a set of $N \times N$ Hermitian matrix valued function on $\mathbb{R}$ and ψ is a $N \times N$ matrix valued function on $\mathbb{R}$ whose matrix entries are SO(9) spinors and $\{\Gamma_i\}_{i=1}^9$ are SO(9) gamma matrices. The dot operation is defined as a derivative $\dot{f}(t) := \frac{\mathrm{d}f}{\mathrm{d}t}(t)$. This matrix model is called the BFSS matrix model and it is postulated to be equivalent to M-theory in light cone infinite momentum frame in the limit $N \to \infty$ [1] and to M-theory in discrete light cone frame in finite N [16]. We will discuss this topic in the following subsection. The basic argument is given as follows. In the light cone infinite momentum frame and the discrete light cone frame, M-theory is basically defined on a certain limit of light like circle. As we saw in the previous subsection, such a theory contains D0-branes as a Kaluza-Klein modes. In the light cone infinite momentum frame and the discrete light cone frame, all the degrees of freedom except for the D0-branes and the massless open strings attached to them would not give a finite contribution to the Hamiltonian of the whole system. The system of D0-branes and massless open strings attached to them is described by 0+1-dimensional supersymmetric Yang-Mills theory. After doing some rescaling and redefining variable, the Lagrangian of the supersymmetric Yang-Mills theory turns into the BFSS matrix model. We can also derive this matrix model from a system of M2-brane, which is summarized in Appendix C.

This matrix model is tested in numerous studies and this model gives correct descriptions of M-theory so far (the classic reference containing such topics is [35]).

B.4 DLCQ M-Theory and the matrix model

In this subsection, we review that the M-theory in discrete light cone quantization (DLCQ) limit is given by a system of multi D0-branes [16]. First, we will denote the 11-th spatial direction X^{10} by y and the timelike coordinate X^0 as t. We will introduce a light cone coordinate as

$$
X^{\pm} := \frac{1}{\sqrt{2}}(t \pm y).
$$

In this frame, let us compactify the X^- to a circle with radius R, i.e.

$$
X^- \sim X^- + R \quad \Leftrightarrow \quad \begin{pmatrix} t \\ y \end{pmatrix} \sim \begin{pmatrix} t \\ y \end{pmatrix} + \frac{R}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{B.6}
$$

Since the canonical momentum of X^- is P^+, this compactification quantizes the P^+ as $P^+ = N/R$ where N is a integer which corresponds to the Kaluza-Klein modes.

Now, let us assume this compactification is a limit of the following compactification

$$\begin{pmatrix} t \\ y \end{pmatrix} \sim \begin{pmatrix} t \\ y \end{pmatrix} + \begin{pmatrix} R/\sqrt{2} \\ -\sqrt{R^2/2 + R_{\mathrm{s}}^2} \end{pmatrix}, \tag{B.7}$$

where $R_{\mathrm{s}} \to 0$ will make this compactification identical to the (B.6). Let us Lorentz boost this system with the parameter β:

$$\begin{pmatrix} t' \\ y' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix},$$

where

$$\beta = \frac{1}{\sqrt{1 + \frac{2R_{\mathrm{s}}^2}{R^2}}} = 1 - \frac{R_{\mathrm{s}}^2}{R^2} + \mathcal{O}(R_{\mathrm{s}}^3)$$

and $\gamma := 1/\sqrt{1-\beta^2}$. Then, the identification (B.7) can be written in this boosted frame as

$$\begin{pmatrix} t' \\ y' \end{pmatrix} \sim \begin{pmatrix} t' \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ R_{\mathrm{s}} \end{pmatrix}.$$

Thus, the compactification becomes a spatial compactification with the radius $R_{\mathrm{s}} \to 0$ in this frame. Therefore, the M-theory in this frame is equivalent to the type IIA superstring theory with parameters

$$g_{\mathrm{s}} = (R_{\mathrm{s}} M_{11})^{3/2}, \quad \alpha' = R_{\mathrm{s}}^{-1} M_{11}^{-3}.$$

Now, let us consider the change of the momentum P^- under a Lorentz boost. P^- transforms as

$$P^- \mapsto P'^- = \gamma(1-\beta)P^- \tag{B.8}$$

and the light cone compactification with radius R in (B.6) will be boosted to yet another light cone compactification with radius $R' = \gamma(1-\beta)R$. Therefore, we conclude that P^- is proportional to the compactification radius R in the light cone compactified frame. For the small R_{s}, (B.8) will be

$$P^- \mapsto P'^- = \gamma(1-\beta)P^- = \left[\frac{1}{\sqrt{2}}\frac{R_{\mathrm{s}}}{R} + \mathcal{O}((R_{\mathrm{s}}/R)^3)\right] P^-.$$

Thus, P'^-, which corresponds to the energy scale in the boosted frame, does not depend on R but proportional to R_{s} in the limit $R_{\mathrm{s}} \to 0$.

The vanishing of the energy scale will be problematic. Therefore, let us fix the energy scale of the M-theory on the light cone compactified space with radius R and M-theory on the boosted spatially compactified space with radius R_{s} as

$$RM_{11} = R_{\mathrm{s}} \tilde{M}_{11},$$

where M_{11} and $\tilde{M}_{11}$ are the 11-dimensional Planck masses of each theory. This means that $\tilde{M}_{11}$ goes to infinity in order to make the energy scale fixed. Thus, the M-theory on the boosted spatially compactified space with radius $R_{\rm s}$ and the 11-dimensional Planck mass $\tilde{M}_{11}$ will be described by the type IIA superstring theory with parameters

$$\begin{aligned} g_{\rm s} &= (R_{\rm s}\tilde{M}_{11})^{3/2} = R_{\rm s}^{3/4}(RM_{11}^2)^{3/4} \\ \alpha' &= R_{\rm s}^{-1}\tilde{M}_{11}^{-3} = R_{\rm s}^{\frac{1}{2}}(RM_{11}^2)^{-3/2} \end{aligned}.$$

In the limit $R_{\rm s} \to 0$ with a finite R and M_{11}, the string coupling is zero and the string energy scale goes to infinity. The system of N Dp-branes will be described by the $p+1$-dimensional supersymmetric Yang-Mills theory with coupling

$$g_{{\rm D}p}^2 = \frac{1}{(2\pi\alpha')^2 T_p} = (2\pi)^{p-2} g_{\rm s} \alpha'^{(p-3)/2} = (2\pi)^{p-2} R_{\rm s}^{p/4} (RM_{11}^2)^{3(1-p/4)}.$$

Therefore, only the system of N D0-branes with massless strings will not be decoupled. This limiting process is consistent with the approximation of DBI action to the supersymmetric Yang-Mills theory because the higher dimensional operators are of order $R_{\rm s}/R$. Therefore, this DLCQ limit of M-theory is equivalent to the system of N D0-branes which is described by 0+1-dimensional Super Yang-Mills theory. After doing some rescaling and redefining variable, the Lagrangian becomes the BFSS matrix model.

C Matrix model from matrix regularization of the M2-brane

In this appendix, we show that the BFSS matrix model can be derived from the matrix regularized theory of a single M2-brane [4].

C.1 Bosonic M2-brane action

Let us consider the theory of a bosonic M2-brane embedded in a pseudo-Riemannian manifold $(M^{1,d}, G)$ with signature $(1, d)$. We denote by $(X^0, X^1, \cdots, X^d)$ the coordinates of $M^{1,d}$ such that $\partial/\partial X^0$ is a time-like vector and $\partial/\partial X^a$ for $a > 0$ are are space-like vectors. Let $(M^{1,2}, g)$ be an embedded pseudo-Riemannian manifold of $(M^{1,d}, G)$ with signature $(1, 2)$, which corresponds to a M2-brane world-volume. We denote by (x^0, x^1, x^2) the coordinates of $M^{1,2}$ such that $\partial/\partial x^0$ is a time-like vector and $\partial/\partial x^i$ for $i = 1, 2$ are space-like vectors. In this subsection, we use Einstein sum convection for the indices of the coordinates of $M^{1,d}$, denoted by $a, b, \cdots$, and the indices of the coordinates of $M^{1,1}$, denoted by $i, j, \cdots$.

The theory of a single M2-brane is naturally described by the world-volume action with the isometric embedding function $X = (X^0, X^1, \cdots, X^d) : M^{1,2} \to \mathbb{R}^{1,d}$. Since a M2-brane electrically couples to 3-form gauge field C_3 on $M^{1,d}$, the action is given by

$$S[X, G, C^3] = -T \int_{M^{1,2}} \mu_g + \int_{M^{1,2}} \tilde{C}_3.$$

Here, the tension is related to the Planck length as $T = 1/l_{11}^3$ and g is the metric on $M^{1,2}$ induced from G and $\tilde{C}_3$ is the 3-form field on $M^{1,2}$ induced from C_3. The first term of action is the Nambu-Goto action and the second term of action represents the electrical coupling to C_3. This is the most general form of the bosonic M2-brane action.

From now on, we assume the flatness $(M^{1,d}, G) = (\mathbb{R}^{1,10}, \eta)$ for $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$ and the absence of 3-form gauge field $C_3 = 0$. Then, the action can be locally written as

$$S_{\mathrm{NG}}[X] = -T \int_{M^{1,2}} \mathrm{d}^3x \sqrt{-\det g},$$

where $\mathrm{d}^3x := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$ and $g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ is given by

$$g_{ij} := \eta_{ab} \frac{\partial X^a}{\partial x^i} \frac{\partial X^b}{\partial x^j}.$$

As we will see in the next subsection, the bosonic part of the BFSS matrix model can be obtained from the matrix regularization of this action. Similar to the case of string theory, we will use the classically equivalent Polyakov action since the actions containing the square root are hard to quantize. Using an auxiliary metric $h = h_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ on $M^{1,2}$, the Polyakov action of a bosonic M2-brane is given by

$$S_{\mathrm{P}}[X, h] = -\frac{T}{2} \int_{M^{1,2}} \mathrm{d}^3x \sqrt{-\det h}(h^{ij} g_{ij} - 1)$$

where (h^{ij}) is the inverse of (h_{ij}). This action is equivalent to S_{NG} if we impose the Euler-Lagrange equation for h as follows. Using $\delta(\det h) = (\det h)h^{ij}\delta h_{ij} = -(\det h)h_{ij}\delta h^{ij}$, we obtain

$$4\frac{\delta S_{\mathrm{P}}[X,h]}{\delta h^{ij}} = T\sqrt{-\det h}\left((h^{i'j'}g_{i'j'}-1)h_{ij} - 2g_{ij}\right) = 0$$
$$\Leftrightarrow \quad (h^{i'j'}g_{i'j'}-1)h_{ij} = 2g_{ij}. \tag{C.1}$$

Multiplying h^{ij} on both sides of (C.1) and use $h_{ij}h^{ij} = \delta^i_i = 3$ leads to

$$g_{ij}h^{ij} = 3. \tag{C.2}$$

By plugging (C.2) into (C.1), we conclude that h satisfying the Euler-Lagrange equation is

$$h = g. \tag{C.3}$$

Thus, we obtain

$$S_{\mathrm{P}}[X, h = g] = S_{\mathrm{NG}}[X].$$

Let us briefly discuss the property of the Polyakov action S_{P}. First, the variation of action with respect to X^a is

$$\delta S_{\mathrm{P}}[X,h] = -T\int_{M^{1,2}} \mathrm{d}^3x\sqrt{-\det h}\,h^{ij}\frac{\partial(\delta X^a)}{\partial x^i}\frac{\partial X^b}{\partial x^j}\eta_{ab} = 0$$
$$\Leftrightarrow \quad \frac{\partial}{\partial x^i}\left(\sqrt{-\det h}\,h^{ij}\frac{\partial X^a}{\partial x^j}\right) = 0.$$

The bosonic M2-brane Polyakov action has a diffeomorphism symmetry of $M^{1,2}$ so does the string Polyakov action. However, unlike the case of string theory, the action of the bosonic M2-brane has no Weyl symmetry i.e. $S_{\mathrm{P}}[X,h] \neq S_{\mathrm{P}}[X, \mathrm{e}^f h]$ for $f \in \mathcal{C}^\infty(M^{1,2},\mathbb{R})$. In the case of the string theory, we can completely fix the gauge of h. There are 3 independent degrees of freedom since h can be represented as a 2×2 symmetric matrix. These degrees of freedom can be completely fixed by 2 diffeomorphism symmetry and 1 Weyl symmetry. In the case of the bosonic M2-brane, there are 6 independent degrees of freedom (3×3 symmetric matrix) and there are 3 diffeomorphism symmetry. It is known that there is a good way to fix the 3 of the 6 degrees of freedom of h:

$$h_{00} = -\frac{4}{\nu^2}\det\tilde{h}, \quad h_{01} = h_{02} = 0,$$

where ν is a positive real parameter and $\tilde{h} := \sum_{i,j=1}^2 h_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$. Then, h in this gauge can be written as

$$h = h_{00}\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \tilde{h},$$

and such a gauge can be globally taken if we have the decomposition $M^{1,2} = \mathbb{R} \times \Sigma$ where $(\Sigma, \tilde{h})$ is a compact 2-dimensional Riemannian manifold [3]. Then, using $\det h = h_{00} \det \tilde{h} = -\frac{4}{\nu^2}(\det \tilde{h})^2$, the Polyakov action in this gauge can be written as

$$S_{\mathrm{P}}[X, h = g] = \frac{T\nu}{4} \int_{\mathbb{R}\times\Sigma} \mathrm{d}^3 x \left(\eta_{ab} \dot{X}^a \dot{X}^b - \frac{4}{\nu^2} \det \tilde{h} \right) \tag{C.4}$$

where we used the constraint $h = g$ in the derivation and the dot represents $\dot{f}(x^0, x^1, x^2) := \frac{\partial f(x^0,x^1,x^2)}{\partial x^0}$.

Since any two-dimensional manifold has a Kahler structure, it is equipped with the symplectic form ω on Σ by

$$\omega := \rho\, \mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

where ρ is a positive constant such that $\int_\Sigma \omega = 1$. Then, the Poisson bracket given in (2.3) induced from ω is

$$\{f_1, f_2\} = -\rho^{-1} \left(\frac{\partial f_1}{\partial x^1} \frac{\partial f_2}{\partial x^2} - \frac{\partial f_1}{\partial x^2} \frac{\partial f_2}{\partial x^1} \right).$$

Then, under the constraint $h = g$, the direct calculation shows

$$\eta_{aa'} \eta_{bb'} \{X^a, X^b\}\{X^{a'}, X^{b'}\} = 2\rho^{-2} \det \tilde{h}. \tag{C.5}$$

Using (C.5), we can rewrite the action (C.4) as

$$S_{\mathrm{P}}[X, h = g] = \frac{T\nu}{4} \int_{\mathbb{R}\times\Sigma} \mathrm{d}^3 x \left(\eta_{ab} \dot{X}^a \dot{X}^b - \frac{2\rho^2}{\nu^2} \eta_{aa'} \eta_{bb'} \{X^a, X^b\}\{X^{a'}, X^{b'}\} \right). \tag{C.6}$$

All the derivatives with respect to σ^1 and σ^2 in the action can be written in terms of the Poisson bracket. The Euler-Lagrange equation for X^a can be expressed in terms of the Poisson bracket in this gauge by

$$\ddot{X}^a = \frac{2\rho^2}{\nu^2} \eta_{bc} \{\{X^a, X^b\}, X^c\}.$$

The constraint (C.3) in this gauge is

$$\eta_{ab} \dot{X}^a \frac{\partial X^b}{\partial x^i} = 0, \tag{C.7}$$

for $i = 1, 2$ and

$$\eta_{ab} \dot{X}^a \dot{X}^b = -\frac{2\rho^2}{\nu^2} \eta_{aa'} \eta_{bb'} \{X^a, X^b\}\{X^{a'}, X^{b'}\}. \tag{C.8}$$

Note that the constraint (C.7) will also implies

$$\eta_{ab} \{\dot{X}^a, X^b\} = 0. \tag{C.9}$$

[3] It is evident that there are other choices of $M^{1,2}$ allowing such a gauge. If $M^{1,2}$ is a noncompact manifold, it is possible to have decompositions $M^{1,2} = \mathbb{R} \times \Sigma$ or $S^1 \times \Sigma$ for a noncompact Riemannian manifold Σ. If $M^{1,2}$ is a compact manifold, it is also possible to have a decomposition $M^{1,2} = S^1 \times \Sigma$ for a compact Riemannian manifold Σ. Similar discussion can be done if $M^{1,2}$ has boundaries. For the application to the BFSS matrix model, we restrict our discussion to $M^{1,2} = \mathbb{R} \times \Sigma$ for a compact manifold Σ.

C.2 Bosonic M2-brane in light cone gauge and its matrix regularization

As shown in the previous subsections, the bosonic M2-brane with geometry $\mathbb{R} \times \Sigma$ embedded in the flat background $\mathbb{R}^{1,10}$ is described by the action (C.6) with constraints (C.7) and (C.8). Even though the collection of constraints seem to be difficult to solve, it is known that we can solve them in so-called light cone gauge.

First, we introduce the light cone coordinate of $\mathbb{R}^{1,10}$ by

$$X^{\pm} := \frac{X^0 \pm X^{10}}{\sqrt{2}}.$$

In this frame, the target space metric is expressed as

$$\eta_{+-} = \eta_{-+} = -1, \quad \eta_{ab} = \delta_{ab}, \quad \eta_{+a} = \eta_{a+} = \eta_{-a} = \eta_{a-} = 0,$$

for $a, b = 1, 2, \cdots, 9$. In this subsection Then, the light cone gauge is a gauge which imposes

$$X^{+}(x^0, x^1, x^2) = x^0.$$

In this subsection, we use indices $a, b, \cdots = 1, 2, \cdots, 9$ and $i, j, \cdots = 1, 2$ and we use Einstein summation convention for these indices. In this gauge, the constraints (C.7) and (C.8) can be solved explicitly. The constraint (C.7) and can be written as

$$\frac{\partial X^-}{\partial x^i} = \dot{X}^a \frac{\partial X^a}{\partial x^i}, \tag{C.10}$$

$$\dot{X}^- = \frac{1}{2}(\dot{X}^a)^2 + \frac{\rho^2}{\nu^2}\{X^a, X^b\}^2, \tag{C.11}$$

and the residual constraint (C.9) becomes

$$\{\dot{X}^a, X^a\} = 0. \tag{C.12}$$

Using the constraints (C.10) and (C.11), one can determines X^- explicitly in terms of X^a up to a constant. On the other hand, the residual constraint (C.12) imposes further constraint on X^a. The action (C.6) in the light cone gauge is

$$S_{\mathrm{LC}}[X] = \frac{T\nu}{4} \int_{\mathbb{R}\times\Sigma} \mathrm{d}^3 x \left(-2\dot{X}^- + (\dot{X}^a)^2 - \frac{2\rho^2}{\nu^2}\{X^a, X^b\}^2 \right).$$

The conjugate momentum of X^- is

$$P^+ = -\frac{\delta S_{\mathrm{LC}}[X]}{\delta \dot{X}^-} = \frac{T\nu}{2},$$

which means that the parameter ν parametrizes the momentum P^+. The other conjugate momenta are given by

$$P^- = -\frac{\delta S_{\mathrm{LC}}[X]}{\delta \dot{X}^+} = 0,$$

$$P^a = \frac{\delta S_{\mathrm{LC}}[X]}{\delta \dot{X}_i} = \frac{\delta S_{\mathrm{LC}}[X]}{\delta \dot{X}^a} = \frac{T\nu}{2}\dot{X}^a.$$

Therefore, the Hamiltonian of the bosonic M2-brane in light cone gauge is given as

$$H = \frac{T\nu}{4\rho}\int_\Sigma \omega\left((\dot{X}^a)^2 + \frac{2\rho^2}{\nu^2}\{X^a, X^b\}^2\right). \tag{C.13}$$

Note that the Hamiltonian does not depend on X^- and the system is only described by the transverse coordinate X^i and the only remaining constraint is therefore (C.12).

Since the Hamiltonian of the bosonic M2-brane in the light cone gauge is written completely in terms of the symplectic structure, we can apply the matrix regularization to this system. Using (1.1) (1.2) and (1.3), the Hamiltonian (C.13) can be regularized as

$$H_{\mathrm{MM}}[X] = T\pi\,\mathrm{Tr}\left((\dot{X}^a)^2 - \frac{1}{2}[X^a, X^b]^2\right).$$

where X^a is a function of $t = x^0 \in \mathbb{R}$ with value in $N\times N$ Hermitian matrices. Here, we set $\nu = 4\pi\rho N$. The constraint (C.12) is regularized as

$$[\dot{X}^a, X^a] = 0.$$

This Hamiltonian is a obvious time independent $\mathrm{U}(N)$ gauge symmetry $X^i \mapsto UX^iU^\dagger$ which corresponds to the symplectomorphism symmetry of (C.13). The Hamilton's equation is now given as

$$\ddot{X}^a + [[X^a, X^b], X^b] = 0.$$

C.3 Matrix regularization for supersymmetric M2-brane

So far, we restricted our discussion to the case of bosonic M2-brane. In this subsection, we generalize the discussion given in the previous subsections to the supersymmetric M2-brane. In the case of the string theory, there are mainly two approaches to introduce supersymmetry. One approach is the Neveu-Schwarz-Ramond (NSR) approach and the other is the Green-Schwarz (GS) approach. In the NSR approach, we introduce world-sheet fermions which make the action manifestly supersymmetric in the sense of the world-sheet but not manifestly in the target space. In order to extend the supersymmetry in the target space, it is known that one needs extra procedure called GSO projection. On the other hand, in the GS approach, we introduce target space fermions which make the action manifestly supersymmetric in the sense of the target space. In order to make the supersymmetry in the world-sheet, we need to introduce a local fermionic symmetry called κ-symmetry. Even though the both approaches are equivalent in the analysis of the superstring theory, it is known that the GS approach is applicable to the higher dimensional membranes.

In the GS approach of the string theory, the dimension of the target space is restricted to D=3,4,6,10 by the κ-symmetry in the classical level. In the case of the M2-brane, the dimension of the target space is also restricted to D=4,5,7,11 by the κ-symmetry in the classical level. The basic procedure of GS approach for the M2-brane is the following. In the bosonic case, the action is a functional of the embedding function X^a and the Lagrangian density is written by the combination $\frac{\partial X^a}{\partial x^i}$. In the GS approach, we replace $\frac{\partial X^a}{\partial x^i}$ in the bosonic action with

$$\Pi_i^a := \frac{\partial X^a}{\partial x^i} + \bar{\psi}\Gamma^a \frac{\partial \psi}{\partial x^i},$$

where ψ is a 16-component Majorana spinor of SO(9) and Γ^a are SO(9) gamma matrices. Using this technique, the Polyakov action of the supersymmetric M2-brane is

$$S_{\mathrm{P}}[X, \psi, h] = -\frac{T}{2} \int_{M^{1,2}} \mathrm{d}^3x \sqrt{-\det h}(\eta_{ab} h^{ij} \Pi_i^a \Pi_j^b - 1).$$

This action has a global supersymmetry on the target space:

$$\delta\psi = \epsilon, \quad \delta X^a = -\bar{\epsilon}\Gamma^a \theta,$$

where ϵ is a constant spinor. However, it still does not have a local fermionic κ-symmetry. In order to introduce a local fermionic κ-symmetry, one needs the additional terms

$$\begin{aligned} S[X, \psi, h] = &-\frac{T}{2} \int_{M^{1,2}} \mathrm{d}^3x \sqrt{-\det h}(\eta_{ab} h^{ij} \Pi_i^a \Pi_j^b - 1) \\ &+ \frac{T}{2} \int_{M^{1,2}} \mathrm{d}^3x \, \epsilon^{ijk} \bar{\psi}\Gamma_{[a}\Gamma_{b]} \frac{\partial \psi}{\partial x^k} \left\{ \frac{1}{2} \frac{\partial X^a}{\partial x^i} \Pi_j^b + \frac{1}{6} \left(\bar{\psi}\Gamma^a \frac{\partial \psi}{\partial x^i} \right) \left(\bar{\psi}\Gamma^b \frac{\partial \psi}{\partial x^j} \right) \right\}. \end{aligned} \tag{C.14}$$

Here, ϵ^{ijk} is the completely skew-symmetric tensor with $\epsilon^{123} = 1$ and $\Gamma_a := \eta_{ab}\Gamma^b$ and $\Gamma_{ab} := \Gamma_{[a}\Gamma_{b]}$. This action is a local fermionic supersymmetry

$$\delta\psi = (1 - \Gamma)\kappa, \quad \delta X^\mu = \bar{\kappa}(1 - \Gamma)\Gamma^\mu \psi,$$

where κ is a local fermionic generator and Γ is given by

$$\Gamma := \frac{\epsilon^{ijk}}{6\sqrt{-\det h}} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{[a}\Gamma_b\Gamma_{c]}.$$

As we did in the bosonic M2-brane, we can properly fix the gauge of h if $M^{1,2} = \mathbb{R} \times \Sigma$ and take light cone gauge. Then, the action (C.14) can also be written completely in terms of the symplectic structure of Σ. After a proper matrix regularization and the recaling of the parameters, one obtains the BFSS matrix model (B.5).

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