Agent behaviors and optimal designs in double-ended queueing systems

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NGUYEN QUOC HUNG

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Graduate School of Science and Technology

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Abstract

This dissertation deals with a specific class of queueing systems — double-ended queueing systems — which frequently appear in reality. The main purpose is to model behaviors of every agent population involved in the system, and to develop policies from these that optimize the system. Since a double-ended queue often involves more than one agent population, a multi-population game theoretical framework specifically adapted to queueing games with multiple agent types is developed as a general framework. Six queueing game models with different settings are considered in this dissertation, each with its own challenges and special features that cannot be solved by existing frameworks in the literature. The results show that infinitely many Nash equilibria exist in the queueing games with observable system states, but, among those equilibria, there is only one rational outcome. In the system where the states are made invisible to agents, multiple equilibria may exist simultaneously, and the system can settle into any of the equilibria. For several models, explicit performance measures and optimizations are obtained; in the more complex models, such results are illustrated numerically. The findings of this dissertation provide some tools and insights for social planners to model and optimize real-world queueing systems that involve heterogeneous, strategic populations of agents.

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An Englishman, even if he is alone, forms an orderly queue of one.

George Mikes

1 Introduction

Waiting is part of life.

1.1 Optimal designs in queueing systems: A brief introduction to queueing theory

Waiting is part of life. People wait in queues for service, whether they are at a grocery store, a restaurant, the post office, a ticket counter, or anywhere else where the resource capacity cannot serve customers at once. Americans spend, on average, a staggering 118 hours per year waiting in queues[2]. From an economic perspective, these waiting times may trigger a huge opportunity cost for the economy because they take away from the times during which people can do other tasks that produce actual value. Therefore, optimal queueing system designs are of great importance.

Scheduling queueing tasks would not be complicated, were they deterministic: for example, if customer arrivals or service times are identical or can be exactly determined beforehand. However, that is rarely the case in reality. By nature, the two processes in a queueing system — the arrival process and the service process —are often stochastic. Customers usually arrive at random times, and the times that they spend on the service also vary case-by-case. From a management perspective, one may want to estimate the optimal number of servers that minimizes customers' mean waiting time for a better service, or balances between queueing time and operating costs.

Queueing theory captures the randomness in queueing systems and provides tools for deriving the relationship between input parameters and performance measures. From this relationship, planners can design a set of parameters that optimize targeted measures.

In queueing theory, a queueing system is characterized by six main factors that are described with the following notations (also known as the Kendall–Lee Notation[58]):

$$\mathrm{A/B/C/D/E/F},$$

which respectively specify the nature of the arrival process, the nature of the service process, the number of identical servers, the maximum number of customers allowed in the system, the queue discipline, and the size of the population from which customers are drawn. If the buffer capacity is infinite, the queue discipline is first-come-first-served (FCFS) and the population size is infinite, and factors D, E, and F are usually omitted. For example, M/M/1 indicates a queue whose arrival process is a Poisson process ('M' stands for 'Markovian'), whose service times follow an exponential distribution, and that has one server.

It should be noted that A/B/C/D/E/F are only the basic characteristics of a typical queue. In fact, many further variants are distinguished with more disciplines or policies. Some queueing systems involve different types of agents, and hence require more detailed description. An example of this is a preemptive priority queue in which customers are classified as either primary or secondary. In this case, not only there are two arrival processes (one for each customer type), but there also needs to be a set of rules to govern the occupation of servers. Another example is a double-ended queues (to be explained further in Section 1.3.3) in which agents arrive at both ends of the queue, with the agents on one end acting as the servers. In such situations, the number of servers varies during the operation and cannot be described with a fixed number.

This dissertation considers double-ended queues with a first-come-first-served discipline, and infinite populations. The other four characteristics of the queue vary on a case-by-case basis, resulting in the several distinct models addressed. Furthermore, the term "server," while appropriate in normal queueing systems, will be replaced by "matching points" when considering double-ended queues.

1.2 Agent behavior: Strategic queueing

1.2.1 The abstract idea

Most of the early queueing theory studies did not consider agents' strategic behaviors. Arriving customers were, by default, assumed to join and stay in the queue until they were served. Queueing theory thus came under criticism of both theoreticians and practitioners in the mid-19th century, partly because of its shortcomings regarding such a cramped horizon and their impracticality[5]. Over the course of time, a variety of new horizons and new methods were developed to tackle increasingly more complicated problems which are more closely capture the complexity of modern, realistic systems. Strategic queueing is one of those noticeable and interesting subfields that have attracted considerable attention along the evolution of Queueing theory.

Humans are impatient by nature. In stochastic queueing systems, waiting is uncertain, and psychology studies reveal that such uncertainty triggers irritation[35,47]. Balking, reneging and jockeying soon result. Balking is when a customer arrives at a queue but decides not to enter it. Reneging is deciding to leave the queue before being served. Jockeying is when a customer joins one queue, only to switch to another. These three strategic behaviors are observed in queues in reality.

Modeling agents as rational entities, via decision theory, has led to the incorporation of strategic behaviors into queueing problems, since the pioneering study of Naor [38]. This research shows that, when customers arriving at an M/M/1 queue can observe the system state, their strategy is based on some threshold queue length, above which no one is willing to join the queue. Later, Edelson & Hildebrand [13] studied the case in which customers make strategic decisions without knowing the queue length. These two classic models have paved the way for a stream of research in strategic queueing that considers a wide range of model variants, thoroughly recapped in [17,19]. Several works have sought to bridge this research stream to the formal terminology used in game theory: Hassin & Haviv [18], Economou [11] provided formal definitions of payoff functions and agent strategies, representing them in a vector that encodes the joining probabilities of agents at all possible states in observable queues (rather than acknowledging a threshold policy from the beginning). Subsequently, Haviv & Oz [22] attempted to define the set of players in a class of queueing problems.

1.2.2 Levels of information

In decision theory, how agents behave depends on their expected utility of each action. The expected utility of joining a queue depends directly on agents' expected waiting time, which is a function of various system parameters (which will be introduced in Section 1.3.1) and system states.

The level of information reflects how much information on the system state is revealed to agents. In an *observable* setting, agents can fully observe the system state. In *unobservable* settings, the system state is fully hidden from agents, so strategic agents evaluate their expected waiting time from given system parameters which are supposed to be disclosed by the system administrator. In hybrid settings, the *partially observable* models, the system state information is only partially revealed. For example, if the system state that describes the Markov chain modeling the system is multi-dimensional, agents may be informed about only some of those dimensions. The studies on the M/M/1 queue by Naor [38] and Edelson &

Hildebrand[13] are the two seminal papers, respectively, corresponding to the observable and unobservable cases. Multiple partially observable variants of the M/M/1 queue followed. Economou & Kanta[12], Guo & Zipkin[15] studied an M/M/1 system in which the queue is divided into compartments of the same size, and customers are informed about either the compartment number or the compartment position. Simhon et al.[52], Kim & Kim[31] considered a system in which information on the queue length is disclosed based on a threshold policy: customers are informed about the queue length only if the queue length is below a threshold. The research showed that the throughput, in equilibrium, is monotonically increasing in the threshold.

In the current dissertation, we focus on (fully) observable and (fully) unobservable models.

1.3 Parameters, performance measures and queueing regulations

1.3.1 System parameters and performance measures

System parameters prescribe the characteristics of a queueing system (as shown in Section 1.1). Typical system parameters that can be named include:

- Arrival rate: The mean number of arrivals to the queue per unit time.
- Service rate: The mean number of customers being served per server, per unit time.
- Number of servers: The fixed number of servers—who are presumed to be identical—in a queueing system. This parameter reflects the service capacity of the system.
- Buffer capacity: The maximum queue length. This parameter reflects the physical constraint of the system's waiting room/space.
- Matching rate: The mean number of pairs of agents that match with each other per unit time. This parameter can be seen in matching queues or double-ended queueing systems, which will be introduced in Section 1.4.

There are two main system performance measures: mean queue length and the agents' mean waiting time. Often, the probability distributions of the queue length and the waiting time are also of interest. Understanding the stochasticity of those performance measures helps planners in answering practical questions such as: If the restaurant manager wants to ensure that only 1% of patrons will have to wait more than 10 minutes for a meal, how many tables should be set up?

1.3.2 Economic parameters and performance measures

Economic parameters and performance measures focus on the monetary aspect of the system. Typical economic parameters include:

- Service value (also called *reward*): The monetary value that a customer agent receives after completion of the service. For simplicity, this parameter is often set identically for all customers, though some studies have also considered the service value as a random variable[33,37].

- Waiting cost rate: The cost to an agent for staying in the system per unit time. This cost can be also interpreted as an opportunity cost, or the monetary value of one time unit. This parameter is usually assumed identical for all agents in a single population. Balachandran & Schaefer[3],[4] considered a system with different types of customers differentiated by their service values and time cost, and showed that only one type of customer—with the highest ratio between their service value and time cost—join the system in equilibrium.

- Service fee: The cost of the service per customer.

- Toll fee: The additional charge on each agent by the platform managers or the government, for a specific purpose.

- Subsidy: The additional subsidy on each agent by the platform managers or the government, for a specific purpose.

Economic measures include:

- Mean social welfare: The mean total welfare, in monetary units, of all agents in the system per unit time. This is obtained by summing up service value of all agents in the system per unit time, less their average time cost.

- Mean revenues: The mean total revenue from toll fee/service fee collected from customers per unit time.

1.3.3 Queueing regulations

From a social perspective, the ultimate goal of social planners is to optimize social welfare. In other cases, the final goal may differ, for example, it may be to maximize revenue (in case of business platforms). It is common knowledge that multiple self-optimizations of each agent rarely coincide with an overall optimization. Naor [38] proved that, in an observable M/M/1 queue, customer self-optimizations always lead to a longer maximum queue length than is socially desired.

Within the scope of strategic queueing, a queueing system is often seen as a miniature version of a market or society. Queueing agents always try to optimize their own utility. The question, then, is how a queueing system should be regulated so that a targeted performance measure can be optimized even when each agent acts selfishly.

Optimal designs aim to optimize performance measures by tuning, at the outset, for a best set of system parameters. In other words, system parameters are often fixed at the stage of system design and are difficult to change once the system commences operation. Furthermore, when agent behaviors follow decision theory, the most straightforward approach to regulate their strategic behaviors is to adjust economic parameters. Naor [38] proposed imposing a toll fee on each customer joining the queue. This fee is actually a money transfer from customers to the system administrator; thus this amount itself adds neither cost nor benefit to social welfare. However, the toll fee does affect customer behavior. As a toll fee is directly subtracted from the expected customer utility, the customer joining rate would decrease. Naor proved that the social welfare function is discretely unimodal with respect to the fee; therefore, there exists an optimal fee range that maximizes social welfare. This practical policy is then followed by multiple studies in this line of research. Another approach, which applies when the queue includes both supply and demand sides, is to adjust the service price. The mechanism is basically similar: the adjusted price affects the agents' joining behavior. An increase in the price is a money transfer from the demand side to the supply side, and vice versa. Depending on the situation, there may be other policies that do not involve money transfers.

Haviv & Oz[20],[21] thoroughly reviewed and proposed regulation mechanisms in observable and unobservable queues. In this dissertation, we focus mainly on money transfer policies.

1.4 Double-ended queueing systems

Double-ended queueing systems (also called *double-sided* or *two-sided* queueing systems in some other studies) are queueing systems at which agents arrive at both ends of the queue for matching. The topic has attracted increasing attention in industrial research since it applies well to various social and service systems in reality. Some real-world examples of a double-ended queue can be listed as follows.

- *Passenger taxi stand*. This is the archetypal real-world motivation for double-ended queueing systems. Passengers and taxis arrive at opposite ends of the queue for matching. The idea of such a model was initiated by Kendall[29], who studied the probability distribution (mean and variance) of the queue length. Kashyap[26] considered a more general case in which passengers are served in batches of a fixed, size and passenger arrivals follow a k-Erlang distribution. Shi & Lian[49],[50] bridged the strategic queueing framework to a double-ended queueing model by assuming that the passengers are rational. Several subsequent works then added real-world policies: Wang et al.[56] extended the model by incorporating a gated policy. In[25], the same system setting was considered in the context of customer loss aversion. In[57], different levels of information were considered. Diamant & Baron[10] considered a matching queue with two types of customers arriving at each end and differentiated by their priority. All of the aforementioned papers neglect the boarding time (or matching time) of passengers so, as a result, at a certain time, either passengers or taxis are present in the queue. The Markov chains that model those systems are, therefore, represented one-dimensionally by the queue length. Furthermore, in the strategic queueing scenario, taxis are assumed not to be strategic.

- *E-hailing service*: This is an app-based service where customers use a smartphone to book a vehicle or taxi for a ride. Existing platforms include the popular Uber, Grab, Lyft and Didi Chuxing. Instead of queueing up at a physical taxi stand, e-hailing customers (and taxis) place an order and wait in a virtual queue. Like the passenger–taxi queueing systems, an e-hailing service can also be modeled as a double-ended queue with passengers and taxis arriving at each side. Following this direction, Xu et al. [60] investigated the backward bending of the supply curve and suggested a price discrimination policy to avoid it. Jacob & Roet-Green [24] studied passenger strategy and optimal policy to maximize revenue.

- *Organ transplantation*. This context involves two independent streams: a stream of patients arriving for transplantation and a stream of donor organs. This medical application of queueing was first mentioned by Conolly et al. [9], who emphasized the "impatient" characteristic of enqueued agents in medical applications: both the donated organs and the demand for them expire after a certain time. Boxma et al. [6] then modeled organ transplantation as a double-ended queue in which both sides are impatient: the health of patients may become worse while waiting, and organs cannot be preserved for long. Elalouf et al. [14] proposed a policy to allocate live organs to candidates and studied the dilemma of whether to store the organs or not by analyzing performance measures under a double-ended queueing framework. Khademi & Liu [30] studied a multiclass matching system in which the patients' status may change, reclassifying them into a different class.

- *Customer–inventory system*. These systems also involve two arrival processes: arrivals of customers and arrival of (usually perishable) items into the inventory. These arrival streams were first modeled as independent processes in a queueing system by Kaspi & Perry [27],[28]. Various extensions of this model followed. Perry & Stadje [46] derived the stationary distribution of the system and the cost functionals of an inventory system with perishable commodities and impatient demand. Afèche et al. [1] considered an inventory problem with batch arrival and abandonment. Lee et al. [32] studied a system with back orders and customer abandonment, focusing on optimal control of the production rate.

- *Assembly facilities*. The double-ended queueing model can also be applied to assembly facilities producing multiple parts. A product's component parts arrive independently, and are assembled when all the parts are available. Noteworthy papers on this topic include Som et al.[53], Takahashi et al.[54].

Double-ended queues also occur in a variety of other fields including market trading[34], finance, parallel processing, database concurrency control, and communication protocols. In most cases, a double-ended queue represents a matching platform between two populations of agents—a demand side and a supply side—which represents a two-sided market where agents on each side often make strategic decisions to optimize their own utility. Studying agent behaviors enables social planners to design the system optimally, developing policies to maximize desired system performance measures. While economists focus mainly on the supply–demand equilibrium of the market led by the price, it is not only price, but also queueing time (which can be measured in equivalent monetary units), that impacts upon the agents' decision making.

A double-ended queue involves at least two distinct agent populations, and the population motivations can make the queueing game complicated. This class of problems poses several challenges. On the one hand, except for a few special settings, the presence of more than one population of agents usually leads to multiple variables to describe system states of the Markov chain that models the system. On the other hand, in a game setting with multiple strategic agent types, the strategy adopted by an agent depends on the strategies of other agents not only in the same population but also in other populations, and a Nash equilibrium must contain the strategies of every population rather than of just one population as in one-sided strategic queueing problems. (It is already challenging to analyze the equilibria of one-sided strategic queueing problems in which the system states are multi-dimensional.) Tang et al.[55] considered an unobservable queueing system with two heterogeneous customer types, differentiated by their delay sensitivity and service time. In this study, states representing the system's Markov chain can be described with only one variable, and the joining strategies of agents in the unobservable case are also represented with a single variable (that is, the agents' joining probability); therefore, the agents' expected waiting time could be derived explicitly. By determining where the two best-response functions intersect, it was not complicated to obtain the equilibrium. Meanwhile, in the observable case, a number of variables (that is, agents' joining probabilities at all possible states) are unknown at the beginning. Consequently, this class of observable strategic queueing problem often presents a higher level of complexity, especially when the agents' strategy is not guaranteed to be threshold-based.

Another gap in the literature on strategic queueing models in general and agent behaviors in double-ended queues specifically is that most studies exploit variants of the M/M/1 queue in which performance measures can be derived in a closed form. This study aims to fill in this gap by tackling several multi-dimensional problems in which, although the results may not be derived explicitly, rigorous proofs of agent behaviors can still be obtained.

1.5 Contributions and organization of this dissertation

1.5.1 Contributions

Theoretical contributions

The theoretical contributions of the current dissertation are as follows:

- A multi-population game theoretical framework is developed as a general framework to derive equilibria of queueing games in double-ended queueing systems.
- Agents' waiting times in queueing systems, modeled by a multi-dimensional Markov chain, are derived using first-hitting-time analysis. The monotone properties of the waiting time function with respect to each dimension in the system state are proved by induction.

Practical contributions

Industrial and service systems in modern society are becoming more sophisticated in tune with the development of technology. The models considered in this dissertation incorporate more complicated and realistic characteristics of real-world systems. First, in reality, both the supply side and the demand side can be strategic, which makes the queueing game become multi-population. Secondly, the matching process often takes time and is not negligible. These characteristics add more challenges to the analysis.

By successfully modeling such systems with high complexity, optimal designs and optimal policies can be obtained.

1.5.2 Organization

This dissertation contains five chapters, including this introduction. The remaining four are as follows.

Chapter 2

This chapter presents a multi-population, game-theoretical, queueing system framework that acts as a general theoretical base for the analysis of the models in the following chapters.

Chapter 3

This chapter considers three double-ended queueing system models with zero matching times and two strategic populations of agents. In the first model, the buffer capacity of both ends is assumed finite, and all Nash equilibrium patterns are derived. In the second and the third models, one end is assumed infinite, and the system state is made observable and unobservable to agents, respectively. We focus on the rational outcome of the queueing game and derive policies that optimize social welfare. The models in this chapter are based on the two published papers in which the author of this dissertation is the first author[42,43].

Chapter 4

This chapter considers three double-ended queueing system models with nonzero matching times. The first model considers a system in which one side is strategic while the other is not. In the second model, both sides are strategic. The third model involves three populations: two agent types on the demand side and one on the supply side. The models in this chapter are based on the two published papers and one under-review paper in which the author of this dissertation is the first author[40,41].

Chapter 5

This chapter concludes the dissertation and suggests directions for further research.

2
Theoretical frameworks

THIS CHAPTER provides theoretical frameworks.

2.1 First-step analysis

First-step analysis is a technique that conditions on the first step of the Markov chain to solve numerous problems such as computing mean first-passage times and hitting probabilities. Within the scope of this study, the mean waiting time of an enqueued agent located at a specific system state is of particular interest because it is one of the main determinants of agent behavior. In what follows, we recall first-step analysis under a specific framework of a continuous-time Markov chain that models the waiting time of agents in a queueing system.

Consider a tagged enqueued agent currently observing a state $\mathbf{s}$ of a continuous-time Markov chain that models a queueing system. Denote by $\mathbf{s}_{\text{absorb}}$ the absorbing state at which the agent enters the service. Let $W(\mathbf{s})$ denote the mean time taken from state $\mathbf{s}$ to state $\mathbf{s}_{\text{absorb}}$. Equivalently, $W(\mathbf{s})$ is the agent's mean waiting time. Assume that there are K possibilities of the next state the agent can encounter after state $\mathbf{s}$, denoted $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K$, with corresponding transition rates from $\mathbf{s}$ denoted by $\gamma_1, \gamma_2, \ldots, \gamma_K$. The Markov chain is diagrammed in Figure 2.1.1.

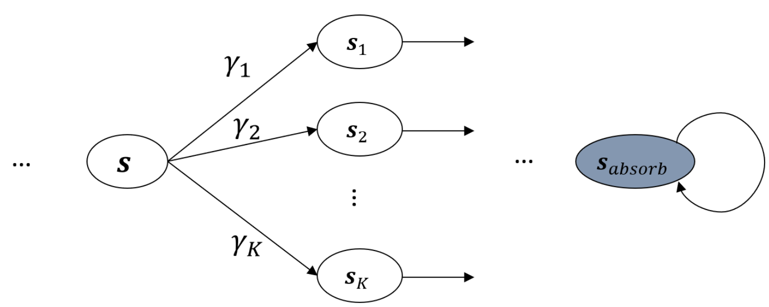


Figure 2.1.1: Diagram of the Markov chain modeling mean waiting times.

The mean waiting time of the tagged agent from state $\mathbf{s}$ is recursively derived as follows:

$$T(\mathbf{s}) = \frac{1}{\sum_{k=1}^{K} \gamma_k} + \sum_{k=1}^{K} \frac{\gamma_k}{\sum_{k=1}^{K} \gamma_k} T(\mathbf{s}_k),$$

where the first term is the mean time for which the agent stays in state $\mathbf{s}$, and the second sum accumulates the mean waiting times from the next encountered state, weighted by the corresponding transition probability.

Definition 2.2.1 (Populations, Strategies, and Profiles). *Consider a society* $\mathbb{P} = \{1, 2, \ldots, M\}$ *that consists of* M populations of agents *arriving at a queueing system. Agents can choose to join or balk the queue upon arrival. Agents receive a reward of* 0 *if they choose to balk. Let* $\mathbb{A} = \{a_1, a_2\}$ *be the set of* pure strategies *of each agent, where* a_1 *represents "joining" and* a_2 *represents "balking" upon arrival. The* strategy *of an arbitrary individual in population* i *is denoted* $\boldsymbol{\sigma}^{(i)}$ *(*$i = 1, 2, \ldots, M$*).*

- *In the observable case,* $\boldsymbol{\sigma}^{(i)}$ *gives probabilities* $\sigma_s^{(i)}$ *(*$s \in \mathbb{S}$*) with which pure strategy* a_1 *is played by that agent at state* s*.*
- *In the unobservable case,* $\boldsymbol{\sigma}^{(i)}$ *gives the probability* $\sigma^{(i)}$ *with which pure strategy* a_1 *is played by that agent.*

The state space is denoted $\mathbb{S}$*. Let a vector* $\mathbf{x}^{(i)} \in [0, 1]^{\mathrm{card}(\mathbb{S})}$ *(*$i = 1, 2, \ldots, M$*) denote the* population profile *of population* i*, which yields the probabilities* $x_s^{(i)}$ *with which the strategy* joining *is played at each state* s *in population* i*.*

A social profile*, defined as* $\mathbf{X} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(M)}\}$*, consists of the strategy profiles of* M *populations.*

Definition 2.2.2 (Limiting Probabilities, Recurrent States, Absorbing States, and Transient States). *(Hassin & Haviv[18])*
Let L_t *be the system state at the arrival time of the* t^{th} *agent. The limiting probability of* $L_t = s$ *when* $t \to +\infty$*, given a social profile* $\mathbf{X}$ *and* $L_0 = s$*, is denoted* $\pi(\mathbf{X}, s)$*. For a given social profile* $\mathbf{X}$*, a state* s *is* recurrent *if* $\pi(\mathbf{X}, s) > 0$*. If* $\pi(\mathbf{X}, s) = 1$*, then the recurrent state* s *is* absorbing*. If* $\pi(\mathbf{X}, s) = 0$*, the state* s *is* transient*.*

Definition 2.2.3 (Payoffs and Best Responses). *The* payoff *to a focal individual in population* i *who adopts a strategy* $\boldsymbol{\sigma}^{(i)}$*, denoted* $U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X})$*, is defined as follows.*

- *In the observable case*

$$U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X}) = \sum_{n \in \mathbb{S}} \left[\sigma_s^{(i)} U_i(a_1|\mathbf{X}, s) + (1 - \sigma_s^{(i)}) U_i(a_2|\mathbf{X}, s) \right], \tag{2.2.1}$$

where $U_i(a_j|\mathbf{X}, s)$ $(j = 1, 2)$ *represents the payoff to a population-i agent who adopts a pure strategy* a_j *at the system state s.*

- *In the unobservable case*

$$U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X}) = \sigma^{(i)} U_i(a_1|\mathbf{X}) + (1 - \sigma^{(i)}) U_i(a_2|\mathbf{X}), \tag{2.2.2}$$

where $U_i(a_j|\mathbf{X})$ $(j = 1, 2)$ *represents the payoff to a population-i agent who adopts pure strategy* a_j*. By definition,* $U_i(a_2|\mathbf{X}) = 0$ *(agents receive a zero payoff if balking), while* $U_i(a_1|\mathbf{X})$ *is determined by subtracting the waiting cost from the service value.*

A strategy $\boldsymbol{\sigma}^{(i)}$ *of a focal individual in population i is called a* best response *against a social profile* $\mathbf{X}$ *if*

$$\boldsymbol{\sigma}^{(i)} \in BR^{(i)}(\mathbf{X}) = \arg\max_{\boldsymbol{\sigma}^{(i)}} U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X}), \tag{2.2.3}$$

where $BR^{(i)}(\mathbf{X})$ *denotes the set of best responses against* $\mathbf{X}$ *of an arbitrary individual in population i.*

Definition 2.2.4 (Nash Equilibria). *A social profile* $\bar{\mathbf{X}} = \left(\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}, \ldots, \bar{\mathbf{x}}^{(M)}\right)$ *is in a* Nash equilibrium *if all agents in each population respond optimally to* $\bar{\mathbf{X}}$*, that is,*

$$NE = \{\bar{\mathbf{X}} : \bar{\boldsymbol{\sigma}}^{(i)} \in BR^{(i)}(\bar{\mathbf{X}}) \textit{ for all } i \in \mathcal{P}\}, \tag{2.2.4}$$

where $\bar{\boldsymbol{\sigma}}^{(i)}$ *is the strategy that generates the population profile* $\bar{\mathbf{x}}^{(i)}$ *(*$\bar{\boldsymbol{\sigma}}^{(i)} = \bar{\mathbf{x}}^{(i)}$*).*

Definition 2.2.5 (Subgame Perfect Nash Equilibria). *A social profile* $\bar{\mathbf{X}}^* = \left(\bar{\mathbf{x}}^{*(1)}, \bar{\mathbf{x}}^{*(2)}, \ldots, \bar{\mathbf{x}}^{*(M)}\right)$ *is in a* subgame perfect Nash equilibrium *(SPNE) if all agents in each population respond optimally to* $\bar{\mathbf{X}}^*$ *at each state in the state space* $\mathcal{S}$*, that is,*

$$SPNE = \{\bar{\mathbf{X}}^* : \bar{\sigma}_s^{*(i)} \in BR^{(i)}(\bar{\mathbf{X}}^*, s) \textit{ for all } i \in \mathcal{P} \textit{ and } s \in \mathcal{S}\}, \tag{2.2.5}$$

where $\bar{\sigma}^{*(i)}$ *is the strategy that generates the population profile* $\bar{\mathbf{x}}^{*(i)}$ *(*$\bar{\sigma}^{*(i)} = \bar{\mathbf{x}}^{*(i)}$*), and* $BR^{(i)}(\mathbf{X}, s)$ *denotes the set of best responses against a population profile* $\mathbf{X}$ *at state s, calculated as*

$$BR^{(i)}(\mathbf{X}, s) = \arg\max_{\sigma_s^{(i)}} U_i(\sigma^{(i)}|\mathbf{X}, s).$$

Remark 2.2.1. *In game theory, a subgame perfect Nash equilibrium can be derived by backward induction: agents make their decisions about a certain system state by reasoning backward from strategies at future states that may be reached from the current state that they observe. The behavior in Definition 2.2.5 ensures that the backward induction is executed since optimizing at* all *possible states in the state space is a stricter condition than optimizing at future states that may be reached from the current state.*

The notation σ refers to a mixed strategy adopted by a tagged individual agent, while $\mathbf{x}$ refers to the proportion of agents who choose the action, "joining". As we are considering only symmetric strategies, i.e., a strategy mutually adopted by the whole population, therefore, the two notations can be used interchangeably.

Definition 2.2.6 (Threshold strategies)**.** *Consider a society in which members of population i join a queueing system, adopting strategy $\boldsymbol{\sigma}^{(i)}$. Assume that the Markov chain modeling a queueing system is prescribed by a D-dimensional vector $\mathbf{s} = (x_1, x_2, \ldots, x_D)$. Consider a possible strategy in which, corresponding to a fixed set $\mathbf{s}_d = (x_1, x_2, \ldots, x_{d-1}, x_{d+1}, \ldots, x_D)$ and a certain nonnegative integer $\nu_d^{(i)}$, $\sigma_{\mathbf{s}}^{(i)} = 1$ for all $x_d \leq \nu_d^{(i)}$ and $\sigma_{\mathbf{s}}^{(i)} = 0$ for all $x_i > \nu_d^{(i)}$. Such strategies are defined as threshold strategies with respect to dimension x_d.*

In the M/M/1 example in the original research by Naor [38], the system state is represented one-dimensionally by the queue length, and it was proved that customers follow a threshold strategy with respect to the queue length (with an assumption that customers who expect a zero payoff join the system).

3

Two-population games in double-ended queues with zero matching times

This chapter deals with modeling double-ended queues in which matching times are assumed zero. This assumption is practically valid in several cases in which the matching times are relatively small and can be dismissed as negligible; for example, the order matching time on rail hailing applications.

The models in this chapter deal with double-ended queueing systems where two populations adopt strategic behaviors. In systems with only one type of strategic agents which are common in literature, agents follow the crowd and form a common strategy adopted by all at the evolutionary endpoint. When both sides are strategic, however, the strategy adopted by any individual on one side depends not only on what those of his own side are doing, but also on the other side, which forms a Nash equilibrium between the two sides. Further optimizations thus become more complicated since the social welfare functions are now multivariate rather than being optimized on a single decision variable.

The following settings apply to the three models in this chapter. Consider a society $\mathbb{P} = \{1, 2\}$ that consists of two populations of agents arriving at a double-ended queueing system based on Poisson processes with rates λ_1 and λ_2. The two populations of agents represent a market with a supply side (population-1) and a demand side (population-2). Matching is performed on a first-come-first-served basis by a pair of a population-1 agent and a population-2 agent in zero unit time. The reward upon the completion of a service and the waiting cost per unit time of a population-i agent are denoted by R_i and C_i ($i = 1, 2$), respectively.

Agents can choose to join or balk the queue upon arrival. Agents receive a reward of 0 if they choose to balk. Let $\mathbb{A} = \{a_1, a_2\}$ be the set of *pure strategies* of each agent, where a_1 represents "joining" and a_2 represents "balking" upon arrival.

3.1 Model 1: Nash equilibria of a queueing game in an observable queueing system with finite ends

In this section, we will derive all possible Nash equilibrium pattern of a system in which the buffer capacity of both ends are finite. This class of system can be seen in most of real-life physical queues in which the waiting area is limited.

This model is based on the following paper: Nguyen, H. Q. & Phung-Duc, T. (2022). A two-population game in observable double-ended queuing systems. Operations Research Letters, 50(4), 407–414[43].

3.1.1 Preliminaries

The buffer capacity of population-i is denoted N_i ($i = 1, 2$). The state space is denoted $\mathbb{S} = \{-N_1, -N_1 + 1, ..., N_2\}$, where a state $s < 0$ prescribes a queue with population-1 agents, while a state $s > 0$ prescribes a queue with population-2 agents, and $s = 0$ prescribes an empty system. The payoff to a focal individual in population i, who adopts a strategy $\boldsymbol{\sigma}^{(i)}$ is given by

$$\begin{aligned} U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X}) &= \sum_{s \in \mathbb{S}} \pi(\mathbf{X}, s) U_i(\boldsymbol{\sigma}^{(i)}|\mathbf{X}, s) \\ &= \sum_{s \in \mathbb{S}} \pi(\mathbf{X}, s) \left[\sigma_s^{(i)} U_i(a_1|\mathbf{X}, s) + (1 - \sigma_s^{(i)}) U_i(a_2|\mathbf{X}, s)\right] \\ &= \sum_{s \in \mathbb{S}} \pi(\mathbf{X}, s) \sigma_s^{(i)} U_i(a_1|\mathbf{X}, s), \end{aligned}$$

where $U_i(a_j|\mathbf{X}, s)$ ($j = 1, 2$) denotes the payoff to the focal individual in the population-i, who adopts the pure strategy a_j upon observing state s. By definition, $U_i(a_2|\mathbf{X}, s) = 0$, while $U_i(a_1|\mathbf{X}, s)$ is obtained by subtracting the waiting cost from the service value.

The index "$\tilde{i}$" is also used to refer to a population other than i. In other words, $\tilde{i} = 2$ if $i = 1$, and vice versa.

Furthermore, let

$$s_i^{(\mathsf{e})} = \max\left\{n : R_i - C_i \frac{n}{\lambda_{\tilde{i}}} \geq 0, n \in \mathbb{N}\right\},$$

and $\mathcal{N}_i = \min\{N_i, s_i^{(\mathsf{e})}\}$. In this paper, $s_i^{(\mathsf{e})}$ is referred to as 'Naor's threshold'.

Remark 3.1.1 (Discussion on Naor's result in the case of finite buffer size)**.** *In addition to considering two strategic populations, the model presented in this paper differs slightly from Naor's in the sense that we assume a finite buffer size for both sides, whereas an infinite buffer size was considered in Naor's model. It should be noted that Naor's result remains the same under a setting with an arbitrarily large finite buffer limit, as long as it is greater than Naor's threshold level. In the observable strategic M/M/1 model, if there is a limit for the buffer size and it is smaller than Naor's threshold, then the threshold adopted by customers is identical to* *that buffer limit.*

3.1.2 Nash equilibria

The Nash equilibria of this game is of interest, defined as $\bar{\mathbf{X}} = \left(\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}\right)$, where $\bar{\mathbf{x}}^{(i)}$ is generated from strategies $\bar{\boldsymbol{\sigma}}^{(i)}$, defined as

$$\bar{\boldsymbol{\sigma}}^{(i)} = \left(\bar{\sigma}^{(i)}_{-N_1}, \bar{\sigma}^{(i)}_{-N_1+1}, \ldots, \bar{\sigma}^{(i)}_{-1}, \bar{\sigma}^{(i)}_{0}, \bar{\sigma}^{(i)}_{1}, \ldots, \bar{\sigma}^{(i)}_{N_2-1}, \bar{\sigma}^{(i)}_{N_2}\right), i = 1, 2.$$

It immediately follows that $\bar{\sigma}^{(1)}_{-N_1} = \bar{\sigma}^{(2)}_{N_2} = 0$.

By defining the equilibria in (2.2.3) and (2.2.4), for $i = 1, 2$, the following condition must be satisfied:

$$U_i\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right) = \max_{\boldsymbol{\sigma}^{(i)}} U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right). \tag{3.1.1}$$

In equilibrium, the system can be modeled as shown in Figure 3.1.1. The dashed arrows indicate that the corresponding transition rates may be equal to 0.

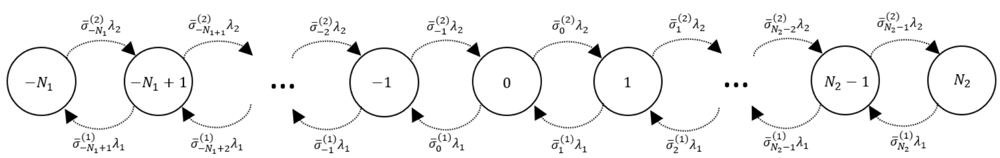


Figure 3.1.1: Transition diagram of the system in equilibrium.

The steady state balance equations are given by

$$\bar{\sigma}^{(2)}_{s}\lambda_2\pi(\bar{\mathbf{X}}, s) = \bar{\sigma}^{(1)}_{s+1}\lambda_1\pi(\bar{\mathbf{X}}, s+1) \tag{3.1.2}$$

for $s = -N_1, -N_1+1, \ldots, N_2-1$.

Lemma 3.1.1. *In equilibrium,*
(1) If there exists a recurrent state $s < 0$, then $\bar{\sigma}^{(2)}_{s} = 1$, $\bar{\sigma}^{(1)}_{s+1} > 0$, and $s+1$ is also a recurrent state.
(2) If there exists a recurrent state $s > 0$, then $\bar{\sigma}^{(1)}_{s} = 1$, $\bar{\sigma}^{(2)}_{s-1} > 0$, and $s-1$ is also a recurrent state.

Proof. *(1)* Assume that $\bar{\sigma}_s^{(2)} < 1$. Then, an arbitrary population-2 agent who adopts a strategy $\boldsymbol{\sigma}^{(2)}$, where $\sigma_j^{(2)} = \bar{\sigma}_j^{(2)}$ for all $j \neq s$, and $\sigma_s^{(2)} = 1$, would find the expected payoff of

$$\begin{aligned}
U_2\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}}\right) &= \sum_{j\neq s} \pi(\bar{\mathbf{X}},j) U_2\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}},j\right) + \pi(\bar{\mathbf{X}},s) U_2\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}},s\right) \\
&= \sum_{j\neq s} \pi(\bar{\mathbf{X}},j) U_2\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}},j\right) + \pi(\bar{\mathbf{X}},s) R_2 \\
&> \sum_{j\neq s} \pi(\bar{\mathbf{X}},j) U_2\left(\boldsymbol{\sigma}^{(2)}|\bar{\mathbf{X}},j\right) + \pi(\bar{\mathbf{X}},s)\bar{\sigma}_s^{(2)} R_2 \text{ (as } \bar{\sigma}_s^{(2)} < 1 \text{ and } \pi(\bar{\mathbf{X}},s) > 0), \\
&= U_2\left(\bar{\boldsymbol{\sigma}}^{(2)}|\bar{\mathbf{X}}\right),
\end{aligned}$$

which contradicts the definition of the best responses and equilibria in (2.2.3) and (2.2.4). This means that $\bar{\sigma}_s^{(2)} = 1$. Because $\pi(\bar{\mathbf{X}},s) > 0$ and $\sigma_s^{(2)} = 1$, the left-hand side of (3.1.2) is positive, implying that $\bar{\sigma}_{s+1}^{(1)} \lambda_1 \pi(\bar{\mathbf{X}}, s+1) > 0$. Thus, $\bar{\sigma}_{s+1}^{(1)} > 0$ and $\pi(\bar{\mathbf{X}}, s+1) > 0$. This also means that $s+1$ is recurrent.
(2) This can be similarly proved.

□

From Lemma 3.1.1, the following important result can be obtained by induction: In equilibrium, if there exists a recurrent state $s < 0$, then all states $s+1, s+2, \ldots, 0$ are also recurrent, and $\sigma_j^{(2)} = 1$ for all $j = s, s+1, \ldots, 0$. Similarly, if there exists a recurrent state $s > 0$ in equilibrium, then all states $s-1, s-2, \ldots, 0$ are also recurrent, and $\sigma_j^{(1)} = 1$ for all $j = s, s-1, \ldots, 0$.

For deriving the joining strategy of a tagged agent, it is necessary to obtain the expected waiting time with respect to the state observed upon arrival, prescribed one-dimensionally by $s \in \mathbb{S}$, which also encodes the number of agents in the same population in front of the tagged agent. This expected waiting time is 0 if there is currently a queue of agents in the opponent population. If a tagged agent arrives and observes a queue of agents in the same population, one more variable that represents the number of agents behind the tagged agent is required to derive the expected waiting time of the tagged agent. This is because the arrival of other agents in the same population behind the tagged agent affects the strategy of agents in the opponent population. According to first-step analysis, the expected waiting time $T_i(u, v)$ of a tagged population-i agent who is at position $u > 0$ (that is, there are $u-1$ other population-

i agents in front of the tagged agent) and observes v other population-i agents behind him ($u + v \leq N_i$) is

$$T_i(u,v) = \begin{cases} \frac{1}{\bar{\sigma}_w^{(i)}\lambda_i + \bar{\sigma}_w^{(\tilde{i})}\lambda_{\tilde{i}}} + \frac{\bar{\sigma}_w^{(i)}\lambda_i}{\bar{\sigma}_w^{(i)}\lambda_i + \bar{\sigma}_w^{(\tilde{i})}\lambda_{\tilde{i}}} T_i(u, v+1) + \frac{\bar{\sigma}_w^{(\tilde{i})}\lambda_{\tilde{i}}}{\bar{\sigma}_w^{(i)}\lambda_i + \bar{\sigma}_w^{(\tilde{i})}\lambda_{\tilde{i}}} T_i(u-1, v) & \text{if } u + v < N_i, \\ \frac{1}{\bar{\sigma}_w^{(\tilde{i})}\lambda_{\tilde{i}}} + T_i(u-1, v) & \text{if } u + v = N_i, \end{cases} \tag{3.1.3}$$

where $T_i(0, v) = 0$ is the bound for the recursion, $w = -(u + v)$ if $i = 1$, and $w = u + v$ if $i = 2$. The joining strategy of a population-i agent upon a state s is based on the expected waiting time upon arrival, that is, $T_i(|s|, 0)$.

By induction, it is easy to determine that $T_i(u, v) \geq \frac{u}{\lambda_{\tilde{i}}}$. Intuitively, $\frac{u}{\lambda_{\tilde{i}}}$ is the expected waiting time of a population-i agent at position u, in an "ideal" scenario that population-$\tilde{i}$ is not strategic (that is, population-$\tilde{i}$ agents always join the system with probability 1). Therefore, when population-$\tilde{i}$ is strategic, a population-i agent should expect a longer waiting time.

In equilibrium, let

$$s_1 = \min\left\{s : -N_1 \leq s \leq 0, \pi\left(\bar{\mathbf{X}}, s\right) > 0\right\},$$

and

$$s_2 = \max\left\{s : 0 \leq s \leq N_2, \pi\left(\bar{\mathbf{X}}, s\right) > 0\right\}.$$

Consider the following 4 cases.

Case 1: $s_1 < 0$ and $s_2 > 0$. Induced from Lemma 3.1.1, the following results can be obtained:

- $\bar{\sigma}_s^{(2)} = 1$ and $\bar{\sigma}_{s+1}^{(1)} > 0$ for all $s_1 \leq s < 0$, and
- $\bar{\sigma}_s^{(1)} = 1$ and $\bar{\sigma}_{s-1}^{(2)} > 0$ for all $0 < s \leq s_2$, and
- any state s satisfying $s_1 \leq s \leq s_2$ is recurrent.

From (3.1.2), we have $\bar{\sigma}_{s_1-1}^{(2)}\lambda_2\pi(\bar{\mathbf{X}}, s_1 - 1) = \bar{\sigma}_{s_1}^{(1)}\lambda_1\pi(\bar{\mathbf{X}}, s_1)$, which implies that $\bar{\sigma}_{s_1}^{(1)} = 0$ (as $\pi(\bar{\mathbf{X}}, s_1) > 0$ and $\pi(\bar{\mathbf{X}}, s_1 - 1) = 0$). Similarly, $\bar{\sigma}_{s_2}^{(2)} = 0$ can be obtained.

For any $s_1 < s \leq 0$, the condition $R_1 - C_1 T_1(|s| + 1, 0) \geq 0$ must be satisfied because if $R_1 - C_1 T_1(|s| + 1, 0) < 0$, then an arbitrary population-1 agent who adopts a strategy $\boldsymbol{\sigma}^{(1)}$, where $\sigma_s^{(1)} = 0$, and $\sigma_j^{(1)} = \bar{\sigma}_j^{(1)}$ for $j \neq s$, would find a payoff of

$$
\begin{aligned}
U_1\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}\right) &= \sum_{j \neq s} \pi(\bar{\mathbf{X}}, j) U_1\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}, j\right) + \pi(\bar{\mathbf{X}}, s) U_1\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}, s\right) \\
&= \sum_{j \neq s} \pi(\bar{\mathbf{X}}, j) U_1\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}, j\right) \\
&> \sum_{j \neq s} \pi(\bar{\mathbf{X}}, j) U_1\left(\boldsymbol{\sigma}^{(1)}|\bar{\mathbf{X}}, j\right) + \pi(\bar{\mathbf{X}}, s) \bar{\sigma}_s^{(1)} (R_1 - C_1 T_1(|s| + 1, 0)) \\
&(\text{as } \bar{\sigma}_s^{(1)} > 0 \text{ and } \pi(\bar{\mathbf{X}}, s) > 0) \\
&= U_1\left(\bar{\boldsymbol{\sigma}}^{(1)}|\bar{\mathbf{X}}\right),
\end{aligned}
$$

which contradicts the definition of "best response" and "equilibria" in (2.2.3) and (2.2.4). Furthermore, if $R_1 - C_1 T_1(|s| + 1, 0) > 0$, then it is easily implied that $\sigma_s^{(1)} = 1$. If $R_1 - C_1 T_1(|s| + 1, 0) = 0$, then $\sigma_s^{(1)} \in (0, 1)$.

Consequently,

$$|s_i| \leq \mathcal{N}_i.$$

Consider the case in which $i = 1$. The above condition is obvious when $N_1 \leq s_1^{(\mathrm{e})}$ (which is equivalent to $\mathcal{N}_1 = N_1$) because s_1 cannot exceed the buffer capacity of the population-1 queue (N_1). When $N_1 > s_1^{(\mathrm{e})}$ (that is, $\mathcal{N}_1 = s_1^{(\mathrm{e})}$), this can be proved by contradiction. Assume that $|s_1| > s_1^{(\mathrm{e})}$, then

$$R_1 - C_1 T_1(s_1^{(\mathrm{e})} + 1, 0) \leq R_1 - C_1 \frac{s_1^{(\mathrm{e})} + 1}{\lambda_2} < 0,$$

which contradicts the definition of $s_1^{(\mathrm{e})}$.

Similarly, for any $s_2 \leq \mathcal{N}_2$ and $0 \leq s < s_2$, the condition $R_2 - C_2 T_2(s + 1, 0) \geq 0$ must be satisfied. If $R_2 - C_2 T_2(s + 1, 0) > 0$, then $\bar{\sigma}_s^{(2)} = 1$. If $R_2 - C_2 T_2(s + 1, 0) = 0$, then $\bar{\sigma}_s^{(2)} \in (0, 1)$.

Furthermore, the condition $R_i - C_i T_i(|s_i| + 1, 0) \leq 0$ ($i = 1, 2$) must also be satisfied. Assuming that $R_i - C_i T_i(|s_i| + 1, 0) > 0$, then an arbitrary population-i agent who adopts a strategy $\boldsymbol{\sigma}^{(i)}$, where $\sigma^{(i)}_{s_i} > 0$ and $\sigma^{(i)}_j = \bar{\sigma}^{(i)}_j$ for $j \neq s_i$, would find a payoff of

$$
\begin{aligned}
U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right) &= \sum_{j\neq s} \pi(\bar{\mathbf{X}},j) U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right) + \pi(\bar{\mathbf{X}},s) U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},s_i\right) \\
&= \sum_{j\neq s_i} \pi(\bar{\mathbf{X}},j) U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right) + \pi(\bar{\mathbf{X}},s_i)\sigma^{(i)}_{s_i}(R_i - C_i T_i(|s_i|+1,0)) \\
&> \sum_{j\neq s_i} \pi(\bar{\mathbf{X}},j) U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}},j\right) \text{ (as } \sigma^{(i)}_{s_i} > 0 \text{ and } \pi(\bar{\mathbf{X}},s_i) > 0), \\
&= U_i\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right),
\end{aligned}
$$

which contradicts the definition of "best response" and "equilibria" in (2.2.3) and (2.2.4). In summary, the system equilibrium in this case is

$$\bar{\mathbf{X}} = \left(\bar{\boldsymbol{\sigma}}^{(1)}, \bar{\boldsymbol{\sigma}}^{(2)}\right),$$

defined with

$$\bar{\boldsymbol{\sigma}}^{(1)} = \left(0, \bar{\sigma}^{(1)}_{-N_1+1}, \ldots, \bar{\sigma}^{(1)}_{s_1-1}, 0, \bar{\sigma}^{(1)}_{s_1+1}, \ldots, \bar{\sigma}^{(1)}_{-1}, \bar{\sigma}^{(1)}_0, 1, 1, \ldots, 1, \bar{\sigma}^{(1)}_{s_2+1}, \ldots, \bar{\sigma}^{(1)}_{N_2-1}, \bar{\sigma}^{(1)}_{N_2}\right),$$

and

$$\bar{\boldsymbol{\sigma}}^{(2)} = \left(\bar{\sigma}^{(i)}_{-N_1}, \bar{\sigma}^{(i)}_{-N_1+1}, \ldots, \bar{\sigma}^{(i)}_{s_1-1}, 1, 1, \ldots, 1, \bar{\sigma}^{(i)}_0, \bar{\sigma}^{(i)}_1, \ldots, \bar{\sigma}^{(i)}_{s_2-1}, 0, \bar{\sigma}^{(i)}_{s_2+1}, \ldots \bar{\sigma}^{(i)}_{N_2-1}, 0\right),$$

where $0 < |s_i| \leq \mathcal{N}_i$, and the unidentified $\bar{\sigma}^{(i)}_s$ take an arbitrary value in $[0,1]$ and satisfy the following conditions.

- $\bar{\sigma}^{(1)}_s > 0$ for all $s_1 < s \leq 0$; and $\bar{\sigma}^{(2)}_s > 0$ for all $0 \leq s < s_2$;
- $T_1(|s|+1,0) \leq \frac{R_1}{C_1}$; if $T_1(|s|+1,0) < \frac{R_1}{C_1}$, then $\sigma^{(1)}_s = 1$; otherwise, $\sigma^{(1)}_s \in (0,1)$, for all $s_1 < s \leq 0$;
  $T_2(|s|+1,0) \leq \frac{R_2}{C_2}$; if $T_2(|s|+1,0) < \frac{R_2}{C_2}$, then $\sigma^{(2)}_s = 1$; otherwise, $\sigma^{(2)}_s \in (0,1)$, for all $0 \leq s < s_2$;

- $T_i(|s_i| + 1, 0) \geq \frac{R_i}{C_i}$;

This pattern of equilibria can be illustrated in Figure 3.1.2.

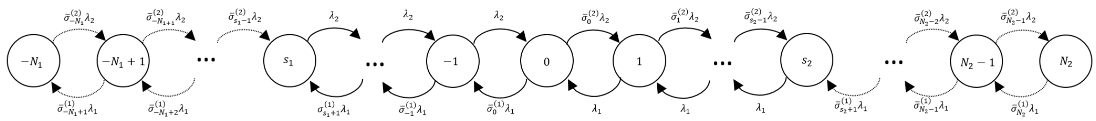


Figure 3.1.2: Equilibrium pattern 1.

Case 2: If $s_1 = s_2 = 0$, then the only existing recurrent state, $s = 0$, becomes the only absorbing state. This implies that $\pi(\bar{\mathbf{X}}, 0) = 1$ and $\pi(\bar{\mathbf{X}}, s) = 0$ for all $s \neq 0$. Now, the solutions of $\bar{\sigma}_0^{(1)}$ and $\bar{\sigma}_0^{(2)}$ must be acquired to obtain the equilibrium strategies of agents in the only recurrent state of the system. From (3.1.2), $\bar{\sigma}_0^{(2)}\lambda_2\pi(\bar{\mathbf{X}}, 0) = \bar{\sigma}_1^{(1)}\lambda_1\pi(\bar{\mathbf{X}}, 1)$ can be obtained, which is equivalent to $\bar{\sigma}_0^{(2)}\lambda_2 = 0$ (because $\pi(\bar{\mathbf{X}}, 1) = 0$), implying that $\bar{\sigma}_0^{(2)} = 0$. Similarly, $\bar{\sigma}_0^{(1)} = 0$ can be obtained. Now, the payoff to a population-i who follows the crowd and adopts strategy $\bar{\boldsymbol{\sigma}}^{(i)}$ is

$$U_i\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}\right) = \sum_s \pi(\bar{\mathbf{X}}, s)U_i\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}, s\right) = \pi(\bar{\mathbf{X}}, 0)U_i\left(\bar{\boldsymbol{\sigma}}^{(i)}|\bar{\mathbf{X}}, 0\right) = 0.$$

Consider an arbitrary population-i who adopts a strategy $\boldsymbol{\sigma}^{(i)}$ where $\sigma_0^{(i)} > 0$. The payoff to this focal population-i agent is

$$U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}\right) = \sum_s \pi(\bar{\mathbf{X}}, s)U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}, s\right) = \pi(\bar{\mathbf{X}}, 0)U_i\left(\boldsymbol{\sigma}^{(i)}|\bar{\mathbf{X}}, 0\right) = \sigma_0^{(i)}\left(R_i - C_iT_i(1, 0)\right).$$

For $\bar{\mathbf{X}}$ to be a state of equilibrium, the condition (3.1.1) is necessary, meaning that the focal population-i agent should find a non-positive payoff. Thus, the set of $\bar{\sigma}_s^{(i)}$ ($s \neq 0$) must satisfy $T_i(1, 0) \geq \frac{R_i}{C_i}$. Otherwise, $R_i - C_iT_i(1, 0) > 0$, then

$$\bar{\sigma}_0^{(i)} \in \arg\max_{\sigma_0^{(i)}} \sigma_0^{(i)}\left(R_i - C_iT_i(1, 0)\right) = \{1\},$$

which contradicts $\bar{\sigma}_0^{(i)} = 0$.

In summary, the system equilibrium in this case is

$$\bar{\mathbf{X}} = \left(\bar{\boldsymbol{\sigma}}^{(1)}, \bar{\boldsymbol{\sigma}}^{(2)}\right),$$

where

$$\bar{\boldsymbol{\sigma}}^{(1)} = \left(0, \bar{\sigma}^{(1)}_{-N_1+1}, \ldots, \bar{\sigma}^{(1)}_{-1}, 0, \bar{\sigma}^{(1)}_{1}, \ldots, \bar{\sigma}^{(1)}_{N_2-1}, \bar{\sigma}^{(1)}_{N_2}\right),$$

and

$$\bar{\boldsymbol{\sigma}}^{(2)} = \left(\bar{\sigma}^{(2)}_{-N_1}, \bar{\sigma}^{(2)}_{-N_1+1}, \ldots, \bar{\sigma}^{(2)}_{-1}, 0, \bar{\sigma}^{(2)}_{1}, \ldots, \bar{\sigma}^{(2)}_{N_2-1}, 0\right),$$

with all $\bar{\sigma}^{(i)}_s$ $(s > 0)$ satisfying $T_2(1,0) \geq \frac{R_2}{C_2}$, and all $\bar{\sigma}^{(i)}_s$ $(s < 0)$ satisfying $T_1(1,0) \geq \frac{R_1}{C_1}$. This equilibrium pattern is illustrated in Figure 3.1.3.

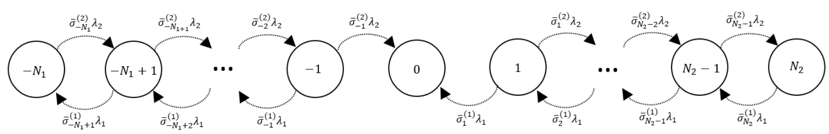


Figure 3.1.3: Equilibrium pattern 2.

Case 3&4: $s_i = 0$ and $s_{\tilde{i}} \neq 0$. These cases can be treated similarly to *Case 1* and *Case 2*. The patterns of equilibria are illustrated in Figure 3.1.4 and Figure 3.1.5. In these equilibrium patterns, there exists only one population of agents in the queue.

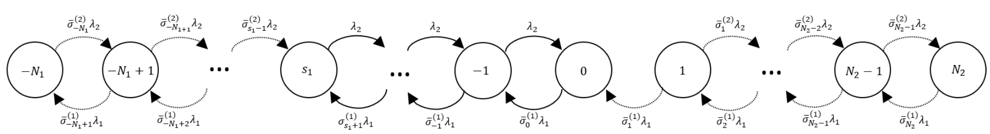


Figure 3.1.4: Equilibrium pattern 3.

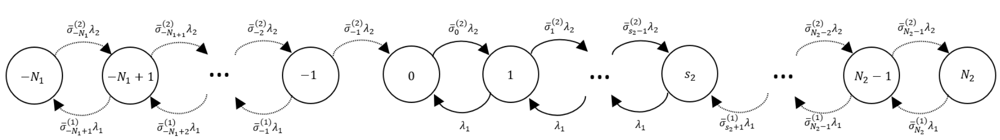


Figure 3.1.5: Equilibrium pattern 4.

In summary, in all four cases, the system may end up at a Nash equilibrium at which the length of each population's buffer does not exceed a certain threshold (prescribed by $|s_i|$, in the case of population-i), and such thresholds do not exceed Naor's threshold. A social profile $\bar{\mathbf{X}} = \left(\bar{\boldsymbol{\sigma}}^{(1)}, \bar{\boldsymbol{\sigma}}^{(2)}\right)$ is in equilibrium if all of the following conditions are satisfied:

- There exist states $s_1 \leq 0$ and $s_2 \geq 0$ such that $\bar{\sigma}^{(i)}_{s_i} = 0, 0 \leq |s_i| \leq \mathcal{N}_i$.
  If $s_1 \neq 0, \bar{\sigma}^{(1)}_s > 0$ for all $s_1 < s \leq 0$. If $s_2 \neq 0, \bar{\sigma}^{(2)}_s > 0$ for all $0 \leq s < s_2$.

- $T_1(|s|+1, 0) \leq \frac{R_1}{C_1}$; if $T_1(|s|+1, 0) < \frac{R_1}{C_1}$, then $\sigma^{(1)}_s = 1$; otherwise, $\sigma^{(1)}_s \in (0,1)$, for all $s_1 < s \leq 0$;
  $T_2(|s|+1, 0) \leq \frac{R_2}{C_2}$; if $T_2(|s|+1, 0) < \frac{R_2}{C_2}$, then $\sigma^{(2)}_s = 1$; otherwise, $\sigma^{(2)}_s \in (0,1)$, for all $0 \leq s < s_2$.

- $T_i(|s_i|+1, 0) \geq \frac{R_i}{C_i}$.

Proposition 3.1.1 (Sensitivity of agents' strategies against buffer capacity). *An arbitrary social profile under the setting of buffer capacity $(N_i, N_{\tilde{i}})$ ($i = 1, 2$), is denoted $\mathbf{X}^{(N_i, N_{\tilde{i}})} = \left(\boldsymbol{\sigma}^{(i),(N_i,N_{\tilde{i}})}, \boldsymbol{\sigma}^{(\tilde{i}),(N_i,N_{\tilde{i}})}\right)$, defined over $[0,1]^{N_1+N_2+1} \times [0,1]^{N_1+N_2+1}$.*

Let $\mathbf{X}^{(N_i+k, N_{\tilde{i}})} = \left(\boldsymbol{\sigma}^{(i),(N_i+k,N_{\tilde{i}})}, \boldsymbol{\sigma}^{(\tilde{i}),(N_i+k,N_{\tilde{i}})}\right)$, defined over $[0,1]^{N_1+N_2+k+1} \times [0,1]^{N_1+N_2+k+1}$, be a social profile with buffer capacity $(N_i+k, N_{\tilde{i}})$, where k is an arbitrary positive integer. All vector elements are indexed by the corresponding system states.

If $\sigma^{(i),(N_i,N_{\tilde{i}})}_s = \sigma^{(i),(N_i+k,N_{\tilde{i}})}_s$ for all $s = -N_1, -N_1+1, ..., N_2$ and $\mathbf{X}^{(N_i,N_{\tilde{i}})}$ is not an equilibrium under the setting $(N_i, N_{\tilde{i}})$ by violating condition (i) when $N^{(\mathrm{e})}_i \leq N_i$, or violating condition (ii), then $\mathbf{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium under the setting $(N_i+k, N_{\tilde{i}})$.

Proof. Consider the following two cases.

- A state s_i satisfying $\sigma^{(i),(N_i,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq \mathcal{N}_i$ does not exist (violation of condition (i)) and $s^{(\mathrm{e})}_i \leq N_i$. As a result, in the setting $(N_i+k, N_{\tilde{i}})$, a state s_i that satisfies $\sigma^{(i),(N_i+k,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq s^{(\mathrm{e})}_i = \min\left\{s^{(\mathrm{e})}_i, N_i+k\right\}$ does not exist, which implies that $\mathbf{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium.

- Two states s_1 and s_2 satisfying $\sigma^{(i),(N_i,N_{\tilde{i}})}_{s_i} = 0$ and $|s_i| \leq \mathcal{N}_i$ exist; however, at least one of the equilibrium conditions concerning the expected waiting times is violated (violation of condition (ii)). It is implied from (3.1.3) that expected waiting times at all states within the these two states do not change when the buffer size increases from N_i to $N_i + k$. Therefore, the corresponding equilibrium conditions for the expected waiting times in the $(N_i + k, N_{\tilde{i}})$ setting are violated, which implies that $\mathbf{X}^{(N_i+k,N_{\tilde{i}})}$ is not an equilibrium state.

$\square$

Remark 3.1.2 (Discussion on the case of infinite buffer size)**.** *Consider a system with an infinite buffer size on both sides of the queue. Similarly to the results in Proposition 3.1.1, a social profile denoted*

$$\mathbf{X} = \left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\right) = \left(\left(\ldots, \sigma^{(1)}_{-1}, \sigma^{(1)}_0, \sigma^{(1)}_1, \ldots\right), \left(\ldots, \sigma^{(2)}_{-1}, \sigma^{(2)}_0, \sigma^{(2)}_1, \ldots\right)\right)$$

is not in equilibrium if:

- *A state s_i satisfying $\sigma^{(i)}_{s_i} = 0$ and $|s_i| \leq s_i^{(\mathrm{e})}$ does not exist (violation of condition (i)), or*
- *Two states s_1 and s_2 satisfying $\sigma^{(i)}_{s_i} = 0$ and $|s_i| \leq s_i^{(\mathrm{e})}$ exist; however, at least one of the equilibrium conditions concerning the expected waiting times is violated (violation of condition (ii)). The recursion in the case of infinite buffer size is slightly different from* (3.1.3) *with regard to the bound. Without a bounding state N_i, the recursion for the calculation of the expected waiting times (under the same notations as* (3.1.3)*) is*

  $$T_i(u,v) = \frac{1}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} + \frac{\sigma^{(i)}_w \lambda_i}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} T_i(u, v+1) + \frac{\sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}}{\sigma^{(i)}_w \lambda_i + \sigma^{(\tilde{i})}_w \lambda_{\tilde{i}}} T_i(u-1, v).$$

  However, if there exists a state s_i satisfying $\sigma^{(i)}_{s_i} = 0$, it is implied that expected waiting times at all states bounded by s_i are calculable and insensitive to the buffer size (s_i becomes the bound of the recursion over all states within this state). Therefore, the result in Proposition 3.1.1 similarly holds when $k \to +\infty$, which is equivalent to the case of an infinite buffer size on both sides.

On another note, whether a state $s < 0$ (if $s > 0$ can be similarly analyzed) is recurrent or transient is directly controlled by population-1 agents' decisions (because the queue length of population-1 agents is the consequence of their decision to join or balk); however, such decisions depend on the strategy of population-2 agents. Because a transient state may never be observed (it occurs with probability 0), the strategy of population-2 agents in transient states does not affect their expected payoff; therefore, they can choose to join at those states with arbitrary probabilities. Meanwhile, the strategy of population-1 agents in those transient states would depend on their beliefs about the strategy of population-2 agents. If population-1 agents hold the belief that population-2 agents join the queue at such transient states $s < 0$ with probabilities smaller than 1, the system may end up at an equilibrium at which the maximal buffer of population-1 agents is smaller than $\mathcal{N}_1$. When population-1 agents believe that population-2 agents optimize their payoff at every state (including transient states) and vice versa, the outcome of the system is a subgame perfect Nash equilibrium derived in the next section.

3.1.3 Subgame perfect Nash equilibrium

In this section, the subgame perfect Nash equilibria of this game is derived, defined as $\bar{\mathbf{X}}^* = \left(\bar{\boldsymbol{\sigma}}^{*(1)}, \bar{\boldsymbol{\sigma}}^{*(2)}\right)$, where

$$\bar{\boldsymbol{\sigma}}^{*(i)} = \left(\bar{\sigma}^{*(i)}_{-N_1}, \bar{\sigma}^{*(i)}_{-N_1+1}, \ldots, \bar{\sigma}^{*(i)}_{-1}, \bar{\sigma}^{*(i)}_{0}, \bar{\sigma}^{*(i)}_{1}, \ldots, \bar{\sigma}^{*(i)}_{N_2-1}, \bar{\sigma}^{*(i)}_{N_2}\right).$$

It immediately follows that $\bar{\sigma}^{*(1)}_{-N_1} = \bar{\sigma}^{*(2)}_{N_2} = 0$.

At any state $s > 0$, we have

$$\bar{\sigma}^{*(1)}_s \in \arg\max_{\bar{\sigma}^{*(1)}_s} U_1\left(\bar{\sigma}^{*(1)}_s | \mathbf{X}^*, s\right) = \arg\max_{\bar{\sigma}^{*(1)}_s} \bar{\sigma}^{*(1)}_s R_1 = \{1\}.$$

In other words, $\bar{\sigma}^{*(1)}_s = 1$ for all $0 < s \leq N_2$.

Similarly, at any state $s < 0$, we have

$$\bar{\sigma}^{*(2)}_s \in \arg\max_{\bar{\sigma}^{*(2)}_s} U_2\left(\bar{\sigma}^{*(2)}_s | \mathbf{X}^*, s\right) = \arg\max_{\bar{\sigma}^{*(2)}_s} \bar{\sigma}^{*(2)}_s R_2 = \{1\}.$$

In other words, $\bar{\sigma}^{*(2)}_s = 1$ for all $-N_1 \leq s < 0$. It then easily follows from (3.1.3) by induction that $T_i(u,0) = \frac{u}{\lambda_{\bar{i}}}$. Therefore, at any state $s \le 0$, we have

$$\bar{\sigma}_s^{*(1)} \in \underset{\bar{\sigma}_s^{*(1)}}{\arg\max}\, U_1\left(\bar{\sigma}_s^{*(1)}|\mathbf{X}^*, s\right) = \underset{\bar{\sigma}_s^{*(1)}}{\arg\max}\left(\bar{\sigma}_s^{*(1)}\left(R_1 - C_1\frac{|s|+1}{\lambda_2}\right)\right)$$

$$= \begin{cases} \{0\} & \text{if } R_1 - C_1\frac{|s|+1}{\lambda_2} < 0, \\ [0,1] & \text{if } R_1 - C_1\frac{|s|+1}{\lambda_2} = 0, \\ \{1\} & \text{if } R_1 - C_1\frac{|s|+1}{\lambda_2} > 0, \end{cases}$$

which is equivalent to

$$\bar{\sigma}_s^{*(1)} = \begin{cases} 0 & \text{if } -N_1 \le s \le -\mathcal{N}_1, \\ p_2 & \text{if } s = -\mathcal{N}_1 + 1, \\ 1 & \text{if } -\mathcal{N}_1 + 2 \le s \le 0, \end{cases}$$

where $p_2 = 1$ if $R_1 - C_1\frac{\mathcal{N}_1}{\lambda_2} > 0$, and p_2 takes any value on $[0,1]$ if $R_1 - C_1\frac{\mathcal{N}_1}{\lambda_2} = 0$. Similarly, at any state $s \ge 0$,

$$\bar{\sigma}_s^{*(2)} = \begin{cases} 0 & \text{if } \mathcal{N}_2 \le s \le N_2, \\ p_1 & \text{if } s = \mathcal{N}_2 - 1, \\ 1 & \text{if } 0 \le s \le \mathcal{N}_2 - 2, \end{cases}$$

where $p_1 = 1$ if $R_2 - C_2\frac{\mathcal{N}_2}{\lambda_1} > 0$, and p_1 takes any value on $[0,1]$ if $R_2 - C_2\frac{\mathcal{N}_2}{\lambda_1} = 0$. This subgame perfect Nash equilibrium can be illustrated in Figure 3.1.6.

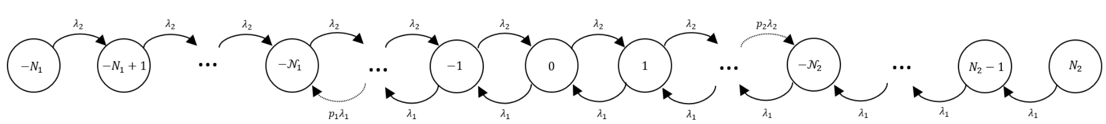


Figure 3.1.6: Subgame perfect Nash equilibrium.

This is a special case of the equilibrium pattern in *Case 1* considered in the previous section, where $s_i = \mathcal{N}_i$ for $i = 1, 2$.

Remark 3.1.3 (Discussion on the case of infinite buffer size). *Under an infinite buffer size setting on both sides of the queue, the subgame perfect Nash equilibrium can still be derived with the same method. If an assumption that agents choose to join when expecting a zero payoff is added, then the conclusion on the subgame perfect Nash equilibrium becomes identical to that in Naor's setting: population-i agents join the queue if they observe a queue of population-i agents with the length of* $s_i^{(\mathsf{e})} - 1$ *or less, and balk otherwise.*

3.2 Model 2: The rational outcome and optimal designs of queueing game in an observable queueing system with one finite end and one infinite end

In this section, we will derive Nash equilibria and optimize social welfare of a system in which the buffer capacity of the supply side is finite while that of the demand side is infinite, which is motivated from the cases in which the supply side takes relatively much more physical space than the demand side does. Real-life examples include passenger-taxi stations at which the parking capacity is limited, or e-commerce platforms on which customers place their order in a virtual queue while suppliers need actual space to store available items. This model is based on the following paper: Nguyen, H. Q., & Phung-Duc, T. (2022). Supply–demand equilibria and multivariate optimization of social welfare in double-ended queueing systems. Computers Industrial Engineering, 170, Article no. 108306[42].

3.2.1 Preliminaries

The buffer capacity of population-(1) is denoted N. The state space is denoted

$$\mathbb{S} = \{-N, -N+1, ..., 0, 1, ...\},$$

where a state $s < 0$ prescribes a queue with population-1 agents, while a state $s > 0$ prescribes a queue with population-2 agents, and $s = 0$ prescribes an empty system. Population-i agents ($i = 1, 2$) arrive according to a Poisson process with arrival rate λ_i. Assume $\lambda_1 > \lambda_2$. Let $\rho = \frac{\lambda_2}{\lambda_1}$.

We decompose the rewards of demanders and suppliers into components as follows,

$$R_2 = R_2^{'} - \mathsf{p},$$

and

$$R_1 = \mathsf{p} - C_f,$$

where p denotes the service fee, R_2' denotes the reward that demanders obtain before subtracting the fee, and C_f represents fixed costs for suppliers.

3.2.2 The rational outcome

We now try to find the strategy profiles of both populations at a SPNE, respectively denoted by

$$\bar{\boldsymbol{\sigma}}^{*(1)} = \left(\bar{\sigma}^{*(1)}_{-N}, \bar{\sigma}^{*(1)}_{-N+1}, \dots, \bar{\sigma}^{*(1)}_{-1}, \bar{\sigma}^{*(1)}_{0}, \bar{\sigma}^{*(1)}_{1}, \dots\right),$$

and

$$\bar{\boldsymbol{\sigma}}^{*(2)} = \left(\bar{\sigma}^{*(2)}_{-N}, \bar{\sigma}^{*(2)}_{-N+1}, \dots, \bar{\sigma}^{*(2)}_{-1}, \bar{\sigma}^{*(2)}_{0}, \bar{\sigma}^{*(2)}_{1}, \dots\right).$$

The social profile is then defined as $\mathbf{X}^* = \left(\bar{\boldsymbol{\sigma}}^{*(1)}, \bar{\boldsymbol{\sigma}}^{*(2)}\right)$.

At any state $s < 0$, we have

$$\bar{\sigma}^{*(2)}_s \in \arg\max_{\sigma^{(2)}_s} U_2\left(\sigma^{(2)}_s | \mathbf{X}^*, s\right) = \arg\max_{\sigma^{(2)}_s} \sigma^{(2)}_s R_2 = \{1\}.$$

Similarly, at any state $s > 0$, we have

$$\bar{\sigma}^{*(1)}_s \in \arg\max_{\sigma^{(1)}_s} U_1\left(\sigma^{(1)}_s | \mathbf{X}^*, s\right) = \arg\max_{\sigma^{(1)}_s} \sigma^{(1)}_s R_1 = \{1\}.$$

In other words, $\bar{\sigma}^{*(2)}_s = 1$ for all $-N \leq s \leq -1$ and $\bar{\sigma}^{*(1)}_s = 1$ for all $s \geq 1$. The expected waiting time for any population-1 agent who enters the system at a state $s > 0$ is then $\frac{|s|}{\lambda_2}$, and the expected waiting time of any population-2 agent who enters the system at a state $s < 0$ is $\frac{|s|}{\lambda_1}$. It then follows that, for any $s \geq 0$,

$$\bar{\sigma}^{*(2)}_s \in \arg\max_{\bar{\sigma}^{*(2)}_s} U_2\left(\bar{\sigma}^{*(2)}_s | \mathbf{X}^*, s\right) = \arg\max_{\bar{\sigma}^{*(2)}_s} \left(\bar{\sigma}^{*(2)}_s \left(R_1 - C_1 \frac{|s|+1}{\lambda_2}\right)\right)$$
$$= \begin{cases} \{0\}, & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} < 0, \\ [0,1], & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} = 0, \\ \{1\}, & \text{if } R_1 - C_1 \frac{|s|+1}{\lambda_2} > 0. \end{cases}$$

This can be rewritten as

$$\bar{\sigma}_s^{*(2)} = \begin{cases} 0, & \text{if } s \geq n_2^{(\mathsf{s})}, \\ p_2, & \text{if } s = n_2^{(\mathsf{s})} - 1, \\ 1, & \text{if } 0 \leq s \leq n_2^{(\mathsf{s})} - 2. \end{cases}$$

Here $p_2 = 1$ if $R_2 - C_2 \frac{n_2^{(\mathsf{s})}}{\lambda_1} > 0$, and p_2 takes any value in $[0,1]$ if $R_2 - C_2 \frac{n_2^{(\mathsf{s})}}{\lambda_1} = 0$. Since the case $R_2 - C_2 \frac{n_2^{(\mathsf{s})}}{\lambda_1} = 0$ rarely happens, we assume that $R_2 - C_2 \frac{n_2^{(\mathsf{s})}}{\lambda_1} > 0$. Then, we can conclude that in the unique subgame perfect equilibrium, population-2 agents adopt a threshold strategy: population-2 agents join the system if they observe a system state $s \leq n_2^{(\mathsf{s})} - 1$ upon their arrival, and balk if they observe a population-2 agent queue length larger than $n_2^{(\mathsf{s})} - 1$. Similarly, with the assumption that $R_1 - C_1 \frac{n_1^{(\mathsf{s})}}{\lambda_2} > 0$, the population-1 agents' strategy can be derived as follows: the population-1 agents join the system if they observe a system state $s \geq -n_1^{(\mathsf{s})} + 1$ upon their arrival, and balk if they observe a queue length larger than $n_1^{(\mathsf{s})} - 1$.

In summary, there is a unique subgame perfect Nash equilibrium of the game, at which both the populations adopt their own threshold strategy as follows.

$$\left(n_1^{(\mathsf{s})}, n_2^{(\mathsf{s})}\right) = \left(\min\left\{N, \left\lfloor \frac{\lambda_2 R_1}{C_1} \right\rfloor\right\}, \left\lfloor \frac{\lambda_1 R_2}{C_2} \right\rfloor\right).$$

The state transition diagram of the system in equilibrium is illustrated in Figure 3.2.1.

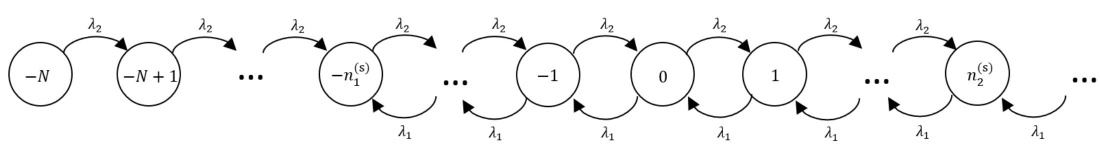


Figure 3.2.1: Transition diagram of the subgame perfect Nash equilibrium.

Remark 3.2.1. *The subgame perfect Nash equilibrium derived above is one among the many possible outcomes of the game. However, we assume that the system ends up at the subgame perfect Nash equilibrium, which is the most rational outcome of the game under* solution concept[8].

The solution concept of subgame perfect Nash equilibrium allows us to eliminate unstable outcomes (often referred to as subgame imperfection*).*

3.2.3 Optimization

Let n_1 and n_2 $(0 \leq n_1 \leq N, 0 \leq n_2)$ respectively denote arbitrary threshold strategies adopted by the population-1 agents and population-2 agents. With $\rho = \frac{\lambda_2}{\lambda_1} < 1$, the state space is given by

$$\mathbb{S} = \{-n_1, -n_1 + 1, ..., -1, 0, 1, ..., n_2\}.$$

Obviously, when either of n_1 or n_2 equals 0, the system would not exist and the social welfare will be 0. Therefore, we consider the case in which $n_1 \geq 1$ and $n_2 \geq 1$.

The steady-state probabilities are

$$\pi_{-n_1} = \frac{1-\rho}{1-\rho^{n_1+n_2+1}},$$

and

$$\pi_s = \rho^{n_1+s}\pi_{-n_1}, \quad s = -n_1 + 1, ..., -1, 0, 1, ..., n_2.$$

According to the PASTA property, the balking probability of population-2 agents (i.e., the probability that a population-2 agent arrives and balks) is π_{n_2}, so the mean number of population-2 agents who actually join the queue is

$$\lambda_2(1 - \pi_{n_2}) = \lambda_2 \frac{1-\rho^{n_1+n_2}}{1-\rho^{n_1+n_2+1}}.$$

The balking probability of population-1 agents (i.e., the probability that a population-1 agent arrives and balks) is π_{-n_1}, so the mean number of population-1 agents which actually join the queue is

$$\lambda_1(1 - \pi_{-n_1}) = \lambda_2 \frac{1-\rho^{n_1+n_2}}{1-\rho^{n_1+n_2+1}},$$

which equals the actual joining rate of population-2 agents. We denote both these joining rates by ξ.

The mean queue lengths of population-1 and population-2 agents are

$$L_1 = \sum_{s=-n_1}^{0} (-s)\pi_0 = \frac{1}{1-\rho^{n_1+n_2+1}} \left(n_1 - \frac{\rho - \rho^{n_1+1}}{1-\rho} \right),$$

and

$$L_2 = \sum_{s=0}^{n_2} s\pi_0 = \frac{\rho^{n_1}}{1-\rho^{n_1+n_2+1}} \left(-n_2\rho^{n_2+1} + \frac{\rho - \rho^{n_2+1}}{1-\rho} \right),$$

respectively.

The mean social welfare is then given by

$$\begin{aligned}
SW(n_1, n_2) &= \xi(R_1 + R_2) - C_1L_1 - C_2L_2 \\
&= \frac{1}{1-\rho^{n_1+n_2+1}} \Bigg[\lambda_2(1-\rho^{n_1+n_2})(R_1+R_2) \\
&\quad - C_2\rho^{n_1} \left(-n_2\rho^{n_2+1} + \frac{\rho-\rho^{n_2+1}}{1-\rho} \right) - C_1 \left(n_1 - \frac{\rho - \rho^{n_1+1}}{1-\rho} \right) \Bigg].
\end{aligned}$$

Now we need to maximize the social welfare function $SW(n_1, n_2)$ of two integer variables n_1 and n_2, under the constraints $1 \leq n_1 \leq N$ and $n_2 \geq 1$.

For any fixed value of n_1, consider the difference between social welfare when the queue length threshold of population-2 agents is set at n_2 and $n_2 + 1$ as follows.

$$\begin{aligned}
SW(n_1, n_2) - SW(n_1, n_2+1) &= \frac{\rho^{n_1+n_2+1}}{(1-\rho^{n_1+n_2+1})(1-\rho^{n_1+n_2+2})} [-\lambda_2(R_1+R_2)(1-\rho)^2 \\
&\quad + C_2\rho \left((n_2+1)(1-\rho) - \rho^{n_1+1} \left(1-\rho^{n_2+1}\right) \right) \\
&\quad - C_1\rho \left(n_1 - n_1\rho - \rho + \rho^{n_1+1} \right)].
\end{aligned}$$

Consider the following continuous function defined over $[1, +\infty)$

$$\begin{aligned}
u(\nu) &= -\lambda_2(R_1+R_2)(1-\rho)^2 + C_2\rho \left((\nu+1)(1-\rho) - \rho^{n_1+1} \left(1-\rho^{\nu+1}\right) \right) \\
&\quad - C_1\rho \left(n_1 - n_1\rho - \rho + \rho^{n_1+1} \right) \\
&= C_2\rho^{n_1+2}\rho^{\nu+1} + C_2\rho(1-\rho)\nu + H,
\end{aligned}$$

where $H = -\lambda_2(R_1+R_2)(1-\rho)^2 + C_2\rho\left(1-\rho-\rho^{n_1+1}\right) - C_1\rho\left(n_1 - n_1\rho - \rho + \rho^{n_1+1}\right)$.

We obtain the monotonic property of the function $u(\nu)$ as below.

Lemma 3.2.1. *$u(\nu)$ is increasing in ν.*

Proof. Computing the first derivative of $u(\nu)$ we have

$$u'(\nu) = C_2\rho\left[\rho^{\nu+n_1+2}ln(\rho) + (1-\rho)\right].$$

Consider the following continuous function defined over $(0,1)$.

$$w(\rho) = \rho^{\nu+n_1+2}ln(\rho) + (1-\rho),$$

where $\nu, n_1 \geq 1$. We have

$$w'(\rho) = (\nu+n_1+2)\rho^{\nu+n_1+2}ln(\rho) + \rho^{\nu+n_1+2} - 1 < 0,$$

for $\rho \in (0,1)$. Therefore $w(\rho)$ is decreasing in ρ. Thus $w(\rho) > w(1) = 0,\ \forall \rho \in (0,1)$, which implies $u'(\nu) > 0$. As such, we can conclude that $u(\nu)$ is increasing in ν. □

Furthermore, we also have

$$SW(n_1,n_2) - SW(n_1,n_2+1) = \frac{\rho^{n_1+n_2+1}}{(1-\rho^{n_1+n_2+1})(1-\rho^{n_1+n_2+2})}u(n_2). \tag{3.2.1}$$

We use the above mathematical results to find the socially optimal value of population-2 agents' strategy threshold given population-1 agents' strategy.

Theorem 3.2.1. *For each fixed value of n_1, we find the optimal value of n_2 which maximizes social welfare as follows.*
(I) *If $u(1) \geq 0$, the optimal value of n_2 is* 1.
(II) *If $u(1) < 0$, the optimal value of n_2 is $\lfloor \nu^{(o)} \rfloor$, where $\nu^{(o)}$ is the unique value satisfying $u(\nu^{(o)}) = 0$ with $\lfloor x \rfloor$ denoting the largest integer number not exceeding x.*

Proof. (I) If $u(1) \geq 0$, because of the increasing property of $u(\nu)$ in ν obtained from Lemma 3.2.1, we have $u(\nu) \geq 0$ for all $\nu \geq 1$. From (3.2.1), we obtain $SW(n_1,n_2) \geq SW(n_1,n_2+1)$

for all $n_2 \geq 1$, which implies

$$\arg\max_{n_2} SW(n_1, n_2) = 1.$$

(II) If $u(1) < 0$, because of the increasing property of $u(\nu)$ in ν, there must exist a unique $\nu^{(o)} > 1$ such that $u(\nu^{(o)}) = 0$ because $u(\nu)$ is continuous in $[1, +\infty)$ and $\lim_{\nu \to +\infty} u(\nu) = +\infty$. Additionally, we have $u(\nu) < 0$ for any $1 \leq \nu < \nu^{(o)}$; and $u(\nu) > 0$ for any $\nu > \nu^{(o)}$. It then can be implied from (3.2.1) that $SW(n_1, n_2) \leq SW(n_1, n_2 + 1)$ for any $n_2 \leq \lfloor \nu^{(o)} \rfloor$, and $SW(n_1, n_2) \geq SW(n_1, n_2 + 1)$ for any $n_2 > \lfloor \nu^{(o)} \rfloor$, from which we obtain

$$\arg\max_{n_2} SW(n_1, n_2) = \lfloor \nu^{(o)} \rfloor.$$

$\square$

Since $n_1 \in [1, N]$, the result in Theorem 3.2.1 allows us to obtain N optimal values of social welfare corresponding to N values of n_1. By comparing those N optimal values of social welfare in N cases of n_1, we obtain $\left(n_1^{(o)}, n_2^{(o)}\right)$ which yields the highest social welfare.

The equation $u(\nu) = 0$ can be rewritten as

$$\rho^{n_1+2}\rho^{\nu} + (1 - \rho)\nu + \frac{H}{C_2 \rho} = 0.$$

Let $x = \nu \ln(\rho)$, $a = \rho^{n_1+2}$, $b = \frac{1-\rho}{\ln(\rho)}$ and $c = \frac{H}{C_2 \rho}$. The equation becomes

$$a\mathrm{e}^{x} + bx + c = 0.$$

Substitute $bx + c = y$ in the above equation. After some calculations, we obtain

$$-\frac{y}{b}\mathrm{e}^{-\frac{y}{b}} = \frac{a}{b}\mathrm{e}^{-\frac{c}{b}}. \tag{3.2.2}$$

We can see that the left hand side of (3.2.2) takes the inverse form of a *Lambert W function*.

The equation (3.2.2) has one unique real root as follows

$$-\frac{y}{b} = W_0\left(\frac{a}{b}\mathrm{e}^{-\frac{c}{b}}\right),$$

which finally yields

$$\nu^{(\mathrm{o})} = \frac{-W_0\left(\frac{a}{b}\mathrm{e}^{-\frac{c}{b}}\right) - \frac{c}{b}}{\ln(\rho)}.$$

Since the solution to the Lambert W function can be instantly found by computer programs such as MATLAB, this formula allows us to explicitly express the optimal value of n_2 for each fixed value of n_1. To sum up, for each value of $n_1 = k$ ranging from 1 to N, we obtain a unique optimal value of n_2, denoted n_{2_k}. Finally, we obtain the socially optimal values of (n_1, n_2) given by

$$\left(n_1^{(\mathrm{o})}, n_2^{(\mathrm{o})}\right) = \arg\max_{0\le k\le N} SW\left(k, n_{2_k}\right).$$

3.2.4 Optimal policies

Two typical policies are introduced. One is to impose a fee or grant a subsidy to each agent, and another is to intervene on price. Since there are two types of agents in the system, the first approach requires a two-sided policy that treats each side differently. The second approach interferes with only one parameter and simultaneously regulates the behaviors of both population-1 agents and population-2 agents. However, it can be limitedly implemented under certain specific conditions.

Toll fee/Subsidy

First, we consider imposing fees or granting subsidies to population-1 agents and population-2 agents denoted by θ_1 and θ_2 (if positive, these amounts are defined as a fee; otherwise, they are defined as a subsidy). The fee or subsidy to population-1 agents should satisfy

$$\begin{cases} R_1 - C_1\dfrac{n_1^{(\mathrm{o})}}{\lambda_2} - \theta_1 > 0, \\ R_1 - C_1\dfrac{n_1^{(\mathrm{o})}+1}{\lambda_2} - \theta_1 < 0. \end{cases}$$

This yields

$$R_1 - C_1\frac{n_1^{(o)}+1}{\lambda_2} < \theta_1 < R_1 - C_1\frac{n_1^{(o)}}{\lambda_2}. \tag{3.2.4}$$

Similarly, the fee or subsidy to population-2 agents should satisfy

$$R_2 - C_2\frac{n_2^{(o)}+1}{\lambda_1} < \theta_2 < R_2 - C_2\frac{n_2^{(o)}}{\lambda_1}. \tag{3.2.5}$$

It should be noted that any values of θ_1 or θ_2 within the optimal ranges derived above yield the same optimal value for social welfare since they do not affect the joining rates of agents. The same applies to any ranges of optimal fees/subsidies/prices derived afterward.

Intervention pricing

Assume that the price p can be intervened and adjusted properly. Then, the service fee should simultaneously satisfy the following conditions

$$\begin{cases} R_2' - \mathsf{p} - C_2\dfrac{n_2^{(o)}}{\lambda_1} > 0, \\ R_2' - \mathsf{p} - C_2\dfrac{n_2^{(o)}+1}{\lambda_1} < 0, \\ \mathsf{p} - C_f - C_1\dfrac{n_1^{(o)}}{\lambda_2} > 0, \\ \mathsf{p} - C_f - C_1\dfrac{n_1^{(o)}+1}{\lambda_2} < 0. \end{cases}$$

It finally yields

$$\mathsf{p} \in \left(R_2' - C_2\frac{n_2^{(o)}+1}{\lambda_2}, R_2' - C_2\frac{n_2^{(o)}}{\lambda_2}\right) \cap \left(C_f + C_1\frac{n_1^{(o)}}{\lambda_2}, C_f + C_1\frac{n_1^{(o)}+1}{\lambda_2}\right). \tag{3.2.6}$$

This is the price range that guarantees that the system equilibrium at which the social welfare is maximized exists. Note that if either $R_2' - C_2\frac{n_2^{(o)}}{\lambda_2} \le C_f + C_1\frac{n_1^{(o)}}{\lambda_2}$ or $C_f + C_1\frac{n_1^{(o)}+1}{\lambda_2} \le R_2' - C_2\frac{n_2^{(o)}+1}{\lambda_2}$, an optimal price p would not exist.

A price scheme, if existent, is similar to collecting a fee from one side of agents and redistributing the same amount to the other side as a subsidy. A fee/subsidy policy is more flexible in the sense that it can be applied in case agents of both the sides need a subsidy, or both the sides were levied a toll (in such a case, a price scheme does not exist). Furthermore, the collected fee from one side of agents (if applicable) might not necessarily be wholly transferred as a subsidy to the other side, but can be used for redistribution of income or other social purposes.

Remark 3.2.2. *Depending on the features of results, there might be other practical policies. For example, if* $n_1^{(o)} \leq n_1^{(s)}$ *and* $n_2^{(o)} \leq n_2^{(s)}$, *one applicable policy could be to limit the buffer capacity of both queue sides.*

3.2.5 Numerical analysis

In the following experiment, we show a sensitivity analysis of social welfare and agents' strategy with respect to parking capacity parameter N in the observable case. Set $\lambda_2 = 4.4$, $\lambda_2 = 4.5$, $R_1 = 15$, $R_2 = 10$, $C_1 = 3$, $C_2 = 2$, $C_f = 5$, $\mathsf{p} = 15$, $R_2' = 25$ and let N varies from 2 to 20. The result is illustrated in Figure 3.2.2.

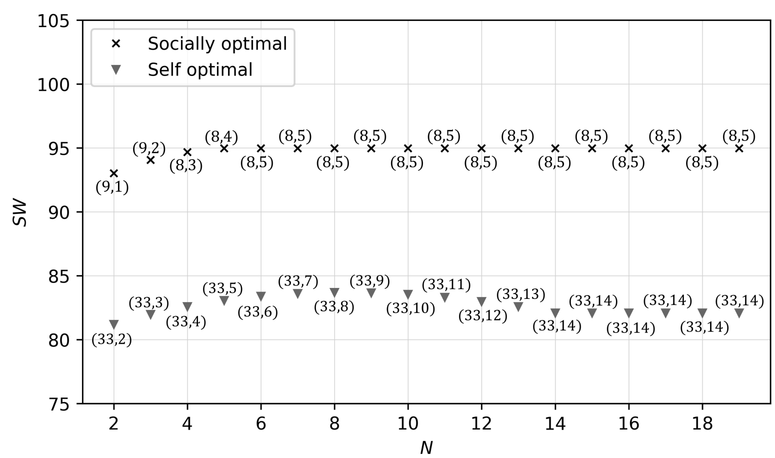


Figure 3.2.2: Social welfare and population-2 agents' strategy with respect to parking capacity in the observable case.

In the above figure, at each value point of social welfare, we include the strategies of agents that trigger the corresponding social welfare value (the first value represents the threshold strategy adopt by population-2 agents, while the second one represents the threshold strategy adopt by population-1 agents). In this example, the socially optimal welfare first increases, then becomes insensitive with N when N gets larger. Furthermore, as N increases, the socially optimal threshold of population-2 agents is non-increasing, while the socially optimal threshold of population-1 agents is increasing from 1 to 5 and remains unchanged at 5 when N is larger.

Meanwhile, a unimodal pattern is observed in case of self-optimization of agents. Furthermore, while the self-optimal threshold of population-2 agents remained unchanged at 33, the self-optimal threshold of population-1 agents increases with N, before remaining unchanged when $N \geq 14$.

3.3 Model 3: Nash equilibria and optimal designs of queueing game in an unobservable queueing system with one finite and one infinite end

In this section, we will derive Nash equilibria and optimize social welfare of a similar system as in Section 3.2, but the system state is unobservable to agents.

This model is based on the following paper: Nguyen, H. Q., & Phung-Duc, T. (2022). Supply–demand equilibria and multivariate optimization of social welfare in double-ended queueing systems. Computers Industrial Engineering, 170, Article no. 108306[42].

3.3.1 The outcomes

Let $\bar{\mathbf{X}} = \left(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}\right) \in [0,1]^2$ be the social profile in equilibrium, where $\bar{\sigma}^{(1)}$ and $\bar{\sigma}^{(2)}$ respectively denote the joining probability of population-1 agents and population-2 agents. The actual joining rate of population-1 agents is $\bar{\lambda}_1 = \bar{\sigma}^{(1)}\lambda_1$, and that for the population-2 agents is $\bar{\lambda}_2 = \bar{\sigma}^{(2)}\lambda_2$. Let $\bar{\rho} = \frac{\bar{\lambda}_2}{\bar{\lambda}_1}$.

First, notice that $\bar{\mathbf{X}} = \left(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}\right) = (0,0)$ is a Nash equilibrium. Under this social profile, any agent who adopts a positive joining probability would receive an infinitely negative expected payoff, so the best response is not to join.

Now we consider the case where $\bar{\sigma}^{(2)} > 0$. Notice that the condition $\rho\bar{\sigma}^{(2)} < \bar{\sigma}^{(1)}$ (or equivalently $\bar{\rho} < 1$) must be satisfied. Otherwise, the system is unstable and an arbitrary population-2 agent who adopts a strategy $\sigma^{(2)} = 0$ would receive a payoff of 0, which is clearly larger than that expected by any population-2 agent adopting $\bar{\sigma}^{(2)}$ and expecting an infinitely negative payoff. Therefore, in equilibrium, $\bar{\rho} < 1$ (also, $\bar{\sigma}^{(1)} > 0$), meaning that the system is stable. By modeling the system in equilibrium as an M/M/1 queue, the expected waiting time of a population-2 agent is

$$W_2(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}) = \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})}.$$

The expected waiting time of a population-1 agent is

$$W_1(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}) = \frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})}.$$

By definition of the Nash equilibria in (2.2.4),

$$\bar{\sigma}^{(2)} \in BR^{(2)}(\bar{\mathbf{X}}) = \begin{cases} \{0\}, & \text{if } R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} < 0, \\ [0,1], & \text{if } R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} = 0, \\ \{1\}, & \text{if } R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} > 0. \end{cases}$$

Since $\bar{\sigma}^{(2)} > 0$, we can remove the first case leading to

$$\bar{\sigma}^{(2)} \in \begin{cases} \{1\}, & \text{if } R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} > 0 & (3.3.1) \\ (0,1], & \text{if } R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} = 0. & (3.3.2) \end{cases}$$

Similarly,

$$\bar{\sigma}^{(1)} \in \begin{cases} \{1\}, & \text{if } R_1 - C_1 \left(\frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})} \right) > 0 & (3.3.3) \\ (0,1], & \text{if } R_1 - C_1 \left(\frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})} \right) = 0. & (3.3.4) \end{cases}$$

Consider the following four cases.

Case 1: Combining (3.3.1) and (3.3.3), we have

$$\bar{\sigma}^{(1)} = \bar{\sigma}^{(2)} = 1$$

under the two conditions $R_2 - C_2 \frac{\rho^N}{\lambda_2(1-\rho)} > 0$ and $R_1 - C_1 \left(\frac{N}{\lambda_2} - \frac{\rho - \rho^{N+1}}{\lambda_2(1-\rho)} \right) > 0$. In this equilibrium, both the populations of agents join the system at full potential rates and find positive payoffs.

Case 2: Combining (3.3.2) and (3.3.3), we have $\bar{\sigma}^{(1)} = 1$ and

$$R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} = 0. \tag{3.3.5}$$

Consider the function

$$U_2(\bar{\sigma}^{(2)}) = R_2 - C_2 \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})},$$

where $\bar{\sigma}^{(1)} = 1, 0 \leq \bar{\sigma}^{(2)} \leq 1, \bar{\rho} = \frac{\bar{\sigma}^{(2)}}{\bar{\sigma}^{(1)}} < 1$.

We have

$$U_2'(\bar{\sigma}^{(2)}) = -C_2 \frac{(N\lambda_1 - (N-1)\lambda_2\bar{\sigma}^{(2)})\bar{\sigma}^{(2)^{N-1}}}{(\lambda_1 - \lambda_2\bar{\sigma}^{(2)})^2}.$$

Since $\lambda_1 > \lambda_2$ and $0 \leq \bar{\sigma}^{(2)} \leq 1$, we have $N\lambda_1 - (N-1)\lambda_2\bar{\sigma}^{(2)} > 0$, and thus, $U_2'(\bar{\sigma}^{(2)}) < 0$ for $0 < \bar{\sigma}^{(2)} \leq 1$. Additionally, $U_2'(\bar{\sigma}^{(2)}) = 0$ for $\bar{\sigma}^{(2)} = 0$. Therefore, the function $U_2(\bar{\sigma}^{(2)})$ is decreasing in $\bar{\sigma}^{(2)}$ in $[0,1]$. This result indicates that the expected utility of each population-2 agent decreases if population-2 agents join the system at a higher rate. This is intuitive because when population-2 agents join the system more frequently, the expected population-2 agent queue length increases. Thus, the expected waiting time of the population-2 agents in the system increases.

Since $U_2(\bar{\sigma}^{(2)})$ is decreasing in $[0,1]$, we have $U_2(1) \leq U_2(\bar{\sigma}^{(2)}) = 0$. Also, because $U_2(0) = R_2 > 0$, equation (3.3.5) must have a unique root $\tilde{\sigma}^{(2)}$ that satisfies $0 < \tilde{\sigma}^{(2)} \leq 1$.

This equilibrium comes under the condition $R_1 - C_1\left(\frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho}-\bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})}\right) > 0$, where $\bar{\lambda}_2 = \tilde{\sigma}^{(2)}\lambda_2$ and $\bar{\rho} = \frac{\tilde{\sigma}^{(2)}\lambda_2}{\lambda_1}$.

In this equilibrium, the population-1 agents join the system at the full potential rate, while the population-2 agents join at a rate at which their expect payoff is 0.

Case 3: Combining (3.3.1) and (3.3.4), we have $\bar{\sigma}^{(2)} = 1$ and

$$R_1 - C_1\left(\frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})}\right) = 0 \tag{3.3.6}$$

when $\bar{\sigma}^{(2)} = 1$.

Consider the function

$$U_1(\bar{\sigma}^{(1)}) = R_1 - C_1\left(\frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})}\right),$$

where $\bar{\sigma}^{(2)} = 1, 0 \leq \bar{\sigma}^{(1)} \leq 1, \bar{\rho} = \frac{\bar{\sigma}^{(2)}\lambda_2}{\bar{\sigma}^{(1)}\lambda_1} < 1$.

Note that

$$U_1'(\bar{\sigma}^{(1)}) = -C_1 \frac{\rho[\bar{\sigma}^{(1)}(\bar{\sigma}^{(1)^N} - \rho^N) - N\rho^N(\bar{\sigma}^{(1)} - \rho)]}{\lambda_2(\bar{\sigma}^{(1)} - \rho)^2\bar{\sigma}^{(1)^{N+1}}}.$$

Consider the function $h(\bar{\sigma}^{(1)}) = \bar{\sigma}^{(1)}(\bar{\sigma}^{(1)^N} - \rho^N) - N\rho^N(\bar{\sigma}^{(1)} - \rho)$ defined over $[\rho, 1]$. We have $h^{'}(\bar{\sigma}^{(1)}) = (N+1)(\bar{\sigma}^{(1)^N} - \rho^N) > 0$ for $\bar{\sigma}^{(1)} \in (\rho, 1]$, which implies that $h(\bar{\sigma}^{(1)})$ is an increasing function with respect to $\bar{\sigma}^{(1)}$ on $[\rho, 1]$. Therefore, $h(\bar{\sigma}^{(1)}) > h(\rho) = 0$ for any $\bar{\sigma}^{(1)} \in (\rho, 1]$, which implies that $U_1^{'}(\bar{\sigma}^{(1)}) < 0$ on $(\rho, 1]$. Thus, $U_1(\bar{\sigma}^{(1)})$ is decreasing in $\bar{\sigma}^{(1)}$ on $[\rho, 1]$.

This equilibrium comes under the condition $R_2 - C_2\frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})} > 0$, where $\bar{\lambda}_1 = \tilde{\sigma}^{(1)}\lambda_1$ and $\bar{\rho} = \frac{\lambda_2}{\tilde{\sigma}^{(1)}\lambda_1}$.

In this equilibrium, population-2 agents join the system at the full potential rate, while population-1 agents join at a rate at which their expected payoff is 0.

Case 4: Combining (3.3.2) and (3.3.4), we have

$$\begin{cases} \frac{R_2}{C_2} & = \frac{\bar{\rho}^N}{\bar{\lambda}_1(1-\bar{\rho})}, \\ \frac{R_1}{C_1} & = \frac{N}{\bar{\lambda}_2} - \frac{\bar{\rho}-\bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})}, \end{cases} \tag{3.3.7}$$

with $0 < \bar{\sigma}^{(1)}, \bar{\sigma}^{(2)} \leq 1$, which yields

$$\frac{R_1C_2}{R_2C_1} = \frac{N}{\bar{\lambda}_2} \cdot \frac{\bar{\lambda}_1(1-\bar{\rho})}{\bar{\rho}^N} - \frac{\bar{\rho} - \bar{\rho}^{N+1}}{\bar{\lambda}_2(1-\bar{\rho})} \cdot \frac{\bar{\lambda}_1(1-\bar{\rho})}{\bar{\rho}^N}.$$

Let $\varphi = \frac{1}{\bar{\rho}}$, $\varphi > 1$. After some calculations, we obtain

$$N\varphi^{N+1} - (N+1)\varphi^N + 1 - \frac{R_1C_2}{R_2C_1} = 0.$$

Consider the function $g(\varphi) = N\varphi^{N+1} - (N+1)\varphi^N + 1 - \frac{R_1C_2}{R_2C_1}$ defined over $[1, +\infty)$. We have $g^{'}(\varphi) = N(N+1)\varphi^{N-1}(\varphi - 1) > 0$ on $(1, +\infty)$. Therefore, $g(\varphi)$ is continuously increasing in $[1, +\infty)$. Also, notice that $g(1) = -\frac{R_1C_2}{R_2C_1} < 0$ and $\lim_{\varphi \to +\infty} f(\varphi) = +\infty$. Thus, (3.3.7) always has a unique solution $\tilde{\varphi}$ in $(1, +\infty)$. Now, set $\tilde{\rho} = \frac{1}{\tilde{\varphi}}$, then $\bar{\rho} = \tilde{\rho}$. Substituting into (3.3.7), and replacing $\bar{\lambda}_1 = \bar{\sigma}^{(1)}\lambda_1$ and $\bar{\lambda}_2 = \bar{\sigma}^{(2)}\lambda_2$, we obtain

$$\bar{\sigma}^{(1)} = \widehat{\sigma}^{(1)} = \frac{C_2}{R_2} \cdot \frac{\tilde{\rho}^N}{1-\tilde{\rho}} \cdot \frac{1}{\lambda_1},$$

and

$$\bar{\sigma}^{(2)} = \widehat{\sigma}^{(2)} = \frac{C_1}{R_1}\left(N - \frac{\tilde{\rho} - \tilde{\rho}^{N+1}}{1-\tilde{\rho}}\right)\frac{1}{\lambda_2}.$$

Furthermore, recalling the conditions $0 < \bar{\sigma}^{(1)}, \bar{\sigma}^{(2)} \leq 1$ and $\rho\bar{\sigma}^{(2)} < \bar{\sigma}^{(1)}$, we should have

- $\frac{C_2}{R_2} \cdot \frac{\tilde{\rho}^N}{1-\tilde{\rho}} \cdot \frac{1}{\lambda_1} \leq 1$,
- $\frac{C_1}{R_1}\left(N - \frac{\tilde{\rho}-\tilde{\rho}^{N+1}}{1-\tilde{\rho}^N}\right)\frac{1}{\lambda_2} \leq 1$, and
- $\frac{C_1}{R_1}\left(N - \frac{\tilde{\rho}-\tilde{\rho}^{N+1}}{1-\tilde{\rho}^N}\right) < \frac{C_2}{R_2} \cdot \frac{\tilde{\rho}^N}{1-\tilde{\rho}}$.

In this equilibrium, both the populations join the system with a probability less than or equal to 1, at which joining agents find an expected payoff of 0.

It should be noted that the three equilibria $(1, 1), (1, \tilde{\sigma}^{(2)},)$ and $(\tilde{\sigma}^{(1)}, 1)$ occur under unique conditions and are mutually exclusive. As a result, there may exist at most three equilibria: $(0, 0)$, $(\widehat{\sigma}^{(1)}, \widehat{\sigma}^{(2)})$ and one of the three equilibria $(1, 1)$, $(1, \tilde{\sigma}^{(2)})$, $(\tilde{\sigma}^{(1)}, 1)$.

Remark 3.3.1 (Equilibrium selection). *Any one of the equilibria may occur. The selection of one equilibrium depends on how the agents are advised and the belief of both sides in the behavior of each other. For example, the equilibrium $(0, 0)$ may occur when the system is suspended and both the agent populations know that they cannot use the service at that platform. The problem of equilibrium selection, which has been thoroughly analyzed in previous works[16], is outside the scope of this research, but will be briefly discussed in a later section (3.3.2).*

Remark 3.3.2 (A Cournot model). *The unobservable case of two-sided strategic queueing can also be modeled as a Cournot game, where the joining probability of one side can be seen as a* reaction function *of the joining probability of the other side. We then reach equilibria by letting two reaction functions cross each other.*

Optimization

Assume that the population-1 agents and population-2 agents adopt strategies $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively. The actual joining rate of population-1 agents is $\widehat{\lambda}_1 = \sigma^{(1)}\lambda_1$, and that of population-1 agents is $\widehat{\lambda}_2 = \sigma^{(2)}\lambda_2$. Assume that the utility of balking population-1 agents and population-2 agents is negligible. The expected social welfare is then a function of two variables $\sigma^{(1)}$ and $\sigma^{(2)}$ as follows.

$$SW(\sigma^{(1)}, \sigma^{(2)}) = (R_1 + R_2)\lambda_2\sigma^{(2)} - C_2L_1(\sigma^{(1)}, \sigma^{(2)}) - C_1L_2(\sigma^{(1)}, \sigma^{(2)}),$$

where L_1 and L_2 denote the expected queue lengths of population-1 and population-2 agents per unit time. Note that for any $\sigma^{(1)}, \sigma^{(2)}$ such that $\rho\sigma^{(2)} \geq \sigma^{(1)}$, we have $\widehat{\rho} = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1} > 1$. Thus $L_2(\sigma^{(1)}, \sigma^{(2)}) = +\infty$ and $L_1(\sigma^{(1)}, \sigma^{(2)}) = 0$ and $SW(\sigma^{(1)}, \sigma^{(2)}) = -\infty$. Therefore, we only consider the case where $\rho\sigma^{(2)} < \sigma^{(1)}$. Under this condition, the actual joining rate of population-1 agents is $\widehat{\lambda}_2$[50]. The problem becomes a multivariate optimization problem where we try to maximize

$$SW(\sigma^{(1)}, \sigma^{(2)}) = (R_1 + R_2)\lambda_2\sigma^{(2)} - C_2\frac{\left(\frac{\sigma^{(2)}\lambda_2}{\sigma^{(1)}\lambda_1}\right)^{N+1}}{1 - \frac{\sigma^{(2)}\lambda_2}{\sigma^{(1)}\lambda_1}} - C_1\left(N - \frac{\frac{\sigma^{(2)}\lambda_2}{\sigma^{(1)}\lambda_1} - \left(\frac{\sigma^{(2)}\lambda_2}{\sigma^{(1)}\lambda_1}\right)^{N+1}}{1 - \frac{\sigma^{(2)}\lambda_2}{\sigma^{(1)}\lambda_1}}\right),$$

Under the constraints $0 < \sigma^{(1)}, \sigma^{(2)} \leq 1$, and $\rho\sigma^{(2)} < \sigma^{(1)}$.

Without modifications, the above problem looks complicated and does not seem to be solvable. Therefore, we rewrite the problem as maximizing

$$SW(\widehat{\lambda}_1, \widehat{\rho}) = (R_1 + R_2)\widehat{\rho}\widehat{\lambda}_1 - C_2\frac{\widehat{\rho}^{N+1}}{1 - \widehat{\rho}} - C_1\left(N - \frac{\widehat{\rho} - \widehat{\rho}^{N+1}}{1 - \widehat{\rho}}\right),$$

under the constraints $0 < \widehat{\rho} < 1$, $\widehat{\lambda}_1 \leq \lambda_1$ and $\widehat{\rho}\widehat{\lambda}_1 \leq \lambda_2$.

For any fixed value of $\widehat{\rho}$, we have

$$\frac{\partial SW(\widehat{\lambda}_1, \widehat{\rho})}{\partial \widehat{\lambda}_1} = (R_1 + R_2)\widehat{\rho} > 0,$$

which implies that $SW(\widehat{\lambda}_1, \widehat{\rho})$ is strictly increasing in $\widehat{\lambda}_1$. Since $\widehat{\lambda}_1 \leq \lambda_2$ and $\widehat{\rho}\widehat{\lambda}_1 \leq \lambda_2$, we obtain

$$\max SW(\widehat{\lambda}_1, \widehat{\rho}) = \max SW\left(\min\left\{\lambda_1, \frac{\lambda_2}{\widehat{\rho}}\right\}, \widehat{\rho}\right).$$

This implies

$$\max SW(\widehat{\lambda}_1, \widehat{\rho}) = \max\left\{\max_{\widehat{\rho}\in(0,\rho]} SW(\lambda_1, \widehat{\rho}), \max_{\widehat{\rho}\in[\rho,1)} SW\left(\frac{\lambda_2}{\widehat{\rho}}, \widehat{\rho}\right)\right\}.$$

Remark 3.3.3. *The above result intuitively means that the optimal value of social welfare occurs only if either one of the two sides join the system at the full rate.*

Since $SW(\lambda_1, \widehat{\rho})$ and $SW\left(\frac{\lambda_2}{\widehat{\rho}}, \widehat{\rho}\right)$ are continuous with respect to $\widehat{\rho}$ on $(0, \rho]$ and $[\rho, 1)$ respectively, we can find their maximum by comparing all of their values at critical points which are obtained by solving

$$\begin{aligned}&\frac{dSW(\lambda_1, \widehat{\rho})}{d\widehat{\rho}} = \\ &\frac{(C_1 + C_1)N\widehat{\rho}^{N+1} - (C_1 + C_2)(N+1)\widehat{\rho}^N + (R_1 + R_2)\lambda_1\widehat{\rho}^2 - 2(R_1 + R_2)\lambda_1\widehat{\rho} + (R_1 + R_2)\lambda_1 + C_1}{(1-\widehat{\rho})^2} \\ &= 0\end{aligned}$$

on $(0, \rho]$, and

$$\frac{dSW\left(\frac{\lambda_2}{\widehat{\rho}}, \widehat{\rho}\right)}{d\widehat{\rho}} = \frac{(C_1 + C_2)N\widehat{\rho}^{N+1} - (C_1 + C_2)(N+1)\widehat{\rho}^N + C_1}{(1-\widehat{\rho})^2} = 0$$

on $[\rho, 1)$.

We can see that both equations above are equivalent to univariate polynomial equations and are computationally solvable. After finding the optimal values of $\widehat{\lambda}_1$ and $\widehat{\rho}$, we can obtain the optimal $({\sigma^{(1)}}^{(o)}, {\sigma^{(2)}}^{(o)})$ by assigning $\sigma^{(1)} = \frac{\widehat{\lambda}_1}{\lambda_1}$ and $\sigma^{(2)} = \frac{\widehat{\rho}\widehat{\lambda}_1}{\lambda_2}$.

3.3.2 Optimal policies

Toll fee/Subsidy

As mentioned in Remark 3.3.3, the socially optimal strategies $(\sigma^{(1)^{(o)}}, \sigma^{(2)^{(o)}})$ may take one of the two forms: $(\sigma^{(1)^{(o)}}, 1)$ or $(1, \sigma^{(2)^{(o)}})$. Assume that social welfare is maximized at $(\sigma^{(1)^{(o)}}, 1)$ (the other case can be treated similarly).

First, we consider imposing fees or granting subsidies to population-1 agents and population-2 agents denoted by θ_1 and θ_2, respectively. If positive, these amounts are defined as a fee. Otherwise, they are defined as a subsidy. For $(\sigma^{(1)^{(o)}}, 1)$ to become the system equilibrium, θ_1 and θ_2 must satisfy

$$R_1 - C_1 W_1(\sigma^{(1)^{(o)}}, 1) - \theta_1 = 0$$

and

$$R_2 - C_2 W_2(\sigma^{(1)^{(o)}}, 1) - \theta_2 \geq 0.$$

In other words, we have

$$\begin{cases} \theta_1 = R_1 - C_1 W_1(\sigma^{(1)^{(o)}}, 1), \\ \theta_2 \leq R_2 - C_2 W_2(\sigma^{(1)^{(o)}}, 1). \end{cases}$$

Intervention pricing

Assume that the price p can be intervened and adjusted properly. Then, the service price should simultaneously satisfy the following conditions

$$\begin{cases} R_2' - \mathsf{p} - C_2 W_2(\sigma^{(1)^{(o)}}, 1) \geq 0, \\ \mathsf{p} - C_f - C_1 W_1(\sigma^{(1)^{(o)}}, 1) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathsf{p} = C_f + C_1 W_1(\sigma^{(1)^{(o)}}, 1), \\ \mathsf{p} \leq R_2' - C_2 W_2(\sigma^{(1)^{(o)}}, 1). \end{cases}$$

In this approach, there exists a single optimal value for p at which social welfare is maximized. However, if $C_f + C_1 W_1(\sigma^{(1)^{(o)}}, 1) > R_2' - C_2 W_2(\sigma^{(1)^{(o)}}, 1)$, this approach becomes infeasible.

Remark 3.3.4 (A Stackelberg model). *We have discussed a socially optimal price range including a price ceiling and a price floor set by the platform manager (or the government). Considering that sellers are the ones who decide the fare (under regulations, if any), this can be modeled similarly as a two-stage Stackelberg game. In the first stage, the sellers decide the price within the price range set in advance. In the second stage, the sellers and buyers choose whether to join the system or not. Since the joining probabilities of the sellers and buyers are unchanged within the given price range, with the advantage of the first mover, in the first stage, the sellers will choose the price at the upper bound to maximize their utility. Consequently, in this scenario, the equilibrium price is the ceiling price.*

Brief discussion on how the system equilibrates

Note that the above policies only provide a necessary condition for the socially optimal strategies $(\sigma^{(1)^{(o)}}, \sigma^{(2)^{(o)}})$ to become an equilibrium, but not guarantee that both populations of agents choose to act on that equilibrium. There might exist other equilibria even if the optimal fees/subsidies/prices are set from the beginning.

Assume that at an arbitrary time, the system is at social profile $(\sigma^{(1)}, \sigma^{(2)})$, where $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively, show the proportion of population-1 and population-2 agents joining the system at the time being considered. In case there exist multiple Nash equilibria, the convergence of the system depends on the initial social profile.

Let us take an example. Assume there exist 3 equilibria, $(1, \tilde{\sigma}_1)$, $(\bar{\sigma}^{*(1)}, \bar{\sigma}^{*(2)})$, and $(0, 0)$ where $0 < \bar{\sigma}^{*(2)} < 1$ and $0 < \bar{\sigma}^{*(1)} < \tilde{\sigma}^{(1)}$ (later we can see that this case can actually happen in a numerical example). Consider an arbitrary social profile $(\sigma^{(2)}, \sigma^{(1)})$ such that $\bar{\sigma}^{*(2)} < \sigma^{(2)} < 1$, $0 < \sigma^{(1)} < \bar{\sigma}^{*(1)}$. If $U_2(\sigma^{(2)}, \sigma^{(1)}) < 0$ and $U_1(\sigma^{(1)}, \sigma^{(2)}) > 0$, the population-2 agents will keep balking while the population-1 agents keep joining. This makes the equilibrium at $(\bar{\sigma}^{*(1)}, \bar{\sigma}^{*(2)})$ more likely to occur. Contrarily, for a social profile $(\sigma^{(1)}, \sigma^{(2)})$ where $\bar{\sigma}^{*(2)} < \sigma^{(2)} < 1$, $\bar{\sigma}^{*(1)} < \sigma^{(1)} < \tilde{\sigma}^{(1)}$, $U_1(\sigma^{(1)}, \sigma^{(2)}) > 0$, and $U_2(\sigma^{(1)}, \sigma^{(2)}) > 0$, it is more likely that the system may end up at the equilibrium $(\tilde{\sigma}^{(1)}, 1)$.

Practically, the equilibrium at $(1, 1)$ is relatively less difficult to attain if sufficient incentives and promotions are provided. Also, taking advantage of the feature stated in Remark 3.3.3, the socially optimal welfare is obtained when one of the two sides joins the system at full potential rate. One may think about attaining the socially optimal equilibrium converging from the social profile $(1, 1)$ by implementing the following two-step strategy.

Step 1: Attract all potential agents at the launching state of the platform. Technically, similar proposed policies can be implemented to ensure that $U_1(1, 1) > 0$ and $U_2(1, 1) > 0$. Platform companies, such as Uber and Lyft, offer generous discounts to both the customers and riders, in the combination of various promotion campaigns, at the time of market entrance. This also enables the evaluation of λ_1 and λ_2, which represent the full potential supply and demand, respectively.

Step 2: Adjust the price/subsidy/fee (as proposed in Section 3.3.2 and Section 3.3.2) such that $U_1(\tilde{\sigma}^{(1)}, 1) = 0$ and $U_2(\tilde{\sigma}^{(1)}, 1) > 0$. Consequently, this makes $U_1(1, 1) < 0$, so the joining probability converges from 1 to $\tilde{\sigma}^{(1)}$, while population-2 agents still adopt $\sigma^{(2)} = 1$ as they still find non-negative utility when the joining rate of population-1 agents decreases to $\tilde{\sigma}^{(1)}$.

3.3.3 Numerical analysis

In this section, we present results of numerical examples that help to have a practical view about the agent behavior in different scenarios and propose some recommendations for policies that can be made to optimize social welfare in the example.

In the example, we numerically show all results that were derived in the previous sections. Set $\lambda_1 = 4.5$, $\lambda_2 = 4$, $N = 25$, $R_1 = 15$, $R_2 = 10$, $C_1 = 3$, $C_2 = 2$, $C_f = 5$, $\mathsf{p} = 15$, $R_2' = 25$. Results are summarized in Table 3.1.

From the results, we can draw the following conclusions.

- In the observable case, the strategy thresholds of socially concerned population-1 and population-2 agents drivers are smaller than when they act selfishly, which means we need to impose extra fees θ_1 and on population-1 agents and θ_2 on population-2 agents.

System setting	Type of behavior	Strategy of population 1	Strategies of population 2	Social welfare
Observable	Selfishly optimal	$n_1 = 13$	$n_2 = 33$	76.538
	Socially optimal	$n_1 = 3$	$n_2 = 11$	89.876
Unobservable	Selfishly optimal	$\sigma^{(1)} = 0.951$	$\sigma^{(2)} = 1$	54.711
		$\sigma^{(1)} = 0.509$	$\sigma^{(2)} = 0.557$	-0.042
		$\sigma^{(1)} = 0$	$\sigma^{(2)} = 0$	0
	Socially optimal	$\sigma^{(1)} = 0.946$	$\sigma^{(2)} = 1$	55.296

Table 3.1: Strategies of the two populations and social welfare in different scenarios.

Using results (3.2.4) and (3.2.5), we can identify the range of those fees as below

$$7 < \theta_1 < 7.75,$$

and

$$9.667 < \theta_2 < 10.111.$$

As discussed, there is another approach that aims to adjust the service price. Using result (3.2.6), we obtain

$$\mathsf{p} \in (24.667, 25.111) \cap (7.25, 8) = \emptyset,$$

which means that this approach cannot be applied in this case.

Another applicable policy is to limit the population-2 agent buffer at 11, and limit the parking capacity at 3 (according to Remark 3.2.2).

- In the unobservable case, there exist three equilibria of the system when the behavior of agents is selfishly optimal. Social welfare is maximized when all potential population-2 agents join the system and population-1 agents join with a probability of 0.946. There are two ways to make $({\sigma^{(1)}}^{(o)}, {\sigma^{(2)}}^{(o)}) = (0.946, 1)$ become an equilibrium. First, plugging in numerical values of the parameters into (3.3.8), we obtain $\theta_1 = 0.472$

and $\theta_2 \leq 13.353$. However, as $U_2(0.946, 1) > 0$, it is not necessary to impose any fee or grant any subsidy to population-2 agents. Meanwhile, each population-1 agent should be levied a fixed fee of 0.472.

The second solution is to fix a price at which $(\sigma^{(1)^{(o)}}, \sigma^{(2)^{(o)}}) = (0.946, 1)$ becomes an equilibrium. Using result (3.2.6), we obtain

$$\begin{cases} \mathsf{p} = 14.528, \\ \mathsf{p} \leq 28.352, \end{cases}$$

which finally yields $\mathsf{p} = 14.528$. This is the fixed price which is the necessary condition to maximize the social welfare.

- Social welfare in the observable case is higher than that in the unobservable case, regardless of agent behavior (selfishly optimal or socially optimal). Therefore, an observable system is recommended if the priority is the total utility of all agents participating in the system.

In the following example, we show a sensitivity analysis of the social welfare and agent strategy with respect to the parking capacity parameter N in the unobservable case. Set $\lambda_1 = 4.4$, $\lambda_2 = 4.5$, $R_1 = 15$, $R_2 = 10$, $C_1 = 2$, $C_2 = 3$, $C_f = 5$, $\mathsf{p} = 15$, $R_2' = 25$ and let N vary from 2 to 40. The result is illustrated in Figure 3.3.1.

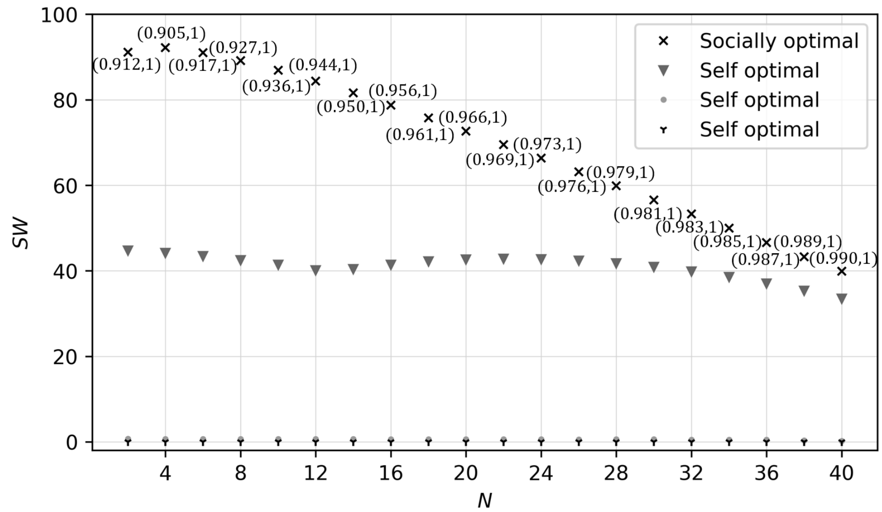


Figure 3.3.1: Social welfare and population-2 agent strategy with respect to the parking capacity in the unobservable case.

For each value of N, there always exist three Nash equilibria, of which one equilibrium is reached when no agents join the system and triggers zero social welfare. Since there are multiple Nash equilibria in each case, we focus on the strategies that optimize the social welfare. In the above figure, at each value point of the optimal social welfare, we include the optimal strategies of agents that trigger the corresponding social welfare value. We see that the social welfare is maximized when no population-1 agents balk in all the considered cases. The maximum optimal social welfare that can be reached is observed at $N = 4$, corresponding to the smallest joining probability of the population-2 agents. As N becomes larger, the optimal social welfare is obtained with a larger joining probability of population-2 agents.

4

Population games in double-ended queues with nonzero matching times

This chapter deals with modeling double-ended queues in which matching times are nonzero, which applies in systems in which the matching times cannot dismissed as negligible; for example, passenger-taxi stations at which passenger often come with bulky luggage and need time to communicate with taxi drivers.

The incorporation of matching times in the models considered in this chapter requires the addition of at least one more dimension in states of the Markov chains that describe the system. Although if a social profile is a Nash equilibrium can still be verified by referring to Definition 2.2.4, some explicit features of Nash equilibria as in Section 3.1 may become impossible to derive. In this chapter, we focus on deriving the rational outcome, i.e., the Subgame perfect Nash equilibria of the queueing games in the observable setting.

4.1 Model 4: The rational outcome and optimal designs of a one-population queueing game in an observable queueing system with multiple matching points, one finite end and one infinite end

This model is based on the following paper: Nguyen, H. Q. & Phung-Duc, T. (2022). Strategic customer behavior and optimal policies in a passenger–taxi double-ended queueing system with multiple matching points and nonzero matching times. Queueing Systems, 102, 481–508[41].

4.1.1 Preliminaries

Consider a society $\mathcal{P} = \{1, 2\}$ that consists of two populations of agents arriving at a double-ended queueing system containing S identical matching points, based on Poisson processes with rates λ_1 and λ_2. The two populations of agents represent a market with a supply side (population-1) and a demand side (population-2). Assume that the demand side is strategic and the supply side is not. Matching is performed on a first-come-first-served basis by a pair of a population-1 agent and a population-2 agent in a random time that follows an exponential distribution with rate μ. The reward upon the completion of a service and the waiting cost per unit time of a population-i agent are denoted by R_i and C_i $(i = 1, 2)$, respectively. Assume that

$$R_2 \geq \frac{C_2}{\mu}.$$

This guarantees that population-2 agents are willing to join an empty queue.

The waiting area (including S matching points) can accommodate at most K population-1 agents at the same time ($K \geq S$). When the waiting area reaches its maximum capacity, the arrivals of population-1 agents are blocked, and they leave immediately. We assume that the buffer capacity of population-2 agents is infinite. If a population-2 agent arrives when all matching points are busy or there are no population-1 agents available for matching, the population-2 agent will wait in the queue under a FCFS (first-come-first-served) service discipline.

The Markov chain describing the system is described by two variables, $X_1(t)$ and $X_2(t)$, that, respectively, represent the number of population-1 agents and the number of population-2 agents present in the system at time t. The state space is given by $\mathbb{S} = \{\mathbf{s} = (x_1, x_2) | (x_1, x_2) \in \mathbb{N}^2\}$. In a non-strategic queueing scenario, the system's transition diagram can be illustrated as in Figure 4.1.1.

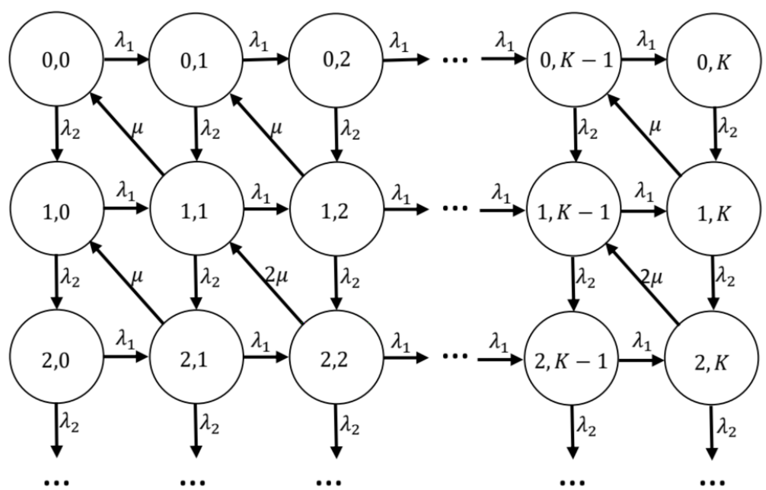


Figure 4.1.1: Transition diagram in the non-strategic scenario.

A single-server version of this model was thoroughly investigated in[51], in which the authors derived the stability condition and sojourn time distributions of both population-1 agents and population-2 agents. Furthermore, the stability condition of the multiserver system with the same setting was derived in[36] as follows.

$$\lambda_2 < \left(\sum_{x_1=0}^{S-1} x_1 \pi_{x_1} + S \sum_{x_1=S}^{K} \pi_{x_1} \right) \mu,$$

where π_{x_1} denotes the probability that there are x_1 population-1 agents in an M/M/S/K queue with arrival rate λ_1 and service rate μ.

In the current research, we add economic parameters and study the system in equilibrium.

4.1.2 The rational outcome

In this section, we derive the strategic behavior of population-2 agents. Let

$$\boldsymbol{\sigma}^{(2)} = \begin{pmatrix} \sigma^{(2)}_{(0,0)} & \sigma^{(2)}_{(0,1)} & \cdots \\ \sigma^{(2)}_{(1,0)} & \sigma^{(2)}_{(1,1)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $\sigma^{(2)}_{(x_1,x_2)}$ denotes the joining probability of population-2 agents when they observe state (x_1, x_2).

The transition diagram in case population-2 agents are strategic can be illustrated as in Figure 4.1.2.

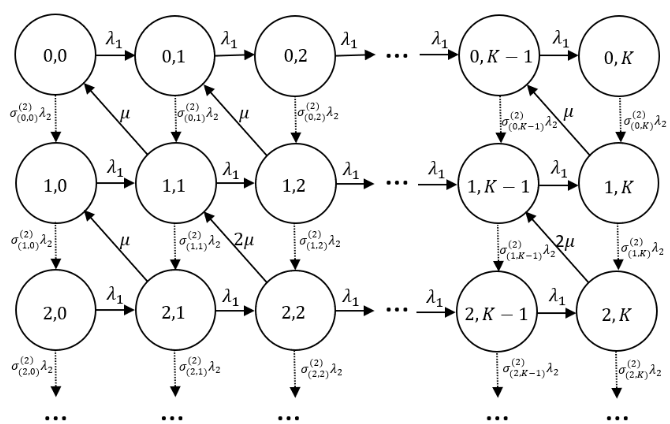


Figure 4.1.2: Transition diagram in the strategic scenario.

The dashed lines in the figure indicate that the corresponding transition rate remains unknown (since the transition rates depend on the agents' joining strategies).

Note that the sojourn time can be decomposed into waiting time and service time, where the expected service time is constant at $\frac{1}{\mu}$, while the expected waiting time depends on the current system state. Next, since population-1 agents are not strategic, the waiting time of a population-2 agent does not depend on the joining strategy of later comers. Denote by $T(\hat{x}_2, x_1)$ the expected waiting time of a population-2 agent observing a system state $\mathbf{s} = (\hat{x}_2, x_1)$ upon arrival, where x_1 represents the current number of population-1 agents in the system, and $\hat{x}_2$ represents the current "position" of the population-2 agent. If $\hat{x}_2 = 0$ then the population-2 agent is currently matching at a matching point. If $\hat{x}_2 > 0$ then the population-2 agent is $\hat{x}_2$ steps away from the matching points; in other words, $\hat{x}_2 - 1$ is the current queue length (which does not include population-2 agents in matching) being observed. We need to calculate all the values of $T(\hat{x}_2, x_1)$ to investigate the strategic behavior of population-2 agents. To this end, we use first-step analysis with one-step transitions from state $(\hat{x}_2, x_1)$ being illustrated in Fig. 4.1.3.

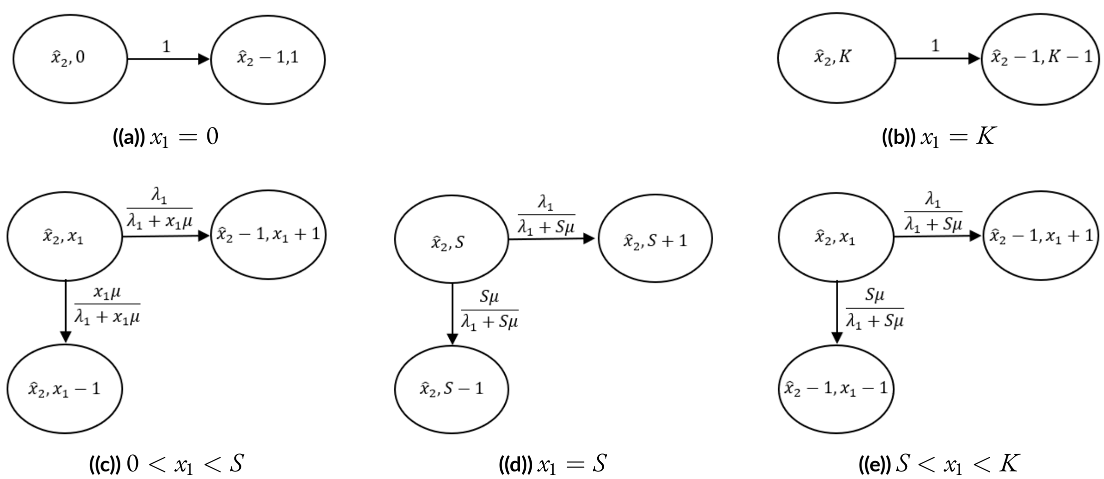


Figure 4.1.3: Transition diagrams from state $(\hat{x}_2, x_1)$ $(\hat{x}_2 > 0)$ with transition probabilities

It immediately follows that $T(0, x_1) = 0$ for $x_1 > 0$. When $\hat{x}_2 > 0$, we can derive

$T(\hat{x}_2, x_1)$ as follows.

$$T(\hat{x}_2, x_1) = \begin{cases} \frac{1}{\lambda_1} + T(\hat{x}_2 - 1, x_1 + 1) & \text{if } x_1 = 0, \\ \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(\hat{x}_2 - 1, x_1 + 1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(\hat{x}_2, x_1 - 1) & \text{if } 0 < x_1 < S, \\ \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(\hat{x}_2, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(\hat{x}_2, x_1 - 1) & \text{if } x_1 = S, \\ \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(\hat{x}_2, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(\hat{x}_2 - 1, x_1 - 1) & \text{if } S < x_1 < K, \\ \frac{1}{S\mu} + T(\hat{x}_2 - 1, x_1 - 1) & \text{if } x_1 = K. \end{cases} \tag{4.1.1}$$

In what follows, we derive population-2 agents' strategies in the form of a multithreshold vector of maximum positions (at which they are willing to join) corresponding to a specific number of population-1 agents present in the system in Theorem 4.1.1, which is obtained by Propositions 4.1.1 and 4.1.3. We present four propositions that show noticeable properties of expected waiting times and the derived threshold strategy. To be more specific, Proposition 4.1.1 shows the monotone property of $T(\hat{x}_2, x_1)$ with respect to $\hat{x}_2$. Proposition 4.1.2 shows the monotone property of $T(\hat{x}_2, x_1)$ with respect to x_1, which is obtained by Lemmas 4.1.1–4.1.3. Proposition 4.1.3 shows that the expected waiting time reaches infinity as $\hat{x}_2$ reaches infinity. Proposition 4.1.4 shows the monotone property of thresholds.

Proposition 4.1.1 (Monotone property of expected waiting times with respect to the position). $T(\hat{x}_2, x_1) \leq T(\hat{x}_2 + 1, x_1)$ *for any fixed value of* x_1.

Proof. We will prove Proposition 4.1.1 by induction on $\hat{x}_2$. The statement is equivalent to

$$T(\hat{x}_2, x_1) \leq T(\hat{x}_2 + 1, x_1), \tag{4.1.2}$$

for any fixed values of x_1.

Since $T(0, x_1) = 0$, it is obviously implied from the recursive formulas that $T(0, x_1) \leq T(1, x_1)$; thus, (4.1.2) holds with $\hat{x}_2 = 0$. Assuming that (4.1.2) holds with $\hat{x}_2 = q - 1$ for any integer $q \geq 1$, which indicates, for any fixed value of x_1,

$$T(q - 1, x_1) \leq T(q, x_1). \tag{4.1.3}$$

We show that it holds with $\hat{x}_2 = q$, which indicates that we need to prove that, for any fixed value of x_1,

$$T(q, x_1) \leq T(q+1, x_1),$$

by considering the following 5 cases.

- When $x_1 = K$, from (4.1.1) we have

$$T(q, x_1) = \frac{1}{S\mu} + T(q-1, K-1), \tag{4.1.4}$$

  and

$$T(q+1, x_1) = \frac{1}{S\mu} + T(q, K-1). \tag{4.1.5}$$

  Since $T(q-1, K-1) \leq T(q, K-1)$ by assumption (4.1.3), from (4.1.4) and (4.1.5), we obtain

$$T(q, x_1) \leq T(q+1, x_1) \text{ for } x_1 = K. \tag{4.1.6}$$

- When $S < x_1 < K$, from (4.1.1) we have

$$T(q, x_1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, x_1 - 1),$$

  and

$$T(q+1, x_1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q+1, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(q, x_1 - 1).$$

  Now, due to (4.1.6), it is seen that the inequality $T(q, x_1) \leq T(q+1, x_1)$ holds for $x_1 = K - 1$ because

$$\begin{aligned} T(q, K-1) &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, K) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, K-2) \\ &\leq \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q+1, K) + \frac{S\mu}{\lambda_1 + S\mu} T(q, K-2) \\ &= T(q+1, K-1). \end{aligned}$$

Then, it is easily obtained by induction on x_1, that

$$T(q, x_1) \leq T(q+1, x_1) \text{ for } S < x_1 < K. \tag{4.1.7}$$

- When $x_1 = 0$, from (**??**) we have

$$T(q, 0) = \frac{1}{\lambda_1} + T(q-1, 1), \tag{4.1.8}$$

and

$$T(q+1, 0) = \frac{1}{\lambda_1} + T(q, 1). \tag{4.1.9}$$

Since $T(q-1, 1) \leq T(q, 1)$ because of the inductive assumption, from (4.1.8) and (4.1.9) we obtain

$$T(q, x_1) \leq T(q+1, x_1) \text{ for } x_1 = 0. \tag{4.1.10}$$

- When $0 < x_1 < S$, from (4.1.1) we have

$$T(q, x_1) = \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(q-1, x_1+1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(q, x_1 - 1),$$

and

$$T(q+1, x_1) = \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(q, x_1+1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(q+1, x_1 - 1),$$

Now, due to (4.1.10), it is seen that the inequality $T(q, x_1) \leq T(q+1, x_1)$ holds for $x_1 = 1$ because

$$\begin{aligned} T(q, 1) &= \frac{1}{\lambda_1 + \mu} + \frac{\lambda_1}{\lambda_1 + \mu} T(q-1, 2) + \frac{\mu}{\lambda_1 + \mu} T(q, 0) \\ &\leq \frac{1}{\lambda_1 + \mu} + \frac{\lambda_1}{\lambda_1 + \mu} T(q, 2) + \frac{\mu}{\lambda_1 + \mu} T(q+1, 0) \\ &= T(q+1, 1). \end{aligned}$$

Then, it is easily obtained by induction on x_1, that

$$T(q, x_1) \leq T(q+1, x_1) \text{ for } 0 < x_1 < S. \tag{4.1.11}$$

- Last, when $x_1 = S$, from (4.1.1) we have

$$T(q, x_1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, S+1) + \frac{S\mu}{\lambda_1 + S\mu} T(q, S-1), \tag{4.1.12}$$

  and

$$T(q+1, x_1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q+1, S+1) + \frac{S\mu}{\lambda_1 + S\mu} T(q+1, S-1). \tag{4.1.13}$$

  However, note that $T(q, S+1) \leq T(q+1, S+1)$ and $T(q, S-1) \leq T(q+1, S-1)$ (implied from results (4.1.7) and (4.1.11)). Therefore, from (4.1.12) and (4.1.13), we obtain

$$T(q, x_1) \leq T(q+1, x_1) \text{ for } x_1 = S. \tag{4.1.14}$$

Equations (4.1.6), (4.1.7), (4.1.10), (4.1.11), (4.1.14) complete our proof. □

This result is intuitive in the sense that with the same number of population-1 agents in the system, the farther population-2 agents are away from the matching points, the longer they need to wait. However, looking at the recursive formulas, we can see that such a relationship between $T(\hat{x}_2, x_1)$ and $T(\hat{x}_2 + 1, x_1)$ is not mathematically trivial because the waiting time function is a piecewise-defined function with two variables.

Lemma 4.1.1.

$$T(1, x_1) = \begin{cases} \frac{1}{\lambda_1} & \text{if } 0 \leq x_1 \leq S-1, \\ \frac{\lambda_1^2 + (S\mu)^2 + \lambda_1 (S\mu)}{\lambda_1 (S\mu)(\lambda_1 + S\mu)} & \text{if } x_1 = S, \\ \frac{1}{S\mu} & \text{if } S+1 \leq x_1 \leq K. \end{cases}$$

Proof. We prove Lemma 4.1.1 by induction on x_1.

First, note that

$$T(1, 0) = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1 + S\mu} T(0, 1) = \frac{1}{\lambda_1},$$

and

$$T(1, K) = \frac{1}{S\mu} + \frac{S\mu}{\lambda_1 + S\mu} T(0, K-1) = \frac{1}{S\mu}.$$

By induction on x_1, we have

$$\begin{aligned} T(1, x_1) &= \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(0, x_1 - 1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(1, x_1 - 1) \\ &= \frac{1}{\lambda_1 + x_1\mu} + \frac{x_1\mu}{\lambda_1 + x_1\mu} \cdot \frac{1}{\lambda_1} \\ &= \frac{1}{\lambda_1}, \end{aligned}$$

for $1 \leq x_1 \leq S - 1$; and

$$\begin{aligned} T(1, x_1) &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(1, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(0, x_1 - 1) \\ &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} \cdot \frac{1}{S\mu} \\ &= \frac{1}{S\mu}, \end{aligned}$$

for $S + 1 \leq x_1 \leq K - 1$.

Finally,

$$\begin{aligned} T(1, S) &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(1, S + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(1, S - 1) \\ &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} \cdot \frac{1}{S\mu} + \frac{S\mu}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} \\ &= \frac{\lambda_1^2 + (S\mu)^2 + \lambda_1 (S\mu)}{\lambda_1 (S\mu) (\lambda_1 + S\mu)}. \end{aligned}$$

It can also be noted that

$$
\begin{aligned}
T(1,S) - T(1,S-1) &= \frac{\lambda_1^2 + (S\mu)^2 + \lambda_1 (S\mu)}{\lambda_1 (S\mu) (\lambda_1 + S\mu)} - \frac{1}{\lambda_1} \\
&= \frac{S\mu}{\lambda_1(\lambda_1 + S\mu)} > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
T(1,S) - T(1,S+1) &= \frac{\lambda_1^2 + (S\mu)^2 + \lambda_1 (S\mu)}{\lambda_1 (S\mu) (\lambda_1 + S\mu)} - \frac{1}{S\mu} \\
&= \frac{\lambda_1}{S\mu(\lambda_1 + S\mu)} > 0.
\end{aligned}
$$

□

This result explicitly shows expected waiting times when population-2 agents are one step away from the matching points. When there are fewer than S population-1 agents, population-2 agents are expected to wait for $\frac{1}{\lambda_1}$ units of time. When there are more than S population-1 agents, population-2 agents are expected to wait for $\frac{1}{S\mu}$ units of time. Meanwhile, the expected waiting time for those who observe exactly S population-1 agents in the system is higher at $\frac{\lambda_1^2+(S\mu)^2+\lambda_1(S\mu)}{\lambda_1(S\mu)(\lambda_1+S\mu)}$ units of time.

We use this result to prove the following two lemmas.

Lemma 4.1.2. *For any fixed value of* $\hat{x}_2$*,*

$$
T(\hat{x}_2, x_1) \leq \frac{1}{x_1\mu} + T(\hat{x}_2, x_1 - 1),
$$

for $1 \leq x_1 \leq S - 1$.

Proof. We prove this lemma by induction on $\hat{x}_2$. We can easily see that it holds with $\hat{x}_2 = 0$ and $\hat{x}_2 = 1$ due to Lemma 4.1.1. Assume that it holds with $\hat{x}_2 = q - 1$ for any integer $q \geq 2$. Additionally, assume that when $\hat{x}_2 = q - 1$, the inequality holds in the case of $x_1 = S$ (we show that, under the same inductive assumptions, it also holds when $\hat{x}_2 = q$ and $x_1 = S$ later in Proposition 4.1.2). Then, from assumptions we have

$$T(q-1,x_1) \leq \frac{1}{x_1\mu} + T(q-1,x_1-1), \text{ for } 1 \leq x_1 \leq S.$$

We show that the inequality holds with $\hat{x}_2 = q$, which indicates that we need to prove that

$$T(q,x_1) \leq \frac{1}{x_1\mu} + T(q,x_1-1), \text{ for } 1 \leq x_1 \leq S-1. \tag{4.1.15}$$

Assume there exists $1 \leq x_1 \leq S-1$ such that

$$T(q,x_1) > \frac{1}{x_1\mu} + T(q,x_1-1). \tag{4.1.16}$$

From (4.1.1) we have

$$\begin{aligned} T(q,x_1) &= \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(q-1,x_1+1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(q,x_1-1) \\ &< \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} T(q-1,x_1+1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} \left(T(q,x_1) - \frac{1}{x_1\mu} \right) \\ &= \frac{\lambda_1}{\lambda_1 + x_1\mu} T(q-1,x_1+1) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(q,x_1), \end{aligned}$$

which is equivalent to

$$T(q,x_1) < T(q-1,x_1+1). \tag{4.1.17}$$

On the other hand, we also have

$$T(q-1,x_1+1) < \frac{1}{(x_1+1)\mu} + T(q-1,x_1), \tag{4.1.18}$$

according to the inductive assumption. From (4.1.16), (4.1.17) and (4.1.18), we obtain

$$\frac{1}{x_1\mu} + T(q,x_1-1) < \frac{1}{(x_1+1)\mu} + T(q-1,x_1+1),$$

which implies

$$T(q,x_1-1) < T(q-1,x_1). \tag{4.1.19}$$

Additionally, from (4.1.1) we have

$$\begin{aligned}
&T(q,x_1-1)\\
&=\frac{1}{\lambda_1+(x_1-1)\mu}+\frac{\lambda_1}{\lambda_1+(x_1-1)\mu}T(q-1,x_1)+\frac{(x_1-1)\mu}{\lambda_1+(x_1-1)\mu}T(q,x_1-2)\\
&>\frac{1}{\lambda_1+(x_1-1)\mu}+\frac{\lambda_1}{\lambda_1+(x_1-1)\mu}T(q,x_1-1)+\frac{(x_1-1)\mu}{\lambda_1+(x_1-1)\mu}T(q,x_1-2)
\end{aligned}$$

(due to (4.1.19)). This implies

$$T(q,x_1-1)>\frac{1}{(x_1-1)\mu}+T(q,x_1-2).$$

By induction on x_1, it finally implies

$$T(q,1)>\frac{1}{\mu}+T(q,0). \tag{4.1.20}$$

However,

$$\begin{aligned}
T(q,1)-T(q,0)&=\left(\frac{1}{\lambda_1+\mu}+\frac{\lambda_1}{\lambda_1+\mu}T(q-1,2)+\frac{\mu}{\lambda_1+\mu}T(q,0)\right)-T(q,0)\\
&=\frac{1}{\lambda_1+\mu}+\frac{\lambda_1}{\lambda_1+\mu}T(q-1,2)-\frac{\lambda_1}{\lambda_1+\mu}T(q,0)\\
&=\frac{1}{\lambda_1+\mu}+\frac{\lambda_1}{\lambda_1+\mu}T(q-1,2)-\frac{\lambda_1}{\lambda_1+\mu}\left(\frac{1}{\lambda_1}+T(q-1,1)\right)\\
&=\frac{\lambda_1}{\lambda_1+\mu}\left(T(q-1,2)-T(q-1,1)\right)\\
&\leq\frac{\lambda_1}{\lambda_1+\mu}\cdot\frac{1}{2\mu}\qquad\text{(due to the inductive assumption)}\\
&<\frac{1}{\mu},
\end{aligned}$$

which contradicts (4.1.20). This indicates that (4.1.15) holds and thus completes the proof. $\square$

Lemma 4.1.3. *For any fixed value of $\hat{x}_2$,*

$$T(\hat{x}_2, x_1 - 1) \leq \frac{1}{\lambda_1} + T(\hat{x}_2, x_1),$$

for $S + 2 \leq x_1 \leq K$.

Proof. We prove this lemma by induction on $\hat{x}_2$. We can easily see that it holds with $\hat{x}_2 = 0$ and $\hat{x}_2 = 1$ due to Lemma 4.1.1. Assume that it holds with $\hat{x}_2 = q - 1$ for any integer $q \geq 2$. Additionally, assume that when $\hat{x}_2 = q - 1$, the inequality holds in the case of $x_1 = S$ (we will show that, under the same inductive assumptions, it also holds when $\hat{x}_2 = q$ and $x_1 = S$ later in Proposition 4.1.2). Then, from assumptions we have

$$T(q-1, x_1 - 1) \leq \frac{1}{\lambda_1} + T(q-1, x_1), \text{ for } S+1 \leq x_1 \leq K.$$

We show that the inequality holds with $\hat{x}_2 = q$, which indicates that we need to prove that

$$T(q, x_1 - 1) \leq \frac{1}{\lambda_1} + T(q, x_1), \text{ for } S+2 \leq x_1 \leq K. \tag{4.1.21}$$

First, notice that

$$\begin{aligned}
&T(q, K-1) - T(q, K) \\
&= \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, K) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, K-2) \right) - T(q, K) \\
&= \frac{1}{\lambda_1 + S\mu} - \frac{S\mu}{\lambda_1 + S\mu} T(q, K) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, K-2) \\
&= \frac{1}{\lambda_1 + S\mu} - \frac{S\mu}{\lambda_1 + S\mu} \left(\frac{1}{S\mu} + T(q-1, K-1) \right) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, K-2) \\
&= \frac{S\mu}{\lambda_1 + S\mu} \left(T(q-1, K-2) - T(q-1, K-1) \right) \\
&\leq \frac{S\mu}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} (\text{due to the inductive assumptions}) \\
&< \frac{1}{\lambda_1},
\end{aligned}$$

which indicates that (4.1.21) holds with $x_1 = K$. Now, we make an inductive assumption on

x_1; and for any $S+2 \leq x_1 \leq K-1$, consider the following

$$
\begin{aligned}
& T(q, x_1 - 1) - T(q, x_1) \\
&= \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, x_1) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, x_1 - 2) \right) \\
&- \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, x_1 + 1) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, x_1 - 1) \right) \\
&= \frac{\lambda_1}{\lambda_1 + S\mu} \left(T(q, x_1) - T(q, x_1 + 1) \right) + \frac{S\mu}{\lambda_1 + S\mu} \left(T(q-1, x_1 - 2) - T(q-1, x_1 - 1) \right) \\
&\leq \frac{\lambda_1}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} + \frac{S\mu}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} \text{ (due to the inductive assumptions)} \\
&= \frac{1}{\lambda_1}.
\end{aligned}
$$

□

We use these results to prove the following proposition.

Proposition 4.1.2 (Monotone property of expected waiting times with respect to the observed number of population-1 agents)**.** *$T(\hat{x}_2, x_1) \leq T(\hat{x}_2, x_1 + 1)$ for $x_1 = 0, 1, \ldots, S-1$, and $T(\hat{x}_2, x_1) \geq T(\hat{x}_2, x_1 + 1)$ for $S, S+1, \ldots, K-1$.*

Proof. We prove Proposition 4.1.2 by induction on $\hat{x}_2$. The statement is equivalent to the following inequalities.

$$T(\hat{x}_2, x_1) \leq T(\hat{x}_2, x_1 + 1), \text{ for } 0 \leq x_1 \leq S-1, \tag{4.1.22}$$

and

$$T(\hat{x}_2, x_1) \geq T(\hat{x}_2, x_1 + 1), \text{ for } S \leq x_1 \leq K-1. \tag{4.1.23}$$

We already showed that (4.1.22) and (4.1.23) hold with $\hat{x}_2 = 0$ and $\hat{x}_2 = 1$ in Lemma 4.1.1. Assuming that (4.1.22) and (4.1.23) hold with $\hat{x}_2 = q-1$ for any integer $q \geq 2$, which indicates that

$$T(q-1, x_1) \leq T(q-1, x_1 + 1), \text{ for } 0 \leq x_1 \leq S-1,$$

and

$$T(q-1,x_1) \geq T(q-1,x_1+1), \text{for } S \leq x_1 \leq K-1.$$

We show that it holds with $\hat{x}_2 = q$, which indicates that we need to prove that

$$T(q,x_1) \leq T(q,x_1+1), \text{for } 0 \leq x_1 \leq S-1,$$

and

$$T(q,x_1) \geq T(q,x_1+1), \text{for } S \leq x_1 \leq K-1.$$

by considering the following 5 cases.

- When $x_1 = 0$, consider the following

$$\begin{aligned}
T(q,1) - T(q,0) &= \left(\frac{1}{\lambda_1+\mu} + \frac{\lambda_1}{\lambda_1+\mu}T(q-1,2) + \frac{\mu}{\lambda_1+\mu}T(q,0)\right) - T(q,0) \\
&= \frac{1}{\lambda_1+\mu} + \frac{\lambda_1}{\lambda_1+\mu}T(q-1,2) - \frac{\lambda_1}{\lambda_1+\mu}T(q,0) \\
&= \frac{1}{\lambda_1+\mu} + \frac{\lambda_1}{\lambda_1+\mu}T(q-1,2) - \frac{\lambda_1}{\lambda_1+\mu}\left(\frac{1}{\lambda_1} + T(q-1,1)\right) \\
&= \frac{\lambda_1}{\lambda_1+\mu}\left(T(q-1,2) - T(q-1,1)\right) \\
&\geq 0 \qquad \text{(due to the inductive assumption)},
\end{aligned}$$

which indicates that

$$T(q,0) \leq T(q,1). \tag{4.1.24}$$

- When $1 \leq x_1 \leq S-2$, from (4.1.1) we have

$$T(q,x_1) = \frac{1}{\lambda_1+x_1\mu} + \frac{\lambda_1}{\lambda_1+x_1\mu}T(q-1,x_1+1) + \frac{x_1\mu}{\lambda_1+x_1\mu}T(q,x_1-1), \tag{4.1.25}$$

and

$$T(q, x_1 + 1) = \frac{1}{\lambda_1 + (x_1 + 1)\mu} + \frac{\lambda_1}{\lambda_1 + (x_1 + 1)\mu} T(q - 1, x_1 + 2)$$
$$+ \frac{(x_1 + 1)\mu}{\lambda_1 + (x_1 + 1)\mu} T(q, x_1). \tag{4.1.26}$$

We prove $T(q, x_1) \leq T(q, x_1 + 1)$ by contradiction. Assuming $\exists x_1,\ T(q, x_1) > T(q, x_1 + 1)$, combining with (4.1.26) we have

$$T(q, x_1) > \frac{1}{\lambda_1 + (x_1 + 1)\mu} + \frac{\lambda_1}{\lambda_1 + (x_1 + 1)\mu} T(q-1, x_1+2) + \frac{(x_1 + 1)\mu}{\lambda_1 + (x_1 + 1)\mu} T(q, x_1),$$

which is equivalent to

$$T(q, x_1) > \frac{1}{\lambda_1} + T(q - 1, x_1 + 2).$$

However, we also have $T(q - 1, x_1 + 2) \geq T(q - 1, x_1 + 1)$ (due to the inductive assumption), so

$$T(q, x_1) > \frac{1}{\lambda_1} + T(q - 1, x_1 + 1). \tag{4.1.27}$$

From (4.1.25) and (4.1.27) we obtain

$$T(q, x_1) < \frac{1}{\lambda_1 + x_1\mu} + \frac{\lambda_1}{\lambda_1 + x_1\mu} \left(T(q, x_1) - \frac{1}{\lambda_1} \right) + \frac{x_1\mu}{\lambda_1 + x_1\mu} T(q, x_1 - 1),$$

which is equivalent to

$$T(q, x_1) < T(q, x_1 - 1).$$

By induction on x_1 (by repeating the same procedure), it finally implies

$$T(q, 1) < T(q, 0),$$

which contradicts (4.1.24) that is proved above. This contradiction indicates that

$$T(\hat{x}_2, x_1) \geq T(\hat{x}_2, x_1 + 1), \text{for } 1 \leq x_1 \leq S - 2. \tag{4.1.28}$$

- When $x_1 = K-1$, consider the following

$$
\begin{aligned}
&T(q,K-1) - T(q,K)\\
&= \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu}T(q,K) + \frac{S\mu}{\lambda_1 + S\mu}T(q-1,K-2)\right) - T(q,K)\\
&= \frac{1}{\lambda_1 + S\mu} + \frac{S\mu}{\lambda_1 + S\mu}T(q-1,K-2) - \frac{S\mu}{\lambda_1 + S\mu}T(q,K)\\
&= \frac{1}{\lambda_1 + \mu} + \frac{S\mu}{\lambda_1 + S\mu}T(q-1,K-2) - \frac{S\mu}{\lambda_1 + \mu}\left(\frac{1}{S\mu} + T(q-1,K-1)\right)\\
&= \frac{S\mu}{\lambda_1 + S\mu}\left(T(q-1,K-2) - T(q-1,K-1)\right)\\
&\geq 0 \qquad \text{(due to the inductive assumption)},
\end{aligned}
$$

  which indicates that

$$T(q,K-1) \geq T(q,K). \tag{4.1.29}$$

- When $S+1 \leq x_1 \leq K-2$, from (4.1.1) we have

$$T(q,x_1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu}T(q,x_1+1) + \frac{S\mu}{\lambda_1 + S\mu}T(q-1,x_1-1),$$

  and

$$T(q,x_1+1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu}T(q,x_1+2) + \frac{S\mu}{\lambda_1 + S\mu}T(q-1,x_1).$$

  Due to the inductive assumption, we have $T(q-1,x_1-1) \geq T(q-1,x_1)$; therefore, due to (4.1.29), the inequality $T(q,x_1) \geq T(q,x_1+1)$ holds for $x_1 = K-2$. By induction on x_1, we obtain

$$T(q,x_1) \geq T(q,x_1+1) \text{ for } S+1 \leq x_1 \leq K-2.$$

Next, we prove that $T(q,S) \geq T(q,S+1)$. From (4.1.1) we have

$$T(q,S) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q,S+1) + \frac{S\mu}{\lambda_1 + S\mu} T(q,S-1), \tag{4.1.30}$$

and

$$T(q,S+1) = \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q,S+2) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1,S).$$

To prove the above inequality, we show that

$$T(q,S-1) \geq T(q-1,S). \tag{4.1.31}$$

From (4.1.1) we have

$$\begin{aligned}
&T(q,S-1) \\
&= \frac{1}{\lambda_1 + (S-1)\mu} + \frac{\lambda_1}{\lambda_1 + (S-1)\mu} T(q-1,S) + \frac{(S-1)\mu}{\lambda_1 + (S-1)\mu} T(q,S-2) \\
&\geq \frac{1}{\lambda_1 + (S-1)\mu} + \frac{\lambda_1}{\lambda_1 + (S-1)\mu} T(q-1,S) \\
&+ \frac{(S-1)\mu}{\lambda_1 + (S-1)\mu} \left(T(q,S-1) - \frac{1}{(S-1)\mu} \right) \\
&\text{(due to Lemma 4.1.2)} \\
&= \frac{\lambda_1}{\lambda_1 + (S-1)\mu} T(q-1,S) + \frac{(S-1)\mu}{\lambda_1 + (S-1)\mu} T(q,S-1),
\end{aligned}$$

which implies that (4.1.31) is true. Therefore,

$$T(q,S) \geq T(q,S+1). \tag{4.1.32}$$

From (4.1.30) and (4.1.32), we have

$$T(q,S) \leq \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q,S) + \frac{S\mu}{\lambda_1 + S\mu} T(q,S-1),$$

which implies

$$T(q, S-1) + \frac{1}{S\mu} \geq T(q, S),$$

and this also completes the proof of Lemma 4.1.2.

- Finally, we prove that $T(q, S) \geq T(q, S-1)$. To show the above equality, first note that

$$\begin{aligned}
&T(q, S-1) \\
&= \frac{1}{\lambda_1 + (S-1)\mu} + \frac{\lambda_1}{\lambda_1 + (S-1)\mu} T(q-1, S) + \frac{(S-1)\mu}{\lambda_1 + (S-1)\mu} T(q, S-2) \\
&\leq \frac{1}{\lambda_1 + (S-1)\mu} + \frac{\lambda_1}{\lambda_1 + (S-1)\mu} T(q-1, S) + \frac{(S-1)\mu}{\lambda_1 + (S-1)\mu} T(q, S-1),
\end{aligned}$$

due to (4.1.28), and this implies

$$T(q, S-1) - T(q-1, S) \leq \frac{1}{\lambda_1}. \tag{4.1.33}$$

Now, due to (4.1.33), Lemma 4.1.3 and the inductive assumptions, we have

$$\begin{aligned}
&T(q, S) - T(q, S+1) \\
&= \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, S+1) + \frac{S\mu}{\lambda_1 + S\mu} T(q, S-1) \right) \\
&- \left(\frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, S+2) + \frac{S\mu}{\lambda_1 + S\mu} T(q-1, S) \right) \\
&= \frac{\lambda_1}{\lambda_1 + S\mu} \left(T(q, S+1) - T(q, S+2) \right) + \frac{S\mu}{\lambda_1 + S\mu} \left(T(q, S-1) - T(q-1, S) \right) \\
&\leq \frac{\lambda_1}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} + \frac{S\mu}{\lambda_1 + S\mu} \cdot \frac{1}{\lambda_1} \\
&= \frac{1}{\lambda_1},
\end{aligned}$$

which indicates that

$$T(q, S) \leq \frac{1}{\lambda_1} + T(q, S+1). \tag{4.1.34}$$

(Note that this conclusion also completes the proof of Lemma 4.1.3).

Now, due to (4.1.34), we have

$$\begin{aligned} T(q,S) &= \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} T(q, S+1) + \frac{S\mu}{\lambda_1 + S\mu} T(q, S-1) \\ &\geq \frac{1}{\lambda_1 + S\mu} + \frac{\lambda_1}{\lambda_1 + S\mu} \left(T(q,S) - \frac{1}{\lambda_1} \right) + \frac{S\mu}{\lambda_1 + S\mu} T(q, S-1) \\ &= \frac{\lambda_1}{\lambda_1 + S\mu} T(q,S) + \frac{S\mu}{\lambda_1 + S\mu} T(q, S-1), \end{aligned}$$

which implies $T(q,S) \geq T(q,S-1)$.

□

Intuitively, at the same arbitrary position, in case the current number of population-1 agents in the system is greater than or equal to S, expected waiting times become shorter if there are more population-1 agents in the system. Nevertheless, we see an opposite association between the number of population-1 agents and waiting times when there are S or fewer than S population-1 agents in the system. This is because when the number of population-1 agents in the systems is smaller than or equal to the number of matching points, the number of population-2 agents being served at the matching points is equal to the number of population-1 agents. In this case, more population-1 agents indicates that more population-2 agents occupy the matching points at the time of arrival. Regardless of the number of population-1 agents in the system (which is less than or equal to S), the population-2 agent at position $\hat{x}_2$ needs a fixed number of $\hat{x}_2$ more population-1 agents for his turn to be served. If more population-2 agents match at matching points, it is more likely that the matching points becomes more "congested", which may slow down the expected waiting time.

Proposition 4.1.3 (Infinity limit of waiting times)**.**

$$\lim_{\hat{x}_2 \to +\infty} T(\hat{x}_2, x_1) = +\infty$$

for all $x_1 = 0, 1, ..., K$.

Proof. First we prove that

$$T(\hat{x}_2, 0) \geq \frac{\hat{x}_2}{\lambda_1}, \tag{4.1.35}$$

for all $\hat{x}_2 = 1, 2, \ldots$.

This inequality holds for $\hat{x}_2 = 1$ because $T(1, 0) = \frac{1}{\lambda_1}$. Assume that it also holds for $\hat{x}_2 = q \geq 1$, indicating that $T(q, 0) \geq \frac{q}{\lambda_1}$. We have

$$\begin{aligned} T(q+1, 0) &= \frac{1}{\lambda_1} + T(q, 1) \\ &\geq \frac{1}{\lambda_1} + T(q, 0) \qquad \text{(due to Proposition 4.1.2)} \\ &\geq \frac{q+1}{\lambda_1}. \end{aligned}$$

Therefore, by induction on $\hat{x}_2$, we obtain that (4.1.35) is true. $\lim_{\hat{x}_2 \to +\infty} \frac{\hat{x}_2}{\lambda_1} = +\infty$, which implies

$$\lim_{\hat{x}_2 \to +\infty} T(\hat{x}_2, 0) = +\infty.$$

By induction on x_1 using formula (4.1.1), we can easily obtain

$$\lim_{\hat{x}_2 \to +\infty} T(\hat{x}_2, x_1) = +\infty.$$

for all $x_1 = 0, 1, 2, \ldots, K$. □

This result indicates that the expected waiting time diverges to infinity if the current position is infinitely far from the matching points. We use this result to prove the following theorem.

Theorem 4.1.1 (Equilibrium strategy of population-2 agents). *population-2 agents who arrive at the system adopt a threshold strategy represented by the vector* $\boldsymbol{\eta}^{(\mathrm{s})} = (n_0^{(\mathrm{s})}, n_1^{(\mathrm{s})}, \ldots, n_K^{(\mathrm{s})})$, *where* $n_{x_1}^{(\mathrm{s})}$ *is the maximum position at which population-2 agents are willing to join the system when they observe* x_1 *population-1 agents upon arrival.*

Proof. This can be obtained from the monotonically nondecreasing property of $T(\hat{x}_2, x_1)$ with regard to $\hat{x}_2$ obtained in Proposition 4.1.1. Since $R_2 - \frac{C_2}{\mu} \geq 0$ by assumption and

$R_2 - C_2 T(\hat{x}_2, x_1) - \frac{C_2}{\mu} \to -\infty$ for all $x_1 = 0, 1, \ldots, K$ when $\hat{x}_2 \to +\infty$ (due to the result in Proposition 4.1.3), for each fixed value of x_1, there must exist $n_{x_1}^{(\mathrm{s})}$ such that $R_2 - C_2 T(n_{x_1}^{(\mathrm{s})}, x_1) - \frac{C_2}{\mu} \geq 0$ and $R_2 - C_2 T(n_{x_1}^{(\mathrm{s})} + 1, x_1) - \frac{C_2}{\mu} < 0$. $n_{x_1}^{(\mathrm{s})}$ is the threshold strategy corresponding to each fixed value of the number of population-1 agents observed upon arrival. □

Proposition 4.1.4 (Monotone property of thresholds). *$n_k^{(\mathrm{s})} \geq n_{k+1}^{(\mathrm{s})}$ for $k = 0, 1, \ldots, S-1$, and $n_k^{(\mathrm{s})} \leq n_{k+1}^{(\mathrm{s})}$ for $k = S, S+1, \ldots, K-1$.*

Proof. This can be proved by contradiction.

First, consider the case when $0 \leq x_1 \leq S-1$. We will prove $n_{x_1}^{(\mathrm{s})} \geq n_{x_1+1}^{(\mathrm{s})}$ Assume that $n_{x_1}^{(\mathrm{s})} < n_{x_1+1}^{(\mathrm{s})}$, which implies $n_{x_1}^{(\mathrm{s})} + 1 \leq n_{x_1+1}^{(\mathrm{s})}$. Since $n_{x_1}^{(\mathrm{s})}$ and $n_{x_1+1}^{(\mathrm{s})}$ are both decision thresholds, we must have $T(n_{x_1}^{(\mathrm{s})} + 1, x_1) > \frac{R}{C} - \frac{1}{\mu}$ and $T(n_{x_1+1}^{(\mathrm{s})}, x_1 + 1) \leq \frac{R}{C} - \frac{1}{\mu}$, which imply $T(n_{x_1}^{(\mathrm{s})} + 1, x_1) > T(n_{x_1+1}^{(\mathrm{s})}, x_1 + 1)$. Additionally, by the monotonic properties of $T(\hat{x}_2, x_1)$ on $\hat{x}_2$ and x_1 (obtained in Propositions 4.1.1 and 4.1.2) and the earlier assumption, we have $T(n_{x_1+1}^{(\mathrm{s})}, x_1+1) \geq T(n_{x_1}^{(\mathrm{s})}+1, x_1+1) \geq T(n_{x_1}^{(\mathrm{s})}+1, x_1)$. This contradiction indicates $n_{x_1}^{(\mathrm{s})} \geq n_{x_1+1}^{(\mathrm{s})}$ for $x_1 = 0, 1, \ldots, S-1$.

Second, consider the case when $S \leq x_1 \leq K-1$. We will prove $n_{x_1}^{(\mathrm{s})} \leq n_{x_1+1}^{(\mathrm{s})}$. Assume that $n_{x_1}^{(\mathrm{s})} > n_{x_1+1}^{(\mathrm{s})}$, which implies $n_{x_1}^{(\mathrm{s})} \geq n_{x_1+1}^{(\mathrm{s})} + 1$. Since $n_{x_1}^{(\mathrm{s})}$ and $n_{x_1+1}^{(\mathrm{s})}$ are both decision thresholds, we must have $T(n_{x_1}^{(\mathrm{s})}, x_1) \leq \frac{R}{C} - \frac{1}{\mu}$ and $T(n_{x_1+1}^{(\mathrm{s})}+1, x_1+1) > \frac{R}{C} - \frac{1}{\mu}$, which imply $T(n_{x_1+1}^{(\mathrm{s})} + 1, x_1 + 1) > T(n_{x_1}^{(\mathrm{s})}, x_1)$. Additionally, by the monotonic properties of $T(\hat{x}_2, x_1)$ on $\hat{x}_2$ and x_1 and the earlier assumption, we have $T(n_{x_1+1}^{(\mathrm{s})} + 1, x_1 + 1) \leq T(n_{x_1}^{(\mathrm{s})}, x_1 + 1) \leq T(n_{x_1}^{(\mathrm{s})}, x_1)$. This contradiction indicates $n_{x_1}^{(\mathrm{s})} \leq n_{x_1+1}^{(\mathrm{s})}$ for $x_1 = S, S+1, \ldots, K-1$. □

This result indicates that the strategy threshold decreases as the number of population-1 agents increases within the range from 0 to S. population-2 agents adopt a greater threshold when there are more population-1 agents in the system and the number of population-1 agents is larger than the number of matching points.

4.1.3 Overall optimization

Let ξ_1 and ξ_2 denote the expected number of population-1 agents and population-2 agents being diverted from the service station per unit time, respectively. Let L_1 and L_2 denote the expected queue lengths of population-1 agents and population-2 agents. Additionally, denote by C_1 the cost of staying in the system per unit time of taxi drivers and R_2 the reward that taxi drivers receive after completing serving a population-2 agent. Similarly, denote by C_2 the cost of staying in the system per unit time of population-2 agents and R_2 the reward (service value) that population-2 agents receive after being served.

Expected social welfare per unit time of all entities joining the system is then

$$SW = (\lambda_1 - \xi_1)R_1 + (\lambda_2 - \xi_2)R_2 - C_1L_1 - C_2L_2. \tag{4.1.36}$$

Since social welfare cannot be explicitly expressed in terms of threshold $\boldsymbol{\eta}$, we need to use a brute-force search method to find the maximum value of social welfare. The traditional approach is to search for a socially optimal threshold strategy in the first place and then derive a corresponding optimal fee range that adjusts the self-optimal threshold to the socially optimal threshold. Such an approach is not feasible in this multidimensional case for several reasons. First, to perform an exhaustive search to find the socially optimal threshold, it is necessary that the number of cases being considered is finite, which requires an upper limit for the population-2 agent buffer size, while we do not have this assumption. Second, even if we set a maximum buffer size of population-2 agents at m, the number of cases to be considered is $(m+1)^{K+1}$, which becomes massively large when K and m are large. Finally, even if we manage to find a socially optimal threshold strategy, there is no guarantee that it can be shifted from the original self-optimal threshold strategy by implementing a fixed value for fee θ since the strategy being considered is a vector of $K+1$ values.

In this paper, we introduce a heuristic algorithm to find an optimal policy. Assume that the administrator of the system levies a toll fee θ on each population-2 agent joining the system. We want to find an optimal range of θ that maximizes expected social welfare. With a toll fee of θ, the expected individual utility of population-2 agents becomes

$$\mathsf{E}(U_2) = R_2 - \theta - C_2\mathsf{E}(W).$$

Similar to the analysis in the previous section, when a toll fee is imposed, population-2 agents still self-optimize and adopt a threshold strategy $\boldsymbol{\eta} = (n_0, n_1, \ldots, n_K)$, which satisfies the property in Proposition 4.1.4. $\boldsymbol{\eta}$ remains unchanged as θ gradually increases within a certain fee range. When θ exceeds the upper bound of the range, some threshold element(s) n_{x_1}, which together with its corresponding number of population-1 agents x_1 yield the longest expected waiting time $T(n_{x_1}, x_1)$, decreases by 1. We then obtain a new $\boldsymbol{\eta}$ and repeat this procedure until all of the elements in $\boldsymbol{\eta}$ converge to 0, which is also the case where the taxi service becomes too expensive and population-2 agents have no incentive to engage. Based on this property, we develop an algorithm to derive ranges of toll fees and strategy thresholds that population-2 agents adopt accordingly.

Algorithm 1 Deriving fee ranges and threshold strategies

$\mathcal{T} \leftarrow \left\{ T(n_{x_1}^{(\mathsf{s})}, x_1) \middle|\, x_1 = 0, 1, \ldots, K \right\}$
$\tau \leftarrow \max \mathcal{T}$
$\delta_1 \leftarrow 0$ ▷ lower bound of fee range (initial)
$\delta_2 \leftarrow R_2 - C_2\tau - \frac{C_2}{\mu}$ ▷ upper bound of fee range (initial)
$\Delta \leftarrow [\delta_1, \delta_2]$ ▷ fee range (initial)
$\boldsymbol{\eta} = (n_0, n_1, \ldots, n_K) \leftarrow (n_0^{(\mathsf{s})}, n_1^{(\mathsf{s})}, \ldots, n_K^{(\mathsf{s})})$ ▷ threshold corresponding to fee range (initial)
$\mathcal{O} \leftarrow \{(\Delta, \eta)\}$ ▷ initial output: an initial pair of fee range and threshold
while $\boldsymbol{\eta} \neq 0$ **do** ▷ 0: zero vector
  for $T(\hat{x}_2, x_1)$ in $\mathcal{T}$ **do** ▷ updating $\boldsymbol{\eta}$
    if $T(\hat{x}_2, x_1) = \tau$ **then**
      $\mathcal{T} \leftarrow \mathcal{T} \setminus \{T(\hat{x}_2, x_1)\} \cup \{T(\hat{x}_2 - 1, x_1)\}$
      $n_{x_1} \leftarrow n_{x_1} - 1$
    end if
  end for
  $\tau \leftarrow \max \mathcal{T}$
  $\delta_1 \leftarrow \delta_2$
  $\delta_2 \leftarrow R_2 - C_2\tau - \frac{C_2}{\mu}$
  $\Delta \leftarrow (\delta_1, \delta_2]$ ▷ updating Δ
  $\mathcal{O} \leftarrow \mathcal{O} \cup \{(\Delta, \eta)\}$ ▷ updating set of output
end while
return $\mathcal{O}$

For each pair of (Δ, η) obtained from Algorithm 2, we can correspondingly derive a value of social welfare by the following procedure.

We then obtain the infinitesimal generator $\mathcal{Q}$ of the Markov chain modeling the system in equilibrium as below.

$$\mathcal{Q} = \begin{pmatrix} \mathcal{B}^{(0)} & \mathcal{C}^{(0)} & & & & \\ \mathcal{A}^{(1)} & \mathcal{B}^{(1)} & \mathcal{C}^{(1)} & & & \\ & \mathcal{A}^{(2)} & \mathcal{B}^{(2)} & \mathcal{C}^{(2)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathcal{A}^{(K-1)} & \mathcal{B}^{(K-1)} & \mathcal{C}^{(K-1)} \\ & & & & \mathcal{A}^{(K)} & \mathcal{B}^{(K)} \end{pmatrix},$$

where block matrices $\mathcal{A}^{(x_1)}$, $\mathcal{B}^{(x_1)}$ and $\mathcal{C}^{(x_1)}$ are dimensionally nonhomogeneous and defined as follows.

- $\mathcal{Q}$ has neither zero columns nor zero rows.
- $\mathcal{C}^{(x_1)}, \mathcal{B}^{(x_1+1)}, \mathcal{A}^{(x_1+2)}$ have the same number of columns for $x_1 = 0, 1, ..., K-2$. $\mathcal{B}^{(0)}$ and $\mathcal{A}^{(1)}$ have the same number of columns. $\mathcal{C}^{(K-1)}$ and $\mathcal{B}^{(K)}$ have the same number of columns.
- $\mathcal{A}^{(x_1)}, \mathcal{B}^{(x_1)}, \mathcal{C}^{(x_1)}$ have the same number of rows for $x_1 = 1, 2, ..., K-1$. $\mathcal{B}^{(0)}$ and $\mathcal{C}^{(0)}$ have the same number of rows. $\mathcal{A}^{(K)}$ and $\mathcal{B}^{(K)}$ have the same number of rows.
- $\mathcal{C}^{(x_1)}_{x_2,x_2} = \lambda_1$, for

$$x_2 = 0, 1, ..., \max\left\{x_1 + \max_{l \geq x_1}\left(\min\{l, S\} + n_l - l\right), \max_{l \leq x_1}\left(\min\{l, S\} + n_l\right)\right\}.$$

- $\mathcal{A}^{(x_1)}_{x_2,x_2-1} = \min(x_2\mu, x_1\mu, S\mu)$, for

$$x_2 = 1, 2, ..., \max\left\{x_1 + \max_{l \geq x_1}\left(\min\{l, S\} + n_l - l\right), \max_{l \leq x_1}\left(\min\{l, S\} + n_l\right)\right\}.$$

- $\mathcal{B}^{(x_1)}_{x_2,x_2+1} = \lambda_2$, for $x_2 = 0, 1, ..., \min\{x_1, S\} + n_{x_1} - 1$.
- All other elements of $\mathcal{Q}$ which do not lie on the main diagonal are 0.

- $\mathcal{Q}_{x_2,x_2} = -\sum_{x_1 \neq x_2} \mathcal{Q}_{x_2,x_1}$.

We can then derive the steady state probabilities defined as $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, ..., \boldsymbol{\pi}_K)$, where $\boldsymbol{\pi}_{x_1} = (\pi_{x_1,0}, \pi_{x_1,1}, ...)$ is a vector encoding all probabilities when there are x_1 population-1 agents in the system at the steady state by solving the following equations:

$$\begin{cases} \boldsymbol{\pi}\mathcal{Q} = 0, \\ \boldsymbol{\pi}\mathbf{e} = 1, \end{cases}$$

where 0 is a zero vector of appropriate dimension, and $\mathbf{e}$ is a vector of appropriate dimension with all elements equal to 1.

The balking probability of population-1 agents is then given by $\boldsymbol{\pi}_K\mathbf{e}$, and the balking probability of population-2 agents is

$$\sum_{x_1=0}^{K} \sum_{x_2 \geq n_{x_1}+\min\{x_1,S\}} \pi_{x_1,x_2}.$$

The numbers of population-1 agents and population-2 agents diverted from the system per unit time are, respectively given by

$$\xi_1 = \lambda_1 \boldsymbol{\pi}_K \mathbf{e},$$

and

$$\xi_2 = \lambda_2 \sum_{x_1=0}^{K} \sum_{x_2 \geq n_{x_1}+\min\{x_1,S\}} \pi_{x_1,x_2}.$$

The mean lengths of the population-1 and population-2 agent queue are, respectively given by

$$L_1 = \sum_{x_1=0}^{K} \sum_{x_2} x_1 \pi_{x_1,x_2},$$

$$L_2 = \sum_{x_1=0}^{K} \sum_{x_2} x_2 \pi_{x_1,x_2}.$$

Substituting ξ_1, ξ_2, L_1, L_2 into (4.1.36), we obtain the value of social welfare with respect to the fee range Δ and threshold strategy $\boldsymbol{\eta}$ of population-2 agents. Comparing all obtained values of social welfare, we acquire the maximum social welfare together with the corresponding fee range and threshold vector, which yield that optimal value.

4.1.4 Revenue maximization

We examine the case in which the owner of the platform aims to maximize their revenue by imposing a toll fee of θ on each population-2 agent. We consider the following two scenarios. In the first scenario, the platform owner collects a fixed fee for a seasonal toll pass from taxi companies. In this case, revenue maximization is equivalent to maximizing revenue from population-2 agents. Thus, the objective function of the platform owner is given by

$$M_1 = (\lambda_2 - \xi_2)\theta.$$

In another scenario, the platform owner also levies a toll fee for each entrance of a population-1 agent, denoted θ_1. Assume that this amount is already fixed in advance. The objective function in this case is given by

$$M_2 = (\lambda_2 - \xi_2)\theta + (\lambda_1 - \xi_1)\theta_1.$$

In both scenarios, it is easily seen that the optimal value of the revenue is attained at one of the fee range upper bounds because all other parameters remain unchanged within the fee range. Since we already obtained all possible fee ranges and corresponding parameters in the previous section, it is possible to compare the revenue in all cases and find the maximum revenue similarly.

4.1.5 Numerical analysis

In this section, we illustrate the results with a specific numerical example. Set $\lambda_1 = 6, \lambda_2 = 7, \mu = 12, S = 4, K = 15, R_1 = 18, R_2 = 20, C_1 = 5, C_2 = 5, \theta_1 = 10$. Calculated results show that

- When there is no intervention from the administrators, population-2 agents adopt threshold strategy

$$\eta^{(s)} = (23, 23, 23, 23, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 33)$$

corresponding to the number of population-1 agents being observed upon arrival ranging from 0 to 15. This results in an expected social welfare of 136.221.

- Figure 4.1.4 shows that social welfare is discretely unimodal with respect to fee ranges, and peaks at 203.122 when a fee ranging in $(16.250, 16.854]$ is imposed on each entrance. Within this fee range, population-2 agents adopt strategy

$$\eta^{\circ} = (3, 3, 3, 3, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14).$$

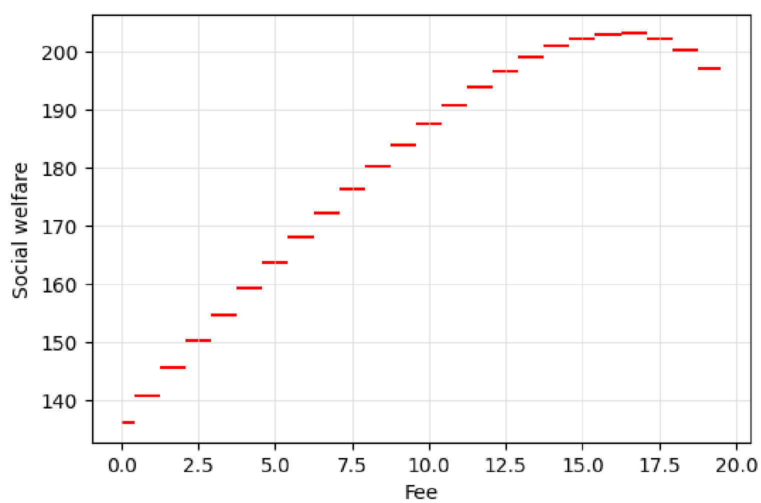


Figure 4.1.4: Social welfare with respect to imposed fee

- The graph in Fig. 4.1.5, which is noncontinuous (only continuous within each range of toll fees), represents the relationship between the revenue from population-2 agents and the toll fee levied on them. The maximum revenue from population-2 agents is

114.893 when the platform charges a toll fee of 19.479 monetary units per population-2 agent entrance.

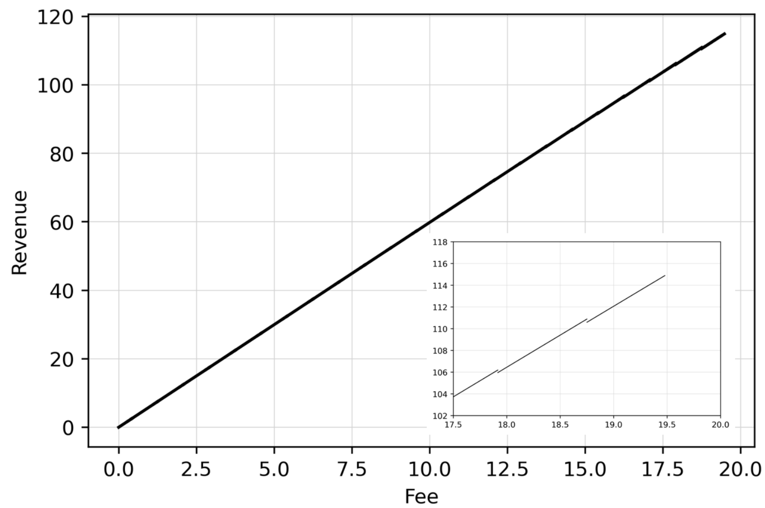


Figure 4.1.5: Revenue from population-2 agents with respect to imposed fee

- Figure 4.1.6 shows that the total revenue is maximum at 173.875 when the platform charges a toll fee of 19.479 monetary units per entrance of population-2 agents.

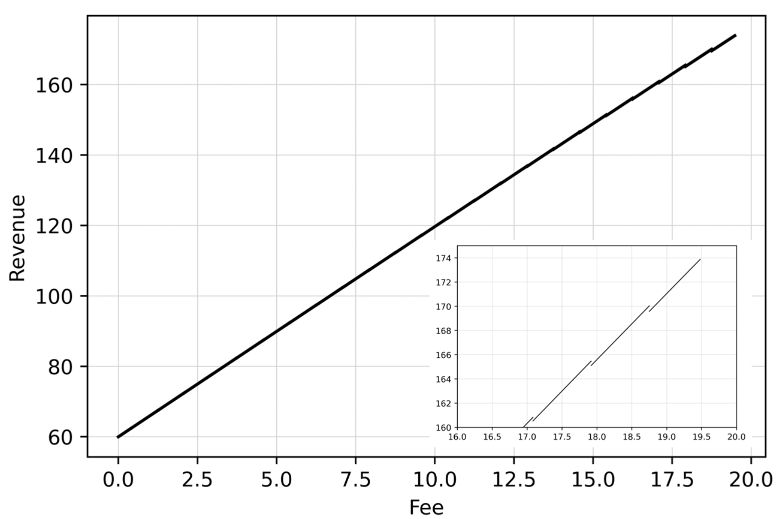


Figure 4.1.6: Platform's revenue with respect to imposed fee

When the fee is larger than 19.479, population-2 agents have no incentive to join the system since their expected utility becomes negative regardless of the number of population-1 agents observed in the system.

- Finally, we present a sensitivity analysis of population-2 agents' strategic behavior with respect to nonnegligible matching times. Except for the mean matching time, which is let vary, all parameters remain the same as in the previous experiments. The results are shown in Fig. 4.1.7.

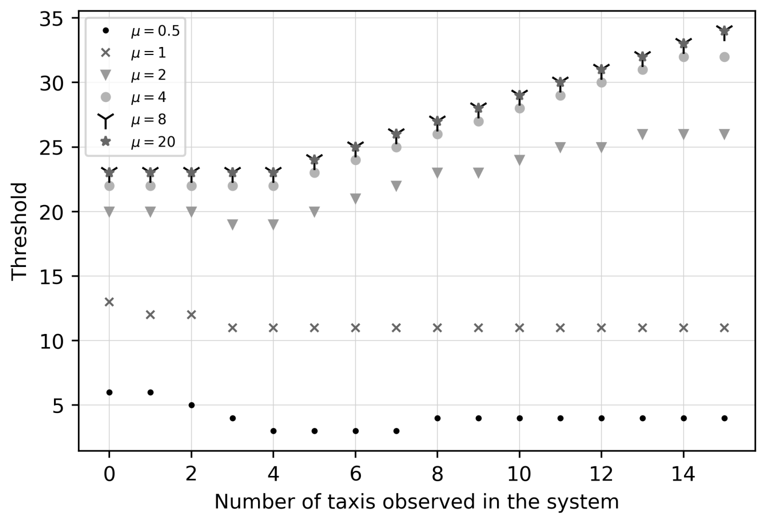


Figure 4.1.7: Sensitivity of population-2 agents' strategic behavior with respect to mean matching time

It can be observed that the thresholds adopted by population-2 agents increase with an increased matching rate μ at first and then remain unchanged as μ becomes larger. Intuitively, as matching times becomes smaller, population-2 agents' expected sojourn times also decrease, so they are willing to join a longer queue.

4.2 Model 5: The rational outcome and optimal designs of a two-population queueing game in an observable queueing system with a single matching point and two infinite ends

In this section, we consider an observable, double-ended queueing system in which both agent populations are strategic and perform matching in nonzero times. Since there is a time axis along which agents arrival at the queue, this game can be categorized as a combination of many sequential matching games, and poses several challenges beyond those posed by traditional sequential games and one-population queueing games. Compared to well-known sequential games such as Tic-Tac-Toe, Pay-raise Voting Game and Entrant–Incumbent game, the current game setting entails the involvement of many more players beyond the two main players who are supposed to match with each other. Furthermore, in most basic one-population queueing game settings, later comers' strategy does not affect the expected waiting times of those who are already enqueued, whereas a newcomer in the current two-population setting may need to account for the joining strategies of all agents arriving afterward until he can successfully match. For example, in the M/M/1 observable queueing game[38], whether or not later comers join the queue, the expected waiting time of an enqueued agent at position n is always $\frac{n}{\mu}$ (given that service times follow an exponential distribution with rate μ). In the model considered in Section 4.1, although the expected waiting times are derived two-dimensionally with respect to the position and the number of agents in the other population in the system, those waiting times are not affected by any strategic later-arriving agents, since one population is not strategic. Meanwhile, in the present two-population matching game, the expected waiting time of a tagged agent arriving at an empty system depends on the joining strategy of the soonest arriving agent in the counterpart population. As such, to derive conditional expected waiting times of a tagged agent, we need to adopt one more dimension, that is, the number of agents enqueued behind him. The waiting time function also contains the (unknown) joining probabilities of agents at future states which prescribe joining strategies of later-coming agents, which increases the problem's complexity.

4.2.1 Preliminaries

Consider a double-ended queueing system with two agent populations, with one population arriving at each side of the queue according to Poisson processes, for matching. population-1 agents arrive with rate λ_1, while population-2 agents arrive with rate λ_2. Matching is performed between a pair (comprising a population-1 agent and a population-2 agent) in a random time that follows an exponential distribution with rate μ, on a first-come, first-served basis. The buffer capacity on each side is infinite. The Markov chain describing the system is described by two variables, $X_1(t)$ and $X_2(t)$, that, respectively, represent the number of population-1 agents and the number of population-2 agents present in the system at time t. The state space is given by $\mathcal{S} = \{\mathbf{s} = (x_1, x_2) | \, (x_1, x_2) \in \mathbb{N}^2\}$. In a non-strategic queueing scenario, the system's transition diagram can be illustrated as in Figure 4.2.1.

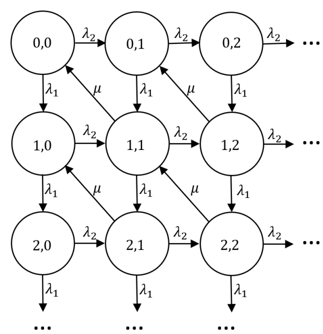


Figure 4.2.1: Transition diagram in the non-strategic scenario.

Let V_i denote the service value (before applying a fee) of population-i agents, C_i denote the waiting cost rate of population-i agents ($i = 1, 2$), and p denote the service price. We consider a scenario in which the two populations of agents represent a market with a supply side (population-1) and a demand side (population-2). Let R_i denote the net reward of population-i agents. It then follows that $R_1 = V_1 + p$ and $R_2 = V_2 - p$. The price p can be also interpreted as a transfer payment, which does not impact the equilibrium analysis, but will be important in the social welfare setting.

Assume that $R_i - C_i \left(\frac{1}{\lambda_{\tilde{i}}} + \frac{1}{\mu} \right) > 0$.

We use the index "$\tilde{i}$" to refer to a population other than i. In other words, $\tilde{i} = 2$ if $i = 1$, and vice versa.

4.2.2 Expected waiting times

Let

$$\boldsymbol{\sigma}^{(i)} = \begin{pmatrix} \sigma^{(i)}_{(0,0)} & \sigma^{(i)}_{(0,1)} & \cdots \\ \sigma^{(i)}_{(1,0)} & \sigma^{(i)}_{(1,1)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $\sigma^{(i)}_{(x_1,x_2)}$ denotes the joining probability of population-i agents when they observe state (x_1, x_2).

Denote by $T_i(x_1, x_2)$ the conditional expected waiting time of a population-i agent encountering a system state (x_1, x_2) (for example, $T_1(x_1, x_2)$ prescribes the expected waiting time of a population-1 agent choosing to join the queue and observing x_1 population-1 agents excluding himself). $T_i(x_1, x_2)$ can be derived by first-hitting-time analysis. However, to this end, we need to employ one more variable, that is, the number of population-i agents enqueued behind our tagged agent, denoted y_i. Also, to track the position of a tagged population-i agent, it is more convenient to use a position variable, denoted $\hat{x}_i$, in place of x_i. This variable represents how far the tagged population-i agent is from the matching point. We thus let $W_i(\hat{x}_i, x_{\tilde{i}}, y_i)$ denote the expected waiting time of a population-i agent, positioned at $\hat{x}_i$, observing $x_{\tilde{i}}$ population-$\tilde{i}$ agents in the system, and y_i population-i)agents behind him. It immediately follows that $W_i(0, x_{\tilde{i}}, y_i) = 0$. According to the first-hitting-time analysis, the expected waiting time of a population-i agent who encounters state $(\hat{x}_i, x_{\tilde{i}}, y_i)$, where $\hat{x}_i > 0$, can be recursively calculated as

$$W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = \begin{cases} \frac{1}{\sigma^{(i)}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\sigma^{(i)}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_i}{\sigma^{(i)}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}, y_i+1) & \\ + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\sigma^{(i)}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i-1, x_{\tilde{i}}+1, y_i) & \text{if } x_{\tilde{i}} = 0, \\ \frac{1}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} + \frac{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i, x_{\tilde{i}}, y_i+1) & \\ + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i, x_{\tilde{i}}+1, y_i) & \\ + \frac{\mu}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i, x_{\tilde{i}}-1, y_i) & \text{if } x_{\tilde{i}} = 1, \\ \frac{1}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} + \frac{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i, x_{\tilde{i}}, y_i+1) & \\ + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i, x_{\tilde{i}}+1, y_i) & \\ + \frac{\mu}{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i+\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}+\mu} W_i(\hat{x}_i-1, x_{\tilde{i}}-1, y_i) & \text{if } x_{\tilde{i}} > 1. \end{cases} \tag{4.2.1}$$

To derive an agent's joining strategy, it is necessary to obtain that agent's expected waiting time upon first arriving, i.e. with no one else queueing behind. To this end, notice that

$$T_i(x_1, x_2) = W_i(x_i + 1 - \min\{x_{\tilde{i}}, 1\}, x_{\tilde{i}}, 0).$$

Remark 4.2.1. *It is not our purpose to derive all possible Nash equilibria (among which there exist subgame imperfections that are not rational outcomes). Even without doing so, however, it is possible to show that multiple Nash equilibria exist in specific examples of this game. Given a social profile in a Nash equilibrium (for example, the equilibrium in which no one joins an empty system, or the subgame perfect Nash equilibrium which will be derived in the next section), we can assign arbitrary values to transient states as long as these transient states do not have any effects on the expected waiting time at recurrent states.*

Consider possible Nash equilibria in which, corresponding to the number of population-$\tilde{i}$ agents in the system, $\sigma^{(i)}_{(x_i,x_{\tilde{i}})} = 1$ for all $x_i \leq \nu^{(i)}_{x_{\tilde{i}}}$, and $\sigma^{(i)}_{(x_i,x_{\tilde{i}})} = 0$ for all $x_i > \nu^{(i)}_{x_{\tilde{i}}}$. We define such strategies as *threshold strategies*. A threshold strategy of population-i agents is that, when there are $x_{\tilde{i}}$ population-$\tilde{i}$ agents in the system, a population-i agent balks if he observes more than $\nu^{(i)}_{x_{\tilde{i}}}$ other population-i in the system, and joins otherwise. In terms of notation, if $\boldsymbol{\nu}^{(i)} = (\nu^{(i)}_0, \nu^{(i)}_1, \ldots)$ is the threshold strategy of population-i agents in equilibrium, we can write $\boldsymbol{\nu}^{(i)} = \mathcal{F}_i(\boldsymbol{\nu}^{(1)}, \boldsymbol{\nu}^{(2)})$.

4.2.3 The rational outcome

An example of the sequential game

In this section, we present an example of the sequential game in which a backward induction is adopted to find the subgame perfect Nash equilibrium of the game. First, we consider how backward induction works in an example of the game. Consider a tagged population-i agent who arrives at an empty system, that is, he encounters state $(0, 0)$ upon arrival. The joining strategy of this agent depends directly on the joining strategy of the soonest arriving population-$\tilde{i}$ agent who will match with the tagged agent. This game can be diagrammed as in Figure 4.2.2.

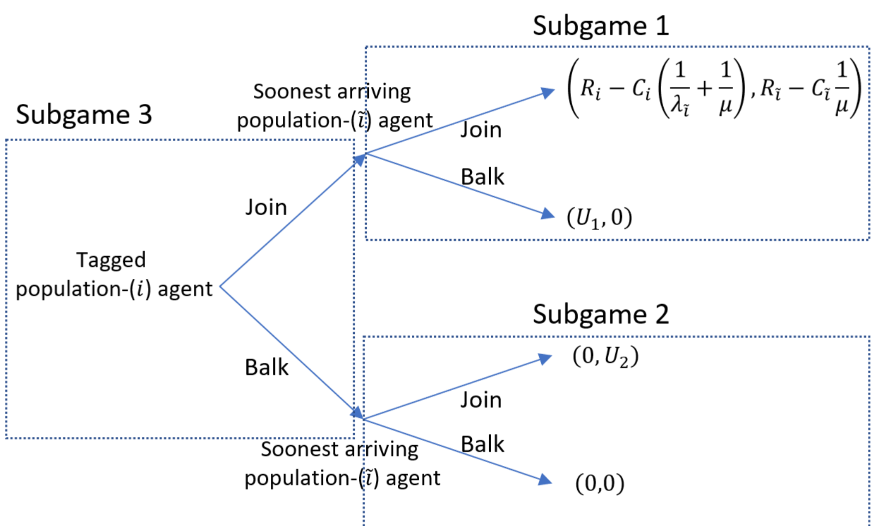


Figure 4.2.2: A sequential game at state $(0, 0)$.

There are three subgames in this example. Using backward induction, we can induce the agents' actions as follows:

- Subgame 1: The soonest arriving population-$\tilde{i}$ agent will join the queue since his expected payoff of joining the system is $R_{\tilde{i}} - C_{\tilde{i}} \frac{1}{\mu} > 0$. This is because, no matter how many other population-i agents arrive in between the soonest-arriving population-$\tilde{i}$ agent and the tagged population-i agent, the soonest-arriving population-$\tilde{i}$ agent can always start matching immediately without waiting. In this subgame, the expected payoff of the tagged population-i agent is $R_i - C_i \left(\frac{1}{\lambda_{\tilde{i}}} + \frac{1}{\mu} \right)$ because he expects to wait for $\frac{1}{\lambda_{\tilde{i}}}$ units of time for the soonest arrival of the population-$\tilde{i}$ agent.

  Note that, if the soonest arriving population-$\tilde{i}$ agent decides to balk, the tagged population-i agent may expect a lower payoff, denoted U_1 (since the expected waiting time becomes longer). However, this outcome is not a subgame perfect Nash equilibrium.

- Subgame 2: There is temporarily not enough information to conclude how the soonest arriving population-$\tilde{i}$ agent will behave if the tagged population-i agent balks, because, in this case, the population-$\tilde{i}$ agent needs further reasoning about the strategy of the next arriving population-i agent, and that becomes another sequential game. However, the tagged population-i agent always obtains zero payoff for his balking decision regardless of the what the soonest arriving population-$\tilde{i}$ agent decides.

- Subgame 3: The tagged population-i agent decides whether to join the system by comparing the the two actions' expected payoffs: $R_i - C_i \left(\frac{1}{\lambda_{\tilde{i}}} + \frac{1}{\mu} \right) > 0$.

As the rational outcome of this game, a population-i agent joins at state $(0, 0)$, and a population-$\tilde{i}$ joins at state $(1, 0)$ if $i = 1$, or $(0, 1)$ if $i = 2$.

The solution of the game

The above example illustrates how agents rationally behave according to backward induction. From the overall perspective, agents make their decision about a certain system state by reasoning backward from strategies at future states that may be reached from the current state that they observe. Since the formula for calculating expected waiting times is recursive, the procedure to derive expected waiting times is identical to an inductive process: agents base the unknown expected waiting time, at a certain state, on the adjacent states at which the expected waiting times and joining strategies of agents are already identified.

In the first place, it can be induced that, even when the buffer capacity is infinite, the maximum length of the buffer on each side is finite. Consider a tagged population-i agent who encounters system state (x_1, x_2) upon arrival. This customer's waiting time can be decomposed into two components: the waiting time for x_i other population-i agents to complete their matching process, and the possible waiting time for population-$\tilde{i}$ agents to arrive for matching. Since the expected value of the first time component is $\frac{x_i}{\mu}$, it is implied that $T_i(x_1, x_2) \geq \frac{x_i}{\mu}$. Because $\frac{x_i}{\mu} \to +\infty$ as $x_i \to +\infty$, we have $\lim_{x_i \to +\infty} T_i(x_1, x_2) = +\infty$. This means that a population-i agent joining at an infinite position would expect a payoff at $-\infty$, which is not rational. Therefore, there exist finite values that the population-i agent buffer length never exceeds. Let

$$N_i = \max \left\{ n : R_i - C_i \frac{n}{\mu} \geq 0, n \in \mathbb{N} \right\}.$$

As such, $\sigma^{(i)}_{(x_1,x_2)} = 0$ for all $x_i = N_i$, and any states (x_1, x_2) in which $x_i > N_i$ are transient. It should be noted that joining strategies of agents at those transient states do not affect their joining strategies at recurrent states in this backward-sequential game setting. We can now derive joining strategies at the remaining states that follow in the system's transition diagram as in Figure 4.2.3.

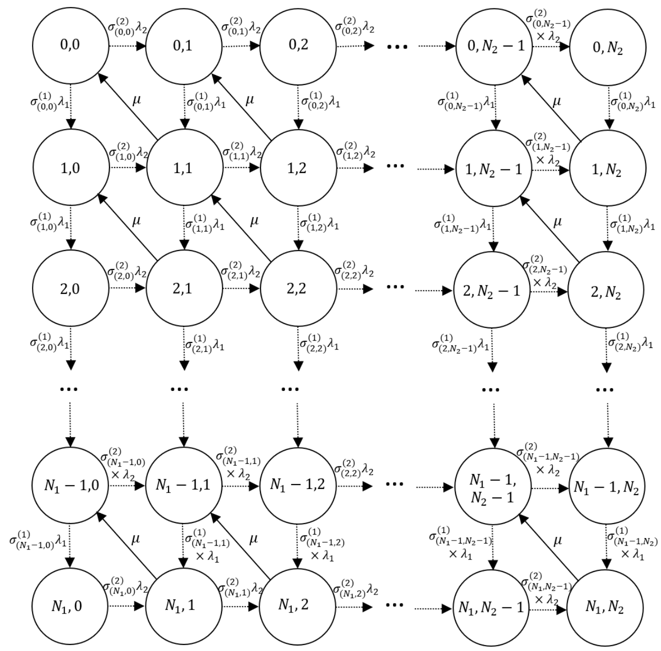


Figure 4.2.3: Transition diagram in the strategic scenario.

The dashed lines in the figure indicate that the corresponding transition rate remains unknown (since the transition rates depend on the agents' joining strategies). Technically, we need to calculate the expected waiting times of agents at every state in the above diagram, and update agents' joining strategies accordingly.

First, notice that $T_1(0, x_2) = 0$ for all $x_2 \geq 1$, and $T_2(x_1, 0) = 0$ for all $x_1 \geq 1$. Therefore, the expected payoff to population-i agents who observe $x_{\tilde{i}} > 0$ population-$\tilde{i}$ agents, and no other population-i agents, in the system upon arrival is $R_i - \frac{C_i}{\mu} > 0$, which implies that

$$\sigma^{(1)}_{(0,x_2)} = 1;\ \sigma^{(2)}_{(x_1,0)} = 1, \tag{4.2.2}$$

for $x_1 > 0$ and $x_2 > 0$.

It then follows that

$$W_1(1, 0, N_1 - 1) = \frac{1}{\sigma^{(2)}_{(N_1,0)}\lambda_2} = \frac{1}{\lambda_2}.$$

For any $y_1 < N_1 - 1$, we have

$$W_1(1,0,y_1) = \frac{1}{\sigma^{(1)}_{(1+y_1,0)}\lambda_1 + \sigma^{(2)}_{(1+y_1,0)}\lambda_2} + \frac{\sigma^{(1)}_{(1+y_1,0)}\lambda_1}{\sigma^{(1)}_{(1+y_1,0)}\lambda_1 + \sigma^{(2)}_{(1+y_1,0)}\lambda_2} W_1(1,0,y_1+1).$$

Since $\sigma^{(2)}_{(1+y_1,0)} = 1$ for all $0 \le y_1 < N_1 - 1$, it is easily obtained by induction that $W_1(1,0,0) = \frac{1}{\lambda_2}$, which implies that $T_1(0,0) = \frac{1}{\lambda_2}$. Similarly, we obtain $W_2(1,0,y_2) = \frac{1}{\lambda_1}$ for all $0 \le y_2 \le N_2 - 1$, and $T_2(0,0) = W_2(1,0,0) = \frac{1}{\lambda_1}$.

Since $R_i - C_i\left(T_i(0,0) + \frac{1}{\mu}\right) = R_i - C_i\left(\frac{1}{\lambda_{\tilde{i}}} + \frac{1}{\mu}\right) > 0$ by assumption, population-i agents always join an empty system. This is also the conclusion of the example in Section 4.2.3. In other words,

$$\sigma^{(i)}_{(0,0)} = 1. \tag{4.2.3}$$

Lemma 4.2.1. *$T_i(x_1,x_2) = \frac{x_i}{\mu}$ for all $x_i < x_{\tilde{i}}$, $x_i \le N_i$.*

Proof. Observe that $W_1(1,N_2,N_1-2) = \frac{1}{\mu}$. For any $y_1 < N_1 - 2$, we have

$$W_1(1,N_2,y_1) = \frac{1}{\sigma^{(1)}_{(2+y_1,N_2)}\lambda_1 + \mu} + \frac{\sigma^{(1)}_{(2+y_1,0)}\lambda_1}{\sigma^{(1)}_{(2+y_1,0)}\lambda_1 + \mu} W_1(1,N_2,y_1+1).$$

It is then easily obtained by induction on y_1 that $W_1(1,N_2,y_1) = \frac{1}{\mu}$ for all $0 \le y_1 \le N_1 - 2$, which implies that $T_1(1,N_2) = W_1(1,N_2,0) = \frac{1}{\mu}$. Then, by induction on x_2, it is easily obtained that $W_1(1,x_2,y_1) = \frac{1}{\mu}$ for $1 \le x_2 \le N_2$ and $0 \le y_1 \le N_1 - 2$. Similarly, $W_i(1,x_{\tilde{i}},y_i) = \frac{1}{\mu}$, for $1 < x_{\tilde{i}} \le N_{\tilde{i}}$ and $0 \le y_i \le N_i - 2$. Also by induction, for any $\hat{x}_i < x_{\tilde{i}}$, we can prove that

$$W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = \frac{\hat{x}_i}{\mu}.$$

It then follows that $T_1(x_1,x_2) = \frac{x_1}{\mu}$ for all $x_1 < x_2$ and $T_2(x_1,x_2) = \frac{x_2}{\mu}$ for all $x_2 < x_1$. $\square$

Remark 4.2.2. *Lemma 4.2.1 can be proved in another way that is more intuitive. Notice that, when there are x_i population-i agents and $x_{\tilde{i}}$ population-$\tilde{i}$ agents in the system ($x_i < x_{\tilde{i}}$), a tagged population-i agent who decides to join the system will find his counterpart population-$\tilde{i}$ agent for* *matching already present on the other side of the queue. Therefore, his waiting time only consists of the matching times of the other x_i enqueued population-i agents, with an expected value of $\frac{x_i}{\mu}$.*

Results in Lemma 4.2.1 suggest that the joining strategies of population-i agents at states where $1 \leq x_i < x_{\tilde{i}}, x_i \leq N_i$ can be identified as follows:

$$\sigma^{(i)}_{(x_1,x_2)} = \begin{cases} 1 & \text{if } x_i < N_i \\ p_i & \text{if } x_i = N_i - 1, \\ 0 & \text{if } x_i = N_i, \end{cases} \tag{4.2.4}$$

where $p_i = 1$ if $R_i - C_i \frac{N_i}{\mu} > 0$, and p_i takes any value on $[0, 1]$ if $R_i - C_i \frac{N_i}{\mu} = 0$.

Now, we will derive agents' strategies upon state $(1, 1)$. First, observe that

$$\begin{aligned} W_i(1,1,N_i-2) &= \frac{1}{\mu + \sigma^{(\tilde{i})}_{(N_i,1)}\lambda_{\tilde{i}}} + \frac{1}{\mu + \sigma^{(\tilde{i})}_{(N_i,1)}\lambda_{\tilde{i}}} W_i(1,0,N_i-2) \\ &+ \frac{\sigma^{(\tilde{i})}_{(N_i,1)}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(N_i,1)}\lambda_{\tilde{i}}} W_i(1,2,N_i-2) \\ &= \frac{1}{\mu + \lambda_{\tilde{i}}} + \frac{\mu}{\mu + \lambda_{\tilde{i}}} \cdot \frac{1}{\lambda_{\tilde{i}}} + \frac{\lambda_{\tilde{i}}}{\mu + \lambda_{\tilde{i}}} \cdot \frac{1}{\mu} \\ &= \frac{\mu^2 + \lambda_{\tilde{i}}^2 + \mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu + \lambda_{\tilde{i}})}. \end{aligned}$$

By induction from Eq. (4.2.1), for any $0 \leq y_i \leq N_i - 2$, we obtain

$$W_i(1,1,y_i) = \frac{\mu^2 + \lambda_{\tilde{i}}^2 + \mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu + \lambda_{\tilde{i}})},$$

which implies that $T_i(1,1) = \frac{\mu^2 + \lambda_{\tilde{i}}^2 + \mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu + \lambda_{\tilde{i}})}$. Then, the joining strategies of population-i agents at state $(1,1)$ can be identified as follows:

$$\bar{\sigma}^{(i)}_{(1,1)} = \begin{cases} 1 & \text{if } R_i - C_i\left(\frac{\mu^2+\lambda_{\tilde{i}}^2+\mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu+\lambda_{\tilde{i}})} + \frac{1}{\mu}\right) > 0, \\ p_i & \text{if } R_i - C_i\left(\frac{\mu^2+\lambda_{\tilde{i}}^2+\mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu+\lambda_{\tilde{i}})} + \frac{1}{\mu}\right) = 0, \\ 0 & \text{if } R_i - C_i\left(\frac{\mu^2+\lambda_{\tilde{i}}^2+\mu\lambda_{\tilde{i}}}{\mu\lambda_{\tilde{i}}(\mu+\lambda_{\tilde{i}})} + \frac{1}{\mu}\right) < 0 \end{cases} \tag{4.2.5}$$

where p_i takes any value on $[0,1]$.

Lemma 4.2.2. *$W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = W_i(\hat{x}_i, x_{\tilde{i}}, y_i + 1)$ for $\hat{x}_i = x_{\tilde{i}} > 1$, $0 \le y_i \le N_i - \hat{x}_i - 1$.*

Proof. We already showed that Lemma 4.2.2 holds for $\hat{x}_i = 1$. Assume that Lemma 4.2.2 holds until $\hat{x}_i = k_i - 1$ ($k_i \ge 2$), which means that $W_i(k_i - 1, x_{\tilde{i}}, y_i) = W_i(k_i - 1, x_{\tilde{i}}, y_i + 1)$ for $x_{\tilde{i}} = k_i - 1 \ge 1$, $0 \le y_i \le N_i - k_i$.

First, we will show that $W_i(k_i, x_{\tilde{i}}, N_i - k_i - 2) = W_i(k_i, x_{\tilde{i}}, N_i - k_i - 1)$ for $x_{\tilde{i}} \ge 1$. According to the recursion in Eq. (4.2.1), we have

$$\begin{aligned} &(\mu + \sigma^{(i)}_{(N_i-1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(N_i-1,x_{\tilde{i}})}\lambda_{\tilde{i}})W_i(k_i, x_{\tilde{i}}, N_i - k_i - 2) \\ =&1 + \mu W_i(k_i - 1, x_{\tilde{i}} - 1, N_i - k_i - 2) + \sigma^{(i)}_{(N_i-1,x_{\tilde{i}})}\lambda_i W_i(k_i, x_{\tilde{i}}, N_i - k_i - 1) \\ &+ \sigma^{(\tilde{i})}_{(N_i-1,x_{\tilde{i}})}\lambda_{\tilde{i}} W_i(k_i, x_{\tilde{i}} + 1, N_i - k_i - 2), \end{aligned}$$

and

$$\begin{aligned} (\mu + \sigma^{(\tilde{i})}_{(N_i,x_{\tilde{i}})}\lambda_{\tilde{i}})W_i(k_i, x_{\tilde{i}}, N_i - k_i - 1) = 1 &+ \mu W_i(k_i - 1, x_{\tilde{i}} - 1, N_i - k_i - 1) \\ &+ \sigma^{(\tilde{i})}_{(N_i,x_{\tilde{i}})}\lambda_{\tilde{i}} W_i(k_i, x_{\tilde{i}} + 1, N_i - k_i - 1), \end{aligned}$$

which imply that

$$\left(\mu + \sigma^{(i)}_{(N_i-1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(N_i-1,x_{\tilde{i}})}\lambda_{\tilde{i}}\right)\left(W_i(k_i, x_{\tilde{i}}, N_i - k_i - 2) - W_i(k_i, x_{\tilde{i}}, N_i - k_i - 1)\right) = 0$$

(because $W_i(k_i - 1, x_{\tilde{i}} - 1, N_i - k_i - 2) = W_i(k_i - 1, x_{\tilde{i}} - 1, N_i - k_i - 1)$ by inductive assumption, $\sigma^{(\tilde{i})}_{(N_i-1,x_{\tilde{i}})} = \sigma^{(\tilde{i})}_{(N_i,x_{\tilde{i}})} = 1$, and $W_i(k_i, x_{\tilde{i}} + 1, N_i - k_i - 2) = W_i(k_i, x_{\tilde{i}} + 1, N_i - k_i - 1) = \frac{k_i}{\mu}$).

It then follows that $W_i(k_i, x_{\tilde{i}}, N_i - k_i - 2) = W_i(k_i, x_{\tilde{i}}, N_i - k_i - 1)$. Similarly, by induction on y_i, we obtain $W_i(k_i, x_{\tilde{i}}, y_i) = W_i(k_i, x_{\tilde{i}}, y_i + 1)$ for $0 \leq y_i \leq N_i - k_i - 1$, which means Lemma 4.2.2 also holds for $\hat{x}_i = k_i$. This completes the proof. $\square$

From Lemma 4.2.2, recalling Eq. (4.2.1), we obtain

$$\begin{aligned} W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = & \frac{1}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} \\ & + \frac{\mu}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i - 1, x_{\tilde{i}} - 1, y_i) \\ & + \frac{\sigma^{(i)}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_i}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}, y_i + 1) \\ & + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}} + 1, y_i), \end{aligned}$$

which is equivalent to

$$W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = \frac{1}{\mu + \lambda_{\tilde{i}}} + \frac{\mu}{\mu + \lambda_{\tilde{i}}} W_i(\hat{x}_i - 1, x_{\tilde{i}} - 1, y_i) + \frac{\lambda_{\tilde{i}}}{\mu + \lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}} + 1, y_i).$$

Since $W_i(\hat{x}_i, x_{\tilde{i}} + 1, y_i) = \frac{\hat{x}_i}{\mu}$, $W_i(\hat{x}_i, x_{\tilde{i}}, y_i)$ can then be calculated inductively. This result suggests that the joining strategies of population-i agents at states where $1 \leq x_i = x_{\tilde{i}}, x_i \leq N_i$ can be identified as follows:

$$\bar{\sigma}^{(i)}_{(x_1,x_2)} = \begin{cases} 1 & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) > 0, \\ p_i & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) = 0, \\ 0 & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) < 0, \end{cases} \tag{4.2.6}$$

where p_i can take any value on $[0, 1]$.

Lemma 4.2.3. *$W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = W_i(\hat{x}_i, x_{\tilde{i}}, y_i + 1)$ for $\hat{x}_i > x_{\tilde{i}}$, $\hat{x}_i \geq 1$, $0 \leq y_i \leq N_i - \hat{x}_i - 1$.*

Proof. We already showed that Lemma 4.2.3 holds for $\hat{x}_i = 1$. Assume that Lemma 4.2.3 holds until $k_i - 1$ ($k_i \geq 2$), which means that $W_i(k_i - 1, x_{\tilde{i}}, y_i) = W_i(k_i - 1, x_{\tilde{i}}, y_i + 1)$ for

$k_i - 1 > x_{\tilde{i}}, k_i - 1 \geq 1, 0 \leq y_i \leq N_i - k_i$.

First, we will show that $W_i(k_i, k_i - 1, N_i - k_i - 2) = W_i(k_i, k_i - 1, N_i - k_i - 1)$ for $k_i \geq 2$. According to the recursion in Eq. (4.2.1), we have

$$\begin{aligned}
&(\mu + \sigma^{(i)}_{(N_i-1,k_i-1)}\lambda_i + \sigma^{(\tilde{i})}_{(N_i-1,k_i-1)}\lambda_{\tilde{i}}) W_i(k_i, k_i - 1, N_i - k_i - 2) \\
&= 1 + \mu W_i(k_i - 1, k_i - 2, N_i - k_i - 2) + \sigma^{(i)}_{(N_i-1,k_i-1)}\lambda_i W_i(k_i, k_i - 1, N_i - k_i - 1) \\
&+ \sigma^{(\tilde{i})}_{(N_i-1,k_i-1)}\lambda_{\tilde{i}} W_i(k_i, k_i, N_i - k_i - 2),
\end{aligned}$$

and

$$\begin{aligned}
&(\mu + \sigma^{(\tilde{i})}_{(N_i,k_i-1)}\lambda_{\tilde{i}}) W_i(k_i, k_i - 1, N_i - k_i - 1) \\
&= 1 + \mu W_i(k_i - 1, k_i - 2, N_i - k_i - 1) + \sigma^{(\tilde{i})}_{(N_i,k_i-1)}\lambda_{\tilde{i}} W_i(k_i, k_i, N_i - k_i - 1),
\end{aligned}$$

which imply

$$\begin{aligned}
&\left(\mu + \sigma^{(i)}_{(N_i-1,k_i-1)}\lambda_i + \sigma^{(\tilde{i})}_{(N_i-1,k_i-1)}\lambda_{\tilde{i}}\right)(W_i(k_i, k_i - 1, N_i - k_i - 2) - W_i(k_i, k_i - 1, N_i - k_i - 1)) \\
&= 0
\end{aligned}$$

(because $W_i(k_i - 1, k_i - 2, N_i - k_i - 2) = W_i(k_i - 1, k_i - 2, N_i - k_i - 1)$ by inductive assumption, $\sigma^{(\tilde{i})}_{(N_i-1,k_i-1)} = \sigma^{(\tilde{i})}_{(N_i,k_i-1)} = 1$, and $W_i(k_i, x_{\tilde{i}}+1, N_i-k_i-2) = W_i(k_i, x_{\tilde{i}}+1, N_i-k_i-1) = \frac{k_i}{\mu}$). It then follows that $W_i(k_i, k_i - 1, N_i - k_i - 2) = W_i(k_i, k_i - 1, N_i - k_i - 1)$. Similarly, by induction on y_i, we obtain $W_i(k_i, k_i - 1, y_i) = W_i(k_i, k_i - 1, y_i + 1)$ for $0 \leq y_i \leq N_i - k_i - 1$.

Then, by induction on $\hat{x}_i$, we obtain $W_i(k_i, x_{\tilde{i}}, y_i) = W_i(k_i, x_{\tilde{i}}, y_i + 1)$, which means that Lemma 4.2.3 also holds for $\hat{x}_i = k_i$. This completes the proof. $\square$

From Lemma 4.2.3, recalling Eq. (4.2.1), we obtain

$$\begin{aligned}W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = &\frac{1}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} \\ &+ \frac{\mu}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i - 1, x_{\tilde{i}} - 1, y_i) \\ &+ \frac{\sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}, y_i + 1) \\ &+ \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu + \sigma^{(i)}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_i + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}} + 1, y_i),\end{aligned}$$

which is equivalent to

$$\begin{aligned}W_i(\hat{x}_i, x_{\tilde{i}}, y_i) = &\frac{1}{\mu + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i - 1, x_{\tilde{i}} - 1, y_i) \\ &+ \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(\hat{x}_i+y_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}} + 1, y_i).\end{aligned}$$

Since $W_i(\hat{x}_i, \hat{x}_i, y_i)$ can already be calculated using the result in Lemma 4.2.2, $W_i(\hat{x}_i, x_{\tilde{i}}, y_i)$ for $\hat{x}_i > x_{\tilde{i}}$ can then be inductively calculated starting from $x_{\tilde{i}} = \hat{x}_i - 1$. This result suggests that the joining strategies of population-i agents, at states where $1 \leq x_i > x_{\tilde{i}}, x_i \leq N_i$, can be identified as follows:

$$\bar{\sigma}^{(i)}_{(x_1,x_2)} = \begin{cases} 1 & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) > 0, \\ p_i & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) = 0, \\ 0 & \text{if } R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}, 0) + \frac{1}{\mu}\right) < 0 \end{cases} \tag{4.2.7}$$

where p_i can take any value on $[0, 1]$.

Remark 4.2.3. *Lemma 4.2.2 and Lemma 4.2.3 can also be explained intuitively. The condition $\hat{x}_i \geq x_{\tilde{i}}$ implies that there are currently more population-i agents than population-$\tilde{i}$ agents in the system. If another population-i agent arrives, that condition still holds. The results in Eq.*

(4.2.4) *indicate that, as long as there are more population-i agents than population-$\tilde{i}$ agents in the system, a newly arriving population-$\tilde{i}$ agent always joins when their population has not exceeded $N_{\tilde{i}}$. Therefore, the waiting time of the tagged population-i agent is insensitive to the number of agents behind him.*

Results (4.2.3), (4.2.4), (4.2.5), (4.2.6) and (4.2.7) complete the derivation of agents' strategies at all states in equilibrium. Furthermore, it should be noted from the existing results that, in subgame perfection, the joining strategy of later comers does not affect the expected waiting time of already enqueued agents of the same population. From now on, denote $W_i(\hat{x}_i, x_{\tilde{i}}) = W_i(\hat{x}_i, x_{\tilde{i}}, 0)$, which represents the expected waiting time of a population-i agent arriving at position $\hat{x}_i$ and observing $x_{\tilde{i}}$ population-$\tilde{i}$ agents upon arrival.

Lemma 4.2.4. *$W_i(\hat{x}_i, x_{\tilde{i}})$ is non-decreasing in $\hat{x}_i$.*

Proof. The expected waiting time of a tagged population-i agent joining at position $\hat{x}_i$ when there are $x_{\tilde{i}}$ population-$\tilde{i}$ agents can be rewritten as follows:

$$W_i(\hat{x}_i, x_{\tilde{i}}) = \begin{cases} \frac{\hat{x}_i}{\mu} \\ \text{if } \hat{x}_i < x_{\tilde{i}}, \\ \frac{1}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i-1, x_{\tilde{i}}-1) + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}+1) \\ \text{if } \hat{x}_i \geq x_{\tilde{i}} > 1, \\ \frac{1}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}-1) + \frac{\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu+\sigma^{(\tilde{i})}_{(\hat{x}_i+1,x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(\hat{x}_i, x_{\tilde{i}}+1) \\ \text{if } \hat{x}_i \geq x_{\tilde{i}} = 1, \\ \frac{1}{\sigma^{(\tilde{i})}_{(\hat{x}_i,x_{\tilde{i}})}\lambda_{\tilde{i}}} + W_i(\hat{x}_i-1, x_{\tilde{i}}+1) \\ \text{if } x_{\tilde{i}} = 0. \end{cases}$$

We will show that $W_i(\hat{x}_i+1, x_{\tilde{i}}) \geq W_i(\hat{x}_i, x_{\tilde{i}})$. It is easily seen that this holds for $\hat{x}_i = 0$. Assume that this inequality also holds for $\hat{x}_i = q_i - 1\,(q_i \geq 1)$, which means that $W_i(q_i, x_{\tilde{i}}) \geq W_i(q_i - 1, x_{\tilde{i}})$ for $0 \leq x_{\tilde{i}} \leq N_{\tilde{i}}$. Also, as $W_i(1,1) > \frac{1}{\mu}$, we assume that $W_i(q_i, q_i) \geq \frac{q_i}{\mu}$.

Consider the following five cases.

- If $x_{\tilde{i}} = 0$, we have

$$\begin{aligned}
&W_i(q_i+1,0) - W_i(q_i,0) \\
&= \left(\frac{1}{\sigma^{(\tilde{i})}_{(q_i+1,0)}\lambda_{\tilde{i}}} + W_i(q_i,1)\right) - \left(\frac{1}{\sigma^{(\tilde{i})}_{(q_i,0)}\lambda_{\tilde{i}}} + W_i(q_i-1,1)\right) \\
&= W_i(q_i,1) - W_i(q_i-1,1) \text{ (since } \sigma^{(\tilde{i})}_{(q_i+1,0)} = \sigma^{(\tilde{i})}_{(q_i,0)} = 1) \\
&\geq 0 \text{ (by inductive assumption),}
\end{aligned}$$

which implies $W_i(q_i+1,0) \geq W_i(q_i,0)$.

- If $x_{\tilde{i}} > 0$ and $q_i < x_{\tilde{i}} - 1$, we have

$$W_i(q_i+1,x_{\tilde{i}}) - W_i(q_i,x_{\tilde{i}}) = \frac{q_i+1}{\mu} - \frac{q_i}{\mu} = \frac{1}{\mu} > 0,$$

which implies $W_i(q_i+1,x_{\tilde{i}}) > W_i(q_i,x_{\tilde{i}})$.

- If $x_{\tilde{i}} > 1$ and $q_i = x_{\tilde{i}} - 1$, we have

$$\begin{aligned}
W_i(q_i+1,x_{\tilde{i}}) = &\frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} \\
&+ \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} W_i(q_i,q_i) + \frac{\sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} W_i(q_i+1,q_i+2) \\
&\geq \frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} \cdot \frac{q_i}{\mu} + \frac{\sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,\tilde{i})}\lambda_{\tilde{i}}} \cdot \frac{q_i+1}{\mu} \\
&\text{(because } W_i(q_i,q_i) \geq \frac{q_i}{\mu} \text{ by inductive assumption, and } W_i(q_i+1,q_i+2) = \frac{q_i+1}{\mu}) \\
&= \frac{q_i+1}{\mu},
\end{aligned}$$

which implies that $W_i(q_i+1,x_{\tilde{i}}) > W_i(q_i,x_{\tilde{i}})$ when $q_i = x_{\tilde{i}} - 1$.

- If $x_{\tilde{i}} > 1$ and $q_i \geq x_{\tilde{i}}$, we have

$$\begin{aligned}
&W_i(q_i+1, x_{\tilde{i}}) - W_i(q_i, x_{\tilde{i}}) \\
&= \left(\frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+2, x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+2, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(q_i, x_{\tilde{i}}-1) + \frac{\sigma^{(\tilde{i})}_{(q_i+2, x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+2, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(q_i+1, x_{\tilde{i}}+1) \right) \\
&- \left(\frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(q_i-1, x_{\tilde{i}}-1) + \frac{\sigma^{(\tilde{i})}_{(q_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+1, x_{\tilde{i}})}\lambda_{\tilde{i}}} W_i(q_i, x_{\tilde{i}}+1) \right)
\end{aligned}$$

  For $q_i \geq x_{\tilde{i}}$, we have $\sigma^{(\tilde{i})}_{(q_i+2, x_{\tilde{i}})} = \sigma^{(\tilde{i})}_{(q_i+1, x_{\tilde{i}})}$. Also, by inductive assumption, $W_i(q_i, x_{\tilde{i}} - 1) \geq W_i(q_i - 1, x_{\tilde{i}} - 1)$. Therefore, if $W_i(q_i + 1, x_{\tilde{i}} + 1) \geq W_i(q_i, x_{\tilde{i}} + 1)$, we can obtain $W_i(q_i + 1, x_{\tilde{i}}) \geq W_i(q_i, x_{\tilde{i}})$. This holds by induction because we already have $W_i(q_i + 1, x_{\tilde{i}}) > W_i(q_i, x_{\tilde{i}})$ when $q_i = x_{\tilde{i}} - 1$.

- If $x_{\tilde{i}} = 1$, we have

$$\begin{aligned}
W_i(q_i+1, 1) = &\frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+2,1)}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+2,1)}\lambda_{\tilde{i}}} W_i(q_i+1, 0) \\
&+ \frac{\sigma^{(\tilde{i})}_{(q_i+2,1)}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+2,1)}\lambda_{\tilde{i}}} W_i(q_i+1, 2),
\end{aligned}$$

  and

$$W_i(q_i, 1) = \frac{1}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,1)}\lambda_{\tilde{i}}} + \frac{\mu}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,1)}\lambda_{\tilde{i}}} W_i(q_i, 0) + \frac{\sigma^{(\tilde{i})}_{(q_i+1,1)}\lambda_{\tilde{i}}}{\mu + \sigma^{(\tilde{i})}_{(q_i+1,1)}\lambda_{\tilde{i}}} W_i(q_i, 2).$$

  For $q_i \geq 1$, $\sigma^{(\tilde{i})}_{(q_i+1,1)} = \sigma^{(\tilde{i})}_{(q_i+2,1)} = 1$. Also, we already have $W_i(q_i+1, 0) \geq W_i(q_i, 0)$ and $W_i(q_i + 1, 2) \geq W_i(q_i, 2)$ from the earlier proofs. Therefore, $W_i(q_i + 1, 1) \geq W_i(q_i, 1)$.

The above five cases complete the proof of Lemma 4.2.4. $\square$

Noting that $R_i - C_i \left(W_i(N_i, x_{\tilde{i}}) + \frac{1}{\mu} \right) < 0$, Lemma 4.2.4 implies that, given $x_{\tilde{i}}$, there exists a threshold level $\eta^{(i)}_{x_{\tilde{i}}}$ of the position above at which population-i agents balk the queue.

Also, in what follows, we assume that, if $R_i - C_i\left(W_i(\hat{x}_i, x_{\tilde{i}}) + \frac{1}{\mu}\right) = 0$, a population-$i$ agent chooses to join at that state. Then, in subgame perfect Nash equilibrium, population-i agents follows a threshold strategy prescribed by a vector

$$\boldsymbol{\eta}^{(i)} = \left(\eta_0^{(i)}, \eta_1^{(i)}, \dots, \eta_{N_{\tilde{i}}}^{(i)}\right),$$

where $\eta_j^{(i)} - 1 + \min\{j, 1\}$ is the maximum position at which a population-i agent chooses to join when there are j population-$\tilde{i}$ agents in the system.

Remark 4.2.4. *Let $\eta_{max}^{(i)} = \max \boldsymbol{\eta}^{(i)} \leq N_i$. $\eta_{max}^{(i)}$ is not necessarily the actual maximum number of population-i agents in the system. For example, there may exist a value $m < \min\{\eta_{max}^{(i)}, \eta_{max}^{(\tilde{i})}\}$ such that $R_i - C_i\left(W_i(m, m) + \frac{1}{\mu}\right) < 0$. In this case, the maximum number of both populations in the system is m. As such, any states (x_1, x_2) such that $x_i > m$ are transient.*

Let $m^{(i)}$ denote the actual maximum number of population-i agents in the system in subgame perfection. Then, the threshold strategy that population-i agents follow can be irreducibly rewritten as

$$\boldsymbol{\eta}^{(i)*} = \left(\eta_0^{(i)}, \eta_1^{(i)}, \dots, \eta_{m^{(\tilde{i})}}^{(i)}\right).$$

Another subgame perfect Nash equilibrium computation technique: A loop algorithm

In this section, we propose a loop algorithm to compute the subgame perfect Nash equilibrium which does not require the backward induction procedure.

Algorithm 2 Deriving equilibrium threshold strategies

$\boldsymbol{\eta}^{(1)} \leftarrow \boldsymbol{\eta}_0^{(1)} = \left(\eta_0^{(1)}, \eta_1^{(1)}, \ldots, \eta_{N_2}^{(1)}\right)$ ▷ arbitrary initial threshold strategy of population 1
$\boldsymbol{\eta}^{(2)} \leftarrow \boldsymbol{\eta}_0^{(2)} = \left(\eta_0^{(2)}, \eta_1^{(2)}, \ldots, \eta_{N_1}^{(2)}\right)$ ▷ arbitrary initial threshold strategy of population 2
$\boldsymbol{\nu}^{(1)} \leftarrow$ an arbitrary $(N_2 + 1)$-dim vector different from $\boldsymbol{\eta}^{(1)}$
$\boldsymbol{\nu}^{(2)} \leftarrow$ an arbitrary $(N_1 + 1)$-dim vector different from $\boldsymbol{\eta}^{(2)}$
while $\boldsymbol{\eta}^{(1)} \neq \boldsymbol{\nu}^{(1)}$ or $\boldsymbol{\eta}^{(2)} \neq \boldsymbol{\nu}^{(2)}$ **do**
    $\boldsymbol{\nu}^{(1)} \leftarrow \boldsymbol{\eta}^{(1)}$
    $\boldsymbol{\eta}^{(1)} \leftarrow \mathcal{F}_1(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)})$
    $\boldsymbol{\nu}^{(2)} \leftarrow \boldsymbol{\eta}^{(2)}$
    $\boldsymbol{\eta}^{(2)} \leftarrow \mathcal{F}_2(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)})$
end while
return $\left(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}\right)$ ▷ equilibrium threshold strategy

The above algorithm is based on multi-variable fixed-point iteration. In numerical examples in a later section (Section 4.2.5), we will show that Algorithm 2 converges to the exactly same result as derived in the previous section, regardless of the initial threshold strategy selected at the beginning. From an intuitive standpoint, this algorithm simulates how agents keep updating their behaviors sequentially based on the strategies of other agents.

Note that the above algorithm is based on several premises. First, agents are assumed to follow threshold strategies. Second, the threshold strategy of population-i agents is a $(N_{\tilde{i}}+1)$-dimensional vector. This is based on the similar fact induced from the previous section that, in an ideal scenario for population-i where there are always population-$\tilde{i}$ agents on the opposite side of the queue to match with, the number of population-i agents in the system never exceeds N_i. Since population-$\tilde{i}$ agents are strategic, the maximum number of population-i agents should be less than or equal to N_i, thus, any states in which $x_i > N_i$ are transient, so the joining probabilities of agents at those states do not need to be considered.

4.2.4 Performance measures in equilibrium

We construct the infinitesimal generator Q of the Markov chain modeling the system in equilibrium where the number of population-1 agents (i.e., x_1) is the level, and the number of population-2 agents (i.e., x_2) is the phase, with x_i satisfying $x_i \leq \eta^{(i)}_{x_{\bar{i}}}$. The steady-state probabilities are defined as $\boldsymbol{\pi} = \left(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_{\eta^{(1)}_{m^{(2)}}}\right)$, where $\boldsymbol{\pi}_{x_1} = \left(\pi_{(x_1,0)}, \pi_{(x_1,1)}, \ldots\right)$ is a vector encoding all probabilities when there are x_1 population-1 agents in the system at the steady state. These probabilities are obtained by solving the following equations:

$$\begin{cases} \boldsymbol{\pi} \mathcal{Q} = 0, \\ \boldsymbol{\pi} \mathbf{e} = 1, \end{cases}$$

where 0 is a zero vector of appropriate length, and **e** is a unit vector of appropriate length.

The mean number of population-i agents in the system, denoted L_i, is given by

$$L_i = \sum_{x_1} \sum_{x_2} x_i \pi_{(x_1,x_2)}.$$

The balking rates of population-i agents, denoted ξ_i, is given by

$$\xi_i = \sum_{x_{\bar{i}}} \sum_{x_i = \eta^{(i)}_{x_{\bar{i}}}} \pi_{(x_1,x_2)} \lambda_i,$$

and the joining rate for population i is $\lambda_i - \xi_i$.

Social welfare is given by

$$\mathrm{SW} = (\lambda_1 - \xi_1)R_1 + (\lambda_2 - \xi_2)R_2 - C_1 L_1 - C_2 L_2.$$

Social welfare can be separated into two parts: the welfare of the demand side, and the welfare of the supply side, which are, respectively, given by

$$\mathrm{SW}^{(1)} = (\lambda_1 - \xi_1)R_1 - C_1 L_1,$$

and

$$\mathrm{SW}^{(2)} = (\lambda_2 - \xi_2)R_2 - C_2 L_2.$$

4.2.5 Numerical examples

In this section, we present numerical examples of the analyses in previous sections.

Example 1 (Equilibrium analysis). Set $\lambda_1 = 1$, $\lambda_2 = 1$, $\mu = 20$, $R_1 = R_2 = 7.5$ and $C_1 = C_2 = 7$. As a result, $N_1 = N_2 = 21$, and the threshold strategies of agents are

$$\boldsymbol{\eta}^{(1)} = \boldsymbol{\eta}^{(2)} = (1, 2, 3, 4, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21).$$

However, as $R_i - C_i\left(T_i(4,4) + \frac{1}{\mu}\right) < 0$, neither population-1 agents nor population-2 agents are willing to join the system at state $(4,4)$ (although, if a population-i agent decided to join at state $(4,4)$, the next-arriving population-$\tilde{i}$ agent would definitely join). Therefore, the maximum number of agents in each population is capped at 4. The irreducible form of the threshold strategies of the two sides of agents is

$$\boldsymbol{\eta}^{(1)*} = \boldsymbol{\eta}^{(2)*} = (1, 2, 3, 4, 4).$$

The transition diagram of the system in subgame perfect Nash equilibrium is illustrated in Figure 4.2.4

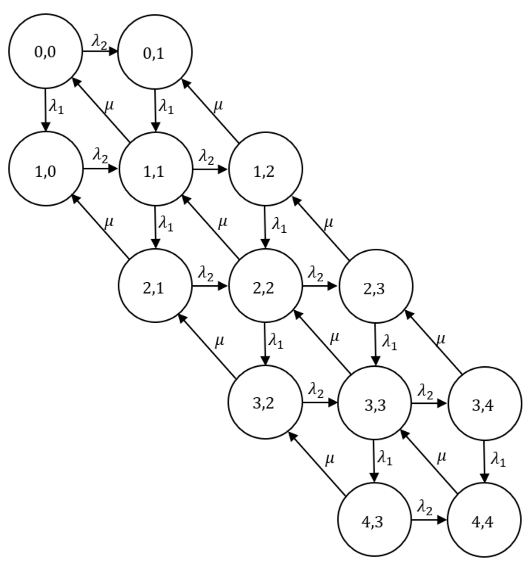
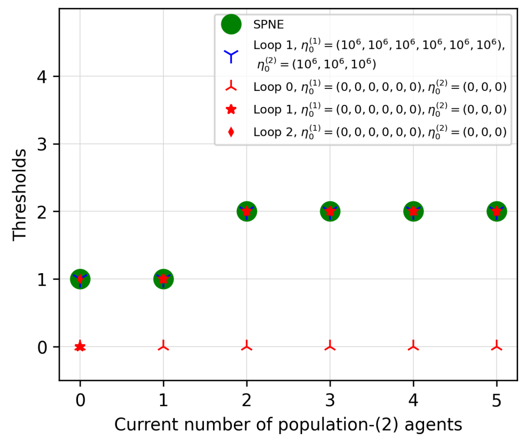


Figure 4.2.4: Transition diagram in subgame perfect Nash equilibrium.

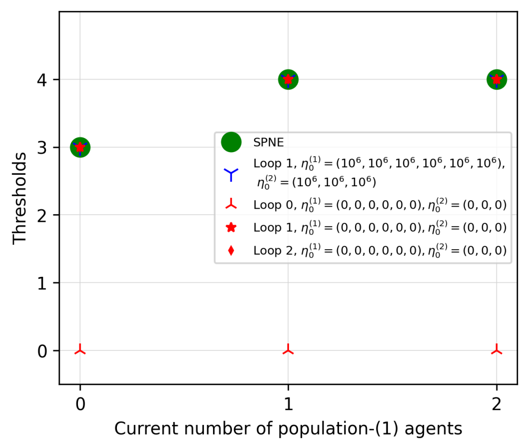
This example illustrates Remark 4.2.4.

Example 2 (Convergence of Algorithm 2 and equilibrium analysis). In this example, we examine the performance of Algorithm 2. Set $\lambda_1 = 5$, $\lambda_2 = 4$, $\mu = 6$, $R_1 = 3$, $R_2 = 6$ and $C_1 = C_2 = 7$. It follows that $N_1 = 2$ and $N_2 = 5$, and threshold strategies of the two populations are represented by vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ of lengths $N_1 + 1$ and $N_2 + 1$, respectively.

We consider two cases differentiated by the initial threshold policy followed by each population. In the first experiment, all elements in both threshold vectors are set relatively large at 10^6, which represents the case where agents are supposed non-strategic. In the second experiment, both threshold vectors are set to 0, representing the case in which agents do not join the system at first. The convergence of Algorithm 2 in the two cases is illustrated in Figure 4.2.5.



((a)) population-1.



((b)) population-2.

Figure 4.2.5: Convergence to subgame perfect Nash equilibrium of Algorithm 2.

It can be seen that, in the first experiment, it took only one iteration for the algorithm to converge to the subgame perfect Nash equilibrium in the first experiment, while two iterations were needed in the second experiment. We also conducted a large number of similar experiments on different initial values for $\boldsymbol{\eta}_0^{(1)}$ and $\boldsymbol{\eta}_0^{(2)}$, and on different set of system parameters, and found that Algorithm 2 *always* converges to the subgame perfect outcome.

Returning to the above example, we can derive the reduced form of the equilibrium threshold strategies as $\boldsymbol{\eta}^{(1)*} = (1, 1, 2, 2, 2)$ and $\boldsymbol{\eta}^{(2)*} = (3, 4, 4)$.

Example 3 (Sensitivity analyses and optimal designs). In this example, set $\lambda_1 = 4$, $\lambda_2 = 5$, $\mu = 6$, $C_1 = C_2 = 7$, $V_1 = 20$ and $V_2 = 5$. The price is tuned for optimal system designs.

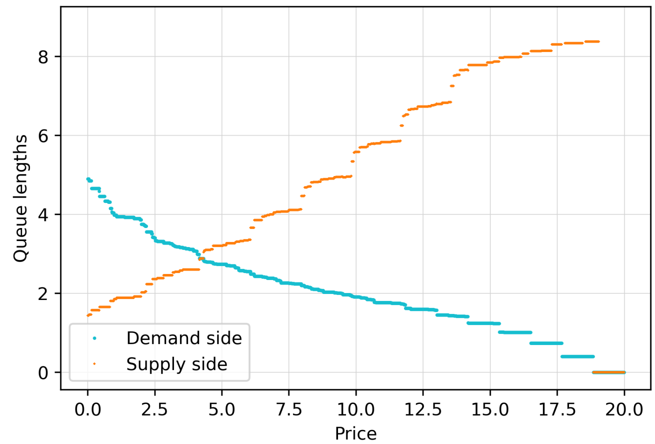


Figure 4.2.6: Queue lengths with respect to price.

Figure 4.2.6 illustrates the sensitivity of mean queue lengths for both populations. A higher price means that more agents on the supply side are willing to wait, while fewer agents on the demand side find incentive to stay enqueued for long. However, when the price exceeds an upper threshold at which no demanders find a positive payoff even if they do not have to wait in the queue, the both queue lengths drop to zero. This is because, if all demanders do not join the system, joining suppliers would wait forever.

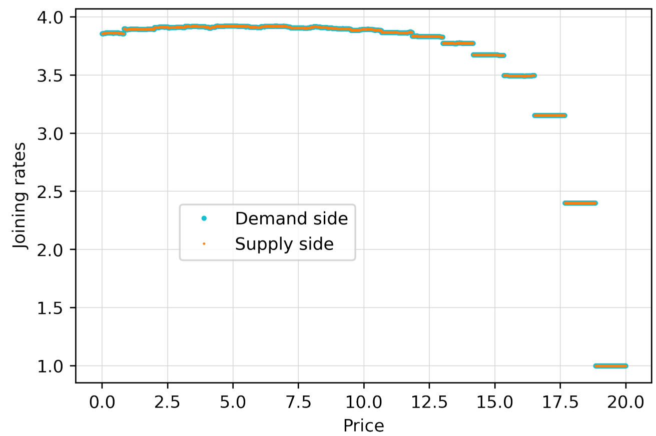


Figure 4.2.7: Joining rates with respect to price.

Figure 4.2.7 shows the joining rates of both populations with respect to the price. Since this is a matching queue, it is intuitive that the mean supply and demand quantities balance in equilibrium, regardless of the price. Furthermore, as the price increases to the upper limit at which no demanders join, both joining rates also drop to zero.

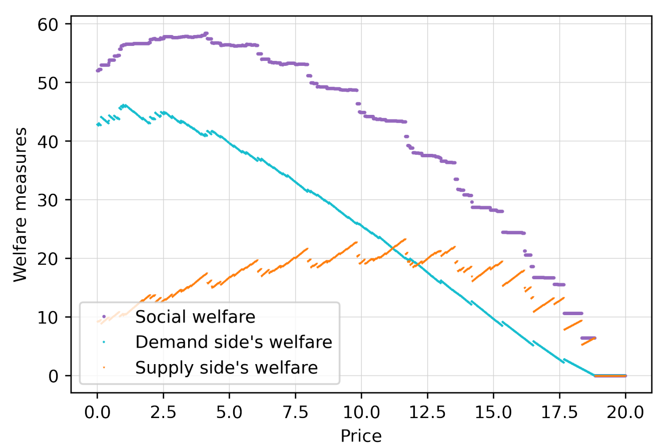


Figure 4.2.8: Welfare measures with respect to price.

Figure 4.2.8 shows that mean social welfare is discretely unimodal with respect to the price. It can be observed that social welfare remains unchanged within each price range. If the price increases but does not change the threshold strategies of both agent populations, it does not affect social welfare because the fare is similar to a transfer payment from the demand side to the supply side. In this example, the maximum social welfare is 58.43, obtained at prices within $[4.09, 4.15)$. In a social application, this optimal social welfare could be achieved through interference pricing from the government. The mean welfare of the demand side peaks at 46.16 when $p = 0.98$, and the mean welfare of the supply side peaks at 23.29 when $p = 11.66$.

4.3 Model 6: Nash equilibria of a three-population queueing game in an unobservable queueing system with multiple matching points, one finite end and one infinite end

The model in this section is motivated from a passenger-taxi queueing system in which there are two types of passengers, differentiated by their mean matching times with taxis. This feature captures the fact that there may be multiple types of passengers whose matching times with taxis are not identical. For example, domestic passengers can match with local taxi drivers more quickly, while it may take a longer time for foreign visitors to communicate with taxis drivers. Another consideration is that the parking space can only accommodate a limited numbers of cars in relative comparison with the capacity of the waiting area for passengers, so it is reasonable to consider a queue with infinite capacity of the demand end and finite capacity of the supply end.

This model is based on the following two papers:

- Nguyen, H. Q. & Phung-Duc, T. (2022). Queueing analysis and Nash equilibria in an unobservable taxi-passenger system with two types of passenger. In International Conference on Operations Research and Enterprise Systems (ICORES) (pp. 48–55)[40]; and
- Nguyen, H. Q. & Phung-Duc, T. (2023). Performance analysis and Nash equilibria in a taxi-passenger system with two types of passenger. SN Computer Science, 4(1), 1–13[44].

This model, which contains three populations of strategic agents, is the most complicated one in the series of models. It will focus on the explanation of the existence of different equilibria and numerical analyses only. Another purpose of this model is to illustrate how difficult the analyses become if more strategic populations are added to the model, which raises the need for other approaches such as simulation-based or heuristic methods in future work.

4.3.1 Preliminaries

Consider a society $\mathcal{P} = \{1, 2, 3\}$ that consists of three populations of agents arriving at a double-ended queueing system containing S identical matching points. The three populations of agents represent a market with a demand side (population 1 and population 2) and a supply side (population 3). The area (including S matching points) can accommodate at most K population-3 agents at the same time ($K \geq S$). Matching is performed on a first-come-first-served basis by a pair of an agent on the demand side and an agent on the supply side. The reward upon the completion of a service and the waiting cost per unit time of a population-i agent are denoted by R_i and C_i ($i = 1, 2, 3$). Assume that all agents on the demand side have identical service value and waiting cost rate, i.e. $R_1 = R_2 = R$ and $C_1 = C_2 = C$.

In an ideal situation where agents are given enough incentive to join the queue without balking, agents on the demand side and the supply side arrive at the system according to Poisson processes with potential arrival rates Λ and Λ_3, respectively. A matching point receives a population-1 agent with probability ε, and a population-2 agent with probability $1 - \varepsilon$. The matching times of population-1 and population-2 agents follow exponential distributions with parameters μ_1 and μ_2, respectively. Without loss of generality, we can assume that $\mu_1 < \mu_2$, meaning that population-1 agents have a larger mean matching time. When the waiting area reaches its maximum capacity, the arrival of any new population-3 agent is blocked, so that population-3 agent leaves immediately. On the other hand, we assume that there is no limit on the buffer of the demand side.

Let $\sigma^{(1)}$, $\sigma^{(2)}$ and $\sigma^{(3)}$ denote the joining probabilities of population-1 agents, population-2 agents, and population-3 agents, respectively. Let $\lambda = (\sigma^{(1)}\varepsilon + \sigma^{(2)}(1-\varepsilon))\Lambda$, $\lambda_1 = \sigma^{(1)}\varepsilon\Lambda$, $\lambda_2 = \sigma^{(2)}(1 - \varepsilon)\Lambda$, $\lambda_3 = \sigma^{(3)}\Lambda_3$ and $\alpha = \frac{\sigma^{(1)}\varepsilon}{\sigma^{(1)}\varepsilon+\sigma^{(2)}(1-\varepsilon)}$. Then, λ_1, λ_2 and λ_3 are the actual arrival rates of agents; these will be are used to derive performance measures in the following section.

4.3.2 Performance measures

Let $X_1(t), X_2(t)$ and $X_3(t)$ respectively denote the number of population-1 agents being matched at the matching points, the number of population-3 agents in the system, and the total number of passengers in the system, at time t. The process $\{(X_1(t), X_2(t), X_3(t)) \mid t \geq 0\}$ is a continuous-time Markov Chain with the state space $\mathbb{S}$ given by

$$\mathbb{S} = \{(x_1, x_2, x_3) \in \{0, 1, ..., S\} \times \{0, 1, ..., K\} \times \{0, 1, 2, \dots\}\}.$$

Also, as implied by their definitions, it should be noted that $x_1 \leq x_2$ and $x_1 \leq x_3$.

The system can be modeled as a quasi-birth-death process with the infinitesimal generator $\mathcal{Q}$ being expressed as follows.

$$\mathcal{Q} = \begin{pmatrix} \mathcal{B}^{(0)} & \mathcal{C}^{(0)} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \dots & \dots & \dots \\ \mathcal{A}^{(1)} & \mathcal{B}^{(1)} & \mathcal{C}^{(1)} & \mathcal{O} & \mathcal{O} & \dots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{A}^{(2)} & \mathcal{B}^{(2)} & \mathcal{C}^{(2)} & \mathcal{O} & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{A}^{(K)} & \mathcal{B}^{(K)} & \mathcal{C}^{(K)} & \mathcal{O} & \dots & \vdots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{A}^{(K)} & \mathcal{B}^{(K)} & \mathcal{C}^{(K)} & \mathcal{O} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

where $\mathcal{O}$ denotes a zero matrix of appropriate dimension. If we denote by $\mathbb{M}(m, n)$ the set of all $m \times n$-dimensional matrices, and $\mathcal{M}(x_2, x_3)$ $(x_2, x_3 \in \mathbb{Z}^+)$ the element at the x_2^{th} row, x_3^{th} column of a matrix $\mathcal{M}$, then the block matrices in $\mathcal{Q}$ can be defined by the system parameters $\lambda, \lambda_3, \mu_1, \mu_2$ and α, shown as follows.

$\mathrm{A}^{(K)}, \mathcal{B}^{(K)}, \mathcal{C}^{(K)} \in \mathbb{M}\left(\frac{(S+1)(S+2)}{2} + (K-S)(S+1), \frac{(S+1)(S+2)}{2} + (K-S)(S+1)\right)$ such that

$$\mathcal{C}^{(K)} = diag(\lambda, \lambda, ..., \lambda);$$

$$\mathcal{A}^{(K)}\left(\frac{(S+1)(S+2)}{2} + i(S+1) + j, \frac{S(S+1)}{2} + i(S+1) + j\right) = \alpha(j-1)\mu_1 + (1-\alpha)(S-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\left(\frac{(S+1)(S+2)}{2} + i(S+1) + j, \frac{S(S+1)}{2} + i(S+1) + (j+1)\right) = \alpha(S-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\left(\frac{(S+1)(S+2)}{2} + i(S+1) + (j+1), \frac{S(S+1)}{2} + i(S+1) + j\right) = (1-\alpha)j\mu_1,$$

$$\mathcal{B}^{(K)}\left(\frac{S(S+1)}{2}+i(S+1)+j,\frac{(S+1)(S+2)}{2}+i(S+1)+(j+1)\right)=\lambda_3,$$

for $i=0,1,...,K-S$ and $j=1,2,...,S+1$;

$$\mathcal{A}^{(K)}\left(\frac{(i+1)(i+2)}{2}+j,\frac{i(i+1)}{2}+j\right)=(i-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\left(\frac{(i+1)(i+2)}{2}+(j+1),\frac{i(i+1)}{2}+j\right)=j\mu_1,,$$

$$\mathcal{B}^{(K)}\left(\frac{i(i+1)}{2}+j,\frac{(i+1)(i+2)}{2}+j\right)=(1-\alpha)\lambda_3,$$

$$\mathcal{B}^{(K)}\left(\frac{i(i+1)}{2}+j,\frac{(i+1)(i+2)}{2}+(j+1)\right)=\alpha\lambda_3,$$

for $i=0,1,...,S-1$ and $j=1,2,...,i+1$.

For $n<K$,

$$\mathcal{C}^{(n)}\in\mathbb{M}\left(\frac{(n+1)(n+2)}{2}+(K-n)(n+1),\frac{(n+1)(n+2)}{2}+(K-n)(n+2)\right),$$

for $n\geq 1$, and

$$\mathcal{A}^{(n)}\in\mathbb{M}\left(\frac{(n+1)(n+2)}{2}+(K-n)(n+1),\frac{n(n+1)}{2}+(K-(n-1))n\right),$$

$$\mathcal{B}^{(n)}\in\mathbb{M}\left(\frac{(n+1)(n+2)}{2}+(K-n)(n+1),\frac{(n+1)(n+2)}{2}+(K-n)(n+1)\right),$$

such that

$$\mathcal{C}^{(n)}(i,i)=\lambda,$$

for $i=1,2,...,\frac{(n+1)(n+2)}{2}$;

$$\mathcal{C}^{(n)}\left(\frac{(n+1)(n+2)}{2}+i(n+1)+j,\frac{(n+1)(n+2)}{2}+i(n+2)+j\right)=(1-\alpha)\lambda,$$

$$\mathcal{C}^{(n)}\left(\frac{(n+1)(n+2)}{2}+i(n+1)+j,\frac{(n+1)(n+2)}{2}+i(n+2)+(j+1)\right)=\alpha\lambda,$$

$$\mathcal{A}^{(n)}\left(\frac{(n+1)(n+2)}{2}+i(n+1)+j,\frac{n(n+1)}{2}+i(n+1)+j\right)=(n-(j-1))\mu_2,$$

$$\mathcal{A}^{(n)}\left(\frac{(n+1)(n+2)}{2}+i(n+1)+(j+1),\frac{n(n+1)}{2}+i(n+1)+j\right)=j\mu_1,$$

$$\mathcal{B}^{(n)}\left(\frac{n(n+1)}{2}+i(n+1)+j,\frac{(n+1)(n+2)}{2}+i(n+1)+(j+1)\right)=\lambda_3,$$

for $i=0,1,\ldots,K-n$ and $j=1,2,\ldots,n+1$;

$$\mathcal{A}^{(n)}\left(\frac{(i+1)(i+2)}{2}+j,\frac{i(i+1)}{2}+j\right)=(i-(j-1))\mu_2,$$

$$\mathcal{A}^{(n)}\left(\frac{(i+1)(i+2)}{2}+(j+1),\frac{i(i+1)}{2}+j\right)=j\mu_1,$$

$$\mathcal{B}^{(n)}\left(\frac{i(i+1)}{2}+j,\frac{(i+1)(i+2)}{2}+j\right)=(1-\alpha)\lambda_3,$$

$$\mathcal{B}^{(n)}\left(\frac{i(i+1)}{2}+j,\frac{(i+1)(i+2)}{2}+(j+1)\right)=\alpha\lambda_3,$$

for $i=0,1,\ldots,n-1$ and $j=1,2,\ldots,i+1$; and $n\geq 1$.

Finally,

$$\mathcal{B}^{(0)}(i,i)=-\sum_{j}\mathcal{A}^{(0)}(i,j)-\sum_{j\neq i}\mathcal{B}^{(0)}(i,j),$$

and

$$\mathcal{B}^{(n)}(i,i)=-\sum_{j}\left(\mathcal{A}^{(n)}(i,j)+\mathcal{C}^{(n)}(i,j)\right)-\sum_{j\neq i}\mathcal{B}^{(n)}(i,j)$$

for $n=1,\ldots,K$.

Letting $\mathcal{Q}^*=\mathcal{A}^{(K)}+\mathcal{B}^{(K)}+\mathcal{C}^{(K)}$, we can then derive the stability condition of the system by simultaneously solving the following equations for $\boldsymbol{\gamma}$.

$$\boldsymbol{\gamma}\mathcal{Q}^*=0,$$

and

$$\boldsymbol{\gamma}\mathbf{e} = 1,$$

where $\boldsymbol{\gamma}$ is the row vector representing the stationary distribution of the infinitesimal generator $\mathcal{Q}^*$, 0 is a row vector with all elements equal to 0 and $\mathbf{e}$ is a column vector with all elements equal to 1. The stability condition, then, is

$$\boldsymbol{\gamma}\mathcal{C}^{(K)}\mathbf{e} < \boldsymbol{\gamma}\mathcal{A}^{(K)}\mathbf{e}. \tag{4.3.1}$$

If stability condition (4.3.1) is not satisfied, the system becomes an $\mathrm{M}/\mathrm{H}_2/S/K$ queue of population-3 agents in which agents on the demand side become "servers" (H_2 denotes a hyper-exponential distribution with two phases). The mean waiting times of population-1 and population-2 agents are given by

$$W_1 = W_2 = +\infty.$$

When the stability condition holds, we can derive the steady state probabilities $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \pi_1, \pi_2, \ldots)$, where $\boldsymbol{\pi}_{x_3} = (\pi_{0,0,x_3}, \pi_{1,0,x_3}, \ldots, \pi_{S,K,x_3})$ is the vector encoding all probabilities when there are x_3 demanders in the system at the steady state. For $x_3 \geq K$, there exists a constant matrix $\mathcal{R}$ such as

$$\boldsymbol{\pi}_{x_3} = \boldsymbol{\pi}_K \mathcal{R}^{x_3 - K},$$

where $\mathcal{R}$ satisfies

$$\mathcal{C}^{(K)} + \mathcal{R}\mathcal{B}^{(K)} + \mathcal{R}^2\mathcal{A}^{(K)} = \mathcal{O}. \tag{4.3.2}$$

The solution of the matrix equation (4.3.2) is obtained by the Matrix Geometric Method proposed by Neuts[39]. Furthermore, for $1 \leq x_3 \leq K$, we also have

$$\boldsymbol{\pi}_{x_3} = \boldsymbol{\pi}_{x_3 - 1}\mathcal{R}^{(x_3)},$$

where $\mathcal{R}^{(K)} = \mathcal{R}$, and $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots, \mathcal{R}^{(K-1)}; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_K$ are recursively calculated as

$$\mathcal{R}^{(x_3)} = -\mathcal{C}^{(x_3)}(\mathcal{B}^{(x_3)} + \mathcal{R}^{(x_3+1)}\mathcal{A}^{(x_3+1)})^{-1}.$$

$\boldsymbol{\pi}_0$ is determined by solving

$$\boldsymbol{\pi}_0(\mathcal{B}^{(0)} + \mathcal{R}^{(1)}\mathcal{A}^{(1)})^{-1} = 0,$$

and

$$\boldsymbol{\pi}_0 \left(\mathcal{I} + \sum_{x_3=1}^{S-1} \prod_{x_2=1}^{x_3} R^{(x_3)} + \left(\prod_{x_2=1}^{S} \mathcal{R}^{(x_3)} \right) (\mathcal{I} - \mathcal{R})^{-1} \right) \mathbf{e} = 1,$$

where $\mathcal{I}$ denotes an identity matrix of appropriate dimension.

Next, we derive performance measures of the system, including average queue lengths, and average waiting times of demanders and suppliers. The average number of demanders in the waiting line is given by

$$\begin{aligned}
\mathcal{L} &= \sum_{x_3=0}^{\infty} \sum_{x_2=0}^{K} \sum_{l=0}^{S} (x_3 - \min\{x_2, x_3, S\}) \pi_{x_1,x_2,x_3} \\
&= \sum_{x_3=0}^{K-1} x_3 \boldsymbol{\pi}_{x_3} \mathbf{e}_{x_3} - \sum_{x_3=0}^{K-1} \boldsymbol{\pi}_{x_3} \mathbf{g}_{x_3} + \boldsymbol{\pi}_K (\mathcal{I} - \mathcal{R})^{-1} [K\mathcal{I} + (\mathcal{I} - \mathcal{R})^{-1} \mathcal{R}] \mathbf{e}_K - \pi_K (\mathcal{I} - \mathcal{R})^{-1} \mathbf{g}_K,
\end{aligned} \tag{4.3.3}$$

where

- $\mathbf{e}_{x_3}$ is a vector of the same dimension as π_{x_3}, will all elements equal to 1.
- $\mathbf{g}_{x_3} = (g_{0,0,x_3}, g_{1,0,x_3}, \ldots, g_{S,K,x_3})$, where $g_{x_1,x_2,x_3} = \min\{x_2, x_3, S\}$.

Note that the summation in (4.3.3) excludes (x_1, x_2, x_3) not existing in the state space. The same rule applies to all later summations and products.

The average number of population-3 agents in the system is given by

$$\begin{aligned}
L_3 &= \sum_{x_3=0}^{\infty} \sum_{x_2=0}^{K} \sum_{l=0}^{S} x_2 \pi_{x_1,x_2,x_3} \\
&= \sum_{x_3=0}^{K-1} \pi_{x_3} \mathbf{f}_{x_3} + \pi_K (\mathcal{I} - \mathcal{R})^{-1} \mathbf{f}_K.
\end{aligned} \tag{4.3.4}$$

Here, $\mathbf{f}_{x_3} = (f_{0,0,x_3}, f_{1,0,x_3}, \ldots, f_{S,K,x_3})$, where $f_{x_1,x_2,x_3} = x_2$.

Corresponding to (4.3.3) and (4.3.4), the average sojourn time population-3 agents can be calculated via Little's Law as follows

$$W_3 = \frac{L_3}{\lambda_3(1 - P_b)},$$

where P_b is the blocking probability of population-3 agents, calculated by

$$P_b = \sum_{x_3=0}^{K-1} \boldsymbol{\pi}_{x_3} \mathbf{u}_{x_3} + \boldsymbol{\pi}_K (\mathcal{I} - \mathcal{R})^{-1} \mathbf{u}_K.$$

Here, $\mathbf{u}_{\mathbf{x}_3}$ is a vector with the same dimension as $\boldsymbol{\pi}_{x_3}$, in which the last $\min(x_3 + 1, S + 1)$ elements equal 1 and the other elements equal 0.

The average number of demanders in the system is given by

$$\begin{aligned} L &= \sum_{x_3=0}^{\infty} x_3 \boldsymbol{\pi}_{x_3} \mathbf{e}_{x_3} \\ &= \sum_{x_3=0}^{K-1} x_3 \boldsymbol{\pi}_{x_3} \mathbf{e}_{x_3} + \boldsymbol{\pi}_K (\mathcal{I} - \mathcal{R})^{-1} [K\mathcal{I} + (\mathcal{I} - \mathcal{R})^{-1} \mathcal{R}] \mathbf{e}_K. \end{aligned}$$

The average sojourn time of all demanders is given by

$$W = \frac{L}{\lambda}.$$

Since a demander knows his own type before entering the system, the expected sojourn times of a population-1 and a population-2 agent are estimated as

$$W_1 = \frac{\mathcal{L}}{\lambda} + \frac{1}{\mu_1}, \tag{4.3.5}$$

and

$$W_2 = \frac{\mathcal{L}}{\lambda} + \frac{1}{\mu_2}. \tag{4.3.6}$$

Finally, social welfare, which equals the total utility of all agents in the system per time unit, is given by

$$SW = \lambda R + \lambda_3(1 - P_b)R_3 - CL - C_3L_3.$$

For further analysis, we acknowledge the following properties, which are supported by intuition and numerous simulation results.

Axiom 4.3.1. *The expected sojourn time of an arbitrary agent depends on the strategies of agents in their own population and the other populations. In other words, W_1, W_2 and W_3 are functions of $\sigma^{(1)}, \sigma^{(2)}$ and $\sigma^{(3)}$. The monotonic properties of these functions with respect to each variable are given as follows.*
Under the stability condition given in (4.3.1)*,*
(1) W_1 is continuously increasing in $\sigma^{(1)}$ and $\sigma^{(2)}$, and decreasing in $\sigma^{(3)}$.
(2) W_2 is continuously increasing in $\sigma^{(1)}$ and $\sigma^{(2)}$, and decreasing in $\sigma^{(3)}$.
(3) W_3 is continuously decreasing in $\sigma^{(1)}$ and $\sigma^{(2)}$, and increasing in $\sigma^{(3)}$.

4.3.3 Nash Equilibria

In this section, we derive all possible Nash equilibria at which agents make a best response to the strategies of other agents. The social profile, which is represented by a triplet $\left(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\right)$, is denoted $\mathbf{X}$. Let $\bar{\mathbf{X}} = \left(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}, \bar{\sigma}^{(3)}\right)$ be the social profile in equilibrium.

The payoff of an arbitrary population-1 agent who adopts a strategy $\sigma^{(1)}$ against the social profile $\bar{\mathbf{X}}$ is given by

$$U_1\left(\sigma^{(1)}|\bar{\mathbf{X}}\right) = \sigma^{(1)}\left(R - CW_1\left(\bar{\mathbf{X}}\right)\right).$$

By definition, $\bar{\sigma}^{(1)}$ is a best response against the social profile in equilibrium, which means

$$\bar{\sigma}^{(1)} \in \underset{\sigma^{(1)}}{\arg\max}\, U_1\left(\sigma^{(1)}|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R - CW_1\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R - CW_1\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R - CW_1\left(\bar{\mathbf{X}}\right) > 0. \end{cases} \tag{4.3.7}$$

If we similarly define $U_2\left(\sigma^{(2)}|\bar{\mathbf{X}}\right)$ and $U_3\left(\sigma^{(3)}|\bar{\mathbf{X}}\right)$ for the other two populations, we also have

$$\bar{\sigma}^{(2)} \in \underset{\sigma^{(2)}}{\arg\max}\, U_2\left(\sigma^{(2)}|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R - CW_2\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R - CW_2\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R - CW_2\left(\bar{\mathbf{X}}\right) > 0, \end{cases} \tag{4.3.8}$$

and

$$\bar{\sigma}_3 \in \underset{\sigma^{(3)}}{\arg\max}\, U_3\left(\sigma^{(3)}|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R_3 - C_3W_3\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R_3 - C_3W_3\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R_3 - C_3W_3\left(\bar{\mathbf{X}}\right) > 0. \end{cases} \tag{4.3.9}$$

(4.3.7), (4.3.8) and (4.3.9) lead to 27 possible combinations. However, we can reduce the number of cases to consider by noting that $W_1\left(\bar{\mathbf{X}}\right) > W_2\left(\bar{\mathbf{X}}\right)$, and considering the following special cases. First, if $\bar{\sigma}^{(1)} = 0$, meaning that population-3 agents do not join the system, then it is easily seen that $\bar{\sigma}^{(1)} = \bar{\sigma}^{(2)} = 0$ since the best response of demanders is not to join the system either. On the other hand, if $\bar{\sigma}^{(2)} = 0$, meaning that population-2 agents have no incentive to join the system, then it is implied that $\bar{\sigma}^{(1)} = 0$ (since population-1 agents always expect longer sojourn times than population-2 agents), which leads to $\bar{\sigma}_3 = 0$. In other words, $\left(\bar{\sigma}^{(1)}, \bar{\sigma}^{(2)}, \bar{\sigma}_3\right) = (0,0,0)$ is an equilibrium and is the only equilibrium where $\bar{\sigma}^{(2)} = 0$ or $\bar{\sigma}_3 = 0$. When $\bar{\sigma}^{(2)} > 0, \bar{\sigma}_3 > 0$ and the stability condition (4.3.1) is satisfied, all possible equilibria can be derived as shown in Table 4.1.

Table 4.1: Equilibria and corresponding conditions.

Equilibria	**Conditions**
$(1,1,1)$	$R - CW_1(1,1,1) > 0$, $R - CW_2(1,1,1) > 0$, $R_3 - C_3W_3(1,1,1) > 0$.
$(\bar{\sigma}^{(1)},1,1)$	$R - CW_1(\bar{\sigma}^{(1)},1,1) = 0$, $R - CW_2(\bar{\sigma}^{(1)},1,1) > 0$, $R_3 - C_3W_3(\bar{\sigma}^{(1)},1,1) > 0$.
$(0,1,1)$	$R - CW_1(0,1,1) < 0$, $R - CW_2(0,1,1) > 0$, $R_3 - C_3W_3(0,1,1) > 0$.
$(0,\bar{\sigma}^{(2)},1)$	$R - CW_1(0,\bar{\sigma}^{(2)},1) < 0$, $R - CW_2(0,\bar{\sigma}^{(2)},1) = 0$, $R_3 - C_3W_3(0,\bar{\sigma}^{(2)},1) > 0$.
$(1,1,\bar{\sigma}^{(3)})$	$R - CW_1(1,1,\bar{\sigma}^{(3)}) > 0$, $R - CW_2(1,1,\bar{\sigma}^{(3)}) > 0$, $R_3 - C_3W_3(1,1,\bar{\sigma}^{(3)}) = 0$.
$(\bar{\sigma}^{(1)},1,\bar{\sigma}^{(3)})$	$R - CW_1(\bar{\sigma}^{(1)},1,\bar{\sigma}^{(3)}) = 0$, $R - CW_2(\bar{\sigma}^{(1)},1,\bar{\sigma}^{(3)}) > 0$, $R_3 - C_3W_3(\bar{\sigma}^{(1)},1,\bar{\sigma}^{(3)}) = 0$.
$(0,1,\bar{\sigma}^{(3)})$	$R - CW_1(0,1,\bar{\sigma}^{(3)}) < 0$, $R - CW_2(0,1,\bar{\sigma}^{(3)}) > 0$, $R_3 - C_3W_3(0,1,\bar{\sigma}^{(3)}) = 0$.
$(0,\bar{\sigma}^{(2)},\bar{\sigma}^{(3)})$	$R - CW_1(0,\bar{\sigma}^{(2)},\bar{\sigma}^{(3)}) < 0$, $R - CW_2(0,\bar{\sigma}^{(2)},\bar{\sigma}^{(3)}) = 0$, $R_3 - C_3W_3(0,\bar{\sigma}^{(2)},\bar{\sigma}^{(3)}) = 0$.

Equilibria #1 and #3 can be verified by simply checking their corresponding conditions. The other equilibria are derived by solving their corresponding conditional equations and double-checking other conditions. The solutions to those equations are not explicit but are computationally solvable. We will illustrate results in several numerical examples in the following section.

4.3.4 Numerical Analysis

In this section, we present the analysis in specific numerical examples. First, we assume that agents are not rational, and numerically examine the variation of some performance measures with respect to system parameters and actual joining rates. In the following examples, we set $\mu_1 = 2, \mu_2 = 5, \alpha = 0.3, R_1 = R_2 = R = 15, R_3 = 20, C_1 = C_2 = 4$ and $C_3 = 5$. Results are illustrated in Figures 4.3.1 to 4.3.8.

Figures 4.3.1 to 4.3.4 verify the monotonic properties of demanders' and population-3 agents' waiting times with respect to the agents' actual joining rates. The results are intuitive and follow exactly as in Axiom 4.3.1.

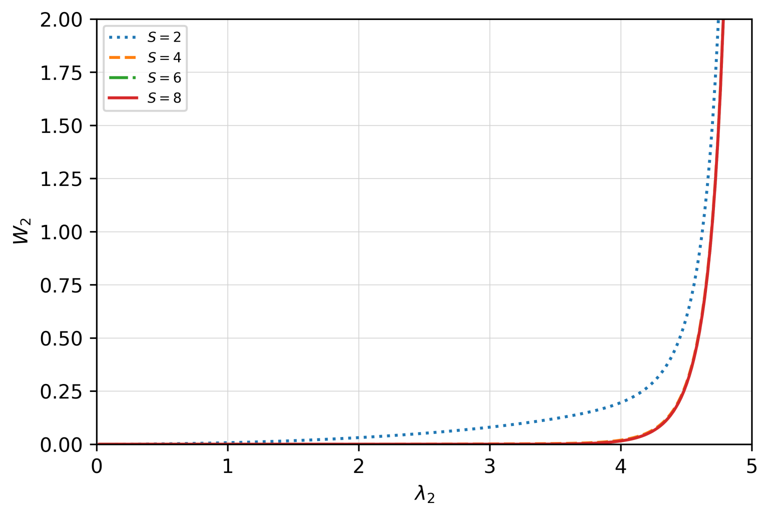


Figure 4.3.1: W_2 with respect to λ_2 ($\lambda_1 = 5, K = 20$).

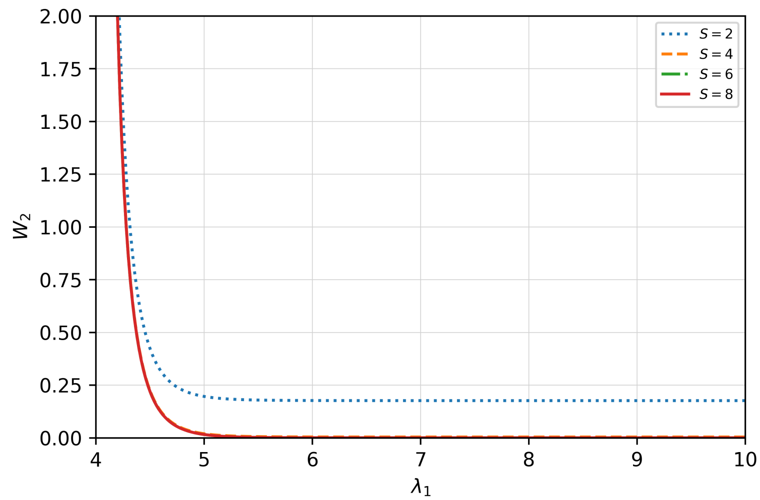


Figure 4.3.2: W_2 with respect to λ_1 ($\lambda_2 = 4, K = 20$).

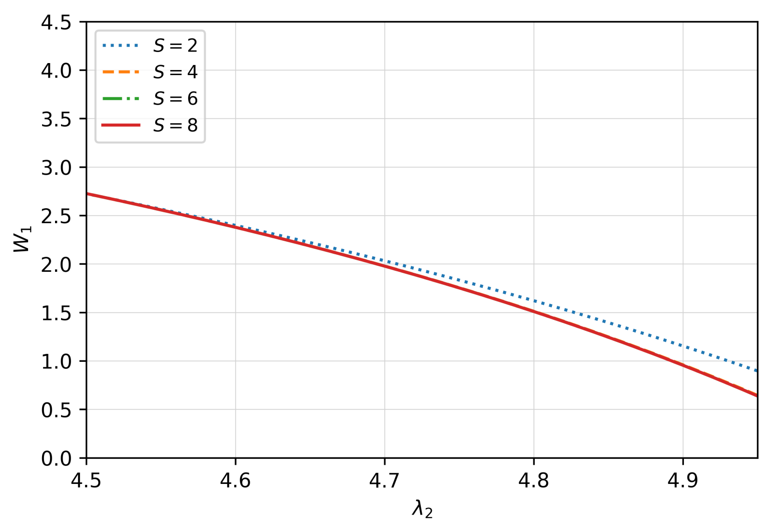


Figure 4.3.3: W_3 with respect to λ_2 ($\lambda_1 = 5, K = 20$).

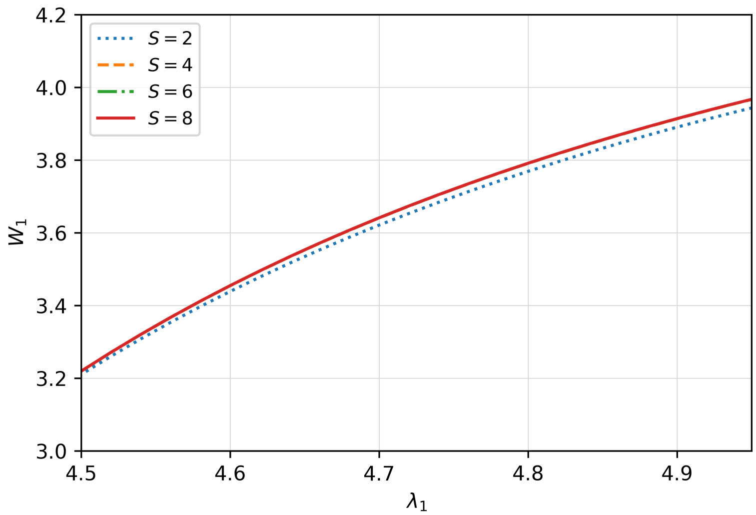


Figure 4.3.4: W_3 with respect to λ_1 ($\lambda_2 = 4, K = 20$).

Figures 4.3.5 and 4.3.6 show that the social welfare function is unimodal with respect to the joining rates of demanders and suppliers. There exists a value of passengers' (or population-3 agents') joining rate at which social welfare reaches its maximum. These patterns suggest that the platform manager can control the arrival rate of one side of agents in case the other side is not strategic, to maximize social welfare. More details about applicable control policies can be found in[21]. For example, when population-3 agents are not strategic and join the queue with rate $\lambda_1 = 5$, policy makers can interfere in the demanders' service value to adjust their arrival rate at around $\lambda_2^* = 4.7$, which yields the highest social welfare. When $\lambda_2 < \lambda_2^*$, demanders need more incentive to join the queue, so a fixed subsidy (such as a discount or coupon) can be granted. On the contrary, when $\lambda_2 > \lambda_2^*$, a toll fee can be applied to reduce the joining rate of demanders. The same policies apply in the case where suppliers are strategic and demanders are not.

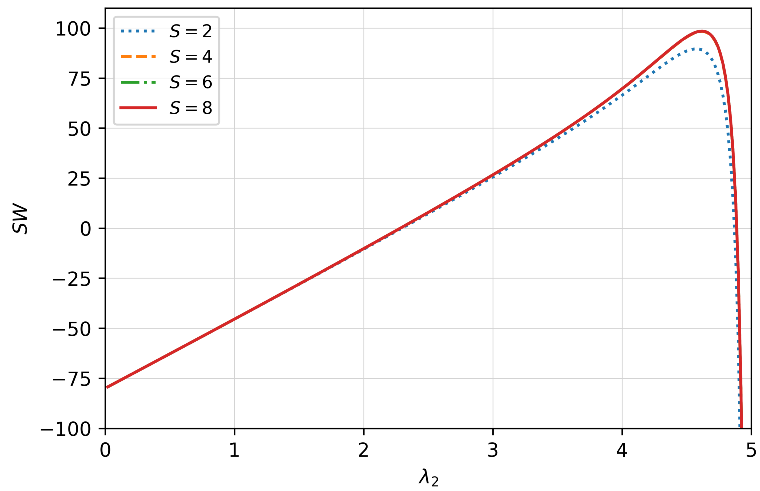


Figure 4.3.5: SW with respect to λ_2 ($\lambda_1 = 5, S = 3, K = 20$).

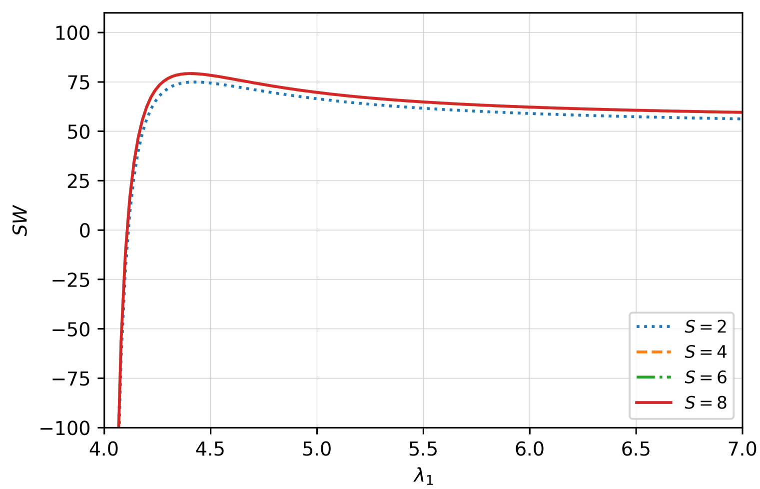


Figure 4.3.6: SW with respect to λ_1 ($\lambda_2 = 4, S = 3, K = 20$).

Figures 4.3.7 and 4.3.8 illustrate how social welfare varies according to the two system parameters S and K. In this experiment, social welfare increases quickly at first, then remains almost unchanged as S becomes larger. It can be observed that 5 matching points are enough and optimal in this example (considering that a larger waiting a may consume more budget for construction and management). On the contrary, social welfare decreases with increased buffer capacity. This phenomenon may stem from the fact that the buffer capacity already exceeds a particular "threshold," above which the queue length of population-3 agents gets longer and leads to inefficient waiting times, thus reducing social welfare.

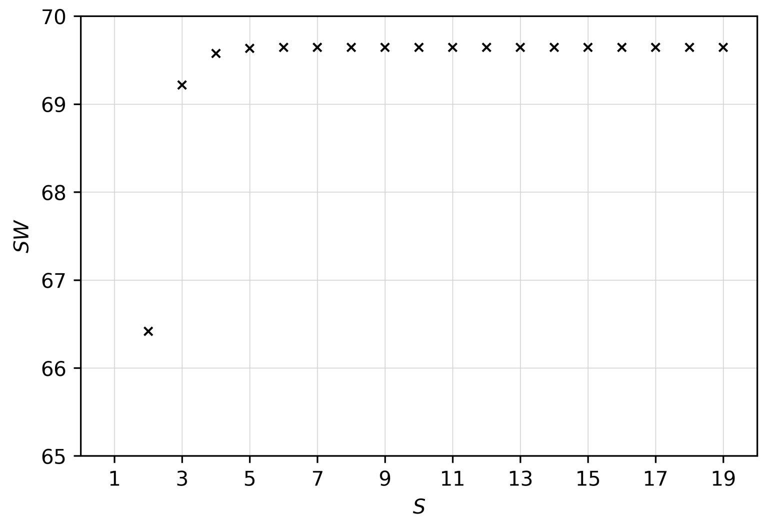


Figure 4.3.7: SW with respect to S ($\lambda_2 = 4, \lambda_1 = 5, K = 20$).

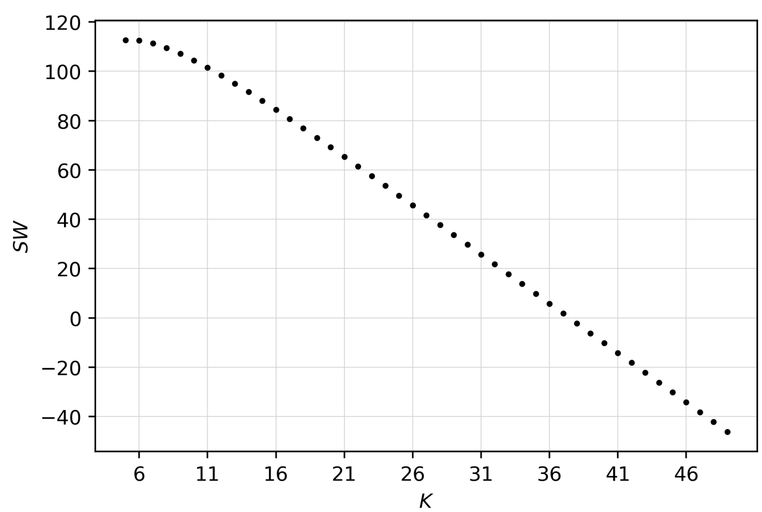


Figure 4.3.8: SW with respect to K ($\lambda_2 = 4, \lambda_1 = 5, S = 3$).

In what follows, we derive joining probabilities of agents and calculate social welfare in the case where agents are strategic. For this, we set $\Lambda_2 = 4$, $\Lambda_1 = 5$, $\mu_1 = 1$, $\mu_2 = 5$, $S = 3$, $K = 8$, $\varepsilon = 0.3$, $C_2 = 5$ and $C_1 = 4$.

First, it can be seen that the equilibrium $(0, 0, 0)$ exists in any setting of parameters. In reality, the system may end up at the equilibrium $(0, 0, 0)$ in extreme situations, for example, when the system is terminated. In the following example, we derive other equilibria from the situations in Table 4.1.

Example 1. Assume $R = 15$ and $R_3 = 20$. In this case, two equilibria exist: $(1, 1, 1)$ (and $(0, 0, 0)$). This means either that potential agents all join, or that none join at all.

Example 2. Assume $R = 15$ and $R_3 = 4$. In this example, we keep the service value of passengers unchanged while reducing the service value of population-3 agents. This make the equilibrium $(1, 1, 1)$ no longer exist since population-3 agents expect a negative payoff when they join the system at full rate. Instead, there exists an equilibrium at $(0, 0.9099, 0.5499)$, at which population-1 agents choose not to join the system at all, while both population-2 agents and population-3 agents join the system at a rate smaller than the corresponding potential rate.

Example 3. Assume $R = 2.5$ and $R_3 = 4$. In this example, we found the equilibrium $(0, 1, 1)$, meaning that population-1 agents choose not to join the system at all, while both population-2 agents and population-3 agents join the system at full potential rates.

5
Conclusion

This chapter covers the following points:

- Summarize the key findings of the study
- Discuss shortcomings of the study
- Present recommendations for future research

5.1 Summary of the key findings

This research proposed a theoretical framework for multi-population queueing games that employs queueing theory and game theory to model agents' joining behaviors and applies that framework to optimally design double-ended queueing systems according to such agent behaviors.

Six different models of queueing games in double-ended queueing systems, categorized into two main streams—systems with zero matching times (Chapter 3) and systems with nonzero matching times (Chapter 4)—have been considered in this dissertation. All six models shared the same game theoretical framework from Chapter 2, but each model poses its own difficulties to be solved. It has been noticed that, in models with zero matching time, only one agent population is present in a non-empty queue at a time. When the matching times are nonzero, in contrast, the queueing system may witness agents on both ends at some point. Technically, dismissing the matching time allowed for fewer dimensions in the system states of the Markov chain modeling the system, which reduced the complexity of the mathematical analysis. As such, most of the results in Chapter 3 could be derived in their closed form. Chapter 4 presented models with higher complexity, but at the cost of the results no longer being derived explicitly.

In observable queues (Sections 3.1, 3.2, 4.1, 4.2), it can be observed that the agents' strategies are subject to a threshold policy: there exist maximum queue lengths above which agents stop joining the queue. This feature is based on a key point: the monotone property of the waiting function with respect to the queue length in subgame perfect Nash equilibrium. In Section 3.2, the conditional expected waiting time and its monotonicity could be obtained instantly. However, in Sections 4.1 and 4.2, the expected waiting times could not be calculated explicitly: they are instead derived recursively by first-hitting-time analysis. As such, their monotone properties are not trivial to obtain. Rigorous proofs by induction have been presented, which suggests an effective method for similar problems.

Another common feature to be observed in most of the models is that the mean social welfare is usually discreetly unimodal with respect to the fee/subsidy/price being implemented; in other words, there always exists a fee/subsidy/price range in which mean social welfare is maximized. This feature was proved in the models in Sections 3.2 and 3.3 and was illustrated by numerical examples in Sections 4.1 and 4.2.

Regarding the proposed optimal policies, two typical policies have been adopted from literature: to impose toll fees or to grant a subsidy, and price intervention. Although the implementations are similar across the literature, the context varies: it is necessary to regulate the behavior of multiple agent populations rather than one population only. In that sense, price intervention offers the advantage of simultaneously regulating the behaviors of both the supply and demand sides, although it may not always be applicable.

In summary, this study proposed a neat solution to a series of queueing games and their optimal designs with at most three populations of agents. Although the theoretical framework was much more general, allowing for an arbitrary n strategic agent populations, it is likely that a heuristic scheme or a simulation approach is needed to solve problems of higher complexity.

5.2 Shortcomings and recommendations for future research

This study leaves several shortcomings which open corresponding directions for future research, as follows.

First, in the observable queue settings considered in Sections 3.1, 3.2, 4.1 and 4.2, it was shown that multiple equilibria exist in those queueing games. Irrational outcomes can be eliminated, which leaves only one subgame perfect Nash equilibrium remaining. However, in the unobservable queue settings (Sections 3.3 and 4.3), multiple equilibria exist and it is not possible to conclude about the rationality of each outcome. It also remains open to determine the situations in which the system ends up at a specific equilibrium. This problem of equilibrium selection deserves further investigation.

Second, in Section 4.2, a fixed-point iteration based algorithm to compute the rational outcome of the game was proposed. Although Algorithm 2 was shown to converge to the subgame perfect outcome in all presented numerical examples, its validation needs further investigation: 1) Is there a rigorous proof of the convergence of the algorithm? 2) Does a similar loop algorithm work in other population games? In fact, the same idea has been applied to another model (Nguyen & Phung-Duc [45]) in which customer arrivals affect the on/off status of servers, thereby changing the expected waiting times of those who were already enqueued before. The successful applications of the algorithm suggest that it may apply in other queueing models in which agents can observe the system state, and the expected waiting times of enqueued agents are affected by the joining strategies of later comers. Under these circumstances, agents keep updating their joining strategies until an equilibrium is reached, meaning that agents do not find any better strategies than following the crowd.

Third, technically, the six models in this dissertation are differentiated by the following main features: the observability of system states (observable or unobservable), matching times (zero or nonzero) and the number of (strategic) populations. The models are intentionally chosen because each model has its own difficulties and hence requires different techniques to solve. However, the combination of the aforementioned features may lead to many more models which have not been considered in the current study for several reasons (for example, similarity with considered models, complexity—too trivial or too complicated to be solved by the current framework, etc). A totally new model can be obtained by varying one of the features. Regarding the level of information that is revealed to agents, it is of interest to investigate partially observable queues. For example, in Model 4 (Section 4.1) and Model 5 (Section 4.2), we can consider a setting in which one of the two dimensions of the system state is invisible to agents. A challenge posed in this setting is a concrete mathematical proof of the possible monotone property of the conditional expected waiting time with respect to the observable dimension which is the precursor to a threshold strategy. Another direction for future research is to incorporate the heterogeneity of agents in reality by expanding the set of agent populations and generalizing the results with arbitrary numbers of agents on both sides of the queue, which greatly increases the complexity of the problem.

Another assumption that can be relaxed is the linear waiting cost. In all the models, the waiting cost rates were assumed constant and the waiting times were always non-increasing with respect to the position of agents. As a matter of fact, there was no *reneging* behavior, as agents always found nonnegative payoff in the queue over time (once they joined). In fact, due to several factors (such as psychology), the waiting cost may be nonlinear. In such a case, if the expected waiting cost at a specific state happens to exceed the service value, the tagged agent would leave the system. This phenomenon would expand the action space and increase the complexity of the problem.

Furthermore, the analyses in the current dissertation rely on several assumptions that might not be realistic in some applications. It is always of interest to incorporate policies that characterize real-life double-ended queueing systems. Some suggestions for possible further investigations are as follows:

- Batch arrivals and batch services can be taken into consideration. Furthermore, one can allow for a more flexible matching mechanism: one-to-many or many-to-many, instead of the one-to-one regime considered throughout the study. These considerations are included in Xu et al.[59], Chai et al.[7], but not under a population game theoretical setting.

- Some passenger–taxi systems may adopt a gated policy which blocks the entrance of an arriving taxi during the idle state and allows taxis to join during the busy period. This policy was captured in Wang et al.[56] under a one-population game setting in passenger–taxi queues with zero matching time.

- The current game theoretical framework is based on a latent assumption that agents are extremely "smart"; however, it is quite impractical that agents can compute their expected waiting times using a complicated formula as in (4.1.1), (4.2.1). To overcome this flaw, a random error term can be added to agents' estimated waiting time, which is known as *bounded rationality*[23].

Last but not least, the main scope of the current study is optimal designs, i.e., to set the system parameters before an operation. The measures of the systems are quantified in their mean values—which represent the system performance in the long run. This is when queueing theory and a Markov process formalism come in handy. In fact, there can be many other approaches to a social problem tied to queueing systems in particular. For example, multi-agent simulation is an alternative powerful technique that can, theoretically, deal with much more universal and complex settings of the system, including the general non-Poisson inputs[48]. However, this method still poses several weaknesses compared to the framework in the current study. On the one hand, it requires numerous rounds of simulation to obtain the mean value of a system measure, which may be time-consuming and costly. On the other hand, while it is usually difficult to explain the patterns resulting from simulation, mathematical modeling allows for some interpretation of the system (especially in case of closed-form results). Another related formalism is the Markov decision process, which is particularly applicable in the scenario of *optimal control*, in which decision variables can be dynamically adjusted during the operation. In conclusion, depending on the specific class of problem, purpose and even resource, one or a mixture of different methods can be chosen for a social problem.

A
Symbols and Notations

List of symbols and notations that are used across sections

Symbol	Definition	Remark
i	A population of agents	index
$\tilde{i}$	The population other than population i (in case of 2 populations)	index
$\mathbb{N}$	Set of natural number $\{0, 1, 2, ...\}$	set
$\mathbb{P}$	Society (Set of populations)	set
$\mathbb{A}$	Set of actions	set
$\mathbb{S}$	State space	set
$\mathrm{card}(\mathbb{S})$	The number of elements in set $\mathbb{S}$	integer
λ_i	Arrival rate of population-i agents	ppl/time
ξ_i	Balking rate of population-i agents	ppl/time
μ	Matching rate (in case of 2 populations)	pairs of ppl/time
S	Number of matching points	integer
p	Price (service fee)	monetary
θ	Toll fee/Subsidy	monetary
C_i	Waiting cost rate of population-i agents	monetary/time
R_i	Service value of population-i agents	monetary
SW	Mean social welfare per unit time	monetary/time
$\mathbf{s}$	A multi-dimensional system state	vector
s	A one-dimensional system state	integer
$\boldsymbol{\sigma}^{(i)}$	Strategy of population-i agents	vector
$\bar{\boldsymbol{\sigma}}^{(i)}$	Strategy of population-i agents in equilibrium	vector
$\sigma_{\mathbf{s}}^{(i)}$	Joining probability of population-i agents at state $\mathbf{s}$ (in observable case)	scalar
$\sigma^{(i)}$	Joining probability of population-i agents (in unobservable case)	scalar
$\bar{\sigma}^{(i)}$	Joining probability of population-i agents in equilibrium (in unobservable case)	scalar
(s)	self-optimal value	index
(o)	overall (socially) optimal value	index
$\lfloor x \rfloor$	The largest integer number not exceeding x	integer

Table A.1: List of symbols used across sections

B
List of publications

B.1 Publications in international journals

[1] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). Strategic customer behavior and optimal policies in a passenger–taxi double-ended queueing system with multiple access points and nonzero matching times. *Queueing Systems*, 102, 481–508. (IF: 1.402, Q2)
[2] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). Supply–demand equilibria and multivariate optimization of social welfare in double-ended queueing systems. *Computers & Industrial Engineering*, 170, Article no. 108306. (IF: 7.180, Q1)
[3] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). A two-population game in observable double-ended queuing systems. *Operations Research Letters*, 50 (4), 407–414. (IF: 1.151, Q1)
[4] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). To wait or not to wait: Strategic behaviors in an observable batch-service queueing system. *Operations Research Letters*, 50 (3), 343–346. (IF: 1.151, Q1)

[5] Morozov, E., Rogozin, S., **Nguyen, H. Q.**, & Phung-Duc, T. (2022). Modified Erlang loss system for cognitive wireless networks. *Mathematics*, 10 (12), Article no. 2101. (IF: 2.592, Q1)
[6] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). Performance Analysis and Nash Equilibria in a Taxi-Passenger System with Two Types of Passenger. *SN Computer Science: Advances on Operations Research and Enterprise Systems*, 4, Article no. 73.

B.2 Publications in international conference proceedings

[1] **Nguyen, H. Q.** & Phung-Duc, T. (2023). M/M/c/Setup queues: Conditional mean waiting times and a loop algorithm to derive customer equilibrium threshold strategy. In Computer Performance Engineering: 18th European Workshop, EPEW 2022, Santa Pola, Spain, September 21–23, 2022, Proceedings (pp. 86–99).: Springer.
[2] **Nguyen, H. Q.**, & Phung-Duc., T. (2022). Queueing analysis and Nash equilibria in an unobservable taxi-passenger system with two types of passenger. In *Proceedings of the 11th international conference on operations research and enterprise systems - ICORES*, (p. 48-55). SciTePress. https://doi: 10.5220/ 0010825200003117
[3] **Nguyen, H. Q.**, & Phung-Duc, T. (2021). Mixture Density Networks as a general framework for estimation and prediction of waiting time distributions in queueing systems. In: Ballarini, P., Castel, H., Dimitriou, I., Iacono, M., Phung-Duc, T., Walraevens, J. (eds) Performance Engineering and Stochastic Modeling. EPEW ASMTA 2021. Lecture Notes in Computer Science (148–161), vol 13104. Springer, Cham. https://doi.org/10.1007/978-3-030-91825-5_9

B.3 Presentations at international conferences (abstract submission)

[1] **Nguyen, H. Q.**, & Phung-Duc, T. (2021). Equilibria of supply and demand in double-ended queueing systems. 31st European Conference on Operational Research, Athens, Greece, 11-14 July, 2021.

B.4 Presentations at domestic conferences

[1] **Nguyen, H. Q.**, & Phung-Duc, T. (2023). The rational outcome of a two-population game in a matching queue. In *Proceedings of the 39th (2022) Queue Symposium - Probabilistic Model and Its Applications*, 1-10.
[2] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). Equilibrium behavior in a double-ended queueing system with positive matching times. In *Proceedings of the 38th (2021) Queue Symposium - Probabilistic Model and Its Applications*, 14-23.
[3] **Nguyen, H. Q.**, & Phung-Duc, T. (2021). Strategic customer behavior and optimal policies in a passenger–taxi double-ended queueing system with multiple access points and nonzero matching times. In *Proceedings of the 37th (2020) Queue Symposium - Probabilistic Model and Its Applications*, 69-78.
[4] **Nguyen, H. Q.**, & Phung-Duc, T. (2021). Mixture Density Networks (MDNs) as a general framework for estimation of waiting time distributions in queueing systems: Two case studies. In *Proceedings of the 37th (2020) Queue Symposium - Probabilistic Model and Its Applications*, 130-131.

B.5 Submissions under review/revision (as of January 2023)

[1] **Nguyen, H. Q.**, & Phung-Duc, T. (2022). The Rational Outcome of a Two-population Game in a Matching Queue.

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