Saturated ideals over higher cardinals

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1 Introduction

Behind many researches of modern set theory is Gödel's program. Gödel proposed the study of set theory using large cardinal axioms. This proposal is known as Gödel's program and realized in some ways now. One is Foreman's generalized large cardinal. Our thesis is on Foreman's philosophy.

First, we recall the history of set theory and explain what Gödel's program means in our context. Cantor's continuum hypothesis (CH) is the statement of $2^{\aleph_0} = \aleph_1$. CH is known as Hilbert's first problem in 1900 and is an independent proposition from ZFC. The consistency of CH was shown by Gödel [23] in 1940. After Gödel's work, Cohen proved the consistency of $\neg$CH by discovering a method of forcing [3], [4] in 1963.

Gödel's proof of the consistency of CH consists of two parts. First, he introduced a new axiom $V = L$ and proved its consistency. Then, he proved that $V = L$ implies CH. $V = L$ is an axiom that claims the universe of set theory is minimum. In [24], he conjectured that some axiom, that claims the universe of set theory is maximum, implies the negation of CH. The author understands Gödel's program as an exploration of what this axiom is. In [24], large cardinal axioms seem to have been expected to become a central notion in the program.

On the other hand, Gödel pointed out that some large cardinal axioms (like inaccessible, Mahlo, weakly compact cardinals, ...) are not enough because these axioms are compatible with $V = L$. Thus, these axioms do not negate CH. He thought stronger axioms are needed. One of them is definitely a measurable cardinal.

In 1961, Scott [36] proved that the measurable cardinal axiom implies $V \neq L$. Therefore, a universe of set theory with a measurable cardinal is not minimum. It seems that we should study a model with a measurable cardinal. Now, it is known that measurable cardinal axiom does not decide the truth value of CH, though. This result was shown by Levy–Solovay [31] after Cohen's work. The author does not think that Gödel's program has been well-studied before Cohen's work. The breakthrough is definitely Cohen's forcing method.

In modern set theory, generic extensions of models with large cardinals are often considered. Typical examples can be found in Foreman–Magidor–Shelah's papers. In part I [12] and part II [13], they studied MM and saturated ideals, respectively. MM is a maximal form of forcing axiom for $\aleph_1$. A saturated ideal is an ideal I over $Z \subseteq \mathcal{P}(X)$ such that $\mathcal{P}(Z)/I$ has the $|X|^+$-c.c. The author calls these axioms "generic" large cardinal axioms. These axioms decide many classical independent propositions. For example, MM implies $2^{\aleph_0} = \aleph_2$. In the author's opinion, Gödel's program can be realized in the study of "generic" large cardinal axioms. One of the most successful examples of Gödel's program is probably a study of MM. It seems that MM is *a* solution of Gödel's program.

In [17, Section 11], Foreman proposed the study of generalized large cardinal axioms, based on part II rather than part I. These axioms assert the existence of generic elementary embeddings. Many large cardinals are the critical point $\mathrm{crit}(j)$, that is least cardinal α with $j(\alpha) > \alpha$, of an elementary embedding $j : V \to M$ with the conditions (Cl) and (W). Here, (Cl) and (W) describes how *closed* M is and *where* $\mathrm{crit}(j)$ is sent by j, respectively. For example, measurable cardinal is characterized as the critical point of an elementary embedding j. Foreman added a new parameter (F), which denotes a type of posets that force the existence of j. His generalized large cardinal axioms are defined by three parameters (Cl), (W), and (F). A generalized large cardinal is the critical point of some elementary embedding $j : V \to M$ which is definable in some generic extension $V[G]$, while a usual large cardinal is the critical point of some elementary embedding which is definable in V.

The existence of generalized large cardinals can be described by the existence of precipitous ideals. The Boolean algebra $\mathcal{P}(Z)/I$ defined by a precipitous ideal I over Z forces the existence of an elementary embedding with its domain V. The completeness of I and Z decide (Cl) and (W). Therefore we are interested in forcing properties of $\mathcal{P}(Z)/I$ to know (F). In the consistency proofs of MM and other "generic" large cardinal axioms, generalized large cardinals appear.

In [28], Kunen obtained a model in which $\aleph_1$ carries a saturated ideal I by using a huge cardinal. A saturated ideal is precipitous and that forces the critical point of $\dot{j}$ is $\aleph_1^V$. Kunen introduced the method of universal collapse. MM has not been extended to higher cardinals (like $\aleph_2, \aleph_3, \aleph_{\omega+1}$,...) yet. But the universal collapse was extended to higher cardinals and used widely. For every regular cardinals μ below a huge cardinal κ, there are generic extensions in which

- (Laver [29]) μ^+ carries a strongly saturated ideal.
- (Foreman–Laver [19]) μ^+ carries a centered ideal.
- (Foreman–Magidor–Shelah [13]) μ^+ carries a strongly layered ideal.
- (Eskew [9]) μ^+ carries a dense ideal.

For the definitions of these ideals, see Section 2. These properties are strengthenings of saturation. For saturated ideals, the following implications are known.

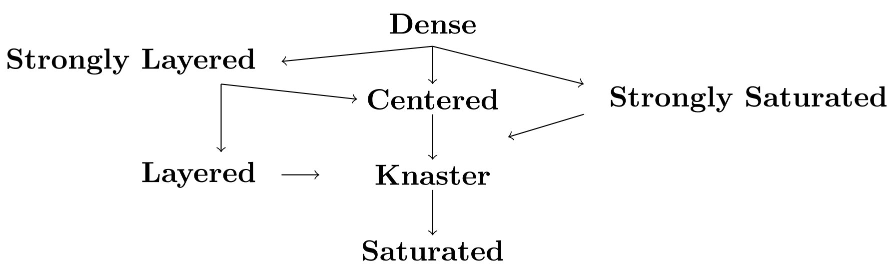


Figure 1: Saturation properties

The study of ideals over higher cardinals enables us to explore Gödel's program on higher cardinals. This thesis is a study of generalized large cardinal axioms on higher cardinals through saturated ideals. Thus, we focus saturation properties as (F).

All of the consistencies of each ideals over μ^+ (in the sense of Figure 1) are proved since dense ideal can exist over μ^+ for every regular μ. Then, we are interested in an ideal over μ^+ in the case of μ is singular. Eskew [10] pointed out that μ^+ cannot carry a dense ideal if μ is singular. Foreman proved that many kinds of Prikry-type forcings at μ forces μ^+ carries a saturated ideal $\overline{I}$ if μ^+ carries a saturated ideal I in the ground. Since Prikry-type forcings make μ into singular, we get a model in which μ^+ carries a saturated ideal for some singular μ. Moreover, This is the unique method to obtain a model in which a successor of a singular cardinal carries a saturated ideal, as far as the author know.

It is easy to see that, for a saturation property Ψ in Figure 1, if Prikry-type forcings force $\overline{I}$ is Ψ then I is Ψ in the ground. Therefore, Our interest is reduced to two problems.

(I) How strong saturation property can an ideal over a successor of a measurable cardinal have?

(II) How strong saturation property do Prikry-type forcings preserve?

For (I), to use Prikry-type forcings at μ, μ need to be measurable. Unfortunately, it is still open that a successor of a measurable cardinal can carry a dense ideal. Kunen, Laver, and Foreman–Laver's poset can make μ to measurable. So we have a model in which μ^+ carries a saturated ideal for some measurable μ. These posets are so interested.

These posets are simplified by Shioya [42]. He showed that some product forcings work as well, contrary to original ones are iterated forcings. We simplify Kunen's poset by Shioya's argument and Magidor's trick. We give a model in which μ^+ carries a saturated ideal and study the extent of saturation. Indeed,

Theorem 5.5 (Tsukuura [44]). *Suppose that j is an almost-huge embedding with critical point κ and $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals. Then $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that there is a saturated ideal I over $\mathcal{P}_\kappa\lambda$ with the following properties:*

1. *I is $(\lambda^+, \lambda^+, < \mu)$-saturated.*
2. *I is not (λ^+, μ, μ)-saturated. In particular, I is not strongly saturated.*
3. *I is layered if and only if $j(\kappa)$ is Mahlo in V.*
4. *I is not centered. In particular, I is not strongly layered.*

Here, $P(\mu,\kappa)$ is a diagonal product of Levy collapses. Note that we can regard an ideal over $\mathcal{P}_\kappa\kappa$ is an ideal over κ. Our investigations work for Kunen's original saturated ideal. We prove

Theorem 5.24 (Tsukuura [46]). *Suppose that $j : V \to M$ is a huge embedding with critical point κ and $f : \kappa \to \mathrm{Reg} \cap \kappa$ satisfies $j(f)(\kappa) \geq \kappa$. For regular cardinals $\mu < \kappa \leq \lambda = j(f)(\kappa) < j(\kappa)$, there is a P such that $P * \dot{S}(\lambda, j(\kappa))$ forces $\mu^+ = \kappa$ and $\lambda^+ = j(\kappa)$ and $\mathcal{P}_\kappa\lambda$ carries a saturated ideal I with the following properties:*

1. *I is $(\lambda^+, \lambda^+, < \mu)$-saturated.*
2. *I is not (λ^+, μ, μ)-saturated. In particular, I is not strongly saturated.*
3. *(Foreman–Magidor–Shelah [13]) I is layered.*
4. *(Foreman–Laver [19]) I is not centered. In particular, I is not strongly layered.*

If we put $f = \mathrm{id}$, $\lambda = \kappa$ and $\mu = \aleph_0$, then P and the ideal are the same as those in Kunen's theorem [28] for a saturated ideal. The negation of centeredness is claimed by Foreman–Laver [19] without proof. We showed explicitly. We also study the extent of saturation of ideals in [29], [42] and [41] in Theorems 5.27, and 5.30, 5.16, respectively.

For (II), we study the preservation of saturation properties via Prikry-type forcings. Foreman proved that, if P is μ-centered and I is a saturated ideal over μ^+ then P forces that the ideal $\overline{I}$, that is generated by I, is saturated. Many kinds of Prikry-type forcings are μ-centered. He also claimed that "saturated" in this can be replaced by "centered" without proof. We prove Foreman's claim in greater generality. We also study other saturation properties. We prove

Theorem 4.3 (Tsukuura; 1 and 3 are in [45]). *Suppose that $2^\mu = \mu^+$, μ is measurable, and U is a normal ultrafilter over μ. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If I is (λ^+, ν', ν')-saturated then $\mathcal{P}_U \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
2. *If I is λ-centered then $\mathcal{P}_U \Vdash \overline{I}$ is λ-centered.*
3. *If I is λ^+-productively saturated then $\mathcal{P}_U \Vdash \overline{I}$ is λ^+-productively saturated.*
4. *If $Z \subseteq \mathcal{P}_\kappa(X)$ and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda^+}_{\geq\mu^+}$.*
5. *If $Z \subseteq [X]^\kappa$, I is λ-dense, and λ is a successor cardinal then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda}_{\geq\mu^+}$.*

We also show the same thing holds if we replace Prikry forcing with Woodin's modification and Magidor forcing, respectively. For these posets, see Section 2.4.2. Theorems 4.6 and 4.7 are analogies of Theorem 4.3 for them. Productive saturation is in between usual saturation and Knasterness. Theorem 4.3 draws the following picture.

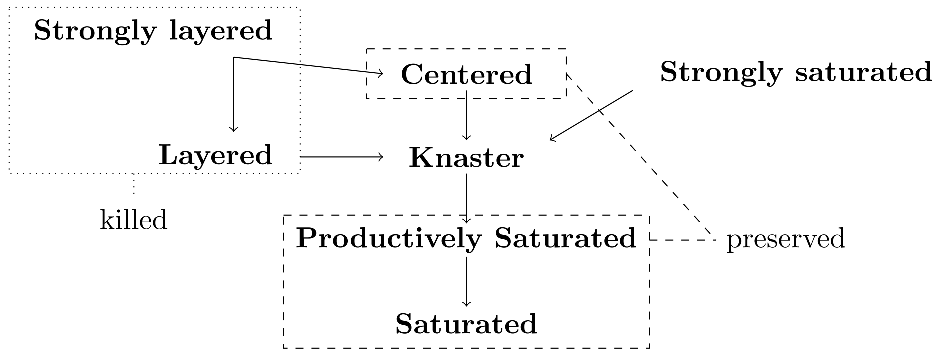


Figure 2. preservation of saturation properties via Prikry-type forcings

The preservation of Knasterness and strong saturation are still unknown. For a μ^+-c.c. poset P and a Knaster ideal I over μ^+, we introduce the $(\mu^{++}, \mu^{++}, 2)$-nice property for projections. By Theorem 3.14, we can show that $\overline{I}$ is forced to be Knaster if and only if some projection is $(\mu^{++}, \mu^{++}, 2)$-nice.

We treat applications of saturated ideals to combinatorics in this thesis. Historically, strengthenings of a saturated ideal were introduced on the context of combinatorics sometimes. For example, the notion of a strongly saturated ideal was introduced to obtain a model in which the polarized partition relation $\binom{\mu^{++}}{\mu^+} \to \binom{\mu^+}{\mu^+}_\mu$ holds. The first model of a centered ideal [19] also was used to show the consistency of a reflection principle of the chromatic number of graphs $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$. For the definitions of these properties, see Section 6. We study them in the extension by Prikry-type forcings. We will show the following theorems.

Theorem 6.19 (Tsukuura [45])**.** *Suppose that there is a supercompact cardinal below an almost-huge cardinal. Then there is a poset which forces that*

1. $\aleph_{\omega+1}$ *carries an ideal* I *that is centered but* not *layered, and*
2. I *is* $(\aleph_{\omega+2}, \aleph_n, \aleph_n)$*-saturated for all* $n < \omega$.
3. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \to \binom{\aleph_n}{\aleph_{\omega+1}}_{\aleph_\omega}$ *for all* $n < \omega$*, and,*
4. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \not\to \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}}_{\aleph_\omega}$.

Theorem 6.30 (Tsukuura [45])**.** *Suppose that a supercompact cardinal exists below a huge cardinal. Then there is a poset which forces that*

1. $[\aleph_{\omega+3}]^{\aleph_{\omega+1}}$ *carries a normal, fine,* $\aleph_{\omega+1}$*-complete* $\aleph_{\omega+2}$*-centered ideal.*
2. $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+3}, \aleph_{\omega+1})$.

The structure of this thesis is as follows:

- In Section 2, we recall basic materials for forcings, saturation properties, saturated ideals, and generic ultrapowers. We introduce an important theorem, that is known as Foreman's duality theorem [18], as Theorem 2.20. The duality theorem works as a central role when we study the saturation properties of $\overline{I}$ in some extension. We also introduce some posets and study them. One is variations of Levy collapses. The others are Prikry-type forcings, like Prikry forcing, Woodin's modification and Magidor forcing.
- In Section 3, we study saturation properties of certain posets. One is the term forcing. In section 5, we define many projections using the basic lemma of the term forcing. The other is the quotient forcing. By the duality theorem, our studies are reduced to saturation properties of some quotient forcing. We give sufficient conditions for the quotient forcings to have nice saturation properties. We also give examples that do not satisfy some nice saturation properties.

- In Section 4, we investigate problem (II). The first half of this section, we study Foreman's lemmas for the preservation of saturation and centeredness. By the duality theorem and the investigations in Section 3, we generalize it as Theorem 4.1. The rest is devoted to proofs of Theorems 4.3, 4.6, and 4.7.

- In Section 5, we study problem (I) through giving models with saturated ideals and studying the extent of saturation of these ideals. First, we present a method of giving a model with a saturated ideal using an almost-huge cardinal.

  We give a model in which $\mathcal{P}_\kappa\lambda$ carries a saturated ideal using the diagonal product of Levy collapses and study the extent of saturation of the ideal. We also give a model in which $[\lambda^+]^{\kappa^+}$ carries a saturated ideal. In the similar way, we also give a model in which $\mathcal{P}_\kappa\lambda$ carries a saturated ideal for all regular $\lambda \geq \kappa$.

  By the contents in Section 4, we study the extent of saturation of ideals over $\aleph_{\omega+1}$. To study, we need to know an ideal in the ground model. We adopt Shioya's model in which $\mathcal{P}_\kappa\lambda$ carries a centered ideal. First, we study the extent of saturation of the ideal. Then we apply Theorems, that were shown in Section 4, to Prikry-type forcings and the ideal.

  In addition, we give a model in which $\aleph_1$ carries a saturated ideal that is not Knaster and study the extent of saturation of ideals in [29], [42] and [41].

- In Section 6, we study the polarized partition relations and reflection principles for the chromatic number of graphs.

  We show that some of polarized partition relations follow from the existence of a saturated ideal or Chang's conjecture. We show the polarized partition relations in the extension by Prikry forcing. Here, we introduce Hajnal–Juhasz's Theorem. We use Hajnal–Juhasz's theorem in a model that is introduced in Section 5. Then we obtain a model in which $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ for all $\nu < \mu$ but $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$. Garti asked whether there is a model in which $\binom{\aleph_2}{\aleph_1} \to \binom{n}{\aleph_1}_{\aleph_0}$ for all $n < \omega$ but $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}_{\aleph_0}$ in [20, Question 1.11]. This question has been solved yet but our model is the simplest in them. We also give a model in which $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \to \binom{\aleph_n}{\aleph_{\omega+1}}_{\aleph_\omega}$ for all $n < \omega$ but $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \not\to \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}}_{\aleph_\omega}$.

  A coloring that is defined in Hajnal–Juhasz's theorem is probably the only known example of $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$. Foreman used this theorem when he show the mutually inconsistency of ideals. But it seems that his proof in [17, Corollary 5.38] does not work well. We introduce the notion of Hajnal–Juhasz coloring and study them. We give a proof of [17, Corollary 5.38] and improve it.

  We study reflection principles for the chromatic number of graphs. We show that the existence of a λ-centered ideal over $[\lambda^+]^{\kappa^+}$ implies $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \kappa^+)$. By Section 5, we have a model in which $[\aleph_{\omega+3}]^{\aleph_{\omega+1}}$ carries a $\aleph_{\omega+2}$-centered ideal, and thus, $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+3}, \aleph_{\omega+1})$ holds.

  The consistency of $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+2}, \aleph_{\omega+1})$ remains open. We introduce Erdős–Hajnal's graph $G(\lambda^+, \mu^+)$. It is known that $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$ holds if the chromatic number $\mathrm{Chr}(G(\mu^{++}, \mu^+))$ is less than μ^+. We show $\binom{\mu^{+++}}{\mu^{++}} \to \binom{2}{\mu^{++}}_{\mu^+}$ if $\mathrm{Chr}(G(\mu^{++}, \mu^+)) \leq \mu^+$. This improves Theorem 6.34, which is due to Erdős–Hajnal. We also evaluate the value of $\mathrm{Chr}(G(\mu^{++}, \mu^+))$ in some models by finding out a μ^{++}-Kurepa tree and ultrafilter D over μ^{++} with $|{}^{\mu^{++}}\mu/D| = \mu^{++}$.

2 Preliminaries

In this thesis, we use [26] as a reference for set theory in general.

2.1 Forcings and saturation properties

In this section, let us see basic facts of forcings and saturation properties. Let us make a list of our notations in this thesis.

- We use κ, λ to denote a regular cardinal unless otherwise stated. We also use μ and ν to denote an infinite cardinal and (possibly finite) cardinal unless otherwise stated, respectively.
- For $\kappa < \lambda$, E^λ_κ, $E^\lambda_{>\kappa}$ and $E^\lambda_{\leq\kappa}$ denote the set of all ordinals below λ of cofinality κ, $> \kappa$ and $\leq \kappa$, respectively.
- By Reg and ON, we mean the class of regular cardinals and the class of ordinals, respectively.
- We write $[\kappa, \lambda) = \{\xi \in \mathrm{ON} \mid \kappa \leq \xi < \lambda\}$.
- For regular θ, $\mathcal{H}_\theta$ is the set of all x with $|\mathrm{trcl}(x)| < \theta$. Here, $\mathrm{trcl}(x)$ is $x \cup (\bigcup x) \cup (\bigcup\bigcup x) \cup \cdots = \bigcup_n \bigcup^n x$.
- By P, Q, and R, we denote posets unless otherwise stated. We write like 1_P and $\leq_P$ for a maximal element and an order relation of P, respectively. We often omit subscripts from these if it is clear from the context.

Throughout this thesis, we identify a poset P with its separative quotient. Thus, $p \leq q \leftrightarrow \forall r \leq p(r \| q) \leftrightarrow p \Vdash q \in \dot{G}$, where $\dot{G}$ is the canonical name of a (V, P)-generic filter.

- A projection $\pi : Q \to P$ is an order-preserving mapping with the following properties:
  - $q \leq_P \pi(p)$ implies $\exists r \leq_Q p(\pi(r) \leq_P q)$.
  - $\pi(1_Q) = 1_P$.
- A complete embedding $\tau : P \to Q$ is an order preserving mapping with the following property:
  - For every maximal anti-chain $\mathcal{A} \subseteq P$, $\tau``\mathcal{A}$ is a maximal anti-chain in Q.
  - $\tau(1_P) = 1_Q$.
- If the inclusion mapping $i : P \to Q$ is complete then we say that P is a complete suborder of Q and write $P \lessdot Q$.
- For a poset P, we denotes its completion by $\mathcal{B}(P)$. That is, $\mathcal{B}(P)$ is a complete Boolean algebra and P is a dense subset and a complete suborder of $\mathcal{B}(P) \setminus \{0\}$. $\mathcal{B}(P)$ is unique up to isomorphism.
- If $\mathcal{B}(P) \simeq \mathcal{B}(Q)$ in the sense of Boolean algebra then We say that P and Q are forcing equivalent and write $P \simeq Q$.
- For $X \subseteq P$, by $\prod X$ and $\sum X$, we mean the least upper bound and the greatest lower bound of X, respectively. For $p, q \in P$, $p \cdot q = \prod\{p, q\}$ and $p + q = \sum\{p, q\}$.
- We say that P is well-met if $\prod X \in P$ for all $X \subseteq P$ with $\prod X \neq 0$ in $\mathcal{B}(P)$. Every Boolean algebra is well-met.
- P is $< \nu$-Baire if P adds no new sequences of length $< \mu$.

Lemma 2.1. *For an order preserving mapping $\tau : P \to Q$ with the conditions of $\tau(1_P) = 1_Q$ and $p\|_{\mathrm{P}} q \to \tau(p)\|_{\mathrm{Q}}\tau(q)$, the following are equivalent:*

1. *$\tau : P \to Q$ is a complete embedding.*

2. *For every $q \in Q$, there is a $p \in P$ such that $\tau(r) \parallel_Q q$ for every $r \leq_P p$.*

We call p of 2 a reduct of q in P. Thus, $\tau : P \to Q$ is complete if and only if every $q \in Q$ has a deduct in P.

Lemma 2.2. *The following are equivalent:*

1. *There is a complete embedding $\tau : P \to \mathcal{B}(Q)$.*

2. *There is a projection $\pi : Q \to \mathcal{B}(P)$.*

Proof. For a given complete embedding $\tau : P \to \mathcal{B}(Q)$, let $\pi(q) = \sum\{p \in P \mid p \text{ is a reduct of } q\}$. By Lemma 2.1, this set is non-empty. By the completeness of $\mathcal{B}(P)$, $\pi(q) \in \mathcal{B}(P)$. It is easy to see that π is a projection.

Conversely, for a given projection $\pi : Q \to \mathcal{B}(P)$, let $\tau(p) = \sum\{q \mid \pi(q) \leq p\}$. By the completeness of $\mathcal{B}(P)$, $\tau(p) \in \mathcal{B}(P)$. It is easy to see that $\pi(q)$ is a reduct of q in the sense of τ for every $q \in Q$. □

Note that, in the proof of Lemma 2.2, the completeness of $\mathcal{B}(Q)$ and $\mathcal{B}(P)$ are not used to define π and τ, respectively. Therefore, for a given complete embedding (resp. projection) from P to Q, we can always define a projection (resp. complete embedding) from Q to $\mathcal{B}(P)$. Note that $\pi(\tau(p)) = p$ and $\tau(\pi(q)) \geq q$ for all $p \in P$ and $q \in Q$.

The following lemma is often used in computations of Boolean values.

Lemma 2.3. *If $\tau : P \to Q$ is a complete embedding between complete Boolean algebras, then the following holds:*

1. *For every $A \subseteq P$, $\tau(\prod A) = \prod \tau``A$.*

2. *If τ is defined by a projection $\pi : Q \to P$, $\pi(\tau(p) \cdot q) = p \cdot \pi(q)$.*

Proof. 1. It is easy to see $\tau(\prod A) \leq \prod \tau``A$. Let us see $\prod \tau``A \leq \tau(\prod A)$. By separativity, it suffices to show that $q \cdot \tau(\prod A) \neq 0$ for all $q \leq \prod \tau``A$. Let $p \in P$ be a reduct of q. For every $r \leq_P p$ and $a \in A$, $\tau(r) \cdot \tau(a) \geq \tau(r) \cdot q \neq 0$. Thus, $r \cdot a \neq 0$ for all $r \leq_P p$, especially $p \leq a$. Therefore, $p \leq \prod A$ and $q \cdot \tau(\prod A) \geq q \cdot \tau(p) \neq 0$.

2. Observe that $\pi(\tau(p) \cdot q) \leq \pi(\tau(p)) \cdot \pi(q) = p \cdot \pi(q)$. To show $p \cdot \pi(q) \leq \pi(\tau(p) \cdot q)$, we check $\forall r \leq \pi(q) \cdot p(r \cdot (p \cdot \pi(q)) \neq 0)$. For any $r \leq \pi(q) \cdot p$, there is an $s \leq q$ with $\pi(s) \leq r$. By $q \cdot \tau(p) = q \cdot \sum\{x \mid \pi(x) \leq p\} = \sum\{q \cdot x \mid \pi(x) \leq p\}$, $s = q \cdot s \leq q \cdot \tau(p)$. Therefore $\pi(s) \leq \pi(q \cdot \tau(p)) \cdot r$. □

For a subset $F \subseteq P$, P/F is a suborder $\{p \in P \mid \forall q \in F(q \not\parallel p)\}$ of P. Note that P/F is not separative in general. Since we identify posets with its separative quotient, P/F is some quotient algebra. For a later purpose, we demonstrate what this order is in the case of P is a Boolean algebra and F is a filter. Let I be the dual ideal of F. Then $P/F = P \setminus I$. Indeed,

$$p \notin P/F \Leftrightarrow \exists r \in F(r \cdot p = 0) \Leftrightarrow \exists r' \in I(p \leq r') \Leftrightarrow p \in I.$$

For $p, q \in P/F$, we have the following translation.

$$\begin{aligned} p \leq_{P/F} q &\Leftrightarrow \forall r \in P \setminus I(r \leq_P q \to r\|_{P/F} q) \\ &\Leftrightarrow \forall r \in P \setminus I(r \leq_P q \to r \cdot q \notin I). \end{aligned}$$

We will define a quotient algebra $\mathcal{P}(Z)/I$ by an ideal I over Z. This observation enables us to identify $\mathcal{P}(Z)/I$ with the separative quotient of $\mathcal{P}(Z) \setminus I$.

Lemma 2.4. *For a projection $\pi : Q \to P$,*

1. *For every dense subset D in P, $\pi^{-1}D$ is dense in Q. It follows that $Q \Vdash \pi``\dot{H}$ generates a (V, P)-generic filter, where $\dot{H}$ is the canonical name of (V, Q)-generic filter.*

2. *The mapping $\tau : Q \to P * Q/\pi``\dot{G}$ defined by $\tau(q) = \langle \pi(q), \hat{q} \rangle$ is a dense embedding. Here, $\hat{q}$ is a $P$-name with $P \Vdash \pi(q) \in \dot{G} \to \hat{q} = q$ and $\pi(q) \notin \dot{G} \to \hat{q} = 1$. In particular, $Q \simeq P * Q/\pi``\dot{G}$.*

We say that $Q/\pi``\dot{G}$ is quotient forcing of Q (by $\dot{G}$). We write $Q/\dot{G}$ if π is clear from the context.
Let us list definitions of saturation properties we will deal in this thesis.

- We say that P is $(\lambda, <\nu)$-centered if $P = \bigcup_{\alpha<\lambda} P_\alpha$ for some $<\nu$-centered subsets $P_\alpha \subseteq P$. A $<\nu$-centered subset is a $C \subseteq P$ such that Z has a lower bound in P for all $Z \in [C]^{<\nu}$. We call a sequence $\langle P_\alpha \mid \alpha < \lambda \rangle$ of centered subsets as above a centering family of P. By λ-centered, we mean $(\lambda, <\omega)$-centered.

- For a stationary subset $S \subseteq \lambda$, we say that P is S-layered if, for any sufficiently large regular θ, there is a club $C \subseteq [\mathcal{H}_\theta]^{<\lambda}$ such that $M \cap P \lessdot P$ for all $M \in C$ with $\sup(M \cap \lambda) \in S$.

- We say that P has the $(\lambda, \kappa, <\nu)$-c.c. if, for every $X \in [P]^\lambda$, there is a $Y \in [X]^\kappa$ such that Z has a lower bound in P for every $Z \in [Y]^{<\nu}$. By (λ, κ, ν)-c.c., we mean $(\lambda, \kappa, <\nu^+)$-c.c.

- We say that P is λ-dense if P has a dense subset of size λ.

Lemma 2.5. 1. *If P is λ-dense then P is $(\lambda, <\lambda^+)$-centered.*

2. *If P is $(\lambda, <\nu)$-centered then P has the $(\lambda^+, \lambda^+, <\nu)$-c.c.*

3. *If P is layered then P has the $(\lambda^+, \lambda^+, 2)$-c.c.*

4. *If P has the $(\lambda^+, \lambda^+, 2)$-c.c. then P has the λ^+-c.c.*

Proof. 1,2, and 4 are trivial. For 3, we refer to [5, Section 3.3]. □

The following are basic facts for centeredness.

Lemma 2.6. 1. *The following are equivalent:*

   (a) *P is $(\lambda, <\nu)$-centered.*
   (b) *There is a function $f : P \to \lambda$ such that $f^{-1}\{\alpha\}$ is a $<\nu$-centered subset for each $\alpha < \lambda$.*

2. *If P is λ-centered then $|P| \leq 2^\lambda$.*

3. *If P is well-met then the following are equivalent:*

   (a) *P is $(\lambda, <\nu)$-centered.*
   (b) *There is a sequence $\langle F_\alpha \mid \alpha < \lambda \rangle$ of $<\nu$-complete filters of P such that $\bigcup_{\alpha<\lambda} = P$.*

We call f in 1.(b) a centering function of P.

Proof. 1 and 3 are easy. For 2, let us define an injection from P to $\mathcal{P}(\lambda)$. For a centering family $\{P_\alpha \mid \alpha < \lambda\}$, a mapping $p \mapsto \{\alpha \mid p \in P_\alpha\}$ is an injection. □

The following lemma is a standard way to obtain the centeredness.

Lemma 2.7. *Suppose that $\lambda^{<\lambda} = \lambda$ and $\{P_\alpha \mid \alpha \in K\}$ is $(\lambda, <\nu)$-centered posets. If $|K| \leq 2^\lambda$ then $\prod^{<\lambda}_{\alpha\in K} P_\alpha$ is $(\lambda, <\nu)$-centered.*

Proof. Our proof is based on the proof in [19, Lemma 4]. For each $\alpha \in K$, let $F_\alpha : P_\alpha \to \lambda$ be a centering function, that is, $F_\alpha^{-1}\{\xi\}$ is a $< \nu$-centered subset of P_α for all $\xi < \lambda$. Let $D : K \to {}^\lambda 2$ be an injection.

For each $p \in \prod_{\alpha\in K}^{<\lambda} P_\alpha$, there is a $\delta < \lambda$ such that $D(\alpha) \restriction \delta \neq D(\beta) \restriction \delta$ for all $\alpha \neq \beta$ in $\mathrm{dom}(p)$. For $D(\alpha) \restriction \delta$ with $\alpha \in \mathrm{dom}(p)$, define a function J_p by $J_p(D(\alpha) \restriction \delta) = F_\alpha(p(\alpha))$. Note that $J_p \in \bigcup_{\delta<\lambda}\{{}^d\lambda \mid d \in [2^\delta]^{<\lambda}\}$. By $\lambda^{<\lambda} = \lambda$, $X = \bigcup_{\delta<\lambda}\{{}^d\lambda \mid d \in [2^\delta]^{<\lambda}\}$ is of size λ. For each $J \in X$, let $C_J = \{q \in \prod_{\alpha\in K}^{<\lambda} \mid J_q = J\}$. It is easy to see that each C_J is a $< \nu$-centered subset and $\bigcup_J C_J = \prod_{\alpha\in K}^{<\lambda} P_\alpha$. □

We will consider the S-layeredness of complete Boolean algebra P. Note that $M \cap P$ is a Boolean subalgebra of P but is not necessarily a complete Boolean subalgebra of P even if $M \cap P \lessdot P$.

Lemma 2.8. *For a stationary subset $S \subseteq \lambda$ and poset P of size $\leq \lambda$, the following are equivalent:*

1. *P is S-layered.*
2. *There is an $\subseteq$-increasing sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$ with the following properties:*
   (a) *$P = \bigcup_{\alpha<\lambda} P_\alpha$.*
   (b) *$P_\alpha \lessdot P$ and $|P_\alpha| < \lambda$ for all $\alpha < \lambda$.*
   (c) *There is a club $C \subseteq \lambda$ such that $\forall\alpha \in S \cap C(P_\alpha = \bigcup_{\beta<\alpha} P_\alpha)$.*
3. *There is an $\subseteq$-increasing continuous sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$ with the following properties:*
   (a) *$P = \bigcup_{\alpha<\lambda} P_\alpha$.*
   (b) *$P_\alpha \subseteq P$ and $|P_\alpha| < \lambda$ for all $\alpha < \lambda$.*
   (c) *There is a club $C \subseteq \lambda$ such that $\forall\alpha \in S \cap C(P_\alpha \lessdot P)$.*
4. *For every $\subseteq$-increasing continuous sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$, the following hold:*
   (a) *$P = \bigcup_{\alpha<\lambda} P_\alpha$.*
   (b) *$P_\alpha \subseteq P$ and $|P_\alpha| < \lambda$ for all $\alpha < \lambda$.*
   (c) *There is a club $C \subseteq \lambda$ such that $\forall\alpha \in S \cap C(P_\alpha \lessdot P)$.*

Proof. For the equivalence between 2 and 3, we refer to [44]. The equivalence of 3 and 4 is shown in [5]. It is easy to see that 1 and 3 are equivalent. □

We call a sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$ a filtration of P if it is $\subseteq$-increasing continuous and $\bigcup_\alpha P_\alpha = P$.

The original definition of S-layeredness of P by Shelah is 3. If we define the S-layeredness by 3. then $\mathcal{B}(P)$ is not necessarily S-layered even if P is. By our definition, the S-layeredness of P is equivalent with that of $\mathcal{B}(P)$.

Lemma 2.9. *Suppose that there is a complete embedding $\tau : P \to Q$.*

1. *If Q has the $(\lambda, \lambda, < \nu)$-c.c. then so does P.*
2. *If Q is S-layered for some stationary $S \subseteq \lambda$, then so is P.*
3. *If Q is $(\lambda, < \nu)$-centered, then so is P.*
4. *If Q is λ-dense then so is P.*

Proof. We may assume that P and Q are Boolean algebras (not necessarily complete). We show only 2 . It suffices to show that $Q \cap M \lessdot Q$ implies $P \cap M \lessdot P$ for club many $M \in [\mathcal{H}_\theta]^{<\lambda}$. We fix $M \prec \mathcal{H}_\theta$ with $P, Q, \tau \in M$. Suppose $Q \cap M \lessdot Q$. Let $p \in P$ be arbitrary. $\tau(p)$ has a reduct q in $Q \cap M$. By the elementarity of M, we can choose a reduct $p_0 \in P \cap M$ of q (in the sense of τ). For every $r \in P \cap M$ with $r \leq p_0$, $\tau(r) \cdot q \neq 0$. Since q is a reduct of $\tau(p)$, $\tau(r) \cdot q \cdot \tau(p) \neq 0$, which in turn implies $r \cdot p \neq 0$ in P. Therefore $p_0 \in P \cap M$ is a reduct of $p \in P$. □

Lastly, we introduce the notion of Laver's indestructibly supercompact. We often use this.

Theorem 2.10 (Laver [30]). *If μ is supercompact then there is a poset P such that*

1. *$P \subseteq V_\mu$,*
2. *$P \Vdash \mu$ is supercompact.*
3. *For every P-name $\dot{Q}$ with $P \Vdash \dot{Q}$ is μ-directed closed, $P * \dot{Q} \Vdash \mu$ is supercompact.*

We say that a supercompact cardinal μ is indestructible if, for every μ-directed closed poset Q, $Q \Vdash \mu$ is supercompact. If μ is supercompact and $\kappa > \mu$ is huge then we can force μ to be indestructible without destroying the hugeness of κ.

2.2 Generic ultrapowers and saturated ideals

In this section, we recall basic properties of saturated ideals and precipitous ideals. For an ideal I over Z,

- I is non-principal if $\{z\} \in I$ for every $z \in Z$.
- The dual filter of I is $I^* = \{Z \setminus z \mid z \in I\}$.
- An I-positive set is $A \subseteq Z$ with $A \notin I$. We write I^+ for the set of all I-positive sets, that is, $I^+ = \mathcal{P}(Z) \setminus I$.
- I is κ-complete if $\bigcup_{\alpha<\nu} A_\alpha \in I$ for every $\{A_\alpha \mid \alpha < \nu\} \in [I]^{<\kappa}$. By countably complete, we mean ω_1-complete.
- The completeness $\mathrm{comp}(I)$ of I is the least cardinal κ such that I is not κ^+-complete.
- I is exactly and uniformly κ-complete if $\mathrm{comp}(I \restriction A) = \kappa$ for all $A \in I^+$. Here, $I \restriction A$ is an ideal $I \cap \mathcal{P}(A)$ over A.

In this thesis, we always assume that every ideal is non-principal.

For $A, B \in \mathcal{P}(Z)$, define $A \simeq_I B$ by $A \mathbin{\triangle} B \in I$. $\simeq_I$ is an equivalence relation. Then the Boolean algebra $\mathcal{P}(Z)/{\simeq_I}$ is induced as the quotient algebra of a set algebra $\langle \mathcal{P}(Z), \cup, \cap, \emptyset, Z, \subseteq \rangle$ by $\simeq_I$. For $A, B \in \mathcal{P}(Z)$, $A \subseteq B$ modulo $\simeq_I$ is equivalent with $A \setminus B \in I$. Therefore, $\mathcal{P}(Z)/{\simeq_I}$ and $\mathcal{P}(Z)/I^*$ are the same things in the sense of posets by the investigation in the previous section. In particular, $\mathcal{P}(Z)/{\simeq_I}$ is a separative quotient of $\langle I^+, \subseteq \rangle$. We write this Boolean algebra by $\mathcal{P}(Z)/I$. $\mathcal{P}(Z)/I$ forces $\dot{G}$ is V-ultrafilter over Z such that $\dot{G} \cap I = \emptyset$. In the extension, we can take the ultrapower mapping $\dot{i} : V \to \mathrm{Ult}(V, \dot{G})$.

We say that I is precipitous if it is forced by $\mathcal{P}(Z)/I$ that $\mathrm{Ult}(V, \dot{G})$ is well-founded. Then $\mathrm{Ult}(V, \dot{G})$ can be Mostowski collapsed to some inner model $\dot{M}$ and thus we have an elementary embedding $\dot{j} : V \to M \simeq \mathrm{Ult}(V, \dot{G}) \subseteq V[\dot{G}]$. We call $\dot{j}$ the generic ultrapower mapping by $\dot{G}$.

Proposition 2.11. *If I is a precipitous ideal over Z. Let $\dot{j}$ be a $\mathcal{P}(Z)/I$-name of the ultrapower mapping by $\dot{G}$. Then*

1. *The following are equivalent:*

(a) $A \Vdash \mathrm{crit}(\dot{j}) = \kappa$

(b) $\{B \leq A \mid \mathrm{comp}(I \upharpoonright B) = \kappa\}$ *is a* $\mathcal{P}(Z)/I$*-dense subset below* A.

2. $\mathrm{comp}(I) \geq \aleph_1$.

Proof. 2. follows from 1. Let us show 1. Assume (a). We claim that $\mathrm{comp}(I \upharpoonright A) \geq \kappa$. For every partition $\{B_\alpha \mid \alpha < \mu\}$ with $\mu < \kappa$, $A \Vdash [\mathrm{id}] \in \dot{j}(A) = \bigcup_{\alpha<\mu} \dot{j}(B_\alpha)$. Then, we have

$$A = ||[\mathrm{id}] \in \textstyle\bigcup_{\alpha<\mu} \dot{j}(B_\alpha)|| = \sum_{\alpha<\mu} ||\mathrm{id} \in B_\alpha|| = \sum_{\alpha<\mu} B_\alpha$$

in the sense of $\mathcal{P}(Z)/I$. Therefore $B_\alpha \in I^+$, as desired. For every $C \leq A$, let us find $B \leq C$ such that $\mathrm{comp}(I \upharpoonright B) = \kappa$. Since $C \Vdash \kappa < j(\kappa)$, we can choose $F : Z \to \kappa$ such that $B \Vdash [F] = \kappa$. Define $B_\alpha = \{z \in Z \mid F(z) = \alpha\}$ for each $\alpha < \kappa$. It is easy to see that $B_\alpha = ||[F] = \alpha|| \in I$.

Lastly, let us show (b). Let $B \leq A$ and κ' be arbitrary such that $B \Vdash \mathrm{crit} j = \kappa'$. By the first half of this proof, $\kappa' \leq \mathrm{comp}(I \upharpoonright B)$ and there is a $\langle B_\alpha \mid \alpha < \kappa' \rangle$ such that $\bigcup_\alpha B_\alpha = B$ but $B_\alpha \in I$. By the assumption, we can choose $C \leq B$ such that I $\upharpoonright$ C $= \kappa$. If $\kappa > \kappa'$ then $B_\alpha \in I^+$ for some α. Therefore $\kappa = \kappa'$, as desired. □

If I is exactly and uniformly κ-complete precipitous ideal over Z then $\mathcal{P}(Z)/I \Vdash \mathrm{crit}(\dot{j}) = \kappa$ by Proposition 2.11. Note that every κ-complete ideal over κ, $\mathcal{P}_\kappa\lambda$ or $[\lambda]^\kappa$ is exactly and uniformly κ-complete.

When we consider an ideal over Z, $Z \subseteq \mathcal{P}(X)$ for some X sometimes. κ, $\mathcal{P}_\kappa\lambda$ and $[\lambda]^\kappa$ are typical examples of Z. Then we add some definitions.

- I is fine if $\{z \in Z \subseteq x \in z\} \in I^*$ for all $x \in X$.
- For $A \subseteq Z$, a regressive function $f : A \to X$ is a function with $f(a) \in a$ for all $a \in A$.
- I is normal if, for every regressive function $f : A \to X$, if $A \in I^+$ then there is an $x \in X$ such that $f^{-1}\{x\} \in I^+$.
- For a sequence $A = \langle A_x \mid x \in X\rangle$ of subsets of Z, the diagonal union of A is $\{z \in Z \mid z \in \bigcup_{x\in z} A_x\}$ and we write $\bigtriangledown_{x\in X} A_x$ or $\bigtriangledown A$.

Proposition 2.12. *For an ideal* I *over* $Z \subseteq \mathcal{P}(X)$*, the following are equivalent.*

1. *I is normal.*
2. *I closed under diagonal unions.*

Proof. First, let us see the forward direction. For a sequence $\langle A_x \mid x \in X\rangle$ of a element in I. If $A = \bigtriangledown_{x\in X} A_x \in I^+$. For each $z \in A$, there is a $f(z) \in z$ such that $z \in A_{f(z)}$. By the normality of I, there is an $x \in X$ such that $f^{-1}\{x\} \in I^+$. By the definition of f, $f^{-1}\{x\} \subseteq A_x \in I$. This is a contradiction.

For the inverse direction, we fix $A \in I^+$ and a regressive function $f : A \to X$. Since f is regressive, $\bigtriangledown_{x\in X} f^{-1}\{x\} = A$. By 2, there is an $x \in X$ with $f^{-1}\{x\} \in I^+$. □

For $\kappa \leq \lambda$, we always assume I is a normal, fine, κ-complete when we consider an ideal I over $\mathcal{P}_\kappa\lambda$ unless otherwise stated. Note that we identify an ideal over κ with one over $\mathcal{P}_\kappa\kappa$

For an ideal I over Z and a saturation property Ψ, we say that I is Ψ if $\mathcal{P}(Z)/I$ is Ψ. I is λ-saturated (resp. $(\lambda, \kappa, < \mu)$-saturated, (λ, κ, μ)-saturated) if $\mathcal{P}(Z)/I$ has the λ-c.c. (resp. $(\lambda, \kappa, < \mu)$-c.c., the (λ, κ, μ)-c.c.).

For $\kappa \leq \lambda$ and an ideal I over $\mathcal{P}_\kappa\lambda$, we omit parameters as follows.

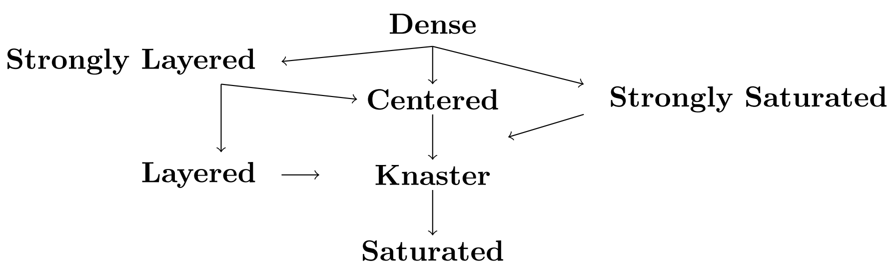
- I is saturated if I is λ^+-saturated.

- I is Knaster if I is λ^+-Knaster, that is, $(\lambda^+, \lambda^+, 2)$-saturated.
- I is strongly saturated if I is $(\lambda^+, \lambda^+, < \lambda)$-saturated.
- I is centered if I is λ-centered.
- I is layered if I is S-layered for some stationary subset $S \subseteq E^{\lambda^+}_{\geq\lambda}$.
- I is strongly layered if I is $E^{\lambda^+}_{\geq\lambda}$-layered.
- I is dense if $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I$ is λ-dense.

The following result is not used in this thesis but we introduce it here.

Theorem 2.13 (Shelah [37])**.** *A strongly layered ideal over $\mathcal{P}_\kappa\lambda$ is centered.*

By this theorem and Lemma 2.5, we have implications in Figure 1. Let us repost Figure 1 here.



Proposition 2.14. *If I is a normal, fine, precipitous ideal over $Z \subseteq \mathcal{P}(X)$. Then*

1. *$\mathcal{P}(Z)/I \Vdash \dot{j}``X = [\mathrm{id}] \in \dot{M}$. Here, $\dot{j}$ is a $\mathcal{P}(Z)/I$-name for the generic ultrapower mapping from V to $\dot{M}$ induced by $\dot{G}$.*
2. *If I is $|X|^+$-saturated then I satisfies the disjointing property. That is, for every $\langle A_\alpha \mid \alpha < \kappa\rangle \subseteq I^+$, if $A_\alpha \cap A_\beta \in I$ for all $\alpha < \beta$ then there is a $\langle B_\alpha \mid \alpha < \kappa\rangle \subseteq I^+$ such that*
   - *$B_\alpha \cap B_\beta = \emptyset$ for all $\alpha < \beta$.*
   - *$B_\alpha \simeq_I A_\alpha$ for all α.*
3. *If I satisfies the disjointing property then $\mathcal{P}(Z)/I \Vdash {}^{|X|^V}\dot{M} \subseteq M$.*

Proof. For 1, first, we check $\Vdash j``X \subseteq [\mathrm{id}]$. For any $x \in X$, since I is fine, $\{z \in Z \mid x \in z\} = ||\dot{j}(x) \in [id]||$ in $\mathcal{P}(Z)/I$. For any $A \Vdash [f] \in [\mathrm{id}]$, we may assume that $f(z) \in z$ for all $z \in Z$. By the normality of I, we have $B \leq A$ and x such that $f(z) = x$ for all $z \in B$. Then B forces $[f] = \dot{j}(x) \in \dot{j}``X$, as desired.

For 2 and 3, see [17, Proposition 2.23] and [17, Theorem 2.25], respectively. □

We introduce

Theorem 2.15. *If I is an ideal over Z that satisfies the disjointing property, then $\mathcal{P}(Z)/I$ is a complete Boolean algebra.*

Proof. See [17, Theorem 2.16] □

Proposition 2.16. *Suppose that I is a normal, fine, exactly and uniformly κ-complete $|X|^+$-saturated ideal over $Z \subseteq \mathcal{P}(X)$. Let $\dot{j}$ be a $\mathcal{P}(Z)/I$-name for the generic ultrapower mapping $\dot{j} : V \to \dot{M}$. Then the following holds:*

1. *If $Z \subseteq \mathcal{P}_\kappa X$ and $\kappa = \mu^+$ then $\mathcal{P}(Z)/I$ forces that $\dot{j}(\kappa) = |X|^+$.*

2. *If $Z \subseteq [X]^\kappa$ then $\mathcal{P}(Z)/I$ forces that $\dot{j}(\kappa) = |X|$ and $\dot{j}(\kappa^+) = |X|^+$.*

3. *If $2^\kappa = \kappa^+$, $\kappa = \mu^+$, and $Z \subseteq \mathcal{P}_\kappa X$ then $2^{|X|} = |X|^+$.*

Proof. Let G be a $(V, \mathcal{P}(Z)/I)$-generic and $j : V \to M$ be the generic ultrapower induced by G. Let us show 1. If $Z \subseteq \mathcal{P}_\kappa X$ then ${}^{|X|^V}M \cap V[G] \subseteq M$. Therefore there is no any cardinal between $|X|^V$ and $|X|^+$ in M. By the $|X|^+$-saturation of I, $|X|^+$ is a cardinal in M. By $\{z \in Z \mid |z| < \kappa\} \in I^*$,

$$\mu \leq |X|^M = |j``X|^M = |[id]|^M < j(\kappa) \leq |X|^+.$$

Therefore $j(\kappa) = |X|^+$.

For 2, we consider in the case of $Z \subseteq [X]^\kappa$. By $\{z \in Z \mid |z| = \kappa\} \in I^*$,

$$|X|^{V[G]} = |X|^M = |j``X|^M = |[\mathrm{id}]| = j(\kappa) < (|X|^+)^M = (|X|^+)^{V[G]}.$$

Therefore, in M, $j(\kappa^+) = (|X|^+)^{V[G]}$.

Lastly, let us see 3 By 1, $j(\kappa) = |X|^+$. By ${}^{|X|^V}M \cap V[G] \subseteq M$,

$$|\mathcal{P}^V(X)|^V \leq |\mathcal{P}(X)|^{V[G]} = |\mathcal{P}(j``X)|^M \leq |\mathcal{P}(\mu)|^M = j(\mu^+) = |X|^+.$$

Since I is $|X|^+$-saturated, $|\mathcal{P}^V(X)|^V \leq |X|^+$ holds in V, as desired. □

For Section 6.4, we introduce the notion of weakly normal. An ideal I over $X \subseteq \mathcal{P}(\lambda)$ is weakly normal if there is an $\alpha < \lambda$ such that $\{z \in Z \mid f(z) < \alpha\} \in I^*$ for every regressive function $f : Z \to \lambda$. Note that the weak normality does not follow from the normality.

Proposition 2.17. *If I is a normal λ-saturated ideal over $Z \subseteq \mathcal{P}(\lambda)$ then I is weakly normal.*

Proof. Let f be a regressive function on Z. Let A be a maximal subset $A \subseteq \lambda$ such that $f^{-1}\{\alpha\} \in I^+$ for all $\alpha \in A$. Since I is normal, $\bigcup_{\alpha \in A} f^{-1}\{\alpha\} \in I^*$. By the λ-saturation of I, $|A| < \lambda$. Then $\{z \in Z \mid f(z) < (\sup A + 1)\} \in I^*$, as desired. □

2.3 Duality theorem

For a precipitous ideal I over Z and a poset P, we can consider a P-name $\overline{I}$ for the ideal generated by I. That is, $P \Vdash \overline{I} = \{A \subseteq Z \mid \exists B \in I(A \subseteq B)\}$. For example, Kakuda [25] proved that if $Z = \kappa$, I is a κ-complete precipitous ideal, and P has the κ-c.c. then $\overline{I}$ is forced to be precipitous by P. The completeness and the normality are also preserved by a poset which has the chain condition. Indeed,

Proposition 2.18. *Suppose that I is a κ-complete ideal over Z and P has the κ-c.c. Then,*

1. *$P \Vdash \overline{I}$ is κ-complete.*

2. *If $Z \subseteq \mathcal{P}(X)$ and I is normal, then $P \Vdash \overline{I}$ is normal.*

Proof. First, let us check the following claim.

Claim 2.19. *If $p \Vdash \dot{A} \in \overline{I}$ then there is a $B \in I$ such that $p \Vdash \dot{A} \subseteq B$.*

Proof of Claim. Let $\mathcal{A}$ be a maximal anti-chain such that there is a $B_q \in I$ such that $q \Vdash \dot{A} \subseteq B_q$ for all $q \in \mathcal{A}$. By the κ-c.c. of P, $|\mathcal{A}| < \kappa$. Since I is κ-complete, $B = \bigcup_{q \in \mathcal{A}} B_q \in I$. B works as a witness. □

To show 1, let $p \Vdash \{\dot{A}_\xi \mid \xi < \mu\} \subseteq I$ be arbitrary. For each ξ, by the claim, there is a $B_\xi \in I$ such that $p \Vdash \dot{A}_\xi \subseteq B_\xi$. Then $p \Vdash \bigcup_\xi \dot{A}_\xi \subseteq \bigcup_\xi B_\xi \in I$.

By Proposition 2.12, the similar proof shows 2. □

On the other hand, saturation may be destroyed by c.c.c. poset (For example, see [17, Theorem 8.54]). Here, we introduce one of variations of Foreman's duality theorem that enables us to study $\mathcal{P}(Z)/\overline{I}$ in some extension. Theorem 2.20 will work as a central role in our study.

Theorem 2.20 (Foreman [18]). *For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$) and μ^+-c.c. P, there is a dense embedding d such that:*

$$\begin{array}{cccc} d: & P * \dot{\mathcal{P}}(Z)/\overline{I} & \longrightarrow & \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P)) \\ & \cup & & \cup \\ & \langle p, \dot{A}\rangle & \longmapsto & \tau(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{A})|| \end{array}$$

Here, $\tau(p) = \langle 1, \dot{j}(p)\rangle$ is a complete embedding from P to $\mathcal{P}(Z)/I * \dot{j}(P)$ and $\dot{j} : V \to \dot{M}$ denotes the generic ultrapower mapping by $\mathcal{P}(Z)/I$. In particular, $P \Vdash \dot{\mathcal{P}}(Z)/\overline{I} \simeq \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P)/\tau``\dot{H}_0)$. Here, $\dot{H}_0$ is the canonical P-name for a generic filter.

Proof. We may assume that P is a complete Boolean algebra. Note that it follows that τ is complete since P has the μ^+-c.c. and $\mathrm{crit}(\dot{j}) = \mu^+$. Indeed, for every maximal anti-chain $\mathcal{A} \subseteq P$, by $|\mathcal{A}| < \mu^+$,

$$\begin{aligned} \textstyle\sum \tau``\mathcal{A} &= \textstyle\sum_{p\in\mathcal{A}} ||\langle 1, \dot{j}(p)\rangle \in \dot{H}|| = \sum_{p\in\mathcal{A}} ||\langle 1, \dot{j}(p)\rangle \in \dot{H}|| \\ &= ||j``\mathcal{A} \cap \dot{H} \neq \emptyset|| = ||j(\mathcal{A}) \cap \dot{H} \neq \emptyset|| \\ &= 1 \end{aligned}$$

Here, $\dot{G} * \dot{H}$ is the canonical $\mathcal{P}(Z)/I * \dot{j}(P)$-name for a generic filter.

Our proof consists of two parts. First, we will give a P-name $\dot{J}$ and a dense embedding $d_0 : P * \dot{\mathcal{P}}(Z)/\dot{J} \to \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P))$. After that, we will see $P \Vdash \dot{J} = \overline{I}$ and $d_0 = d$.

Let $\dot{J}$ be a P-name defined by $P \Vdash \dot{J} \subseteq \dot{\mathcal{P}}(Z)$ and

$$A \in \dot{J} \text{ if and only if } \mathcal{P}(Z)/I * \dot{j}(P)/\tau``\dot{H}_0 \Vdash [\mathrm{id}] \notin \dot{j}(\dot{A}).$$

It is easy to see that $\dot{J}$ is forced to be an ideal. Define $d_0 : P * \dot{\mathcal{P}}(Z)/\dot{J} \to \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P))$ by $d_0(p, \dot{A}) = \tau(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{A})||$. By the definition of $\dot{J}$, if $P \nVdash \dot{A} \in \dot{J}$ then by some element then $||[\mathrm{id}] \in \dot{j}(\dot{A})|| \neq 0$.

Let us see the range of d_0 is a dense subset. Let $\langle B, \dot{q}\rangle \in \mathcal{P}(Z)/I * \dot{j}(P)$ be an arbitrary element. Since I is λ^+-saturated, by the disjointing property of I, we can choose $f : Z \to P$ such that $B \Vdash \dot{q} = [f]$. Since τ is complete, $\langle B, \dot{q}\rangle$ has a reduct $p \in P$. For every $r \leq p$, $\tau(r) \cdot \langle B, \dot{q}\rangle \neq 0$ and this forces $\dot{j}(f)([\mathrm{id}]) = [f] = \dot{q} \in \dot{H} = \dot{j}(\dot{H}_0)$. Therefore p forces that

$$\mathcal{P}(\mu^+)/I * \dot{j}(P)/\tau``\dot{H}_0 \nVdash [\mathrm{id}] \notin \dot{j}(\{x \in B \mid f(x) \in \dot{H}_0\}).$$

Thus, there is a P-name $\dot{A}$ such that $P \Vdash \dot{A} \in \dot{J}^+$ and $p \Vdash \dot{A} = \{x \in B \mid f(x) \in \dot{H}_0\}$. It is easy to see $d(p, \dot{A}) = \tau(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{A})|| \leq \langle B, \dot{q}\rangle$, as desired.

Lastly, we claim that $P \Vdash \dot{J} = \overline{I}$. $P \Vdash \overline{I} \subseteq \dot{J}$ is clear. To show $P \Vdash \dot{J} \subseteq \overline{I}$, let us consider $p \Vdash \dot{C} \in \overline{I}^+$. We let $D = \{x \in Z \mid ||x \in \dot{C}||_P \cdot p \neq 0\} \in I^+$. D forces $\dot{j}(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{C})||^{\dot{M}}_{\dot{j}(P)} \neq 0$. Let $\dot{q}$ be a $\mathcal{P}(Z)/I$-name such that $\Vdash \dot{q} \in \dot{j}(P)$ and $D \Vdash \dot{q} = \dot{j}(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{C})||^{\dot{M}}_{\dot{j}(P)}$. Let r be a reduct of $\langle D, \dot{q}\rangle \in \mathcal{P}(Z)/I * \dot{j}(P)$. It is easy to see that $r \leq p$ and $r \Vdash$ "$\langle D, \dot{q}\rangle \leq ||[\mathrm{id}] \in \dot{j}(\dot{C})||$ in the quotient forcing". By the definition of $\dot{J}$, $r \Vdash \dot{C} \in \dot{J}^+$, as desired. Of course, $d = d_0$. The proof is completed. □

Corollary 2.21 (Baumgartner–Taylor [2]). *For a saturated ideal I over μ^+ and μ^+-c.c. P, the following are equivalent:*

1. *$P \Vdash \overline{I}$ is saturated.*

2. *$\mathcal{P}(\mu^+)/I \Vdash \dot{j}(P)$ has the $(\mu^{++})^V$-c.c.*

In particular, for a saturated ideal I over μ^+, if P is μ-centered then $P \Vdash \overline{I}$ is saturated. Some of Prikry-type forcings are μ-centered. Therefore $\overline{I}$ is forced to be saturated by these posets. Using Theorem 2.20, we can get necessary conditions of $\overline{I}$ to become centered or strongly saturated. The following is a motivation for Section 4.

Corollary 2.22. *For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$), μ-centered poset P, and $\nu < \lambda$, if $\lambda^\mu = \lambda$ then the following holds.*

1. *If $P \Vdash \overline{I}$ is $(\lambda, < \nu)$-centered then so is I.*
2. *If $P \Vdash \overline{I}$ is $(\lambda^+, \lambda^+, < \nu)$-saturated then so is I.*

Proof. We may assume that P is a Boolean algebra (not necessarily complete).

Let $\tau : P \to (\mathcal{P}(Z)/I * \dot{j}(P))$ be a complete embedding given in Theorem 2.20. Then $P \Vdash \mathcal{P}(Z)/\overline{I} \simeq \mathcal{P}(Z)/I * \dot{j}(P)/\dot{G}$. For every $A \in I^+$, $P \Vdash \langle A, \dot{1} \rangle \in \mathcal{P}(Z)/I * \dot{j}(P)/\dot{G}$. Indeed, for every $p \in P$, $\tau(p) \cdot \langle A, 1 \rangle = \langle A, \dot{j}(p) \rangle \in \mathcal{P}(Z)/I * \dot{j}(P)$. It is easy to see that $\mathcal{P}(Z)/I$ is completely embedded in $\mathcal{P}(Z)/I * \dot{j}(P)/\dot{G}$ by a mapping $A \mapsto \langle A, \dot{1} \rangle$.

We check 1. Let $\langle P_\alpha \mid \alpha < \lambda \rangle$ be a centering family of P and let $\dot{f}$ be a P-name for a centering function of $(\mathcal{P}(Z)/I)^V$. We may assume that each P_α is a filter. For each $A \in I^+$, define $f(A) = \langle \xi \mid \alpha < \lambda, \exists q \in P_\alpha (q \Vdash \dot{f}(A) = \xi) \rangle$. By $\lambda^\mu = \lambda$, we identify the range of f with λ. It is easy to see that f works as a centering function in V. Therefore I is centered.

Let us see 2. Similarly, P forces $(\mathcal{P}(Z)/I)^V$ has the $(\lambda^+, \lambda^+, < \nu)$-c.c. For every $X \in [I^+]^{\lambda^+}$, there is a P-name $\dot{Y}$ such that $P \Vdash \dot{Y} \in [X]^{\lambda^+}$ and $\forall Z \in [\dot{Y}]^{<\nu}(\bigcap Z \in I^+)$. Since P is μ-centered, $|P| \leq 2^\mu \leq \lambda$. Therefore there is an $Y \in [X]^{\lambda^+}$ and $p \in P$ such that $p \Vdash Y \in [\dot{Y}]^{\lambda^+}$. Y works as a witness. ☐

2.4 Collapses and Prikry-type forcings

2.4.1 Collapses

In this thesis, we use some collapsing posets. First, we introduce Levy collapses $\mathrm{Coll}(\kappa, < \lambda)$ and its diagonal product $P(\kappa, \lambda)$. In the half of this section, we study these posets. In the rest, we introduce the nested product of Levy collapses $R(\kappa, \lambda)$, the Easton collapse $E(\kappa, \lambda)$, the Silver collapse $S(\kappa, \lambda)$ and Laver collapse $L(\kappa, \lambda)$.

We use a slight modification of Levy collapse. We write $[\kappa, \lambda)_{\mu\text{-cl}}$ for the set of all μ-closed cardinal in $[\kappa, \lambda)$. Here, a μ-closed cardinal is a cardinal γ with $\gamma^{<\mu} = \gamma$. For $\mu < \kappa$, $\mathrm{Coll}(\mu, < \kappa)$ is the $< \mu$-support product $\prod^{<\mu}_{\gamma \in [\mu^+, \kappa)_{\mu\text{-cl}}} {}^{<\mu}\gamma$. Note that our Levy collapse $\mathrm{Coll}(\mu, < \kappa)$ is forcing equivalent to the usual one if κ is inaccessible.

Lemma 2.23. *For regular cardinals $\mu < \kappa$,*

1. $\mathrm{Coll}(\mu, < \kappa)$ *is μ-directed closed.*
2. *If κ is inaccessible then* $\mathrm{Coll}(\mu, < \kappa)$ *has the $(\kappa, \kappa, < \mu)$-c.c.*

Proof. 1 is easy. 2 is included in Lemma 2.24. ☐

For regular cardinals $\mu < \kappa$, the diagonal product of Levy collapses is $P(\mu, \kappa) = \prod^{<\mu}_{\alpha \in [\mu, \kappa) \cap \mathrm{Reg}} \mathrm{Coll}(\alpha, < \kappa)$.

Lemma 2.24. *For regular cardinals $\mu < \kappa$,*

1. *$P(\mu, \kappa)$ is $< \mu$-directed closed.*
2. *If κ is inaccessible, then $P(\mu, \kappa)$ has the $(\kappa, \kappa, < \mu)$-c.c. In particular, $P(\mu, \kappa)$ forces $\mu^+ = \kappa$.*

Proof. 1 is easy. Let us see 2. For any $X \in [P(\mu,\kappa)]^\kappa$, the usual Δ-system argument takes $Y \in [X]^\kappa$ and r with the following properties:

- $\{\mathrm{supp}(p) \mid p \in Y\}$ is a Δ-system with its root r.
- $r \subseteq \kappa$ is bounded by some regular cardinal $\eta < \kappa$.

For each $\alpha \in r$, we can see that $p(\alpha)$ is a partial function from $\alpha \times [\alpha,\kappa)_{\alpha^+\text{-cl}}$ to κ. Note that $|\bigcup_{\alpha\in r}\{\alpha\} \times \mathrm{dom}(p(\alpha))| < \eta$. Again, the usual Δ-system argument takes $Y' \in [Y]^\kappa$, r' and q such that

- $\{\{\alpha\} \times \mathrm{dom}(p(\alpha)) \mid \alpha \in Y'\}$ is a Δ-system with its root r'.
- $q \in P(\mu,\kappa)$.
- For all $p \in Y'$ and $\alpha \in r$, $p(\alpha) \restriction \{\xi \mid \langle\alpha,\xi\rangle \in r'\} = q(\alpha)$.

It is easy to see that Y' works. □

For a later purpose, we study the layeredness of $P(\mu,\kappa)$.

Lemma 2.25. *For inaccessible κ and regular $\mu < \kappa$,*

1. *If κ is Mahlo, then $P(\mu,\kappa)$ is $[\mu,\kappa) \cap \mathrm{Reg}$-layered.*
2. *If κ is not Mahlo, then $P(\mu,\kappa)$ is not S-layered for all stationary subsets $S \subseteq \kappa$.*

Lemma 2.25 follows from

Lemma 2.26. *For inaccessible κ and regular $\mu < \kappa$,*

1. *$P(\mu,\delta) \lessdot P(\mu,\kappa)$ for all $\delta < \kappa$.*
2. *There is a club C such that $\bigcup_{\eta<\delta} P(\mu,\eta) \lessdot P(\mu,\kappa)$ if and only if δ is regular for all $\delta \in C$.*

Proof. 1 is easy. Let us see 2. It is easy to see $P(\mu,\delta) \supseteq \bigcup_{\eta<\delta} P(\mu,\eta)$.

Let $C = \{\delta < \kappa \mid \forall\eta < \delta(\eta^{<\eta} < \delta)$ and δ is a limit cardinal$\}$. C is a club in κ. Note that $\sup[\alpha^+,\delta)_{\alpha\text{-cl}} = \delta$ for each $\delta \in C$ and $\alpha < \delta$.

In the case of $\delta \in C$ regular, $P(\mu,\delta) = \bigcup_{\eta<\delta} P(\mu,\eta) \lessdot P(\mu,\kappa)$, by 1. If $\delta \in C$ is singular, there is a regular cardinal α with $\mathrm{cf}(\delta) < \alpha < \delta$. Then $\sup[\alpha^+,\delta)_{\alpha\text{-cl}} = \delta$. Let $\{\delta_i \mid i < \mathrm{cf}(\delta)\} \subseteq [\alpha^+,\delta)_{\alpha\text{-cl}}$ be a sequence which converges to δ. Define $p \in P(\mu,\delta)$ by,

- $\mathrm{supp}(p) = \{\alpha\}$.
- $p(\alpha) \in \mathrm{Coll}(\alpha,<\delta)$ is such that
  - $\mathrm{dom}(p(\alpha)) = \{\delta_i \mid i < \mathrm{cf}(\delta)\}$, and
  - $p(\alpha)(\delta_i) = \begin{cases} \{\langle 0,\delta_{i-1}\rangle\} & i \text{ is successor ordinal} \\ \{\langle 0,0\rangle\} & \text{otherwise} \end{cases}$

It is easy to see $p(\alpha) \in \mathrm{Coll}(\alpha,<\delta) \setminus \bigcup_{\eta<\delta}\mathrm{Coll}(\alpha,<\eta)$. In particular, p does not have a reduct in $\bigcup_{\eta<\delta} P(\mu,\eta)$. □

Proof of Lemma 2.25. Let C be a club in Lemma 2.26. For 1, by Lemma 2.26, $P(\mu,\kappa)$ is $[\mu,\kappa) \cap \mathrm{Reg}$-layered witnessed by $\langle P(\mu,\delta) \mid \delta < \kappa\rangle$.

For 2, by the assumption, there is a club $D \subseteq C$ such that every element in D are singular. Define $Q_\delta = \bigcup_{\eta<\delta} P(\mu,\eta)$. $\langle Q_\delta \mid \delta < \kappa\rangle$ is a filtration of $P(\mu,\kappa)$. By Lemma 2.26, $Q_\delta \not\lessdot P(\mu,\kappa)$ for all $\delta \in D$. By Lemma 2.8, $P(\mu,\kappa)$ is not S-layered for all stationary subsets $S \subseteq \kappa$. □

The following lemma is contained in the proof of 2 of Lemma 2.26, and is used in the proof of Claim 5.7.

Lemma 2.27. *For inaccessible κ and regular $\mu < \kappa$, let C be a club in 2 of Lemma 2.26. For every singular $\delta \in C$, there is a $p \in P(\mu, \delta) \setminus \bigcup_{\eta<\delta} P(\mu, \eta)$ with the following properties:*

- $\mathrm{supp}(p) \cap (\lambda + 1) = \emptyset$, *and,*
- *For every $q \in \bigcup_{\eta<\delta} P(\mu, \eta)$, there is an $r \in \bigcup_{\eta<\delta} P(\mu, \eta)$ such that $\mathrm{dom}(r) \cap (\lambda + 1) = \emptyset$, $r \perp p$ in $P(\mu, \delta)$ and $r \cdot q \in \bigcup_{\eta<\delta} P(\mu, \eta)$.*

Proof. The condition p which was defined in the proof of Lemma 2.26 works. □

The following property of Levy collapses is used in the proof of Claim 5.8.

Lemma 2.28. *For inaccessible λ and regular $\kappa < \alpha < \lambda$, $\mathrm{Coll}(\kappa, < \lambda)$ forces $\mathrm{Coll}^V(\alpha, < \lambda)$ is not κ-centered.*

Proof. We show by contradiction. We may assume that $\mathrm{Coll}(\kappa, < \lambda)$ forces that $\mathrm{Coll}^V(\alpha, < \lambda)$ is κ-centered. Let $\langle \dot{F}_\xi \mid \xi < \kappa \rangle$ be a $\mathrm{Coll}(\kappa, < \lambda)$-name for a centering. We may assume that it is forced that each $\dot{F}_\xi$ is a filter because $\prod X \in \mathrm{Coll}(\kappa, < \lambda)$ for every $X \subseteq \mathrm{Coll}(\kappa, < \lambda)$ with X has a lower bound.

For each $\xi < \kappa$ and $q \in \mathrm{Coll}(\alpha, < \lambda)$, let $\rho(q, \xi)$ be the least cardinal δ such that there is a maximal anti-chain $\mathcal{A} \subseteq \mathrm{Coll}(\kappa, < \delta)$ with $\forall p \in \mathcal{A}(p \text{ decides } q \in \dot{F}_\xi)$. Let $D \subseteq \lambda$ be a club generated by a mapping $\delta \mapsto \sup\{\rho(q, \xi) \mid \xi < \kappa \wedge q \in \mathrm{Coll}(\alpha, < \delta)\} \cup \{\delta^{<\delta}\}$.

Fix a $\delta \in D \cap E^\lambda_{\geq\kappa} \cap E^\lambda_{<\alpha}$. The following hold now.

- $|\mathrm{Coll}(\kappa, < \delta)| = \delta$, in particular, $\mathrm{Coll}(\kappa, < \delta) \Vdash (\delta^+)^V \geq \kappa^+$.
- $\mathrm{Coll}(\alpha, < \delta)$ has an anti-chain of size $\delta^{\mathrm{cf}(\delta)} \geq \delta^+$.

The first item follows from the standard cardinal arithmetic. Let us define an anti-chain for $\mathrm{Coll}(\alpha, < \delta)$ of size $\delta^{\mathrm{cf}(\delta)}$. Note that we can choose a sequence $\langle \delta_i \mid i < \mathrm{cf}(\delta) \rangle \subseteq [\alpha^+, \delta)_{\alpha\text{-cl}}$ which converges to δ. For each $i < \mathrm{cf}(\delta)$, ${}^{<\alpha}\alpha_i$ has an anti-chain $\{p^i_\xi \mid \xi < \alpha_i\}$ of size α_i. For each $f \in \prod_{i<\mathrm{cf}(\delta)} \alpha_i$, define $p_f \in \mathrm{Coll}(\alpha, < \lambda)$ as follows:

- $\mathrm{supp}(p_f) = \{\alpha_i \mid i < \mathrm{cf}(\delta)\}$.
- $p_f(\alpha_i) = p^i_{f(i)}$.

It is easy to see that $\{p_f \mid f \in \prod_{i<\mathrm{cf}(\delta)} \alpha_i\}$ witnesses.

Let G be an arbitrary $(V, \mathrm{Coll}(\kappa, < \lambda))$-generic. G can be factored as $G = G_0 \times G_1$ where G_0 is a $(V, \mathrm{Coll}(\kappa, < \delta))$-generic. Let us discuss in $V[G_0]$. Letting $Q = (\bigcup_{\zeta<\delta} \mathrm{Coll}(\alpha, < \zeta))^V$. Let $F_\xi = \dot{F}^G_\xi$ in $V[G]$, note that $F_\xi \cap Q \in V[G_0]$ by $\delta \in D$. In particular, Q has a centering $\langle F_\xi \cap Q \mid \xi < \kappa \rangle$ in $V[G_0]$. Define $H_\xi = \{q \in \mathrm{Coll}^V(\alpha, < \delta) \mid \forall \alpha \in \mathrm{supp}(q)(q \restriction (\mathrm{supp}(q) \cap \alpha) \in F_\xi \cap Q)\}$. We claim that $\langle H_\xi \mid \xi < \kappa \rangle$ is a centering for $\mathrm{Coll}^V(\alpha, < \delta)$. It is easy to see that each H_ξ is a filter. For each $q \in \mathrm{Coll}^V(\alpha, < \delta)$, in $V[G]$, there is a ξ such that $q \in F_\xi$. For every $\alpha < \mathrm{supp}(q)$, $q \restriction (\mathrm{supp}(q) \cap \alpha) \in Q \cap F_\xi$. This has held in $V[G_0]$ yet, and thus, $q \in H_\xi$ in $V[G_0]$.

We showed that $\mathrm{Coll}^V(\alpha, < \delta)$ is κ-centered, which in turn implies the κ^+-c.c. But $\mathrm{Coll}^V(\alpha, < \delta)$ has a maximal anti-chain of size $(\delta^{\mathrm{cf}(\delta)})^V \geq \kappa^+$ as we have seen. This is a contradiction. □

On the other hand,

Lemma 2.29. *For inaccessible λ and regular $\alpha \leq \kappa$, $\mathrm{Coll}(\kappa, < \lambda)$ forces $\mathrm{Coll}^V(\alpha, < \lambda)$ is κ-centered.*

Proof. We discuss in the extension by $\mathrm{Coll}(\kappa, <\lambda)$. For all $\gamma \in [\alpha^+, \lambda)_{\alpha\text{-cl}}$, because of $|({}^{<\alpha}\gamma)^V| \leq \kappa$, $({}^{<\alpha}\gamma)^V$ is κ-centered. By Lemma 2.7, it follows that $\prod^{<\alpha}_{\gamma\in[\alpha^+,\lambda)}({}^{<\alpha}\gamma)^V$ is κ-centered. In particular, $\mathrm{Coll}^V(\alpha, <j(\kappa))$ is κ-centered. □

For a later purpose, we introduce the nested product of Levy collapses, that was introduced by Shioya [41]. For $\kappa < \lambda$, the nested product of Levy collapses is the full support product $R(\kappa, \lambda) = \prod_{n<\omega} R^n(\kappa, \lambda)$. Here, $R^0(\kappa, \lambda) = \mathrm{Coll}(\kappa, <\lambda)$ and $R^{n+1}(\kappa, \lambda) = \prod^{<\kappa}_{\alpha\in[\kappa,\lambda)\cap\mathrm{SR}} R^n(\alpha, \lambda)$. SR is the class of all strong regular cardinals, that is a cardinal γ with $\gamma^{<\gamma} = \gamma$. Note that $P(\kappa, \lambda) = R^1(\kappa, \lambda)$.

Lemma 2.30 (Shioya [41])**.** *For regular cardinals $\mu \leq \kappa < \lambda$,*

1. *$R(\kappa, \lambda)$ is κ-directed closed.*
2. *$R(\mu, \lambda) \lessdot R(\kappa, \lambda)$.*
3. *If λ is Mahlo then $R(\kappa, \lambda)$ is $\lambda \cap \mathrm{Reg}$-layered.*
4. *If λ is Mahlo and $\kappa > \aleph_0$ then $R(\kappa, \lambda) \Vdash R(\mu, \lambda)$ is κ-centered.*

Proof. 1 and 2 are easy. Let us show 3. By induction on $n < \omega$, we show that $R^{n+1}(\alpha, \lambda)$ is $\lambda\cap\mathrm{Reg}$-layered for all $\alpha < \lambda$.

By Lemma 2.25, $R^0(\alpha, \lambda)$ and $R^1(\alpha, \lambda)$ are $\lambda \cap \mathrm{Reg}$-layered for all regular $\alpha < \lambda$. Suppose $R^n(\alpha, \lambda)$ is $[\kappa, \lambda) \cap \mathrm{Reg}$-layered for all $\alpha < \lambda$. For each $\alpha \in \lambda \cap \mathrm{Reg}$, there is a sequence $\langle R^\alpha_\xi \mid \xi < \lambda\rangle$ of complete suborders of $R^n(\alpha, \kappa)$ and a club $C_\alpha \subseteq \lambda$ such that $\forall \xi \in C_\alpha \cap \mathrm{Reg}(P^\alpha_\xi = \bigcup_{\zeta<\xi} P^\alpha_\zeta)$. Let $P_\xi = \prod^{<\kappa}_{\alpha\in[\kappa,\xi)\cap\mathrm{Reg}} P^\alpha_\xi$ and $C^n = \triangle_\alpha C_\alpha$. Then $\langle P^n_\xi \mid \xi < \lambda\rangle$ is a sequence of complete suborders of $R^{n+1}(\kappa, \lambda)$ and C^n is a club. It is easy to see that $\forall \xi \in C^n \cap \mathrm{Reg}(P^n_\xi = \bigcup_{\zeta<\xi} P^n_\zeta)$. The similar proof shows that $\bigcap_n C^n$ and $\langle \prod_n P^n_\xi \mid \xi < \lambda\rangle$ witnesses $[\kappa, \lambda) \cap \mathrm{Reg}$-layeredness of $R(\kappa, \lambda)$.

Lastly, we check 4. By Lemma 2.7 and the induction on $n < \omega$, we can show $R^{n+1}(\kappa, \lambda)$ forces that $R^n(\alpha, \lambda)$ is κ-centered for all $n < \omega$ as we saw in Lemma 2.29. By 2., $R(\kappa, \lambda)$ forces $R^n(\alpha, \lambda)$ is κ-centered for all $n < \omega$. Then Lemma 2.7 shows $R(\kappa, \lambda)$ forces $R(\mu, \lambda)$ is κ-centered. □

We also recall Easton collapse, that was also introduced by Shioya [42]. For a sequence of posets $\langle Q_\gamma \mid \gamma \in K\rangle$ with $K \subseteq \mathrm{Reg}$, $\prod^E_{\gamma\in K} Q_\gamma$ is $\bigcup\{\prod_{\gamma\in d} Q_\gamma \mid d \subseteq K \text{ is Easton}\}$. Here, Easton subset is a $d \subseteq \mathrm{Reg}$ such that $\sup d \cap \alpha < \alpha$ for all regular $\alpha < \kappa$. That is ordered by a standard way. For $\kappa < \lambda$, the Easton collapse $E(\kappa, \lambda)$ is $\prod^E_{\gamma\in[\kappa,\lambda)\cap\mathrm{SR}} {}^{<\kappa}\gamma$.

Lemma 2.31. *For regular cardinals $\kappa < \lambda$,*

1. *$E(\kappa, \lambda)$ is λ-directed closed.*
2. *If λ is Mahlo then $E(\kappa, \lambda)$ has the $(\lambda, \lambda, <\mu)$-c.c. for all $\mu < \lambda$.*
3. *If λ is Mahlo then $E(\kappa, \lambda)$ is $\lambda \cap \mathrm{Reg}$-layered.*

Proof. 1 is easy. 2 follows from the usual Δ-system argument. The similar proof of 3 in Lemma 2.30 show 3. □

For cardinals $\kappa < \lambda$, the Silver collapse $S(\kappa, \lambda)$ is the set of all p with the following properties:

- $p \in \prod^{\leq\kappa}_{\gamma\in[\kappa^+,\lambda)\cap\mathrm{Reg}} {}^{<\kappa}\gamma$.
- There is a $\xi < \kappa$ such that $\mathrm{dom}(p(\gamma)) \subseteq \xi$ for all $\gamma \in \mathrm{dom}(p)$.

$S(\kappa, \lambda)$ is ordered by reverse inclusion. The following properties are well known.

Lemma 2.32. *For regular cardinals $\kappa < \lambda$,*

1. *$S(\kappa, \lambda)$ is κ-directed closed.*
2. *If λ is inaccessible, then $S(\kappa, \lambda)$ has the $(\lambda, \lambda, \kappa)$-c.c.*
3. *If $\mu < \lambda$ then $S(\kappa, \mu) \lessdot S(\kappa, \lambda)$.*

Proof. 1 and 3 trivially hold. 2 follows from the usual Δ-system argument. □

The following lemma will be used in Section 5.5.

Lemma 2.33. *For a cardinal δ and a regular cardinal $\kappa < \delta$,*

1. *If $\mathrm{cf}(\delta) \leq \kappa$ then $S(\kappa, \delta)$ has an anti-chain of size δ^+.*
2. *If $\mathrm{cf}(\delta) > \kappa$ and $\delta^\kappa = \delta$ then $S(\kappa, \delta)$ forces $(\delta^+)^V \geq \kappa^+$.*

Proof. Let us show 1. By $\mathrm{cf}(\delta) < \kappa$, we can fix an increasing sequence of regular cardinals $\langle \delta_i \mid i < \mathrm{cf}(\delta) \rangle$ which converges to δ. For $f \in \prod_{i<\mathrm{cf}(\delta)} \delta_i$, define $p_f \in S(\kappa, \delta)$ by $\mathrm{dom}(p_f) = \{\delta_i \mid i < \mathrm{cf}(\delta)\}$ and $p_f(\delta_i) = \{\langle 0, f(i) \rangle\}$. It is easy to see that $f \neq g$ implies $p_f \perp p_g$. Therefore $\{p_f \mid f \in \prod_{i<\mathrm{cf}} \delta_i\}$ is an anti-chain of size $(\delta^{\mathrm{cf}\delta}) \geq \delta^+$, as desired.

2 follows from a standard cardinal arithmetic. Indeed, the assumption implies $|S(\kappa, \delta)| = \delta$. Therefore $S(\kappa, \delta)$ has the δ^+-c.c. □

For cardinals $\kappa < \lambda$, the Laver collapse $L(\kappa, \lambda)$. $L(\kappa, \lambda)$ is the set of all p such that

- $p \in \prod^{<\lambda}_{\gamma \in [\kappa^+, \lambda) \cap \mathrm{Reg}} {}^{<\kappa}\gamma$.
- There is a $\xi < \kappa$ such that $\mathrm{dom}(p(\gamma)) \subseteq \xi$ for all $\gamma \in \mathrm{dom}(p)$.
- $\mathrm{dom}(p) \subseteq \lambda$ is an Easton subset.

$L(\kappa, \lambda)$ is ordered by reverse inclusion. It is easy to see that

Lemma 2.34. *For regular cardinals $\kappa < \lambda$,*

1. *$L(\kappa, \lambda)$ is κ-directed closed.*
2. *If λ is Mahlo, then $L(\kappa, \lambda)$ has the $(\lambda, \lambda, < \mu)$-c.c. for all $\mu < \lambda$.*

Proof. 1 trivially holds. 2 follows from the usual Δ-system argument. □

2.4.2 Prikry-type forcings

Modifications of Prikry forcing are called Prikry-type forcings. Original Prikry forcing was introduced by Prikry [34]. For a given filter F over μ, $\mathcal{P}_F$ is $[\mu]^{<\omega} \times F$ ordered by $\langle a, X \rangle \leq \langle b, Y \rangle$ if and only if $a \supseteq b$, $a \cap (\max b + 1) = b$ and $a \setminus b \cup X \subseteq Y$.

Lemma 2.35. 1. *$\mathcal{P}_F$ is $(\mu, < \mathrm{comp}(F))$-centered.*

2. *$\mathcal{P}_F \Vdash \mathrm{cf}(\mu) = \omega$.*

Proof. 1. Let $P_a = \{\langle a, X \rangle \mid X \in F\}$ then $\mathcal{P}_F = \bigcup_{a \in [\mu]^{<\omega}} P_a$. Each P_a is $< \mathrm{comp}(F)$-centered. 2. Let $\dot{g}$ be a $(V, \mathcal{P}_F)$-name for a subset $\bigcup\{a \mid \exists X(\langle a, X \rangle \in \dot{G})\}$. It is easy to see that $\Vdash \sup \dot{g} = \mu$ and $\dot{g} \in [\mu]^\omega$. □

Prikry forcing is $\mathcal{P}_U$ for some normal ultrafilter U. Prikry forcing preserves all cardinals and forces $\mathrm{cf}(\mu) = \omega$.

For a family $\{X_a \mid a \in [\kappa]^{<\omega}\} \subseteq U$, the diagonal intersection $\triangle_a X_a$ is $\{\xi < \kappa \mid \forall a \in [\xi]^{<\omega}(\xi \in X_a)\}$. Since U is normal, $\triangle_a X_a \in U$. The diagonal intersection gives "fusion-like" conditions as follows.

Lemma 2.36. *For every $\{X_a \mid a \in [\kappa]^{<\omega}\} \subseteq U$ and $a \in [\kappa]^{<\omega}$, every extension of $\langle a, \triangle_b X_b\rangle$ is compatible with $\langle a, X_a\rangle$.*

Proof. Take an arbitrary extension $\langle c, Y\rangle \leq \langle a, \triangle_b X_b\rangle$. Then for any $\xi \in c$ with $\max(a) < \xi$, $\xi \in X_a$. Thus, $\langle c, Y \cap X_a\rangle$ is a common extension of $\langle c, Y\rangle$ and $\langle a, X_a\rangle$. □

We call Lemma 2.37 Prikry lemma. We will see Prikry lemmas for other variations of Prikry forcing. Since our proof is a prototype of proofs of them, we describe here.

Lemma 2.37. *Suppose that U is a normal ultrafilter over μ. For every $a \in [\mu]^{<\mu}$ and statement σ in the forcing language of $\mathcal{P}_U$, there is $Z \in U$ such that $\langle a, Z\rangle$ decides σ. That is, $\langle a, Z\rangle \Vdash \sigma$ or $\langle a, Z\rangle \Vdash \neg\sigma$.*

Proof. For every $b \in [\kappa]^{<\omega}$ with $a \subseteq_e b$, let $X_b \in U$ be one of the following sets

- $X_b^0 = \{\xi < \kappa \mid \exists Y \in U\ \langle b \cup \{\xi\}, Y\rangle \Vdash \sigma \wedge \max b < \xi\}$.
- $X_b^1 = \{\xi < \kappa \mid \exists Y \in U\ \langle b \cup \{\xi\}, Y\rangle \Vdash \neg\sigma \wedge \max b < \xi\}$.
- $X_b^2 = \kappa \setminus (X_b^0 \cup X_b^1)$.

Otherwise, let $X_b = \kappa$.

For every $b \in [\kappa]^{<\omega}$, let $Y_b = Y$ if there is a $Y \in U$ such that $\langle b, Y\rangle$ decides σ. Otherwise, let $Y_b = \kappa$. Let $Z = \triangle_b X_b \cap \triangle_b Y_b$. We claim that Z is as desired. Take an arbitrary extension $\langle c, Y\rangle \leq \langle a, Z\rangle$ that decides σ. We may assume that $\langle c, Y\rangle$ forces σ, that $c = b \cup \{\xi\}$, and that $\max c = \xi$ with $a \subseteq_e b$. We claim that $\langle b, Y\rangle$ also forces σ.

Since $\xi \in X_b$ and $\langle b \cup \{\xi\}, Y\rangle \Vdash \sigma$, $X_b = X_b^0$. In particular, for every $\zeta \in X_b$, $\langle b \cup \{\zeta\}, Y_{b\cup\{\zeta\}}\rangle \Vdash \sigma$. For any $\langle d, Y'\rangle \leq \langle b, Y\rangle$ with $d \neq b$, by $\max(b) < \eta = \min(d \setminus b) \in Z$, $\eta \in X_b$. This yields the following result.

$$\langle b \cup \{\eta\}, Y_{b\cup\{\eta\}}\rangle \Vdash \sigma.$$

Moreover, by Lemma 2.36, $\langle b \cup \{\eta\}, Y_{b\cup\{\eta\}}\rangle$ and $\langle d, Y'\rangle$ have a common extension that forces σ. Therefore, $\langle b, Y\rangle \Vdash \sigma$. Repeating this argument yields $\langle a, Y\rangle \Vdash \sigma$. In particular, $\langle a, Z\rangle \Vdash \sigma$. □

We have

Theorem 2.38 (Prikry)**.** *Suppose that U is a normal ultrafilter over μ. Then $\mathcal{P}_U$ preserves all cardinals. Therefore $\mathcal{P}_U \Vdash \mu$ is a singular cardinal of cofinality ω.*

Proof. Since $\mathcal{P}_U$ has the μ^+-c.c., it suffices to show that $\mathcal{P}_U$ adds no new bounded subset of μ. Let $\langle a, X\rangle \Vdash \dot{A} \subseteq \eta$ be arbitrary for some $\eta < \mu$. By Lemma 2.37, for each $\alpha < \eta$, there is an $X_\alpha \in U$ such that $\langle a, X_\alpha\rangle$ decides $\alpha \in \dot{X}$. Let $B = \{\alpha < \eta \mid \langle a, X_\alpha\rangle \Vdash \alpha \in \dot{A}\}$. Then $\langle a, X \cap \bigcap_{\alpha<\eta} X_\alpha\rangle \Vdash \dot{A} = B$, as desired. □

For a later purpose, we introduce Rowbottom's theorem.

Corollary 2.39 (Rowbottom [35])**.** *Suppose that μ is a measurable cardinal and U is a normal ultrafilter over μ. Then, for every $f : [X]^{<\omega} \to \nu$ with $X \in U$ and $\nu < \mu$, there is an $H \in U$ such that $|f``[H]^n| \leq 1$ for all $n < \omega$ and $H \subseteq X$.*

Proof. Let $\dot{g}$ be a $(V,\mathcal{P}_U)$-name in the proof of Lemma 2.35. Note that $\langle\emptyset,X\rangle\Vdash\dot{g}\subseteq X$. Let $\dot{g}_n$ be a $(V,\mathcal{P}_U)$-name for the set of the first n-th elements in $\dot{g}$. By Lemma 2.37, we have $\alpha_n<\nu$ and $Z_n\in U$ such that $\langle\emptyset,Z_n\rangle\Vdash f(\dot{g}_n)=\alpha_n$. It is easy to see that, for all $a\in[Z_n]^n$, $\langle a,Z_n\rangle\leq\langle\emptyset,Z_n\rangle$ forces $a=\dot{g}_n$ and thus $f(a)=f(\dot{g}_n)=\alpha_n$. Therefore $f``[Z_n]^n=\{\alpha_n\}$. $H=\bigcap_n Z_n$ works as a witness. □

We often use the following variation of Lemma 2.37.

Lemma 2.40. *Suppose that U is a normal ultrafilter over μ and $\mathcal{A}\subseteq\mathcal{P}_U$ is a maximal anti-chain below $\langle a,X\rangle$. Then there are n and $X\supseteq Z\in U$ such that $\{\langle b,Y\rangle\in\mathcal{A}\mid|b|=n\}$ is a maximal anti-chain below $\langle a,Z\rangle$.*

Proof. Suppose that $\mathcal{A}$ is a maximal anti-chain below $\langle a,X\rangle$. For each $n<\omega$, by Lemma 2.37, there is a $Z_n\in U$ such that $\langle a,Z_n\rangle$ decides $\exists\langle b,Y\rangle\in\dot{G}\cap\mathcal{A}(|b|=n)$. $Z=X\cap\bigcap_n Z_n$ works. □

Lemma 2.41. *For posets $P\lessdot Q$, let $\dot{U}$ and $\dot{W}$ be a P-name and a Q-name for a filter over μ, respectively. If $Q\Vdash\dot{U}\subseteq\dot{W}$ and $\dot{W}$ is a normal ultrafilter over μ, then the following are equivalent:*

1. *$P*\mathcal{P}_{\dot{U}}\lessdot Q*\mathcal{P}_{\dot{W}}$.*
2. *$P\Vdash\dot{U}$ is ultrafilter.*

Proof. We may assume that P and Q are Boolean algebras.

Let us show the forward direction. We show contraposition. Suppose that there are $p\in P$ and $\dot{X}$ such that $p\Vdash\dot{X}\notin\dot{U}$ and $\mu\setminus\dot{X}\notin\dot{U}$. Then there is an extension $q\in Q$ of p which decides $\dot{X}\in\dot{W}$. We may assume that q forces $\dot{X}\in\dot{W}$. We claim that there is no reduct of $\langle q,\langle\emptyset,\dot{X}\rangle\rangle$ in $P*\mathcal{P}_{\dot{U}}$.

For any $\langle r,\langle a,\dot{Y}\rangle\rangle\in P*\mathcal{P}_{\dot{U}}$, if r is not a reduct of q (in the sense of $P\lessdot Q$), there is nothing to do. Suppose that r is a reduct of q. Then we have $r\leq p$. By $r\Vdash\dot{X}\notin\dot{U}$, $r\Vdash|\dot{Y}\setminus\dot{X}|=\mu$. Choose $r'\leq r$ and α with $r'\Vdash\alpha\in\dot{Y}\setminus\dot{X}\cup(\max a+1)$. Then $\langle r',\langle a\cup\{\alpha\},\dot{Y}\rangle\rangle\leq\langle r,\langle a,\dot{Y}\rangle\rangle$ does not meet with $\langle q,\langle\emptyset,\dot{X}\rangle\rangle$, as desired.

The inverse direction follows from Lemma 2.40. For a maximal anti-chain $\mathcal{A}\subseteq P*\mathcal{P}_{\dot{U}}$, consider P-name $\dot{\mathcal{B}}$ such that $P\Vdash\dot{\mathcal{B}}=\{\langle a,X\rangle\mid\exists p\in\dot{G}(\langle p,\langle a,X\rangle\rangle\in\mathcal{A})\}$. $\dot{\mathcal{B}}$ is forced to be a maximal anti-chain. It is enough to prove that $Q\Vdash\dot{\mathcal{B}}$ is maximal anti-chain below $\mathcal{P}_{\dot{W}}$. For every $p\Vdash\langle a,\dot{X}\rangle\in\mathcal{P}_{\dot{W}}$, because of $P\Vdash\dot{\mathcal{B}}$ is maximal anti-chain below $\langle a,\emptyset\rangle$, there are $p'\leq p$, n, and, P-name $\dot{Z}$ such that $p'\Vdash\{\langle b,Y\rangle\in\dot{\mathcal{B}}\mid|b|=n\}$ is maximal anti-chain below $\langle a,\dot{Z}\rangle\in\mathcal{P}_{\dot{U}}$. If $n\leq|a|$, there is a $\dot{Y}$ such that $p'\Vdash\langle b,\dot{Y}\rangle\in\dot{\mathcal{B}}\wedge a\setminus b\subseteq Y)$. Here, b is the first n-th elements in a. Thus, it is forced that $\langle a,\dot{X}\cap\dot{Y}\rangle\leq\langle b,\dot{Y}\rangle,\langle a,\dot{X}\rangle$.

If $n>|a|$, we can choose $p''\leq p'$ and $\alpha_0,...,\alpha_{n-|a|-1}$ with $p''\Vdash\{\alpha_i\mid i<n-|a|\}\in[(\dot{X}\cap\dot{Z})\setminus(\max a+1)]^{n-|a|}$. Let $c=a\cup\{\alpha_i\mid i<n-|a|\}$. p'' forces that $\langle c,\dot{Z}\rangle\leq\langle a,\dot{Z}\rangle$ meets with $\dot{\mathcal{B}}$. Because of $|c|=n$, there is a $\dot{Y}$ with $p''\Vdash\langle c,\dot{Y}\rangle\in\dot{\mathcal{B}}$. In particular, it is forced that $\langle c,\dot{Y}\cap\dot{X}\rangle$ is a common extension of $\langle c,\dot{Y}\rangle$ and $\langle a,\dot{X}\rangle$, as desired. □

We introduce two forcing notions of variations of Prikry forcing. We call them "Prikry-type".

The first one is Woodin's modification [22], which changes a measurable cardinal into $\aleph_\omega$. For a normal ultrafilter U over μ, let j_U denote the ultrapower mapping $j_U:V\to M_U\simeq\mathrm{Ult}(V,U)$. It is easy to see $|j_U(\mu)|=\mu^+$ if $2^\mu=\mu^+$. This shows

Lemma 2.42. *Suppose that μ is measurable, U is a normal ultrafilter over μ, and $2^\mu=\mu^+$. Then there is a $(M_U,\mathrm{Coll}(\mu^+,<j_U(\mu))^{M_U})$-generic filter $\mathcal{G}$.*

Proof. Since $\mathrm{Coll}(\mu^+,<j_U(\mu)))^{M_U}$ has the $j_U(\mu)$-c.c. in M_U and $|j_U(\mu)^{<j_U(\mu)}|=|j_U(\mu)|=\mu^+$, we can enumerate $\mathrm{Coll}(\mu^+,<j_U(\mu)))^{M_U}$ anti-chain belongs to M_U as $\langle\mathcal{A}_\alpha\mid\alpha<\mu^+\rangle$. Because $\mathrm{Coll}(\mu^+,<j_U(\mu)))^{M_U}$ is μ^+-closed, the standard argument takes a filter $\mathcal{G}$ that meets with any $\mathcal{A}_\alpha$. □

We call this $\mathcal{G}$ a guiding generic of U. $\mathcal{P}_{U,\mathcal{G}}$ is the set of $\langle a, f, X, F\rangle$ such that

- $a = \{\alpha_1, ..., \alpha_{n-1}\} \in [\Psi]^{<\omega}$.
- $f = \langle f_0, ..., f_{n-1}\rangle \in \prod_{i<n} \mathrm{Coll}(\alpha_i^+, <\alpha_{i+1})$. But α_0 and α_n denote ω and μ, respectively.
- $X \in U$ and $X \subseteq \Psi$.
- $F \in \prod_{\alpha \in X} \mathrm{Coll}(\alpha^+, <\mu)$ and $[F] \in \mathcal{G}$.

Here, $\Psi = \{\alpha < \mu \mid \alpha$ is an inaccessible and $2^\alpha = \alpha^+\}$. $\mathcal{P}_{U,\mathcal{G}}$ is ordered by $\langle a, f, X, F\rangle \leq \langle b, g, Y, H\rangle$ if and only if $\langle a, X\rangle \leq \langle b, Y\rangle$ in $\mathcal{P}_U$, $\forall i \in [|b|, |a|)(h(i) \supseteq F(\beta_i))$, and $\forall \alpha \in X(F(\alpha) \supseteq H(\alpha))$. Let LP be the set of all $\langle a, f\rangle$ with $\langle a, f, Z, F\rangle$ for some Z and F.

Lemma 2.43. *Suppose that μ is measurable, U is a normal ultrafilter over μ, and $\mathcal{G}$ is a guiding generic of U. Then*

1. *$\mathcal{P}_{U,\mathcal{G}}$ is $(\mu, <\mu)$-centered.*
2. *The mapping from $\mathcal{P}_{U,\mathcal{G}}$ to $\mathcal{P}_U$ that sends $\langle a, f, X, F\rangle$ to $\langle a, X\rangle$ is a projection.*

Proof. Easy. □

Let us introduce an analogie of Lemma 2.36.

Lemma 2.44. *Suppose that μ is measurable, U is a normal ultrafilter over μ, $\mathcal{G}$ is a guiding generic of U. For every sequence $\langle\langle a, f, X_{a,f}, F_{a,f}\rangle \mid \langle a, f\rangle \in \mathrm{LP}\rangle \subseteq \mathcal{P}_{U,\mathcal{G}}$, there are Z^* and H^* such that, for every $\langle a, f\rangle \in \mathrm{LP}$, every extension of $\langle a, f, Z^*, H^*\rangle$ are compatible with $\langle a, f, X_{a,f}, F_{a,f}\rangle$.*

Proof. By $|\mathrm{LP}| = \kappa$, we can choose a lower bound $[H] \in \mathcal{G}$ in $\{[F_{a,f}] \mid \langle a, f\rangle \in \mathrm{LP}\}$. Fix $Y_{a,f} = \{\alpha < \kappa \mid H(\alpha) \supseteq F_{a,f}(\alpha)\} \in U$. For a sequence $\langle Z_{a,f} \mid \langle a, f\rangle \in \mathrm{LP}\rangle$ of elements in U, let $\triangle_{a,f} Z_{a,f} = \{\alpha \in \Psi \mid \forall\langle a, f\rangle \in \mathrm{LP}(\max(a) < \alpha \wedge \bigcup \mathrm{rng}(f) \subseteq \alpha \to \alpha \in Z_{a,f})\}$. Then $\triangle_{a,f} Z_{a,f} \in U$.

Let $Z^* = \triangle_{a,f} X_{a,f} \cap Y_{a,f}$ and $H^* = H \restriction Z^*$. Z^* and H^* work as witnesses. □

Lemma 2.45 and 2.46 are analogies of Lemma 2.37 and 2.40 for $\mathcal{P}_{U,\mathcal{G}}$, respectively. By Lemma 2.44, we can show Lemma 2.45 as well as the proof of Lemma 2.37.

Lemma 2.45. *Suppose that μ is measurable, U is a normal ultrafilter over μ, $\mathcal{G}$ is a guiding generic of U. For any $\langle a, f, X, F\rangle$ and σ, there is a $\langle a, f, Z, I\rangle$ such that, if $\langle b, g, Y, G\rangle \leq \langle a, f, Z, I\rangle$ decides σ then $\langle a, g \restriction |a|, Z, I\rangle$ decides σ.*

Lemma 2.46. *Suppose that μ is measurable, U is a normal ultrafilter over μ, and $\mathcal{G}$ is a guiding generic of U. For any $\langle a, f, X, F\rangle$ and maximal anti-chain $\mathcal{A}$ below p, there are n, f', Z, I such that $\{\langle b, g, Y, H\rangle \in \mathcal{A} \mid |b| = n\}$ is a maximal anti-chain below $\langle a, f', Z, I\rangle$.*

We have

Theorem 2.47 (Woodin)**.** *Suppose that μ is measurable, U is a normal ultrafilter over μ, and $\mathcal{G}$ is a guiding generic of U. Then $\mathcal{P}_{U,\mathcal{G}}$ forces that*

1. *For each $n < \omega$, $\dot{G}_n = \{f \mid \exists a, X, F(\langle a, f, X, F\rangle \in \dot{G} \wedge |f| = n+1)\}$ is $(V, (\mathrm{Coll}(\omega, <\dot{g}_0) \times \prod_{i<n} \mathrm{Coll}(\dot{g}_i^+, <\dot{g}_{i+1}))^V)$-generic. Here, $\dot{g}_i$ is the i-th element of $\bigcup\{a \mid \exists f, X, F(\langle a, f, X, F\rangle \in \dot{G})\}$.*
2. *If A is bounded subset of some μ then $A \in V[\dot{G}_n]$ for some $n < \omega$.*
3. *$\mu = \aleph_\omega$.*

Proof. 1 follows from a density argument. Let us show 2. Let $\langle a, f, X, F\rangle \Vdash \dot{A} \subseteq \eta$ be arbitrary for some $\eta < \mu$. For each $\alpha < \eta$, applying Lemma 2.45 to $\langle a, f, X, F\rangle$ and the statement $\alpha \in \dot{A}$, there are X_α and F_α that satisfy the consequence of Lemma 2.45. Let $\langle a, f, Y, G\rangle$ be a common extension of $\{\langle a, f, X_\alpha, F_\alpha\rangle \mid \alpha < \eta\}$. This condition forces that

$$\dot{A} = \{\alpha < \eta \mid \exists g \in \dot{G}_{|a|}(\langle a, g, Y, G\rangle \Vdash \alpha \in \dot{A})\} \in V[\dot{G}_{|a|}].$$

3 follows from 1 and 2, as desired. □

Magidor forcing was introduced in [33]. Magidor forcing uses a sequence of normal ultrafilters over μ instead of a single normal ultrafilter. For normal ultrafilters U, U' over μ, $U \lhd U'$ iff $U \in M \simeq \mathrm{Ult}(V, U')$. Let $U = \langle U_\alpha \mid \alpha < \nu\rangle$ be a $\lhd$-increasing sequence with $\nu < \mu$. The Mitchell order $\mathrm{o}(\mu)$ of μ is the height of $\lhd$. Note that $\mathrm{o}(\mu) = (2^\mu)^+$ if μ is supercompact.

For measurable μ with $\mathrm{o}(\mu) \geq \nu$, we fix a sequence $F = \langle F^\alpha_\beta \mid \beta < \alpha < \nu\rangle$ of functions $F^\alpha_\beta \in {}^\mu V$ such that $[F^\alpha_\beta]_{U_\alpha} = U_\beta$ for each $\alpha < \beta < \mu$. We call a pair of U and F coherent system (of length ν) over μ. For each $\alpha < \nu$, define

$$A_\alpha = \{\delta < \mu \mid \forall\beta < \alpha\forall\gamma < \beta(F^\alpha_\gamma(\delta) \lhd F^\alpha_\beta(\delta) \text{ are normal ultrafilters over } \delta)\}.$$
$$B_\alpha = \{\delta \in A_\alpha \setminus (\nu + 1) \mid \forall\beta < \alpha\forall\gamma < \beta([F^\beta_\gamma \restriction \delta]_{F^\alpha_\beta(\delta)} = F^\alpha_\gamma(\delta))\}.$$

Note that $B_\alpha \in U_\alpha$. Magidor forcing $\mathcal{M}_{U,F}$ is the set of pairs $\langle a, X\rangle$ such that

- a is an increasing function such that
  - $\mathrm{dom}(a) \in [\nu]^{<\omega}$ and $\forall\alpha \in \mathrm{dom}(a)(a(\alpha) \in B_\alpha)$.
- X is a function such that
  - $\mathrm{dom}(X) = \nu \setminus \mathrm{dom}(a)$ and $\forall\alpha \in \mathrm{dom}(X)(X(\alpha) \subseteq B_\alpha)$,
  - For every $\alpha \in \mathrm{dom}(X)$, if $\mathrm{dom}(a) \setminus (\alpha + 1) = \emptyset$, $X(\alpha) \in U_\alpha$. Otherwise, $X(\alpha) \in F^\beta_\alpha(a(\rho))$ where $\beta = \min(\mathrm{dom}(a) \setminus (\alpha + 1))$.

$\mathcal{M}_{U,F}$ is ordered by $\langle a, X\rangle \leq \langle b, Y\rangle$ iff $b \subseteq a$, $\forall\alpha \in \mathrm{dom}(X)(X(\alpha) \subseteq Y(\alpha))$ and $\forall\alpha \in \mathrm{dom}(a) \setminus \mathrm{dom}(b)(a(\alpha) \in Y(\alpha))$. $\mathcal{M}_{U,F}$ is $(\mu, <\nu)$-centered. $\mathcal{M}_{U,F}$ preserves all cardinals above μ but changes the cofinality of μ like Prikry forcing. Let $\dot{g}$ be an $\mathcal{M}_{U,F}$-name such that $\mathcal{M}_{U,F} \Vdash \dot{g} = \bigcup\{a \mid \exists X\langle a, X\rangle \in \dot{G}\}$, where $\dot{G}$ is the canonical $\mathcal{M}_{U,F}$-name for a generic filter.

For each $\beta < \nu$, We let $(\mathcal{M}_{U,F})_\beta = \{\langle a, X\rangle_\beta \mid \langle a, X\rangle \in \mathcal{M}_{U,F}\}$ and $(\mathcal{M}_{U,F})^\beta = \{\langle a, X\rangle^\beta \mid \langle a, X\rangle \in \mathcal{M}_{U,F}\}$. Here, $\langle a, X\rangle_\beta$ and $\langle a, X\rangle^\beta$ are $\langle a \restriction (\beta + 1), X \restriction (\beta + 1)\rangle$ and $\langle a \restriction (\nu \setminus (\beta + 1)), X \restriction (\nu \setminus (\beta + 1))\rangle$ respectively. The orders on $(\mathcal{M}_{U,F})_\beta$ and $(\mathcal{M}_{U,F})^\beta$ are naturally defined by that on $\mathcal{M}_{U,F}$. $\mathcal{M}_{U,F}$ can be factored as follows.

Lemma 2.48. *Suppose that μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. For a coherent system $\langle U, F\rangle$ of length ν. For every $\langle a, X\rangle \in \mathcal{M}_{U,F}$ and $\beta \in \mathrm{dom}(a) \cup \{-1\}$, we have*

$$\mathcal{M}_{U,F}/\langle a, X\rangle \simeq (\mathcal{M}_{U,F})_\beta/\langle a, X\rangle_\beta \times (\mathcal{M}_{U,F})^\beta/\langle a, X\rangle^\beta.$$

Note that $(\mathcal{M}_{U,F})_\beta/\langle a, X\rangle_\beta$ has the $a(\beta)^+$-c.c. if $\beta \in \mathrm{dom}(a)$. Lemmas 2.49 and 2.50 are analogues of Lemmas 2.36 and 2.40 for Magidor forcing respectively. See [33] for proofs.

Lemma 2.49. *Suppose that μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. For a coherent system $\langle U, F\rangle$ of length ν. For $\langle a, X\rangle \in \mathcal{M}_{U,F}$, let $\{\langle b, X_b\rangle \mid b \in \mathrm{LP}\}$ be a set of extensions of $\langle a, X\rangle$. Here, $\mathrm{LP} = \{b \mid \exists Y(\langle b, Y\rangle \in \mathcal{M}_{U,F})\}$. Then there is a Z such that $\langle a, Z\rangle \in \mathcal{M}_{U,F}$ and every extension of $\langle b, Y\rangle$ is compatible with $\langle b, X_b\rangle$ if $\langle b, Y\rangle \leq \langle a, Z\rangle$.*

Lemma 2.50 (Prikry lemma). *Suppose that μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. For a coherent system $\langle U, F \rangle$ of length ν. For every $\langle a, X \rangle \in \mathcal{M}_{U,F}$ and statement σ of the forcing language, $\beta \in \mathrm{dom}(a) \cup \{-1\}$, there is a Z such that*

- $\langle a, Z \rangle \leq \langle a, X \rangle$ *and* $\langle a, Z \rangle_\beta = \langle a, X \rangle_\beta$.
- *If* $\langle b, Y \rangle \leq \langle a, Z \rangle$ *decides* σ, *then* $\langle b, Y \rangle_\beta ^\frown \langle a, Z \rangle^\beta$ *decides* σ.

We introduce an analogie of Corollary 2.39 for Magidor forcing.

Corollary 2.51. *Suppose that μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. $\langle U, F \rangle$ is a coherent system of length ν. Then, for every $f : \mathrm{LP} \to 2$, there is a sequence H of length μ such that*

1. $H(\alpha) \in U_\alpha$ *and* $H(\alpha) \subseteq B_\alpha$.
2. *For every* $x \in [\nu]^{<\omega}$, $|f``\{a \in \mathrm{LP} \mid \mathrm{dom}(a) = x \text{ and } \forall \alpha \in x(a(\alpha) \in H(\alpha))\}| \leq 1$.

Proof. By Lemma 2.50, for every x, we have $\langle \emptyset, H_x \rangle \in \mathcal{M}_{U,F}$ such that $\langle \emptyset, H_x \rangle$ decides $f(\dot{g} \restriction x) = 0$ and $f(\dot{g} \restriction x) = 1$. It is easy to see that $|f``\{a \in \mathrm{LP} \mid \mathrm{dom}(a) = x \text{ and } \forall \alpha \in x(a(\alpha) \in H_x(\alpha))\}| \leq 1$. Let H be the coordinate-wise intersection of $\{H_x \mid x \in [\nu]^{<\omega}\}$. H witnesses, as desired. □

Here is the fundamental theorem of Magidor forcing:

Theorem 2.52 (Magidor [33]). *Suppose that μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. For a coherent system $\langle U, F \rangle$ of length ν, the following holds.*

1. $\mathcal{M}_{U,F}$ *adds no new subset to* ν. *Thus, the regularities below* ν *are preserved.*
2. $\mathcal{M}_{U,F}$ *preserves all cardinals.*
3. $\mathcal{M}_{U,F} \Vdash \mathrm{cf}(\mu) = \nu$.

Proof. 1 follows from the proof of 2. We let to show 2. Since $\mathcal{M}_{U,F}$ has the μ^+-c.c., it is enough to show that every cardinal below μ is preserved. For a cardinals $\theta_0 < \theta_1 < \mu$, let $\langle a, X \rangle \Vdash \dot{f} : \theta_0 \to \theta_1$ be an arbitrary. We may assume that θ_1 is a successor cardinal and there is a $\beta \in \mathrm{dom}(a)$ with $a(\beta) < \theta_1 < a(\beta^{+a})$. Here, β^{+a} is $\min a \setminus (\beta + 1)$. By Lemma 2.50, for each $\xi < \theta_0$ and $\eta < \theta_1$, there is a $Z_{\xi,\eta}$ such that

- $\langle a, Z_{\xi,\eta} \rangle_\beta = \langle a, X \rangle_\beta$.
- $\langle a, Z_{\xi,\eta} \rangle \leq \langle a, X \rangle$.
- If $\langle b, Y \rangle \leq \langle a, Z \rangle$ decides $\dot{f}(\xi) = \eta$ then $\langle b, Y \rangle_\beta {}^\frown \langle a, Z_{\xi,\eta} \rangle^\beta$ decides $\dot{f}(\xi) = \eta$.

Let Z be a coordinate-wise union of $\langle Z_{\xi,\eta} \mid \xi < \theta_0, \eta < \theta_1 \rangle$. Since every component in $Z_{\xi,\eta} \restriction \beta$ is in some $a(\beta^+)$-complete ultrafilter, which in turn implies $\langle a, Z \rangle \in \mathrm{M}_{U,F}$. Then,

$$\langle a, Z \rangle \Vdash \dot{f} = \{\langle \xi, \zeta \rangle \mid \exists p \in \dot{G}_\beta (p^\frown \langle a, Z \rangle \Vdash \dot{f}(\xi) = \zeta)\}.$$

Here, $\dot{G}_\beta$ is an $\mathcal{M}_{U,F}$-name for $\dot{G} \cap (\mathcal{M}_{U,F})_\beta$. Note that $\dot{G}_\beta$ is forced to be a $(V, (\mathcal{M}_{U,F})_\beta / \langle a, Z \rangle_\beta)$-generic. By the $a(\beta)^+$-c.c. of $(\mathcal{M}_{U,F})_\beta \restriction \langle a, Z \rangle_\beta$, the range of $\dot{f}$ is forced to be bounded in θ_1, as desired. If we assume $\theta_1 < \nu$, this proof shows 1.

3 follows from 1 and the usual density argument. □

3 Saturation properties of certain forcings

3.1 Term forcing

We use the notion of the term forcing. For a poset P and a P-name $\dot{Q}$ for a poset, the term forcing $T(P,\dot{Q})$ is a complete set of representatives from $\{\dot{q} \mid \Vdash \dot{q} \in \dot{Q}\}$ with respect to the canonical equivalence relation. $T(P,\dot{Q})$ is ordered by $\dot{q} \leq \dot{q}' \leftrightarrow \Vdash \dot{q} \leq \dot{q}'$. The following is known as the basic lemma of the term forcing.

Lemma 3.1 (Laver)**.** *$\mathrm{id} : P \times T(P,\dot{Q}) \to P * \dot{Q}$ is a projection. In particular, $P \Vdash$ there is a projection from $T(P,\dot{Q})$ to $\dot{Q}$.*

Proof. Easy. □

For a later purpose, let us see

Lemma 3.2. *Suppose that P has the κ-c.c. and $|P| \leq \kappa$. Then, for every inaccessible $\lambda > \kappa$, $\mathrm{Coll}(\kappa,<\lambda) \simeq T(P,\dot{\mathrm{Coll}}(\kappa,<\lambda))$ by a continuous dense embedding τ_0. That is, $\prod \tau_0``Z = \tau_0(\prod Z)$ for all $Z \subseteq \mathrm{Coll}(\kappa,<\lambda)$ with $\prod Z \neq 0$.*

Lemma 3.2 follows from

Lemma 3.3. *Suppose that P has the κ-c.c. and $|P| \leq \kappa$. Then the following holds:*

1. *If γ is κ-closed then there is a dense embedding from ${}^{<\kappa}\gamma$ to $T(P,\dot{{}^{<\kappa}\gamma})$.*
2. *If $\langle \dot{Q}_\gamma \mid \gamma \in I\rangle$ is a sequence of P-names of a poset, then there is a dense embedding from $\prod^{<\kappa}_{\gamma\in I} T(P,\dot{Q}_\gamma)$ to $T(P,\prod^{<\kappa}_{\gamma\in I}\dot{Q}_\gamma)$*

Proof. 1. Note that $D = \{\dot{q} \in T(P,\dot{{}^{<\kappa}\gamma}) \mid \exists\delta(\Vdash \mathrm{dom}(\dot{q}) = \delta)\}$ is dense in $T(P,\dot{{}^{<\kappa}\gamma})$. For each $\dot{p} \in T(P,\dot{{}^{<\kappa}\gamma})$, by the κ-c.c. of P, there is a $\delta < \kappa$ with $\Vdash \mathrm{dom}(\dot{p}) < \delta$. The usual density argument takes $\dot{q} \in D$ with $\Vdash \dot{q} \leq \dot{p}$.

By the assumption, there is a sequence $\langle \dot{\tau}_\alpha \mid \alpha < \gamma\rangle$ of P-names for ordinals below γ with the following properties:

- $\Vdash \dot{\tau} \in \gamma$ implies $\Vdash \dot{\tau} = \dot{\tau}_\alpha$ for some α.
- $\not\Vdash \dot{\tau}_\alpha = \dot{\tau}_\beta$ for all $\alpha < \beta$.

For each $p \in {}^{<\kappa}\gamma$, the mapping which sends p to $\langle \dot{\tau}_{p(\xi)} \mid \xi \in \mathrm{dom}(p)\rangle$ is an isomorphism between ${}^{<\kappa}\gamma$ and D. This is a required embedding.

2. Note that $E = \{\dot{q} \in T(P,\prod^{<\kappa}_{\gamma\in I}\dot{Q}_\gamma) \mid \exists d \subseteq I(\Vdash \mathrm{supp}(\dot{q}) = d)\}$ is dense in $T(P,\prod^{<\kappa}_{\gamma\in I}\dot{Q}_\gamma)$ by the similar proof of 1. The natural isomorphism from $\prod^{<\kappa}_{\gamma\in I} T(P,\dot{Q}_\gamma)$ onto E works. □

Proof of Lemma 3.2. We remark that P does not change the class of all κ-closed cardinals. The required embedding τ_0 follows from,

$$\begin{aligned}\mathrm{Coll}(\kappa,<\lambda) &= \textstyle\prod^{<\kappa}_{\gamma\in[\kappa^+,<\lambda)_{\kappa\text{-cl}}} {}^{<\kappa}\gamma \\ &\to \textstyle\prod^{<\kappa}_{\gamma\in[\kappa^+,<\lambda)_{\kappa\text{-cl}}} T(P,\dot{{}^{<\kappa}\gamma}) \\ &\to T(P,\textstyle\prod^{<\kappa}_{\gamma\in[\kappa^+,<\lambda)_{\kappa\text{-cl}}} \dot{{}^{<\kappa}\gamma}) \\ &= T(P,\dot{\mathrm{Coll}}(\kappa,<\lambda)).\end{aligned}$$

The second line and the third line follow from 1 and 2 in Lemma 3.3, respectively. Of course, τ_0 is a dense embedding.

For $p \in \mathrm{Coll}(\kappa,<\lambda)$, let $\dot{p}$ be a P-name such that $\pi_0(p) = \dot{p}$. Then $\dot{p}$ is a P-name such that

- $P \Vdash \mathrm{dom}(\dot{p}) = \mathrm{dom}(p) \subseteq \lambda \times [\kappa^+, \lambda)_{\kappa\text{-cl}}$.
- $P \Vdash \dot{p}(\xi, \zeta) = \dot{\tau}^{\zeta}_{p(\xi,\zeta)}$ for all $\langle \xi, \zeta \rangle \in \mathrm{dom}(p)$).

Here, $\langle \dot{\tau}^{\zeta}_{\alpha} \mid \alpha < \zeta \rangle$ is a sequence of P-names defined in Lemma 3.3 and we identify an element of $\mathrm{Coll}(\kappa, < \lambda)$ with a partial function from $\kappa \times [\kappa^+, \lambda)_{\kappa\text{-cl}}$ to λ.

Consider a set $Z = \{p_i \mid i < \nu\} \subseteq P \times \mathrm{Coll}(\kappa, < \lambda)$ such that $\prod Z \in \mathrm{Coll}(\kappa, < \lambda)$. We remark that $\prod Z = \bigcup_i p_i$. We let $p = \bigcup_i p_i$. Our goal is showing $\pi_0(p) = \prod \pi_0 “Z$. Note that

- $P \Vdash \dot{p}_i(\xi, \zeta) = \dot{\tau}^{\zeta}_{p_i(\xi,\zeta)} = \dot{\tau}^{\zeta}_{p(\xi,\zeta)} = \dot{p}(\xi, \zeta)$ for each $\langle \xi, \zeta \rangle \in \mathrm{dom}(p_i)$.
- $P \Vdash \mathrm{dom}(\dot{p}) = \mathrm{dom}(p) = \bigcup_{i<\nu} \mathrm{dom}(p_i) = \bigcup_{i<\nu} \mathrm{dom}(\dot{p}_i)$.

Therefore, $\Vdash \dot{p} = \bigcup_i \dot{p}_i$. In particular, we have $\pi_0(p) = p = \prod \pi_0 “Z$. □

For a later purpose, we introduce the following lemma that is an analogie of 2 in Lemma 3.3. for Easton support product.

Lemma 3.4. *Suppose P has the κ-c.c. and $|P| \leq \kappa$. If $\langle \dot{Q}_\gamma \mid \gamma \in K \rangle$ is a sequence of P-names of a poset such that $K \subseteq \mathrm{Reg}$, then there is a dense embedding from $\prod^E_{\gamma \in K} T(P, \dot{Q}_\gamma)$ to $T(P, \prod^E_{\gamma \in K} \dot{Q}_\gamma)$.*

The term forcings of Levy collapses are Levy collapses, this is very explicit. On the other hand, the term forcing is not clear to study. For example, the term forcing $T(P, \dot{Q})$ may not have the chain condition unless $\dot{Q}$ is forced to have the chain condition. Let us study the λ-c.c. of term forcings.

Lemma 3.5. *Suppose that λ is weakly compact. Let P be a poset and $\dot{Q}$ be a P-name for a poset. If $|P| < \lambda$ and $P \Vdash \dot{Q}$ has the λ-c.c. then $T(P, \dot{Q})$ has the λ-c.c.*

Proof. Note that the size of the completion of P is at most $2^{|P|} < \lambda$. We may assume that P is a complete Boolean algebra.

Suppose otherwise, let $\{\dot{p}_\alpha \mid \alpha < \lambda\}$ be an anti-chain of $T(P, \dot{Q})$. For all $\alpha < \beta$, it is easy to see $||\dot{p}_\alpha \cdot \dot{p}_\beta = 0|| \neq 0$. By $|P| < \lambda$ and weakly compactness of λ, there are $H \in [\lambda]^\lambda$ and $b \in P$ such that $\forall \alpha, \beta \in H(||\dot{p}_\alpha \cdot \dot{p}_\beta = 0|| = b)$. b forces that $\{\dot{p}_\alpha \mid \alpha \in H\}$ is an anti-chain of $\dot{Q}$. This is a contradiction. □

We need the weakly compactness to show. Indeed,

Proposition 3.6. *If T is a λ-Suslin tree then, for every (non-trivial) poset P, $T(P, \check{T})$ does not have the λ-c.c.*

Proof. By these assumptions, we can choose

- $\langle p_\xi, q_\xi \mid \xi < \lambda \rangle \subseteq T \times T$ is an anti-chain.
- $a, (-a) \in P \setminus \{1\}$.

For each $\xi < \lambda$, let $\dot{r}_\xi$ be a P-name such that

- $a \Vdash \dot{r}_\xi = p_\xi$.
- $(-a) \Vdash \dot{r}_\xi = q_\xi$.

Then $\dot{r}_\xi \in T(P, \check{T})$. If $T(P, \check{T})$ has the λ-c.c. then there are $\xi < \zeta$ such that $\Vdash \dot{r}_\xi \cdot \dot{r}_\zeta \neq 0$. By the definition of $\dot{r}_\xi$, $p_\xi \cdot p_\zeta, q_\xi \cdot q_\zeta \neq 0$, which in turn implies that $\langle p_\xi, q_\xi \rangle \cdot \langle p_\zeta, q_\zeta \rangle \neq 0$ in $T \times T$. This is a contradiction. □

Theorem 3.7. *If λ is weakly compact then there is a poset $\mathcal{T}$ which forces that λ is Mahlo and there is a λ-Suslin tree T. In particular, $\mathcal{T}$ forces that $T(P,T)$ does not have the λ-c.c. but $P \Vdash T$ has the λ-c.c. for all P with $|P| < \lambda$.*

Proof. By the proof of [17, Theorem 7.32], we have a poset $\mathcal{T}$ such that

- $\mathcal{T}$ has the λ-c.c.,
- $\mathcal{T}$ is $< \lambda$-Baire, and,
- $\mathcal{T}$ forces there is a λ-Suslin tree T.

Since $\mathcal{T}$ is $< \lambda$-Baire, $\mathcal{T}$ does not change $\mathrm{Reg} \cap \lambda$. By the λ-c.c. of $\mathcal{T}$, $\mathrm{Reg} \cap \lambda$ remains a stationary subset in the extension by $\mathcal{T}$. Therefore λ is Mahlo in the extension, as desired.

We discuss in the extension. For every poset P with $|P| < \lambda$, we have that $P \Vdash \check{T}$ has the λ-c.c. by $|P| < \lambda$. By Lemma 3.6, $T(P, \check{T})$ does not have the λ-c.c. □

If $\dot{Q}$ is forced to have more strong saturation property, we can omit the weakly compactness as follows.

Proposition 3.8. *For a stationary subset $S \subseteq \lambda$, let P be a poset and $\dot{Q}$ be a P-name for a poset. If $2^{|P|} < \lambda$ and $P \Vdash \dot{Q}$ is S-layered and λ-dense then $T(P, \dot{Q})$ is S-layered and λ-dense.*

Proof. We may assume that the size of $\dot{Q}$ is forced to be less than λ. The standard cardinal arithmetic shows $|T(P, \dot{Q})| \leq \lambda$. Let $\langle Q_\alpha \mid \alpha < \lambda \rangle$ be a filtration of $T(P, \dot{Q})$. Let $\dot{Q}_\alpha$ be a P-name for $Q_\alpha[\dot{G}] = \{\dot{q}^{\dot{G}} \mid \dot{q} \in Q_\alpha\}$.

Since $\dot{Q}$ is forced to be S-layered and λ-dense, we have a club C such that $\forall \alpha \in C \cap S (\Vdash \dot{Q}_\alpha \lessdot \dot{Q})$. This shows that $\forall \alpha \in C \cap S (Q_\alpha \lessdot T(P, \dot{Q}))$, as desired. □

3.2 Quotient analysis

By Theorem 2.20, our studies are reduced to saturation properties of quotient forcings. First, we see that the continuity, that appeared in Lemma 3.2, is useful to study the quotient forcing.

Definition 3.9. *Suppose $\pi : Q \to P$ is a projection between complete Boolean algebras. We say that π is $< \mu$-continuous if $\pi(\prod Z) = \prod \pi``Z$ for all $Z \in [Q]^{<\mu}$ with $\prod^Q Z \neq 0$.*

We also say that π is continuous if π is $< \mu$-continuous for all μ.

For a projection $\pi : Q \to P$ between posets, we say that P is $< \mu$-continuous if the lifting $\pi : \mathcal{B}(Q) \to \mathcal{B}(P)$ is $< \mu$-continuous. It is easy to see that this is equivalent with $\prod \pi``X = \pi(\prod X)$ for every $X \in [P]^{<\mu}$ with $\prod X \neq 0$.

Lemma 3.10. *Suppose that P is $< \mu$-Baire, $\pi : Q \to P$ is a $< \mu$-continuous projection between complete Boolean algebras, and Q has the $(\lambda, \lambda, < \mu)$-c.c. then $P \Vdash Q/\dot{G}$ has the $(\lambda, \lambda, < \mu)$-c.c.*

Proof. Let p and $\{\dot{q}_\alpha \mid \alpha < \lambda\}$ be arbitrary with $p \Vdash \dot{q}_\alpha \in Q/\dot{G}$. For each α, we can take r_α such that $\pi(r_\alpha) \leq p$ and $\pi(r_\alpha) \Vdash r_\alpha \leq \dot{q}_\alpha$ in $Q/\dot{G}$. Since Q has the $(\lambda, \lambda, < \mu)$-c.c., there is a $K \in [\lambda]^\lambda$ such that $\forall Z \in [K]^{<\mu} (\prod_{\alpha \in Z} q_\alpha \neq 0)$. Let $b = ||\,|\{\alpha \in K \mid \pi(q_\alpha) \in \dot{G}\}| = \lambda||$. Since P has the λ-c.c., $b \neq 0$. Let $\dot{K}$ be a P-name for $\{\alpha \in K \mid \pi(q_\alpha) \in \dot{G}\}$.

We claim that $b \leq p$ forces $\forall Z \in [\dot{K}]^{<\lambda} (\prod_{\alpha \in Z} q_\alpha \in Q/\dot{G})$. Consider $q \leq b$ and $\dot{Z}$ such that $q \Vdash \dot{Z} \in [\dot{K}]^{<\mu}$. By the $< \mu$-Baireness of P, we may assume that $q \Vdash \dot{Z} = Z$ for some $Z \in [K]^{<\mu}$. For each $\alpha \in Z$, we have $q \Vdash \pi(r_\alpha) \in \dot{G}$, and thus $q \leq \pi(r_\alpha)$. Because of $q \leq \prod_{\alpha \in Z} \pi(r_\alpha) = \pi(\prod_{\alpha \in Z} r_\alpha)$, q forces $\prod_{\alpha \in Z} r_\alpha \in Q/\dot{G}$. q also forces $\prod_{\alpha \in Z} r_\alpha \leq r_\alpha \leq \dot{q}_\alpha$ for each $\alpha \in Z$, as desired. □

Next, we consider the case of layeredness.

Lemma 3.11. *Suppose that Q is S-layered for some stationary subset $S \subseteq \lambda$, Q is λ-dense, and $\pi : Q \to P$ is a 2-continuous projection. Then $P \Vdash Q/\dot{G}$ is S-layered.*

Proof. We may assume that P and Q are Boolean algebras. Remark that they need not be complete. We may assume that $|Q| = \lambda$. Let $\langle \mathcal{B}_\delta \mid \delta < \lambda \rangle$ be a filtration of Q with each $\mathcal{B}_\delta$ is a Boolean subalgebra of Q. Because P has the λ-c.c., S remains stationary in the extension by P. It is enough to prove that $\mathcal{B}_\delta \lessdot Q$ implies $P \Vdash \mathcal{B}_\delta/\dot{G} \lessdot Q/\dot{G}$ for each δ.

Let $D = \{q \in Q \mid \exists b \in \mathcal{B}_\delta (b \geq q \text{ and } b \text{ is a reduct of } q)\}$. D is dense in Q. For each $q \in Q$, q has a reduct $b \in \mathcal{B}_\delta$. It is easy to see that b is a reduct of $q \cdot b$. Thus, $q \cdot b \in D$ and this extends q.

To show $P \Vdash \mathcal{B}_\delta/\dot{G} \lessdot Q/\dot{G}$, take an arbitrary $p \in P$ and $q \in Q$ with $p \Vdash q \in Q/\dot{G}$. We may assume $q \in D$. Thus, q has a reduct $b \geq q$. Because of $p \leq \pi(q) \leq \pi(b)$, $p \Vdash b \in \mathcal{B}_\delta/\dot{G}$. It suffices to show that $p \Vdash \forall c \in \mathcal{B}_\delta/\dot{G}(c \leq b \to c \cdot q \in Q/\dot{G})$. For any $p' \leq p$ and $c \leq b$ with $p' \Vdash c \in \mathcal{B}_\delta/\dot{G}$, Since b is a reduct of q and π is 2-continuous, $p' \leq \pi(c) \cdot \pi(q) = \pi(c \cdot q)$. Thus, $p' \Vdash c \cdot q \in Q/\dot{G}$. □

Lemma 3.12. *For a projection $\pi : Q \to P$, suppose that π is $< \nu$-continuous, P is $< \nu$-Baire, and $P \Vdash Q$ is $(\lambda, < \nu)$-centered. Then $P \Vdash Q/\dot{G}$ is $(\lambda, < \nu)$-centered.*

Proof. We may assume that P and Q are Boolean algebras. Let G be an arbitrary (V, P)-generic filter. We discuss in $V[G]$. Let $\langle F_\xi \mid \xi < \lambda \rangle$ be a centering of Q. It is enough to prove that $Z \in [F_\xi \cap Q/G]^{<\nu}$ implies $\prod Z \in Q/G$.

Since P is $< \nu$-Baire, $Z \in V$. Note that $\pi(p) \in G$ for each $p \in Z$. Since F_ξ is a centered subset, $\prod Z \neq 0$ in Q. The $< \mu$-continuity implies $\pi(\prod Z) = \prod \pi``Z \in G$, as desired. □

The $(\lambda, \lambda, < \mu)$-c.c. of the quotient can be characterized in term of properties of projections as

Definition 3.13. *For a projection $\pi : Q \to P$ between complete Boolean algebras, we say that π is $(\lambda, \lambda, < \mu)$-nice if, for every $X \in [Q]^\lambda$, there is a $Y \in [Q]^\lambda$ with the following properties:*

- *There is an injection $f : Y \to X$ such that $y \leq f(y)$ for all $y \in Y$.*
- *$\prod Z \neq 0$ and $\pi(\prod Z) = \prod \pi``Z$ for all $Z \in [Y]^{<\mu}$.*

Theorem 3.14. *Suppose that P is $< \mu$-Baire, $\pi : Q \to P$ is a projection between complete Boolean algebras, and Q has the $(\lambda, \lambda, < \mu)$-c.c. Then the following are equivalent.*

1. *π is $(\lambda, \lambda, < \mu)$-nice.*
2. *$P \Vdash Q/\dot{G}$ has the $(\lambda, \lambda, < \mu)$-c.c.*

Proof. The forward direction can be shown as in the proof of Lemma 3.10. We should check the inverse direction. Let $\{q_\alpha \mid \alpha < \lambda\} \subseteq Q$ be arbitrary. We let $b = || |\{\alpha < \lambda \mid \pi(q_\alpha) \in \dot{G}\}| = \lambda ||$ and $\dot{K}$ be a P-name for $\{\alpha < \lambda \mid \pi(q_\alpha) \in \dot{G}\}$. Since P has the λ-c.c., $b \neq 0$. By the definition of quotient forcing, $b \Vdash \{q_\alpha \mid \alpha \in \dot{K}\} \subseteq Q/\dot{G}$. Because P forces that $Q/\dot{G}$ has the $(\lambda, \lambda, < \mu)$-c.c., we can choose $\dot{K}'$ such that $b \Vdash \dot{K}' \in [\dot{K}]^\lambda$ and $\prod_{\alpha \in Z} q_\alpha \neq 0$ for all $Z \in [\dot{K}']^{<\mu}$.

By the λ-c.c. of P, $K = \{\alpha < \lambda \mid b \cdot ||\alpha \in \dot{K}'|| \neq 0\}$ is of size λ. Define $p_\alpha = b \cdot ||\alpha \in \dot{K}'||$ for each $\alpha \in K$. There is a $K' \in [K]^\lambda$ with $\forall Z \in [K']^{<\omega}(\prod_{\alpha \in K'} p_\alpha \neq 0)$. Observe that for every $Z \in [K']^{<\mu}$, $\prod_{\alpha \in Z} p_\alpha$ forces $\prod_{\alpha \in Z} q_\alpha \in Q/\dot{G}$, and thus, $\prod_{\alpha \in Z} p_\alpha = \prod_{\alpha \in Z} p_\alpha \cdot \pi(\prod_{\alpha \in Z} q_\alpha)$.

Let $r_\alpha = q_\alpha \cdot e(p_\alpha)$, where e is a complete embedding induced by π. We claim that $\prod_{\alpha \in Z} \pi(q_\alpha) = \pi(\prod_{\alpha \in Z} q_\alpha)$ for every $Z \in [K']^{<\mu}$. This follows from:

$$\begin{aligned}\textstyle\prod_{\alpha \in Z} \pi(r_\alpha) &= \textstyle\prod_{\alpha \in Z} p_\alpha \\ &= \textstyle\prod_{\alpha \in Z} p_\alpha \cdot \pi(\prod_{\alpha \in Z} q_\alpha) \\ &= \textstyle\pi(\prod_{\alpha \in Z} q_\alpha \cdot \prod_{\alpha \in Z} e(p_\alpha)) \\ &= \textstyle\pi(\prod_{\alpha \in Z} q_\alpha \cdot e(p_\alpha)) = \pi(\prod_{\alpha \in Z} r_\alpha).\end{aligned}$$

Thus, $\{r_\alpha \mid \alpha \in K'\}$ witnesses to $(\lambda, \lambda, <\mu)$-nice. □

In particular, Knasterness of the quotient forcing can be characterized in term of projections as follows.

Corollary 3.15. *Suppose that $\pi : Q \to P$ is a projection between complete Boolean algebras and Q is λ-Knaster. Then the following are equivalent.*

1. *π is $(\lambda, \lambda, 2)$-nice.*
2. *$P \Vdash Q/\dot{G}$ is λ-Knaster.*

We will show that Corollary 3.15 is not meaningless, that is, 2 does not hold unconditionally. To see this, we use Todorčević's construction of a Suslin tree from a Cohen real.

Lemma 3.16 (Todorčević [43])**.** *There is an $\langle e_\alpha : \alpha \to \omega \mid \alpha < \omega_1\rangle$ with the following properties:*

1. *$\{\xi < \alpha \mid e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite for all $\alpha < \beta$.*
2. *$\{\xi < \alpha \mid e_\alpha(\xi) \leq n\}$ is finite for all $n < \omega$.*

Proposition 3.17. *There is a projection $\pi : Q \to P$ between $\aleph_1$-Knaster posets such that $P \Vdash Q/\dot{G}$ is not $\aleph_1$-Knaster. In particular, π is not $(\aleph_1, \aleph_1, 2)$-nice.*

Proof. Let C be a Cohen forcing, that is, $C = {}^{<\omega}\omega$. C is ordered by reverse inclusion. Let $\dot{c}$ be a C-name such that $C \Vdash \dot{c} = \bigcup \dot{G}$. Todorčević showed that C forces that the poset $\dot{T} = \{\dot{c} \circ e_\alpha \restriction \beta \mid \beta \leq \alpha < \omega_1\}$, ordered by reverse inclusion, has the $\aleph_1$-c.c. and is not $\aleph_1$-Knaster. We refer to [43] for more details.

Let $P = C$, $Q = C * \dot{T}$ and $\pi : Q \to P$ be a natural projection. Of course, $P \Vdash Q/\dot{G} \simeq \dot{T}$ is not $\aleph_1$-Knaster.

It remains to show that Q is $\aleph_1$-Knaster. Let $X = \{\langle p_i, \dot{c} \circ e_{\alpha_i} \restriction \beta_i\rangle \mid i < \omega_1\}$ be arbitrary. Shrinking X, there are $K \in [\omega_1]^{\omega_1}$ and p such that $p_i = p$ for all $i \in K$. For each $i \in K$, $a_i = \{\xi < \alpha_i \mid e_{\alpha_i}(\xi) \leq |p|\}$ is finite. The usual Δ-system argument takes $K' \in [\omega_1]^{\omega_1}$ and r such that $a_i \cap a_j = r$ for each $i < j$ in K'. Note that the number of functions that has a form of $e_\alpha \restriction r$ is ω at most. There is a $K'' \in [K']^{\omega_1}$ such that $e_{\alpha_i} \restriction a = e_{\alpha_j} \restriction a$ for each $i < j$ in K''. We claim that any two elements in $Y = \{\langle p_i, \dot{c} \circ e_{\alpha_i} \restriction \beta_i\rangle \mid i \in K''\}$ are compatible.

Fix a pair $i < j$ in K''. For every ξ, if $e_{\alpha_i}(\xi), e_{\alpha_j}(\xi) < |p|$ then $\xi \in r$, which in turn implies $e_{\alpha_i}(\xi) = e_{\alpha_j}(\xi)$. This ensures us, for every ξ with $e_{\alpha_i}(\xi) \neq e_{\alpha_j}(\xi)$, one of the following holds:

- $e_{\alpha_i}(\xi), e_{\alpha_j}(\xi) \geq |p|$.
- $e_{\alpha_i}(\xi) \geq |p|$ and $e_{\alpha_j}(\xi) < |p|$.
- $e_{\alpha_j}(\xi) \geq |p|$ and $e_{\alpha_i}(\xi) < |p|$.

Since $\Delta = \{\xi \mid e_{\alpha_i}(\xi) \neq e_{\alpha_i}(\xi)\}$ is finite, $m = \max(e_{\alpha_i}``\Delta) \cup (e_{\alpha_j}``\Delta) + 1$ is a natural number. Define $q \in {}^m\omega$ by

$$q(n) = \begin{cases} p(n) & n < |p| \\ p(e_{\alpha_j}(\xi)) & \text{there is a } \xi \text{ such that } n = e_{\alpha_i}(\xi) \text{ and } e_{\alpha_j}(\xi) < |p| \\ p(e_{\alpha_i}(\xi)) & \text{there is a } \xi \text{ such that } n = e_{\alpha_j}(\xi) \text{ and } e_{\alpha_i}(\xi) < |p| \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $\langle q, \dot{c} \circ e_{\alpha_k} \restriction \beta_k\rangle$ is a common extension of $\langle p, \dot{c} \circ e_{\alpha_i} \restriction \beta_i\rangle$ and $\langle p, \dot{c} \circ e_{\alpha_j} \restriction \beta_j\rangle$, here k is i or j such that $\beta_i, \beta_j \leq \beta_k$. □

We saw that continuity is useful to study the quotient forcing. We will use these in Section 5.1. By the way, continuity does not hold in many cases. Let us introduce the following lemmas that work even if a projection is not continuous.

Lemma 3.18. *Suppose that $\tau : P \to Q$ is a complete embedding between posets and Q has the (λ, ν, ν)-c.c. Then P forces that $Q/\dot{G}$ has the (λ, ν, ν)-c.c.*

Proof. We may assume that P and Q are complete Boolean algebras. Let $p \Vdash \{\dot{q}_\alpha \mid \alpha < \lambda\} \subseteq Q/\dot{G}$ be arbitrary. For each $\alpha < \lambda$, there are $p_\alpha \leq p$ and $q_\alpha \in Q$ such that $p_\alpha \Vdash \dot{q}_\alpha = q_\alpha$. By the (λ, ν, ν)-c.c. of Q, there is a $Z \in [\lambda]^\nu$ such that $\prod_{\alpha \in Z} \tau(p_\alpha) \cdot q_\alpha \neq 0$. It is easy to see that

$$\prod_{\alpha \in Z} \tau(p_\alpha) \cdot q_\alpha = \prod_{\alpha \in Z} \tau(p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha = \tau(\prod_{\alpha \in Z} p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha.$$

Let r be a reduct of $\tau(\prod_{\alpha \in Z} p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha$. Then $r \leq \prod_{\alpha \in Z} p_\alpha \leq p$ and this forces that $\prod_{\alpha \in Z} q_\alpha \in Q/\dot{G}$ is a lower bound of $\{\dot{q}_\alpha \mid \alpha \in Z\}$, as desired. □

Lemma 3.19. *Suppose that P is $(\mu, < \nu)$-centered, Q is $(\lambda, < \nu)$-centered, and $\dot{R}$ is a Q-name for a $(\mu, < \nu)$-centered poset. We also assume that a mapping $\tau : P \to Q * \dot{R}$, which has the form of $\tau(p) = \langle 1, f(p) \rangle$, is complete and there is a $\langle P_\alpha, \dot{R}_\alpha \mid \alpha < \mu \rangle$ such that P_α is a filter, $P = \bigcup_{\alpha < \lambda} P_\alpha$, $\dot{R}_\alpha$ is a Q-names for a $< \nu$-complete filter, and $Q \Vdash f``P_\alpha \subseteq \dot{R}_\alpha$ and $\bigcup_\alpha \dot{R}_\alpha = \dot{R}$. If $\lambda^\mu = \lambda$ then the term forcing $T(P, Q * \dot{R}/\dot{G})$ is $(\lambda, < \nu)$-centered. In particular, if P is $< \nu$-Baire then $P \Vdash Q * \dot{R}/\dot{G}$ is $(\lambda, < \nu)$-centered.*

Proof. We may assume that Q is a complete Boolean algebra. Let $F : Q \to \lambda$ be a centering function.

We want to define a centering function $l : T(P, Q * \dot{R}/\dot{G}) \to \lambda$. For each $\dot{p} \in T(P, Q * \dot{R}/\dot{G})$, we have a maximal anti-chain $\mathcal{A}_{\dot{p}} \subseteq P$ such that every $p \in \mathcal{A}_{\dot{p}}$ forces $\dot{p} = \langle q, \dot{r} \rangle$ for some $\langle q, \dot{r} \rangle \in Q * \dot{R}$. Note that p is a reduct of $\langle q, \dot{r} \rangle$.

Define $l(p)$ by $\langle F(q \cdot ||\dot{r} \in \dot{R}_\alpha||) \mid p \in \mathcal{A}_{\dot{q}}, \alpha < \mu, p \Vdash \dot{p} = \langle q, \dot{r} \rangle \rangle$. Note that the size of μ-centered posets is at most 2^μ. By the assumption, the number of $\mathcal{A}_p$ is at most λ. This observation enables us to identify the range of l with λ. Suppose $l(\dot{p}_0) = \cdots = l(\dot{p}_i) = \cdots$ $(i < \zeta < \nu)$. Put $\mathcal{A} = \mathcal{A}_{\dot{p}}$. It is enough to show that each $p \in \mathcal{A}$ forces $\prod_i \dot{p}_i \in Q * \dot{R}/\dot{G}$. Fix $p \in \mathcal{A}$. Then, for each $i < \zeta$, there is a $\langle q_i, \dot{r}_i \rangle \in Q * \dot{R}$ such that p forces $\dot{p}_i = \langle q_i, \dot{r}_i \rangle$.

For every $r \leq p$, since p is a reduct of $\langle q_i, \dot{r}_i \rangle$, there is an $\alpha < \mu$ such that $q_i \cdot ||\dot{r}_i \cdot f(r) \in \dot{R}_\alpha|| \neq 0$ for some (any) $i < \zeta$. By $Q \Vdash f``P_\alpha \subseteq \dot{R}_\alpha$, $r \in P_\alpha$. Note that $\prod_i q_i \cdot ||f(r) \cdot \prod_i \dot{r}_i \in \dot{R}_\alpha|| = \prod_i q_i \cdot ||\dot{r}_i \in \dot{R}_\alpha|| \neq 0$. p forces that

$$\begin{aligned}
\tau(r) \cdot \textstyle\prod_i \dot{p}_i &= \tau(r) \cdot \textstyle\prod_i \langle q_i, \dot{r}_i \rangle \\
&\geq \langle 1, f(r) \rangle \cdot \textstyle\prod_i \langle q_i \cdot ||\dot{r}_i \in \dot{R}_\alpha||, \dot{r}_i \rangle \\
&= \langle 1, f(r) \rangle \cdot \langle \textstyle\prod_i q_i \cdot || \prod_i \dot{r}_i \in \dot{R}_\alpha||, \prod_i \dot{r}_i \rangle \\
&= \langle \textstyle\prod_i q_i \cdot || \prod_i \dot{r}_i \cdot f(r) \in \dot{R}_\alpha||, \prod_i \dot{r}_i \cdot r \rangle \\
&\neq 0.
\end{aligned}$$

The translation of lines three to four follows from $||f(r) \in \dot{R}_\alpha|| = 1$ and $|| \prod_i \dot{r}_i \in \dot{R}_\alpha|| \cdot ||f(r) \in \dot{R}_\alpha|| = || \prod_i \dot{r}_i \cdot f(r) \in \dot{R}_\alpha|| \leq || \prod_i \dot{r}_i \cdot f(r) \neq 0||$. Therefore p is a reduct of $\prod_i \langle q_i, \dot{r}_i \rangle$. p forces $\prod_i \langle q_i, \dot{r}_i \rangle = \prod_i \dot{p}_i \in Q * \dot{R}/\dot{G}$. In particular, $\prod_i \dot{p}_i$ in the term forcing and it is a lower bound of $\dot{p}_i$'s. By Lemma 3.1, if P is $< \nu$-Baire then $P \Vdash Q * \dot{R}/\dot{G}$ is $(\lambda, < \nu)$-centered. □

Lastly, we let to describe a sufficient condition for the quotient forcing *not* to be S-layered. We say that Q is nowhere S-layered if $Q \upharpoonright q$ is not S-layered for all $q \in Q$.

Lemma 3.20. *Suppose that Q is nowhere S-layered for some $S \subseteq E^{\lambda^+}_{\geq \kappa}$, and Q is of size λ^+. We also assume that there is a complete embedding τ from κ-c.c. P to Q. Then $P \Vdash Q/\dot{G}$ is not S-layered.*

Proof. Suppose otherwise. That is, there is a $p \in P$ which forces that $Q/\dot{G}$ is S-layered. By Lemma 2.8, we can fix P-names $\dot{R}_\alpha$ such that

- $p \Vdash \dot{R}_\alpha \lessdot Q/\dot{G}$ for each $\alpha < \lambda^+$.
- $p \Vdash \alpha < \beta \to \dot{R}_\alpha \subseteq \dot{R}_\beta$.
- $p \Vdash$ there is a club $\dot{C} \subseteq \lambda^+$ such that $\forall \alpha \in \dot{C} \cap S(\dot{R}_\alpha = \bigcup_{\beta<\alpha} \dot{R}_\beta)$.

By the κ-c.c. of P, there is a club D such that $p \Vdash D \subseteq \dot{C}$. We claim that $P \restriction p * (Q/\dot{G})$ is S-layered. Let $Q_\alpha = P * \dot{R}_\alpha \restriction \langle p, \dot{1}\rangle$. It is easy to see that $Q_\alpha \lessdot P * (Q/\dot{G}) \restriction \langle p, \dot{1}\rangle$. For $\alpha \in C \cap S$, choose $\langle p_0, \dot{q}_0\rangle \in Q_\alpha$ then $p_0 \leq p$ and $P \Vdash \dot{q}_0 \in \dot{R}_\alpha$. Since $\mathrm{cf}(\alpha) \geq \kappa$ and P has the κ-c.c., there is an $\beta < \alpha$ such that $P \Vdash \dot{q}_0 \in P_\beta$. Therefore $\langle p_0, \dot{q}_0\rangle \in R_\beta$, as desired.

$\mathcal{B}(Q)$ has a dense subset which is isomorphic to $P * (Q/\dot{G})$. Since $P * (Q/\dot{G}) \restriction \langle p, \dot{1}\rangle$ is S-layered, Q is *not* nowhere S-layered. This is a contradiction. □

4 Preservation theorems of saturation

4.1 Generalization of Foreman's lemmas

For a given saturated ideal I, We are interested in saturation properties of $\overline{I}$ in some extension as we have seen in Section 2.3. For an ideal I over μ^+, in [14], Foreman proved that it is forced that $\overline{I}$ is saturated by any μ-centered poset if I is saturated. Foreman also claimed that saturated can be replaced by centered in [16] without proof. In this section, we prove them by giving generalizations. Let us show

Lemma 4.1. *For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$) and a $(\mu, <\nu)$-centered poset P,*

1. *If I is (λ^+, ν', ν')-saturated then $P \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \nu$.*
2. *If I is $(\lambda, <\nu)$-centered, P is $<\nu$-Baire, and $\lambda^\mu = \lambda$ then $P \Vdash \overline{I}$ is $(\lambda, <\nu)$-centered.*

Proof. Let $\dot{j}$ be a $\mathcal{P}(Z)/I$-name for the generic ultrapower mapping. By Lemma 2.20, we have a complete embedding $\tau : P \to \mathcal{P}(Z)/I * \dot{j}(P)$ that sends p to $\langle 1, \dot{j}(p)\rangle$ such that $P \Vdash \mathcal{P}(Z)/I * \dot{j}(P)/\dot{G} \simeq \dot{\mathcal{P}}(Z)/\overline{I}$. Let $\dot{M}$ denote the range of $\dot{j}$, then $\mathcal{P}(Z)/I \Vdash {}^{\lambda}\dot{M} \subseteq \dot{M}$. This shows that $\dot{j}(P)$ is $(\lambda, <\nu)$-centered.

First, we check 1. Since $\mathcal{P}(Z)/I$ forces that $\dot{j}(P)$ is $(\lambda, <\nu)$-centered, and thus, $\dot{j}(P)$ has the (λ, ν', ν')-c.c. Therefore $\mathcal{P}(Z)/I * \dot{j}(P)$ has the (λ, ν', ν')-c.c. By Lemma 3.18, $P \Vdash \mathcal{P}(Z)/I * \dot{j}(P)/\dot{G}$ has the (λ, ν', ν')-c.c.

Lastly, we check 2. Let $\langle P_\alpha \mid \alpha < \lambda\rangle$ be a centering family of P. We may assume that each P_α is a filter. Let $\dot{R}_\alpha$ be a $\mathcal{P}(Z)/I$-name for $\dot{j}(P_\alpha)$. Then $\mathcal{P}(Z)/I \Vdash \dot{j}(P) = \bigcup_{\alpha<\mu} R_\alpha$ and $\dot{j}``P_\alpha \subseteq R_\alpha$.

Since $\mathcal{P}(Z)/I$ is $(\lambda, <\nu)$-centered and $\lambda^\mu = \lambda$, we can apply Lemma 3.19 to $\tau : P \to \mathcal{P}(Z)/I * \dot{j}(P)$ and $\langle P_\alpha, \dot{R}_\alpha \mid \alpha < \mu\rangle$. By the $<\nu$-Baireness of P, P forces that $\mathcal{P}(Z)/I * \dot{j}(P)/\dot{G} \simeq \dot{\mathcal{P}}(Z)/\overline{I}$ is $(\lambda, <\nu)$-centered. □

Corollary 4.2. *Let $\nu < \mu < \lambda$ be cardinals. μ and λ are regular. Let Z be one of $\mathcal{P}_{\mu^+}\lambda$, $[\lambda]^{\mu^+}$ or $[\lambda^+]^{\mu^+}$. If I is a normal, fine, μ^+-complete ideal over Z and P is $(\mu, <\nu)$-centered. Then,*

1. *If I is (λ^+, ν', ν')-saturated then $P \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \nu$.*
2. *If I is $(\lambda, <\nu)$-centered, P is $<\nu$-Baire and $\lambda^\mu = \lambda$ then $P \Vdash \overline{I}$ is $(\lambda, <\nu)$-centered.*

Note that the λ^+-saturation is the $(\lambda^+, 2, 2)$-saturation.

4.2 Preservation and destruction via Prikry-type forcings

We studied saturation properties of $\overline{I}$ in the extension by P. In this section, we focus in the case of P is a Prikry-type forcing. Here, we deal Prikry forcing, Woodin's modification, and Magidor forcing. Let us introduce the notion of productive λ-c.c. We say that P has the productive λ-c.c. if $P \times P$ has the λ-c.c. This property lies between λ-Knasterness and the λ-c.c. We say that an ideal I over Z is productively λ-saturated if $\mathcal{P}(Z)/I$ has the productive λ-c.c.

First, we only focus on Prikry forcing since our strategy of proofs in the case of Prikry forcing works as well in other Prikry-type forcings. Theorems 4.6 and 4.7 are analogies of Theorem 4.3 for Woodin's modification and Magidor forcing, respectively.

Theorem 4.3. *Suppose that $2^\mu = \mu^+$, μ is measurable, and U is a normal ultrafilter over μ. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If I is (λ^+, ν', ν')-saturated then $\mathcal{P}_U \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*

2. *If I is λ-centered then $\mathcal{P}_U \Vdash \overline{I}$ is λ-centered.*

3. *If I is λ^+-productively saturated then $\mathcal{P}_U \Vdash \overline{I}$ is λ^+-productively saturated.*

4. *If $Z \subseteq \mathcal{P}_\kappa(X)$ and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda^+}_{\geq\mu^+}$.*

5. *If $Z \subseteq [X]^\kappa$, I is λ-dense, and λ is a successor cardinal then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda}_{\geq\mu^+}$.*

Lemma 4.4. *Suppose that $2^\mu = \mu^+$, μ is measurable, and U is a normal ultrafilter over μ. Suppose that P be productive λ-c.c., $\dot{W}$ is P-name for a uniform ultrafilter over μ with $\Vdash U \subseteq \dot{W}$. If $2^\mu < \lambda$ and the mapping from $\mathcal{P}_U$ to $P * \mathcal{P}_{\dot{W}}$ that sends p to $\langle 1, p\rangle$ is complete then $\mathcal{P}_U \Vdash P * \mathcal{P}_{\dot{W}}/\dot{G}$ has the productive λ-c.c.*

Proof. Let $\langle a, X\rangle \Vdash \langle \dot{p}_\alpha, \dot{q}_\alpha\rangle \in (P * \mathcal{P}_{\dot{W}}/\dot{G})^2$ be arbitrary. By $2^\mu = \mu^+$, we may assume that, for every $\alpha < \lambda$, there are $\langle p_\alpha, \langle b, \dot{Y}_\alpha\rangle\rangle$ and $\langle q_\alpha, \langle c, \dot{Z}_\alpha\rangle\rangle \in P * \mathcal{P}_{\dot{W}}$ such that

- $p \Vdash \dot{p}_\alpha = \langle p_\alpha, \langle b, \dot{Y}_\alpha\rangle\rangle$.
- $p \Vdash \dot{q}_\alpha = \langle q_\alpha, \langle c, \dot{Z}_\alpha\rangle\rangle$.

It is easy to see that a end-extends b and c, that is, $a \supseteq b \cup c$, $a \cap (\max b + 1) = b$ and $a \cap (\max c + 1) = c$. Let us find $\alpha < \beta$ and $X_0, X_1 \in U$ with the following properties.

- $\langle a, X_0\rangle \Vdash \langle p_\alpha \cdot p_\beta, \langle b, \dot{Y}_\alpha \cap \dot{Y}_\beta\rangle\rangle \in P * \mathcal{P}_{\dot{W}}/\dot{G}$.
- $\langle a, X_1\rangle \Vdash \langle q_\alpha \cdot q_\beta, \langle c, \dot{Z}_\alpha \cap \dot{Z}_\beta\rangle\rangle \in P * \mathcal{P}_{\dot{W}}/\dot{G}$.

First, let π be a projection from $P * \mathcal{P}_{\dot{W}}$ to $\mathcal{B}(\mathcal{P}_U)$ induced by τ. Then $\langle a, X\rangle \leq \pi(\langle p_\alpha, \langle b, \dot{Y}_\alpha\rangle\rangle)$. Thus,

$$\langle a, X\rangle = \langle a, X\rangle \cdot \pi(\langle p_\alpha, \langle b, \dot{Y}_\alpha\rangle\rangle) = \pi(\langle p_\alpha \cdot ||a \setminus b \subseteq \dot{Y}_\alpha||, \langle a, \dot{Y}_\alpha \cap X\rangle\rangle).$$

Let $p'_\alpha = p_\alpha \cdot ||a \setminus b \subseteq \dot{Y}_\alpha||$ and $q'_\alpha = q_\alpha \cdot ||a \setminus c \subseteq \dot{Z}_\alpha||$. By the productive λ-c.c. of P, there are $\alpha < \beta$ such that $p'_\alpha \cdot p'_\beta \neq 0$ and $q'_\alpha \cdot q'_\beta \neq 0$.

Define $c : [X \setminus \max a + 1]^{<\omega} \to 2$ by

$$c(\xi_0, ..., \xi_n) = \begin{cases} 0 & p'_\alpha \cdot p'_\beta \cdot ||\xi_0, ..., \xi_n \in \dot{Y}_\alpha \cap \dot{Y}_\beta|| \neq 0 \\ 1 & \text{o.w.} \end{cases}$$

By Corollary 2.39, there is an $X_0 \in U$ such that $|c``[X_0]^n| \leq 1$ for all $n < \omega$. We claim that $c``[X_0]^n = \{0\}$ for all n. Suppose otherwise. Let n be the least counterexample. Let ξ_i be the i-th element in X_0.

By the choice of n, $r = p'_\alpha \cdot p'_\beta \cdot ||\xi_0, ..., \xi_{n-2} \in \dot{Y}_\alpha \cap \dot{Y}_\beta|| \neq 0$. For all $\xi \in X_0 \setminus (\xi_{n-2} + 1)$, $p'_\alpha \cdot p'_\beta \cdot ||\xi_0, ..., \xi_{n-2}, \xi \in \dot{Y}_\alpha \cap \dot{Y}_\beta|| = 0$, which in turn implies $r \Vdash \xi \notin \dot{Y}_\alpha \cap \dot{Y}_\beta$. Therefore $r \Vdash \dot{Y}_\alpha \cap \dot{Y}_\beta \cap X_0 = \{\xi_0, ..., \xi_{n-2}\} \in \dot{W}$. This is a contradiction.

In particular, $\langle a, X_0\rangle$ is a reduct of $\langle p_\alpha \cdot p_\beta, \langle b, \dot{Y}_\alpha \cap \dot{Y}_\beta\rangle\rangle$. Note that $\langle a, X_0\rangle \leq \langle a, X\rangle$. The similar argument takes $X_1 \in U$. Then $\langle a, X_0 \cap X_1\rangle \Vdash \langle p_\alpha \cdot p_\beta, \langle b, \dot{Y}_\alpha \cap \dot{Y}_\beta\rangle\rangle, \langle q_\alpha \cdot q_\beta, \langle c, \dot{Y}_\alpha \cap \dot{Y}_\beta\rangle\rangle \in (P * \mathcal{P}_{\dot{W}}/\dot{G})^2$, as desired. □

Lemma 4.5. *Suppose that P has the λ^+-c.c., $P \Vdash \mu$ is measurable and $\dot{W}$ is normal ultrafilter over μ. If $P \Vdash \mu^+ = \lambda^+$ then $P * \mathcal{P}_{\dot{W}}$ is nowhere S-layered for all stationary $S \subseteq \lambda^+$.*

Proof. We let $Q = P * \mathcal{P}_{\dot{W}}$. Let us fix sufficiently large regular θ and $M \prec \mathcal{H}_\theta$ such that $|M| = \lambda$, $\lambda \subseteq M$, and M contains all relevant elements.

It is enough to show that $Q \cap M \not\lessdot Q$. If $P \cap M \not\lessdot P$ then there is nothing to do. Assume $P \cap M \lessdot P$. Then we may assume that $P \cap M \Vdash |M| = \mu$.

Let $\dot{F}$ be a $P \cap M$-name for the filter generated by $\{X \in M[\dot{G}] \mid \exists q \in \dot{G} \cap M(q \Vdash X \in \dot{W}\}$. It is easy to see that $Q \cap M$ is dense in $(P \cap M) * \mathcal{P}_{\dot{F}}$. By $P \cap M \Vdash |M[\dot{G}]| = |M| < 2^\mu$, $\mathcal{P}(\mu^+)/I \cap M \Vdash \dot{F}$ is not an ultrafilter. By Lemma 2.41, $Q \cap M \simeq (P \cap M) * \mathcal{P}_{\dot{F}} \not\lessdot P * \mathcal{P}_{\dot{W}} = Q$, as desired. □

Proof of Theorem 4.3. By Lemma 4.1, 1 and 2 holds. By Lemma 2.20, the mapping $\tau : \mathcal{P}_U \to \mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)$ that sends $\langle a, X\rangle$ to $\langle 1, \langle a, X\rangle\rangle$ is a complete embedding such that $\mathcal{P}_U \Vdash \mathcal{P}(Z)/\overline{I} \simeq \mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)/\dot{G}$. Note that $\mathcal{P}(Z)/I \Vdash \dot{j}(\mathcal{P}_U) = \mathcal{P}_{\dot{j}(U)}$ and $\dot{j}(U)$ is a normal ultrafilter over μ.

For 3, by Lemma 4.4, if I is λ^+-productively saturated then $\mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)/\dot{G}$ is forced to have the productively λ^+-c.c. Therefore $\Vdash \overline{I}$ is λ^+-productively saturated.

Let us see 4. By Lemma 4.5, $\mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)$ is nowhere S-layered for all stationary $S \subseteq \lambda^+$. We note that $2^\lambda = \lambda^+$ follows from $2^\mu = \mu^+$. This implies $|\mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)| = \lambda^+$ by combining the assumption of $\lambda^{<\kappa} = \lambda$. By Lemma 3.20, $\mathcal{P}_U$ forces that $\mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_U)/\dot{G}$ is *not* S-layered for all stationary $S \subseteq \dot{E}^{\lambda^+}_{\geq\mu^+}$.

5 follows from the same proof for 4. □

Note that we cannot omit the assumption of λ is a successor cardinal from 5 in Theorem 4.3. We are going to give a model in which $[\lambda]^{\mu^+}$ carries a $\lambda \cap \mathrm{Reg}$-layered ideal for some Mahlo λ and singular μ. This model and the ideal are obtained as an extension by the Prikry forcing and the generated ideal, respectively. See Proposition 5.20.

Theorem 4.6. *Suppose that $2^\mu = \mu^+$, μ is measurable, U is a normal ultrafilter over μ, $\mathcal{G}$ is a guiding generic of U. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If I is (λ^+, ν', ν')-saturated then $\mathcal{P}_U \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
2. *If I is λ-centered then $\mathcal{P}_U \Vdash \overline{I}$ is λ-centered.*
3. *If I is λ^+-productively saturated then $\mathcal{P}_U \Vdash \overline{I}$ is λ^+-productively saturated.*
4. *If $Z \subseteq \mathcal{P}_\kappa(X)$ and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_{U,\mathcal{G}} \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda^+}_{\geq\mu^+}$.*
5. *If $Z \subseteq [X]^\kappa$, I is λ-dense, and λ is a successor cardinal then $\mathcal{P}_{U,\mathcal{G}} \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda}_{\geq\mu^+}$.*

Proof. By Lemma 4.1, 1 and 2 holds. By Lemma 2.20, the mapping $\tau : \mathcal{P}_{U,\mathcal{G}} \to \mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_{U,\mathcal{G}})$ that sends $\langle a, f, X, F\rangle$ to $\langle 1, \langle a, f, X, F\rangle\rangle$ is a complete embedding such that $\mathcal{P}_{U,\mathcal{G}} \Vdash \mathcal{P}(Z)/\overline{I} \simeq \mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_{U,\mathcal{G}})/\dot{G}$. Note that $\mathcal{P}(Z)/I \Vdash \dot{j}(\mathcal{P}_{U,\mathcal{G}}) = \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}$, $\dot{j}(U)$ is a normal ultrafilter over μ, and $\dot{j}(\mathcal{G})$ is a guiding generic of $\dot{j}(U)$.

First, we check 4. Note that $2^\lambda = \lambda^+$ holds by the assumption. By 2 in Lemma 2.43, there is a projection from $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}$ to $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U)}$. As we have seen in the proof of Theorem 4.3, $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U)}$ is nowhere S-layered for all stationary $S \subseteq \lambda^+$ and so is $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}$. The size of $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}$ is λ. By Lemma 3.20, $\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}/\dot{G}$ is forced to be *not* S-layered for all stationary $S \subseteq \dot{E}^{\lambda^+}_{\geq\mu^+}$. Therefore $\overline{I}$ is *not* S-layered. 5 follows from the same proof.

Lastly, we check 3. We put $Q = \mathcal{P}(Z)/I * \dot{j}(\mathcal{P}_{U,\mathcal{G}})$. Let $\langle a, f, X, F\rangle \Vdash \langle \dot{p}_\alpha, \dot{q}_\alpha\rangle \in (\mathcal{P}(Z)/I * \mathcal{P}_{\dot{j}(U),\dot{j}(\mathcal{G})}/\dot{G})^2$ be arbitrary. By $2^\mu = \mu^+$, we may assume that, for every $\alpha < \lambda^+$, there are $\langle p_\alpha, \langle b, g, \dot{Y}_\alpha, \dot{G}_\alpha\rangle\rangle$ and $\langle q_\alpha, \langle c, h, \dot{Z}_\alpha, \dot{H}_\alpha\rangle\rangle$ in Q such that

- $p \Vdash \dot{p}_\alpha = \langle p_\alpha, \langle b, g, \dot{Y}_\alpha, \dot{G}_\alpha \rangle\rangle$.
- $p \Vdash \dot{q}_\alpha = \langle q_\alpha, \langle c, h, \dot{Z}_\alpha, \dot{H}_\alpha \rangle\rangle$.

a end-extends b and c. Let us find $\alpha < \beta$, $X_0, X_1 \in U$, $\dot{Y}_0, \dot{Z}_0, \dot{G}_0$, and $\dot{H}_0$ with the following properties.

- $\Vdash \dot{Y}_0 \subseteq \dot{Y}_\alpha \cap \dot{Y}_\beta$ and $\forall \xi \in \dot{Y}_0(\dot{G}_0(\xi) \supseteq \dot{G}_\alpha(\xi) \cup \dot{G}_\beta(\xi))$.
- $\Vdash \dot{Z}_0 \subseteq \dot{Z}_\alpha \cap \dot{Z}_\beta$ and $\forall \xi \in \dot{Z}_0(\dot{H}_0(\xi) \supseteq \dot{H}_\alpha(\xi) \cup \dot{H}_\beta(\xi))$.
- $\langle a, f, X_0, F \upharpoonright X_0 \rangle \Vdash \langle p_\alpha \cdot p_\beta, \langle b, g, \dot{Y}_0, \dot{G}_0 \rangle\rangle \in Q/\dot{G}$.
- $\langle a, f, X_1, F \upharpoonright X_1 \rangle \Vdash \langle q_\alpha \cdot q_\beta, \langle c, \dot{Z}_\alpha \cap \dot{Z}_\beta \rangle\rangle \in Q/\dot{G}$.

By the proof of Lemma 4.4, we may assume that $p_\alpha \leq |a \setminus b \subseteq \dot{Y}_\alpha||$ and $q_\alpha \leq ||a \setminus c \subseteq \dot{Z}_\alpha||$. By the productive λ-c.c. of P, there are $\alpha < \beta$ such that $p_\alpha \cdot p_\beta \neq 0$ and $q_\alpha \cdot q_\beta \neq 0$.

Let $\dot{Y}_0$ and $\dot{G}_0$ be $\mathcal{P}_{U,\mathcal{G}}$-names for $\dot{Y}_\alpha \cap \dot{Y}_\beta \cap \{\xi \mid \dot{G}_\alpha(\xi) \cup \dot{G}_\beta(\xi)$ is a function$\}$ and $\dot{G}_\alpha \cup \dot{G}_\beta \upharpoonright \dot{Y}_0$, respectively. Let $\dot{Z}_0$ and $\dot{G}_0$ be defined as well.

Define $c : [X \setminus \max a + 1]^{<\omega} \to 2$ by

$$c(\xi_0, ..., \xi_n) = \begin{cases} 0 & p_\alpha \cdot p_\beta \cdot ||\xi_0, ..., \xi_n \in \dot{Y}_0|| \neq 0 \\ 1 & \text{o.w.} \end{cases}$$

By Corollary 2.39, there is an $X_0 \in U$ such that $|c``[X_0]^n| \leq 1$ for all $n < \omega$. We claim that $c``[X_0]^n = \{0\}$ for all n as in the proof of Theorem 4.4.

We claim that $\langle a, f, X_0, F \upharpoonright X_0 \rangle$ is a reduct of $\langle p_\alpha \cdot p_\beta, \langle b, g, \dot{Y}_0, \dot{G}_0 \rangle\rangle$. For every extension $\langle a \cup \{\xi_0, ..., \xi_n\}, f^\frown\langle f_0, ..., f_n \rangle, X', F' \rangle\rangle \leq \langle a, f, X_0, F \upharpoonright X_0 \rangle$, by $c(\xi_0, ..., \xi_n) = 0$,

$$\begin{aligned} 0 &\neq p_\alpha \cdot p_\beta \cdot ||\xi_0, ..., \xi_n \in \dot{Y}_0|| \cdot \textstyle\prod_i ||f_i \supseteq \dot{G}_\alpha(\xi_i)|| \cdot \prod_i ||f_i \supseteq \dot{G}_\beta(\xi_i)|| \\ &= p_\alpha \cdot p_\beta \cdot ||\xi_0, ..., \xi_n \in \dot{Y}_0|| \cdot \textstyle\prod_i ||f_i \supseteq \dot{G}_0(\xi_i)||. \end{aligned}$$

The similar argument takes X_1. We can choose a common extension of $\langle a, f, X_0, F \upharpoonright X_0 \rangle$ and $\langle a, f, X_1, F_1 \upharpoonright X_1 \rangle$. This forces that $\langle \dot{p}_\alpha \cdot \dot{p}_\beta, \dot{q}_\alpha \cdot \dot{q}_\beta \rangle \in (Q/\dot{G})^2$, as desired. □

Theorem 4.7. *Suppose that $2^\mu = \mu^+$, μ is measurable with $\mathrm{o}(\mu) \geq \nu$ and $\nu < \mu$ is regular. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If I is (λ^+, ν', ν')-saturated then $\mathcal{M}_{U,F} \Vdash \overline{I}$ is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
2. *If I is λ-centered then $\mathcal{M}_{U,F} \Vdash \overline{I}$ is λ-centered.*
3. *If I is λ^+-productively saturated then $\mathcal{M}_{U,F} \Vdash \overline{I}$ is λ^+-productively saturated.*
4. *If $Z \subseteq \mathcal{P}_\kappa(X)$ and $\lambda^{<\kappa} = \lambda$ then $\mathcal{M}_{U,F} \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda^+}_{\geq\mu^+}$.*
5. *If $Z \subseteq [X]^\kappa$, I is λ-dense, and λ is a successor cardinal then $\mathcal{M}_{U,F} \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda}_{\geq\mu^+}$.*

Proof. 1 and 2 follows from Lemma 4.1. By Lemma 2.20, the mapping $\tau : \mathcal{M}_{U,F} \to \mathcal{P}(Z)/I * \mathcal{M}_{\dot{j}(U),\dot{j}(F)}$ that sends $\langle a, X \rangle$ to $\langle 1, \langle a, X \rangle\rangle$ is a complete embedding such that $\mathcal{M}_{U,F} \Vdash \mathcal{P}(Z)/\overline{I} \simeq \mathcal{P}(Z)/I * \dot{j}(\mathcal{M}_{U,F})/\dot{G}$. Note that $\mathcal{P}(Z)/I \Vdash \dot{j}(\mathcal{M}_{U,F}) = \mathcal{M}_{\dot{j}(U),\dot{j}(F)}$.

To show 3, it is enough to prove that $\mathcal{P}(Z)/I * \mathcal{M}_{\dot{j}(U),\dot{j}(F)}/\dot{G}$ has the productive λ^+-c.c in the extension. Note that we have an analogie of Corollary 2.39 for Magidor forcing. An point of the proof of Lemma 4.4

was Corollary 2.39. By Corollary 2.51, $\mathcal{P}(Z)/I * \mathcal{M}_{\dot{j}(U),\dot{j}(F)}/\dot{G}$ is forced to have the productive λ^+-c.c. if I is productively λ^+-saturated, as desired.

Let $Q = \mathcal{P}(Z)/I * \mathcal{M}_{\dot{j}(U),\dot{j}(F)}$. By $2^\mu = \mu^+$, we have $2^\lambda = \lambda^+$. For 4 and 5, since the same proof works, we only show 4. By Lemma 3.20, it is enough to prove that Q is nowhere S-layered for all stationary $S \subseteq \lambda^+$.

Let θ be sufficiently large regular. We fix $M \prec \mathcal{H}_\theta$ with $|M| = \lambda$, $\lambda \subseteq M$, and M contains all relevant elements. We claim that $Q \cap M \lessdot Q$. If $\mathcal{P}(Z)/I \cap M \not\lessdot \mathcal{P}(Z)/I$ then there is nothing to do. Assume $\mathcal{P}(Z)/I \cap M \lessdot \mathcal{P}(Z)/I$. We may assume that $\mathcal{P}(Z)/I \cap M \Vdash |M| = \mu$. Let $\dot{F}$ be a $\mathcal{P}(Z)/I$-name for the filter generated by $\{X \in M[\dot{G}] \mid \exists q \in \dot{G} \cap M(q \Vdash X \in \dot{j}(U_0))\}$. It is forced that $\dot{F}$ does not generate an ultrafilter over μ by $|M| < \mu^+$. We can choose $A \in \mathcal{P}(Z)/I$ and $\dot{X}$ such that $A \Vdash \dot{X} \in \dot{j}(U_0)$, $\dot{X} \notin \dot{F}$ and $\mu \setminus \dot{X} \notin \dot{F}$. Then $\langle A, \langle \emptyset, \langle \dot{X} \cap \dot{B}_0 \rangle ^\frown \langle \dot{B}_\alpha \mid \alpha > 0 \rangle \rangle \rangle \in Q$ does not have a reduct in $Q \cap M$, as we have seen in the proof of Lemma 2.41. The proof is completed. $\square$

5 Models with saturated ideals

In this section, we present some models with a saturated ideals. First, we introduce basic proofs to obtain a model with a saturated ideal. These were originally due to Kunen [28].

Lemma 5.1. *Suppose that j is an almost-huge embedding with critical point κ and $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals. If P is a poset with the following properties:*

- *$P \subseteq V_\kappa$ has the κ-c.c. and $P \Vdash \kappa = \mu^+$.*
- *$j(P)$ has the $j(\kappa)$-c.c.*
- *There is a projection from $\pi : j(P) \to P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ such that $\pi(p) = \langle p, \dot{1}\rangle$ for all $p \in P$.*

Then there is a $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$-name $\dot{I}$ such that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces the following.

1. *$\dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.*
2. *$\mathcal{P}(\mathcal{P}_\kappa\lambda)/I \simeq j(P)/\dot{G} * \dot{H}$.*

Proof. Let $G * H$ be an arbitrary $(V, P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)))$-generic filter. First, we give a saturated ideal on $\mathcal{P}_\kappa\lambda$ in $V[G][H]$. Let $\overline{G}$ be an arbitrary $(V, j(P))$-generic with $\pi``\overline{G} \subseteq G * H$. Note that $j``G = G \subseteq \overline{G}$, which in turn implies that j lifts to $j : V[G] \to M[\overline{G}]$ in $V[\overline{G}]$ such that $j(G) = \overline{G}$. By the $j(\kappa)$-c.c. of $j(P)$, ${}^{<j(\kappa)}M[\overline{G}] \subseteq M[\overline{G}]$. Let m_α be the coordinate-wise union of $j``(H \cap \mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < \alpha))$. $m_\alpha \in \mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < jj(\kappa))$ for all $\alpha < j(\kappa)$ by the closure property of $M[G]$ and the directed closedness of $\mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < jj(\kappa))$. By the $j(\kappa)$-c.c., we can choose a list $\langle X_\alpha \mid \alpha < j(\kappa)\rangle$ of $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$-names of all subset in $\mathcal{P}_\kappa\lambda$. There is a descending sequence $\langle s_\alpha \mid \alpha < j(\kappa)\rangle$ with the following properties:

- $s_\alpha \leq m_\alpha$.
- s_α decides $j``\lambda \in j(\dot{X}_\alpha)$.

Let $U = \{\dot{X}^{G*H} \mid \exists\beta(s_\beta \Vdash j``\lambda \in j(\dot{X}))\}$. $U$ is a $V[G][H]$-normal $V[G][H]$-ultrafilter over $\mathcal{P}_\kappa\lambda$. Because $\overline{G}$ was an arbitrary $(V, j(P))$-generic with $\pi``\overline{G} \subseteq G * H$, we can take a $j(P)/G * H$-name $\dot{U}$ for such ultrafilter. Let I be define by

$$X \in I \text{ if and only if } j(P)/G * H \Vdash \mathcal{P}_\kappa\lambda \setminus X \in \dot{U}.$$

The standard argument shows that I is a normal and fine ideal over $\mathcal{P}_\kappa\lambda$. Towards a showing $j(\kappa)$-saturation of I, let $\langle X_\xi \mid \xi \in K\rangle$ be an anti-chain in $\mathcal{P}(\mathcal{P}_\kappa\lambda)$, we have the following:

- $||X_\xi \in \dot{U}|| \cdot ||X_\zeta \in \dot{U}|| = ||X_\xi \cap X_\zeta \in \dot{U}|| = 0$ for each $\xi \neq \zeta$ in K.
- $||X_\xi \in \dot{U}|| \neq 0$ for each $\xi \in K$.

It follows that $\{||X_\xi \in \dot{U}|| \mid \xi \in K\}$ is an anti-chain in $\mathcal{B}(j(P)/G * H)$. Note that each $||X_\xi \in \dot{U}||$ is a $\mathcal{B}(j(P)/G * H)$-value. By the $j(\kappa)$-c.c. of $j(P)$ and Lemma 3.18, $j(P)/G * H$ has the $j(\kappa)$-c.c. Therefore $|K| < j(\kappa)$, as desired.

Let us show $j(P)/G * H \simeq \mathcal{P}(\mathcal{P}_\kappa\lambda)/I$.

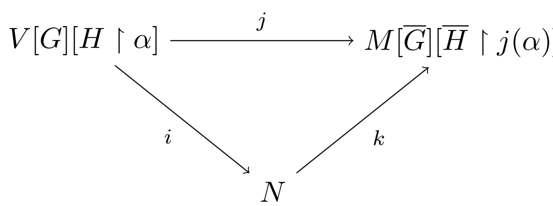
The proof is based on Foreman–Magidor–Shelah [13]. As in the previous argument, let us consider a mapping $\tau : \mathcal{P}(\mathcal{P}_\kappa\lambda)/I \to \mathcal{B}(j(P)/G * H)$ that sends X to $||X \in \dot{U}||$. The standard argument shows that τ is a complete embedding and $\dot{U}$ is a $j(P)/G * H$-name for $(V[G][H], \mathcal{P}(\mathcal{P}_\kappa\lambda)/I)$-generic filter generated by $\tau^{-1}\dot{\overline{G}}$. Here, $\dot{\overline{G}}$ is the canonical name of $(V[G][H], j(P)/G * H)$-generic filter. It is enough to prove that τ is a dense embedding.

We claim that there is an f_q such that $||\{a \in \mathcal{P}_\kappa\lambda \mid f_q(a) \in G\} \in \dot{U}|| = q$ for every $q \in j(P)/G * H$. It follows that the range of τ is a dense subset in $\mathcal{B}(j(P)/G * H)$.

Let $\overline{G}$ be an arbitrary $(V, j(P))$-generic filter with $\pi``\overline{G} \subseteq G * H$. Note that $q \in j(P) \cap V_\beta$ for some $\beta < j(\kappa)$. By the elementarity of j and j is almost-huge, we can choose inaccessible $\alpha < j(\kappa)$ with $\alpha > \beta$. Let $U_\alpha = \dot{U}^{\overline{G}} \cap \mathcal{P}(\mathcal{P}_\kappa\lambda)^{V[G][H \restriction \alpha]}$. By the definition of $\dot{U}$, we can choose a $(V[\overline{G}], \mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < jj(\kappa)))$-generic filter $\overline{H}$ such that,

- j lifts to $j : V[G][H \restriction \alpha] \to M[\overline{G}][\overline{H} \restriction j(\alpha)]$ and $j(G) = \overline{G}$.
- $X \in U_\alpha$ if and only if $j``\lambda \in j(X)$ in $M[\overline{G}][\overline{H} \restriction j(\alpha)]$.

Here, $H \restriction \alpha = H \cap \mathrm{Coll}(\lambda, < \alpha)$ and $\overline{H} \restriction j(\alpha) = \overline{H} \cap \mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < j(\alpha))$. We can consider the following commutative diagram of elementary embeddings.



Here, $i : V \to N \simeq \mathrm{Ult}(V[G][H \restriction \alpha], U_\alpha)$ is the ultrapower mapping and k is defined by $k([f]_{\dot{U}_\alpha}) = j(f)(j``\lambda)$. It is easy to see that k is an elementary embedding. We claim $\mathrm{crit}(k) \geq \alpha$. Because α is inaccessible, $\lambda^+ = \alpha$ in $V[G][H \restriction \alpha]$. We remark that $\mathcal{P}(\lambda)^{V[G][H \restriction \alpha]} \subseteq N$ and $i(\kappa) \geq \alpha = \lambda^+$. $\mathcal{P}(\lambda)^{V[G][H \restriction \alpha]} \subseteq N$ follows from, for each x, x can be written as $\{\xi \in \lambda \mid i(\xi) \in i``\lambda \cap i(x)\}$. That is, x is definable in N by the normality of U_α. By $\kappa = \mu^+$ in $V[G][H \restriction \alpha]$, $N \models i(\kappa)$ is the least cardinal greater than μ. N has no cardinals between κ and α. On the other hand, $\mathrm{crit}(k) \geq \kappa$ must be cardinal in N. Therefore $\mathrm{crit}(k) \geq \alpha$.

Let us find a name of q in N. Since $|V_\beta| = \lambda < \alpha$ holds in $V[G][H \restriction \alpha]$, the same thing holds in N. We can enumerate $i(P) \cap V_\beta^N$ as $\langle q_\xi \mid \xi < \lambda \rangle$ in N. By $\mathrm{crit}(k) \geq \alpha > \beta$, k is the identity mapping on V_β, and thus, $k(\langle q_\xi \mid \xi < \lambda \rangle) = \langle q_\xi \mid \xi < \lambda \rangle$. By the elementarity of k, q appears in this sequence, that is there exists a ξ such that $q_\xi = k(q_\xi) = q \in N$. We can choose x with $[x]_{U_\alpha} = q$. Since $\dot{U}$ is $(V[G][H], \mathcal{P}(\mathcal{P}_\kappa\lambda)/I)$-generic, $\overline{G}$ is an arbitrary, and, I is a saturated ideal, there is an $f_q : \mathcal{P}_\kappa\lambda \to V[G][H]$ such that $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I \Vdash q = [f_q]_{\dot{U}_\alpha}$.

Therefore $||\{a \in \mathcal{P}_\kappa\lambda \mid f_q(a) \in G\} \in \dot{U}|| = q$ follows from

$$\begin{aligned}\{a \in \mathcal{P}_\kappa\lambda \mid f_q(a) \in G\} \in U &\Leftrightarrow [f_q]_{U_\alpha} = q \in i(G) (\text{for some } \alpha) \\ &\Leftrightarrow k(q) = q \in k \circ i(G) = \overline{G}.\end{aligned}$$

The proof is completed. □

The similar proof shows

Lemma 5.2. *Suppose that j is a huge embedding with critical point κ and $\mu < \kappa < \lambda < j(\kappa)$ are regular cardinals. We also assume that* GCH *holds. If P is a poset with the following properties:*

- *$P \subseteq V_\kappa$ has the κ-c.c. and $P \Vdash \kappa = \mu^+$.*
- *$j(P)$ has the $j(\kappa)$-c.c.*
- *There is a projection from $\pi : j(P) \to P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ such that $\pi(p) = \langle p, \dot{1} \rangle$ for all $p \in P$.*

Then there is a $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$-name $\dot{I}$ such that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces the following.

1. $\dot{I}$ *is a normal, fine,* κ*-complete* $j(\kappa)$*-saturated ideal over* $[j(\kappa)]^\kappa$*.*

2. $\mathcal{P}([j(\kappa)]^\kappa)/I \simeq j(P)/\dot{G} * \dot{H}$*.*

Proof. Let $G * H$ be a $(V, P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)))$-generic. By the proof of Lemma 5.1, we only give the definition of a required ideal in $V[G][H]$.

Let $\overline{G}$ be an arbitrary $(V, j(P))$-generic with $\pi``\overline{G} \subseteq G*H$. In $V[\overline{G}]$, j lifts to $j : V[G] \to M[\overline{G}]$. By the GCH, we can choose a list $\langle X_\alpha \mid \alpha < j(\kappa)^+\rangle$ of $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$-names of subset of $[j(\kappa)]^\kappa$. Note that ${}^{j(\kappa)}M[\overline{G}] \cap V[\overline{G}] \subseteq M[\overline{G}]$. By $\lambda > \kappa$ and the closure property of $M[\overline{G}]$, $\mathrm{Coll}^{M[\overline{G}]}(j(\lambda), < jj(\kappa))$ is $j(\kappa)^+$-directed closed in $V[\overline{G}]$. Let m be the coordinate-wise union of $j``H$. Then we have $m \in \mathrm{Coll}^{M[G]}(j(\lambda), < jj(\kappa))$ by $|j``H| = j(\kappa)$. We can construct a descending sequence $\langle s_\alpha \mid \alpha < j(\kappa)^+\rangle$ such that

- $s_\alpha \leq m$.
- s_α decides $j``j(\kappa) \in j(\dot{X}_\alpha)$.

Let $U = \{X \subseteq [j(\kappa)]^\kappa \mid s_\alpha \Vdash j``j(\kappa) \in X_\alpha\}$. There is a $j(P)/G * H$-name $\dot{U}$ for U. In $V[G][H]$, let I be an induced ideal by $\dot{U}$, that is, $X \in I$ if and only if $j(P)/G*H \Vdash [j(\kappa)]^\kappa \setminus X \in \dot{U}$. I works as witness. □

The proof of Lemma 5.1 notices us that some poset with nice closure property works instead of Levy collapse, like collapses that we have seen in Section 2.4.1.

Lemma 5.3. *Suppose that* j *is an almost-huge embedding with critical point* κ *and* $\mu < \kappa \leq \lambda < j(\kappa)$ *are regular cardinals. For a poset* P *and* P*-name for a poset* $\dot{Q}$*, suppose the following*

- $P \subseteq V_\kappa$ *has the* κ*-c.c. and* $P \Vdash \kappa = \mu^+$*.*
- $P \Vdash \dot{Q}$ *has the* $j(\kappa)$*-c.c and* $\dot{Q} \subseteq \dot{V}_{j(\kappa)}$*.*
- $j(P) \Vdash \dot{j}(\dot{Q})$ *is* $j(\kappa)$*-directed closed and well-met.*
- $j(P)$ *has the* $j(\kappa)$*-c.c.*
- *There is a projection from* $\pi : j(P) \to P * \dot{Q}$ *such that* $\pi(p) = \langle p, \dot{1}\rangle$ *for all* $p \in P$*.*

Then there is a $P * \dot{Q}$*-name* $\dot{I}$ *such that* $P * \dot{Q}$ *forces the following.*

1. $\dot{I}$ *is a saturated ideal over* $\mathcal{P}_\kappa\lambda$*.*

2. $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I \simeq j(P)/\dot{G} * \dot{H}$*.*

Lemma 5.4. *Suppose that* j *is a huge embedding with critical point* κ *and* $\mu < \kappa < \lambda < j(\kappa)$ *are regular cardinals. We also assume that GCH holds. For a poset* P *and* P*-name for a poset* $\dot{Q}$*, suppose the following*

- $P \subseteq V_\kappa$ *has the* κ*-c.c. and* $P \Vdash \kappa = \mu^+$*.*
- $P \Vdash \dot{Q}$ *has the* $j(\kappa)$*-c.c and* $\dot{Q} \subseteq \dot{V}_{j(\kappa)}$*.*
- $j(P) \Vdash \dot{j}(\dot{Q})$ *is* $j(\kappa)^+$*-directed closed and well-met.*
- $j(P)$ *has the* $j(\kappa)$*-c.c.*
- *There is a projection from* $\pi : j(P) \to P * \dot{Q}$ *such that* $\pi(p) = \langle p, \dot{1}\rangle$ *for all* $p \in P$*.*

Then there is a $P * \dot{Q}$*-name* $\dot{I}$ *such that* $P * \dot{Q}$ *forces the following.*

1. $\dot{I}$ *is a normal, fine,* κ*-complete* $j(\kappa)$*-saturated ideal over* $[j(\kappa)]^\kappa$*.*

2. $\mathcal{P}([j(\kappa)]^\kappa)/I \simeq j(P)/\dot{G} * \dot{H}$*.*

5.1 The extent of saturation of induced ideals

In this section, we give an example of P and π in the assumption of Lemma 5.1

Theorem 5.5. *Suppose that j is an almost-huge embedding with critical point κ and $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals. Then $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that there is a saturated ideal I over $\mathcal{P}_\kappa\lambda$ with the following properties:*

1. *I is $(\lambda^+, \lambda^+, < \mu)$-saturated.*
2. *I is not (λ^+, μ, μ)-saturated. In particular, I is not strongly saturated.*
3. *I is layered if and only if $j(\kappa)$ is Mahlo in V.*
4. *I is not centered. In particular, I is not strongly layered.*

Proof. We let $P = P(\mu,\kappa)$. Since j is almost-almost huge, $j(P) = P(\mu, j(\kappa))$. By Lemma 2.24, P and $j(P)$ has the κ-c.c. and the $j(\kappa)$-c.c., respectively. To use Lemma 5.1, let us define a continuous projection from $\pi : j(P) \to P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ as follows

$$\begin{aligned} j(P) &= \textstyle\prod^{<\mu}_{\alpha\in[\mu,j(\kappa))\cap\mathrm{Reg}} \mathrm{Coll}(\alpha, < j(\kappa)) \\ &\to \textstyle\prod^{<\mu}_{\alpha\in[\mu,\kappa)\cap\mathrm{Reg}} \mathrm{Coll}(\alpha, < \kappa) \times \mathrm{Coll}(\lambda, < j(\kappa)) \\ &\to P \times T(P, \dot{\mathrm{Coll}}(\lambda, < j(\kappa))) \\ &\to P * \dot{\mathrm{Coll}}(\kappa, < \lambda). \end{aligned}$$

The third line follow from Lemma 3.2. The last line follows from Lemma 3.1. Since each component is continuous, π is continuous. Note that $P \subseteq V_\kappa$, $P \lessdot j(P)$, and, π, $\pi(p) = \langle p, \emptyset\rangle$ for all $p \in P$. By Lemma 5.1, there is a $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$-name $\dot{I}$ such that

1. $\dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.
2. $\mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$. Here, $\dot{G} * \dot{H}$ is the canonical name for a $(V, P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)))$-generic filter.

To prove items 1 to 4, let us show the corresponding claims for the quotient forcing $j(P)/\dot{G} * \dot{H}$. Items 1 and 2 follow from Claim 5.6.

Claim 5.6. *$P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that*

(i) *$j(P)/\dot{G} * \dot{H}$ has the $(j(\kappa), j(\kappa), < \mu)$-c.c.*

(ii) *$j(P)/\dot{G} * \dot{H}$ does not have the $(j(\kappa), \mu, \mu)$-c.c.*

Proof of Claim. For (i), We recall that $j(P)$ has the $(j(\kappa), j(\kappa), < \mu)$-c.c. and $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ is μ-closed. By Lemma 3.10 and the continuity of π, it is forced by $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ that $j(P)/\dot{G} * \dot{H}$ has the $(j(\kappa), j(\kappa), < \mu)$-c.c.

We prove (ii) by contradiction. Suppose otherwise. We consider a set $X = \{r_\alpha \mid \alpha \in [\lambda^+, j(\kappa)) \cap \mathrm{Reg}\} \subseteq j(P)$ with $\mathrm{supp}(r_\alpha) = \{\alpha\}$ for every α. By the definition of π, $\pi(r_\alpha) = \langle\emptyset,\emptyset\rangle$. Therefore $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash r_\alpha \in j(P)/\dot{G} * \dot{H}$ for every α. By the assumption, there are $\dot{Z}$, $r \in j(P)$ and $\langle p, \dot{q}\rangle$ such that, $\langle p, \dot{q}\rangle$ forces that

- r is a lower bound in $\{r_\alpha \mid \alpha \in \dot{Z}\}$ in $j(P)/\dot{G} * \dot{H}$, and
- $|\dot{Z}| = \mu$.

Since $|\mathrm{supp}(r)| < \mu$, we can choose β and $\langle p', \dot{q}'\rangle \leq \langle p, \dot{q}\rangle$ such that $\langle p', \dot{q}'\rangle \Vdash \beta \in \dot{Z} \setminus \mathrm{supp}(r)$. Clearly, $\langle p', \dot{q}'\rangle$ does not force that $r \leq r_\beta$ in $j(P)/\dot{G} * \dot{H}$ but this is a contradiction. □

Note that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I}$ is a complete Boolean algebra because $\dot{I}$ is saturated. The poset $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces $|\mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I}| \leq 2^{\lambda^{<\kappa}} = 2^\lambda = j(\kappa)$, and thus, $|\mathcal{B}(j(P)/\dot{G} * \dot{H})| \leq j(\kappa)$.

Claim 5.7. *(i) If $j(\kappa)$ is Mahlo then $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash j(P)/\dot{G} * \dot{H}$ is S-layered for some stationary subset $S \subseteq E^{j(\kappa)}_\lambda$.*

(ii) If $j(\kappa)$ is not Mahlo then $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash j(P)/\dot{G} * \dot{H}$ is not S-layered for any stationary subset $S \subseteq \lambda$.

Proof of Claim. First, we show (i). By Lemma 2.25, $j(P)$ is $[\mu, j(\kappa)) \cap \mathrm{Reg}$-layered. Note that $[\lambda, j(\kappa)) \cap \mathrm{Reg}$ remains stationary in the extension. We also remark that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces $[\lambda, j(\kappa)) \cap \mathrm{Reg}^V \subseteq E^{j(\lambda)}_\lambda$ since $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ has the form of (κ-c.c.) $*$ (λ-closed). By Lemma 3.11 and the continuity of π, $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash j(P)/\dot{G} * \dot{H}$ is $[\lambda, j(\kappa)) \cap \mathrm{Reg}^V$-layered.

For (ii), we let $Q_\delta = \bigcup_{\eta<\delta} P(\mu, \eta)$ and $C \subseteq j(\kappa)$ be a club in Lemma 2.26. $\langle Q_\delta \mid \delta < j(\kappa)\rangle$ is a filtration and $Q_\delta \not\lessdot j(P)$ for all singular $\delta \in C$ by Lemma 2.26. Since $j(\kappa)$ is not Mahlo, there is a club $D \subseteq C$ such that each element in D is singular. Note that it is forced that $\langle Q_\delta/\dot{G} * \dot{H} \mid \delta < j(\kappa)\rangle$ is a filtration of $j(P)/\dot{G} * \dot{H}$. By Lemma 2.8, it is enough to show that $\Vdash Q_\delta/\dot{G} * \dot{H} \not\lessdot j(P)/\dot{G} * \dot{H}$ for all $\delta \in D$. Fix $\delta \in D$, Lemma 2.27 gives a $p \in P(\mu, \delta)$ with certain properties. By $\pi(p) = \langle\emptyset, \emptyset\rangle$, we have $\Vdash p \in P(\mu, \delta)/\dot{G} * \dot{H}$. We claim that $\Vdash p \in P(\mu, \delta)/\dot{G} * \dot{H}$ has no reduct in $Q_\delta/\dot{G} * \dot{H}$.

For any $b \in P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ and $q \in Q_\delta$ with $b \Vdash q \in Q_\delta/\dot{G} * \dot{H}$, there is an $r \in Q_\delta$ such that $\pi(r) = \langle\emptyset, \emptyset\rangle$, $r \perp p$ in $P(\mu, \delta)$, and, $r \cdot q \in Q_\delta$. Then, the following hold:

- $b \leq \pi(q) = \pi(q) \cdot \langle\emptyset, \emptyset\rangle = \pi(q) \cdot \pi(r) = \pi(q \cdot r)$.
- $b \Vdash q \cdot r \leq q$ in $Q_\delta/\dot{G} * \dot{H}$ but $(q \cdot r) \perp p$ in $P(\mu, \delta)/\dot{G} * \dot{H}$.

Thus, $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that $q \in Q_\delta/\dot{G} * \dot{H}$ is not reduct of p, as desired. □

4 follows from Claim 5.8.

Claim 5.8. $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash j(P)/\dot{G} * \dot{H}$ *is not λ-centered.*

Proof of Claim. Define

- $P_* = \prod^{<\mu}_{\alpha \in [\mu, \lambda+1) \cap \mathrm{Reg}} \mathrm{Coll}(\alpha, < j(\kappa))$ and
- $P^* = \prod^{<\mu}_{\alpha \in [\lambda+1, j(\kappa)) \cap \mathrm{Reg}} \mathrm{Coll}(\alpha, < j(\kappa))$.

We have $j(P) \simeq P_* \times P^*$. By the definition of π, it follows that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces $j(P)/\dot{G} * \dot{H} \simeq (P_*/\dot{G} * \dot{H}) \times P^*$. Thus, it suffices that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces P^* is not λ-centered.

Fix $\alpha \in [\lambda^+, j(\kappa)) \cap \mathrm{Reg}$. Lemma 3.2 shows that P forces that $\mathrm{Coll}^V(\alpha, < j(\kappa)) \simeq T(P, \mathrm{Coll}^{V^P}(\alpha, < j(\kappa)))$ is projected to $\mathrm{Coll}^{V^P}(\alpha, < j(\kappa))$. Since P^* is projected to $\mathrm{Coll}^V(\alpha, < j(\kappa))$, in the extension by $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$, if P^* is λ-centered then so is $\mathrm{Coll}^{V^P}(\alpha, < j(\kappa))$.

But, by Lemma 2.28, $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces $\mathrm{Coll}^{V^P}(\alpha, < j(\kappa))$ is not λ-centered for all $\alpha > \lambda$. In particular, $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that P^* is not λ-centered. □

This complete the proof. □

Magidor [32] gave a model with a normal, fine, and countably complete $\aleph_3$-saturated ideal over $[\aleph_3]^{\aleph_1}$ using the universal collapse that we will mention in Section 5.5. We get the same result using the diagonal product of Levy collapses. Indeed,

Theorem 5.9. *Suppose that j is a huge embedding with critical point κ, GCH holds, and $\mu < \kappa < \lambda < j(\kappa)$ are regular cardinals. Then $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that there is a normal, fine, κ-complete ideal I over $[\lambda^+]^\kappa$ with the following properties:*

1. *$\mathcal{P}([\lambda^+]^\kappa)/I$ has the $(\lambda^+,\lambda^+,<\mu)$-c.c.*
2. *$\mathcal{P}([\lambda^+]^\kappa)/I$ does not have the (λ^+,μ,μ)-c.c.*
3. *$\mathcal{P}([\lambda^+]^\kappa)/I$ is S-layered poset for some stationary subset $S \subseteq E^{\lambda^+}_\lambda$.*
4. *$\mathcal{P}([\lambda^+]^\kappa)/I$ is not λ-centered.*
5. *$\mathcal{P}([\lambda^+]^\kappa)/I$ is λ^+-dense.*

Proof. Let $P = P(\mu,\kappa)$. As in the proof of Theorem 5.5, we have a continuous projection $j(P) \to P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ which satisfies the assumption of Lemma 5.2. We have $\dot{I}$ such that

- $P * \dot{\mathrm{Coll}}(\lambda,< j(\kappa)) \Vdash \dot{I}$ is a normal, fine, κ-complete $j(\kappa)$-saturated ideal over $[j(\kappa)]^\kappa$.
- $P * \dot{\mathrm{Coll}}(\lambda,< j(\kappa)) \Vdash \mathcal{P}([j(\kappa)]^\kappa)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$.

We have studied the extent of saturation of $j(P)/\dot{G} * \dot{H}$ yet. By Claims 5.6, 5.7(i), and 5.8, items 1, 2, 3, and 4 hold. By $|j(P)| = j(\kappa)$, the item 5 holds. Therefore $\dot{I}$ is a required ideal. □

In the proof of Claim 5.8, we proved that it is forced by $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\lambda,< j(\kappa))$ that P^* is not λ-centered. On the other hand, $P_*/\dot{G} * \dot{H}$ is λ-centered. Indeed, in the extension by $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$, we have

$$\begin{aligned} P_* &= (\textstyle\prod^{<\mu}_{\alpha\in[\mu,\lambda+1)\cap\mathrm{Reg}} \mathrm{Coll}(\alpha,< j(\kappa)))^V \\ &= \textstyle\prod^{<\mu}_{\alpha\in[\mu,\lambda+1)\cap\mathrm{Reg}^V} \mathrm{Coll}^V(\alpha,< j(\kappa)). \end{aligned}$$

Lemmas 2.7 and 2.29 show P_* is λ-centered. By Lemma 3.12, we have that $P * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$ forces that $P_*/\dot{G} * \dot{H}$ is λ-centered, as desired.

5.2 Generically supercompact cardinal via saturated ideals

We say that κ is super-almost-huge if, for all $\lambda \geq \kappa$, there is an almost-huge embedding $j : V \to M$ with critical point κ and $j(\kappa) > \lambda$. Eskew showed the consistency of a generic supercompact cardinal as follows.

Theorem 5.10 (Eskew)**.** *If a super-almost-huge cardinal exists then it is consistent with* ZFC *that $\mathcal{P}_\kappa\lambda$ carries a dense ideal for all regular $\lambda \geq \kappa$.*

In this section, by using the diagonal product of Levy collapses, we show

Theorem 5.11. *If a super-almost-huge cardinal exists then it is consistent* ZFC *that $\mathcal{P}_\kappa\lambda$ carries a saturated ideal ideal for all regular $\lambda \geq \kappa$.*

Lemma 5.12 (Foreman [14])**.** *Suppose P has the λ-c.c. and Q is λ-closed. If P forces $\mathcal{P}_\kappa\alpha$ carries a λ-saturated ideal and $\alpha^{<\kappa} < \lambda$ then $P \times Q$ forces $\mathcal{P}_\kappa\alpha$ carries a λ-saturated ideal.*

Proof. Let $\dot{I}$ be a P-name for a normal, fine, κ-complete λ-saturated ideal over $\mathcal{P}_\kappa\alpha$. We claim that $P \times Q$ forces $\dot{I}$ witnesses.

Observes that $P \Vdash Q$ is $<\lambda$-Baire, and thus, $\dot{I}$ is a normal, fine, κ-complete ideal over $\mathcal{P}_\kappa\alpha$ by $\alpha^{<\kappa} < \lambda$. Towards showing a contradiction, we assume that $P \times Q$ forces $\dot{I}$ is not λ-saturated. Then there are $\langle p,q\rangle \in P \times Q$ and $\langle \dot{X}_\xi \mid \xi < \lambda\rangle$ such that $\langle p,q\rangle \Vdash \dot{A}_\xi \cap \dot{A}_\zeta \in \dot{I}$, $\dot{A}_\xi \in \dot{I}^+$ for all $\xi \neq \zeta$.

A standard argument takes $\langle q_\xi, \dot{B}_\xi \mid \xi < \lambda\rangle$ such that

- $q_\zeta \leq q_\xi \leq q$ for all $\xi < \zeta$.
- $\langle p, q_\xi \rangle \Vdash \dot{B}_\xi = \dot{A}_\xi$.
- $\dot{B}_\xi$ is P-name.

It is easy to see that $p \Vdash \dot{B}_\xi \cap \dot{B}_\zeta \in \dot{I}$ and $\dot{B}_\xi \in \dot{I}^+$ for all $\xi \neq \zeta$. This is a contradiction. □

Proof of Theorem 5.11. For simplicity, we assume that $V_\delta \models \kappa$ is super-almost-huge for some inaccessible δ. Let $S_0 = \{\alpha < \delta \mid \kappa$ is almost-huge with target $\alpha\}$. Let $S = \mathrm{Succ}(S_0)$. Then $\sup S = \delta$ and $S \subseteq \mathrm{Reg}$. We note that δ does not need to be Mahlo.

Define $P = \prod^E_{\alpha \in [\mu,\kappa)\cap\mathrm{Reg}} \mathrm{Coll}(\alpha, <\kappa)$. Since κ is Mahlo, an usual Δ-system argument shows that P has the κ-c.c. P forces that $\mu^+ = \kappa$ and $P \subseteq V_\kappa$.

Let $f : S \to \mathrm{Reg}$ be defined by

$$f(\alpha) = \begin{cases} \kappa & \text{if } \alpha = \min S \\ \sup(S \cap \alpha) & \text{if } \sup(S \cap \alpha) \text{ is regular.} \\ (\sup(S \cap \alpha))^+ & \text{otherwise.} \end{cases}$$

Our aim is to show that $P * \prod_{\alpha \in S} \dot{\mathrm{Coll}}(f(\alpha), < j(\alpha))$ forces $V_\delta[\dot{G}][\dot{H}]$ is a required model of ZFC, that is, $V_\delta[\dot{G}][\dot{H}] \models \mathcal{P}_\kappa(\lambda)$ carries a saturated ideal for all regular $\lambda \geq \kappa$.

Let $\dot{Q}_\alpha$ and $\dot{R}_\alpha$ be P-names for

- $\prod^E_{\beta \in S \cap (\alpha+1)} \dot{\mathrm{Coll}}(f(\beta), <\beta)$ and
- $\prod^E_{\beta \in S \setminus (\alpha+1)} \dot{\mathrm{Coll}}(f(\beta), <\beta)$, respectively.

Claim 5.13. *For each $\alpha \in S$, P forces the following.*

1. *$\dot{Q}_\alpha$ has the α-c.c.*
2. *$\dot{R}_\alpha$ is α-closed.*
3. *$\prod^E_{\beta \in S} \dot{\mathrm{Coll}}(f(\beta), <\beta) \Vdash f(\alpha)^+ = \alpha$.*

Proof. We discuss in the extension $V[G]$ by P. Let $Q_\alpha = \dot{Q}^G_\alpha$ and $R_\alpha = \dot{R}^G_\alpha$. 2 is easy. Let us prove 1 Note that

$$Q_\alpha = (\textstyle\prod^E_{\beta \in S \cap \alpha} \mathrm{Coll}(f(\beta), <\beta)) \times \mathrm{Coll}(f(\beta), <\alpha).$$

It is easy to see $|\prod^E_{\beta \in S\cap\alpha} \mathrm{Coll}(f(\beta), <\beta)| < \alpha$. By the α-c.c. of $\mathrm{Coll}(f(\alpha), <\alpha)$, Q_α has the α-c.c.

3 follows from 1, 2 and $\prod^E_{\beta \in S} \mathrm{Coll}(f(\beta), <\beta) \simeq Q_\alpha \times R_\alpha$. □

Claim 5.13 shows that δ remains an inaccessible in the extension by $P * \prod^E_{\beta \in S} \dot{\mathrm{Coll}}(f(\beta), <\beta)$. Therefore $\prod^E_{\beta \in S} \dot{\mathrm{Coll}}(f(\beta), <\beta) \Vdash \dot{V}_\delta = V_\delta[\dot{G}][\dot{H}]$ is a model of ZFC. Claim 5.13 shows that $P * \prod^E_{\alpha \in S} \dot{\mathrm{Coll}}(f(\alpha), <\alpha)$ forces

- $V_\delta[\dot{G}][\dot{H}] \models \mathrm{Reg} \setminus \mu^+ = \{f(\alpha) \mid \alpha \in S\}$ and
- $f(\alpha)^+ = \alpha$ for all $\alpha \in S$.

By Lemma 5.12, it is enough to prove that, for all $\alpha \in S$, $P * \dot{Q}_\alpha$ forces $\mathcal{P}_\kappa f(\alpha)$ carries a saturated ideal. Fix $\alpha \in S$. Then there is an almost-huge embedding $j : V \to M$ with critical point κ such that $j(\kappa) = \alpha$.

There is a continuous projection $\pi : j(P) \to P * \dot{Q}_\alpha$ defined as

$$\begin{aligned} j(P) &= \textstyle\prod^E_{\beta \in [\mu,\alpha) \cap \mathrm{Reg}} \mathrm{Coll}(\beta, < \beta) \\ &\to P \times \textstyle\prod^E_{\beta \in S \cap (\alpha+1)} \mathrm{Coll}(f(\beta), < \beta) \\ &\to P \times \textstyle\prod^E_{\beta \in S \cap (\alpha+1)} T(P, \dot{\mathrm{Coll}}(f(\beta), < \beta)) \\ &\to P \times T(P, \textstyle\prod^E_{\beta \in S \cap (\alpha+1)} \dot{\mathrm{Coll}}(f(\beta), < \beta)). \\ &\to P * \textstyle\prod^E_{\beta \in S \cap (\alpha+1)} \dot{\mathrm{Coll}}(f(\beta), < \beta) = P * \dot{Q}_\alpha. \end{aligned}$$

The third line follows from Lemma 3.2. The fourth and fifth lines follow from Lemmas 3.4 and 3.1, respectively. By the definition, it is easy to see that $\pi(p) = \langle p, \emptyset \rangle$ for all $p \in P$. Of course, $j(P)$ forces that $j(\dot{Q}_\alpha)$ is $j(\kappa)$-directed closed and well-met. By Lemma 5.3, $P * \dot{Q}_\alpha$ forces $\mathcal{P}_\kappa f(\alpha)$ carries a saturated ideal, as desired.

By Lemma 5.12, $P * \prod^E_{\alpha \in S} \dot{\mathrm{Coll}}(f(\alpha), < \alpha) = P * (\dot{Q}_\alpha \times \dot{R}_\alpha)$ forces $\mathcal{P}_\kappa f(\alpha)$ carries a saturated ideal $\dot{I}_\alpha$. $\dot{I}_\alpha$ is forced to be saturated in $V_\delta[\dot{G}][\dot{H}]$.

Note that, since π is continuous, $P * \dot{Q}_\alpha$ forces that

1. $\dot{I}_\alpha$ is $(\alpha, \alpha, < \mu)$-saturated.
2. $\dot{I}_\alpha$ is not (α, μ, μ)-saturated.
3. $\dot{I}_\alpha$ is layered if and only if α is Mahlo in V.
4. $\dot{I}_\alpha$ is not centered.

□

5.3 Saturated ideals over $\aleph_{\omega+1}$

In this section, we give an application of Section 4. We show

Theorem 5.14. *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa$ is supercompact, $\nu < \mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ Mahlo. Then there is a poset which forces that*

1. *$[\kappa, \lambda] \cap \mathrm{Reg}$ and $[\omega, \nu] \cap \mathrm{Reg}$ are not changed,*
2. *$\kappa = \mu^+$, $j(\kappa) = \lambda^+$, $\mathrm{cf}(\mu) = \nu$,*
3. *$\mathcal{P}_\kappa \lambda$ carries a saturated ideal I such that*
   - *(a) I is $(\lambda^+, \lambda^+, < \omega)$-saturated but not $(\lambda^+, \lambda^+, < \lambda)$-saturated.*
   - *(b) I is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
   - *(c) I is not layered.*
   - *(d) I is centered.*

First, we introduce Shioya's theorem and give a proof.

Theorem 5.15 (Shioya [41])**.** *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ is Mahlo. Then $R(\mu, \kappa) * R(\lambda, < j(\kappa))$ forces that there is a $(\lambda, < \mu)$-centered ideal I over $\mathcal{P}_\kappa \lambda$.*

Proof. To use Lemma 5.3, let us define a continuous projection from $R(\mu, j(\kappa))$ into $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa))$.

First, by the definition of $R^{n+1}(\mu, j(\kappa))$, we can show that there is a natural projection from $R^{n+1}(\mu, j(\kappa))$ to $R^{n+1}(\mu,\kappa) \times R^n(\lambda, j(\kappa))$, that is continuous. Therefore $R(\mu, j(\kappa))$ is projected to $R(\mu,\kappa) \times R(\lambda, j(\kappa))$ as follows.

$$\begin{aligned} R(\mu, j(\kappa)) &= R^0(\mu, j(\kappa)) \times \textstyle\prod_{n<\omega} R^{n+1}(\mu, j(\kappa)) \\ &\to R^0(\mu,\kappa) \times \textstyle\prod_{n<\omega} R^{n+1}(\mu,\kappa) \times R^n(\lambda, j(\kappa)) \\ &= R(\mu,\kappa) \times R(\lambda, j(\kappa)). \end{aligned}$$

Let us define a continuous projection from $R(\mu,\kappa) \times R(\lambda, j(\kappa))$ to $R(\mu,\kappa) \times \dot{R}(\lambda, j(\kappa))$. Let $P = R(\mu,\kappa)$. By induction on $n < \omega$ and Lemma 3.3, we have a continuous dense embedding as follows.

$$\begin{aligned} R^{n+1}(\lambda, j(\kappa)) &= \textstyle\prod^{<\lambda}_{\alpha\in[\lambda,j(\kappa))\cap\mathrm{Reg}} R^n(\alpha, j(\kappa)) \\ &\to \textstyle\prod^{<\lambda}_{\alpha\in[\lambda,j(\kappa))\cap\mathrm{Reg}} T(P, \dot{R}^n(\alpha, j(\kappa))) \\ &\to T(P, \textstyle\prod^{<\lambda}_{\alpha\in[\lambda,j(\kappa))\cap\mathrm{Reg}} \dot{R}^n(\alpha, j(\kappa))) \\ &= T(P, \dot{R}^{n+1}(\lambda, j(\kappa))) \end{aligned}$$

Therefore, we also have a continuous dense embedding as follows.

$$\begin{aligned} R(\lambda, j(\kappa)) &= \textstyle\prod_n R^n(\alpha, j(\kappa)) \\ &= \mathrm{Coll}(\lambda, < j(\kappa)) \times \textstyle\prod_n R^{n+1}(\alpha, j(\kappa)) \\ &\to T(P, \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \times \textstyle\prod_n T(P, \dot{R}^{n+1}(\lambda, j(\kappa))) \\ &\to T(P, \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \times \textstyle\prod_{n<\omega} \dot{R}^{n+1}(\alpha, j(\kappa))) \\ &= T(P, \dot{R}(\lambda, j(\kappa))) \end{aligned}$$

By Lemma 3.1, there is a continuous projection

$$\begin{aligned} R(\mu, j(\kappa)) &\to R(\mu,\kappa) \times R(\lambda, j(\kappa)) \\ &\to R(\mu,\kappa) \times T(P, \dot{R}(\lambda, j(\kappa))) \\ &\to R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa)). \end{aligned}$$

By Lemma 5.3, there is an $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa))$-name $\dot{I}$ such that

1. $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa)) \Vdash \dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.
2. $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa)) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I} \simeq R(\mu, j(\kappa))/\dot{G} * \dot{H}$.

By the proof of 4 in Lemma 2.30, $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa)) \Vdash (R(\mu, j(\kappa)))^V$ is $(\lambda, <\mu)$-centered. By Lemma 3.12 and the $<\mu$-Baireness of $R(\mu,\kappa) * \dot{R}(\lambda, j(\kappa))$, $R(\mu, j(\kappa))/\dot{G} * \dot{H}$ is forced to be $(\lambda, <\mu)$-centered, which in turn implies that $\dot{I}$ is $(\lambda, <\mu)$-centered. □

Let us the extent of saturation of the ideal in Theorem 5.15.

Theorem 5.16. *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ is Mahlo. Then $R(\mu,\kappa) * R(\lambda, < j(\kappa))$ forces that there is a centered ideal I over $\mathcal{P}_\kappa\lambda$ with the following properties:*

1. *I is $(\lambda^+, \lambda^+, <\mu)$-saturated but not $(\lambda^+, \lambda^+, <\lambda)$-saturated.*
2. *I is layered.*

3. *I is not strongly layered.*

4. *I is* $(\lambda, <\mu)$*-centered.*

Proof. Let $\dot{I}$ be an $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$-name for the ideal in the proof of Theorem 5.15. 4 has been proven and this shows that the $(j(\kappa),j(\kappa),<\mu)$-saturation of $\dot{I}$. The proof of 2 in Claim 5.6 shows that $\dot{I}$ is not $(j(\kappa),j(\kappa),<\lambda)$-saturated.

Note that the projection that we used in the proof of Theorem 5.15 was continuous. By Lemmas 2.30 and 3.11, $R(\mu,j(\kappa))/\dot{G}*\dot{H}$ is $\mathrm{Reg}\cap j(\kappa)$-layered in the extension. Therefore it is forced that $\dot{J}$ is layered. But, it is easy to see that $R(\mu,j(\kappa))$ is not $S=(E^{j(\kappa)}_{\geq\lambda}\setminus\mathrm{Reg})$-layered. S remains a stationary subset of $j(\kappa)$ in the extension by $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$. it is forced that $\dot{I}$ is not S-layered, and thus, $\dot{I}$ is not strongly layered. □

Proof of Theorem 5.14. By Lemma 2.10, we may assume that μ is indestructible. We discuss in the extension V_1 by $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$. Let J be a saturated ideal in Theorem 5.15. Since μ remains supercompact in V_1, we can take a normal ultrafilter U over μ and a coherent system $\langle W,F\rangle$ of length ν.

By Theorems 5.16, 4.3, and 4.7, Both $\mathcal{P}_U$ and $\mathcal{M}_{W,F}$ force

- $\overline{J}$ is (λ^+,ν',ν')-saturated for all $\nu'<\mu$.
- $\overline{J}$ is $(\lambda,<\omega)$-centered. This implies that $\overline{J}$ is $(\lambda^+,\lambda^+,<\omega)$-saturated.
- $\overline{J}$ is not layered.

Let $\dot{U}$ and $\langle\dot{W},\dot{F}\rangle$ be $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$-names for U and $\langle W,F\rangle$, respectively. If $\nu=\omega$ then $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))*\mathcal{P}_{\dot{U}}$ is a required poset. If $\nu>\omega$, $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))*\mathcal{M}_{\dot{W},\dot{F}}$ is a required poset. □

Theorem 5.17. *Suppose that j is an almost-huge embedding with critical point κ, $\mu<\kappa$ is supercompact, $\nu<\mu<\kappa\leq\lambda<j(\kappa)$ are regular cardinals, and $j(\kappa)$ Mahlo. Then there is a poset which forces that*

1. $\kappa=\aleph_{\omega+1}$, $j(\kappa)=\lambda^+$, *and*
2. $\mathcal{P}_\kappa\lambda$ *carries a saturated ideal I such that*
   (a) *I is $(\lambda^+,\lambda^+,<\omega)$-saturated but not $(\lambda^+,\lambda^+,<\lambda)$-saturated.*
   (b) *I is (λ^+,ν',ν')-saturated for all $\nu'<\mu$.*
   (c) *I is not layered.*
   (d) *I is centered.*

Proof. By Lemma 2.10, we may assume that μ is indestructible. We discuss in the extension V_1 by $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$ again. Let J be a saturated ideal in Theorem 5.15. Since μ remains supercompact in V_1, there is a normal ultrafilter U over μ and a guiding generic $\mathcal{G}$ of U. Then $\mathcal{P}_{U,\mathcal{G}}$ forces that

- $\overline{J}$ is (λ^+,ν',ν')-saturated for all $\nu'<\mu$.
- $\overline{J}$ is $(\lambda,<\omega)$-centered. This implies that $\overline{J}$ is $(\lambda^+,\lambda^+,<\omega)$-saturated.
- $\overline{J}$ is not layered.

Let $\dot{U}$ and $\dot{\mathcal{G}}$ be $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))$-names for U and $\mathcal{G}$. $R(\mu,\kappa)*\dot{R}(\lambda,j(\kappa))*\mathcal{P}_{\dot{U},\dot{\mathcal{G}}}$ is a required poset, as desired. □

By Lemma 5.4, the similar proofs show that Theorems 5.18 and 5.19.

Theorem 5.18. *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa$ is supercompact, $\nu < \mu < \kappa < \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ Mahlo. Then there is a poset which forces that*

1. $[\kappa, \lambda] \cap \mathrm{Reg}$ *and* $[\omega, \nu] \cap \mathrm{Reg}$ *are not changed,*
2. $\kappa = \mu^+$, $j(\kappa) = \lambda^+$, $\mathrm{cf}(\mu) = \nu$,
3. $[\lambda^+]^\kappa$ *carries a saturated ideal I such that*
   (a) *I is $(\lambda^+, \lambda^+, < \omega)$-saturated but not $(\lambda^+, \lambda^+, < \lambda)$-saturated.*
   (b) *I is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
   (c) *I is not S-layered for all stationary $S \subseteq \lambda^+$.*
   (d) *I is λ-centered.*

Theorem 5.19. *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa$ is supercompact, $\nu < \mu < \kappa \leq \lambda < \theta$ are regular cardinals, and $j(\kappa)$ Mahlo. Then there is a poset which forces that*

1. $\kappa = \aleph_{\omega+1}$, $j(\kappa) = \lambda^+$, *and*
2. $[\lambda^+]^{\aleph_{\omega+1}}$ *carries a saturated ideal I such that*
   (a) *I is $(\lambda^+, \lambda^+, < \omega)$-saturated but not $(\lambda^+, \lambda^+, < \lambda)$-saturated.*
   (b) *I is (λ^+, ν', ν')-saturated for all $\nu' < \mu$.*
   (c) *I is not S-layered for all stationary $S \subseteq \lambda$.*
   (d) *I is λ-centered.*

Lastly, we give a model that we mentioned after Theorem 4.3.

Proposition 5.20. *Suppose that κ is a huge cardinal and $\mu < \kappa$ is a supercompact cardinal. Then there is a poset that forces that λ is Mahlo, μ is singular, and $Z = [\lambda]^{\mu^+}$ carries a normal, fine, μ^+-complete λ-saturated ideal I such that*

1. *I is S-layered for some stationary $S \subseteq \lambda \cap \mathrm{Reg}$.*
2. *I is λ-dense.*

Proof. We may assume that μ is indestructible supercompact by Theorem 2.10. Let $j : V \to M$ be a huge embedding with critical point κ. Then $\mathrm{Coll}(\mu, <\kappa) \Vdash [j(\kappa)]^\kappa$ carries a normal, fine, and κ-complete ideal I such that $\mathcal{P}([j(\kappa)]^\kappa)/I \simeq \mathrm{Coll}(\mu, < j(\kappa))$ (See [17, Example 7.25]). Let $\dot{U}$ be a $\mathrm{Coll}(\mu, < \kappa)$-name for a normal ultrafilter over μ. By Theorem 2.20, $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \Vdash \mathcal{P}([j(\kappa)]^\kappa)/\overline{I} \simeq \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$. We claim that it is forced that $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \Vdash \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$ is $(\mathrm{Reg} \cap j(\kappa))^V$-layered. It is easy to see that $\overline{I}$ is forced to be $j(\kappa)$-dense and $j(\kappa)$-saturated. It remains to show that $\overline{I}$ is forced to be S-layered for some S.

For $\mathrm{Coll}(\mu, < j(\kappa))$-name $\dot{X}$ for a subset of μ, there is a maximal anti-chain $\mathcal{A}_{\dot{X}}$ such that every $q \in \mathcal{A}_{\dot{X}}$ decides $\dot{X} \in j(\dot{U})$. Let $\rho(\dot{X})$ be the least $\alpha < j(\kappa)$ such that $\mathcal{A}_{\dot{X}} \subseteq \mathrm{Coll}(\mu, < \alpha)$. For $\beta < j(\kappa)$, define $\rho(\beta) < j(\kappa)$ by $\sup\{\rho(\dot{X}) \mid \dot{X}$ is $\mathrm{Coll}(\mu, < \beta)$-name for a subset of $\mu\} \cup \{2^\beta\}$. Let C be a club generated by ρ. For every $\alpha \in C \cap \mathrm{Reg}$, $\mathrm{Coll}(\mu, < \alpha) \Vdash \dot{U}_\alpha := \dot{j}(U) \cap V[\dot{G}_\alpha]$ is an ultrafilter. Here, $\dot{G}_\alpha$ is the canonical name for a generic filter of $\mathrm{Coll}(\mu, < \alpha)$. By Lemma 2.41,

$$\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \lessdot \mathrm{Coll}(\mu, < \alpha) * \mathcal{P}_{\dot{U}_\alpha} \lessdot \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}.$$

Then $\mathrm{Coll}(\mu, < \alpha) * \mathcal{P}_{\dot{U}_\alpha}/\dot{G} * \dot{H} \lessdot \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$ holds in the extension by $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}}$. Let $\dot{P}_\alpha$ be a $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}}$-name for $\mathrm{Coll}(\mu, < f(\alpha)) * \mathcal{P}_{\dot{U}_{f(\alpha)}}$, here $f(\alpha) = \min(C \cap \mathrm{Reg})^V \setminus \alpha$. $\langle \dot{P}_\alpha \mid \alpha < j(\kappa) \rangle$ is forced to satisfy the condition of 2 of Lemma 2.8. □

5.4 Not $\aleph_2$-Knaster ideal

Many well-known saturated ideals are Knaster. For example, see Theorems 5.5. Contrary, Cox [5] constructed a model with a saturated ideal over $\aleph_1$ that absorbs a $\aleph_2$-Suslin tree by studying the theory of universal iterations. This saturated ideal does not have the $(\aleph_2, \aleph_2, 2)$-c.c. Here, We give a model by the two-step iteration of some product forcing and a Levy collapse. Indeed,

Theorem 5.21. *Suppose that j is an almost-huge embedding with critical point κ and $j(\kappa)$ weakly compact. Then there is posets $\langle Q_\alpha \mid \alpha < \kappa\rangle$ such that $\prod^{<\omega}_{\alpha<\kappa} Q_\alpha$ has the κ-c.c. $\prod^{<\omega}_{\alpha<\kappa} Q_\alpha * \dot{\mathrm{Coll}}(\kappa, < j(\kappa))$ forces*

1. *$\kappa = \aleph_1$, $j(\kappa) = \aleph_2$,*
2. *κ carries a saturated ideal I,*
3. *There is a $\aleph_2$-Suslin tree T such that there is a projection from $\dot{\mathcal{P}}(\aleph_1)/I$ to T.*

We need the following fact.

Theorem 5.22. *For $\kappa < \lambda$, if λ is an inaccessible then* $\mathrm{Coll}(\kappa, < \lambda)$ *forces a λ-Suslin tree exists.*

Proof of Theorem 5.21. By induction, let us define $\langle Q_\alpha \mid \alpha < \kappa \text{ inaccessible or } 0\rangle$ by $Q_0 = \mathrm{Coll}(\omega, < \kappa)$ and $Q_\alpha = T(\prod^{<\omega}_{\beta<\alpha} Q_\beta \cap V_\alpha, \dot{\mathrm{Coll}}(\alpha, < \kappa) * \dot{T}_\alpha)$. Here, $\dot{T}_\alpha$ is a $\prod^{<\omega}_{\beta<\alpha} Q_\beta \cap V_\alpha * \dot{\mathrm{Coll}}(\alpha, < \kappa)$-name for a κ-Suslin tree. We may assume that it is forced that the base set of $\dot{T}_\alpha$ is κ.

For each α, by Lemma 3.5, Q_α and $j(Q)_\alpha$ has the κ-c.c. and $j(\kappa)$-c.c., respectively. Let $P = \prod^{<\omega}_{\alpha<\kappa} Q_\alpha$. Since κ and $j(\kappa)$ are weakly compact, P and $j(P)$ has the $j(\kappa)$-c.c. It is easy to see that $P \subseteq V_\kappa$ and $P \Vdash \aleph_1 = \kappa$.

There is a projection π from $j(P)$ to $P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa))$. Note that $Q_\beta \subseteq V_\kappa$ for all $\beta < \kappa$ and thus $j(Q)_\beta \cap V_\kappa = Q_\beta$. Therefore $j(Q_\beta)$ projects to Q_β by identifying Q_β with its completion. $j(P)$ is projected as follows:

$$\begin{aligned} j(P) &\to \textstyle\prod^{<\omega}_{\beta<\kappa} j(Q_\beta) \times j(Q)_\kappa \\ &\to \textstyle\prod^{<\omega}_{\beta<\kappa} Q_\beta \times T(\prod^{<\omega}_{\beta<\kappa} Q_\beta, \dot{\mathrm{Coll}}(\kappa, < j(\kappa)) * j(\dot{T})_\kappa) \\ &= P \times T(P, \dot{\mathrm{Coll}}(\kappa, < j(\kappa)) * \dot{T}_\kappa) \\ &\to P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa)) * j(\dot{T})_\kappa \\ &\to P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa)). \end{aligned}$$

It is easy to see that $\pi(p) = \langle p, \emptyset\rangle$ for all $p \in P$.

By Lemma 5.1, there is a $P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa))$-name for an ideal $\dot{I}$ such that $P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa)) \Vdash \mathcal{P}(\kappa)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$. Note that we can identify an ideal over κ with $\mathcal{P}_\kappa\kappa$. By the definition of π, $P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa))$ forces that there is a projection such that

$$j(P)/\dot{G} * \dot{H} \to (P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa)) * j(\dot{T})_\kappa)/\dot{G} * \dot{H} \simeq j(\dot{T})_\kappa.$$

Of course, $j(\dot{T})_\kappa$ is forced by $P * \dot{\mathrm{Coll}}(\kappa, < j(\kappa))$ to be $j(\kappa)$-Suslin tree. Therefore, it is forced that $\mathcal{P}(\kappa)/\dot{I}$ is projected to $j(\kappa)$-Suslin tree $j(\dot{T})_\kappa$, as desired. □

We can obtain a model with a saturated ideal by starting a model with an almost-huge cardinal as we saw in Theorem 5.5. For an almost-huge cardinal, its target is not Mahlo usually. On the other hand, to show the chain condition of $j(Q)_\alpha$, we need the weakly compactness of $j(\kappa)$. We cannot omit this assumption if we use Lemma 3.5 by Theorem 3.7.

5.5 The extent of saturation of Kunen's and Laver's ideals

The first model in which $\aleph_1$ carries a saturated ideal is due to Kunen.

Theorem 5.23 (Kunen [28]). *If j is a huge embedding with critical point κ. Then there is a poset P such that $P * \dot{S}(\kappa, j(\kappa))$ forces $\kappa = \aleph_1$ carries a saturated ideal.*

In this section, as an application of Section 5.1 we study the extent of saturation of Kunen's ideal. We show

Theorem 5.24. *Suppose that $j : V \to M$ is a huge embedding with critical point κ and $f : \kappa \to \mathrm{Reg} \cap \kappa$ satisfies $j(f)(\kappa) \geq \kappa$. For regular cardinals $\mu < \kappa \leq \lambda = j(f)(\kappa) < j(\kappa)$, there is a P such that $P * \dot{S}(\lambda, j(\kappa))$ forces $\mu^+ = \kappa$ and $\lambda^+ = j(\kappa)$ and $\mathcal{P}_\kappa(\lambda)$ carries a saturated ideal I such that*

1. *I is $(\lambda^+, \lambda^+, < \mu)$-saturated.*
2. *I is not (λ^+, μ, μ)-saturated. In particular, $\dot{I}$ is not strongly saturated.*
3. *I is layered.*
4. *I is not centered.*

Note that, if $\mu = \omega$, $\kappa = \lambda$, and $f = \mathrm{id}$, P and the ideal in Theorem 5.24 are the same as those in Theorem 5.23.

Proof. We may assume that $f(\alpha) \geq \alpha$ for all α. Let $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ be the $< \mu$-support iteration such that

- $P_0 = S(\mu, \kappa)$.
- $P_{\alpha+1} = \begin{cases} P_\alpha * S^{P_\alpha \cap V_\alpha}(f(\alpha), \kappa) & \alpha \text{ is good} \\ P_\alpha & \text{otherwise} \end{cases}$.

Here, we say that α is good if $P_\alpha \cap V_\alpha \lessdot P_\alpha$ has the α-c.c., α is inaccessible, and $\alpha \geq \mu$. The set $P_\alpha * S^{P_\alpha \cap V_\alpha}(\alpha, \kappa)$ is the set of all $\langle p, \dot{q} \rangle$ such that $p \in P_\alpha$ and $\dot{q}$ is a $P_\alpha \cap V_\alpha$-name for an element of $S^{P_\alpha \cap V_\alpha}(\alpha, \kappa)$. For a poset Q, $S^Q(\alpha, \kappa)$ denotes a Q-name for Silver collapse $S(\alpha, \kappa)$.

Let P be the set of all $p \in P_\kappa$ such that $p(\alpha)$ is $P_\alpha \cap V_\alpha$-name for every good $\alpha < \kappa$. The following is a list of certain properties of P:

- P is μ-directed closed and has the $(\kappa, \kappa, < \mu)$-c.c.
- $j(P)$ has the $(j(\kappa), j(\kappa), < \mu)$-c.c.
- $P \subseteq V_\kappa$ and $P \Vdash \mu^+ = \kappa$.
- κ is good for $j(P)$. In particular, $j(P)_\kappa \cap V_\kappa = P \lessdot j(P)_\kappa$.
- There is a complete embedding $\tau : P * \dot{S}(\lambda, j(\kappa)) \to j(P)_{\kappa+1} \lessdot j(P)$ such that $\tau(p, \emptyset) = p$ for all $p \in P$.
- The projection $\pi : j(P) \to \mathcal{B}(P * \dot{S}(\lambda, < j(\kappa)))$ induced by τ is $(j(\kappa), j(\kappa), < \mu)$-nice.

We should check that $j(P)$ has the $(j(\kappa), j(\kappa), < \mu)$-c.c. For every $\{p_\xi \mid \xi < j(\kappa)\} \subseteq j(P)$, the usual Δ-system argument takes an unbounded subset $X \subseteq j(\kappa)$ and $\eta < j(\kappa)$ such that

- $\mathrm{supp}(p_\xi)$ is a Δ-system with its root $r \subseteq \eta$.
- For every good $\alpha \in r$, $P_\alpha \cap V_\alpha$ forces that $\{p_\xi(\alpha) \mid \xi < j(\kappa)\}$ is a Δ-system with its root $\subseteq \eta^3$.

Since $j(P)$ was defined as the $<\mu$-support iteration, for every $Z \in [X]^{<\mu}$, $\prod_{\xi\in Z} p_\xi \neq 0$. The similar proof shows that P has the $(\kappa,\kappa,<\mu)$-c.c.

We check π is $(j(\kappa), j(\kappa), <\mu)$-nice. We have a complete embedding

$$P \overset{\tau_0}{\to} P * \dot{S}(\lambda, j(\kappa)) \overset{\tau}{\to} j(P).$$

Here, τ_0 is a natural embedding. Note that $\tau \circ \tau_0$ is the identity mapping. Let π_0 be a projection induced by $\tau\circ\tau_0$. It is easy to see that $\pi(p) = \langle q, \dot{r}\rangle$ implies $q = \pi_0(p)$ and $q \Vdash \dot{r} = p(\kappa)$. It is easy to see that π_0 is $(j(\kappa), j(\kappa), <\mu)$-nice. For $\{p'_\xi \mid \xi < j(\kappa)\} \subseteq j(P)$, we have an extensions $p_\xi \leq p_\xi$ and $X \in [j(\kappa)]^{j(\kappa)}$ such that $\pi(p_\xi) = \langle \pi_0(p_\xi), p_\xi(\kappa)\rangle$, $\prod_{\xi\in Z} p_\xi \neq 0$ and $\pi_0(\prod_{\xi\in Z} p_\xi) = \prod_{\xi \in Z}\pi_0(p_\xi)$ for all $Z \in [X]^{<\mu}$. By the proof of $(j(\kappa), j(\kappa), <\mu)$-c.c. of $j(P)$, we can take $Y \in [X]^{j(\kappa)}$ such that

- $\mathrm{supp}(p_\xi)$ is a Δ-system with its root $r \subseteq \kappa + 1$.
- For every good $\alpha \in r$, $P_\alpha \cap V_\alpha$ forces that $\{p_\xi(\alpha) \mid \xi < j(\kappa)\}$ is a Δ-system with its root $\subseteq (\kappa+1)^3$.

Then, for every $Z \in [Y]^{<\mu}$, $\Vdash \prod_{\xi\in Z} p_\xi(\kappa) \neq 0$. Therefore, we have

$$\begin{aligned}
\textstyle\prod_{\xi\in Z}\pi(p_\xi) &= \textstyle\prod_{\xi\in Z}\langle \pi_0(p_\xi(\kappa)), p_\xi(\kappa)\rangle \\
&= \textstyle\langle \prod_{\xi\in Z}\pi_0(p_\xi)\cdot \|\prod_{\xi\in Z}\dot{p}_\xi(\kappa) \neq 0\|, \prod_\xi \dot{p}_\xi(\kappa)\rangle \\
&= \textstyle\langle \pi_0(\prod_{\xi\in Z} p_\xi), \dot{p}_\xi(\kappa))\rangle \\
&= \textstyle\pi(\prod_{\xi\in Z} p_\xi).
\end{aligned}$$

By identifying $P * \dot{S}(\lambda, j(\kappa))$ with its completion, we can apply Lemma 5.3 to π. There is a $P * \dot{S}(\lambda, j(\kappa))$-name $\dot{I}$ such that

1. $P * \dot{S}(\lambda, j(\kappa)) \Vdash \dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.
2. $P * \dot{S}(\lambda, j(\kappa)) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$.

Since π is $(j(\kappa), j(\kappa), <\mu)$-nice and P is $<\mu$-Baire, by Lemma 3.14, $j(P)/\dot{G}*\dot{H}$ is forced to have the $(j(\kappa), j(\kappa), <\mu)$-c.c., and thus, $\dot{I}$ is $(j(\kappa), j(\kappa), <\mu)$-saturated.

Note that $j(P)$ was defined as the $<\mu$-support iteration. By the proof of 2 in Claim 5.6, it is forced that $j(P)/\dot{G} * \dot{H}$ is not $(j(\kappa), \mu, \mu)$-c.c. Thus, 2 holds.

For the proof of 3, we refer to [13, Theorem 3]. Lastly, we check the following claim.

Claim 5.25. *$P * \dot{S}(\lambda, j(\kappa))$ forces $j(P)/\dot{G} * \dot{H}$ is not λ-centered.*

Proof of Claim. Note that $\{\alpha < j(\kappa) \mid V_\alpha \cap j(P)_\alpha \lessdot j(P)_\alpha$ has the α-c.c. and α is inaccessible$\}$ is unbounded in $j(\kappa)$. We choose $\alpha > \lambda$ from this set. Note that $j(f)(\alpha) \geq \alpha$ and $j(f)(\alpha)$ is regular. It is enough to prove that $P * \dot{S}(\lambda, j(\kappa))$ forces that $j(P)_{\alpha+1}/\dot{G} * \dot{H}$ is not λ-centered.

We show by contradiction. Suppose that the existence of a centering family $\langle \dot{C}_\xi \mid \xi < \lambda\rangle$ of $j(P)_{\alpha+1}/\dot{G} * \dot{H}$ is forced by some condition. We may assume that each $\dot{C}_\xi$ is forced to be a filter. To simplify notation, we assume $P * \dot{S}(\lambda, j(\kappa))$ forces the existence of such a centering family. By the κ-c.c. of P, for every $\langle p, \dot{q}\rangle \in P * \dot{S}(\lambda, j(\kappa))$, $P \Vdash \dot{q} \in \dot{S}(\lambda, \beta)$ for some $\beta < j(\kappa)$. For each $q \in j(P)_{\alpha+1}$, let $\rho(q)$ be defined by the following way:

For $\xi < \lambda$, let $\mathcal{A}_q^\xi \subseteq P * \dot{S}(\lambda, j(\kappa))$ be a maximal anti-chain such that, for every $r \in \mathcal{A}^\xi_q$, r decides $q \in \dot{C}_\xi$. $\rho(q)$ is the least ordinal $\beta < j(\kappa)$ such that $P \Vdash \dot{q} \in \dot{S}(\lambda, \beta)$ for every $\langle p, \dot{q}\rangle \in \bigcup_\xi \mathcal{A}^\xi_q$.

We put $Q = j(P)_\alpha \cap V_\alpha$. Let $C \subseteq j(\kappa)$ be a club generated by $\beta \mapsto \sup\{\rho(q) \mid q \in Q * S^Q(j(f)(\alpha), \beta)\}$. Since $j(\kappa)$ is inaccessible, we can find a strong limit cardinal $\delta \in C \cap E^{j(\kappa)}_{>\lambda} \cap E^{j(\kappa)}_{\leq\alpha} \setminus (j(f)(\alpha)+1)$.

By the κ-c.c. of P and 2 in Lemma 2.33, $P * \dot{S}(\lambda, \delta) \Vdash (\delta^+)^V \geq \lambda^+$. We will discuss in the extension by $P * \dot{S}(\lambda, \delta)$. Let $\dot{G} * \dot{H}_\delta$ be the canonical $P * \dot{S}(\lambda, \delta)$-name for a generic filter. By Lemma 2.33.1,

$S(j(f)(\alpha),\delta)$ has an anti-chain of size δ^+. This defines a $P * \dot{S}(\lambda,\delta)$-name that is forced to be an anti-chain in $Q * S^Q(j(f)(\alpha),\delta)/\dot{G} * \dot{H}_\delta$ of size $(\delta^+)^V$. Therefore, $P * \dot{S}(\lambda,\delta)$ forces that $Q * S^Q(j(f)(\alpha),\delta)/\dot{G} * \dot{H}_\delta$ does not have the λ^+-c.c. and thus is not λ-centered.

On the other hand, $\langle \dot{C}_\xi \mid \xi < \lambda \rangle$ defines a $P * \dot{S}(\lambda,\delta)$-name of a centering family of $Q * S^Q(j(f)(\alpha),\delta)/\dot{G} * \dot{H}_\delta$. Let $G * H$ be an arbitrary $(V, P * \dot{S}(\lambda, j(\kappa)))$-generic filter. Note that $G * H_\delta = G * H \cap (P * \dot{S}(\lambda,\delta))$ is $(V, P * \dot{S}(\lambda,\delta))$-generic. We discuss in $V[G][H_\delta]$. Let $D_\xi \subseteq Q * S^Q(j(f)(\alpha),\delta)/G * H_\delta$ be defined by

$$\langle p, \dot{q} \rangle \in D_\xi \Leftrightarrow \text{“for every } \beta < \delta \text{ some condition in } G * H_\delta \text{ forces } \langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi\text{”}.$$

It is easy to see that D_ξ is a filter over $Q * S^Q(j(f)(\alpha),\delta)/G * H_\delta$. We claim that $\langle D_\xi \mid \xi < \lambda \rangle$ covers $Q * S^Q(j(f)(\alpha),\delta)/G * H_\delta$. For each $\langle p, \dot{q} \rangle \in Q * \dot{S}(j(f)(\alpha),\delta)/G * H_\delta$, in $V[G][H]$, there is a ξ such that $\langle p, \dot{q} \rangle \in \dot{C}_\xi^{G*H}$. Then $\langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi^{G*H}$ for every $\beta < \delta$, since $\dot{C}_\xi^{G*H}$ is a filter.

Note that $\rho(\langle p, \dot{q} \restriction \beta \rangle) < \delta$ for all $\beta < \delta$ by $\delta \in C$. Therefore, for each $\beta < \delta$, the statement $\langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi$ is decided by $P * \dot{S}(\lambda,\delta)$. In particular, for every $\beta < \delta$ some condition in $G * H_\delta$ forces $\langle p, \dot{q} \restriction \beta \rangle \in \dot{C}_\xi$. By the definition of D_ξ, $\langle p, \dot{q} \rangle \in D_\xi$ in $V[G][H_\delta]$, as desired. This is a contradiction. □

By this claim, it is forced that $\dot{I}$ is not centered. □

In [29], Laver established

Theorem 5.26 (Laver [29]). *Suppose that $j : V \to M$ is a huge embedding with critical point κ. Then there is a poset P such that $P * \dot{L}(\kappa, j(\kappa))$ forces that $\aleph_1$ carries a strongly saturated ideal.*

Theorem 5.27 is an analogie of Theorem 5.24 for Laver's model.

Theorem 5.27. *Suppose that j is a huge embedding with critical point κ and $f : \kappa \to \mathrm{Reg} \cap \kappa$ satisfies $j(f)(\kappa) \geq \kappa$. For regular cardinals $\mu < \kappa \leq \lambda = j(f)(\kappa) < j(\kappa)$, there is a P such that $P * \dot{L}(\lambda, j(\kappa))$ forces that $\mu^+ = \kappa$, $\lambda^+ = j(\kappa)$ and $\mathcal{P}_\kappa(\lambda)$ carries a strongly saturated ideal I such that*

1. *I is $(\lambda^+, \lambda^+, < \lambda)$-saturated and $(\lambda^+, \lambda, \lambda)$-saturated.*
2. *I is not $(\lambda^+, \lambda^+, \lambda)$-saturated.*
3. *I is layered.*

Proof. Let $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ be the Easton support iteration such that

- $P_0 = L(\mu, \kappa)$.
- $P_{\alpha+1} = \begin{cases} P_\alpha * L^{P_\alpha \cap V_\alpha}(f(\alpha), \kappa) & \alpha \text{ is good} \\ P_\alpha & \text{otherwise} \end{cases}.$

Again, we say that α is good if $P_\alpha \cap V_\alpha \lessdot P_\alpha$ has the α-c.c., α is inaccessible, and $\alpha \geq \mu$. For a poset Q, $L^Q(\alpha, \kappa)$ denotes a Q-name for Laver collapse $L(\alpha, \kappa)$.

Let P be the set of all $p \in P_\kappa$ such that $p(\alpha)$ is $P_\alpha \cap V_\alpha$-name for every good $\alpha < \kappa$. The following is a list of certain properties of P:

- P is μ-directed closed and has the $(\kappa, \kappa, < \nu)$-c.c for all $\nu < \kappa$.
- $P \subseteq V_\kappa$ and $P \Vdash \mu^+ = \kappa$.
- κ is good for $j(P)$. In particular, $j(P)_\kappa \cap V_\kappa = P \lessdot j(P)_\kappa$.
- There is a complete embedding $\tau : P * \dot{L}(\lambda, j(\kappa)) \to j(P)_{\kappa+1} \lessdot j(P)$ such that $\tau(p, \emptyset) = p$ for all $p \in P$.

The chain condition is proven from the usual Δ-system argument. Let π be the projection induced by τ. By Lemma 5.3, there is a $P * \dot{L}(\lambda, j(\kappa))$-name $\dot{I}$ such that

1. $P * \dot{L}(\lambda, j(\kappa)) \Vdash \dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.
2. $P * \dot{L}(\lambda, j(\kappa)) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I} \simeq \dot{j}(P)/\dot{G} * \dot{H}$.

For the $(j(\kappa), j(\kappa), < \lambda)$-c.c. of $\dot{j}(P)/\dot{G} * \dot{H}$, we refer to [29, Section 1]. Note that $j(P)$ has the $(j(\kappa), j(\kappa), \mu)$-c.c. for all $\mu < j(\kappa)$ since j is a huge. Therefore $j(P)$ has the $(j(\kappa), \lambda, \lambda)$-c.c. By Lemma 3.18, $j(P)/\dot{G} * \dot{H}$ is forced to have the $(j(\kappa), \lambda, \lambda)$-c.c. Therefore $P * \dot{L}(\lambda, j(\kappa))$ forces that $\dot{I}$ is $(j(\kappa), \lambda, \lambda)$-saturated.

Claim 5.28. *$P * \dot{L}(\lambda, j(\kappa))$ forces that $j(P)/\dot{G} * \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c.*

Proof. Let us show $P * \dot{L}(\lambda, j(\kappa)) \Vdash j(P)/\dot{G} * \dot{H}$ does not has the $(j(\kappa), j(\kappa), \lambda)$-c.c. Let $\{q_\alpha \in j(P) \mid \alpha > \kappa + 1\}$ be an arbitrary such that $\mathrm{supp}(q_\alpha) = \{\alpha\}$ for all α. Note that $\langle \emptyset, \emptyset \rangle \in P * \dot{L}(\lambda, j(\kappa))$. For every $p \in P * \dot{L}(\lambda, j(\kappa))$, $\tau(p) \in j(P)_{\kappa+1}$. Therefore $\tau(p)$ meets with q_α. We have $\Vdash \{q_\alpha \mid \alpha > \kappa + 1\} \subseteq j(P)/\dot{G} * \dot{H}$.

We fix an arbitrary $\dot{A}$ with $\Vdash \dot{A} \in [j(\kappa)]^{j(\kappa)}$. Let us find an $\dot{x}$ such that $\Vdash \dot{x} \in [\dot{A}]^\lambda$ and $\{q_\alpha \mid \alpha \in \dot{x}\}$ has a lower bound in $j(P)/\dot{G} * \dot{H}$. By the $j(\kappa)$-c.c. of $P * \dot{j}(\lambda, j(\kappa))$, there is a club $C \subseteq j(\kappa)$ such that $\Vdash C \subseteq \mathrm{Lim}(\dot{A})$. Since $j(\kappa)$ is Mahlo, there is an inaccessible $\alpha \in C \setminus (\lambda + 1)$. Let $\dot{x}$ be a $P * \dot{L}(\lambda, j(\kappa))$-name for the set $\dot{A} \cap \alpha$. Since $\Vdash \alpha \in \mathrm{Lim}(\dot{A})$, $\sup \dot{x} = \alpha$. Since $\lambda \leq \alpha < j(\kappa)$, $|\dot{x}| = |\alpha| = \lambda$ is forced by $P * \dot{L}(\lambda, j(\kappa))$. We claim that $\Vdash \{q_\alpha \mid \alpha \in \dot{x}\}$ witnesses. Suppose otherwise, there are $p \in P * \dot{L}(\lambda, j(\kappa))$ and $r \in j(P)$ such that $p \Vdash r \in j(P)/\dot{G} * \dot{H} \leq q_\alpha$ for all $\alpha \in \dot{x}$. Note that $\mathrm{supp}(r) \cap \alpha < \beta$ for some $\beta < \alpha$. By $q \Vdash \sup \dot{x} = \alpha$, there are $q \leq p$ and $\gamma \in [\beta^+, \alpha)$ such that $q \Vdash \gamma \in \dot{x}$. Therefore $q \Vdash r \leq q_\gamma$, and thus, $\{\gamma\} \subseteq \mathrm{supp}(r) \cap [\beta^+, \alpha)$. This is a contradiction. □

By this claim, $P * \dot{L}(\lambda, j(\kappa))$ forces that $\dot{I}$ is not $(j(\kappa), j(\kappa), \lambda)$-saturated. For the layeredness of $\dot{I}$, we refer to [13, Theorem 3]. □

We introduce Shioya's theorem for a model of a strongly saturated ideal.

Theorem 5.29 (Shioya [41])**.** *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ is Mahlo. Then $E(\mu, \kappa) * \dot{E}(\lambda, j(\kappa))$ forces that there is a strongly saturated ideal I over $\mathcal{P}_\kappa\lambda$.*

We give an analogie of Theorem 5.27 for Theorem 5.29.

Theorem 5.30. *Suppose that j is an almost-huge embedding with critical point κ, $\mu < \kappa \leq \lambda < j(\kappa)$ are regular cardinals, and $j(\kappa)$ is Mahlo. Then $E(\mu, \kappa) * \dot{E}(\lambda, j(\kappa))$ forces that there is a strongly saturated ideal I over $\mathcal{P}_\kappa\lambda$ with the following properties.*

1. *I is $(\lambda^+, \lambda^+, < \lambda)$-saturated and $(\lambda^+, \lambda, \lambda)$-saturated.*
2. *I is not $(\lambda^+, \lambda^+, \lambda)$-saturated.*
3. *I is layered.*

Proof. We recall the projection $\pi : E(\mu, j(\kappa)) \to E(\mu, \kappa) * \dot{E}(\lambda, j(\kappa))$ that Shioya used in [41], because we need an property of π. For a detail, we refer [41, Proposition 1]. He gave an isomorphism in the sense of poset as follows:

$$E(\mu, j(\kappa)) \simeq E(\mu, \kappa) \times \prod\nolimits^{E}_{\gamma \in \mathrm{SR} \cap [\kappa, \lambda)} {}^{<\mu}\gamma \times E(\lambda, j(\kappa)) \times \prod\nolimits^{E}_{\gamma \in \mathrm{SR} \cap [\lambda, j(\kappa))} {}^{<\mu}\lambda.$$

And the natural projection $\pi_0 : E(\mu, j(\kappa)) \to E(\mu,\kappa) \times E(\lambda, j(\kappa))$ is continuous and $\pi_0(p) = \langle p, \emptyset\rangle$ for all $p \in E(\mu,\kappa)$.

We can define a projection $E(\mu,\kappa) \times E(\lambda, j(\kappa)) \to E(\mu,\kappa) * \dot{E}(\lambda, j(\kappa))$. Let $P = E(\mu,\kappa)$. Then, by Lemmas 3.3.1 and 3.4, there is a continuous dense embedding defined π_1 by

$$\begin{aligned} E(\lambda, j(\kappa)) &= \textstyle\prod^E_{\gamma\in \mathrm{SR}\cap[\lambda,j(\kappa))} {}^{<\lambda}\gamma \\ &\simeq \textstyle\prod^E_{\gamma\in \mathrm{SR}\cap[\lambda,j(\kappa))} T(P, {}^{<\dot{\lambda}}\gamma) \\ &\simeq T(P, \textstyle\prod^E_{\gamma\in \mathrm{SR}\cap[\lambda,j(\kappa))} {}^{<\dot{\lambda}}\gamma) \\ &= T(P, \dot{E}(\lambda, j(\kappa))). \end{aligned}$$

Therefore, we have a continuous projection $\pi = (\mathrm{id} \times \pi_1) \circ \pi_0 : j(P) \to P * \dot{E}(\lambda, j(\kappa))$. By Lemma 5.3, there is an $P * \dot{E}(\lambda, j(\kappa))$-name $\dot{I}$ such that

1. $P * \dot{E}(\lambda, j(\kappa)) \Vdash \dot{I}$ is a saturated ideal over $\mathcal{P}_\kappa\lambda$.
2. $P * \dot{E}(\lambda, j(\kappa)) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/\dot{I} \simeq j(P)/\dot{G} * \dot{H}$.

For strong saturation of $\dot{I}$, we refer to [41].

By Lemma 3.18 and π is a continuous, $j(P)/\dot{G} * \dot{H}$ is forced to have the $(j(\kappa), \lambda, \lambda)$-c.c.

We claim that $P * \dot{E}(\lambda, j(\kappa))$ forces that $j(P)/\dot{G} * \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c. We let $Q = \prod^E_{\gamma\in \mathrm{SR}\cap[\lambda,j(\kappa))} {}^{<\mu}\lambda$. By the definition of π, we have $P * \dot{E}(\lambda, j(\kappa))$ forces that there is a complete embedding $\tau_0 : Q \to j(P)/\dot{G} * \dot{H}$. It is enough to prove that Q does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c. in the extension.

Let $\{q_\alpha \mid \alpha \in \mathrm{Reg} \cap j(\kappa)\} \subseteq Q$ be a family such that $\mathrm{dom}(q_\alpha) = \{\alpha\}$ for each α. For any $p \in P * \dot{E}(\lambda, j(\kappa))$ and $\dot{A}$ with $p \Vdash \dot{A} \in [\nu]^\nu$, let us find $\dot{x}$ such that $p \Vdash \dot{x} \in [\dot{A}]^\lambda$ and $\{q_\alpha \mid \alpha \in x\}$ does not have a lower bound. By the ν-c.c., there is a club C with $p \Vdash C \subseteq \mathrm{Lim}(\dot{A})$. Since ν is Mahlo, there is an inaccessible $\alpha \in C$.

Let $\dot{x}$ be a P-name for the set $\dot{A} \cap \alpha$. Since p forces $\alpha \in \mathrm{Lim}(\dot{A})$, $p \Vdash \sup \dot{x} = \alpha$. Note that $p \Vdash |\dot{x}| = |\alpha| = \lambda$.

We claim that $\dot{x}$ witnesses. Suppose otherwise, there are $q \leq p$ and $r \in Q$ such that $q \Vdash r$ is a lower bound of $\{q_\alpha \mid \alpha \in \dot{x}\}$. Note that $\mathrm{dom}(r) \cap \alpha < \beta$ for some $\beta < \alpha$. By $q \Vdash \sup \dot{x} = \alpha$, there are $q' \leq q$ and $\alpha > \gamma > \beta$ with $q' \Vdash \gamma \in \dot{x}$. This shows $r \leq q_\gamma$, and thus, $\mathrm{dom}(r) \cap [\delta_\gamma, \alpha) \neq \emptyset$. But $\mathrm{dom}(r) \cap [\delta_\gamma, \alpha) \subseteq \mathrm{dom}(r) \cap [\beta, \alpha) = \emptyset$. This is a contradiction.

Therefore $P * \dot{E}(\lambda, j(\kappa))$ forces that $j(P)/\dot{G} * \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c. and thus, $\dot{I}$ is not $(j(\kappa), j(\kappa), \lambda)$-saturated.

Since $j(P) = E(\lambda, j(\kappa))$ is $\mathrm{Reg} \cap j(\kappa)$-layered and π is continuous, by Lemma 3.11, $j(P)/\dot{G} * \dot{H}$ is forced to be $(\mathrm{Reg} \cap j(\kappa))^V$-layered. For the same reason in the proof of Claim 5.7(i), $(\mathrm{Reg} \cap j(\kappa))^V \subseteq \dot{E}^{\lambda^+}_{\geq\lambda}$ is forced. Therefore $\dot{I}$ is forced to be layered. □

6 Applications to combinatorics

Some strengthenings of saturated ideals are founded in the contest of combinatorics. For example, Laver introduced the notion of strong saturation to obtains a model in which $\aleph_1$ carries a strongly saturated ideal and $\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_1}{\aleph_1}_{\aleph_0}$. Foreman–Laver modified Kunen's proof of Theorem 5.23 to obtains a model in which $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$. In Foreman–Laver's model, there is a centered ideal over $\aleph_1$. We define $\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_1}{\aleph_1}_{\aleph_0}$ and $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$ later.

6.1 Polarized partition relations

In the first half of this section, we present some sufficient conditions for polarized partition relations in terms of Chang's conjectures and saturated ideals.

The notion of polarized partition relations was introduced by Erdős–Hajnal–Rado [8]. $\binom{\kappa_0}{\kappa_1} \to \binom{\lambda_0}{\lambda_1}_\theta$ states, for every $f : \kappa_0 \times \kappa_1 \to \theta$ there are $H_0 \in [\kappa_0]^{\lambda_0}$ and $H_1 \in [\kappa_1]^{\lambda_1}$ such that $|f``H_0 \times H_1| \leq 1$. $\binom{\kappa_0}{\kappa_1} \to \binom{\kappa_0}{\kappa_1}_\theta$ is the most strongest form. This form trivially holds sometimes. Indeed, if $\mathrm{cf}(\kappa_0) > \theta^{\kappa_1}$ then $\binom{\kappa_0}{\kappa_1} \to \binom{\kappa_0}{\kappa_1}_\theta$ holds. But under the GCH, the non-trivial case cannot hold:

Theorem 6.1 (Erdős–Hajnal–Rado [8])**.** *If $2^\mu = \mu^+$ then $\binom{\mu^+}{\mu} \not\to \binom{\mu^+}{\mu}_2$.*

We are interested in how strong $\binom{\mu^+}{\mu} \to \binom{\lambda_0}{\lambda_1}_\theta$ can hold under the GCH. If μ is a limit cardinal, $\binom{\mu^+}{\mu} \to \binom{\mu}{\mu}_{<\mu}$ holds sometimes (For example, see [1], [40], and [47]). On the other hand, for a successor cardinal, negative partition relation is known as Theorem 6.2.

Lemma 6.2 (Folklore?)**.** *If there is a Kurepa tree on μ^+ then $\binom{\mu^{++}}{\mu^+} \not\to \binom{2}{\mu^+}_\mu$ holds.*

Proof. Let T be a Kurepa tree on μ^{++}. That is,

- T is a tree of height μ^+ and $|\mathrm{Lev}_\alpha(T)| < \mu^+$ for each $\xi < \mu^+$.
- T has μ^{++}-many cofinal branches.

Let $\{b_\alpha \mid \alpha < \mu^{++}\}$ be a set of cofinal branches with $b_\alpha \neq b_\beta$ for all $\alpha < \beta$. For each $\xi < \mu^+$, we enumerate $\mathrm{Lev}_\xi(T)$ as $\{t_{\xi i} \mid i < \mu\}$. Let $f : \mu^{++} \times \mu^+ \to \mu$ be defined by

$$f(\alpha, \xi) = i \text{ if and only if } b_\alpha \cap \mathrm{Lev}_\xi(T) = \{t_{\xi i}\}$$

For any set $H_1 \in [\mu^+]^{\mu^+}$ and α, β in μ^{++}, if $|f``\{\alpha, \beta\} \times H_1| \leq 1$ then $b_\alpha = b_\beta$. Therefore $\alpha = \beta$. c witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{2}{\mu^+}_\mu$. □

In [7, Problem 27], Erdős–Hajnal asked whether $\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_0}{\aleph_1}_{\aleph_1}$ can hold or not. Laver [29] proved that $\binom{\mu^{++}}{\mu^+} \to \binom{\mu^+}{\mu^+}_\mu$ holds if $2^\mu = \mu^+$ and μ^+ carries a strongly saturated ideal.

We introduce two notions that are Chang's conjectures and pre-saturated ideals. For cardinals $\lambda \geq \lambda'$ and $\kappa \geq \kappa' \geq \mu$, $(\lambda, \lambda') \twoheadrightarrow_\mu (\kappa, \kappa')$, which was introduced by Shelah [39], is the statement that any structure $\langle \lambda, \lambda', \in, ... \rangle$ of a language of size μ has an elementary substructure $\langle X, X \cap \lambda', \in, ... \rangle$ such that $|X| = \kappa$, $|X \cap \lambda'| = \kappa'$, and $\mu \subseteq X$. By $(\lambda, \lambda') \twoheadrightarrow (\kappa, \kappa')$, we mean $(\lambda, \lambda') \twoheadrightarrow_\omega (\kappa, \kappa')$. Note that $(\mu^{++}, \mu^+) \twoheadrightarrow_\mu (\mu^+, \mu)$ and $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ are equivalent. For a detail, we refer to [11, Lemma 14].

A pre-saturated ideal over μ^+ is a precipitous ideal I such that $\mathcal{P}(\mu^+)/I$ preserves the cardinality of μ^{++}. Of course, every saturated ideal is pre-saturated. These imply the polarized partition relation.

Lemma 6.3. *Assume one of the following holds:*

1. $(\mu^{++},\mu^{+}) \twoheadrightarrow_{\mu} (\mu^{+},\mu)$.

2. μ^{+} *carries a pre-saturated ideal.*

Then $\binom{\mu^{++}}{\mu^{+}} \to \binom{n}{\mu^{+}}_{\mu}$ *holds for all* $n<\omega$.

Proof. Let $f:\mu^{++}\times\mu^{+}\to\mu$ be an arbitrary coloring.

First, we assume $(\mu^{++},\mu^{+}) \twoheadrightarrow_{\mu} (\mu^{+},\mu)$. Consider a structure $\mathcal{A}=\langle\mu^{++},\mu^{+},\in,f\rangle$. We can choose a $\mathcal{B}=\langle X, X\cap\mu^{+},\in,f\restriction X\rangle \prec \mathcal{A}$ such that $|X|=\mu^{+}$, $|X\cap\mu^{+}|=\mu$, and $\mu\subseteq X$. Let $\delta=\sup X\in\mu^{+}$. There are $\alpha_0,...,\alpha_{n-1}\in X$ such that $f(\alpha_0,\delta)=\cdots=f(\alpha_{n-1},\delta)=\eta$ for some $\eta\in\mu\subseteq X$.

Then, the elementarity shows

$$\mathcal{B}\models\forall\xi\in\mu^{+}\exists\zeta\geq\xi(f(\alpha_0,\zeta)\wedge\cdots\wedge f(\alpha_{n-1},\zeta)=\eta).$$

Indeed, for every $\xi\in X$, $\zeta\geq\xi$ can be taken as δ in $\mathcal{A}$. In particular, $H_1=\{\xi<\mu^{+}\mid\forall i<n(f(\alpha_i,\xi)=\eta)\}$ is unbounded in μ^{+}. $f``\{\alpha_0,...,\alpha_{n-1}\}\times H_1=\{\eta\}$.

Next, we assume the existence of pre-saturated ideal I. Let G be a $(V,\mathcal{P}(\mu^{+})/I)$-generic filter and $j:V\to M\subseteq V[G]$ be the generic ultrapower mapping. Note that $|j``\mu^{++}|=(\mu^{+})^{V[G]}$ and $\mathrm{crit}(j)=(\mu^{+})^{V}$. Then, in $V[G]$, there are $\alpha_0,...,\alpha_{n-1}\in\mu^{++}$ and η such that $M\models j(f)(j(\alpha_0),(\mu^{+})^{V})=\cdots=j(f)(j(\alpha_{n-1}),(\mu^{+})^{V})=\eta=j(\eta)$.

Again, the elementarity shows that $H_1=\{\xi<\mu^{+}\mid\forall i<n(f(\alpha_i,\xi)=\eta)\}$ is unbounded in μ^{+} and $f``\{\alpha_0,...,\alpha_{n-1}\}\times H_1=\{\eta\}$. □

In the same proof, we have

Proposition 6.4. *For each* $\nu\leq\mu$, *if* μ *carries a pre-saturated ideal* I *that satisfies the condition of* $\mathcal{P}(\mu^{+})/I\Vdash\forall f:\mu^{++}\to\mathrm{ON}\exists X\in[\mu^{++}]^{\nu}(f\restriction X\in V)$, *then* $\binom{\mu^{++}}{\mu^{+}}\to\binom{\nu}{\mu^{+}}_{\mu}$ *holds.*

Proposition 6.5. *Suppose that* P *has the* λ*-c.c. Then the following are equivalent:*

1. $P\Vdash\forall f:\lambda\to\mathrm{ON}\exists X\in[\lambda]^{\nu}(f\restriction X\in V)$.

2. I *is* (λ,ν,ν)*-saturated.*

Proof. The inverse direction is easy. We only show the forward direction. For each $\{p_\alpha\mid\alpha<\lambda\}\subseteq P$, by the μ^{+}-c.c. of P, $|||\{\alpha\mid p_\alpha\in\dot{G}\}|=\lambda||\neq 0$. By the assumption, we have $X\in[\lambda]^{\mu}$ and $f:X\to\lambda$ such that

$$q\Vdash p_{f(\beta)}\text{ is the }\beta\text{-th element in }\{\alpha\mid p_\alpha\in\dot{G}\}$$

for every $\beta\in X$. Then q is a lower bound of $\{p_{f(\alpha)}\mid\alpha\in X\}$, as desired. □

Therefore the existence of a (μ^{++},ν,ν)-saturated ideal implies the polarized partition relation. Note that we can show without using the generic ultrapower as follows.

Lemma 6.6. *If* μ^{+} *carries* (μ^{++},ν,ν)*-saturated ideal for some* $\nu\leq\mu$ *then* $\binom{\mu^{++}}{\mu^{+}}\to\binom{\nu}{\mu^{+}}_{\mu}$ *holds.*

Proof. Let I be a (μ^{++},ν,ν)-saturated ideal.

Let $f:\mu^{++}\times\mu^{+}\to\mu$ be an arbitrary coloring. For each $\alpha<\mu^{++}$, there is an η_α such that $A_\alpha=\{\xi<\mu^{+}\mid f(\alpha,\xi)=\eta_\alpha\}\in F^{+}$. By the (μ^{++},ν,ν)-saturation and μ^{+}-completeness of I, there are $H_0\in[\mu^{++}]^{\nu}$ and η such that $H_1=\bigcap_{\alpha\in H_0}A_\alpha\in F^{+}$ and $\forall\alpha\in H_0(\eta_\alpha=\eta)$. Then $f``H_0\times H_1=\{\eta\}$. □

The existence of (μ^{++},ν,ν)-saturated ideals over μ^{+} is preserved by any $(\mu^{+},<\nu^{+})$-centered poset as we saw in Corollary 4.2. Therefore, under the existence of this ideal, $\binom{\mu^{++}}{\mu^{+}}\to\binom{\nu}{\mu^{+}}_{\mu}$ is also preserved by $(\mu^{+},<\nu^{+})$-centered posets. We can omit the ideal assumption from this fact.

Let us study the preservation of polarized partitions, without using saturated ideals, via Prikry forcing.

Theorem 6.7. *Prikry forcing preserves the following:*

1. $\binom{\mu^{++}}{\mu^+} \to \binom{n}{\mu^+}_\mu$ *for each* $n < \omega$.

2. $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ *for each regular* $\nu < \mu$.

3. $\binom{\mu^{++}}{\mu^+} \not\to \binom{n}{\mu^+}_\mu$ *for each* $n < \omega$.

4. $\binom{\mu^{++}}{\mu^+} \not\to \binom{\nu}{\mu^+}_\mu$ *for each regular* $\nu < \mu$.

5. $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu^+}{\mu^+}_\mu$.

Theorem 6.7 follows from Lemmas 6.8, 6.9, 6.10, and 6.11. Let us begin to prove these.

Lemma 6.8. *If* $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ *holds for some cardinal* $\nu < \mu$ *then any* $(\mu, < \nu^+)$*-centered poset forces the same partition relation.*

Proof. Let P be a $(\mu, < \nu^+)$-centered poset and $\langle P_\alpha \mid \alpha < \mu \rangle$ be a $(\mu, < \nu^+)$-centering family of P. Consider $p \Vdash \dot{f} : \mu^{++} \times \mu^+ \to \mu$. For each $\alpha \in \mu^{++}$ and $\xi < \mu^+$, we can choose $p_{\alpha\xi} \leq p$ and $\eta_{\alpha\xi}$ such that $p_{\alpha\xi} \Vdash \dot{f}(\alpha, \xi) = \eta_{\alpha\xi}$. We fix $\beta_{\alpha\xi} < \mu$ with $p_{\alpha\xi} \in P_{\beta_{\alpha\xi}}$

Define $d(\alpha, \xi) = \langle \beta_{\alpha\xi}, \eta_{\alpha\xi} \rangle$. By $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$, there are $H_0 \in [\mu^{++}]^\nu$ and $H_1 \in [\mu^+]^{\mu^+}$ such that d is monochromatic on $H_0 \times H_1$ with value $\langle \beta, \eta \rangle$.

For each $\xi \in H_1$, q_ξ be a lower bound of $\{p_{\alpha\xi} \mid \alpha \in H_0\}$. By the μ^+-c.c. of P, there is a $q \leq p$ which forces that $|\{\xi \in H_1 \mid q_\xi \in \dot{G}\}| = \mu^+$. Let $\dot{H}_1$ be a P-name for such set. We have $q \Vdash \dot{f}``H_0 \times \dot{H}_1 = \{\eta\}$. □

Lemma 6.9. *If* $\binom{\mu^{++}}{\mu^+} \not\to \binom{n}{\mu^+}_\mu$ *for some regular* $n < \omega$ *then any* μ^+*-Knaster poset forces the same partition relation.*

Proof. Let f be a coloring that witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{n}{\mu^+}_\mu$. For each $p \Vdash \dot{H}_1 \in [\mu^+]^{\mu^+}$ and $H_0 \in [\mu^{++}]^n$, we want to find $q \leq p$ which forces $|f``H_0 \times H_1| \geq 2$. For each $i < \mu^+$, we can choose $q_i \leq p$ which decides the value of the i-th element of $\dot{H}_1$ as ξ_i. There is a $K \in [\mu^+]^{\mu^+}$ such that $\forall i, j \in K (q_i \cdot q_j \neq 0)$. By the property of f, we can choose $\alpha, \beta \in H_0$ and $i, j \in K$ such that $f(\alpha, \xi_i) \neq f(\beta, \xi_j)$. Thus, $q_i \cdot q_j \Vdash |f``H_0 \times \dot{H}_1| \geq 2$. □

Lemma 6.10. *If* $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu^+}{\mu^+}_\mu$ *holds then any* $(\mu^+, \mu^+, < \omega)$*-c.c. poset forces the same partition relation.*

Proof. Let f be a coloring that witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu^+}{\mu^+}_\mu$. For each $p \Vdash \dot{H}_0 \in [\mu^{++}]^{\mu^+}$ and $\dot{H}_0 \in [\mu^+]^{\mu^+}$, we want to find $q \leq p$ which forces $|f``H_0 \times \dot{H}_1| \geq 2$. For each $i < \mu^+$, we can choose $q_i \leq p$ which decides the value of the i-th element of $\dot{H}_0$ and $\dot{H}_1$ as ξ_i, ζ_i. There is a $K \in [\mu^+]^{\mu^+}$ such that $\forall i, i', j, j' \in K (q_i \cdot q_{i'} \cdot q_j \cdot q_{j'} \neq 0)$. By the property of f, we can choose $i, i', j, j' \in K$ such that $f(\xi_i, \zeta_j) \neq f(\xi_{i'}, \zeta_{j'})$. Thus, $q_i \cdot q_{i'} \cdot q_j \cdot q_{j'} \Vdash |f``\dot{H}_0 \times \dot{H}_1| \geq 2$. □

Lemma 6.11. *Suppose that* U *is a normal ultrafilter over* μ*. If* $\binom{\mu^{++}}{\mu^+} \not\to \binom{\nu}{\mu^+}_\mu$ *for some regular* $\nu < \mu$ *then* $\mathcal{P}_U$ *forces the same partition relation.*

Proof. We divide two cases $\nu = \omega$ and $\nu > \omega$. First, we assume $\nu > \omega$.

Let $f : \mu^{++} \times \mu^+ \to \mu$ be a coloring that witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{\nu}{\mu^+}_\mu$. For each $\langle a, X \rangle \Vdash \dot{H}_0 \in [\mu^{++}]^\nu$ and $\dot{H}_1 \in [\mu^+]^{\mu^+}$, we want to find an extension of $\langle a, X \rangle$ that forces $|f``\dot{H}_0 \times \dot{H}_1| \geq 2$. This $\dot{H}_0$ can be shrinked to be in V by

Claim 6.12. *$\mathcal{P}_U$ forces that if $\mu > \mathrm{ot}(A) = \mathrm{cf}(\nu) = \nu > \omega$ then there is $B \in V$ such that $\mathrm{ot}(A) = \mathrm{ot}(B)$ and $B \subseteq A$ for all $A \subseteq \mathrm{ON}$.*

Proof of Claim. Consider $\langle a, X\rangle \Vdash \dot{A} \subseteq \mathrm{ON}$ and $\mathrm{ot}(\dot{A}) = \nu$, $\nu \in [\omega_1, \mu) \cap \mathrm{Reg}$. For every $i < \nu$, let $\mathcal{A}_i$ be a maximal anti-chain below $\langle a, X\rangle$ such that $\forall \langle b, Y\rangle \in \mathcal{A}_i \exists \xi \in \mathrm{ON}(\langle b, Y\rangle \Vdash$ the i-th element in $\dot{A}$ is $\xi)$. By Lemma 2.40, there is a $Z_i \subseteq X$ and $n_i < \omega$ such that $\mathcal{B}_i = \{\langle b, Y\rangle \in \mathcal{A}_i \mid |b| = n_i\}$ is maximal anti-chain below $\langle a, Z_i\rangle$. There are $I \in [\nu]^\nu$ and n such that $n_i = n$ for all $i \in I$. If $n \leq |a|$, letting $B = \{\xi \mid \langle a, Z_i\rangle \Vdash$the i-th element of $\dot{A}$ is $\xi\}$, it is easy to see that $\langle a, \bigcap_{i\in K} Z_i\rangle \Vdash B \subseteq \dot{A}$ and $\mathrm{ot}(B) = \nu$. If $n > |a|$, Let $x \in [\bigcap_{i\in K} Z_i \setminus (\max a+1)]^{n-|a|}$. If we let $b = a \cup x$ then it is forced that $B = \{\xi \mid \langle b, Z_i\rangle \Vdash$the i-th element of $\dot{A}$ is $\xi\}$ works as a a witness by $\langle b, \bigcap_i Z_i\rangle \leq \langle a, \bigcap_{i\in K} Z_i\rangle$, as desired. □

By the claim, there are $q \leq \langle a, X\rangle$ and H_0 such that $q \Vdash H_0 \subseteq \dot{H}_0$ and $\mathrm{ot}(H_0) = \nu$. For each $i < \mu^+$, we can choose $\langle c_i, Z_i\rangle \leq q$ which forces that the i-th value of $\dot{H}_1$ is ξ_i. Then there are $K \in [\mu^+]^{\mu^+}$ and c such that $c_i = c$ for all $i \in K$. By the property of f, we can choose $\alpha < \beta$ in H_0 and $i < j$ in K such that $f(\alpha, \xi_i) \neq f(\beta, \xi_j)$. Now $\langle c, Z_i \cap Z_j\rangle$ forces $f(\alpha, \xi_i), f(\beta, \xi_j) \in f``\dot{H}_0 \times \dot{H}_1$.

Let us show in the case of $\nu = \omega$. Let f be a coloring that witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{\omega}{\mu^+}_\mu$. Let $\langle a, X\rangle \Vdash \dot{H}_0 \in [\mu^{++}]^\omega$ and $\dot{H}_1 \in [\mu^+]^{\mu^+}$. For each $i < \mu^+$, we can choose $\langle c_i, Z_i\rangle \leq \langle a, X\rangle$ which forces that the i-th value of $\dot{H}_1$ is ξ_i. Again, there are $K \in [\mu^+]^{\mu^+}$ and c such that $c_i = c$ for all $i \in K$. There is a $\langle c, Z\rangle \leq \langle a, X\rangle$ which forces that $|\{i \in K \mid \langle c, Z_i\rangle \in \dot{G}\}| = \mu^+$. We claim that $\langle c, Z\rangle$ forces $|f``\dot{H}_0 \times \dot{H}_1| \geq 2$.

First, we assume there are $\langle b, Y\rangle \leq \langle c, Z\rangle$, $J \in [\omega]^\omega$, and $H = \{\alpha_n \mid n \in J\}$ such that $\langle b, Y\rangle \Vdash H \subseteq \dot{H}_0$. Note that $\{i < \mu^+ \mid b \setminus c \subseteq Z_i\}$ is unbounded since $\langle b, Y\rangle \Vdash \{i < \mu^+ \mid \langle c, Z_i\rangle \in \dot{G}\}$ is unbounded. By the property of f, there are $i, j \in K$ and $n, m \in J$ such that $f(\alpha_n, \xi_i) \neq f(\alpha_m, \xi_j)$ and $b \setminus c \subseteq Z_i \cap Z_j$. $\langle b, Z \cap Z_i \cap Z_j\rangle \Vdash f(\alpha_n, \xi_i), f(\alpha_m, \xi_j) \in f``\dot{H}_0 \times \dot{H}_1$.

Next, we assume there is no such $\langle b, Y\rangle$. Towards showing a contradiction, suppose that there is an extension $\langle b, Y\rangle$ of $\langle c, Z\rangle$ which forces $f``\dot{H}_0 \times \dot{H}_1 = \{\eta\}$ for some η. Let $\dot{\alpha}_n$ be a $\mathcal{P}_U$-name of the n-th element of $\dot{H}_0$. By Lemma 2.40, for each $n < \omega$, there are Y_n and $l(n)$ such that $\{\langle b', Y'\rangle \leq \langle b, Y_n\rangle \mid |b'| = l(n)$ and $\langle b', Y'\rangle$ decides the value of $\dot{\alpha}_n\}$ contains maximal anti-chain $\mathcal{A}_n$. Note that, for each $x \in [Y_n \setminus (\max b + 1)]^{l(n)-n}$, there are $\alpha_n^x < \mu^{++}$ and Y_n^x such that $\langle b \cup x, Y_n^x\rangle \Vdash \dot{\alpha}_n = \alpha_n^x$ and $\mathcal{A}_n = \{\langle b \cup x, Y_n^x\rangle \mid x \in [Y_n \setminus (\max b + 1)]^{n-l(n)}\}$.

Then the nice name defined by $\bigcup_{x\in[Y_n\setminus(\max b+1)]^{n-l(n)}} \{\langle \alpha_n^x, \langle b \cup x, Y_n^x\rangle\rangle\}$ denotes $\dot{\alpha}_n$ below $\langle b, Y_n\rangle$.

Again, we note that $\{i \in K \mid b \setminus c \subseteq Z_i\}$ is unbounded. Let θ be a sufficiently large regular. Let $M \prec \mathcal{H}_\theta$ be an elementary substructure with the following conditions:

- ${}^\omega M \cup \mu \subseteq M$ and $f, \{Z_i, \xi_i \mid i \in K\}, \{\langle \alpha_n^x \mid x \in [Y_n \setminus (\max b + 1)]^{n-l(n)}\rangle \mid n < \omega\}, U \in M$.
- $M \cap \mu^+ = \delta < \mu^+$ and $|M| < \mu^+$.

Note that there is a $\delta^* \geq \delta$ such that $b \setminus c \subseteq Z_{\delta^*}$ and $\delta^* \in K$. Because there is no extension of $\langle c, Z\rangle$ which forces $[\dot{H}_0]^\omega \cap V \neq \emptyset$, there is n such that $\{\alpha_n^x \mid x \in [Y_n \cap Z_{\delta^*} \setminus (\max b + 1)]^{n-l(n)}\}$ is of size μ. Fix $\{x_k \mid k < \omega\} \subseteq [Y_n \cap Z_{\delta^*} \setminus (\max b + 1)]^{l(n)}$ with $\alpha_n^{x_k} \neq \alpha_n^{x_l}$ for $k \neq l$. Because $\langle b \cup x_k, Z_{\delta^*} \cap Y_n\rangle$ is a common extension of $\langle c, Z_{\delta^*}\rangle$ and $\langle b \cup x_k, Y_n \cap Y_n^{x_k}\rangle$, that forces $\langle \alpha_n^{x_k}, \xi_{\delta^*}\rangle \in \dot{H}_0 \times \dot{H}_1$. In particular, $f(\alpha_n^{x_k}, \xi_{\delta^*}) = \eta$.

Since ${}^\omega M \subseteq M$, $H_0' = \{\alpha_n^{x_k} \mid k < \omega\} \in M$. In M, the following holds:

$$\forall i < \mu^+ \exists j > i(f``H_0' \times \{\xi_j\} = \{\eta\}).$$

From this, $H_1' = \{\xi_i \mid f``H \times \{\xi_i\} = \eta\}$ is unbounded in $\mu^+$. Thus, $f``H_0' \times H_1' = \{\eta\}$. This contradicts the choice of f. □

Let $\mathrm{Add}(\mu, \lambda)$ be the set of all partial functions from λ to μ of size $< \mu$. The following are basic property.

Lemma 6.13. *For regular cardinals $\mu < \lambda$,*

1. $\mathrm{Add}(\mu, \lambda)$ *is μ-directed closed*
2. *If $\mu^{<\mu} = \mu$ then $\mathrm{Add}(\mu, \lambda)$ has the μ^+-c.c.*
3. *If $\mu^{<\mu} = \mu$ then $\mathrm{Add}(\mu, \mu^+)$ is $(\mu, <\mu)$-centered.*
4. *For a (possibly singular) cardinal $\kappa < \lambda$, $\mathrm{Add}(\mu, \kappa) \lessdot \mathrm{Add}(\mu, \lambda)$.*

Proof. 1 and 4 are easy. 2 follows from the usual Δ-system argument.

We check 3. It is easy to see that $\mathrm{Add}(\mu, \mu^+) \simeq \prod_{\alpha<\mu^+}^{<\mu} 2^{<\mu}$. $2^{<\mu}$ is of size μ, and thus $(\mu, <\mu)$-centered. By Lemma 2.7, the product is $(\mu, <\mu)$-centered. □

We introduce

Theorem 6.14 (Hajnal–Juhasz)**.** *If $\mathrm{Add}(\mu, \mu^+)$ has the μ^+-c.c. then $\mathrm{Add}(\mu, \mu^+)$ forces $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$.*

Proof. See Section 6.2. □

Corollary 6.15. *Suppose that $\mu^{<\mu} = \mu$ and $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ holds for all $\nu < \mu$. Then $\mathrm{Add}(\mu, \mu^+)$ forces that $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ holds for all $\nu < \mu$ but $\binom{\mu^{++}}{\mu^+} \to \binom{\mu}{\mu^+}_2$ fails.*

Proof. Note that $\mathrm{Add}(\mu, \mu^+)$ is $(\mu, <\mu)$-centered. Lemmas 6.8 and 6.14 show $\mathrm{Add}(\mu, \mu^+)$ forces the desired partition relations. □

This answers [20, Question 1.11]. Note that this question has been solved in [48] and [21]. But our proof is the simplest of them. Indeed,

Corollary 6.16. *Suppose that λ is an ω_1-Erdős cardinal. Then $\mathrm{Coll}(\omega_1, <\lambda) \times \mathrm{Add}(\omega, \omega_1)$ forces $\binom{\aleph_2}{\aleph_1} \to \binom{n}{\aleph_1}_{\aleph_0}$ for all $n < \omega$ and $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}_{\aleph_0}$.*

Proof. It is known that $\mathrm{Coll}(\omega_1, <\lambda)$ forces $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ if λ is ω_1-Erdős. By Lemma 6.3, $\mathrm{Coll}(\omega_1, <\lambda)$ forces $\binom{\aleph_2}{\aleph_1} \to \binom{n}{\aleph_1}_{\aleph_0}$ for all $n < \omega$. By Corollary 6.15, $\mathrm{Coll}(\omega_1, <\lambda) \times \mathrm{Add}(\omega, \omega_1) \simeq \mathrm{Coll}(\omega_1, <\lambda) * \dot{\mathrm{Add}}(\omega, \omega_1)$ forces the desired conditions. □

Using an almost-huge cardinal, we can show that

Theorem 6.17. *Suppose that μ is a regular cardinal below an almost-huge cardinal. Then there is a μ-directed closed poset which forces that $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ for all $\nu < \mu$ and $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_\mu$.*

Proof. Let $j : V \to M$ be a huge embedding with critical point $\kappa > \mu$. By Theorem 5.5, $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\kappa, <j(\kappa))$ forces that μ^+ carries a $(\mu^{++}, \mu^{++}, <\mu)$-saturated ideal and $2^\mu = \mu^+$, and thus, $\binom{\mu^{++}}{\mu^+} \to \binom{\nu}{\mu^+}_\mu$ for all $\nu < \mu$ by Lemma 6.6. By Lemmas 6.8 and 6.14, $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\kappa, <j(\kappa)) * \dot{\mathrm{Add}}(\mu, \kappa)$ forces the required polarized partition relations. □

In the extension $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\kappa, <j(\kappa)) * \dot{\mathrm{Add}}(\mu, \kappa)$, there is a saturated ideal I. I is form of $\overline{J}$ for some $P(\mu, \kappa) * \mathrm{Coll}(\kappa, <j(\kappa))$-name of a saturated ideal in Theorem 5.5. We note that I is $(j(\kappa), j(\kappa), <\mu)$-saturated as follows.

Proposition 6.18. *Suppose $j : V \to M$ is an almost-huge embedding with critical point κ. For regular cardinals $\mu < \kappa \leq \lambda < j(\kappa)$, let $\dot{I}$ be a $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\kappa, <j(\kappa))$-name in Theorem 5.5. Then $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\lambda, <j(\kappa)) * \dot{\mathrm{Add}}(\mu, j(\kappa))$ forces $\overline{\dot{I}}$ is a saturated ideal over $\mathcal{P}_\kappa \lambda$ with the following properties:*

1. $\dot{\bar{I}}$ *is* $(\lambda^+, \lambda^+, <\mu)$*-saturated.*

2. $\dot{\bar{I}}$ *is not* (λ^+, μ, μ)*-saturated.*

3. $\dot{\bar{I}}$ *is layered if and only if* $j(\kappa)$ *is Mahlo in* V*.*

4. $\dot{\bar{I}}$ *is not centered.*

Proof. Let $G * H$ be a $(V, (\mu,\kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)))$-generic filter. Let $I = \dot{I}^{G*H}$. We discuss in $V[G][H]$.

By Lemma 2.20, we have a complete embedding $\tau : \mathrm{Add}(\mu,\kappa) \to \mathcal{P}(\mathcal{P}_\kappa\lambda)/I * \dot{j}(\mathrm{Add}(\mu,\kappa))$ such that $\mathrm{Add}(\mu,\kappa) \Vdash \mathcal{P}(\mathcal{P}_\kappa\lambda)/I * \dot{j}(\mathrm{Add}(\mu,\kappa))/\dot{G} \simeq \mathcal{P}(\mathcal{P}_\kappa\lambda)/\overline{I}$. It is easy to see that $\dot{j}(\mathrm{Add}(\mu,\kappa))$ is forced to be $\dot{Add}(\mu, j(\kappa))$. $\mathrm{Add}(\mu,\kappa)$ forces

$$\mathcal{P}(\mathcal{P}_\kappa\lambda)/I * \dot{j}(\mathrm{Add}(\mu,\kappa))/\dot{G} \simeq \mathcal{P}(\mathcal{P}_\kappa\lambda)/I * \dot{\mathrm{Add}}(\mu, j(\kappa)) \simeq \mathcal{P}(\mathcal{P}_\kappa\lambda)/I \times \mathrm{Add}(\mu, j(\kappa)).$$

It is easy to see that this poset has the saturation properties that correspond each items. □

Lastly,

Theorem 6.19. *Suppose that there is a supercompact cardinal below an almost-huge cardinal. Then there is a poset which forces that*

1. $\aleph_{\omega+1}$ *carries an ideal* I *that is centered but* not *layered, and*

2. I *is* $(\aleph_{\omega+2}, \aleph_n, \aleph_n)$*-saturated for all* $n < \omega$*.*

3. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \to \binom{\aleph_n}{\aleph_{\omega+1}}_{\aleph_\omega}$ *for all* $n < \omega$*, and,*

4. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \not\to \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}}_{\aleph_\omega}$*.*

Proof. Starting a model with a supercompact cardinal μ below a huge cardinal κ. By Theorem 2.10, we may assume that μ is indestructible.

Let us discuss in the extension by $R(\mu,\kappa) * \dot{R}(\kappa, j(\kappa))$. By Theorem 5.15, μ^+ carries a saturated ideal J such that

1. J is $(\mu^+, <\mu)$-centered.

2. J is $(\mu^{++}, \mu^{++}, <\mu)$-saturated.

Note that $\mathrm{Add}(\mu,\kappa) = \mathrm{Add}(\mu,\mu^+)$ is $(\mu, <\mu)$-centered and μ-directed closed. Let K be a $(V[G][H], \mathrm{Add}(\mu,\kappa))$-generic. By Lemmas 4.2 and 6.14, the following holds in $V[G][H][K]$.

1. I is $(\mu^+, <\mu)$-centered, which in turn implies I is $(\mu^{++}, \mu^{++}, <\mu)$-saturated.

2. $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$.

We discuss in $V[G][H][K]$. Note that $R(\mu,\kappa) * \dot{R}(\kappa, j(\kappa)) * \dot{Add}(\mu, j(\kappa))$ is $<\mu$-directed closed. Therefore μ remains supercompact. Let U and $\mathcal{G}$ a normal ultrafilter over μ and a guiding generic of U. By Lemmas 4.6 and 6.10, $\mathcal{P}_{U,\mathcal{G}}$ forces that

1. $\mu = \aleph_\omega$,

2. $\overline{I}$ is an ideal over $\mu^+ = \aleph_{\omega+1}$ that is centered but *not* layered.

3. $\overline{I}$ is $(\aleph_{\omega+2}, \aleph_n, \aleph_n)$-saturated, which in turn implies that $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \to \binom{\aleph_n}{\aleph_{\omega+1}}_{\aleph_\omega}$ for all $n < \omega$.

4. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \not\to \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}}_{\aleph_\omega}$.

Let $\dot{U}$ and $\dot{\mathcal{G}}$ be $R(\mu,\kappa) * \dot{R}(\mu, j(\kappa)) * \dot{\mathrm{Add}}(\mu,\mu^+)$-names for U and $\mathcal{G}$. $R(\mu,\kappa) * \dot{R}(\mu, j(\kappa)) * \dot{\mathrm{Add}}(\mu,\mu^+) * \mathcal{P}_{\dot{U},\dot{\mathcal{G}}}$ is a required poset. □

We give more observations for the preservation of polarized partition relations. As we saw in Theorem 6.3, $\binom{\aleph_2}{\aleph_1} \to \binom{2}{\aleph_1}_{\aleph_0}$ follows from Chang's conjecture $(\aleph_2,\aleph_1) \twoheadrightarrow (\aleph_1,\aleph_0)$. It is known that $(\aleph_2,\aleph_1) \twoheadrightarrow (\aleph_1,\aleph_0)$ is c.c.c. indestructible. On the other hand, as known the result of Jensen, the square principle $\square_{\omega_1}$ implies that there is a c.c.c. poset that adds a Kurepa tree on ω_1. Therefore, by Lemma 6.2, $\binom{\aleph_2}{\aleph_1} \to \binom{2}{\aleph_1}_{\aleph_0}$ can be destroyed by c.c.c. poset. Indeed, $\binom{\aleph_2}{\aleph_1} \to \binom{2}{\aleph_1}_{\aleph_0}$ is compatible with $\square_{\omega_1}$ (For example, [17, Theorem 8.54] and Lemma 6.3 show).

6.2 Hajnal–Juhasz's coloring

In this section, we study Theorem 6.14 and its application to saturated ideals. First, we give a proof of Theorem 6.14.

Proof of Theorem 6.14. Let $P = \mathrm{Add}(\mu,\mu^+)$. Let $\langle A_\alpha \mid \alpha < \mu^{++}\rangle$ be an almost disjoint sets of μ^+. That is, for every $\alpha < \beta < \mu^{++}$, $|A_\alpha \cap A_\beta| < \mu^+$. Let $\dot{g}$ be a P-name for the function $\bigcup \dot{G} : \mu^+ \to 2$. Let $\dot{c}$ be a P-name for a function $\mu^{++} \times \mu^+ \to 2$ by $\Vdash \dot{c}(\alpha,\xi) = \dot{g}(\rho^\alpha_\xi)$. Here, ρ^α_ξ is the ξ-th element in A_α.

We claim that, for every P-name $\dot{H}_0$ for a subset of μ^{++} of size μ and $p \in P$, there is a γ_* such that, for all $\xi \geq \gamma_*$ and $q \leq p$, $q \cdot ||\alpha,\beta \in \dot{H}_0|| \cdot ||\dot{c}(\alpha,\xi) \neq \dot{c}(\beta,\xi)|| \neq 0$ for some $\alpha < \beta$.

Since $\mathrm{Add}(\mu,\mu^+)$ has the μ^+-c.c., there is an $H \in [\mu^{++}]^\mu$ such that $p \Vdash \dot{H}_0 \subseteq H$ and, for all $\alpha \in H$, there is a $q \leq p$ which forces $\alpha \in \dot{H}_0$. For each $\alpha \in H$, there is an anti-chain $\mathcal{A}_\alpha \subseteq \mathrm{Add}(\mu,\mu^+)$ below p such that $\sum \mathcal{A}_\alpha = ||\alpha \in \dot{H}_0|| \cdot p$. Let $\gamma_* = \sup\{\sup\mathrm{dom}(p) \mid p \in \mathcal{A}_\alpha, \alpha \in H\} \cup \{\sup A_\alpha \cap A_\beta \mid \alpha,\beta \in H\} < \mu^+$.

For every $\xi \geq \gamma_*$ and $q \leq p$, we can choose $\alpha < \beta$ such that

- $\rho^\alpha_\xi \neq \rho^\beta_\xi$.
- $\rho^\alpha_\xi, \rho^\beta_\xi \notin \mathrm{supp}(q)$.
- $||\alpha,\beta \in \dot{H}_0|| \cdot q \neq 0$.

By the definition of γ_*, $||\alpha,\beta \in \dot{H}_0|| \cdot (q \cup \{\langle \rho^\alpha_\xi, 0\rangle, \langle \rho^\alpha_\xi, 1\rangle\}) \neq 0$, as desired. □

For a matrix $\mathcal{A} = \langle \rho^\alpha_\xi \mid \alpha < \lambda, \xi < \mu^+\rangle \subseteq \mu^+$ and a function $h : \mu^+ \to 2$, Hajnal–Juhasz's coloring $c^{\mathcal{A},h} : \lambda \times \mu^+ \to 2$ is a function defined by $c^{\mathcal{A},h}(\alpha,\xi) = h(\rho^\alpha_\xi)$.

We use an eventually different sequence instead of an almost disjoint sequence. A κ-eventually distinct family (κ-edf) of length λ is a matrix $\mathcal{A} = \langle \rho^\alpha_\xi \mid \alpha < \lambda, \xi < \kappa\rangle \subseteq \kappa$ with the following conditions:

- $\xi < \zeta$ implies $\rho^\alpha_\xi \neq \rho^\alpha_\xi$.
- $\alpha < \beta < \lambda$ implies that $\rho^\alpha_\xi \neq \rho^\beta_\xi$ for all large ξ.

Note that there is a κ-edf of length κ^+ for all regular κ. Let $\dot{g}$ be an $\mathrm{Add}(\mu,\mu^+)$-name for $\bigcup \dot{G}$. By the proof of Theorem 6.14, we have

Lemma 6.20 (Hajnal–Juhasz)**.** *For all μ^+-edf $\mathcal{A}$ of length μ^{++}, if $\mathrm{Add}(\mu,\mu^+)$ has the μ^+-c.c. then $\mathrm{Add}(\mu,\mu^+) \Vdash c^{\mathcal{A},\dot{g}}$ witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$.*

It is essential that $\mathcal{A}$ was taken in the ground. Indeed,

Proposition 6.21. $\mathrm{Add}(\mu,\mu^+)$ *forces that* $c^{\mathcal{A},\dot{g}}$ *is monochromatic for some* μ^+*-edf* $\mathcal{A}$ *of length* μ^{++}*. Therefore,* $c^{\mathcal{A},\dot{g}}$ *does not work as a witness of* $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$.

Proof. Let $\langle B^\alpha_\xi \mid \xi,\alpha<\mu^+\rangle$ be such that

- $\mathrm{ot}(B^\alpha_\xi)=\mu$ and $\mathrm{ot}(\bigcup_{\xi<\mu^+} B^\alpha_\xi)=\mu^+$.
- $B^\alpha_\xi \cap B^\beta_\zeta = \emptyset$ for every $\langle \xi,\alpha\rangle \neq \langle \zeta,\beta\rangle$.
- $B^\alpha_\xi \subseteq \mu^+$.

Let $\langle f_\alpha \in {}^{\mu^+}\mu^+ \mid \alpha<\mu^{++}\rangle$ be functions such that

- f_α is an injection.
- $\alpha<\beta<\lambda$ implies $f_\alpha(\xi)\neq f_\beta(\xi)$ for all large ξ.

We let $A_\alpha = \bigcup_{\xi<\mu^+} B^{f_\alpha(\xi)}_\xi$. Then $|A_\alpha\cap A_\beta|\le\mu$ for all $\alpha<\beta<\lambda$. It is easy to see that $A_\alpha\cap A_\beta = \bigcup\{B^{f_\alpha(\xi)}_\xi \mid \xi<\mu^+ \wedge f_\alpha(\xi)=f_\beta(\xi)\}$ for each $\alpha\neq\beta$. The right-hand side is bounded.

Let $\dot\rho^\alpha_\xi$ be an $\mathrm{Add}(\mu,\mu^+)$-name such that $\mathrm{Add}(\mu,\mu^+)$ forces $\dot\rho^\alpha_\xi$ is the least $\rho\in B^{f_\alpha(\xi)}_\xi$ such that $\dot g(\rho)=1$. By $\Vdash \{\rho^\alpha_\xi \mid \xi<\mu^+\}\subseteq A_\alpha$, it is forced that $\langle \rho^\alpha_\xi \mid \alpha<\mu^{++},\xi<\mu^+\rangle$ is μ^+-edf of length μ^{++}. Let $\dot{\mathcal{A}}$ be an $\mathrm{Add}(\mu,\mu^+)$-name for this μ^+-edf. By the definition of $\dot\rho^\alpha_\xi$, $\Vdash c^{\dot{\mathcal{A}},\dot g}(\alpha,\xi)=1$ for all $\alpha<\mu^{++}$ and $\xi<\mu^+$. □

On the other hand, it is possible that $\mathcal{A}$ is not in the ground but $c^{\dot{\mathcal{A}},\dot g}$ is forced to be a witness.

Proposition 6.22. $\mathrm{Add}(\mu,\mu^+)$ *forces that there is a* μ^+*-edf* $\mathcal{A}\notin V$ *of length* μ^{++} *such that* $c^{\dot{\mathcal{A}},\dot g}$ *witnesses* $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$.

Proof. First, let $\langle B^\alpha_\xi \mid \xi,\alpha<\mu^+\rangle$ be such that

- $\mathrm{ot}(B^\alpha_\xi)=\mu$ and $\mathrm{ot}(\bigcup_{\xi<\mu^+} B^\alpha_\xi)=\mu^+$.
- $B^\alpha_\xi \cap B^\beta_\zeta = \emptyset$ for every $\langle \xi,\alpha\rangle \neq \langle \zeta,\beta\rangle$.
- $B^\alpha_\xi \subseteq \mu^+\cap\mathrm{Lim}$.

Let f_α and $\dot\rho^\alpha_\xi$ be as in the proof of the previous lemma. That is, $\Vdash \dot\rho^\alpha_\xi$ is the least $\rho\in B^{f_\alpha(\xi)}_\xi$ such that $\dot g(\rho)=1$. Let $\dot{\mathcal{A}}$ be an $\mathrm{Add}(\mu,\mu^+)$-name for $\langle \dot\rho^\alpha_\xi+1 \mid \xi<\mu^+,\alpha<\mu^{++}\rangle$. It is easy to see that $\dot{\mathcal{A}}$ is forced to be μ^+-edf.

Let $\dot c$ be an $\mathrm{Add}(\mu,\mu^+)$-name for $c^{\dot{\mathcal{A}},\dot g}$. We claim that $\Vdash \dot c$ witnesses $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$. It suffices to show that there is a $\gamma<\mu^+$ such that $p\Vdash \forall\xi\ge\gamma\exists\alpha,\beta\in\dot H_0(\dot c(\alpha,\xi)\neq\dot c(\beta,\xi))$. For every $p\Vdash \dot H_0\in[\mu^{++}]^\mu$, there is an $H\in[\mu^{++}]^\mu$ such that $p\Vdash \dot H_0\subseteq H$ and for all $\alpha\in H$, there is a $q\le p$ which forces $\alpha\in\dot H_0$. For each $\alpha\in H$, there is an anti-chain $\mathcal{A}_\alpha\subseteq\mathrm{Add}(\mu,\mu^+)$ below p such that $\sum\mathcal{A}_\alpha = ||\alpha\in\dot H_0||\cdot p$. Let $\gamma_* = \sup\{\sup\mathrm{dom}(p)\mid p\in\mathcal{A}_\alpha,\alpha\in H\}<\mu^+$.

Let $\gamma=\sup_{\alpha,\beta\in H}\{\sup(B^{f_\alpha(\xi)}_\xi\cup B^{f_\beta(\xi)}_\xi)\mid \xi<\Delta_{\alpha\beta}\}\cup\{\Delta_{\alpha\beta}\}\cup\{\gamma_*\}+1$. Here, $\Delta_{\alpha\beta}$ is the least ξ such that $f_\alpha\restriction(\xi+1)\neq f_\beta\restriction(\xi+1)$. $\gamma<\mu^+$. Then, for all $\xi\ge\gamma$ and $q\le p$, we can find a pair $\alpha<\beta$ and $r\le q$ such that $r\Vdash \alpha,\beta\in\dot H_0$ and $\dot c(\alpha,\xi)\neq\dot c(\beta,\xi)$. Since $|\mathrm{dom}(q)|<\mu$, we can find $\alpha<\beta$ in H such that

- $||\alpha\in\dot H_0||\cdot||\beta\in\dot H_0||\cdot q\neq 0$.

- $(B_\xi^{f_\alpha(\xi)} \cup \{\rho+1 \mid \rho \in B_\xi^{f_\alpha(\xi)}\}) \cap \mathrm{dom}(q) = \emptyset$ and $(B_\xi^{f_\beta(\xi)} \cup \{\rho+1 \mid \rho \in B_\xi^{f_\beta(\xi)}\}) \cap \mathrm{dom}(q) = \emptyset$.

Let $\rho_0 = \min B_\xi^{f_\alpha(\xi)}$ and $\rho_1 = \min B_\xi^{f_\beta(\xi)}$. By the choice of ξ, $f_\alpha(\xi) \neq f_\beta(\xi)$ and $\rho_0 \neq \rho_1$ are greater than γ_*. Let $r = q \cup \{\langle \rho_0, 1\rangle, \langle \rho_0+1, 0\rangle\} \cup \{\langle \rho_1, 1\rangle, \langle \rho_1+1, 1\rangle\}$. By $\rho_0, \rho_1 \geq \gamma_*$, $||\alpha \in \dot{H}_0|| \cdot ||\beta \in \dot{H}_0|| \cdot r \neq 0$ and this forces $\dot{c}(\alpha, \xi) = 0 \neq 1 = \dot{c}(\beta, \xi)$.

Thus, there is no $\dot{H}_1$ and $q \leq p$ such that $q \Vdash |\dot{c}``\dot{H}_0 \times \dot{H}_1| \leq 1$ and $\dot{H}_0 \in [\mu^+]^{\mu^+}$, as desired. □

In [17, Corollary 5.38], the following was claimed.

Theorem 6.23 (Woodin). *The existence of a normal, fine, countably complete ideal I over $[\aleph_2]^{\aleph_1}$ with $\mathcal{P}([\aleph_2]^{\aleph_1})/I \simeq \mathcal{B}(\mathrm{Coll}(\omega, <\aleph_1) * \dot{\mathrm{Add}}(\omega, \aleph_2^V))$ implies $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}_2$.*

The proof in [17] used $c^{A,\dot{g}}$ and Theorem 6.14. On the other hand, by Proposition 6.21, the proof is not enough to show Theorem 6.23 since they did not study where A is in.

Here, we improve Theorem 6.23 as Theorem 6.24. The proof is due to Monroe Eskew and improved by Toshimichi Usuba. I am grateful to them. We recall the λ-approximation property. We say that P satisfies the λ-approximation property if, for every P-name $\dot{f}$ of function from λ to some cardinal κ, $P \Vdash \forall \alpha < \lambda (f \upharpoonright \alpha \in V)$ implies $f \in V$. It is easy to see that λ-Knaster poset satisfies the λ-approximation property.

Theorem 6.24. *Suppose that I is a normal, fine, μ^+-complete λ^+-saturated ideal over $[\lambda]^{\mu^+}$ such that $\mathcal{P}([\lambda]^{\mu^+})/I \simeq \mathcal{B}(Q * \mathrm{Add}(\mu, \lambda))$ for some Q. If $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$ then $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$ holds.*

Proof. Let G be an arbitrary $(V, \mathcal{P}([\lambda]^{\mu^+})/I)$-generic filter. By the assumption, $G \simeq G_0 \times G_1$ for some $(V, Q * \mathrm{Add}(\mu^+, \lambda))$-generic $G_0 * G_1$. Let $j : V \to M$ be a generic ultrapower associated with G.

Then,

- ${}^\lambda M \cap V[G] \subseteq M$.
- $g = \bigcup G_1 \in M$.
- $j(\mu^+) = \lambda$.
- $j(\mu^{++}) = (\lambda^+)^M = (\lambda^+)^{V[G]}$.

By ${}^\lambda M \cap V[G] \subseteq M$, $(\mathcal{P}_\lambda \lambda)^V \in M$, $\mathcal{P}(\lambda)^{V[G]} = \mathcal{P}(\lambda)^M \in M$ and $(\mathcal{P}(\lambda))^V \subseteq M$.

We let $X = \{x \in \mathcal{P}(\lambda)^{V[G_0][G_1]} \mid \forall \alpha < \lambda (x \cap \alpha \in (\mathcal{P}_\lambda \lambda)^V)\}$. By $\mathcal{P}(\lambda)^{V[G]}, (\mathcal{P}_\lambda \lambda)^V \in M$, we have $X \in M$. Let us discuss in $V[G_0]$. By $\mu^{<\mu} = \mu$, $\mathrm{Add}(\mu, \lambda) = \mathrm{Add}(\mu, \mu^+)$ is μ-centered as we saw in the proof of Corollary 6.15. Therefore $\mathrm{Add}(\mu, \lambda)$ satisfies λ-approximation property. Therefore $X = \{x \in \mathcal{P}(\lambda)^{V[G_0]} \mid \forall \alpha < \lambda (x \cap \alpha \in (\mathcal{P}_\lambda \lambda)^V)\}$. It is easy to see that $\mathcal{P}(\lambda)^V \subseteq X$. Thus $|X| \geq \lambda^+$. It is easy to see that X defines an almost disjoint sequence $\mathcal{A} \in V[G_0] \cap M$ of length λ^+.

By Lemma 6.20, $c = c^{\mathcal{A},g}$ witnesses $\binom{\lambda^+}{\lambda} \not\to \binom{\mu}{\lambda}_2$. By $c \in M$ and the elementarity of j, $\binom{\mu^{++}}{\mu^+} \not\to \binom{\mu}{\mu^+}_2$ holds in V. □

Corollary 6.25. *Under the CH, the existence of a normal, fine, countably complete ideal I over $[\aleph_2]^{\aleph_1}$ with $\mathcal{P}([\aleph_2]^{\aleph_1})/I \simeq \mathcal{B}(\mathrm{Coll}(\omega, <\aleph_1) * \dot{\mathrm{Add}}(\omega, \aleph_2^V))$ implies $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}_2$.*

Corollary 6.26. *For an inaccessible cardinal λ, the existence of a normal, fine, countably complete ideal I over $[\lambda]^{\aleph_1}$ with $\mathcal{P}([\lambda]^{\aleph_1})/I \simeq \mathcal{B}(\mathrm{Coll}(\omega, <\lambda))$ implies $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}_2$.*

Corollary 6.27. *Under the GCH, the following are inconsistent with each other:*

1. *$\aleph_1$ carries a $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal.*

2. $[\lambda]^{\aleph_1}$ *carries a normal, fine, countably complete ideal* I *over* $[\lambda]^{\aleph_1}$ *with* $\mathcal{P}([\lambda]^{\aleph_1})/I \simeq \mathcal{B}(Q \times \mathrm{Add}(\omega, \lambda))$ *for some* Q *and* λ.

We don't have a model with the ideal in the assumption of Corollary 6.25. On the other hand, the assumption of Theorem 6.24 is consistent for not inaccessible λ. For example, we mentioned it in Theorem 5.9.

Proposition 6.28. *Suppose that* j *is a huge embedding with critical point* κ*, GCH holds, and* $\mu < \kappa < \lambda < j(\kappa)$ *are regular cardinals. Let* $\dot{I}$ *be a* $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa))$*-name for the ideal over* $Z = [j(\kappa)]^{\kappa}$ *in Theorem 5.9. Then it is forced that* $\mathcal{P}([j(\kappa)]^{\kappa})/\dot{I} \simeq \mathcal{B}(Q * \dot{\mathrm{Add}}(\kappa, j(\kappa)))$ *for some* Q.

Proof. By Theorem 5.9, $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)) \Vdash \mathcal{P}(Z)/I \simeq P(\mu, j(\kappa))/\dot{G} * \dot{H}$. Let $G * H$ be a $(V, P(\mu, \kappa) * \dot{\mathrm{Coll}}(\lambda, < j(\kappa)))$-generic. We discuss in $V[G][H]$.

For each $p \in \mathrm{Coll}(\mu, < j(\kappa))$, define $\pi_0(p) \in \mathrm{Add}(\mu, j(\kappa))$ by $\mathrm{dom}(\pi_0(p)) = \mathrm{supp}(p)$ and $\pi_0(p)(\alpha) = 0$ if and only if $p(\alpha, 0)$ is odd ordinal. π_0 is a projection. It is easy to see that $\mathrm{Add}(\mu, j(\kappa)) \Vdash \mathrm{Coll}(\mu, < j(\kappa))/\dot{G}_0 \simeq (\mathrm{Coll}(\mu, < j(\kappa)))^V$. Note that $j(P)/G * H \simeq (\prod_{\alpha<\kappa} \mathrm{Coll}(\alpha, < j(\kappa))/G) \times Q_0$ for some Q_0. Then,

$$\begin{aligned} j(P)/G * H &\simeq (\textstyle\prod_{\alpha<\kappa}^{<\mu} \mathrm{Coll}(\alpha, < j(\kappa))/G) \times Q_0 \\ &\simeq \mathrm{Coll}(\mu, < j(\kappa))/(G \cap \mathrm{Coll}(\mu, < j(\kappa))) \times (\textstyle\prod_{\mu<\alpha<\kappa}^{<\mu} \mathrm{Coll}(\alpha, < j(\kappa))/G) \times Q_0 \\ &\simeq \mathrm{Coll}(\mu, < j(\kappa)) \times (\textstyle\prod_{\mu<\alpha<\kappa}^{<\mu} \mathrm{Coll}(\alpha, < j(\kappa))/G) \times Q_0 \\ &\simeq \mathrm{Add}(\mu, j(\kappa)) \times \mathrm{Coll}(\mu, < j(\kappa)) \times (\textstyle\prod_{\mu<\alpha<\kappa}^{<\mu} \mathrm{Coll}(\alpha, < j(\kappa))/G) \times Q_0 \\ &\simeq \mathrm{Add}(\mu, j(\kappa)) \times j(P)/G * H. \end{aligned}$$

In particular,

$$\mathcal{P}(Z)/I \simeq j(P)/G * H \simeq (j(P)/G * H) \times \mathrm{Add}(\mu, j(\kappa)) \simeq (j(P)/G * H) \times \dot{\mathrm{Add}}(\mu, j(\kappa)),$$

as desired. □

6.3 Reflection principles for the chromatic number of graphs

In this section, we study the relation between centered ideals on and $\mathrm{Tr}_{\mathrm{Chr}}(\lambda, \kappa)$. $\mathrm{Tr}_{\mathrm{Chr}}(\lambda, \kappa)$ is the statement that every graph of size and chromatic number λ has a subgraph of size and chromatic number κ.

Shelah [38] proved that $V = L$ implies the existence of a graph $\mathcal{G}$ of size and chromatic number μ^+ with every subgraph of size $\leq \mu$ has countable chromatic number for every cardinal μ. Therefore, in L, $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \mu^+)$ fails for all $\mu < \lambda$. Foreman and Laver [19] proved the consistency of $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \mu^+)$ for each regular $\mu < \lambda$ by using a huge cardinal. Therefore we are interested in $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \mu^+)$ for singular μ.

In Section 5.3, we obtained a model in which $[\aleph_{\omega+3}]^{\aleph_{\omega+1}}$ carries an $\aleph_{\omega+2}$-centered ideal. In this model, $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+3}, \aleph_{\omega+1})$ holds.

Lemma 6.29. *Suppose that* $[\lambda^+]^{\mu^+}$ *carries a normal, fine,* μ^+*-complete* λ*-centered ideal. Then* $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \mu^+)$ *holds.*

Proof. Let $P = \mathcal{P}([\lambda^+]^{\mu^+})/I$ and G be a (V, P)-generic filter. In $V[G]$, there is an elementary embedding $j : V \to M$ such that

- $\mathrm{crit}(j) = \mu^+$.
- $j(\mu^+) = \lambda^+$ and $j(\mu^{++}) = \lambda^{++}$.
- $j``\lambda^+ \in M$.

Let $\mathcal{G} = \langle \lambda^+, E\rangle \in V$ be a graph of chromatic number λ^+. Since $j``\lambda^+ \in M$, $j(\mathcal{G})$ has a subgraph that is isomorphic with $\mathcal{G}$ in M.

We claim that the chromatic number of $\mathcal{G}$ is $j(\mu^+)$ in $V[G]$. Fix a P-name $\dot{c}$ for a coloring $\mathcal{G} \to \mu$ and $p \in P$. Let $F : P \to \lambda$ be a centering function of P. Define $d : \mathcal{G} \to \mu \times \lambda$ by $d(x) = \langle \xi, \alpha\rangle$ if and only if $\exists q \leq p(F(q) = \alpha \wedge q \Vdash \dot{c}(x) = \xi))$. Since the chromatic number of $\mathcal{G}$ is λ^+, there are $x, y \in \mathcal{G}$ such that $x\ E\ y$ and $d(x) = d(y)$. By $d(x) = d(y)$ and the definition of d, we have a $q \leq p$ which forces that $\dot{c}(x) = \dot{c}(y)$. Thus P forces that the chromatic number of $\mathcal{G}$ is $\lambda^+ = \dot{j}(\mu^+)$.

By the elementarity of j, there is a subgraph $\mathcal{G}$ of size and chromatic number of μ^+. The proof is completed. □

Theorem 6.30. *Suppose that there is a supercompact cardinal below a huge cardinal. Then there is a poset which forces that*

- $[\aleph_{\omega+3}]^{\aleph_{\omega+1}}$ *carries a normal, fine,* $\aleph_{\omega+1}$*-complete* $\aleph_{\omega+2}$*-centered ideal.*
- $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+3}, \aleph_{\omega+1})$.

Proof. By Lemma 6.29, Theorem 5.19 gives a required extension. □

By using Magidor forcing, we obtain

Theorem 6.31. *Suppose that κ is a huge cardinal with target θ, $\mu < \kappa$ is a supercompact cardinal. For regular cardinals $\nu < \mu < \kappa < \lambda < \theta$, there is a poset which forces that*

1. $[\kappa, \lambda] \cap \mathrm{Reg}$ *and* $[\omega, \nu] \cap \mathrm{Reg}$ *are not changed,*
2. $\kappa = \mu^+$, $\lambda^+ = \theta$, $\mathrm{cf}(\mu) = \nu$,
3. $[\theta]^\kappa$ *carries a normal, fine,* κ*-complete* λ*-centered ideal, and*
4. $\mathrm{Tr}_{\mathrm{Chr}}(\theta, \kappa)$.

Proof. By Lemma 6.29, Theorem 5.18 gives a required extension. □

We proved the consistency of $\mathrm{Tr}_{\mathrm{Chr}}(\lambda^+, \mu^+)$ for singular μ and regular $\lambda > \mu^+$. We ask

Question 6.32. *Is* $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\omega+2}, \aleph_{\omega+1})$ *consistent?*

6.4 Chromatic number of the Erdős–Hajnal graph

In the previous section, we ask whether $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$ can holds or not for singular μ. We introduce another sufficient condition, that is related to the chromatic number of some specific graphs, which implies $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$. For $\kappa < \lambda$, $G(\lambda, \kappa)$ is a graph $\langle {}^\lambda\kappa, \perp\rangle$ such that $f \perp g$ iff $|\{\xi < \lambda \mid f(\xi) = g(\xi)\}| < \lambda$. This graph was introduced in [6]. We call $G(\lambda, \kappa)$ an Erdős–Hajnal graph.

Lemma 6.33 (Erdős–Hajnal [6])**.** *If* $\mathrm{Chr}(G(\mu^{++}, \mu)) \leq \mu^+$ *then* $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$ *holds.*

Proof. We show contraposition. Suppose that there is a graph $\langle G, E\rangle$ of size and chromatic number μ^{++} such that every small subgraph of G has the chromatic number $\leq \mu$. We may assume that $G = \mu^{++}$

Then there is a $F : G \to G(\mu^{++}, \mu)$ such that $\alpha\ E\ \beta$ implies $F(\alpha) \perp F(\beta)$. For each $\alpha < \mu^{++}$, by the assumption, $\langle \alpha, E \cap [\alpha]^2\rangle$ has a good coloring $g_\alpha : \alpha \to \mu$. Let $F(\alpha) : \mu^{++} \to \mu$ be defined by

$$F(\alpha)(\beta) = \begin{cases} 0 & \beta \leq \alpha, \\ g_\beta(\alpha) & \alpha < \beta. \end{cases}$$

Suppose $\alpha\ E\ \beta$. Then, for all $\gamma \geq \max\{\alpha,\beta\}+1$, $F(\alpha)(\gamma) = g_\gamma(\alpha) \neq g_\gamma(\beta) = F(\beta)(\gamma)$. Therefore $F(\alpha) \perp F(\beta)$.

Therefore, for any good coloring $h : G(\mu^{++},\mu) \to \kappa$, $\alpha \mapsto h(F_\alpha)$ define a good coloring of G. By $\mathrm{Chr}(G) = \mu^{++}$, $\kappa \geq \mu^{++}$ and thus $\mathrm{Chr}(G(\mu^{++},\mu)) \geq \mu^{++}$, as desired. □

The assumption for $\mathrm{Chr}(G(\mu^{++},\mu)) \geq \mu^{++}$ is very strong in the following sense.

Theorem 6.34 (Erdős–Hajnal [6])**.** *Under the GCH, if* $\mathrm{Chr}(G(\aleph_2,\aleph_0)) \leq \aleph_1$ *then there is no* $\aleph_2$*-Kurepa tree.*

By Theorem 6.2, $\binom{\aleph_3}{\aleph_2} \to \binom{2}{\aleph_2}_{\aleph_1}$ is stronger than the *non*-existence of a $\aleph_2$-Kurepa tree. We improve Theorem 6.34 as follows.

Theorem 6.35. *Under the GCH, if* $\mathrm{Chr}(G(\mu^{++},\mu)) \leq \mu^{+}$ *then* $\binom{\mu^{+++}}{\mu^{++}} \to \binom{2}{\mu^{++}}_{\mu^+}$ *holds.*

Proof. We show the contraposition. Suppose $\binom{\mu^{+++}}{\mu^{++}} \not\to \binom{2}{\mu^{++}}_{\mu^+}$. Let c witnesses. For each $\alpha < \mu^{+++}$ and $\xi < \mu^{++}$, define $c_\alpha(\xi) = c(\alpha,\xi)$. Then $\langle c_\alpha \mid \alpha < \mu^{+++}\rangle$ satisfies the following properties:

- $c_\alpha : \mu^{++} \to \mu^+$.
- For all $\alpha < \beta < \mu^{+++}$, $\{\xi < \mu^{++} \mid c_\alpha(\xi) \neq c_\beta(\xi)\}$ is bounded in μ^{++}.

Let $\langle A(i) \mid i < \mu^+\rangle$ be an almost disjoint family of μ. For $\alpha < \beta < \mu^{+++}$, define $f_{\alpha\beta} : \mu^{++} \to \mu$ by $f_{\alpha\beta}(\xi) = \min(A(c_\alpha(\xi)) \setminus A(c_\beta(\xi)))$. Of course, $f_{\alpha\beta} \in G(\mu^{++},\mu)$. We claim that $\{f_{\alpha\beta} \mid \alpha<\beta<\mu^{+++}\}$ is a subgraph of $G(\mu^{++},\mu)$ with its chromatic number $\geq \mu^{++}$. Let $F : \{f_{\alpha\beta} \mid \alpha<\beta<\mu^{+++}\} \to \mu^+$ be a mapping. Our aim is to find $\alpha<\beta<\gamma$ with $f_{\alpha\beta} \perp f_{\beta\gamma}$ but $F(f_{\alpha\beta}) = F(f_{\alpha\beta})$.

By the GCH and Erdős–Rado's theorem, we have $\mu^{+++} \to (\mu^{++})^2_{\mu^+}$. Applying this to $\{\alpha,\beta\} \mapsto F(f_{\alpha\beta})$, we have $\alpha<\beta<\gamma$ such that $F(f_{\alpha\beta}) = F(f_{\beta\gamma})$. By the property of c_α's, for all sufficiently large $\xi < \mu^{++}$, we have $c_\alpha(\xi) \neq c_\beta(\xi)$ and $c_\beta(\xi) \neq c_\gamma(\xi)$, which in turn imply $f_{\alpha\beta}(\xi) \neq f_{\beta\gamma}(\xi)$. Therefore $f_{\alpha\beta} \perp f_{\beta\gamma}$. Thus, F is not good coloring, as desired. □

The following lemma is a way of evaluating the value of $\mathrm{Chr}(G(\mu^{++},\mu))$.

Lemma 6.36. *For any ultrafilter D over λ,* $\mathrm{Chr}(G(\mu^{++},\mu)) \leq |{}^{\lambda}\kappa/D|$.

Proof. Let $\{[f_\alpha] \mid \alpha < |{}^{\lambda}\kappa/D|\}$ be an enumeration of ${}^{\lambda}\kappa/D$. For every $f : \lambda \to \kappa$, there is an α with $[f] = [f_\alpha]$. This defines a good coloring. □

Komjath [27, Theorem 3] pointed out that $\mathrm{Chr}(G(\aleph_2,\aleph_0)) = \aleph_2$ in Magidor's model [32, Corollary 7]. We give another model by using Theorem 5.9.

Theorem 6.37. *Suppose that j is a huge embedding with critical point κ, GCH holds, and $\mu < \kappa$ is a regular cardinal. Then* $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\kappa^+, < j(\kappa)) * {}^{<\dot{\mu}}\kappa$ *forces that* $j(\kappa) = \mu^{++}$ *and* $\mathrm{Chr}(\dot{G}(\mu^{++},\mu)) = \mu^{++}$.

Proof. Note that $j(\kappa)$ is forced to be μ^{++}. By Theorem 6.35 and Lemma 6.36, it is enough to find an ultrafilter D over $j(\kappa)$ such that $|{}^{Z}\mu/D| \leq j(\kappa)$ and a $j(\kappa)$-Kurepa tree in the final model. The first half of our proof is based on the proof of [32, Theorem 6].

Let V_1 be an extension by $P(\mu,\kappa) * \dot{\mathrm{Coll}}(\kappa^+, < j(\kappa))$. By Theorem 5.9, we have a normal, fine, κ-complete $j(\kappa)$-saturated $j(\kappa)$-dense ideal I. Note that I is weakly normal.

We let $P = {}^{<\mu}\kappa$. It is easy to see that $P \Vdash \overline{I}$ is weakly normal. We claim that P forces that, for any ultrafilter D over Z with $\overline{I}^* \subseteq D$, then $|{}^{Z}\mu/D| = j(\kappa)$. For $\alpha < j(\kappa)$. Let $f_\alpha(z) = \mathrm{ot}(z \cap \alpha)$ for each $z \in Z = [j(\kappa)]^\kappa$. We may assume that the domain of P is κ by $\kappa^{<\mu} = \kappa$.

For a P-name $\dot{f}$ of a functions from Z to μ, let us define P-names $\dot{F}_0(\dot{f})$ and $\dot{F}_1(\dot{f})$ be P-name for a functions such that

- $P \Vdash \dot{F}_0(\dot{f}) : Z \to \kappa$ and $\dot{F}_1(\dot{f}) : Z \to j(\kappa)$.
- $P \Vdash \dot{F}_0(\dot{f})(z)$ is the least $\beta < \kappa$ such that $\beta \in \dot{G}$ and $\exists \xi < \mu(\beta \Vdash \dot{f}(z) = \xi)$.
- $P \Vdash \dot{F}_1(\dot{f})(z)$ is the $\dot{F}_0(\dot{f})(z)$-th element in z.

By the definition, $\dot{F}_1(\dot{f})$ is forced to be a regressive. Since $\overline{I}$ is weakly normal and P has the $j(\kappa)$-c.c., we have $\gamma_{\dot{f}}$ such that $P \Vdash \{z \in Z \mid \dot{F}_1(\dot{f})(z) < \gamma_{\dot{f}}\} \in \overline{I}^*$.

For each $\alpha < \kappa$, we have a bijection $\psi_\alpha : \mu \to \alpha$. Define $\langle g_{\xi,\dot{f}} : Z \to \mu \mid \xi < \mu \rangle$ by $g_{\xi,\dot{f}}(z) = \psi_{\mathrm{ot}(z \cap \gamma_{\dot{f}})}(\xi)$. Let us define $\langle h_{\xi,\dot{f}} : Z \to \mu \mid \xi < \mu \rangle$ by

$$h_{\xi,\dot{f}}(z) = \begin{cases} 0 & \text{if there is no } \xi \text{ such that } g_{\xi,\dot{f}}(z) \Vdash \dot{f}(z) = \xi, \\ \xi + 1 & \text{if } g_{\xi,\dot{f}}(z) \Vdash \dot{f}(z) = \xi. \end{cases}$$

Let $l(\dot{f})$ be $\langle \gamma_{\dot{f}}, \langle [h_{\xi,\dot{f}}]_I \mid \xi < \mu \rangle \rangle$. Then $l(\dot{f}) = l(\dot{g})$ implies $\Vdash [\dot{f}]_{\overline{I}} = [\dot{g}]_{\overline{I}}$. We have

$$\Vdash |{}^Z\mu/\overline{I}| \leq j(\kappa) \times (|{}^Z\mu/I|^\mu)^V.$$

Let us see $|{}^Z\mu/I| = j(\kappa)$. For each $f : Z \to \mu$ and $\xi < \mu$, let $A_{\xi,f} = f^{-1}\{\xi\}$. It is easy to see that $[f]_I = [g]_I$ implies $A_{\xi,f} \simeq_I A_{\xi,g}$ for all $\xi < \mu$. Since I is $j(\kappa)$-saturated and $j(\kappa)$-dense, $|\mathcal{P}(Z)/I| \leq j(\kappa)^{<j(\kappa)} = j(\kappa)$. In particular,

$$|{}^Z\mu/I| \leq |\mathcal{P}(Z)/I|^\mu \leq (j(\kappa))^\mu = j(\kappa),$$

as desired.

Lastly, we check P forces $j(\kappa)$-Kurepa tree. In V, let $T = {}^{<j(\kappa)}j(\kappa)$. Then,

- The height of T is $j(\kappa)$.
- For each $\alpha < j(\kappa)$, $|\mathrm{Lev}_\alpha(T)| < j(\kappa)$.
- T has $2^{j(\kappa)} = j(\kappa)^+$ cofinal branches.

These properties remain true in the extension by $P(\mu, \kappa) * \dot{\mathrm{Coll}}(\kappa^+, < j(\kappa)) * {}^{<\dot{\mu}}\kappa$. This means T is a $j(\kappa)$-Kurepa tree in the final model, as desired. □

Theorem 6.38. $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$ *does not imply* $\mathrm{Chr}(G(\mu^{++}, \mu)) \leq \mu^+$.

Proof. Let j be a huge embedding with critical point κ. Then $R(\mu, \kappa) * \dot{R}(\kappa, j(\kappa))$ forces that $\mathrm{Tr}_{\mathrm{Chr}}(\mu^{++}, \mu^+)$. For the proof, we refer to [41]. On the other hand, it is forced that μ^{++}-Kurepa tree exists by the same proof of Theorem 6.37. □

Note that Foreman [15] constructed a model in which $\aleph_2$ carries an ultrafilter D such that $|{}^{\omega_2}\omega/D| = \aleph_1$. In this model, $\mathrm{Chr}(G(\aleph_2, \aleph_0)) = \aleph_1$.

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