Some Constructions of Wavelets, Frames and Orthonormal Bases

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Some Constructions of Wavelets, Frames and Orthonormal Bases

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1 Fourier analysis

1.1 Fourier transform on Euclidean space

Fourier analysis is the most fundamental theory in time-frequency analysis. The Fourier transform of $f \in L^1(\mathbb{R}^n_x)$ and the inverse Fourier transform of $g \in L^1(\mathbb{R}^n_\xi)$ are defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \quad \xi \in \mathbb{R}^n_\xi,$$

$$\mathcal{F}^{-1}[g](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi}d\xi, \quad x \in \mathbb{R}^n_x.$$

The Fourier transform $\mathcal{F}$ maps f on $\mathbb{R}^n_x$ to $\mathcal{F}[f]$ (or simply $\hat{f}$) on $\mathbb{R}^n_\xi$. Furthermore, when $n = 1$, the variables x and ξ are usually interpreted as time and frequency, respectively. Let $BUC(\mathbb{R}^n)$ denote the Banach space consisting of bounded uniformly continuous functions on $\mathbb{R}^n$. Since the Fourier transform of $f \in L^1(\mathbb{R}^n)$ is bounded as $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ and uniformly continuous, we have $\hat{f} \in BUC(\mathbb{R}^n)$. In addition, the following formulas hold:

(F1) $\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g], \quad \alpha, \beta \in \mathbb{C}$.

(F2) $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$.

(F3) $\lim_{|\xi|\to\infty} \mathcal{F}[f](\xi) = 0$.

(F4) $\mathcal{F}_{x\to\xi}\left[f(x-h)\right](\xi) = e^{-ih\cdot\xi}\mathcal{F}_{x\to\xi}[f](\xi)$.

(F5) $\mathcal{F}_{x\to\xi}[e^{ih\cdot x}f(x)](\xi) = \mathcal{F}_{x\to\xi}[f](\xi - h)$.

(F6) $\mathcal{F}_{x\to\xi}[f(ax)](\xi) = |a|^{-n}\mathcal{F}_{x\to\xi}[f](a^{-1}\xi), \quad a \neq 0$.

Formula (F2) shows that the Fourier transform intertwines the convolution and the product. Let $C_0(\mathbb{R}^n)$ denote the Banach space consisting of continuous functions vanishing at infinity on $\mathbb{R}^n$. Then, formula (F3) implies $\hat{f} \in C_0(\mathbb{R}^n)$, which is referred to as the Riemann-Lebesgue lemma. Furthermore, formulas (F4), (F5) and (F6) show the relations between translation, modulation and dilation and the Fourier transform, respectively, which are frequently used in Fourier analysis.

Since $\hat{f} \in BUC(\mathbb{R}^n)$, $\hat{f}$ does not belong to $L^1(\mathbb{R}^n)$ generally. Therefore, we require some additional conditions and techniques to reconstruct f from $\hat{f}$ using the inverse Fourier transform $\mathcal{F}^{-1}$.

Proposition 1.1. *Suppose that $f, \hat{f} \in L^1(\mathbb{R}^n)$. Then, the inversion formula holds:*

$$f(x) = \mathcal{F}^{-1}\left[\mathcal{F}[f]\right](x) \quad a.e.\ x \in \mathbb{R}^n.$$

Now, $L^1(\mathbb{R})$ is a commutative Banach algebra under the convolution satisfying

$$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}.$$

By Proposition 1.1 and formulas (F1), (F2) and (F3), the Fourier transform is an algebra isomorphism of $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$. It is interesting to note that every complex (non-trivial) homomorphism $\varphi\colon L^1(\mathbb{R}) \to \mathbb{C}$ can be written as

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)e^{-ixt}dx$$

for a unique $t \in \mathbb{R}$ (see [33]).

The Fourier transform of f is also defined if f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, and the domain $D(\mathcal{F})$ and the range $R(\mathcal{F})$ can be characterized.

Proposition 1.2. *The Fourier transform $\mathcal{F}$ is a continuous bijection operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Moreover, the inversion formula and Parseval's identity hold:*

$$f(x) = \mathcal{F}^{-1}\left[\mathcal{F}[f]\right](x), \quad x \in \mathbb{R}^n,\ f \in \mathcal{S}(\mathbb{R}^n),$$
$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)}d\xi, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{1}$$

Next, we will generalize the Fourier transform to other function spaces. It is also useful to consider the Fourier transform of a distribution such as the "Dirac delta" δ. By the duality $\mathcal{S}$-$\mathcal{S}'$, the Fourier transform of a tempered distribution T can be defined by

$$(\mathcal{F}[T], \varphi)_{\mathcal{S}'\times\mathcal{S}} = (T, \mathcal{F}[\varphi])_{\mathcal{S}'\times\mathcal{S}}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 1.2 and the duality enable us to characterize the domain and the range.

Proposition 1.3. *The Fourier transform $\mathcal{F}$ is a continuous bijection operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Moreover, the inversion formula holds:*

$$T = \mathcal{F}^{-1}\left[\mathcal{F}[T]\right], \quad T \in \mathcal{S}'(\mathbb{R}^n).$$

By identifying $f \in L^p(\mathbb{R}^n)$ with a tempered distribution for $1 \le p \le \infty$, we can obtain the Fourier transform of an L^p-function. In particular, Hilbert space $L^2(\mathbb{R}^n)$ plays an important role in Fourier analysis.

Now, suppose that $f, g \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, Parseval's identity (1) can be written as an inner product on $L^2(\mathbb{R}^n)$:

$$\langle f, g\rangle = \frac{1}{(2\pi)^n}\langle \hat{f}, \hat{g}\rangle \overset{f=g}{\Longrightarrow} \|f\|_{L^2} = \frac{1}{(2\pi)^{\frac{n}{2}}}\|\hat{f}\|_{L^2}.$$

Then, by the density argument, the Fourier transform $\mathcal{F}$ can be uniquely extended to the bounded linear operator $\mathcal{F}_{L^2}$, which is called L^2-Fourier transform, in $L^2(\mathbb{R}^n)$. The following proposition is obtained immediately.

Theorem 1.4 (Plancherel theorem)**.** *The L^2-Fourier transform $\mathcal{F}_{L^2}$ is a unitary operator on $L^2(\mathbb{R}^n)$, up to $(2\pi)^{-n/2}$.*

Unfortunately, we still only know the existence of the L^2-Fourier transform, so the explicit form of $\mathcal{F}_{L^2}$ remains non-trivial. We want it to have the same form as the Fourier transform on $L^1(\mathbb{R}^n)$, and in fact the following holds: For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\mathcal{F}_{L^2}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad a.e.\ \xi \in \mathbb{R}^n.$$

Using this result, we can determine the explicit L^2-Fourier transform.

Proposition 1.5. *For $f \in L^2(\mathbb{R}^n)$, we have the following:*

$$\mathcal{F}_{L^2}[f](\xi) = \underset{R\to\infty}{\mathrm{l.i.m}} \int_{|x|\le R} f(x)e^{-ix\cdot\xi}dx \tag{2}$$
$$\iff \quad \int_{\mathbb{R}^n}\left|\mathcal{F}_{L^2}[f](\xi) - \int_{|x|\le R} f(x)e^{-ix\cdot\xi}dx\right|^2 d\xi \to 0 \quad \text{as } R \to \infty.$$

Thus, we can calculate the L^2-Fourier transform from the right-hand side of expression (2):

$$\mathcal{F}_{L^2}[f](\xi) = \lim_{R\to\infty}\int_{|x|\le R} f(x)e^{-ix\cdot\xi}dx \quad \text{in } L^2(\mathbb{R}^n).$$

Hereafter, we will rewrite $\mathcal{F}_{L^2}$ simply as $\mathcal{F}$ to avoid confusion. For more details, see e.g. [5, 26, 29, 34].

1.2 Uncertainty principle

By the Plancherel theorem, properties of a function $f \in L^2(\mathbb{R})$ can be translated into properties of its Fourier transform $\hat{f} \in L^2(\mathbb{R})$. Although we can obtain information of all frequencies by integration over $\mathbb{R}$, we can obtain little frequency information of a neighborhood of a point $x = b$. The simplest way to deal with this problem is to multiply $e^{ix\xi}$ by a window function $g \in L^2(\mathbb{R})$ localized in a neighborhood of a point $x = b$.

Definition 1.6 (Windowed Fourier transform)**.** A function $g \in L^2(\mathbb{R})$ is called a window function if it satisfies

$$\int_{\mathbb{R}} |xg(x)|^2 dx < \infty.$$

Then, we define the windowed Fourier transform (WFT) of $f \in L^2(\mathbb{R})$ by the window function g as follows:

$$V_g[f](b, \xi) = \int_{\mathbb{R}} f(x)\overline{u_{b,\xi}(x)} dx,$$

where $u_{b,\xi}(x) = g(x - b)e^{ix\xi}$. If g and $\hat{g}$ are window functions, the operator V_g is called the short-time Fourier transform (STFT).

We note that the definition of the window function yields $|\cdot|^{1/2} g \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, that is, $\hat{g} \in BUC(\mathbb{R}) \cap C_0(\mathbb{R})$. Using (F4) and (F5), we have

$$\widehat{u_{q,p}}(\xi) = e^{-iq(\xi - p)} \hat{g}(\xi - p).$$

Hence, by the Plancherel theorem, we obtain

$$\langle f, u_{q,p} \rangle = \frac{1}{2\pi} \langle \hat{f}, \widehat{u_{q,p}} \rangle. \tag{3}$$

Equality (3) says that the value $\langle f, u_{q,p} \rangle$ in a neighborhood of a point $x = q$ is equal to the value $(2\pi)^{-1} \langle \hat{f}, \widehat{u_{q,p}} \rangle$ in a neighborhood of a point $\xi = p$ by the Fourier transform, which means that information of the pair $(f, \hat{f})$ in a neighborhood of the point $(x, \xi) = (q, p)$ on the time-frequency plane (phase plane in physics) $\mathbb{R}_x \times \mathbb{R}_\xi$ is extracted by the window function g. Therefore, characterizing the localization of the window function g in the time space and the localization of $\hat{g}$ in the frequency space is an important problem.

Definition 1.7. The center x^* and radius $\triangle_g$ of a window function $g \in L^2(\mathbb{R})$ are defined by

$$x^* = \frac{1}{\|f\|_{L^2}^2} \int_{\mathbb{R}} x|f(x)|^2 dx, \qquad \triangle_g = \frac{1}{\|f\|_{L^2}} \left\{ \int_{\mathbb{R}} (x - x^*)^2 |f(x)|^2 dx \right\}^{\frac{1}{2}}.$$

The value $2\triangle_g$ is called the width of g.

Therefore, the interval which is essentially localized by a window function g is $[x^* - \triangle_g, x^* + \triangle_g]$. Since g and $\hat{g}$ are window functions for the STFT, we consider the region called the time-frequency window:

$$[q + x^* - \triangle_g, q + x^* + \triangle_g] \times [p + \xi^* - \triangle_{\hat{g}}, p + \xi^* + \triangle_{\hat{g}}], \tag{4}$$

which is extracted by equality (3). Most importantly, the shape of the time-frequency window (4) depends only on the choice of a window function g, and its measure is $4\triangle_g \triangle_{\hat{g}}$ (Figure 1).

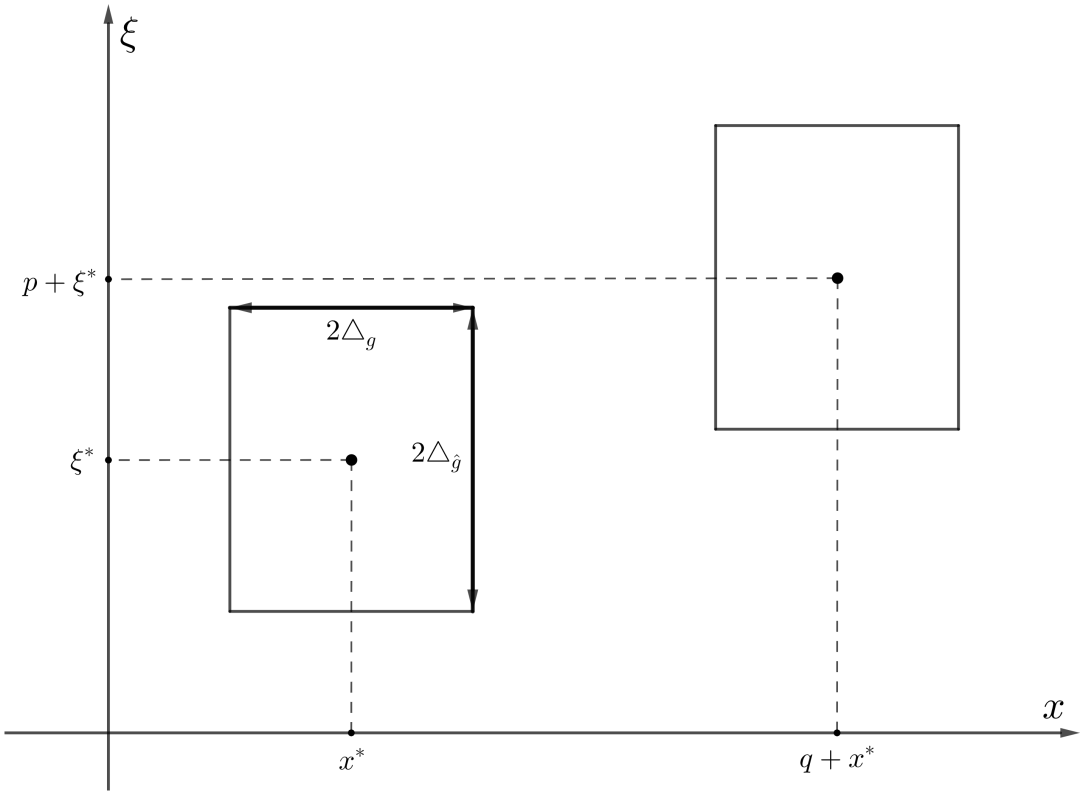


Figure 1: Time-frequency windows of g.

Hence, in order to localize time and frequency information simultaneously, the value $\triangle_g\triangle_{\hat{g}}$ should be as small as possible, but we know that there is a lower bound for this value.

Theorem 1.8 (Heisenberg uncertainty principle). *If $g \in \mathcal{S}(\mathbb{R})$, then $\triangle_g\triangle_{\hat{g}} \geq 1/2$. Equality holds if and only if*

$$g(x) = ce^{iax}g_\alpha(x-b), \quad a, b \in \mathbb{R},\ c \neq 0,$$

where g_α is the Gaussian function

$$g_\alpha(x) = \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{x^2}{4\alpha}}, \quad \alpha > 0.$$

Proof. Using change of variables, we can assume that $x^* = \xi^* = 0$. Then,

$$\begin{aligned}(\triangle_g\triangle_{\hat{g}})^2 &= \left\{\frac{1}{\|g\|_{L^2}^2}\int_{\mathbb{R}} x^2|g(x)|^2dx\right\}\left\{\frac{1}{\|\hat{g}\|_{L^2}^2}\int_{\mathbb{R}} \xi^2|\hat{g}(\xi)|^2d\xi\right\} \\ &= \frac{1}{\|g\|_{L^2}^2\|\hat{g}\|_{L^2}^2}\left\{\int_{\mathbb{R}} x^2|g(x)|^2dx\right\}\left\{\int_{\mathbb{R}} |i\xi\hat{g}(\xi)|^2d\xi\right\} \\ &= \frac{1}{\|g\|_{L^2}^2\|\hat{g}\|_{L^2}^2}\left\{\int_{\mathbb{R}} x^2|g(x)|^2dx\right\}\left\{\int_{\mathbb{R}} \left|\widehat{g'}(\xi)\right|^2d\xi\right\}.\end{aligned}$$

By the Plancherel theorem, we have

$$(\triangle_g\triangle_{\hat{g}})^2 = \frac{1}{\|g\|_{L^2}^4}\left\{\int_{\mathbb{R}} x^2|g(x)|^2dx\right\}\left\{\int_{\mathbb{R}} |g'(x)|^2dx\right\}. \tag{5}$$

Applying the inequality

$$\left\{\int_{\mathbb{R}} |xg(x)|^2dx\right\}\left\{\int_{\mathbb{R}} |g'(x)|^2dx\right\} \geq \left|\mathrm{Re}\int_{\mathbb{R}} xg(x)\overline{g'(x)}dx\right|^2,$$

to (5), we obtain

$$(\triangle_g \triangle_{\hat{g}})^2 \geq \frac{1}{\|g\|_{L^2}^4} \left| \mathrm{Re} \int_{\mathbb{R}} x g(x) \overline{g'(x)} dx \right|^2 = \frac{1}{\|g\|_{L^2}^4} \left| \frac{1}{2} \int_{\mathbb{R}} x \frac{d}{dx} |g(x)|^2 dx \right|^2.$$

Hence, using integration by parts, we obtain

$$\begin{aligned}(\triangle_g \triangle_{\hat{g}})^2 &\geq \frac{1}{4\|g\|_{L^2}^4} \left| \left[x|g(x)|^2 \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} |g(x)|^2 dx \right|^2 \\ &= \frac{1}{4\|g\|_{L^2}^4} \left| \int_{\mathbb{R}} |g(x)|^2 dx \right|^2 \\ &= \frac{1}{4}.\end{aligned}$$

For equality in $\triangle_g \triangle_{\hat{g}} \geq 1/2$, we solve the following equation:

$$\left\{ \int_{\mathbb{R}} |x g(x)|^2 dx \right\} \left\{ \int_{\mathbb{R}} |g'(x)|^2 dx \right\} = \left| \mathrm{Re} \int_{\mathbb{R}} x g(x) \overline{g'(x)} dx \right|^2.$$

The solution is known to be a Gaussian function (see e.g. [5, 13]).

□

Therefore, the measure of the time-frequency window is never less than 2, and the time window and the frequency window cannot be narrowed simultaneously. The STFT with g_α, which has minimal uncertainty, as the window function is given the special name the Gabor transform. In contrast to STFT, the time-frequency window defined for the continuous wavelet transform, which will appear in the next section, changes its shape. Because of this property, a wavelet is often called a "mathematical microscope." For more details about the uncertainty principle, see e.g. [5, 13, 19].

2 Wavelet analysis

Wavelet theory is a field of harmonic analysis and time-frequency analysis, but research on concept of wavelets is not restricted to mathematics; wavelet theory is also important in physics, signal processing and other engineering fields. The idea of the wavelet was first proposed by J. Morlet, who was a French geophysicist and engineer at Elf Aquitaine, for the analysis of seismic data in the early 1980s. After that, Morlet decided to provide the mathematical foundations of the wavelet with theoretical physicist A. Grossmann in Marseille. Then, the group in Marseille (including I. Daubechies) introduced the continuous wavelet transform using the theory of coherent states in quantum mechanics. They initially used the name "wavelets of constant shape" for the wavelet (see [20]). In 1985, Y. Meyer, a mathematician at the Ecole Polytechnique, came to understand a deep relationship between the wavelet's reconstruction formula and Calderón's formula in the Calderón-Zygmund operators and the atomic decompositions, which were central to his research in harmonic analysis. After this, the mathematical aspects of wavelet theory developed rapidly, continuing to the present day. For more of its history, see e.g. [14, 27, 30, 31].

2.1 Continuous wavelet transform

A wavelet is defined by

$$\psi_{ab} = \frac{1}{\sqrt{|a|}} \psi \left(\frac{\cdot - b}{a} \right), \quad a \in \mathbb{R} \backslash \{0\}, \ b \in \mathbb{R} \tag{6}$$

for $\psi \in L^2(\mathbb{R})$, which is called a mother wavelet. The definition (6) can be rewritten as

$$\psi_{ab} = T_b D_a \psi, \quad a \in \mathbb{R} \backslash \{0\}, \ b \in \mathbb{R}, \tag{7}$$

where T_b and D_a are the translation operator and the dilation operator:

$$T_b\psi = \psi(\cdot - b) \quad \text{and} \quad D_a\psi = \frac{1}{\sqrt{|a|}}\psi\left(\frac{\cdot}{a}\right).$$

Using the composition

$$(T_{b'}D_{a'})(T_bD_a)\psi = T_{b'+a'b}D_{a'a}\psi,$$

we consider the $ax+b$ group $\mathbb{R}\backslash\{0\} \ltimes \mathbb{R}$ equipped with the following multiplication:

$$(a', b') \circ (a, b) = (a'a, b' + a'b).$$

In fact, operation (7) is a unitary representation of the $ax + b$ group on $L^2(\mathbb{R})$. Consequently, the role of the wavelet can also be understood from the above geometric observations, in addition to the definition (6).

Now, we define the continuous wavelet transform (CWT) of $f \in L^2(\mathbb{R})$ by a mother wavelet ψ as follows:

$$W_\psi[f](a, b) = \langle f, \psi_{ab}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{ab}(x)}dx, \quad a \in \mathbb{R}\backslash\{0\},\ b \in \mathbb{R}.$$

If a mother wavelet ψ satisfies the admissibility condition

$$C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi < \infty, \tag{8}$$

we get the resolution of the identity:

$$\int_{\mathbb{R}^2} W_\psi[f](a, b)\overline{W_\psi[g](a, b)}\frac{da}{a^2}db = C_\psi\langle f, g\rangle. \tag{9}$$

By a calculation, we get the reconstruction formula of the CWT for any $f \in L^2(\mathbb{R})$:

$$f = \frac{1}{C_\psi}\int_{\mathbb{R}^2} W_\psi[f](a, b)\psi_{ab}\frac{da}{a^2}db \quad \text{in } L^2(\mathbb{R}). \tag{10}$$

Example 2.1 (Mexican hat wavelet)**.** We define the mother wavelet ψ by

$$\psi(x) = -\frac{d^2}{dx^2}e^{-\frac{x^2}{2}} = (1 - x^2)e^{-\frac{x^2}{2}},$$

which is called the Mexican hat wavelet. Figure 2 shows a graph of the Mexican hat wavelet and Figure 3 shows a graph of its Fourier transform.

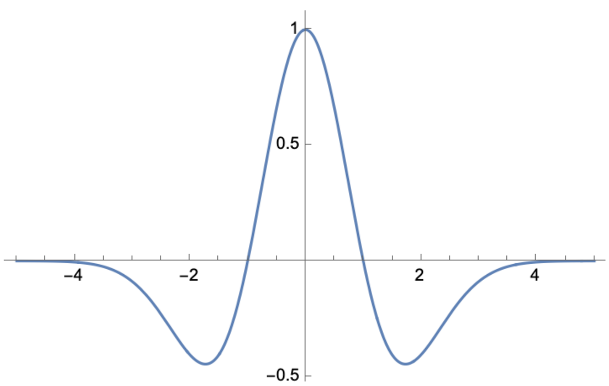


Figure 2: Mexican hat wavelet.

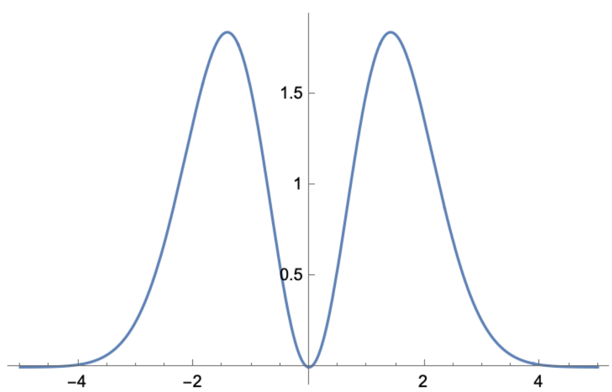


Figure 3: Fourier transform of the Mexican hat wavelet.

We note that the weighted Lebesgue measure $a^{-2}dadb$ on $\mathbb{R}^2$ can be regarded as the left Haar measure on the $ax+b$ group derived by the representation theory of abstract harmonic analysis. Moreover, identity (9) can also be derived by the Duflo-Moore theorem of the square integrable representation (see [6]).

For applications, it is sufficient to restrict the scale parameter a to positive values, that is,

$$\psi_{ab} = \frac{1}{\sqrt{a}}\psi\left(\frac{\cdot - b}{a}\right), \quad a \in \mathbb{R}_{>0},\ b \in \mathbb{R}. \tag{11}$$

Then, if a mother wavelet ψ satisfies the admissibility condition

$$\tilde{C}_\psi := \int_{-\infty}^{0} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi = \int_{0}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty,$$

which is more stringent than (8), we get the corresponding reconstruction formula:

$$f = \frac{1}{\tilde{C}_\psi} \int_{\mathbb{R}} \int_0^\infty W_\psi[f](a,b)\psi_{ab} \frac{da}{a^2} db \quad \text{in } L^2(\mathbb{R}).$$

Next, to observe the time-frequency window in the CWT, let ψ and $\hat{\psi}$ satisfy the assumption of a window function and let $a > 0$. From expression (11), the time window is given by

$$[b + ax^* - a\triangle_\psi, b + ax^* + a\triangle_\psi].$$

Using the Plancherel theorem, we obtain

$$\langle f, \psi_{ab} \rangle = \frac{1}{2\pi} \langle \hat{f}, \widehat{\psi_{ab}} \rangle,$$

where $\widehat{\psi_{ab}}(\xi) = a^{1/2} e^{-ib\xi} \hat{\psi}(a\xi)$. Hence, the frequency window is as follows:

$$[a^{-1}\xi^* - a^{-1}\triangle_{\hat{\psi}}, a^{-1}\xi^* + a^{-1}\triangle_{\hat{\psi}}].$$

Thus, we obtain the time-frequency window as

$$[b + ax^* - a\triangle_\psi, b + ax^* + a\triangle_\psi] \times [a^{-1}\xi^* - a^{-1}\triangle_{\hat{\psi}}, a^{-1}\xi^* + a^{-1}\triangle_{\hat{\psi}}].$$

In contrast to the STFT, the time-frequency window of the CWT is flexible thanks to the parameter a (Figure 4). However, the measure of the time-frequency window always remains $4\triangle_g\triangle_{\hat{g}}$, the same as for the STFT. To be more precise, a short-time observation in the time space can capture high frequencies, while a long-time observation can capture low frequencies. This is why the CWT is said to be a mathematical microscope.

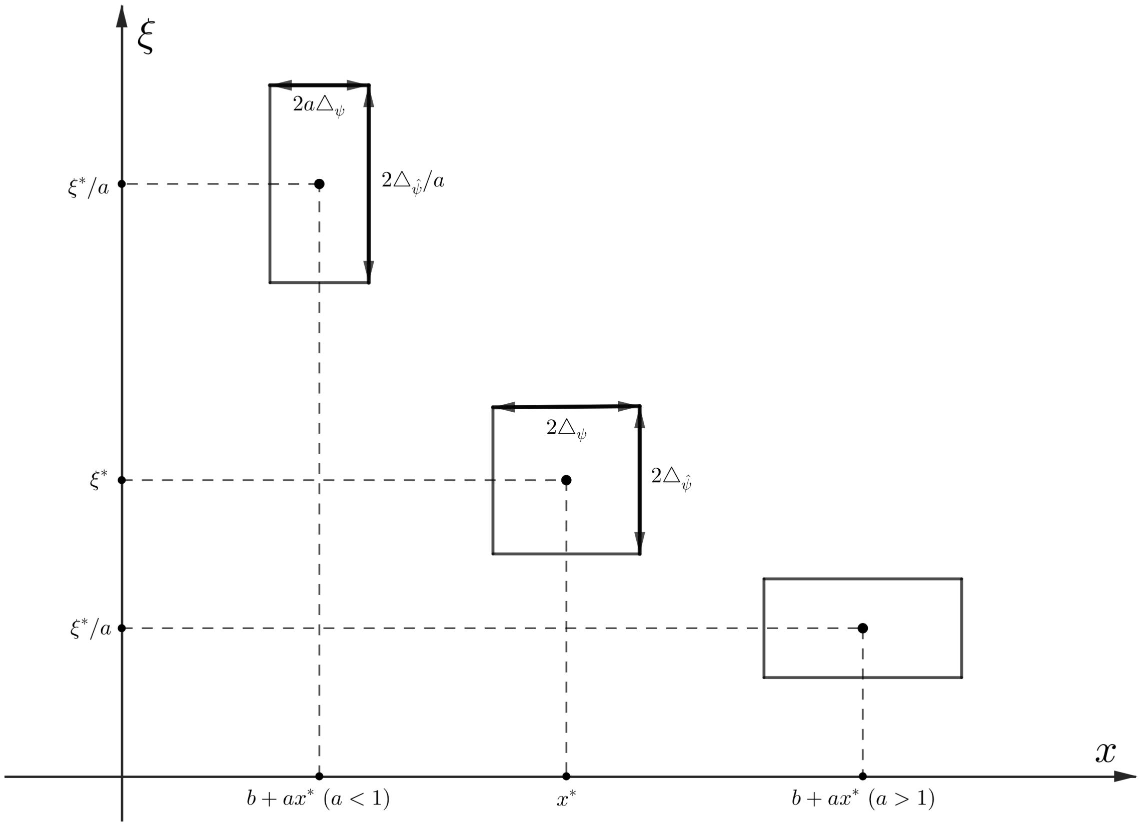


Figure 4: Time-frequency windows of the CWT.

For more details about the continuous wavelet transform, see e.g. [2, 5, 13, 14, 19, 27, 30, 31].

2.2 Discrete wavelet transform

By using continuous parameters (a, b), expansion (10) is redundant. Therefore, we discretize the parameters $a = 2^{-j},\ b = k2^{-j},\ (j, k \in \mathbb{Z})$ and consider the discrete wavelet transform (DWT):

$$W_\psi[f](j,k) = \langle f, \psi_{j,k} \rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k}(x)}dx,$$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}.$$

In this case, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is not necessarily an orthonormal basis (or complete orthonormal system) but is generally a frame for $L^2(\mathbb{R})$.

Definition 2.2 (Frame)**.** Let N be a countable set. A system $\{x_n\}_{n\in N}$ in a separable Hilbert space $\mathcal{H}$ is a frame if there exist constants $A, B > 0$ called frame bounds such that

$$A\|x\|_{\mathcal{H}}^2 \le \sum_{n\in N} |\langle x, x_n \rangle_{\mathcal{H}}|^2 \le B\|x\|_{\mathcal{H}}^2$$

for all $x \in \mathcal{H}$. A frame $\{x_n\}_{n\in N}$ is a tight frame if $A = B$. In particular, it be said a Parseval frame if $A = B = 1$.

By definition, an orthonormal basis is a Parseval frame. Therefore, a Parseval frame is a redundant generalization of an orthonormal basis.

Example 2.3. Let $\{x_n\}_{n\in N}$ and $\{y_m\}_{m\in M}$ be orthonormal bases for $\mathcal{H}$. Then, the union $\{x_n\}_{n\in N} \cup \{y_m\}_{m\in M}$ is a tight frame with $A = B = 2$.

Example 2.4 (Weyl-Heisenberg frame)**.** Given $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$, we define

$$\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i n\beta} g(\cdot - m\alpha)\}_{m,n\in\mathbb{Z}}.$$

If the system $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$, then it is called the Weyl-Heisenberg frame or the Gabor frame.

Even though a frame is more redundant than an orthonormal basis, any $x \in \mathcal{H}$ can be reconstructed with it. In order to obtain the reconstruction formula with a frame $\{x_n\}_{n\in N}$, we define the frame operator S by

$$Sx = \sum_{n\in N} \langle x, x_n\rangle_{\mathcal{H}}\, x_n.$$

Then, the following holds.

Theorem 2.5. *Let $\{x_n\}_{n\in N}$ be a frame for $\mathcal{H}$ with frame bounds A, B. Then, the frame operator $S\colon \mathcal{H} \to \mathcal{H}$ is surjective and is a positive bounded linear operator which has an inverse satisfying*

$$AI_{\mathcal{H}} \le S \le BI_{\mathcal{H}} \quad \overset{def}{\Longleftrightarrow} \quad A\langle x, x\rangle_{\mathcal{H}} \le \langle Sx, x\rangle_{\mathcal{H}} \le B\langle x, x\rangle_{\mathcal{H}},$$

$$\frac{1}{B} I_{\mathcal{H}} \le S^{-1} \le \frac{1}{A} I_{\mathcal{H}}.$$

Furthermore, the system $\{S^{-1}x_n\}_{n\in N}$ is also a frame with frame bounds B^{-1}, A^{-1} and any $x \in \mathcal{H}$ is reconstructed by

$$x = \sum_{n\in\mathbb{N}} \left\langle x, S^{-1}x_n\right\rangle_{\mathcal{H}} x_n = \sum_{n\in\mathbb{N}} \langle x, x_n\rangle_{\mathcal{H}}\, S^{-1}x_n.$$

The above frame $\{S^{-1}x_n\}_{n\in N}$ is called the dual frame.

For the proof of Theorem 2.5, see e.g. [14, 19, 25]. Now, there exists a sufficient condition such that the wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ obtained by discretizing the mother wavelet ψ in the CWT is a frame for $L^2(\mathbb{R})$.

Theorem 2.6. *For $\psi \in L^2(\mathbb{R})$, we define*

$$S(\xi) = \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|, \quad t_m(\xi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2\pi m))}, \quad m \in \mathbb{Z},$$

and

$$\underline{S}_{\psi} = \operatorname*{ess\,inf}_{\xi\in\mathbb{R}} S(\xi), \quad \overline{S}_{\psi} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}} S(\xi),$$

$$\beta_{\psi}[m] = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |t_m(2^k\xi)|.$$

Then, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a frame which has frame bounds A_ψ, B_ψ if it satisfies the following:

$$A_{\psi} = \underline{S}_{\psi} - \sum_{q\in2\mathbb{Z}+1} \{\beta_{\psi}[q]\beta_{\psi}[-q]\}^{\frac{1}{2}} > 0,$$

$$B_{\psi} = \overline{S}_{\psi} + \sum_{q\in2\mathbb{Z}+1} \{\beta_{\psi}[q]\beta_{\psi}[-q]\}^{\frac{1}{2}} < \infty.$$

For the proof of Theorem 2.6, see e.g. [14, 25]. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$, ψ is called an orthonormal wavelet. As suggested by Theorem 2.6, it is nontrivial to obtain an orthonormal wavelet from a mother wavelet in the CWT. Therefore, we regard the CWT and the DWT as separate frameworks and focus on constructing an orthonormal wavelet in the DWT using the method of multiresolution analysis (MRA).

Definition 2.7 (MRA)**.** A closed subspace sequence $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ together with a scaling function $\varphi \in V_0$ is called an MRA if it satisfies the following conditions:

(MRA1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$.

(MRA2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(MRA3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(MRA4) $f(\cdot) \in V_j \iff f(2\cdot) \in V_{j+1}$.

(MRA5) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

The condition (MRA5) can be weakened to a Riesz basis, which is a frame consisting of the image of an invertible bounded operator of some orthonormal basis. We also note that (MRA1), (MRA4) and (MRA5) imply (MRA3).

From (MRA1), V_1 can be decomposed into a direct sum $V_1 = V_0 \oplus W_0$ with $W_0 := V_0^\perp \cap V_1$. Repeating the orthogonal decomposition for $\{V_j\}_{j\geq 2}$ yields

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{n=0}^{\infty} W_n\right), \quad W_n := V_n^\perp \cap V_{n+1}.$$

We also repeat this decomposition for $\{V_j\}_{j\leq 0}$:

$$V_0 = V_{-1} \oplus W_{-1} = V_{-n} \oplus W_{-n} \oplus W_{-n+1} \oplus \cdots \oplus W_{-1}.$$

Then, we obtain the orthogonal direct sum of $L^2(\mathbb{R})$ as

$$L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n, \quad W_n := V_n^\perp \cap V_{n+1}.$$

Given an MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$, a function $\psi \in W_0$ is called an MRA wavelet if $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0. It is easy to see that $\{2^{j/2}\psi(2^j \cdot -k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j as well. Therefore, $\{2^{j/2}\psi(2^j \cdot -k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Definition 2.8 (Low-pass filter)**.** Let $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ be an MRA. The scaling function φ can be expressed as

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k \sqrt{2}\varphi(2x - k), \quad \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Then, we define a 2π-periodic function m_0 called the low-pass filter associated with φ as follows:

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2}} e^{-ik\xi} \in L^2(\mathbb{T}).$$

By using the Fourier transform, we get the formula $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$, which is the reason why it is called low-pass filter. As a sequence, an MRA wavelet ψ is obtained explicitly from an MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$.

Theorem 2.9. *Let $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ be an MRA. Then, the function ψ defined by*

$$\hat{\psi}(\xi) = -e^{-i\xi/2}\overline{m_0\left(\frac{\xi}{2} + \pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right)$$

is an orthonormal wavelet for $L^2(\mathbb{R})$. Furthermore, we have

$$\psi(x) = \sum_{k\in\mathbb{Z}} \overline{c_{1-k}}(-1)^k \sqrt{2}\varphi(2x - k),$$

where $\{c_k\}_{k\in\mathbb{Z}}$ is the sequence of the low-pass filter.

Example 2.10 (Haar wavelet)**.** The Haar MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ is given by

$$V_j = \left\{ f \in L^2(\mathbb{R}) : \text{A function } f \text{ is a piecewise constant on } \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right),\ \forall k \in \mathbb{Z} \right\},\ j \in \mathbb{Z},$$

$$\varphi = \mathbf{1}_{[0,1)}.$$

Then, the Haar wavelet is given by

$$\psi(x) = \sum_{k\in\mathbb{Z}} \overline{c_{1-k}}(-1)^k \sqrt{2}\varphi(2x-k) = \varphi(2x) - \varphi(2x-1).$$

Figure 5 shows a graph of the Haar wavelet and Figure 6 shows a graph of its scaling function.

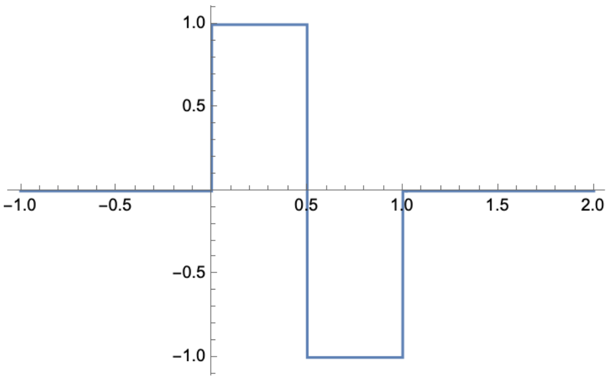


Figure 5: Haar wavelet.

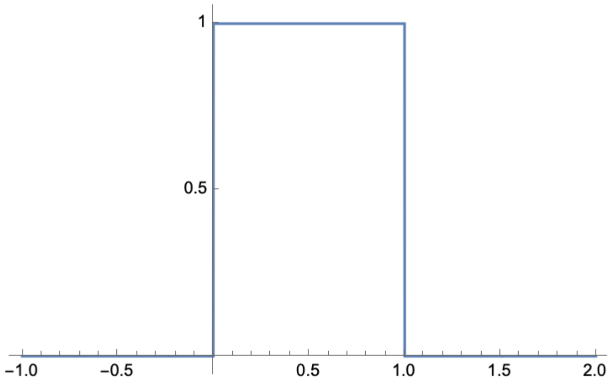


Figure 6: Haar scaling function.

Example 2.11 (Shanon wavelet)**.** The Shannon MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ is given by

$$V_j = \left\{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0,\ |\xi| > 2^j\pi \right\},$$

$$\hat{\varphi}(\xi) = \mathbf{1}_{[-\pi,\pi)} \Longleftrightarrow \varphi(x) = \frac{\sin(\pi x)}{\pi x}.$$

Then, the Shannon wavelet is given by

$$\psi(x) = \frac{\sin \pi(x-1/2) - \sin 2\pi(x-1/2)}{\pi(x-1/2)}.$$

Figure 7 shows a graph of the Shannon wavelet and Figure 8 shows a graph of its scaling function.

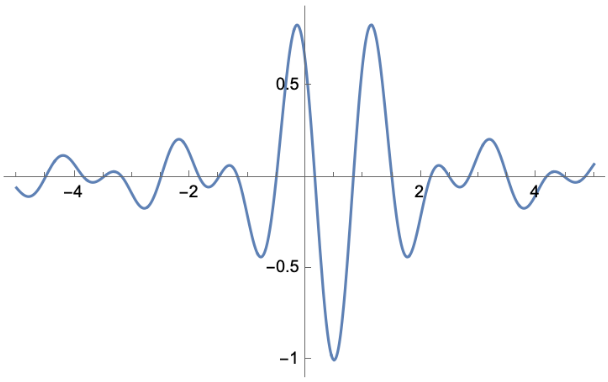


Figure 7: Shannon wavelet.

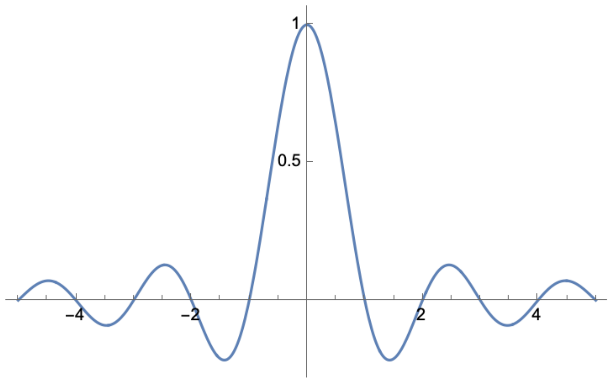


Figure 8: Shannon scaling function.

For more details about the DWT and the MRA, see e.g. [5, 13, 14, 25, 27, 31, 35].

2.3 Construction of the orthonormal wavelet in the Hardy space $H^2(\mathbb{R})$

It is possible to define MRAs for $L^2_E(\mathbb{R})$ by

$$L^2_E(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{a.e. } \xi \in \mathbb{R}\backslash E\},$$

where $E \subset \mathbb{R}$ is a measurable set such that

$$|E| > 0, \quad E = 2E, \quad |\mathbb{R}\backslash E| > 0.$$

In [21], we studied the classical Hardy space

$$H^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{a.e. } \xi \leq 0 \right\},$$

which is the special case when $E = \mathbb{R}_{>0}$. As the name suggests, $H^2(\mathbb{R})$ is realized as the boundary space of the Hardy space on the upper half-plane of $\mathbb{C}$ by the Paley-Wiener theorem (see e.g. [25, 31]).

Example 2.12. For $\ell \in \mathbb{Z}_{\geq 0}$, define the set

$$K_\ell = \left[\frac{2^{\ell+1}}{2^{\ell+1}-1}\pi, 2\pi \right] \cup \left[2^{\ell+1}\pi, \frac{2^{2\ell+2}}{2^{\ell+1}-1}\pi \right] \subset \mathbb{R}_{>0}.$$

Then, $\{\psi_\ell\}_{\ell \geq 0}$ defined by $\hat{\psi}_\ell = \mathbf{1}_{K_\ell}$ are H^2-wavelets. When $\ell = 0$, we obtain the Shannon-type wavelet ψ_0, where $K_0 = [2\pi, 4\pi]$. When $\ell = 2$, we also obtain the Journé-type wavelet ψ_2, where

$$K_2 = \left[\frac{8}{7}\pi, 2\pi \right] \cup \left[8\pi, \frac{64}{7}\pi \right].$$

It is known that ψ_0 is associated with an H^2-MRA, but ψ_ℓ is not associated with any H^2-MRA when $\ell \geq 1$ (see [25]).

The following decay rate at infinity in the frequency space characterizes the H^2-wavelet.

Definition 2.13 (Regularity condition $\mathfrak{R}^0$)**.** We say that a function ψ satisfies the regularity condition $\mathfrak{R}^0$ if $|\hat{\psi}|$ is continuous on $\mathbb{R}$ and satisfies

$$|\hat{\psi}(\xi)| = O\left(\langle \xi \rangle^{-\alpha-1/2} \right) \quad \text{for some } \alpha > 0. \tag{12}$$

In [31], Meyer posed the question of whether there exists an H^2-wavelet belonging to $\mathcal{S}(\mathbb{R})$. In response, Auscher proved that there is no H^2-wavelet satisfying $\mathfrak{R}^0$, which implies that there is no H^2-wavelet ψ with $\psi \in \mathcal{S}(\mathbb{R})$ (see [3, 4]).

Now, from Definition 2.13, we see that square integrability follows from (12), but the inverse is not always true. Then, a natural question arises.

Question 2.14. Does there exist an H^2-wavelet ψ such that $|\hat{\psi}|$ is continuous on $\mathbb{R}$ but does not satisfy the decay rate (12)?

Paying careful attention to the construction of the scaling function, we find a critical decay rate of the Fourier transform of the H^2-wavelet generated by some MRA.

Theorem 2.15. *There exists an H^2-wavelet ψ such that $|\hat{\psi}|$ is continuous on $\mathbb{R}$ and satisfies*

$$|\hat{\psi}(\xi)| = O\left((\log\langle \xi \rangle)^{-1} \right).$$

For the existence of such a wavelet, we construct a scaling function from a concrete low-pass filter. Figure 9 and Figure 10 show graphs of the Fourier transforms of our scaling function φ and our H^2-wavelet ψ, respectively.

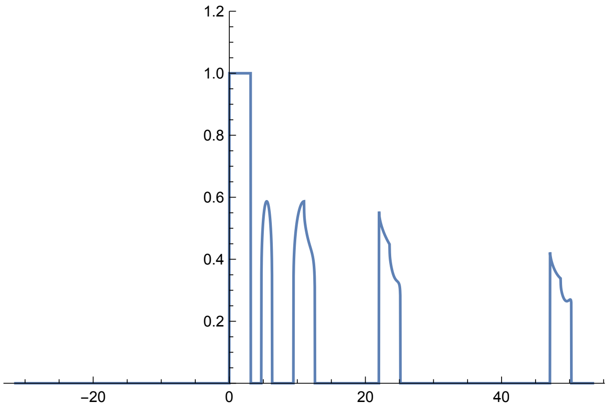


Figure 9: Graph of $\hat{\varphi}$ in frequency space.

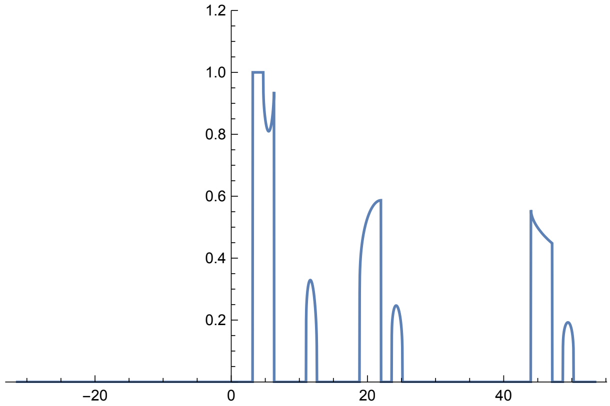


Figure 10: Graph of $|\hat{\psi}|$ in frequency space.

2.3.1 Proof of Theorem 2.15

In this section, we provide an outline of the proof of Theorem 2.15 according to [21]. We first define the function on the frequency space to construct a low-pass filter by setting

$$M(\xi;\varepsilon) := \begin{cases} 0^- & \xi \in [-\pi, -2^{-1}\pi], \\ 1^- & \xi \in (-2^{-1}\pi, -\varepsilon], \\ N_\varepsilon(\xi) & \xi \in (-\varepsilon, 0), \\ 1^+ & \xi \in [0, 2^{-1}\pi), \\ 0^+ & \xi \in [2^{-1}\pi, \pi-\varepsilon], \\ P_\varepsilon(\xi) & \xi \in (\pi-\varepsilon, \pi) \end{cases}$$

for any small $\varepsilon > 0$, where N_ε and P_ε are given by

$$N_\varepsilon(\xi)^2 := 1 - \frac{1}{\log_2\left(1 - \frac{1}{\xi(\xi+\varepsilon)}\right)},$$

$$P_\varepsilon(\xi)^2 := \frac{1}{\log_2\left(1 - \frac{1}{(\xi-\pi)(\xi-\pi+\varepsilon)}\right)}$$

and the superscripts $\pm$ of the constant functions 0 and 1 indicate the positive or negative domain, respectively. There are many choices of ε; however, we shall fix $\varepsilon = \pi/4$ for simplicity. Figure 11 and Figure 12 show graphs of $M(\xi;\pi/4)$ and its square.

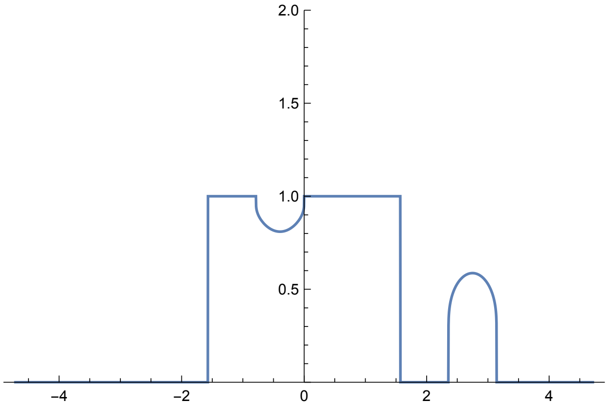


Figure 11: $M(\xi;\pi/4)$.

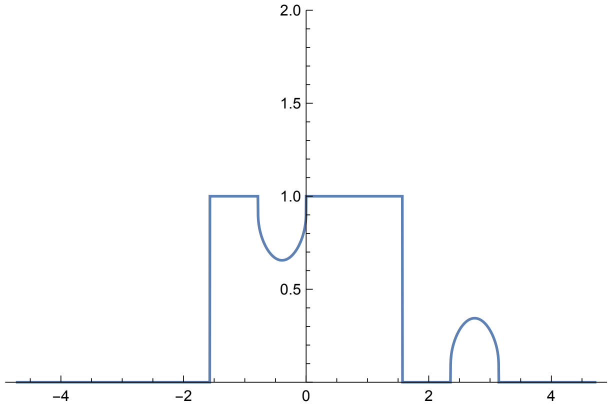


Figure 12: $M^2(\xi;\pi/4)$.

Next, let m_0 be the 2π-periodic extension of $M(\xi) \equiv M(\xi;\pi/4)$ defined by

$$m_0(\xi) := \sum_{k=-\infty}^{\infty} M^{(k)}(\xi),$$

where $M^{(k)}(\xi) := M(\xi - 2\pi k)$, that is,

$$M^{(k)}(\xi) := \begin{cases} 0^- & \xi \in [-\pi + 2\pi k, -2^{-1}\pi + 2\pi k], \\ 1^- & \xi \in (-2^{-1}\pi + 2\pi k, -2^{-2}\pi + 2\pi k], \\ N^{(k)}(\xi) \equiv N_{\pi/4}(\xi - 2\pi k) & \xi \in (-2^{-2}\pi + 2\pi k, 2\pi k), \\ 1^+ & \xi \in [2\pi k, 2^{-1}\pi + 2\pi k), \\ 0^+ & \xi \in [2^{-1}\pi + 2\pi k, 3\cdot 2^{-2}\pi + 2\pi k], \\ P^{(k)}(\xi) \equiv P_{\pi/4}(\xi - 2\pi k) & \xi \in (3\cdot 2^{-2}\pi + 2\pi k, \pi + 2\pi k). \end{cases}$$

Furthermore, let us set the following function:

$$\hat{\varphi}(\xi) := \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right) = \prod_{j=1}^{\infty}\sum_{k=-\infty}^{\infty} M^{(k)}\left(\frac{\xi}{2^j}\right),$$

with

$$m_0\left(\frac{\xi}{2^j}\right) = \begin{cases} 0^- & \xi \in [-2^j\pi + 2^{j+1}\pi k, -2^{j-1}\pi + 2^{j+1}\pi k], \\ 1^- & \xi \in (-2^{j-1}\pi + 2^{j+1}\pi k, -2^{j-2}\pi + 2^{j+1}\pi k], \\ N^{(k)}(2^{-j}\xi) & \xi \in (-2^{j-2}\pi + 2^{j+1}\pi k, 2^{j+1}\pi k), \\ 1^+ & \xi \in [2^{j+1}\pi k, 2^{j-1}\pi + 2^{j+1}\pi k), \\ 0^+ & \xi \in [2^{j-1}\pi + 2^{j+1}\pi k, 3\cdot 2^{j-2}\pi + 2^{j+1}\pi k], \\ P^{(k)}(2^{-j}\xi) & \xi \in (3\cdot 2^{j-2}\pi + 2^{j+1}\pi k, 2^j\pi + 2^{j+1}\pi k). \end{cases} \tag{13}$$

We prove that this m_0 is a low-pass filter associated with an MRA. Therefore, φ becomes a scaling function of the MRA.

Step 1

In order to determine the support of $\hat{\varphi}$, we investigate intervals where $\hat{\varphi}(\xi) = 0$ on the positive domain. We rewrite the set of 0^- in (13)

$$2^{j+1}\pi\left(-\frac{1}{2}+k\right) \le \xi \le 2^{j+1}\pi\left(-\frac{1}{4}+k\right) \quad j = 1,2,\ldots,\ k = 1,2,\ldots$$

as

$$2^{j+1}\pi\left(\frac{1}{2}+k\right) \le \xi \le 2^{j+1}\pi\left(\frac{3}{4}+k\right) \quad j = 1,2,\ldots,\ k = 0,1,\ldots.$$

On the other hand, we do not change the parametrization of the set of 0^+. Hence, we denote the intervals of 0^- and 0^+ on the positive domain by

$$\mathcal{N}_{j,k} \coloneqq \left\{\xi : 2^j\pi + 2^{j+1}\pi k \leq \xi \leq \frac{3}{2}\cdot 2^j\pi + 2^{j+1}\pi k\right\},$$

$$\mathcal{P}_{j,k} \coloneqq \left\{\xi : 2^{j-1}\pi + 2^{j+1}\pi k \leq \xi \leq \frac{3}{2}\cdot 2^{j-1}\pi + 2^{j+1}\pi k\right\},$$

for $j \in \mathbb{N}, k \in \mathbb{N}_{\geq 0}$, and put

$$\mathcal{N}_{\mathrm{all}} \coloneqq \bigcup_{j\in\mathbb{N}}\bigcup_{k\in\mathbb{N}_{\geq 0}} \mathcal{N}_{j,k},$$

$$\mathcal{P}_{\mathrm{all}} \coloneqq \bigcup_{j\in\mathbb{N}}\bigcup_{k\in\mathbb{N}_{\geq 0}} \mathcal{P}_{j,k}.$$

Lemma 2.16. *The union $\mathcal{N}_{\mathrm{all}} \cup \mathcal{P}_{\mathrm{all}}$ is equal to $\mathcal{N}_{\mathrm{all}} \cup \left[\pi, \frac{3}{2}\pi\right]$.*

We note the particular subset of $\mathcal{N}_{\mathrm{all}}$

$$\mathcal{N}_i^\star \coloneqq \begin{cases} \displaystyle\bigcup_{\ell=1}^{i-1} \mathcal{N}_{i-\ell,2^\ell-1} \cup \ \mathcal{N}_{i,0} & (i \geq 2), \\ \mathcal{N}_{i,0} & (i = 1) \end{cases}$$

and put

$$\mathcal{N}^\star \coloneqq \bigcup_{i=1}^{\infty} \mathcal{N}_i^\star.$$

Lemma 2.17. *The set $\mathcal{N}_i^\star$ is equal to $[2^i\pi, (2^{i+1}-1)\pi]$.*

Obviously, $\mathcal{N}^\star \subset \mathcal{N}_{\mathrm{all}}$ holds. To show that $\mathcal{N}^\star = \mathcal{N}_{\mathrm{all}}$, we define

$$\mathcal{R}_i \coloneqq \left((2^{i+1}-1)\pi, 2^{i+1}\pi\right) \quad \text{and} \quad \mathcal{R}_{\mathrm{all}} \coloneqq \bigsqcup_{i=1}^{\infty} \mathcal{R}_i.$$

Thanks to Lemma 2.17, we know that $\mathcal{R}_{\mathrm{all}}$ satisfies

$$[0,\pi) \cup \left(\frac{3}{2}\pi, 2\pi\right) \cup \mathcal{R}_{\mathrm{all}} = \mathbb{R}_{\geq 0} \Big\backslash \left(\mathcal{N}^\star \cup \left[\pi, \frac{3}{2}\pi\right]\right).$$

Lemma 2.18. *The function $\hat{\varphi}$ is non-zero on $\mathcal{R}_{\mathrm{all}}$. Hence, $\mathcal{N}^\star \supset \mathcal{N}_{\mathrm{all}}$.*

Lemmas 2.16, 2.17 and 2.18 prove the following proposition.

Proposition 2.19. *All the intervals where $\hat{\varphi} \neq 0$ are determined by*

$$\operatorname{coz}\hat{\varphi} = [0,\pi) \cup \left(\frac{3}{2}\pi, 2\pi\right) \cup \mathcal{R}_{\mathrm{all}}.$$

Furthermore, $\mathcal{N}^\star = \mathcal{N}_{\mathrm{all}}$ holds, where $\operatorname{coz}\hat{\varphi} \coloneqq \{\xi \in \mathbb{R} : \hat{\varphi}(\xi) \neq 0\}$.

Step 2

In order to know the behavior of $\hat{\varphi}(\xi)$ on $\mathcal{R}_i$, we shall change the parametrization of the sets of 1^- and $N^{(k)}\left(2^{-j}\xi\right)$ in (13):

$$1^-,\ \xi \in (-2^{j-1}\pi + 2^{j+1}\pi k, -2^{j-2}\pi + 2^{j+1}\pi k] \quad j = 1, 2, \dots,\ k = 1, 2, \dots$$

as

$$1^-,\ \xi \in (3 \cdot 2^{j-1}\pi + 2^{j+1}\pi k, 7 \cdot 2^{j-2}\pi + 2^{j+1}\pi k] \quad j = 1, 2, \dots,\ k = 0, 1, \dots,$$

$$N^{(k)}\left(\frac{\xi}{2^j}\right),\ \xi \in (-2^{j-2}\pi + 2^{j+1}\pi k, 2^{j+1}\pi k) \quad j = 1, 2, \dots,\ k = 1, 2, \dots$$

as

$$N^{(k+1)}\left(\frac{\xi}{2^j}\right),\ \xi \in (7 \cdot 2^{j-2}\pi + 2^{j+1}\pi k, 2^{j+1}\pi + 2^{j+1}\pi k) \quad j = 1, 2, \dots,\ k = 0, 1, \dots.$$

For convenience, we define

$$\nu_{j,k}(\xi) \coloneqq N^{(k+1)}\left(\frac{\xi}{2^j}\right), \quad \xi \in (7 \cdot 2^{j-2}\pi + 2^{j+1}\pi k, 2^{j+1}\pi + 2^{j+1}\pi k),$$

$$\rho_{j,k}(\xi) \coloneqq P^{(k)}\left(\frac{\xi}{2^j}\right), \quad \xi \in (3 \cdot 2^{j-2}\pi + 2^{j+1}\pi k, 2^j\pi + 2^{j+1}\pi k).$$

When we need to pay attention to the parameters j, k, we also write 1^- as $1^-_{j,k}$. Then, we obtain two lemmas.

Lemma 2.20. *For $i, j \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 0}$,*

$$\operatorname{coz} \rho_{j,k} \cap \mathcal{R}_i \neq \emptyset$$

if and only if $j = i + 1,\ k = 0$.

Lemma 2.21. *For $i, j \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 0}$,*

$$\operatorname{coz} \nu_{j,k} \cap \mathcal{R}_i \neq \emptyset$$

if and only if $1 \leq j \leq i$ and $k = 2^{i-j} - 1$.

We write 1^+ as

$$1^+_{j,k}, \quad \xi \in [2^{j+1}\pi k, 2^{j-1}\pi + 2^{j+1}\pi k) \quad j = 1, 2, \dots,\ k = 1, 2, \dots.$$

From $\operatorname{coz} 1^+_{j,0} = [0, 2^{j-1}\pi)$, the next lemma is clear.

Lemma 2.22. *For $i, j \in \mathbb{N}$, we obtain $\mathcal{R}_i \subset \operatorname{coz} 1^+_{j,0}$ if $j \geq i + 2$.*

Lemmas 2.20, 2.21 and 2.22 prove the following theorem.

Theorem 2.23. *Let $i \in \mathbb{N}$,*

$$\mathcal{R}_i^{\triangleleft} \coloneqq \left((2^{i+1} - 1)\pi, (2^{i+1} - 2^{-1})\pi\right] \quad \text{and} \quad \mathcal{R}_i^{\triangleright} \coloneqq \left((2^{i+1} - 2^{-1})\pi, 2^{i+1}\pi\right).$$

Define $\Lambda_i(\xi) \coloneqq \prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right)$ for $\xi \in \mathcal{R}_i$. Then, Λ_i on $\mathcal{R}_i = \mathcal{R}_i^{\triangleleft} \sqcup \mathcal{R}_i^{\triangleright}$ is composed of the following two parts:

$$\Lambda_i^{\triangleleft}(\xi) \coloneqq \rho_{i+1,0}(\xi) \prod_{j=2}^{i} \nu_{j,2^{i-j}-1}(\xi), \quad \xi \in \mathcal{R}_i^{\triangleleft},$$

$$\Lambda_i^{\triangleright}(\xi) \coloneqq \rho_{i+1,0}(\xi) \prod_{j=1}^{i} \nu_{j,2^{i-j}-1}(\xi), \quad \xi \in \mathcal{R}_i^{\triangleright},$$

where we remark that if $i = 1$, the product $\prod_{j=2}^{i} \nu_{j,2^{i-j}-1}(\xi)$ in $\Lambda_i^{\triangleleft}(\xi)$ is an empty product, whose value is 1.

Step 3

We are allowed to derive a decay estimate only on $\mathcal{R}_i$ for the case of $\xi > 0$. We know that $\nu_{j,k}(\xi)$ is $2^{i+1}\pi$-periodic (more precisely $2^{j+1}\pi$-periodic) and that $N_{\pi/4}(2^{-j}\xi)$ is defined on $(-2^{j-2}\pi, 0)$. Then, with $\tilde{\xi} := \xi - 2^{i+1}\pi$, the function $\Lambda_i^{\triangleright}$ in Theorem 2.23 is calculated as the product of the following functions on $\mathcal{R}_i^{\triangleright}$:

$$\begin{cases} \rho_{i+1,0}(\xi) = \sqrt{\dfrac{1}{\log_2\left(1 - \frac{1}{(2^{-i-1}\xi - \pi)(2^{-i-1}\xi - 3\pi/4)}\right)}}, \\ \displaystyle\prod_{j=1}^{i} \nu_{j,2^{i-j}-1}(\xi) = \prod_{j=1}^{i} \nu_{j,2^{i-j}-1}(\tilde{\xi}) = \prod_{j=1}^{i} \sqrt{1 - \dfrac{1}{\log_2\left(1 - \frac{1}{2^{-j}\tilde{\xi}(2^{-j}\tilde{\xi} + \pi/4)}\right)}}, \end{cases}$$

where we remark that $\tilde{\xi} \in (-2^{-1}\pi, 0)$ for $\xi \in \mathcal{R}_i^{\triangleright}$. On the other hand, the function $\Lambda_i^{\triangleleft}$ is calculated as the product of the following functions on $\mathcal{R}_i^{\triangleleft}$:

$$\begin{cases} \rho_{i+1,0}(\xi) = \sqrt{\dfrac{1}{\log_2\left(1 - \frac{1}{(2^{-i-1}\xi - \pi)(2^{-i-1}\xi - 3\pi/4)}\right)}}, \\ \displaystyle\prod_{j=2}^{i} \nu_{j,2^{i-j}-1}(\xi) = \prod_{j=2}^{i} \nu_{j,2^{i-j}-1}(\tilde{\xi}) = \prod_{j=2}^{i} \sqrt{1 - \dfrac{1}{\log_2\left(1 - \frac{1}{2^{-j}\tilde{\xi}(2^{-j}\tilde{\xi} + \pi/4)}\right)}}, \end{cases}$$

where we remark that $\tilde{\xi} \in (-\pi, -2^{-1}\pi]$ for $\xi \in \mathcal{R}_i^{\triangleleft}$. Since $\nu_{1,0}(\xi) = N^{(1)}(2^{-1}\xi) \leq 1$, we can combine $\Lambda_i^{\triangleright}$ and $\Lambda_i^{\triangleleft}$ and see that Λ_i $(i \geq 2)$ satisfy

$$\Lambda_i(\xi)^2 \leq \mathscr{P}_i(\xi) \cdot \mathscr{N}_i(\xi), \quad \xi \in \mathcal{R}_i,$$

where

$$\begin{cases} \mathscr{P}_i(\xi) := \dfrac{1}{\log_2\left(1 - \frac{1}{(2^{-i-1}\xi - \pi)(2^{-i-1}\xi - 3\pi/4)}\right)} & \xi \in \mathcal{R}_i, \\ \mathscr{N}_i(\xi) := \displaystyle\prod_{j=2}^{i} \left\{1 - \dfrac{1}{\log_2\left(1 - \frac{1}{2^{-j}\tilde{\xi}(2^{-j}\tilde{\xi} + \pi/4)}\right)}\right\} & \tilde{\xi} \in (-\pi, 0). \end{cases}$$

(i) **Estimate of $\mathscr{N}_i(\xi)$:**

We may suppose that $i \geq 3$. As a result, starting from $j = 3$, we obtain

$$\mathscr{N}_i(\xi) \leq \exp\left\{-\sum_{j=3}^{i} \frac{1}{j + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}\right\}, \tag{14}$$

where $c_{\tilde{\xi}} := -\log_2(-\tilde{\xi})$ and $\gamma_{\tilde{\xi}} := 1 - \log_2\left(2^{-j}\tilde{\xi} + \pi/4\right)$ satisfy

$$c_{\tilde{\xi}} \geq -\log_2(\pi) \quad (> -2),$$

$$(1 <) \quad 1 - \log_2(\pi/4) < \gamma_{\tilde{\xi}} < 1 - \log_2(\pi/8) \quad (< 3).$$

Furthermore, since

$$\sum_{j=3}^{i} \frac{1}{j + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}} \geq \int_3^{i+1} \frac{1}{t + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}} dt = \ln\left(\frac{i + 1 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}{3 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}\right),$$

it follows that

$$\mathscr{N}_i(\xi) \leq \exp\left\{-\ln\left(\frac{i + 1 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}{3 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}\right)\right\} = \frac{3 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}{i + 1 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}. \tag{15}$$

(ii) **Estimate of $\mathscr{P}_i(\xi)$:**

As a result, for $\xi \in \mathcal{R}_i$ $(i \geq 2)$, we obtain

$$\mathscr{P}_i(\xi) \leq \frac{1}{-\log_2\{-2^{-i-1}\xi + \pi\}}.$$

It follows that

$$\mathscr{P}_i(\xi) \leq \frac{1}{-\log_2\{-2^{-i-1}\tilde{\xi}\}} = \frac{1}{i+1+c_{\tilde{\xi}}}. \tag{16}$$

Taking the product of (15) and (16), we arrive at the following estimate:

$$\Lambda_i(\xi) \leq \sqrt{\frac{3 + c_{\tilde{\xi}} + \gamma_{\tilde{\xi}}}{(i+1+c_{\tilde{\xi}})(i+1+c_{\tilde{\xi}}+\gamma_{\tilde{\xi}})}} \leq Ci^{-1}, \quad \xi \in \mathcal{R}_i. \tag{17}$$

The interval $\mathcal{R}_i$ enables us to regard ξ as $2^{i+1}\pi$, that is, $i \sim \log_2 \xi$. Hence, the estimate (17) implies that $\hat{\varphi}(\xi) = O\left((\log_2 \xi)^{-1}\right)$ as $\xi \to \infty$. This can also be rewritten as

$$\hat{\varphi}(\xi) = O\left((\log\langle\xi\rangle)^{-1}\right). \tag{18}$$

We will prove that $\hat{\varphi}(\xi) = 0$ for $\xi < 0$ and that $\|\hat{\varphi}\|_{L^2} < \infty$ in the next step.

Step 4

Theorem 2.24. *Let m_0 be the low-pass filter given in (13). Then, φ defined by*

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

is a scaling function of the MRA for $H^2(\mathbb{R})$, that is, $\langle\varphi, \varphi(\cdot - \ell)\rangle = \delta_{0,\ell}$ for $\ell \in \mathbb{Z}$ and $\hat{\varphi}(\xi) = 0$ for $\xi < 0$.

Proof sketch of Theorem 2.24. For the case of $\xi > 0$, we give a proof according to Theorem 4.8 in Chap. 7 of [25]. Let the function f_n be defined by

$$\hat{f}_n(\xi) = \mathbf{1}_{2^n K}(\xi) \prod_{j=1}^{n} m_0\left(\frac{\xi}{2^j}\right), \quad K = [-\pi, \pi].$$

Then, we observe that $\hat{f}_n \to \hat{\varphi}$ for all $\xi \in \mathbb{R}$.

We show that $\langle f_n, f_n(\cdot - \ell)\rangle = \delta_{0,\ell}$ for any $n \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. As a result, we get the following reduction formula:

$$\langle f_n, f_n(\cdot - \ell)\rangle = \langle f_{n-1}, f_{n-1}(\cdot - \ell)\rangle = \cdots = \langle f_1, f_1(\cdot - \ell)\rangle = \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\ell 2\mu} d\mu = \delta_{0,\ell}.$$

In particular, we get $\|f_n\|_{L^2}^2 = (2\pi)^{-1}\|\hat{f}_n\|_{L^2}^2 = 1$ when $\ell = 0$. Hence, Fatou's lemma yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \liminf_{n\to\infty} |\hat{f}_n(\xi)|^2 d\xi \leq \frac{1}{2\pi}\liminf_{n\to\infty}\int_{-\infty}^{\infty} |\hat{f}_n(\xi)|^2 d\xi = 1,$$

which means that $\varphi \in L^2(\mathbb{R})$.

In order to conclude the orthogonality of $\{\varphi(\cdot - \ell)\}_{\ell\in\mathbb{Z}}$ from $\langle f_n, f_n(\cdot - \ell)\rangle = \delta_{0,\ell}$, it is sufficient to show that $\lim_{n\to\infty} f_n = \varphi$ in the norm of $L^2(\mathbb{R})$. We shall skip showing that $\lim_{n\to\infty}\|\hat{f}_n - \hat{\varphi}\|_{L^2(0,\infty)} = 0$, whose proof is quite similar to that in [25]. Therefore, it remains to show that $\lim_{n\to\infty}\|\hat{f}_n\|_{L^2(-\infty,0)} = 0$, which also implies that $\hat{\varphi}(\xi) = 0$ for $\xi < 0$. We next obtain

$$\left\|\hat{f}_n\right\|_{L^2(-\infty,0)}^2 \leq \int_{-\pi}^{0} \mathscr{N}_n(\xi) d\xi.$$

Now, inequality (14) with ξ instead of $\tilde{\xi}$ also gives

$$\int_{-\pi}^{0} \mathscr{N}_n(\xi)d\xi \leq \int_{-\pi}^{0} \exp\left\{-\sum_{j=3}^{n} \frac{1}{j+c_\xi+\gamma_\xi}\right\} d\xi.$$

Since $j - \log_2(-\xi) + \gamma_\xi > 0$ and

$$\exp\left\{-\sum_{j=3}^{n} \frac{1}{j-\log_2(-\xi)+\gamma_\xi}\right\} \to 0 \quad \text{as } n\to\infty \text{ for } \xi\in(-\pi,0),$$

we arrive at $\|\hat{f}_n\|_{L^2(-\infty,0)} \to 0$, using the monotone convergence theorem.

□

Step 5

Finally, we replace the function M by a continuous function:

$$M_\delta(\xi) := \begin{cases} 0 & \xi\in[-\pi,-2^{-1}\pi], \\ \alpha_\delta(\xi) & \xi\in(-2^{-1}\pi,-2^{-1}\pi+\delta], \\ 1^- & \xi\in(-2^{-1}\pi+\delta,-2^{-2}\pi], \\ N_{\pi/4}(\xi) & \xi\in(-2^{-2}\pi,0), \\ 1^+ & \xi\in[0,2^{-1}\pi), \\ \beta_\delta(\xi) & \xi\in[2^{-1}\pi,2^{-1}\pi+\delta], \\ 0 & \xi\in[2^{-1}\pi+\delta,3\cdot2^{-2}\pi], \\ P_{\pi/4}(\xi) & \xi\in(3\cdot2^{-2}\pi,\pi), \end{cases}$$

where $\delta>0$ is sufficiently small and $\alpha_\delta,\ \beta_\delta$ are continuous functions such that

$$\alpha_\delta^2(\xi)+\beta_\delta^2(\xi+\pi)=1, \quad \xi\in(-2^{-1}\pi,-2^{-1}\pi+\delta].$$

Since Theorem 2.24 still holds for M_δ, we denote its scaling function by φ_δ. Furthermore, this modification does not influence the choice of $\mathcal{N}_{j,k}$. Hence, it does not change $\mathcal{N}_{\text{all}}$; however, $\mathcal{P}_{1,0}$ must be changed. Then, $\mathcal{N}_{\text{all}} \cup \left[\pi, \frac{3}{2}\pi\right]$ in Lemma 2.16 becomes

$$\mathcal{N}_{\text{all}} \cup \left[\pi+2\delta, \frac{3}{2}\pi\right].$$

Thus, we get

$$\operatorname{coz}\hat{\varphi}_\delta = [0,\pi+2\delta) \cup \left(\frac{3}{2}\pi, 2\pi\right) \cup \mathcal{R}_{\text{all}}.$$

From the continuity at both endpoints of $\mathcal{R}_i$, we see that $\hat{\varphi}_\delta$ is smooth except at the origin.

In conclusion, we can construct the wavelet $\hat{\psi}_\delta$ consisting of the continuous low-pass filter $m_0^{(\delta)}$ with M_δ and the scaling function $\varphi_\delta \in H^2(\mathbb{R})$:

$$\hat{\psi}_\delta(\xi) = e^{-i\xi/2}\overline{m_0^{(\delta)}\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_\delta\left(\frac{\xi}{2}\right).$$

At the same time, we redefine

$$\operatorname{coz}\hat{\varphi}_\delta\left(\frac{\cdot}{2}\right) = [0,2\pi+4\delta)\cup(3\pi,4\pi)\cup\mathcal{R}'_{\text{all}},$$

$$\mathcal{R}'_{\text{all}} := \bigsqcup_{i=1}^{\infty}\mathcal{R}'_i \qquad \text{and} \quad \mathcal{R}'_i := \left((2^{i+2}-2)\pi, 2^{i+2}\pi\right)$$

and

$$M_\delta^{(k)}\left(\frac{\xi}{2}+\pi\right) := \begin{cases} 0^- & \xi \in [-4\pi+4\pi k, -3\pi+4\pi k], \\ \alpha_\delta(\xi) & \xi \in (-3\pi+4\pi k, -3\pi+4\pi k+2\delta], \\ 1^- & \xi \in (-3\pi+4\pi k+2\delta, -5\cdot 2^{-1}\pi+4\pi k], \\ N^{(k)}(\xi) & \xi \in (-5\cdot 2^{-1}\pi+4\pi k, -2\pi+4\pi k), \\ 1^+ & \xi \in [-2\pi+4\pi k, -\pi+4\pi k), \\ \beta_\delta(\xi) & \xi \in [-\pi+4\pi k, -\pi+4\pi k+2\delta], \\ 0^+ & \xi \in [-\pi+4\pi k+2\delta, -2^{-1}\pi+4\pi k], \\ P^{(k)}(\xi) & \xi \in (-2^{-1}\pi+4\pi k, 4\pi k). \end{cases} \tag{19}$$

We see that the discontinuity of $\hat{\varphi}_\delta(\xi/2)$ at $\xi=0$ is flattened due to 0^- in (19) for $k=1$. Therefore, from (18), the wavelet ψ_δ has a continuous Fourier transform such that $|\hat{\psi}_\delta(\xi)| = O\left((\log\langle\xi\rangle)^{-1}\right)$.

2.4 Directional frames having Lipschitz continuous Fourier transforms

To apply the method of wavelet analysis to image processing, we must consider a two-dimensional wavelet. The standard way of constructing a two-dimensional wavelet is to take the tensor product of two one-dimensional wavelets. Specifically, if ψ is an orthonormal wavelet for $L^2(\mathbb{R})$,

$$\psi_{j_1,k_1}\otimes\psi_{j_2,k_2}(x_1,x_2) = \psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2)$$

are orthonormal wavelets for $L^2(\mathbb{R}^2)$. In the case of an MRA wavelet ψ associated with $(\{V_j\}_{j\in\mathbb{Z}},\varphi)$, we can define a multiresolution ladder in $L^2(\mathbb{R}^2)$ by

- $\cdots \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \cdots$,
- $\overline{\bigcup_{j\in\mathbb{Z}} \mathbf{V}_j} = L^2(\mathbb{R}^2)$,
- $\bigcap_{j\in\mathbb{Z}} \mathbf{V}_j = \{0\}$,
- $F(\cdot,\cdot)\in\mathbf{V}_j \iff F(2\cdot,2\cdot)\in\mathbf{V}_{j+1}$,

where

$$\mathbf{V}_0 = V_0\otimes V_0 = \overline{\operatorname{span}\{F(x_1,x_2)=f(x_1)g(x_2) : f,g\in V_0\}}$$

and

$$\{\Phi_{0;k_1,k_2} := \varphi(\cdot-k_1)\varphi(\cdot-k_2)\}_{k_1,k_2\in\mathbb{Z}}$$

is an orthonormal basis for $\mathbf{V}_0$. By the decomposition $V_{j+1} = V_j\oplus W_j$, we have

$$\begin{aligned}\mathbf{V}_{j+1} &= V_{j+1}\otimes V_{j+1} \\ &= (V_j\oplus W_j)\otimes(V_j\oplus W_j) \\ &= (V_j\otimes V_j)\oplus[(W_j\otimes V_j)\oplus(V_j\otimes W_j)\oplus(W_j\otimes W_j)] \\ &=: \mathbf{V}_j\oplus\mathbf{W}_j.\end{aligned}$$

Then, $\mathbf{W}_j$ has three orthonormal bases as

$$\Psi^v_{j,k_1,k_2} := \psi_{j,k_1}\otimes\varphi_{j,k_2}, \quad \Psi^h_{j,k_1,k_2} := \varphi_{j,k_1}\otimes\psi_{j,k_2}, \quad \Psi^d_{j,k_1,k_2} := \psi_{j,k_1}\otimes\psi_{j,k_2},$$

where v,h,d stand for "vertical," "horizontal," "diagonal," respectively. Their superscripts are derived from the edges in the image which correspond to the wavelet coefficients. Thus, an orthonormal basis for $L^2(\mathbb{R}^2) = \overline{\bigcup_{j\in\mathbb{Z}}\mathbf{W}_j}$ is given by

$$\{\Psi^\lambda_{j;k_1,k_2} : j\in\mathbb{Z},\ k_1,k_2\in\mathbb{Z},\ \lambda=v,h,d\}.$$

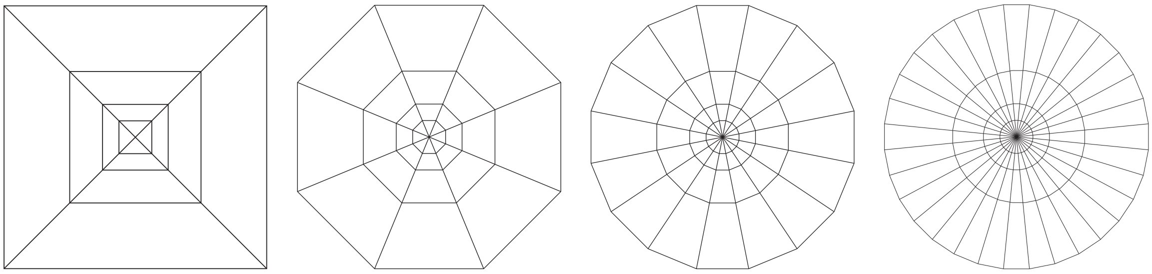
For a more detailed description including more general two-dimensional wavelets, see e.g. [2, 14, 28].

However, a tensor product wavelet is not effective for detecting singularities along curved in an image. In order to overcome these obstacles, the curvelet was introduced by Candès and Donoho (see [10, 11, 12], etc.). Their contribution marked the start of a multiscale method, the so-called geometric multiscale analysis. Over the last 20 years or so, various approaches of geometric multiscale analysis have been proposed, including ridgelet and shearlet. In [18], we presented multidirectional methods based on concentric regular 2^N-sided polygons in the frequency domain.

First, we consider the sets of trapezoids (Figure 13):

$$S_j^{(N)} = \left[-\xi_2 \tan\frac{\pi}{2^N}, \xi_2 \tan\frac{\pi}{2^N}\right] \times [2^{j-1}\pi, 2^j\pi],$$

$$\tilde{S}_j^{(N)} = \left[-\xi_2 \tan\frac{\pi}{2^N}, \xi_2 \tan\frac{\pi}{2^N}\right] \times [0, 2^j\pi].$$



(a) $N = 2$ (b) $N = 3$ (c) $N = 4$ (d) $N = 5$

Figure 13: Polygonal tilings based on trapezoids $S_j^{(N)}$ in the frequency domain.

Then, we define the functions in the frequency domain:

$$\hat{\psi}_{j,0}^{(N)}(\xi) = 2^{-j}\sqrt{\cot\frac{\pi}{2^N}}\,\mathbf{1}_{S_j^{(N)}}(\xi),$$

$$\hat{\phi}_{j,0}^{(N)}(\xi) = 2^{-j}\sqrt{\cot\frac{\pi}{2^N}}\,\mathbf{1}_{\tilde{S}_j^{(N)}}(\xi).$$

Using the inverse Fourier transform, we obtain the following:

$$\psi_{j,0}^{(N)}(x) = \frac{\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+2}\pi^2 x_1}\sum_{\pm}\left\{\pm\frac{e^{i2^j\pi(x_2\mp x_1\tan\frac{\pi}{2^N})} - e^{i2^{j-1}\pi(x_2\mp x_1\tan\frac{\pi}{2^N})}}{x_2\mp x_1\tan\frac{\pi}{2^N}}\right\},$$

$$\phi_{j,0}^{(N)}(x) = \frac{\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+2}\pi^2 x_1}\sum_{\pm}\left\{\pm\frac{e^{i2^j\pi(x_2\mp x_1\tan\frac{\pi}{2^N})} - 1}{x_2\mp x_1\tan\frac{\pi}{2^N}}\right\}.$$

Second, let R_ℓ be the operator of anticlockwise rotation by angle $2^{1-N}\ell\pi$ and define

$$\psi_{j,\ell}^{(N)}(x) = \psi_{j,0}^{(N)}(R_\ell x) \quad\text{and}\quad \phi_{j,\ell}^{(N)}(x) = \phi_{j,0}^{(N)}(R_\ell x).$$

In order to get a real-valued function in the spatial domain, $\psi_{j,0}^{(N)}$ should be coupled with $\psi_{j,2^{N-1}}^{(N)}$. Hence, we have the following:

$$\Psi_{j,0}^{(N)}(x) := \psi_{j,0}^{(N)} + \psi_{j,2^{N-1}}^{(N)} = \frac{\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+1}\pi^2 x_1}\sum_{\pm}\left\{\pm\frac{\cos 2^{j-1}\pi d_x^{\pm} - \cos 2^j\pi d_x^{\pm}}{d_x^{\pm}}\right\}, \tag{20}$$

$$\Phi_{j,0}^{(N)}(x) := \phi_{j,0}^{(N)} + \phi_{j,2^{N-1}}^{(N)} = \frac{\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+1}\pi^2 x_1}\sum_{\pm}\left\{\pm\frac{1 - \cos 2^j\pi d_x^{\pm}}{d_x^{\pm}}\right\},$$

where $d_x^{\pm} := x_2 \pm x_1 \tan\frac{\pi}{2^N}$. Let $N \geq 2$ and

$$p_2 = \frac{1}{\sqrt{2}}, \quad p_3 = \frac{1}{\sqrt{2+\sqrt{2}}}, \cdots, p_N = \frac{1}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}.$$

Then, we see that

$$\cos\frac{\pi}{2^N} = \frac{1}{2p_N},$$

$$\cot\frac{\pi}{2^N} = \left(\tan^2\frac{\pi}{2^N}\right)^{-1/2} = \left(\frac{1}{\cos^2\frac{\pi}{2^N}} - 1\right)^{-1/2} = (4p_N^2 - 1)^{-1/2}.$$

With

$$X_0^{\pm}(x) := x_2\cos\frac{\pi}{2^N} \pm x_1 \sin\frac{\pi}{2^N} \quad \left(= \frac{d_x^{\pm}}{2p_N}\right),$$

formula (20) can be written as

$$\Psi_{j,0}^{(N)}(x) = \frac{\cos\frac{\pi}{2^N}\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+1}\pi^2}\sum_{\pm}\left\{\pm\frac{\cos(2^j\pi p_N X_0^{\pm}(x)) - \cos(2^{j+1}\pi p_N X_0^{\pm}(x))}{x_1 X_0^{\pm}(x)}\right\}.$$

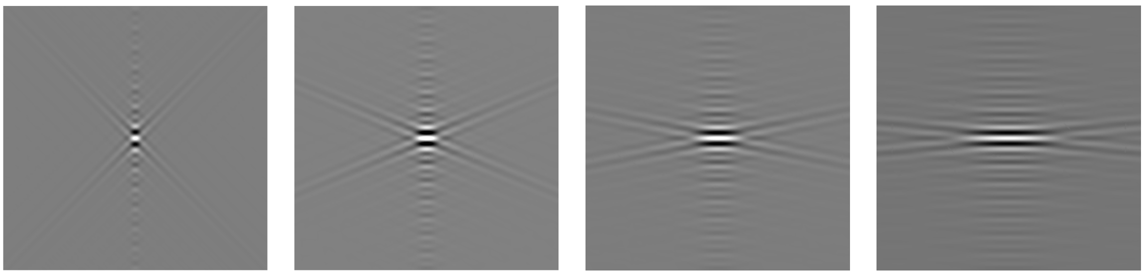
Third, setting

$$X_\ell^{\pm}(x) := X_0^{\pm}(R_\ell x) = x_1 \sin\frac{(2\ell\pm 1)\pi}{2^N} + x_2\cos\frac{(2\ell\pm 1)\pi}{2^N},$$

and noting that $x_1 = x\cdot(1,0)$ can be replaced by $R_\ell x\cdot(1,0)$, we obtain the following (Figure 14):

$$\Psi_{j,\ell}^{(N)}(x) = \frac{\cos\frac{\pi}{2^N}\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+1}\pi^2}\sum_{\pm}\left\{\pm\frac{\cos(2^j\pi p_N X_\ell^{\pm}(x)) - \cos(2^{j+1}\pi p_N X_\ell^{\pm}(x))}{X_\ell^{\pm}(x) R_\ell x\cdot(1,0)}\right\},$$

$$\Phi_{j,\ell}^{(N)}(x) = \frac{\cos\frac{\pi}{2^N}\sqrt{\cot\frac{\pi}{2^N}}}{2^{j+1}\pi^2}\sum_{\pm}\left\{\pm\frac{2\sin^2\frac{2^{j-1}\pi X_\ell^{\pm}(x)}{\cos\frac{\pi}{2^N}}}{X_\ell^{\pm}(x) R_\ell x\cdot(1,0)}\right\}.$$



(a) $N=2$ (b) $N=3$ (c) $N=4$ (d) $N=5$

Figure 14: $\Psi_{0,0}^{(N)}(x)$ on the spatial domain, where $x\in[-30,30]\times[-30,30]$.

Note that $\hat{\Psi}_{j,\ell}^{(N)}$ satisfies the partition of unity in the following sense:

$$\tan\frac{\pi}{2^N}\sum_{j\in\mathbb{Z}}\sum_{1\leq\ell\leq 2^{N-1}} 2^{2j}|\hat{\Psi}_{j,\ell}^{(N)}(\xi)|^2 = 1.$$

Thus, using $\sum_{1\leq\ell\leq 2^{N-1}}$ instead of $\sum_{0\leq\ell\leq 2^{N-1}-1}$, we get the following.

Theorem 2.25. *Let $N \geq 2$, $J \in \mathbb{Z}$, $k' = \left(k_1 \cot \frac{\pi}{2^N}, k_2\right)$ and the real-valued functions $\Psi_{j,\ell}^{(N)}$ and $\Phi_{j,\ell}^{(N)}$ be defined as*

$$\begin{aligned} \Psi_{j,\ell}^{(N)}(x) &= \sum_{\pm} \left\{ \pm \frac{\cos(2^j \pi p_N X_\ell^{\pm}(x)) - \cos(2^{j+1} \pi p_N X_\ell^{\pm}(x))}{2^{j+2}\pi^2 p_N (4p_N^2 - 1)^{1/4} X_\ell^{\pm}(x) R_\ell x \cdot (1,0)} \right\}, \\ \Phi_{j,\ell}^{(N)}(x) &= \sum_{\pm} \left\{ \pm \frac{\sin^2(2^j \pi p_N X_\ell^{\pm}(x))}{2^{j+1}\pi^2 p_N (4p_N^2 - 1)^{1/4} X_\ell^{\pm}(x) R_\ell x \cdot (1,0)} \right\}, \end{aligned}$$

where $X_\ell^{\pm}(x) = x_1 \sin \frac{(2\ell \pm 1)\pi}{2^N} + x_2 \cos \frac{(2\ell \pm 1)\pi}{2^N}$. Then, $f \in L^2(\mathbb{R}_x^2)$ can be expanded as

$$\begin{aligned} f(x) &= \sum_{j \geq J+1} \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \alpha_{j,\ell,k} \Psi_{j,\ell}^{(N)} \left(x - 2^{-j} R_{-\ell} k'\right) \\ &\qquad + \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \beta_{J,\ell,k} \Phi_{J,\ell}^{(N)} \left(x - 2^{-J} R_{-\ell} k'\right), \end{aligned}$$

where

$$\alpha_{j,\ell,k} = \int_{\mathbb{R}_x^2} f(x) \Psi_{j,\ell}^{(N)} \left(x - 2^{-j} R_{-\ell} k'\right) dx, \quad \beta_{j,\ell,k} = \int_{\mathbb{R}_x^2} f(x) \Phi_{j,\ell}^{(N)} \left(x - 2^{-j} R_{-\ell} k'\right) dx,$$

and R_ℓ is the operator of anticlockwise rotation by angle $2^{1-N}\ell\pi$.

2.4.1 Lipschitz continuous type

In order to construct Lipschitz continuous type in the frequency domain, we enlarge the supports which overlap each other.

Step 1

We consider

$$\hat{\psi}_{j,0}^{\natural(N)}(\xi) = 2^{-j} \sqrt{\cot \frac{\pi}{2^{N-1}} \Lambda_j(\xi) \Gamma^{(N)}(\xi)},$$

with non-negative functions Λ_j and $\Gamma^{(N)}$ defined by

$$\begin{aligned} \Lambda_j(\xi) &= \int_0^{2^{2-j}\pi^{-1}\xi_2 - 1} \{2\Delta(2\tau) - \Delta(\tau - 1)\}\, d\tau \quad \text{for } \xi \in \mathbb{R}^2, \\ &= \begin{cases} \int_0^{2^{3-j}\pi^{-1}\xi_2 - 2} \Delta(\tau) d\tau & \xi \in \mathbb{R} \times [2^{j-2}\pi, 2^{j-1}\pi], \\ 1 - \int_1^{2^{2-j}\pi^{-1}\xi_2 - 1} \Delta(\tau - 1) d\tau = 1 - \int_0^{2^{2-j}\pi^{-1}\xi_2 - 2} \Delta(\tau) d\tau & \xi \in \mathbb{R} \times [2^{j-1}\pi, 2^j\pi], \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{21}$$

$$\Gamma^{(N)}(\xi) = \begin{cases} \cos^2\left(2^{N-2} \operatorname{Arctan} \frac{\xi_1}{\xi_2}\right) & \xi \in \left[-|\xi_2| \tan \frac{\pi}{2^{N-1}}, |\xi_2| \tan \frac{\pi}{2^{N-1}}\right] \times \mathbb{R} \backslash \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

where Δ is the hat function $\Delta(\tau) = \max\{1 - |\tau - 1|, 0\}$ for $\tau \in \mathbb{R}$. Then, we find that

$$\begin{aligned} \operatorname{supp} \hat{\psi}_{j,0}^{\natural(N)} &= \left[-\xi_2 \tan \frac{\pi}{2^{N-1}}, \xi_2 \tan \frac{\pi}{2^{N-1}}\right] \times [2^{j-2}\pi, 2^j\pi] \\ &=: S_j^{\natural(N)} = S_{j-1}^{(N-1)} \cup S_j^{(N-1)}. \end{aligned}$$

By construction, Λ_j is piecewise quadratic and degenerate of order 2. Hence, $\sqrt{\Lambda_j}$ is Lipschitz continuous. From [18], we see that $\sqrt{\Gamma^{(N)}}$ is a piecewise rational function. Thus, $\hat{\psi}_{j,0}^{\natural(N)}$ is Lipschitz continuous, and so is the rotated $\hat{\psi}_{j,\ell}^{\natural(N)}$. Now, we consider the case of $(N-1) \geq 2$, which can be reduced to the case of $N \geq 2$ in the previous section.

Step 2

From (21), we get

$$\sum_{j\in\mathbb{Z}}\Lambda_j(\xi)=\begin{cases}1 & \xi\in\mathbb{R}\times(0,\infty),\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$R_\ell=\begin{pmatrix}\cos(2^{1-N}\ell\pi) & -\sin(2^{1-N}\ell\pi)\\ \sin(2^{1-N}\ell\pi) & \cos(2^{1-N}\ell\pi)\end{pmatrix}\quad\text{and}\quad R_{\ell+2^{N-1}}\xi=-R_\ell\xi,$$

it holds that

$$\sum_{j\in\mathbb{Z}}\left\{\Lambda_j(R_\ell\xi)+\Lambda_j(R_{\ell+2^{N-1}}\xi)\right\}=1\quad\text{a.e. }\xi\in\mathbb{R}^2. \tag{22}$$

Furthermore, we proved the following in [18]:

$$\Gamma^{(N)}(R_{\ell+2^{N-1}}\xi)=\Gamma^{(N)}(-R_\ell\xi)=\Gamma^{(N)}(R_\ell\xi), \tag{23}$$

$$\sum_{1\le\ell\le 2^{N-1}}\Gamma^{(N)}(R_\ell\xi)=1\quad\text{for }\xi\in\mathbb{R}^2\backslash\{0\}. \tag{24}$$

Now, we also define

$$\psi^{\natural(N)}_{j,\ell}(x)=\psi^{\natural(N)}_{j,0}(R_\ell x)$$

and the real-valued function

$$\Psi^{\natural(N)}_{j,\ell}(x)=\psi^{\natural(N)}_{j,\ell}(x)+\psi^{\natural(N)}_{j,\ell+2^{N-1}}(x).$$

Thus, (22), (23) and (24) and $\operatorname{supp}\hat{\psi}^{\natural(N)}_{j,\ell}\cap\operatorname{supp}\hat{\psi}^{\natural(N)}_{j,\ell+2^{N-1}}=\emptyset$ give

$$\begin{aligned}\tan\frac{\pi}{2^{N-1}}\sum_{j\in\mathbb{Z}}\sum_{1\le\ell\le2^{N-1}}2^{2j}|\hat{\Psi}^{\natural(N)}_{j,\ell}(\xi)|^2&=\tan\frac{\pi}{2^{N-1}}\sum_{j\in\mathbb{Z}}\sum_{1\le\ell\le2^{N-1}}2^{2j}\left\{|\hat{\psi}^{\natural(N)}_{j,\ell}(\xi)|^2+|\hat{\psi}^{\natural(N)}_{j,\ell+2^{N-1}}(\xi)|^2\right\}\\&=\sum_{j\in\mathbb{Z}}\sum_{1\le\ell\le2^{N-1}}\left\{\Lambda_j(R_\ell\xi)\Gamma^{(N)}(R_\ell\xi)+\Lambda_j(R_{\ell+2^{N-1}}\xi)\Gamma^{(N)}(R_{\ell+2^{N-1}}\xi)\right\}\\&=\sum_{1\le\ell\le2^{N-1}}\sum_{j\in\mathbb{Z}}\left\{\Lambda_j(R_\ell\xi)+\Lambda_j(R_{\ell+2^{N-1}}\xi)\right\}\Gamma^{(N)}(R_\ell\xi)\\&=1\quad\text{a.e. }\xi\in\mathbb{R}^2.\end{aligned}$$

Consequently, the Lipschitz continuous $\hat{\Psi}^{\natural(N)}_{j,\ell}$ satisfies the partition of unity in the following sense:

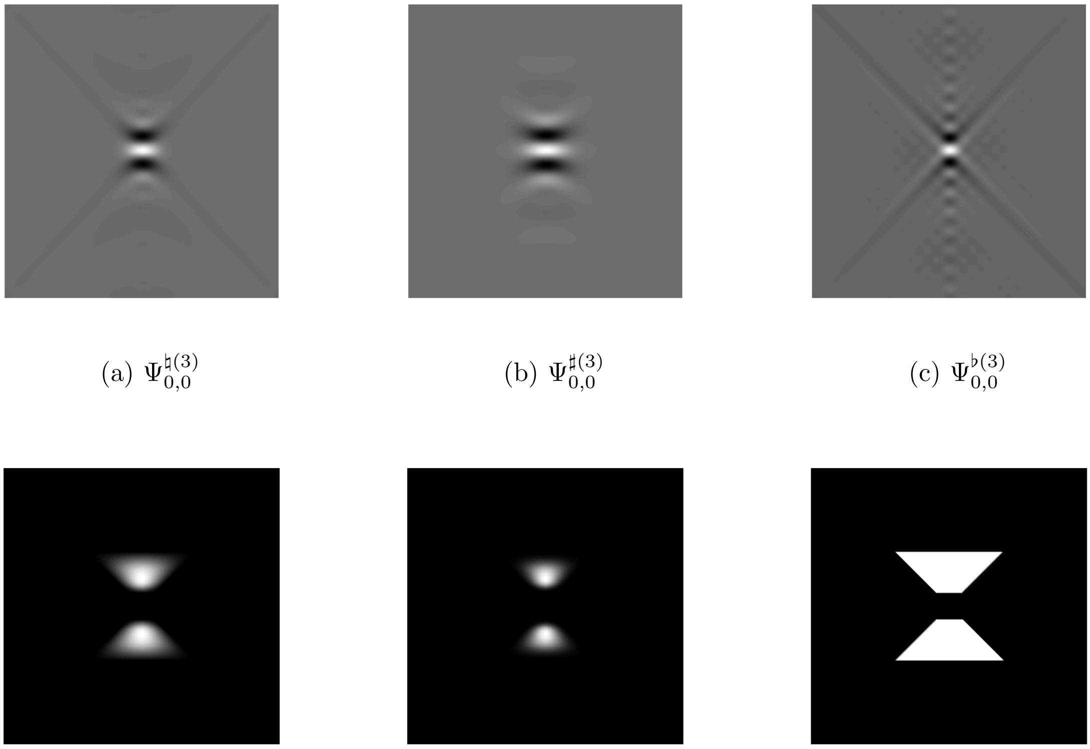
$$\tan\frac{\pi}{2^{N-1}}\sum_{j\in\mathbb{Z}}\sum_{1\le\ell\le2^{N-1}}2^{2j}|\hat{\Psi}^{\natural(N)}_{j,\ell}(\xi)|^2=1\quad\text{a.e. }\xi\in\mathbb{R}^2. \tag{25}$$

In addition, we define two functions:

$$\Psi^{\sharp(N)}_{j,\ell}(x)=\psi^{\sharp(N)}_{j,\ell}(x)+\psi^{\sharp(N)}_{j,\ell+2^{N-1}}(x),\qquad\hat{\psi}^{\sharp(N)}_{j,0}(\xi):=2^{-j}\sqrt{\cot\frac{\pi}{2^{N-1}}}\Lambda_j(\xi)\Gamma^{(N)}(\xi),$$

$$\Psi^{\flat(N)}_{j,\ell}(x)=\psi^{\flat(N)}_{j,\ell}(x)+\psi^{\flat(N)}_{j,\ell+2^{N-1}}(x),\qquad\hat{\psi}^{\flat(N)}_{j,0}(\xi):=2^{-j}\sqrt{\cot\frac{\pi}{2^{N-1}}}\mathbf{1}_{S^{\natural(N)}_j}(\xi),$$

where $S^{\natural(N)}_j=S^{(N-1)}_{j-1}\cup S^{(N-1)}_j$. Figure 15 shows $\Psi^{\natural(N)}_{j,\ell}$, $\Psi^{\sharp(N)}_{j,\ell}$ and $\Psi^{\flat(N)}_{j,\ell}$ with their Fourier transforms.



(a) $\Psi^{\natural(3)}_{0,0}$ (b) $\Psi^{\sharp(3)}_{0,0}$ (c) $\Psi^{\flat(3)}_{0,0}$

(d) $\hat{\Psi}^{\natural(3)}_{0,0}$ (e) $\hat{\Psi}^{\sharp(3)}_{0,0}$ (f) $\hat{\Psi}^{\flat(3)}_{0,0}$

Figure 15: The shapes of $\Psi^{\natural(N)}_{j,\ell}, \Psi^{\sharp(N)}_{j,\ell}$ and $\Psi^{\flat(N)}_{j,\ell}$ at $j=0$, $\ell=0$ and $N=3$.

Obviously, $\hat{\psi}^{\sharp(N)}_{j,0}$ is an almost C^2 regular piecewise rational function. Then, $\psi^{\sharp(N)}_{j,0}$ has a faster decay on the spatial domain. Since $|\hat{\Psi}^{\natural(N)}_{j,\ell}|^2 = \hat{\Psi}^{\flat(N)}_{j,\ell}\hat{\Psi}^{\sharp(N)}_{j,\ell}$, (25) can be replaced by

$$\tan\frac{\pi}{2^{N-1}}\sum_{j\in\mathbb{Z}}\sum_{1\le\ell\le 2^{N-1}} 2^{2j}\hat{\Psi}^{\flat(N)}_{j,\ell}(\xi)\hat{\Psi}^{\sharp(N)}_{j,\ell}(\xi) = 1 \quad \text{a.e. } \xi\in\mathbb{R}^2.$$

Step 3

We define the following three scaling functions:

$$\Phi^{\natural(N)}_{j,\ell} = \phi^{\natural(N)}_{j,\ell} + \phi^{\natural(N)}_{j,\ell+2^{N-1}},$$

$$\Phi^{\sharp(N)}_{j,\ell} = \phi^{\sharp(N)}_{j,\ell} + \phi^{\sharp(N)}_{j,\ell+2^{N-1}},$$

$$\Phi^{\flat(N)}_{j,\ell} = \phi^{\flat(N)}_{j,\ell} + \phi^{\flat(N)}_{j,\ell+2^{N-1}},$$

where

$$\hat{\phi}^{\natural(N)}_{j,0}(\xi) = 2^{-j}\sqrt{\cot\frac{\pi}{2^{N-1}}\tilde{\Lambda}_j(\xi)\Gamma^{(N)}(\xi)},$$

$$\hat{\phi}^{\sharp(N)}_{j,0}(\xi) = 2^{-j}\sqrt{\cot\frac{\pi}{2^{N-1}}}\tilde{\Lambda}_j(\xi)\Gamma^{(N)}(\xi),$$

$$\hat{\phi}^{\flat(N)}_{j,0}(\xi) = 2^{-j}\sqrt{\cot\frac{\pi}{2^{N-1}}}\mathbf{1}_{\tilde{S}^{\natural(N)}_j}(\xi)$$

with non-negative functions $\tilde{\Lambda}_j$ defined by

$$\tilde{\Lambda}_j(\xi) = \begin{cases} 1 - \int_0^{2^{2-j}\pi^{-1}\xi_2 - 2} \Delta(\tau) d\tau & \xi \in \mathbb{R} \times (0, \infty), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\operatorname{supp} \hat{\phi}_{j,0}^{\natural(N)} = \left[-\xi_2 \tan \frac{\pi}{2^{N-1}}, \xi_2 \tan \frac{\pi}{2^{N-1}}\right] \times [0, 2^j \pi] =: \tilde{S}_j^{\natural(N)} = \tilde{S}_j^{(N-1)}.$$

Thus, we get the following.

Theorem 2.26 (Parseval frame)**.** *Let $N \geq 3$, $J \in \mathbb{Z}$ and $k'' = \left(k_1 \cot \frac{\pi}{2^{N-1}}, k_2\right)$. Then, $f \in L^2(\mathbb{R}_x^2)$ can be expanded as*

$$\begin{aligned} f(x) \quad = \quad & \sum_{j \geq J+1} \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \alpha_{j,\ell,k}^{\natural} \Psi_{j,\ell}^{\natural(N)} \left(x - 2^{-j} R_{-\ell} k''\right) \\ & \qquad + \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \beta_{J,\ell,k}^{\natural} \Phi_{J,\ell}^{\natural(N)} \left(x - 2^{-J} R_{-\ell} k''\right), \end{aligned}$$

where

$$\alpha_{j,\ell,k}^{\natural} = \int_{\mathbb{R}_x^2} f(x) \Psi_{j,\ell}^{\natural(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx, \quad \beta_{j,\ell,k}^{\natural} = \int_{\mathbb{R}_x^2} f(x) \Phi_{j,\ell}^{\natural(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx.$$

We also get the following result about the couple of $\{\Psi_{j,\ell}^{\sharp(N)}\}$ and $\{\Psi_{j,\ell}^{\flat(N)}\}$.

Theorem 2.27 (Non-Parseval frame)**.** *Let $N \geq 3$, $J \in \mathbb{Z}$ and $k'' = \left(k_1 \cot \frac{\pi}{2^{N-1}}, k_2\right)$. Then, $f \in L^2(\mathbb{R}_x^2)$ can be expanded as*

$$f(x) = \begin{cases} \displaystyle\sum_{j \geq J+1} \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \alpha_{j,\ell,k}^{\flat} \Psi_{j,\ell}^{\sharp(N)} \left(x - 2^{-j} R_{-\ell} k''\right) \\ \qquad \displaystyle + \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \beta_{J,\ell,k}^{\flat} \Phi_{J,\ell}^{\sharp(N)} \left(x - 2^{-J} R_{-\ell} k''\right) & (\textit{Type I}), \\ \displaystyle\sum_{j \geq J+1} \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \alpha_{j,\ell,k}^{\sharp} \Psi_{j,\ell}^{\flat(N)} \left(x - 2^{-j} R_{-\ell} k''\right) \\ \qquad \displaystyle + \sum_{1 \leq \ell \leq 2^{N-1}} \sum_{k \in \mathbb{Z}^2} \beta_{J,\ell,k}^{\sharp} \Phi_{J,\ell}^{\flat(N)} \left(x - 2^{-J} R_{-\ell} k''\right) & (\textit{Type II}), \end{cases}$$

where

$$\begin{aligned} \alpha_{j,\ell,k}^{\flat} &= \int_{\mathbb{R}_x^2} f(x) \Psi_{j,\ell}^{\flat(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx, \qquad \beta_{j,\ell,k}^{\flat} = \int_{\mathbb{R}_x^2} f(x) \Phi_{j,\ell}^{\flat(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx, \\ \alpha_{j,\ell,k}^{\sharp} &= \int_{\mathbb{R}_x^2} f(x) \Psi_{j,\ell}^{\sharp(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx, \qquad \beta_{j,\ell,k}^{\sharp} = \int_{\mathbb{R}_x^2} f(x) \Phi_{j,\ell}^{\sharp(N)} \left(x - 2^{-j} R_{-\ell} k''\right) dx. \end{aligned}$$

For the proof of Theorem 2.27, see [18].

2.4.2 Numerical simulations

In this section, we compare the quality of four types of expansions in Theorems 2.25, 2.26 and 2.27, which we denote by Normal, Natural, Type I and II, respectively . We set $J = 0$ for all of the numerical experiments, and the highest resolution level is denoted by $j_{\max}$ with respect to the sum for $j \geq J+1$.

Now, we consider the function of pyramid form defined by

$$f(x_1, x_2) = \max\{1 - |x_1| - |x_2|, 0\}.$$

Figure 16 shows expanded f with our four types of reconstruction formulas when $j_{\max} = 3$ and $N = 3$, $N = 4$ and $N = 5$.

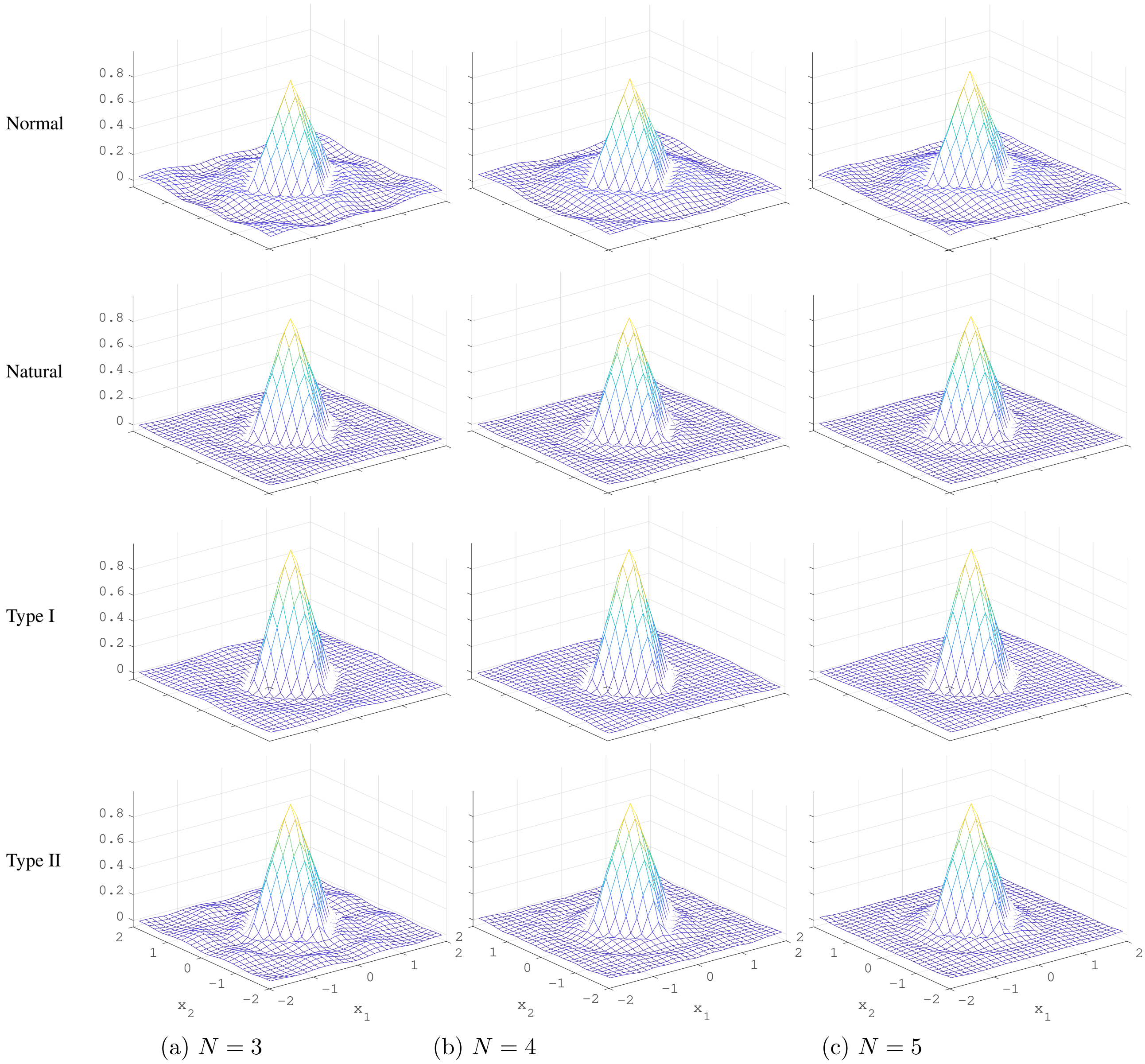


Figure 16: Frame expansions of f with $|k| \leq 2^{j+1}$.

As can be seen from Figure 16, the reconstruction quality improves as the number of orientations N increases. Furthermore, in the graphs of Normal, some major oscillations occur outside the support of f, which should be equal to 0. The same oscillations can also be seen for the case of the other reconstructions, but their appearance is very low. This result is considered to be due to the smoothness of the frames.

In [18], we also considered frames interpolating between Lipschitz continuity and C^∞ and presented more detailed numerical simulations.

3 Radon transform

The Radon transform was introduced by J. Radon in order to reconstruct a differentiable function on $\mathbb{R}^3$ by means of its surface integrals over planes. Surprisingly, A. M. Cormack and G. N. Hounsfield were awarded the Nobel Prize in Physiology or Medicine in 1979 for the development of computed tomography (CT) by applying the Radon transform. This news has taught us the importance of the mutual development of mathematics and engineering.

3.1 Radon transform on Euclidean space

Following [22], we denote the space of all hyperplanes in $\mathbb{R}^n$ by $\mathbb{P}^n$ (i.e. $\dim \xi = n-1$, $\xi \in \mathbb{P}^n$).

Definition 3.1 (Radon transform)**.** Let f be a function on $\mathbb{R}^n$ belonging to a suitable function space. We define the Radon transform $\mathcal{R}$ on a hyperplane ξ by

$$\mathcal{R}[f](\xi) = \int_\xi f(x)dm(x), \quad \xi \in \mathbb{P}^n,$$

where m is the Euclidean measure on ξ.

Now, we define a formal inverse of $\mathcal{R}$ in advance.

Definition 3.2 (Dual Radon transform)**.** Let φ be a function on $\mathbb{P}^n$ belonging to a suitable function space. We define the dual Radon transform $\mathcal{R}^*$ by

$$\mathcal{R}^*[\varphi](x) = \int_{x\in\xi} \varphi(\xi)d\mu(\xi), \quad x \in \mathbb{R}^n,$$

where μ is the normalized measure on the compact set $\{\xi \in \mathbb{P}^n : x \in \xi\}$.

Since each hyperplane $\xi \in \mathbb{P}^n$ can be written as

$$\xi \coloneqq (\omega, p) = \{x \in \mathbb{R} : x \cdot \omega = p,\ \omega \in S^{n-1},\ p \in \mathbb{R}\},$$

a function φ on $\mathbb{P}^n$ can be regarded as a function on $S^{n-1} \times \mathbb{R}$ satisfying $\varphi(\omega, p) = \varphi(-\omega, -p)$. Then, the Radon transform and the dual Radon transform can be written explicitly as

$$\mathcal{R}[f](\omega, p) = \int_{x\cdot\omega=p} f(x)dm(x),$$

$$\mathcal{R}^*[\varphi](x) = \frac{1}{\Omega_n}\int_{S^{n-1}} \varphi(\omega, x\cdot\omega)d\omega,$$

where Ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ given by $2\pi^{n/2}/\Gamma(n/2)$ for the normalization. The following well-known formula holds between the Fourier transform and the Radon transform.

Theorem 3.3 (Fourier slice theorem)**.** *If $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathcal{F}[f](s\omega) = \mathcal{F}_{p\to s}[\mathcal{R}[f](\omega, p)](s).$$

Proof. By the Fourier transform, we obtain

$$\mathcal{F}_{p\to s}[\mathcal{R}[f](\omega, p)](s) = \int_{-\infty}^{\infty} \mathcal{R}[f](\omega, p)e^{-ips}dp.$$

Thus,

$$\begin{aligned}
\mathcal{F}_{p\to s}[\mathcal{R}[f](\omega, p)](s) &= \int_{-\infty}^{\infty} \left\{\int_{x\cdot\omega=p} f(x)dm(x)\right\} e^{-ips}dp \\
&= \int_{-\infty}^{\infty}\int_{x\cdot\omega=p} f(x)e^{-ips}dm(x)dp \\
&= \int_{\mathbb{R}^n} f(x)e^{-is\langle x,\omega\rangle}dx \\
&= \mathcal{F}[f](s\omega).
\end{aligned}$$

□

For the Radon transform of f, we can recover f in a sense by using the Fourier slice theorem, but the standard reconstruction formula is the next theorem.

Theorem 3.4 (Inversion formula)**.** *If $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$f = \frac{1}{C_n}(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*[\mathcal{R}[f]],$$

where C_n is the constant defined by

$$C_n = (4\pi)^{\frac{n-1}{2}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Remark 3.5. For the Laplacian on Euclidean space

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

we define the fractional Laplacian by

$$(-\Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}[|\cdot|^s\mathcal{F}[f]].$$

We also define a (singular) integral operator

$$I^s[f](x) = \frac{1}{H_n(s)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}dy, \quad H_n(s) = \frac{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)},$$

which is called the Riesz potential. Then, we have formally $(-\Delta)^{\frac{s}{2}} = I^{-s}$. Thus, I^s can be regarded as an inverse operator of $(-\Delta)^{\frac{s}{2}}$ under suitable conditions. The proof of Theorem 3.4 uses the following result:

$$\text{For } f \in \mathcal{S}(\mathbb{R}^n), \qquad I^{-k}[I^k[f]] = f, \quad (0 < k < n).$$

For more detailed descriptions of the Riesz potential, see e.g. [22, 23, 24].

Proof sketch of Theorem 3.4 Let $O(n)$ be the orthogonal group in $\mathbb{R}^n$ and let dk be its normalized Haar measure, then the dual Radon transform can also be written as

$$\mathcal{R}^*[\varphi](x) = \int_{O(n)}\varphi(x + k\cdot\xi_0)dk,$$

where ξ_0 is a fixed hyperplane through the origin and dot $\cdot$ denotes the group action of $O(n)$ on $\mathbb{P}^n$. Using this notation, we have

$$\mathcal{R}^*[\mathcal{R}[f]](x) = \int_{O(n)}\left\{\int_{\xi_0} f(x+k\cdot y)dm(y)\right\}dk = \int_{\xi_0}\left\{\int_{O(n)} f(x+k\cdot y)dk\right\}dm(y).$$

Using polar coordinates, we observe the relation

$$\int_{O(n)} f(kx)dk = \frac{1}{\Omega_n}\int_{S^{n-1}} f(r\omega)d\omega.$$

Then,

$$\mathcal{R}^*[\mathcal{R}[f]](x) = \Omega_n\int_0^\infty\left\{\frac{1}{\Omega_n}\int_{S^{n-1}} f(x+r\omega)d\omega\right\}r^{n-2}dr = \frac{\Omega_{n-1}}{\Omega_n}\int_{\mathbb{R}}\frac{f(y)}{|x-y|}dy. \tag{26}$$

Next, the integral on the far right-hand side of (26) can be represented as the Riesz potential

$$\frac{\Omega_{n-1}}{\Omega_n}\int_{\mathbb{R}}\frac{f(y)}{|x-y|}dy = \frac{H_n(n-1)\Omega_{n-1}}{\Omega_n}I^{n-1}f(x) = 2^{n-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)I^{n-1}f(x).$$

Taking the fractional Laplacian $(-\Delta)^{\frac{n-1}{2}}$,

$$(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*[\mathcal{R}[f]](x) = 2^{n-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)(-\Delta)^{\frac{n-1}{2}}(-\Delta)^{-\frac{n-1}{2}}f(x) = 2^{n-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)f(x)$$

holds for $f \in \mathcal{S}(\mathbb{R}^n)$.

$\square$

For more details about the Radon transform, see e.g. [22, 23, 24, 32].

3.2 Wavelet-like orthonormal basis and its application to two-dimensional Radon transforms

There are several known ways to define the CWT on $\mathbb{R}^n$ $(n \geq 2)$. The standard method is to replace the $ax+b$ group by the similitude group $SIM(n)$ defined by

$$SIM(n) = \mathbb{R}^n \rtimes (\mathbb{R}_{>0} \times SO(n)).$$

Then, the unitary irreducible representation in $L^2(\mathbb{R}^n)$

$$[U(b,a,\rho)\psi](x) = a^{-\frac{n}{2}}\psi(a^{-1}\rho^{-1}(x-b))$$

is square integrable and

$$\int_{\mathbb{R}^n} \frac{|\hat{\psi}(\xi)|^2}{|\xi|^n} d\xi < \infty$$

is the admissibility condition, up to some constant (see [1, 2]). In dimension $n = 2$, the CWT is a mathematical tool for image processing. However, the CWT is not a perfect tool for capturing all features of an image. To detect line singularities which the CWT failed to capture, the ridgelet mentioned in Sec. 2.4 was proposed by Candès [7]. Let ψ be a function on $\mathbb{R}^2$ belonging to a suitable function space (e.g. $\psi \in \mathcal{S}(\mathbb{R})$). Then, ridgelets are defined by

$$\psi_{a\theta b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{u_\theta \cdot x - b}{a}\right), \quad a \in \mathbb{R}_{>0},\ u_\theta = (\cos\theta, \sin\theta)^t \in S^1,\ b \in \mathbb{R},$$

and we define the continuous ridgelet transform (CRT) of f as follows:

$$\mathcal{R}_\psi[f](a,\theta,b) = \langle f, \psi_{a\theta b}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{a\theta b}(x)}dx.$$

For more detailed descriptions of the CRT, see e.g. [2, 7, 8, 9].

The CRT is the combination of the Radon transform on $\mathbb{R}^2$ and the CWT on $\mathbb{R}$. In fact, since the Radon transform can be written as

$$\mathcal{R}[f](\omega,p) = \int_{\mathbb{R}^2} f(x)\delta(x\cdot\omega - p)dx$$

in the distribution sense, we see that

$$\begin{aligned}
W_\psi\left[\mathcal{R}[f](u_\theta,\cdot)\right](a,b) &= \int_{\mathbb{R}} \left\{\int_{\mathbb{R}^2} f(x)\delta(x\cdot u_\theta - p)dx\right\} \overline{\frac{1}{\sqrt{a}}\psi\left(\frac{p-b}{a}\right)}dp \\
&= \int_{\mathbb{R}^2} f(x)\left\{\int_{\mathbb{R}} \overline{\frac{1}{\sqrt{a}}\psi\left(\frac{p-b}{a}\right)}\delta(x\cdot u_\theta - p)dp\right\}dx \\
&= \int_{\mathbb{R}^2} f(x)\overline{\frac{1}{\sqrt{a}}\psi\left(\frac{u_\theta\cdot x - b}{a}\right)}dx \\
&= \mathcal{R}_\psi[f](a,\theta,b)
\end{aligned}$$

by formal manipulation. Therefore, the CRT inherits the properties of the Radon transform and the CWT. In [17], we presented a wavelet-like orthonormal basis of $L^2(\mathbb{R}^2)$ with an H^2-wavelet and the Radon transform.

Remark 3.6. In [17], we denoted the Radon transform of g by

$$R[g](\theta,t) = \int_{\theta^\perp} g(t\theta + u)d_\theta u,$$

where $\theta^{\perp} = \{x : x \cdot \theta = 0\}$ is the hyperplane orthogonal to $\theta = \theta_\tau = (\cos\tau, \sin\tau)^t \in S^1$ and passing through the origin, and d_θ is the Euclidean measure on $\theta^{\perp}$. Then, the dual Radon transform R^* is defined only for f on $S^1 \times \mathbb{R}$ satisfying

$$f(\theta, t) = f(-\theta, -t) \quad \text{for } \theta \in S^1,\ t \in \mathbb{R}.$$

However, for F on $\mathbb{R}_\tau \times \mathbb{R}_t$ which is 2π-periodic in τ, we distinguish it from R^* by using $\mathcal{R}^*$. In this section, we unify these two definitions of the dual Radon transform into $\mathcal{R}^*$ without distinguishing between them.

We denote the Sobolev space of order $\sigma \in \mathbb{R}$ by $L^2_\sigma(\mathbb{R}^2)$. The following is the main theorem.

Theorem 3.7. *Let ψ be a continuous H^2-wavelet such that*

$$\hat{\psi} \in L^1(\mathbb{R}) \quad \textit{and} \quad s^{-1}|\hat{\psi}(s)|^2 \in L^1(\mathbb{R}_s).$$

Define

$$B_{\ell,j,k}(x_1, x_2) = (-\Delta_x)^{1/4}\mathcal{R}^*[e^{i\ell\tau}\psi_{j,k}(t)](x_1, x_2).$$

Then, $B_{\ell,j,k} \in L^2(\mathbb{R}^2)$ and $\{B_{\ell,j,k}\}_{(\ell,j,k)\in\mathbb{Z}^3}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$, that is, for any $g \in L^2(\mathbb{R}^2)$,

$$g = \sum_{(\ell,j,k)\in\mathbb{Z}^3} \langle g, B_{\ell,j,k}\rangle_{L^2(\mathbb{R}^2)} B_{\ell,j,k}$$

with convergence in the $L^2(\mathbb{R}^2)$-norm.

3.2.1 Proof of Theorem 3.7

In this section, we provide an outline of the proof of Theorem 3.7 according to [17]. We first prove the following.

Lemma 3.8. *Let ψ be a continuous H^2-wavelet such that*

$$\hat{\psi} \in L^1(\mathbb{R}) \quad \textit{and} \quad s^{-1}|\hat{\psi}(s)|^2 \in L^1(\mathbb{R}_s).$$

Define $F(\tau, t) = e^{i\ell\tau}\psi_{j,k}(t)$. Then, $\mathcal{R}^*[F] \in L^2_{1/2}(\mathbb{R}^2)$ can be represented as

$$\mathcal{R}^*[F](x) = \mathcal{F}^{-1}_{\xi\to x}\left[|\xi|^{-1}\left(\frac{\xi_1 + i\xi_2}{|\xi|}\right)^{\ell}\left\{\mathcal{F}[\psi_{j,k}](|\xi|) + (-1)^{\ell}\mathcal{F}[\psi_{j,k}](-|\xi|)\right\}\right](x).$$

Since $\mathcal{R}^*[F] \in L^2_{1/2}(\mathbb{R}^2)$ from Lemma 3.8, we obtain

$$(-\Delta_x)^{1/4}\mathcal{R}^*[F] = \mathcal{F}^{-1}_{\xi\to x}\left[|\xi|^{-1/2}\left(\frac{\xi_1 + i\xi_2}{|\xi|}\right)^{\ell}\left\{\mathcal{F}[\psi_{j,k}](|\xi|) + (-1)^{\ell}\mathcal{F}[\psi_{j,k}](-|\xi|)\right\}\right] \in L^2(\mathbb{R}^2).$$

Next, we prove the following.

Proposition 3.9. *Let ψ be a continuous H^2-wavelet such that*

$$\hat{\psi} \in L^1(\mathbb{R}) \quad \textit{and} \quad s^{-1}|\hat{\psi}(s)|^2 \in L^1(\mathbb{R}_s).$$

Define $F(\tau, t) = e^{i\ell\tau}\psi_{j,k}(t)$, $\tilde{F}(\tau, t) = e^{i\tilde{\ell}\tau}\psi_{\tilde{j},\tilde{k}}(t)$. Then, $\mathcal{R}^*[F], \mathcal{R}^*[\tilde{F}] \in L^2_{1/2}(\mathbb{R}^2)$ satisfies

$$\left\langle(-\Delta_x)^{1/4}\mathcal{R}^*[F], (-\Delta_x)^{1/4}\mathcal{R}^*[\tilde{F}]\right\rangle_{L^2(\mathbb{R}^2_x)} = \left\langle\frac{F(\tau, t) + F(\tau + \pi, -t)}{2\pi}, \tilde{F}\right\rangle_{L^2((-\pi,\pi)\times\mathbb{R}_t)}.$$

From Proposition 3.9, we can see that $\{B_{\ell,j,k}\}_{(\ell,j,k)\in\mathbb{Z}^3}$ is an orthonormal system. In fact, putting

$$F(\tau,t)=e^{i\ell\tau}\psi_{j,k}(t),\qquad \tilde{F}(\tau,t)=e^{i\tilde{\ell}\tau}\psi_{\tilde{j},\tilde{k}}(t),$$

we get

$$\begin{aligned}
\left\langle B_{\ell,j,k},B_{\tilde{\ell},\tilde{j},\tilde{k}}\right\rangle_{L^2(\mathbb{R}^2)} &= \left\langle(-\Delta_x)^{1/4}\mathcal{R}^*[F],(-\Delta_x)^{1/4}\mathcal{R}^*[\tilde{F}]\right\rangle_{L^2(\mathbb{R}^2_x)}\\
&=\frac{1}{2\pi}\left\langle F(\tau,t)+F(\tau+\pi,-t),\tilde{F}\right\rangle_{L^2((-\pi,\pi)\times\mathbb{R}_t)}\\
&=\frac{1}{2\pi}\left\langle e^{i\ell\tau}\psi_{j,k}(t)+e^{i\ell(\tau+\pi)}\psi_{j,k}(-t),e^{i\tilde{\ell}\tau}\psi_{\tilde{j},\tilde{k}}(t)\right\rangle_{L^2((-\pi,\pi)\times\mathbb{R}_t)}\\
&=\frac{1}{2\pi}\left\langle e^{i\ell\tau},e^{i\tilde{\ell}\tau}\right\rangle_{L^2(-\pi,\pi)}\left\langle\psi_{j,k}(t)+(-1)^\ell\psi_{j,k}(-t),\psi_{\tilde{j},\tilde{k}}(t)\right\rangle_{L^2(\mathbb{R}_t)}\\
&=\delta_{\ell,\tilde{\ell}}\left\{\delta_{j,\tilde{j}}\delta_{k,\tilde{k}}+(-1)^\ell\left\langle\psi_{j,k}(-t),\psi_{\tilde{j},\tilde{k}}(t)\right\rangle_{L^2(\mathbb{R}_t)}\right\}.
\end{aligned}$$

Since $\psi\in H^2(\mathbb{R})$, it holds that

$$\left\langle\psi_{j,k}(-t),\psi_{\tilde{j},\tilde{k}}(t)\right\rangle_{L^2(\mathbb{R}_t)}=\frac{1}{2\pi}\left\langle 2^{-j/2}e^{i2^{-j}ks}\hat{\psi}(-2^{-j}s),2^{-\tilde{j}/2}e^{-i2^{-\tilde{j}}\tilde{k}s}\hat{\psi}(2^{-\tilde{j}}s)\right\rangle_{L^2(\mathbb{R}_s)}=0.$$

This fact enables us to prove

$$\left\langle B_{\ell,j,k},B_{\tilde{\ell},\tilde{j},\tilde{k}}\right\rangle_{L^2(\mathbb{R}^2)}=\delta_{\ell,\tilde{\ell}}\delta_{j,\tilde{j}}\delta_{k,\tilde{k}}.$$

We have already shown that $\{B_{\ell,j,k}\}_{(\ell,j,k)\in\mathbb{Z}^3}$ is an orthonormal system. Therefore, it is sufficient to show that $\{B_{\ell,j,k}\}_{(\ell,j,k)\in\mathbb{Z}^3}$ is a Parseval frame. For this purpose, we need the following:

$$\sum_{(\ell,j,k)\in\mathbb{Z}^3}\left|\langle g,B_{\ell,j,k}\rangle_{L^2(\mathbb{R}^2)}\right|^2=\|g\|^2_{L^2(\mathbb{R}^2_x)}. \tag{27}$$

For the proof of formula (27), see [17]. This completes the proof of Theorem 3.7.

In preprint [17], we considered Theorem 3.7 further and presented results for reconstructing the original function from the Radon transform and the CRT. Furthermore, we also confirmed that our results correctly reconstructed the original function by numerical simulations. In [15, 16], Donoho introduced the orthonormal ridgelet which provides an orthonormal basis of $L^2(\mathbb{R}^2)$. Our result seems similar to this; however, we use the H^2-wavelet for the radial direction and the Fourier basis for the angular direction on the frequency domain.

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