Topological properties of fractals

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Abstract

A fractal is a geometric shape containing involved structure at arbitrarily small scales. Examples of fractals abound, but a class of fractals have attracted particular attention: fractals which occur as attractors of iterated function systems.

An iterated function system (IFS) is a family of contractions f_k on $\mathbb{R}^d$, where $k \in \{1, \ldots, n\}$ and $n \geq 2$. According to Hutchinson [14], there is a unique nonempty compact subset F of $\mathbb{R}^d$, called the attractor of IFS $\{f_1, \ldots, f_n\}$, such that

$$F = \bigcup_{k=1}^{n} f_k(F). \tag{0.1}$$

In particular, F is called a self-similar set, if f_k's are similitudes. Certain self-similar sets are well known: the middle-third Cantor set, the Koch snowflake, the Sierpinski gasket, the Sierpinski carpet and the Heighway dragon. And there are a lot of researches on properties of self-similar sets in the history.

The notion of fractal dimension of a set is central to nearly all fractal studies. Roughly speaking, a fractal dimension is a tool to measure the complexity of a set.

Two fundamental and important fractal dimensions are Box-counting dimension and Hausdorff dimension, the specific definition is in Falconer [11]. Some effort has been devoted to calculating these two dimensions of IFS attractors (c.f Falconer [12]). Thereinto, an important result is about self-similar sets: if a self-similar set satisfies the open set condition (OSC) then the Hausdorff dimension of it satisfies the dimension formula and is equal to the Box-counting dimension of it. The definition of OSC is found in Falconer [11], [12]. And this condition ensures that the components $f_k(F)$ of F do not overlap too much.

Except the class of self-similar sets, there are many other classes of IFS attractors. We focus on a class called fractal necklaces which are generated by the necklace IFSs (NIFSs). The NIFS $\{f_1, \ldots, f_n\}$ is an IFS satisfying that $n \geq 3$ and f_k's are homeomorphisms with

$$f_m(F) \cap f_k(F) = \begin{cases} \text{a singleton} & \text{if } |m-k| = 1 \text{ or } n-1 \\ \emptyset & \text{if } 2 \leq |m-k| \leq n-2 \end{cases} \tag{0.2}$$

for all distinct $m, k \in \{1, \ldots, n\}$. For example, the usual IFS of the Sierpinski gasket is a NIFS and the Sierpinski gasket is a fractal necklace.

Here we discuss a basic remaining topological question for fractal necklaces. We can check that every fractal necklace is path-connected. And it seems like that fractal necklaces have no cut points. We say that a point x of a connected topological space X is a cut point, if $X \setminus \{x\}$ is not connected.

However, counter intuitively, some fractal necklaces have cut points. We first consider which fractal necklaces in $\mathbb{R}^d$ have no cut points. In Chapter 1.1, We give two subclasses of fractal necklaces and prove that every necklace in these two classes has no cut points. Thereinto, one subclass is called good necklaces, another subclass is called stable necklaces

of bounded ramification. These two subclasses are not mutually inclusive. Also, we prove that every stable self-similar necklace in $\mathbb{R}^2$ has no cut points, whilst an analog for self-affine necklaces is false.

We know that distinct IFSs can generate the same attractor. However, by the definition of NIFS, it seems like that the NIFS of a fractal necklace is unique in a certain sense. In Chapter 1.1, some properties of necklace have been given by its NIFS, if we can show the uniqueness of NIFS, these properties only depend on the necklace.

In Chapter 1.2, we prove every good necklace has a unique NIFS in a certain sense. By the same idea, we can get that two good necklaces admit only rigid homeomorphisms and thus the group of self-homeomorphisms of a good necklace is countable. In addition, a certain weaker co-Hopfian property of good necklaces is also obtained. The above rigidity and the weaker co-Hopf property on fractals have been studied by C. Bandt and T. Retta [9].

We conjecture that these theorems in Chapter 1.2 hold for all necklaces. However, it seems very difficult to prove (disprove) this conjecture.

Besides the research of dimensions, we also consider some basic geometric questions for IFS attractors. For example, we are concerned about the convex hulls of IFS attractors.

Let A be a $d \times d$ contractive matrix and $d_i \in \mathbb{R}^d$. The convex hull of the attractor of IFS $\{f_i \mid i = 1, 2, \dots m\}$ with $f_i = Ax + d_i$ is studied by Strichartz-Wang [30]. They observed an important property of extreme points of the convex hull and deduced that the attractor has a polygonal convex hull if and only if there exists a positive integer s such that A^s is a scalar matrix.

Kirat-Kocyigit [20] considered the case that the linear part of f_i may not be identical and proved that, if the attractor has a polygonal convex hull, the vertices must have eventually periodic codings. We make a little progress on Kirat-Kocyigit's result in Chapter 2: Let K be the attractor of an IFS $\{f_i \mid i = 1, 2, \dots m\}$ on the complex plane $\mathbb{C}$ with

$$f_i(z) = a_i z + b_i, \ a_i, b_i \in \mathbb{C}, \ 0 < |a_i| < 1.$$

Suppose K is not a singleton. If eventually periodic word $i_1 i_2 \cdots i_l (j_1 \cdots j_k)^\infty$ in $\{1, 2, \dots m\}^\mathbb{N}$ is a coding of an extreme point of $\mathrm{co}(K)$ then

$$a_{j_1} a_{j_2} \cdots a_{j_k} > 0.$$

Besides, Kirat-Kocyigit [20] also gaves a sufficient and necessary condition such that the attractor of a given IFS has a polygonal convex hull. Moreover, they found a way to check their condition, but the termination is not discussed.

Since the Kirat-Kocyigit's condition is not easy to check, the discussion for the convex hulls of IFS attractors is far from over. We devote some effort to study the convex hulls of dragon curves.

The dragon curves is a family of self-similar fractals in $\mathbb{R}^2$, they can be regard as the attractors of the following IFSs in the complex plane:

$$f_1(z) = az \ \text{ and } \ f_2(z) = 1 - \bar{a}z,$$

$$\text{where}\quad a := a(\eta) = \frac{e^{-i\eta}}{2\cos\eta} \quad\text{and}\quad \eta \in (0, \pi/3).$$

The dragon curve has also been obtained as the limit of the renormalized paperfolding curves in the Hausdorff metric as well; see R. Albers [1] and S. Tabachnikov [31].

When $\eta = \frac{\pi}{4}$, the dragon curve is well-known as Heighway dragon. Heighway dragon has some properties, for examples: Heighway dragon never traverses itself; Heighway dragon can tile the plane; as a non-self-crossing space-filling curve, Heighway dragon has fractal dimension exactly 2. More detailed results are in C. Davis and D.E. Knuth [10].

Motivated by properties of Heighway dragon, Tabachnikov [31], Albers [1], Allouche et al [3], and Kamiya [19] studied similar questions for dragon curve with an arbitrarily fixed angle η,

In Chapter 2, we say that the convex hull of a dragon curve is a polygon. To our knowledge, this is the first example of a parameterized family of fractals whose convex hull is a polygon. In most cases, we can give the values of vertices of polygonal convex hull of a dragon curve. Besides, we are also concerned that if the dragon curves satisfies OSC and when a dragon curve is an arc. These are what we want to study in the future.

Contents

1 Fractal necklaces

1.1 Fractal necklaces with no cut points

1.1.1 Introduction

Let $I = \{1, 2, \cdots, n\}$. For each $k \in I$ let $f_k : \mathbb{R}^d \to \mathbb{R}^d$ be a contractive map satisfying

$$|f_k(x) - f_k(y)| \le c_k |x - y|$$

for all $x, y \in \mathbb{R}^d$, where $c_k \in (0, 1)$. According to Hutchinson [15], there is a unique nonempty compact subset F of $\mathbb{R}^d$, called the attractor of $\{f_1, f_2, \cdots, f_n\}$, such that

$$F = \bigcup_{k=1}^{n} f_k(F). \tag{1.1}$$

We call $\{f_1, f_2, \cdots, f_n\}$ an iterated function system (IFS) of F.

Definition 1. *An attractor F with an IFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$ is called a fractal necklace or a necklace for short, if $n \ge 3$ and f_k's are contractive homeomorphisms of $\mathbb{R}^d$ satisfying*

$$f_m(F) \cap f_k(F) = \begin{cases} \text{a singleton} & \text{if } |m-k| = 1 \text{ or } n-1 \\ \emptyset & \text{if } 2 \le |m-k| \le n-2 \end{cases}$$

for each pair of distinct digits $m, k \in I$. In this case, the ordered family $\{f_1, f_2, \cdots, f_n\}$ is called a necklace IFS or a NIFS. We say that F is self-similar (self-affine), if f_k's are similitudes (affine maps).

Figure 1.1.1 illustrates two planar self-similar necklaces. The first one is generated by 3 similitudes of ratio 1/2 and the second one is generated by 6 similitudes of ratio 1/3. They arise as examples of many papers for various purposes; see for example [25, 27]. Among the results of [27], Tyson and Wu proved that these two necklaces are of conformal dimension 1.

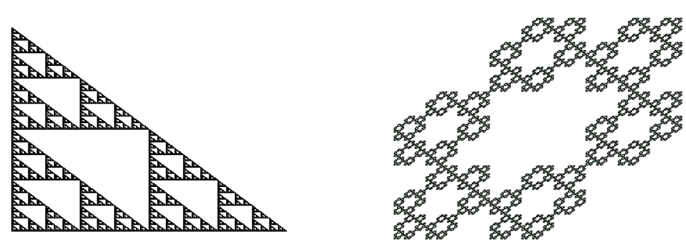


Figure 1: Two self-similar necklaces in $\mathbb{R}^2$.

It is not difficult to see that every fractal necklace is path-connected and locally path-connected; see [14, 17]. It is natural to ask whether every necklace has no cut points. The answer is no; see Section 1.1.2. Hereafter we say that a point x of a connected topological space X is a cut point, if $X \setminus \{x\}$ is not connected. The present paper is devoted to the following question. For a study on cut points of self-affine tiles we refer to [2].

Question 1. *Which necklaces in $\mathbb{R}^d$ have no cut points?*

We start by notations. *From now on denote by F a necklace with a NIFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$, if it is not specified.* For every integer $m \geq 0$ and every word $\sigma = i_1 i_2 \cdots i_m \in I^m$ we write f_σ for $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ and F_σ for $f_\sigma(F)$, where $I^0 = \{\emptyset\}$ and $f_\emptyset = id$. The set F_σ is called an m-level copy of F. Denote by $\mathcal{C}_m(F)$ the family of m-level copies of F and let

$$\mathcal{C}(F) = \bigcup_{m=0}^{\infty} \mathcal{C}_m(F).$$

A copy of F always means a member of $\mathcal{C}(F)$.

For each $k \in I$ denote by z_k the unique common point of the 1-level copies F_k and F_{k+1}. The ordered points $z_1, z_2, \cdots, z_n$ are called main nodes of F. For every subset A of F denote respectively by $\mathrm{int}_F A$ and $\partial_F A$ the interior and the boundary of A in the relative topology of F. Thus $\partial_F F_k = \{z_{k-1}, z_k\}$ for every $k \in I$. Hereafter we prescribe

$$F_{n+1} = F_1 \text{ and } z_0 = z_n.$$

Definition 2. *We say a necklace F is good, if $\partial_F F_k \not\subset F_{kj}$ for any $k, j \in I$.*

Equivalently, a necklace F is good, if F and F_k are the only two copies containing $\partial_F F_k$ for each $k \in I$.

Let $I^* = \cup_{m=0}^{\infty} I^m$ and let $\sigma \in I^*$. Since f_k's have been assumed to be homeomorphisms of $\mathbb{R}^d$, F_σ is a necklace with an induced NIFS

$$\{f_\sigma \circ f_j \circ f_\sigma^{-1} : j \in I\}$$

whose main nodes are $f_\sigma(z_1), f_\sigma(z_2), \cdots, f_\sigma(z_n)$. The phrase, m-level copies of F_σ, is now meaningful.

Definition 3. *We say a necklace F is stable, if for each $k \in I$*

$$\sharp\{F_{kj} : F_{kj} \cap \partial_F F_k \neq \emptyset,\ j \in I\} \geq 2. \tag{1.2}$$

Hereafter $\sharp$ denotes the cardinality.

By the above definitions, every good necklace is stable. Additionally, every necklace F with the condition that z_{k-1} or z_k is a main node of F_k for each $k \in I$ is stable.

For each $z \in F$ and for every integer $m \geq 0$ let

$$\mathcal{C}_m(F, z) = \{A \in \mathcal{C}_m(F) : z \in A\} \tag{1.3}$$

and $\mathcal{C}(F,z) = \cup_{m=0}^{\infty}\mathcal{C}_m(F,z)$. Let

$$c_m(z) := c_m(F,z) := \sharp\,\mathcal{C}_m(F,z) \tag{1.4}$$

denote the number of m-level copies containing z. Thus $c_1(z) = 1$ or 2, and $c_1(z) = 2$ if and only if z is a main node of F. Note that for each $A \in \mathcal{C}_m(F,z)$ there is one or two copies $B \in \mathcal{C}_{m+1}(F,z)$ lying in A. We have

$$c_m(z) \leq c_{m+1}(z) \leq 2c_m(z).$$

It then follows that $\{c_m(z)\}_{m=1}^{\infty}$ is a nondecreasing integer sequence satisfying $1 \leq c_m(z) \leq 2^m$ for each $m \geq 1$.

Definition 4. *We say a necklace F is* of bounded ramification, *if the sequence $\{c_m(z_k)\}_{m=1}^{\infty}$ is bounded for each $k \in I$.*

Equivalently, a necklace F is of bounded ramification, if $\pi^{-1}(x)$ is finite for any $x \in F$, where $\pi : I^{\infty} \to F$ is the code map (see [11]). By the definition, if there is a main node z_k such that it is a main node of each copy $A \in \mathcal{C}(F,z_k)$, then F is not of bounded ramification. Such necklaces can be found in Figure 3 and Figure 4(b).

The main results are as follows.

Theorem 1.1. *Every good necklace in $\mathbb{R}^d$ has no cut points.*

Theorem 1.2. *Every stable necklace of bounded ramification has no cut points.*

An attractor with an IFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$ is said to satisfy the open set condition (OSC), if there is a nonempty bounded open subset V of $\mathbb{R}^d$ such that $f_1(V), f_2(V), \cdots, f_n(V)$ are pairwise disjoint open subsets of V; see [11, 26].

Theorem 1.3. *Every stable self-similar necklace in $\mathbb{R}^d$ with the OSC has no cut points.*

Actually, we shall show that every self-similar necklace in $\mathbb{R}^d$ with the OSC is of bounded ramification, which together with Theorem 1.5 implies Theorem 1.3.

As a corollary of a theorem of Bandt and Rao [8], every self-similar necklace in $\mathbb{R}^2$ satisfies the OSC. Thus, Theorem 1.3 gives the following corollary.

Corollary 1. *Every stable self-similar necklace in $\mathbb{R}^2$ has no cut points.*

Remark 1.1. *A self-similar necklace of bounded ramification in $\mathbb{R}^2$ may have cut points; see Example 1.*

Remark 1.2. *A stable self-affine necklace in $\mathbb{R}^2$ may have cut points; see Example 2.*

Remark 1.3. *Stable necklaces of bounded ramification and good necklaces are not mutually inclusive; see Example 3.*

Without assuming F is self-similar, we have the following result.

Theorem 1.4. *Every planar necklace with no cut points satisfies the OSC.*

The paper is organized as follows. In Section 1.1.2, we give examples of necklaces to show Remarks 1, 2 and 3. Then we prove Theorem 1.1 in Section 1.1.3, Theorem 1.5 in Section 1.1.4, and Theorems 1.3 and 1.4 in Section 1.1.5. In the light of our results we put some further questions in Section 1.1.6.

1.1.2 Examples

We first show by an example that a planar self-similar necklace of bounded ramification may have cut points.

Example 1. We use the complex number notation. Let $\{f_1, \cdots, f_{24}\}$ be a NIFS on the complex plane $\mathbb{C}$ defined by

$$f_j(z) = \begin{cases} \frac{z}{3} + a_j & \text{if } j \in \{1, 7, 13, 19\} \\ \frac{z}{15} + a_j & \text{if } j \in \{1, 2, \cdots, 24\} \setminus \{1, 7, 13, 19\}, \end{cases}$$

where $a_1, a_2, \cdots, a_{24} \in \mathbb{C}$ satisfy

$$\begin{aligned} & f_{24}(1) = f_1 \circ f_{13}(i), \ f_1 \circ f_{13}(1) = f_2(i) \\ & f_6(1+i) = f_7 \circ f_{19}(0), \ f_7 \circ f_{19}(1+i) = f_8(0) \\ & f_{12}(i) = f_{13} \circ f_1(1), \ f_{13} \circ f_1(i) = f_{14}(1) \\ & f_{18}(0) = f_{19} \circ f_7(1+i), \ f_{19} \circ f_7(0) = f_{20}(1+i) \\ & f_j(1+i) = f_{j+1}(0), \ j \in \{2, 4, 9, 11\} \\ & f_j(1) = f_{j+1}(i), \ j \in \{3, 5, 20, 22\} \\ & f_j(i) = f_{j+1}(1), \ j \in \{8, 10, 15, 17\} \\ & f_j(0) = f_{j+1}(1+i), \ j \in \{14, 16, 21, 23\}. \end{aligned}$$

The planar self-similar necklace F generated by $\{f_1, f_2, f_3, \cdots, f_{24}\}$ is illustrated in Figure 2. It has the following properties.

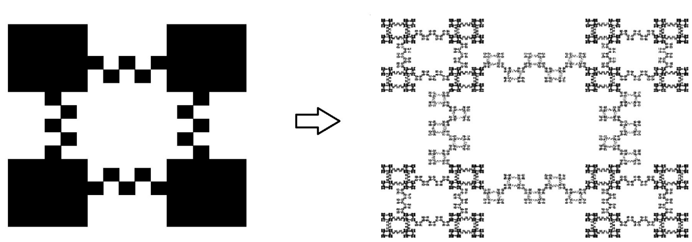


Figure 2: A planar self-similar necklace of bounded ramification and with cut points

(1) F is not stable, in fact, for the 1-level copy F_1 one has

$$\{F_{1j} : F_{1j} \cap \partial_F F_1 \neq \emptyset, j \in \{1, 2, \cdots, 24\}\} = \{F_{1(13)}\},$$

so F does not satisfy (1.2). Here the bracketed number in the subscript emphasises that it is a digit.

(2) F is of bounded ramification, indeed, given a main node z_k and an integer $m \geq 1$, F has only two m-level copies containing z_k.

(3) $(1+i)/4$ is a cut point of F. In fact, $F \setminus F_{1(13)}$ is obviously not connected. By zooming we see that

$$F \setminus F_{1(13)},\ F \setminus F_{1(13)1(13)},\ F \setminus F_{1(13)1(13)1(13)},\ \cdots,$$

are not connected and tend to $F \setminus \{(1+i)/4\}$ increasingly, by which one easily shows that $F \setminus \{(1+i)/4\}$ is not connected, as desired.

Next we give an example of stable planar self-affine necklaces with cut points.

Example 2. Let T_0 and T_1 be two closed solid isosceles triangles sharing a common vertex z_0 and of different sizes, whose angles at z_0 are a pair of vertical angles and whose opposite sides are parallel. Let $T = T_0 \cup T_1$. Let V be the set of the four extremal points of T. Let $\{f_1, f_2, \cdots, f_6\}$ be a family of invertible contractive affine maps of $\mathbb{R}^2$ satisfying the following conditions:

1) $V \subset \cup_{k=1}^{6} f_k(T) \subset T$.
2) $\sharp(f_j(T) \cap V) = 1$ for each $j \in \{1, 2, 5, 6\}$.
3) $V \cap (f_3(T) \cup f_4(T)) = \emptyset$.
4) $f_k(T) \cap f_m(T) = f_k(V) \cap f_m(V)$ if $k \neq m$ and $\{k, m\} \neq \{1, 6\}$.
5) $f_1(T) \cap f_6(T) = \{z_0\}$ and $f_1(z_0) = f_6(z_0) = z_0$.
6) $\sharp\,(f_j(T) \cap f_{j+1}(T)) = 1$ for $j \in \{1, 2, 3, 4, 5\}$.
7) $f_k(T) \cap f_m(T) = \emptyset$ if $|m - k| \geq 2$.

Then $\{f_1, f_2, \cdots, f_6\}$ is a NIFS which generates a self-affine necklace F in $\mathbb{R}^2$. The first step construction of F is illustrated in Figure 3, where the shadow part consists of $f_1(T), f_2(T), \cdots, f_6(T)$. Their connecting points are main nodes of F. z_0 is a main node and a cut point of F.

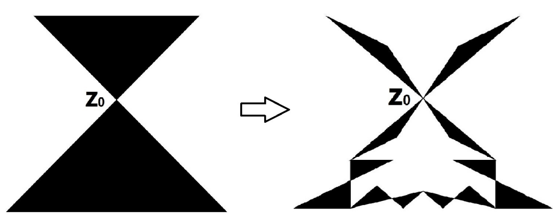


Figure 3: The first step construction of a stable planar self-affine necklace with cut points.

The necklace F is not of bounded ramification, in fact, by condition 5) we have

$$\mathcal{C}_m(F, z_0) = \{F_{i_1 i_2 \cdots i_m} : i_1 i_2 \cdots i_m \in \{1, 6\}^m\} \text{ and } c_m(z_0) = 2^m$$

for each integer $m \geq 0$, so $\{c_m(z_0)\}_{m=0}^{\infty}$ is unbounded.

The necklace F is stable. In fact, given $k \in \{1, 6\}$, we have $z_0 \in (\partial_F F_k) \cap F_{k1} \cap F_{k6}$, so

$$\sharp\{F_{kj} : F_{kj} \cap \partial_F F_k \neq \emptyset,\ j \in \{1, 2, \cdots, 6\}\} \geq 2.$$

On the other hand, given $k \in \{2, 3, 4, 5\}$, we have $\partial_F F_k \subset f_k(V)$ and $\sharp\,(f_k(V) \cap F_{kj}) \leq 1$ for each $j \in \{1, 2, 3, 4, 5, 6\}$, so $\sharp\,((\partial_F F_k) \cap F_{kj}) \leq 1$, and so

$$\sharp\{F_{kj} : F_{kj} \cap \partial_F F_k \neq \emptyset,\ j \in \{1, 2, \cdots, 6\}\} \geq 2.$$

This proves that F is stable.

Finally, we show by examples that stable necklaces of bounded ramification and good necklaces are not mutually inclusive.

Example 3. Let us see Figure 4. The left one is a planar self-similar necklace generated by $\{f_1, f_2, f_3\}$, where

$$f_1(z) = \frac{e^{\pi i/6}\overline{z}}{\sqrt{3}},\ f_2(z) = \frac{z+1}{3},\ f_3(z) = \frac{e^{5\pi i/6}z}{\sqrt{3}} + \frac{2}{3},\ z \in \mathbb{C}.$$

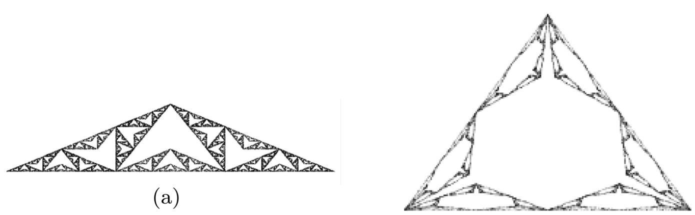
Let F be this necklace and let $I = \{1, 2, 3\}$. Then for each $k \in I$

$$\sharp\{F_{kj} : F_{kj} \cap \partial_F F_k \neq \emptyset,\ j \in I\} = 2,$$

so F is stable. On the other hand, it is of bounded ramification because

$$c_m(z_1) = c_m(z_2) = 3 \ \text{ and } \ c_m(z_3) = 2$$

for each integer $m \geq 1$, where z_1, z_2, z_3 are main nodes of F. In addition, noticing that $F_{13} \supset \partial_F F_1$, we conclude that F is not good.



(a) (b)

Figure 4: Stable necklaces of bounded ramification and good necklaces are not mutually inclusive.

Next let $0 < \alpha < \sqrt{3}/6$ be given. For $z = x + iy \in \mathbb{C}$ let

$$g_1(z) = \frac{x + i\alpha y}{2},\ g_2(z) = g_1(z) + \frac{1}{2},$$

$$g_3(z) = e^{\frac{2\pi i}{3}} g_1(z) + 1,\ g_4(z) = e^{\frac{2\pi i}{3}} g_1(z) + \frac{3+i\sqrt{3}}{4}$$

$$g_5(z) = e^{\frac{\pi i}{3}} \overline{g_1(z)} + \frac{1+i\sqrt{3}}{4},\ g_6(z) = e^{\frac{\pi i}{3}} \overline{g_1(z)}.$$

Then $\{g_1, g_2, \cdots, g_6\}$ is a NIFS which generates is a self-affine necklace. It is illustrated on the right of Figure 4. For each $k \in \{1, 2, \cdots, 6\}$ we easily see that F and F_k are the only two copies containing $\partial_F F_k$ and that

$$c_m(z_k) = 2^m$$

for each $m \geq 1$. Thus F is good but not of bounded ramification.

1.1.3 The Proof of Theorem 1.1

In this section we prove Theorem 1.1: Every good necklaces has no cut points. The following lemma will be used.

Lemma 1.1. *Let X be a connected metric space, E be connected and dense in X, and $x \in X$. If x is a cut point of X, then x belongs to E and is a cut point of E.*

Proof. Suppose $E \setminus \{x\}$ is connected. Since E is dense in X, $E \setminus \{x\}$ is so. It then follows from $E \setminus \{x\} \subset X \setminus \{x\} \subset X$ that $X \setminus \{x\}$ is connected, contradicting the assumption that x is a cut point of X. Thus x belongs to E and is a cut point of E. $\square$

As prescribed, F is a necklace with a NIFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$. Let $x, u \in F$ and let k be a positive integer. We say that a finite sequence $(A_1, A_2, \cdots, A_N)$ of k-level copies of F is a k-level chain from x to u, if

$$x \in A_1 \setminus \bigcup_{j=2}^{N} A_j,\ u \in A_N \setminus \bigcup_{j=1}^{N-1} A_j,$$

and

$$A_j \cap A_m = \begin{cases} \text{a singleton} & \text{if } |j-m| = 1 \\ \emptyset & \text{if } |j-m| \geq 2 \end{cases} \qquad j, m \in \{1, 2, \cdots, N\}.$$

In this case, we also say that $\cup_{j=1}^{N} A_j$ is a k-level chain. By convention we prescribe $\cup_{j=2}^{N} A_j = \cup_{j=1}^{N-1} A_j = \emptyset$, if $N = 1$.

Let $(A_1, A_2, \cdots, A_N)$ be a k-level chain of F. Denote by x_j the unique point of $A_j \cap A_{j+1}$ for each $j \in \{1, 2, \cdots, N-1\}$. We call the ordered points $x_1, x_2, \cdots, x_{N-1}$ the connections of the chain.

By Lemma 1.1, to prove a topological space has no cut points, it suffices to show that it has a connected dense subset with no cut points. We shall show that every good necklace has Property I and that every necklace with Property I has a connected dense subset with no cut points. Here we say that a necklace F has Property I, if each of its 1-level copies F_k has an arc from z_{k-1} to z_k which passes through at least two main nodes of F_k. By convention an arc means a subset homeomorphic with the unit interval $[0,1]$.

The necklace F in Example 1 does not have Property I, indeed, F_1 does not have a wanted arc with Property I. On the other hand, there are necklaces with Property I, but they are not good. The left necklace in Figure 4 is one of such.

Now we have made the preparations to prove Theorem 1.

The proof of Theorem 1.1. Let $x, u \in F$.

Claim 1. For each $k \geq 1$, F has a k-level chain Γ_k from x to u. They satisfy $\Gamma_{k+1} \subset \Gamma_k$ and

$$\{x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}\} \subseteq \{x_1^{(k+1)}, x_2^{(k+1)}, \cdots, x_{N_{k+1}}^{(k+1)}\},$$

where $x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}$ are the connections of Γ_k.

Proof. It is obvious that F has a 1-level chain from x to u.

Suppose F has a k-level chain $(A_1, A_2, \cdots, A_N)$ from x to u for an integer $k \geq 1$. In the case where $N = 1$, one has $x, u \in A_1$. As is known, A_1 has a 1-level chain from x to u. Such a chain of A_1 is clearly a $(k+1)$-level chain of F from x to u. For the case $N > 1$ let $x_1, x_2, \cdots, x_{N-1}$ be the ordered connections of the chain $(A_1, A_2, \cdots, A_N)$. Then A_1 has a 1-level chain from x to x_1, A_j has a 1-level chain from x_{j-1} to x_j for each $j \in \{2, 3, \cdots, N-1\}$, and A_N has a 1-level chain from x_{N-1} to u. These N chains arranged in the evident order yield a $(k+1)$-level chain of F from x to u.

By induction, for each $k \geq 1$, F has a k-level chain from x to u with the additional requirements.

Claim 2. F has an arc from x and u.

Proof. For each $k \geq 1$ let Γ_k be a k-level chain of F from x to u and let $x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}$ be its ordered connections as in Claim 1. Let

$$\gamma = \bigcap_{k=1}^{\infty} \Gamma_k.$$

Then γ is a compact subset of F containing the connections of Γ_k for all k. We are going to show that γ is an arc from x to u.

Let

$$X = \bigcup_{k=1}^{\infty} \{x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}\}.$$

Then X is dense in γ. On the other hand, X is a well ordered set with an ordering induced by those of $\{x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}\}$, $k \geq 1$. We may choose a dense subset

$$Y = \bigcup_{k=1}^{\infty} \{y_1^{(k)}, y_2^{(k)}, \cdots, y_{N_k}^{(k)}\}$$

of the interval $[0,1]$ such that the map $h : X \to Y$ defined by

$$h(x_j^{(k)}) = y_j^{(k)},\ j = 1, 2, \cdots, N_k,\ k \geq 1$$

is an order-preserving homeomorphism. Now we easily see that h can be extended to a homeomorphism of γ onto $[0,1]$.

Claim 3. Every good necklace has Property I.

Proof. By the proof of Claim 2, each chain Γ of F from x to u has an arc from x to u and such an arc contains the connections of Γ.

Suppose F is good. To check Property I, we fix $k \in I$. Note that the connections of every 1-level chain of F_k are main nodes of F_k.

Case 1. Either z_{k-1} or z_k is a main node of F_k. Let Γ be 1-level chain of F_k from z_{k-1} to z_k. Since F is good, Γ contains at least two 1-level copies of F_k, so its connections are nonempty. Let γ be an arc of Γ from z_{k-1} to z_k. Then γ contains at least two main nodes of F_k.

Case 2. Neither z_{k-1} nor z_k is a main node of F_k. In this case, there is a unique pair $l, j \in I$ such that $z_{k-1} \in F_{kl}$ and $z_k \in F_{kj}$. Since F is good, we have $l \neq j$.

Subcase 1. $F_{kl} \cap F_{kj} = \emptyset$. Let Γ be 1-level chain of F_k from z_{k-1} to z_k. Then Γ contains at least three 1-level copies of F_k, so Γ has at least two connections. Let γ be an arc of Γ from z_{k-1} to z_k. Then γ contains at least two main nodes of F_k.

Subcase 2. $F_{kl} \cap F_{kj} \neq \emptyset$. In this subcase, $F_{kl} \cap F_{kj}$ is a singleton whose unique point is denoted by w. Let

$$L = \bigcup_{i \in I, i \neq l, i \neq j} F_{ki}.$$

Then L can be regarded as a 1-level chain of F_k from a to b, where $\{a, b\} = \partial_F L$. And we may assume that $a \in F_{kl}$ and $b \in F_{kj}$. Clearly, a, b are main nodes of F_k. Since F is good, F_{kl} has a 1-level chain A from z_{k-1} to a and F_{kj} has a 1-level chain B from b to z_k such that $w \notin A \cup B$. Thus $A \cap L = \{a\}$, $L \cap B = \{b\}$, and $A \cap B = \emptyset$. Let γ_A be an arc of A from z_{k-1} to a, γ_L be an arc of L from a to b, and γ_B be an arc of B from b to z_k. Then $\gamma_A \cup \gamma_L \cup \gamma_B$ is an arc from z_{k-1} to z_k which contains at least two main nodes of F_k.

Claim 4. Every necklace with Property I has no cut points.

Proof. Suppose F satisfies Property I. For each $k \in I$ let $\gamma^{(k)}$ be an arc of F_k from z_{k-1} to z_k which passes through at least two main nodes of F_k. Let

$$\gamma = \bigcup_{k \in I} \gamma^{(k)}.$$

Then γ is a circle of F passing through all main nodes of F, where a circle means a subset homeomorphic with the geometric circle. Let

$$E = \bigcup_{\sigma \in I^*} f_\sigma(\gamma).$$

Then each $f_\sigma(\gamma)$ is a circle with

$$f_\sigma(\{z_1, z_2, \cdots, z_n\} \subset f_\sigma(\gamma) \subset F_\sigma \tag{1.5}$$

and E is dense in F. In addition, by the construction of γ, we have

$$\sharp(f_\sigma(\gamma) \cap f_{\sigma j}(\gamma)) \geq 2$$

for each $\sigma \in I^*$ and each $j \in I$, from which we easily infer that E is connected and has no cut points. Now, by Lemma 1.1, we get that F has no cut points.

This completes the proof of Theorem 1.1.

Remark 1.4. Let F be a necklace. By Claim 2, F is path-connected. We further conclude that F is locally path-connected, indeed, for each $z \in F$ and each integer $m \geq 1$ the set

$$\bigcup_{A \in \mathcal{C}_m(F,z)} A$$

is a path-connected neighborhood of z, where $\mathcal{C}_m(F, z)$ is a family of m-level copies of F defined by (1.3).

1.1.4 The proof of Theorem 1.2

In this section we prove Theorem 1.2: Every stable necklace of bounded ramification has no cut points.

Let F be a necklace with a NIFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$. Let

$$M_F = \bigcup_{\sigma \in I^*} \{f_\sigma(z_1), f_\sigma(z_2), \cdots, f_\sigma(z_n)\}. \tag{1.6}$$

Then $x \in M_F$ if and only if x is a main node of some copy of F. Also, we use the notations $\mathcal{C}_m(F, z)$ and $c_m(F, z)$ from (1.3) and (1.4). As each copy A of F is a necklace with an induced NIFS, the notations M_A, $\mathcal{C}_m(A, z)$ and $c_m(A, z)$ are self-evident.

Lemma 1.2. *Suppose F is stable. Then every point of $F \setminus M_F$ is not a cut point of F.*

Proof. Fix $z \in F \setminus M_F$. Then, by the definition of M_F, for each $m \geq 1$ there is a unique m-level copy containing z, so $c_m(F, z) = 1$ and $z \in \mathrm{int}_F V_m$, where V_m denotes the unique member of $\mathcal{C}_m(F, z)$. Let

$$U_m = \bigcup_{A \in \mathcal{C}_m(F) \setminus \mathcal{C}_m(F,z)} A. \tag{1.7}$$

Then $U_m \cup V_m = F$ and

$$U_m \cap V_m = \partial_F U_m = \partial_F V_m. \tag{1.8}$$

Furthermore $\{U_m\}_{m=1}^{\infty}$ is increasing with

$$F \setminus \{z\} = \bigcup_{m=1}^{\infty} U_m. \tag{1.9}$$

Let

$$L_m = \bigcup_{B \in \mathcal{C}_1(V_m) \setminus \mathcal{C}_1(V_m, z)} B. \tag{1.10}$$

Then L_m is connected and

$$U_{m+1} = U_m \cup L_m. \tag{1.11}$$

We claim that U_m is connected for every $m \geq 1$. In fact, U_1 is a 1-level chain of F, so it is connected. Assume that U_m is connected for an integer $m \geq 1$. We are going to prove that U_{m+1} is connected.

Since F is stable, we may take two distinct copies $A, B \in \mathcal{C}_1(V_m)$ such that $A \cap \partial_F V_m \neq \emptyset$ and $B \cap \partial_F V_m \neq \emptyset$, so one has $A \cap U_m \neq \emptyset$ and $B \cap U_m \neq \emptyset$ by (1.8). Without loss of generality assume $B \neq V_{m+1}$. Then $B \subset L_m$ by (1.10). Therefore

$$U_m \cap L_m \neq \emptyset. \tag{1.12}$$

Since L_m is connected and U_m has been assumed to be connected, we get from (1.11) and (1.12) that U_{m+1} is connected.

By induction, U_m is connected for every $m \geq 1$, which together with (1.9) implies that $F \setminus \{z\}$ is connected, so z is not a cut point. This completes the proof. □

The proof of Theorem 1.2. Suppose F is stable and of bounded ramification. We are going to prove that F has no cut points. As Lemma 1.2 was proved, it suffices to prove that every point of M_F is not a cut point of F.

Let $z \in M_F$ be given. Then there is a copy E of F such that z is a main node of E. In what follows we assume that E is the biggest copy of F with this property. Then z is a main node of E and $z \in \text{int}_F E$. To show that $F \setminus \{z\}$ is connected, it suffices to prove that $E \setminus \{z\}$ is connected.

Since F is of bounded ramification, $\{c_m(E, z)\}_{m=1}^{\infty}$ is bounded. Thus we may take an integer $k \geq 1$ such that

$$c_m(E, z) = c_k(E, z)$$

for all integers $m \geq k$, which in turn implies that z is not a main node of any copy $A \in \mathcal{C}_k(E, z)$.

Therefore $z \in A \setminus M_A$ for each $A \in \mathcal{C}_k(E, z)$. Since A is stable by the assumption condition, we have by Lemma 1.2 that $A \setminus \{z\}$ is connected.

Now that $A \setminus \{z\}$ is connected for each $A \in \mathcal{C}_k(E, z)$, by which we easily see that $B \setminus \{z\}$ is connected for each $B \in \mathcal{C}_{k-1}(E, z)$. Step by step, we get that $E \setminus \{z\}$ is connected. This completes the proof.

1.1.5 The proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let F be a self-similar necklace with a NIFS $\{f_1, f_2, \cdots, f_n\}$ on $\mathbb{R}^d$ satisfying the OSC. We are going to show that F is of boundary ramification.

Let $\mathcal{C}_m(F,z)$ and $c_m(z)$ be defined as (1.3) and (1.4). We have to show that $\{c_m(z_k)\}_{m=1}^{\infty}$ is bounded for each main node z_k of F.

Given z_k and m, let $\tau \in I^m$ be a word such that

$$F_\tau \in \mathcal{C}_m(F,z_k) \ \text{ and } \ \mathrm{diam}(F_\tau) = \min_{A \in \mathcal{C}_m(F,z_k)} \mathrm{diam}(A).$$

For each $A \in \mathcal{C}_m(F,z_k)$ we may take a copy $\widetilde{A} \in \mathcal{C}(F,z_k)$ such that

$$\widetilde{A} \subseteq A \ \text{ and } \ c_*\mathrm{diam}(F_\tau) < \mathrm{diam}(\widetilde{A}) \le \mathrm{diam}(F_\tau), \tag{1.13}$$

where $c_* = \min_{1\le j\le n} c_j$ and $c_1, c_2, \cdots, c_n \in (0,1)$ are respectively the similarity ratios of $f_1, f_2, \cdots, f_n$. Then we get $c_m(z_k)$ copies of comparable diameters, which are denoted as

$$F_{\sigma_1}, F_{\sigma_2}, \cdots, F_{\sigma_{c_m(z_k)}} \tag{1.14}$$

where $\sigma_1, \sigma_2, \cdots, \sigma_{c_m(z_k)} \in I^*$ are the corresponding words. It follows from (1.13) that

$$\bigcup_{j=1}^{c_m(z_k)} F_{\sigma_j} \subset B(z_k, \mathrm{diam}(F_\tau)),$$

where $B(z_k, \mathrm{diam}(F_\tau))$ is the closed ball of radius $\mathrm{diam}(F_\tau)$ centred at z_k. Since the NIFS satisfies the OSC, there is a nonempty bounded open set V of $\mathbb{R}^d$ such that $f_1(V), f_2(V), \cdots, f_n(V)$ are pairwise disjoint subsets of V. Thus

$$f_{\sigma_1}(V), f_{\sigma_2}(V), \cdots, f_{\sigma_{c_m(z_k)}}(V)$$

are pairwise disjoint. On the other hand, as V is bounded, we may take a constant $H \ge 1$ such that $V \subset B(z_k, H\mathrm{diam}(F))$. Then

$$\bigcup_{j=1}^{c_m(z_k)} f_{\sigma_j}(V) \subset B(z_k, H\mathrm{diam}(F_\tau)). \tag{1.15}$$

By comparing volumes we get from (1.15) that

$$c_m(z_k) \min_{1\le j\le c_m(z_k)} \mathrm{Vol}(f_{\sigma_j}(V)) \le w_d(H\mathrm{diam}(F_\tau))^d, \tag{1.16}$$

where w_d denotes the volume of the d-dimensional unit ball. Since the NIFS consists of similitudes of $\mathbb{R}^d$, one has by (1.13)

$$\frac{\mathrm{Vol}(f_{\sigma_j}(V))}{\mathrm{Vol}(f_\tau(V))} = \left(\frac{\mathrm{diam}(f_{\sigma_j}(V))}{\mathrm{diam}(f_\tau(V))}\right)^d = \left(\frac{\mathrm{diam}(F_{\sigma_j})}{\mathrm{diam}(F_\tau)}\right)^d \ge c_*^d \tag{1.17}$$

for every $j \in \{1, 2, \cdots, c_m(z_k)\}$, which together with (1.16) yields

$$c_m(z_k) c_*^d \mathrm{Vol}(f_\tau(V)) \le w_d(H \mathrm{diam}(F_\tau))^d. \tag{1.18}$$

Therefore

$$c_m(z_k) \le \frac{w_d(H \mathrm{diam}(F_\tau))^d}{c_*^d \mathrm{Vol}(f_\tau(V))} = \frac{w_d(H \mathrm{diam}(F))^d}{c_*^d \mathrm{Vol}(V)}.$$

Thus the sequence $\{c_m(z_k)\}_{m=1}^\infty$ is bounded. This proves that F is of bounded ramification. Now Theorem 1.3 follows by Theorem 1.5.

Proof of Theorem 1.4. Let F be a planar necklace with no cut points. We are going to show that F satisfies the OSC.

By the proof of Theorem 1.1, F has a circle. Thus $\mathbb{R}^2 \setminus F$ has infinitely many bounded components by the definition of F and Jordan's curve theorem. Let U be a fixed bounded component of $\mathbb{R}^2 \setminus F$. Then

$$\partial U \subset F \text{ and } U \cap F = \emptyset.$$

Let $\{f_1, f_2, \cdots, f_n\}$ be a NIFS of F. Since f_k's have been assumed to be contracting homeomorphisms of $\mathbb{R}^2$, we have that, for every $\sigma \in I^*$, the image $f_\sigma(U)$ of U under f_σ is a bounded component of $\mathbb{R}^2 \setminus F_\sigma$ with

$$\partial(f_\sigma(U)) \subset F_\sigma \text{ and } f_\sigma(U) \cap F_\sigma = \emptyset. \tag{1.19}$$

We are going to show that $f_\sigma(U)$ is a bounded component of $\mathbb{R}^2 \setminus F$ for each word $\sigma \in I^*$. First, we have by (1.19) and the definition of F

$$\bigcup_{j=3}^{n-1} F_j \subset f_1(U) \text{ or } \left(\bigcup_{j=3}^{n-1} F_j\right) \cap f_1(U) = \emptyset.$$

Moreover, since $\mathrm{diam}(F) > \mathrm{diam}(F_1)$, we have

$$F_2 \setminus \overline{f_1(U)} \neq \emptyset \text{ or } F_n \setminus \overline{f_1(U)} \neq \emptyset.$$

Next we show that $f_1(U) \cap F = \emptyset$ under the assumption that F has no cut points. In fact, if not, we encounter several different cases and, in each case, there is a digit $k \in \{2, n\}$ such that

$$f_1(U) \cap F_k \neq \emptyset \text{ and } F_k \setminus \overline{f_1(U)} \neq \emptyset,$$

which implies that either z_1 or z_n is a cut point of F, contradicting the assumption on F.

Similarly, for each $k \in I$ we have $f_k(U) \cap F = \emptyset$, which together with (1.19) implies that $f_k(U)$ is a bounded component of $\mathbb{R}^2 \setminus F$. For each word $\sigma \in I^*$, arguing as above step by step, we get $f_\sigma(U) \cap F = \emptyset$, so $f_\sigma(U)$ is a bounded component of $\mathbb{R}^2 \setminus F$.

Let $m, k \in I$ be distinct and let $\sigma, \tau \in I^*$. By the above conclusions, $f_m(f_\sigma(U))$ and $f_k(f_\tau(U))$ are two distinct components of $\mathbb{R}^2 \setminus F$, so

$$f_m(f_\sigma(U)) \cap f_k(f_\tau(U)) = \emptyset. \tag{1.20}$$

Now let

$$V = \bigcup_{\sigma \in I^*} f_\sigma(U).$$

It is obvious that $f_k(V) \subset V$ for every $k \in I$. On the other hand, we see by (1.20) that $f_m(V)$ and $f_k(V)$ are disjoint for distinct $m, k \in I$. This proves that, with the open set V, the NIFS satisfies the OSC. The proof is completed.

1.1.6 Some further questions

• The OSC problem

We just proved that every planar necklace with no cut points satisfies the OSC. However, the proof is invalid for necklaces in $\mathbb{R}^d$, $d \geq 3$. Actually, we easily check that every necklace is of topological dimension 1. Therefore, for a necklace F in $\mathbb{R}^d$, $d \geq 3$, we see that $\mathbb{R}^d \setminus F$ does not have any bounded components. We do not know if every necklace with no cut points satisfies the OSC in the higher dimensional case. It is open even for self-similar necklaces.

• Conformal dimension of self-similar necklaces

Tyson and Wu [27] proved that the two necklaces in Figure 1 are of conformal dimension 1. We thus ask: Can one develop a unified method to prove that a big class of self-similar necklaces are of conformal dimension 1?

1.2 Topological Rigidity of good fractal necklaces

1.2.1 Introduction

The fractal necklaces had been introduced by the author in [24], where some conditions for fractal necklaces with no cut points are obtained. The present paper is devoted to studying the topological rigidity of good fractal necklaces. Roughly speaking, a subset of $\mathbb{R}^d$ is rigid in a certain sense if the group of its related automorphisms is small. We refer to [6, 7] for the quasisymmetric rigidity of Schotty sets and square carpets.

A map $f : \mathbb{R}^d \to \mathbb{R}^d$ is contracting, if there exists $0 < c < 1$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in R^d$. Let $\{f_1, f_2, \ldots, f_n\}$ be a family of contracting maps of $\mathbb{R}^d$. According to Hutchinson [14], there is a unique nonempty compact subset F of $\mathbb{R}^d$, called the attractor of $\{f_1, f_2, \ldots, f_n\}$, such that $F = \cup_{k=1}^n f_k(F)$.

The attractor F is called a *fractal necklace or a necklace*, if $n \geq 3$ and $f_1, f_2, \ldots, f_n$ are contracting homeomorphisms of $\mathbb{R}^d$ satisfying

$$f_m(F) \cap f_k(F) = \begin{cases} \text{a singleton} & \text{if } |m-k| = 1 \text{ or } n-1 \\ \emptyset & \text{if } 2 \leq |m-k| \leq n-2 \end{cases} \tag{1.21}$$

for all distinct $m, k \in \{1, 2, \ldots, n\}$. In this case, the ordered family $\{f_1, f_2, \ldots, f_n\}$ is called *a necklace iterated function system (NIFS)*.

Let $I = \{1, 2, \ldots, n\}$. For every integer $m \geq 0$ and every sequence $i_1 i_2 \ldots i_m \in I^m$ write $f_{i_1 i_2 \ldots i_m}$ for the composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ and $F_{i_1 i_2 \ldots i_m}$ for $f_{i_1 i_2 \ldots i_m}(F)$, where we prescribe $I^0 = \{\varepsilon\}$ and $f_\varepsilon = id$. We call $F_{i_1 i_2 \ldots i_m}$ *an m-level copy of F*. Denote by $\mathcal{C}_m(F)$ the collection of m-level copies of F and let $\mathcal{C}(F) = \cup_{m=0}^{\infty} \mathcal{C}_m(F)$. From now on a copy of F means an m-level copy of F for some $m \geq 0$. By the definition, two distinct copies A, B of F have one of the following four relationships:

$$A \subset B;\ B \subset A;\ A \cap B = \emptyset;\ A \cap B \text{ is a singleton.} \tag{1.22}$$

For every $k \in I$ denote by z_k the unique common point of the 1-level copies F_k and F_{k+1}. We call the ordered points $z_1, z_2, \ldots, z_n$ *the main nodes of F*. We say that two main nodes z_k and z_m are *adjacent*, if $|k - m| = 1$ or $n - 1$. For a subset A of F denote respectively by $\partial_F A$ and $\text{int}_F A$ the boundary and the interior of A in the relative topology of F. Then we have $\partial_F F_k = \{z_{k-1}, z_k\}$ for each $k \in I$ and $\sharp\, \partial_F A \geq 2$ for each $A \in \cup_{m \geq 1} \mathcal{C}_m(F)$. Hereafter denote by $\sharp$ the cardinality and prescribe

$$F_{n+1} = F_1 \text{ and } z_0 = z_n. \tag{1.23}$$

We say that a fractal necklace F with a NIFS $\{f_1, f_2, \ldots, f_n\}$ is *good*, if $\sharp(F_{kj} \cap \partial_F F_k) \leq 1$ for all $k, j \in I$. In this case, we also say that the NIFS is good. Equivalently, F is good if and only if F_k is the smallest copy containing $\{z_{k-1}, z_k\}$ for each $k \in I$.

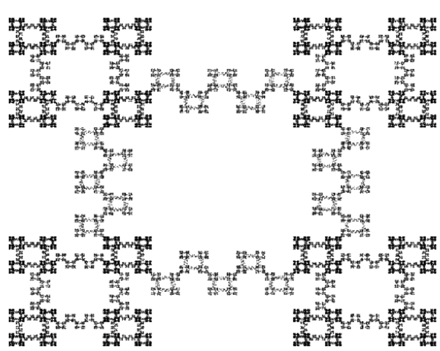


Figure 5: A necklace that is not good but has cut points.

Let F be a fractal necklace. Then F is path-connected and locally path-connected. Moreover, if F is good then it has no cut points. Figure 1 presents a necklace that is not good and has cut points. All of these can be found in [24]. J. Kigami [18, 19] established harmonic calculus on p. c. f. self-similar sets. It is clear that our fractal necklaces satisfy the p. c. f. property.

It should be mentioned that necklaces can be defined by language of fractal structures used by C. Penrose [22], C. Bandt and K. Keller [5], and C. Bandt and T. Retta [9]. In the present paper, some questions are the same as that of [9].

For a necklace, its copies and main nodes and the goodness have been defined by its given NIFS. Since two distinct NIFSs may generate the same necklace, it is natural to ask whether or not these properties of necklaces are independent of their NIFSs.

Let τ and s be two permutations of I, where

$$\tau(k) = \begin{cases} k+1 & \text{if } 1 \leq k < n \\ \\ 1 & \text{if } k = n \end{cases} \tag{1.24}$$

and $s(k) = n - k + 1$ for all $k \in I$. Let $\mathcal{G}_n$ be the group generated by τ and s. Then $\mathcal{G}_n$ is a dihedral group of $2n$ elements. Let F be a necklace with a NIFS $\{f_1, f_2, \ldots, f_n\}$ on $\mathbb{R}^d$ and $\sigma \in \mathcal{G}_n$. We easily see that $\{f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(n)}\}$ remains to be a NIFS of F.

We shall prove that every good necklace has a unique NIFS in the following sense.

Theorem 1.5. *Let F be a necklace with a good NIFS $\{f_1, f_2, \ldots, f_n\}$ on $\mathbb{R}^d$. Then for each NIFS $\{g_1, g_2, \ldots, g_m\}$ of F we have*

(1) $m = n$ and

(2) There is a permutation $\sigma \in \mathcal{G}_n$ such that $g_k(F) = f_{\sigma(k)}(F)$ for each $k \in I$.

Remark 1.5. By Theorem 1.5, we see that, if F is a necklace with a good NIFS, then all NIFSs of F are good and its copies and main nodes are actually independent of the choice of its NIFSs. Note also that the conclusion (2) here does not imply $g_k = f_{\sigma(k)}$.

Definition 5. *We say that a homeomorphism of two necklaces F and G is rigid, if the image of every m-level copy of F is an m-level copy of G for every $m \geq 0$.*

Denote by $h(F, G)$ the family of homeomorphisms of F onto G.

Theorem 1.6. *Let F and G be two topologically equivalent good fractal necklaces in $\mathbb{R}^d$. Then every homeomorphism of F onto G is rigid. Furthermore, $h(F, G)$ is countable, in particular, the group $h(F, F)$ of homeomorphisms of F is countable.*

A topological space X is co-Hopfian, if every topological embedding of X into itself is onto; see [13, 21, 23]. By contrast, we prove that every good fractal necklace has a weaker co-Hopfian property as follows.

Theorem 1.7. *Let F and G be two topologically equivalent good necklaces in $\mathbb{R}^d$ and let h be a topological embedding of F into G. Then $h(F)$ is a copy of G.*

Remark 1.6. In Theorem 1.7, the assumption that F and G are topologically equivalent can not be removed off. Indeed, a good necklace F may have a subset that is a good necklace, but it is not any copy of F. The readers easily see this from the standard Sierpinski gasket.

Remark 1.7. The above rigidity and the weaker co-Hopf property on fractals have been studied by C. Bandt and T. Retta [9]. For finite-to-one and good necklaces, our Theorems 1.6 and 1.7 can be obtained by Theorem 5.1 of [9]. However, it is easy to construct a good necklace that is not finite-to-one, for example, Figure 2 illustrates a good self-affine necklace with six main nodes in a triangle, where the vertices and midpoints of sides of the triangle are its main nodes. Thus our results and those of [9] are not completely overlapped. Besides, our strategy in the proof of these results is different from that of [9].

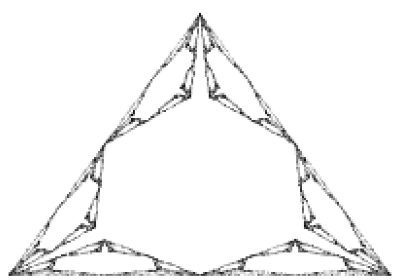


Figure 6: A necklace that is good but not finite-to-one.

We shall introduce and characterize extremal 2-cuts for good necklaces in Section 2. Using this characterization and some related properties of extremal 2-cuts, we shall prove the above theorems in Section 3. We conjecture that these theorems hold for all necklaces. However, since the extremal 2-cuts for general necklaces are more elusive, it seems very difficult to prove (disprove) this conjecture.

1.2.2 2-cuts of necklaces with no cut points

In this section we discuss the 2-cuts of necklaces with no cut points and the related topological invariants.

Let X be a connected topological space and $A \subset X$. We say that A is *a cut of X*, if $X \setminus A$ is not connected and $X \setminus B$ is connected for each $B \subsetneq A$. A cut consisting of k points is called *a k-cut*. A 1-cut is also called a cut point.

For each subset A of X define

$$N(A, X) = \sup\{\mathrm{ncp}(\overline{C}) : C \text{ is a component of } X \setminus A\}, \tag{1.25}$$

where $\overline{C}$ is the closure of C in X, $\mathrm{ncp}(\overline{C})$ denotes the number of cut points of $\overline{C}$, a component means a maximal connected subset. We say that a component C of $X \setminus A$ is *extremal*, if $\mathrm{ncp}(\overline{C}) = N(A, X)$.

For each integer $k \geq 1$ define

$$N_k(X) := \sup\{N(A, X) : A \text{ is a } k\text{-cut of } X\}. \tag{1.26}$$

We say that a k-cut A of X is *extremal*, if $N(A, X) = N_k(X)$.

Lemma 1.3. *Let $h : X \to Y$ be a homeomorphism of two connected topological spaces. Then we have the followings.*

(1) Let A be a cut of X. Then $h(A)$ is a cut of Y and

$$N(h(A), Y) = N(A, X).$$

Moreover, C is an extremal component of $X \setminus A$ if and only if $h(C)$ is an extremal component of $Y \setminus h(A)$.

(2) Let $k \geq 1$ be an integer. Then $N_k(Y) = N_k(X)$. Moreover, A is an extremal k-cut of X if and only if $h(A)$ is an extremal k-cut of Y.

Proof. It is immediate. □

Lemma 1.4. *Let X be a connected and locally connected metric space. Let A be a k-cut of X, where $k \geq 1$ is an integer. Then we have*

1) $\partial_X C = A$ for every component C of $X \setminus A$, and

2) $U \setminus A$ is not connected for each neighborhood U of A in X, where a subset U of X is called a neighborhood of A, if $A \subset \mathrm{int}_X U$.

Proof. 1) Let C be a component of $X \setminus A$. Then C is closed in $X \setminus A$. Since X is locally connected and A is finite, $X \setminus A$ is locally connected, so C is also open in $X \setminus A$. Thus $\partial_X C \subseteq A$.

Next we prove $A \subseteq \partial_X C$. Since C is open in $X \setminus A$, we easily see that C is open in $(A \setminus \partial_X C) \cup (X \setminus A)$. On the other hand, since C is closed in $X \setminus A$, we have

$$C = (C \cup \partial_X C) \cap (X \setminus A).$$

As $\partial_X C \subseteq A$ is proved, we have $(C \cup \partial_X C) \cap (A \setminus \partial_X C) = \emptyset$, so

$$C = (C \cup \partial_X C) \cap ((A \setminus \partial_X C) \cup (X \setminus A)).$$

Thus C is also closed in $(A \setminus \partial_X C) \cup (X \setminus A)$.

It then follows that $(A \setminus \partial_X C) \cup (X \setminus A)$ is not connected. Since A is a cut of X, we get $A \setminus \partial_X C = \emptyset$, so $A \subseteq \partial_X C$.

2) Let U be a neighborhood of A. Suppose $U \setminus A$ is connected. Then $X \setminus A$ has a component C with $C \supset U \setminus A$, so

$$A \subset \mathrm{int}_X (C \cup A) \tag{1.27}$$

As mentioned, C is open in $X \setminus A$, which together with (1.27) implies that $C \cup A$ is open in X. On the other hand, as was shown, one has $\partial_X C = A$, which together with (1.27) implies that $\partial_X(C \cup A) = \emptyset$, so $C \cup A$ is closed in X. Since X is connected, we then get $C \cup A = X$, which yields $C = X \setminus A$, a contradiction. $\square$

Remark 1.8. Under the condition of Lemma 1.4, if x is a cut point of X and U is a connected neighborhood of x then x is a cut point of U.

From now on denote by F a necklace with a NIFS $\{f_1, f_2, \dots, f_n\}$ on $\mathbb{R}^d$ and by $z_1, z_2, \dots, z_n$ its ordered main nodes. As mentioned, we prescribe $z_0 = z_n$. The main results of this section are the following theorems.

Theorem 1.8. *If F has no cut points, then $N_2(F) = n - 2$ and $\{z_{k-1}, z_k\}$, $k \in I$, are extremal 2-cuts of F. Moreover, if F is good, then $\{z_{k-1}, z_k\}$, $k \in I$, are the only extremal 2-cuts of F.*

Theorem 1.9. *If F has no cut points and $k \in I$, then $F \setminus F_k$ is an extremal component of $F \setminus \{z_{k-1}, z_k\}$. Moreover, if F is good, then $F \setminus F_k$ is the only extremal component of $F \setminus \{z_{k-1}, z_k\}$.*

The assumption that F is good can not be removed off for the related results in Theorem 1.8 and Theorem 1.9.

Example 4. Let T be a closed solid triangle of vertices $0, 1, v$ in the complex plane, whose corresponding angles α, β, γ satisfy $4\beta < 2\alpha < \gamma$. Appropriately choosing a real number $a \in (0, 1)$, we may construct a planar self-similar necklace F by 4 similarity maps as in Figure 3, such that its ordered main nodes z_1, z_2, z_3, z_4 are $0, a, a + (1-a)v, v$ respectively. This necklace is not good and has no cut points (see [24], Theorem 2). By the first implication of Theorem 1.3 we have $N_2(F) = 2$. We easily check that

$$\{a + a(1-a), a + a(1-a) + (1-a)^2 v\}$$

is an extremal 2-cut of F, but it is not equal to $\{z_{k-1}, z_k\}$ for any $k \in \{1, 2, 3, 4\}$. On the other hand, $F \setminus \{z_2, z_3\}$ has three components, two of which are extremal.

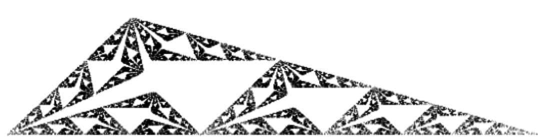


Figure 7: A necklace that is not good and has no cut points.

Let $i_1 i_2 \ldots i_m \in I^m$. Since $f_1, f_2, \ldots, f_n$ have been assumed to be homeomorphisms of $\mathbb{R}^d$, we easily see that $F_{i_1 i_2 \ldots i_m}$ is a necklace with an induced NIFS

$$\{f_{i_1 \ldots i_m} \circ f_1 \circ f^{-1}_{i_1 \ldots i_m},\ f_{i_1 \ldots i_m} \circ f_2 \circ f^{-1}_{i_1 \ldots i_m},\ \ldots,\ f_{i_1 \ldots i_m} \circ f_n \circ f^{-1}_{i_1 \ldots i_m}\}$$

whose main nodes are $f_{i_1 \ldots i_m}(z_1),\ f_{i_1 \ldots i_m}(z_2),\ \ldots,\ f_{i_1 \ldots i_m}(z_n)$ and whose 1-level copies are $F_{i_1 \ldots i_m 1},\ F_{i_1 \ldots i_m 2},\ \ldots,\ F_{i_1 \ldots i_m n}$. Let

$$M_F = \bigcup_{m=0}^{\infty} \bigcup_{i_1 \ldots i_m \in I^m} \{f_{i_1 \ldots i_m}(z_1),\ f_{i_1 \ldots i_m}(z_2),\ \ldots,\ f_{i_1 \ldots i_m}(z_n)\}.$$

Therefore $z \in M_F$ if and only if z is a main node of some copy of F.

From now on we assume that F has no cut points. Thus each copy of F has no cut points.

The proof of Theorems 1.8 and 1.9 will occupy the rest part of this section. The connectedness and local connectedness of a necklace F and the assumption that F has no cut points will be used frequently.

Lemma 1.5. *Suppose F has no cut points. Let $k, m \in I$, $k \neq m$. Then we have the following statements.*

1) $\{z_k, z_m\}$ is a cut of F.

2) If z_k and z_m are not adjacent then $F \setminus \{z_k, z_m\}$ has exactly two components and $N(\{z_k, z_m\}, F) < n - 2$.

3) $F \setminus F_k$ is a component of $\overline{F \setminus \{z_{k-1}, z_k\}}$.

4) The set of cut points of $\overline{F \setminus F_k}$ is $\{z_1, z_2, \ldots, z_n\} \setminus \{z_{k-1}, z_k\}$.

5) For each $z \in \{z_1, z_2, \ldots, z_n\} \setminus \{z_{k-1}, z_k\}$, $\overline{F \setminus F_k} \setminus \{z\}$ has exactly two components, one containing z_{k-1} and the other containing z_k.

Proof. It is obvious. □

Lemma 1.6. *Suppose F has no cut points and $\{z, w\}$ is a cut of F. Let $A = F_{i_1 i_2 \ldots i_m} \in \mathcal{C}(F)$ be the smallest copy such that $\{z, w\}$ is a cut of $F_{i_1 i_2 \ldots i_j}$ for each $0 \leq j \leq m$. Then we have*

1) Both z and w are main nodes of A, and

2) $A \setminus \{z, w\}$ has exactly two components with $N(\{z, w\}, A) \leq n - 2$.

Proof. 1) Under the assumption, since

$$\lim_{m \to \infty} \max_{C \in \mathcal{C}_m(F)} \operatorname{diam}(C) = 0,$$

the smallest copy $A = F_{i_1 i_2 \ldots i_m}$ such that $\{z, w\}$ is a cut of $F_{i_1 i_2 \ldots i_j}$ for each $0 \leq j \leq m$ does exist.

To show that z and w are main nodes of A, it suffices to prove

$$\{z, w\} \cap \operatorname{int}_A B = \emptyset$$

for each $B \in \mathcal{C}_1(A)$. In fact, suppose there is a copy $B \in \mathcal{C}_1(A)$ such that $\{z,w\} \cap \mathrm{int}_A B \neq \emptyset$. Without loss of generality assume $z \in \mathrm{int}_A B$. As z is a cut point of $A \setminus \{w\}$, it follows from Lemma 1.4 that z is a cut point of $B \setminus \{w\}$, so $w \in B$ and $\{z,w\}$ is a cut of B, contradicting the minimality of A.

2) There are two cases as follows.

Case 1. z and w are nonadjacent main nodes of A. By Lemma 1.5, A has exactly two components with $N(\{z,w\},A) < n-2$.

Case 2. z and w are adjacent main nodes of A. Let $B \in \mathcal{C}_1(A)$ be the copy such that $\partial_A B = \{z,w\}$. By Lemma 1.5, $A \setminus B$ is a component of $A \setminus \{z,w\}$ whose closure has exactly $n-2$ cut points. On the other hand, by the minimality of A we see that $B \setminus \{z,w\}$ is another component of $A \setminus \{z,w\}$ whose closure has no cut points. Thus $A \setminus \{z,w\}$ has exactly two components with $N(\{z,w\},A) = n-2$ in this case. □

Lemma 1.7. *Suppose F has no cut points, $\{z,w\}$ is a cut of F, and $i \in I$. If $\{z,w\}$ is a cut of F_i then*

$$N(\{z,w\},F) \leq \max\{n-2, N(\{z,w\},F_i)\}. \tag{1.28}$$

Proof. There are two possible cases.

Case 1. $\{z,w\} = \{z_{i-1},z_i\}$. In this case, $F \setminus F_i$ is a component of $F \setminus \{z,w\}$ whose closure has exactly $n-2$ cut points and the other components of $F \setminus \{z,w\}$ are those of $F_i \setminus \{z,w\}$. Thus (1.28) holds.

Case 2. $\{z,w\} \neq \{z_{i-1},z_i\}$. Then, by Lemma 1.4, $F \setminus F_i$ is not a component of $F \setminus \{z,w\}$. So there is a component A of $F \setminus \{z,w\}$ with

$$A \supsetneq F \setminus F_i.$$

Clearly, $F_i \setminus \{z,w\}$ has at most two components intersecting $\{z_{i-1},z_i\}$.

Subcase 1. $F_i \setminus \{z,w\}$ has only one component B with

$$B \cap \{z_{i-1},z_i\} \neq \emptyset.$$

In this subcase, one has

$$A = (F \setminus F_i) \cup B \text{ and } \{z_{i-1},z_i\} \subset \overline{B}.$$

Then, by the statement 5) of Lemma 1.5, we see that the cut points of $\overline{A}$ belong to those of $\overline{B}$, so $\mathrm{ncp}(\overline{A}) \leq \mathrm{ncp}(\overline{B})$.

Subcase 2. $F_i \setminus \{z,w\}$ has two components C and D with

$$C \cap \{z_{i-1},z_i\} = \{z_{i-1}\} \text{ and } D \cap \{z_{i-1},z_i\} = \{z_i\}.$$

In this subcase,

$$A = (F \setminus F_i) \cup C \cup D.$$

By Lemma 1.4, one has $\{z,w\} = \overline{C} \cap \overline{D}$, so $\overline{C \cup D}$ is connected, which together with the statement 5) of Lemma 1.5 implies that the cut points of $\overline{A}$ belong to those of $\overline{C \cup D}$. As F_i has no cut points, we easily see that $\overline{C \cup D}$ has no cut points, so $\mathrm{ncp}(\overline{A}) = 0$.

Thus, for both subcases we have $\mathrm{ncp}(\overline{A}) \leq N(\{z,w\}, F_i)$. As the other components of $F \setminus \{z,w\}$ belong to those of $F_i \setminus \{z,w\}$, we get $N(\{z,w\}, F) \leq N(\{z,w\}, F_i)$, so (1.28) holds in Case 2. □

Proof of the first implication of Theorem 1.8. Suppose F has no cut points. We are going to show that $N_2(F) = n-2$ and that $\{z_{k-1}, z_k\}$, $k \in I$, are extremal 2-cuts of F.

First, we show

$$N(\{z,w\}, F) \leq n-2 \tag{1.29}$$

for each cut $\{z,w\}$ of F.

Let $F_{i_1 i_2 \ldots i_m}$ be the smallest copy such that $\{z,w\}$ is a cut of $F_{i_1 i_2 \ldots i_j}$ for each $0 \leq j \leq m$. If $m = 0$, (1.29) follows from Lemma 1.6 directly. If $m \geq 1$, by Lemma 1.7 we have

$$N(\{z,w\}, F_{i_1 i_2 \ldots i_{j-1}}) \leq \max\{n-2, N(\{z,w\}, F_{i_1 i_2 \ldots i_j})\}$$

for all $1 \leq j \leq m$, which together with Lemma 1.6 implies

$$N(\{z,w\}, F) \leq \max\{n-2, N(\{z,w\}, F_{i_1 i_2 \ldots i_m})\} \leq n-2.$$

This proves (1.29), and thus we have $N_2(F) \leq n-2$.

Secondly, by Lemma 1.5, given $k \in I$, $\{z_{k-1}, z_k\}$ is a 2-cut of F and $F \setminus F_k$ is a component of $F \setminus \{z_{k-1}, z_k\}$ with $\mathrm{ncp}(\overline{F \setminus F_k}) = n-2$, so

$$N(\{z_{k-1}, z_k\}, F) \geq n-2,$$

which yields $N_2(F) \geq n-2$.

To sum up, we have $N_2(F) = N(\{z_{k-1}, z_k\}, F) = n-2$ for each $k \in I$. This completes the proof.

Suppose F has no cut points. Then $F \setminus F_k$ is connected with

$$\mathrm{ncp}(\overline{F \setminus F_k}) = n-2$$

for each $k \in I$. However, $F \setminus F_{i_1 i_2}$ may not be connected for $i_1 i_2 \in I^2$; see for example the necklace in Figure 1. And, in the case $F \setminus F_{i_1 i_2}$ is connected, it is possible that $\mathrm{ncp}(\overline{F \setminus F_{i_1 i_2}}) > n-2$; see for example the Sierpinski triangle. By contrast, we have the following lemma.

Lemma 1.8. *Suppose F is good. Then we have*

1) $F \setminus A$ is connected for each $A \in \cup_{m=1}^{\infty} \mathcal{C}_m(F)$, and

2) ncp$(\overline{F \setminus A}) < n-2$ for each $A \in \cup_{m=2}^{\infty} \mathcal{C}_m(F)$ with $\sharp\, \partial_F A = 2$.

Proof. 1) The assumption implies that F has no cut points; see [24]. So $F \setminus F_k$ is connected for each $k \in I$. Let $lj \in I^2$. Then $F \setminus F_l$ and $F_l \setminus F_{lj}$ are connected.

Since F is good, we may take a point $x \in (\partial_F F_l) \setminus F_{lj}$. Then $(F \setminus F_l) \cup \{x\}$ is connected. Observing

$$((F \setminus F_l) \cup \{x\}) \cap (F_l \setminus F_{lj}) = \{x\}$$

and

$$F \setminus F_{lj} = (F \setminus F_l) \cup \{x\} \cup (F_l \setminus F_{lj}),$$

we see that $F \setminus F_{lj}$ is connected. Inductively, we get that $F \setminus A$ is connected for each $A \in \cup_{m=1}^{\infty} \mathcal{C}_m(F)$.

2) We first prove that $\mathrm{ncp}(\overline{F \setminus A}) < n-2$ for each $A \in \mathcal{C}_2(F)$ with $\sharp\, \partial_F A = 2$. Let such a copy A be given. We may write $A = F_{lk}$ and $\partial_F F_{lk} = \{z, w\}$, where $lk \in I^2$. Then we have the following facts.

(a) $\partial_{F_l} F_{lk} = \{z, w\}$, so z and w are two adjacent main nodes of F_l.
(b) $\overline{F_l \setminus F_{lk}}$ is connected with $\mathrm{ncp}(\overline{F_l \setminus F_{lk}}) = n - 2$.
(c) $\overline{F_l \setminus F_{lk}} = \cup_{j \in I,\, j \neq k} F_{lj}$.
(d) $F \setminus F_{lk} = (F \setminus F_l) \cup (F_l \setminus F_{lk})$.
(e) $\partial_F (F \setminus F_l) \subset \overline{F_l \setminus F_{lk}}$.
(f) Let u be a main node of F_l with $u \notin \{z, w\}$. Then u is a cut point of $\overline{F_l \setminus F_{lk}}$ and $\overline{F_l \setminus F_{lk}} \setminus \{u\}$ has exactly two components, one containing z and the other containing w.

We only prove (e). Since $\partial_F (F \setminus F_l) = \partial_F F_l = \{z_{l-1}, z_l\}$, the task is to show $\{z_{l-1}, z_l\} \subset \overline{F_l \setminus F_{lk}}$. Suppose it is false, say $z_l \notin \overline{F_l \setminus F_{lk}}$. Then one has $z_l \in F_{lk}$, so $z_l \in \partial_F F_{lk}$. By (a), z_l is a main node of F_l, so there is a copy F_{lj} such that $\{z_l\} = F_{lj} \cap F_{lk}$. Then by (c) we get $z_l \in \overline{F_l \setminus F_{lk}}$, a contradiction.

Now, since $\overline{F \setminus F_{lk}} = \overline{F \setminus F_l} \cup \overline{F_l \setminus F_{lk}}$, we see from the statement 5) of Lemma 1.5 and (e) that the cut points of $\overline{F \setminus F_{lk}}$ belong to those of $\overline{F_l \setminus F_{lk}}$. On the other hand, since F is good, one has $(\partial_F F_l) \setminus F_{lj} \neq \emptyset$ for each $j \in I$, so by (c) and (e) there are two distinct $j_1, j_2 \in I \setminus \{k\}$ such that $z_{l-1} \in F_{lj_1}$ and $z_l \in F_{lj_2}$, which together with (f) implies that $\overline{F_l \setminus F_{lk}}$ has a cut point that is not any cut point of $\overline{F \setminus F_{lk}}$. It then follows from (b) that $\mathrm{ncp}(\overline{F \setminus F_{lk}}) < n-2$.

Next let $A = F_{i_1 i_2 \ldots i_m}$, where $i_1 i_2 \ldots i_m \in I^m$, $m \geq 2$, and $\sharp\, \partial_F A = 2$. Then $F \setminus A$, $F \setminus F_{i_1}$ and $F_{i_1} \setminus A$ are connected,

$$\overline{F \setminus A} = \overline{F \setminus F_{i_1}} \cup \overline{F_{i_1} \setminus A}, \text{ and } \partial_F (F \setminus F_{i_1}) = \partial_F F_{i_1} \subset \overline{F_{i_1} \setminus A}.$$

Thus we have by the statement 5) of Lemma 1.5 that the cut points of $\overline{F \setminus A}$ belong to those of $\overline{F_{i_1} \setminus A}$, so

$$\mathrm{ncp}(\overline{F \setminus A}) \leq \mathrm{ncp}(\overline{F_{i_1} \setminus A}). \tag{1.30}$$

Since $\sharp\, \partial_{F_{i_1 \cdots i_{m-1}}} A = 2$, the assumption $\sharp\, \partial_F A = 2$ implies

$$\partial_F A = \partial_{F_{i_1}} A = \cdots = \partial_{F_{i_1 \cdots i_{m-1}}} A.$$

We may repeatedly apply (1.30) to get

$$\mathrm{ncp}(\overline{F \setminus A}) \leq \mathrm{ncp}(\overline{F_{i_1 i_2 \ldots i_{m-2}} \setminus A}).$$

Since $F_{i_1i_2\ldots i_{m-2}} \setminus A$ and $F \setminus F_{i_{m-1}i_m}$ are topologically equivalent, we then get

$$\mathrm{ncp}(\overline{F \setminus A}) \leq \mathrm{ncp}(\overline{F \setminus F_{i_{m-1}i_m}}).$$

As $\mathrm{ncp}(\overline{F \setminus F_{i_{m-1}i_m}}) < n-2$ is proved, we get $\mathrm{ncp}(\overline{F \setminus A}) < n-2$. □

Proof of the second implication of Theorem 1.8. Suppose F is good. We are going to show that $\{z_{k-1}, z_k\}$, $k \in I$, are the only extremal 2-cuts of F. As the first implication of Theorem 1.3 is proved, it suffices to show

$$N(\{z, w\}, F) < n-2 \tag{1.31}$$

for each 2-cut $\{z, w\}$ of F with $\{z, w\} \notin \{\{z_{k-1}, z_k\} : k \in I\}$.

Let such a 2-cut $\{z, w\}$ of F be given. Let $A = F_{i_1i_2\ldots i_m} \in \mathcal{C}(F)$ be the smallest copy such that $\{z, w\}$ is a cut of $F_{i_1i_2\ldots i_j}$ for each $0 \leq j \leq m$. By Lemma 1.6, z and w are main nodes of A and $A \setminus \{z, w\}$ has exactly two components, which will be denoted by B and C. Thus there is a subset J of I such that

$$\overline{B} = \bigcup_{j\in J} F_{i_1i_2\ldots i_m j} \quad \text{and} \quad \overline{C} = \bigcup_{j\in I\setminus J} F_{i_1i_2\ldots i_m j}. \tag{1.32}$$

In addition, $F \setminus A$ is connected by 1) of Lemma 1.8. And we have

$$F \setminus \{z, w\} = (F \setminus A) \cup B \cup C.$$

Case 1. $A = F$. By the assumption on $\{z, w\}$, we see that z and w are actually two nonadjacent main nodes of F. The inequality (1.31) follows by Lemma 1.5.

Case 2. $A \in \cup_{m=1}^{\infty} \mathcal{C}_m(F)$ and $\partial_F A = \{z, w\}$. By the assumption on $\{z, w\}$ we actually have $A \in \cup_{m=2}^{\infty} \mathcal{C}_m(F)$. In this case, $F \setminus A$, B, and C are the only three components of $F \setminus \{z, w\}$. By 2) of Lemma 1.8, we have

$$\mathrm{ncp}(\overline{F \setminus A}) < n-2.$$

On the other hand, since F is good, we see that z, w are actually two nonadjacent main nodes of A, which implies

$$\mathrm{ncp}(\overline{B}) < n-2 \quad \text{and} \quad \mathrm{ncp}(\overline{C}) < n-2.$$

Then the inequality (1.31) follows.

Case 3. $A \in \cup_{m=1}^{\infty} \mathcal{C}_m(F)$ and $\partial_F A \neq \{z, w\}$. Since $\partial_F(F \setminus A) = \partial_F A$, we have by Lemma 1.4 that $F \setminus A$ is not a component of $F \setminus \{z, w\}$. Thus $F \setminus A$ meets exactly one of B and C. Without loss of generality assume that $(F \setminus A) \cap B \neq \emptyset$. Then $(F \setminus A) \cup B$ and C are the only two components of $F \setminus \{z, w\}$ and we have $\partial_F A \subset \overline{B}$. Let J be the subset of I such that (1.32) holds. Then $\sharp J \leq n-1$. Since F is good, one also has $\sharp J \geq 2$, so

$$\mathrm{ncp}(\overline{C}) = \sharp(I \setminus J) - 1 < n-2.$$

Since $\partial_F(F \setminus A) = \partial_F A \subset \overline{B}$, arguing as we did in the proof of 2) of Lemma 1.8, we have that the cut points of $\overline{(F \setminus A) \cup B}$ belong to those of $\overline{B}$ and that $\overline{B}$ has at least one cut point that is not any cut point of $\overline{(F \setminus A) \cup B}$. Thus

$$\mathrm{ncp}(\overline{(F \setminus A) \cup B}) < \mathrm{ncp}(\overline{B}) \leq \sharp J - 1 \leq n - 2.$$

Then the inequality (1.31) follows. This completes the proof.

Proof of Theorem 1.9. Suppose F has no cut points. Fix $k \in I$. Then we have $N_2(F) = n - 2$ by the first implication of Theorem 1.3. Since $F \setminus F_k$ is a component of $F \setminus \{z_{k-1}, z_k\}$ with $\mathrm{ncp}(\overline{F \setminus F_k}) = n - 2$, it is an extremal component of $F \setminus \{z_{k-1}, z_k\}$.

Now suppose F is good. Let C be a component of $F \setminus \{z_{k-1}, z_k\}$ with $C \neq F \setminus F_k$. Then C is a component of $F_k \setminus \{z_{k-1}, z_k\}$. In the case $C = F_k \setminus \{z_{k-1}, z_k\}$, one has $\mathrm{ncp}(\overline{C}) = 0$. In the other case, since F is good, there is no copy F_{kj} containing $\{z_{k-1}, z_k\}$, so z_{k-1} and z_k are two nonadjacent main nodes of F_k, which implies $\mathrm{ncp}(\overline{C}) < n - 2$. This proves that $F \setminus F_k$ is the only extremal component of $F \setminus \{z_{k-1}, z_k\}$.

1.2.3 The proof of main results

Proof of Theorem 1.5. Let F be a fractal necklace with a good NIFS $\{f_1, f_2, \ldots, f_n\}$ on $\mathbb{R}^d$. Then F has no cut points. Let $\{g_1, g_2, \ldots, g_m\}$ be an arbitrary NIFS of F. We do not know if this NIFS is good at this stage. As the first implication of Theorem 1.3 is valid for all necklaces with no cut points, we have

$$N_2(F) = n - 2 = m - 2,$$

which yields $n = m$. We are going to show that there is a permutation $\sigma \in \mathcal{G}_n$ such that $g_k(F) = f_{\sigma(k)}(F)$ for each $k \in I$.

Let $z_1, z_2, \ldots, z_n$ be the ordered main nodes of F under $\{f_1, \ldots, f_n\}$ and $w_1, w_2, \ldots, w_n$ be those of F under $\{g_1, \ldots, g_n\}$. Thus

$$\{z_{k-1}, z_k\} = \partial_F(f_k(F)) \text{ and } \{w_{k-1}, w_k\} = \partial_F(g_k(F)).$$

By the first implication of Theorem 1.3, $\{w_{k-1}, w_k\}$, $k \in I$, are extremal 2-cuts of F. By the second implication of Theorem 1.3, $\{z_{k-1}, z_k\}$, $k \in I$, are the only extremal 2-cuts of F. Thus we have

$$\{\{w_{k-1}, w_k\} : k \in I\} = \{\{z_{k-1}, z_k\} : k \in I\}, \tag{1.33}$$

which implies

$$\{w_1, w_2, \ldots, w_n\} = \{z_1, z_2, \ldots, z_n\} \text{ as sets.}$$

Let $j \in I$ satisfy $\{w_1, w_2\} = \{z_{j-1}, z_j\}$. Then we have

$$w_1 = z_j \text{ and } w_2 = z_{j-1} \tag{1.34}$$

or

$$w_1 = z_{j-1} \text{ and } w_2 = z_j. \tag{1.35}$$

First consider the case (1.34). Let $\sigma = \tau^{n-j}s \in \mathcal{G}_n$ be a permutation of I. By the definitions of τ and s in Section 1 we have $\sigma(k) = j - k + 1$ for each $k \in I$, hereafter we identify an integer l with an integer $k \in I$ if $|l - k| = 0$ or n. Thus (1.34) can be written as $w_1 = z_{\sigma(1)}$ and $w_2 = z_{\sigma(2)}$, which implies $w_k = z_{\sigma(k)}$ for each $k \in I$ by using (1.33).

Fix $k \in I$. We then have

$$\{w_{k-1}, w_k\} = \{z_{\sigma(k-1)}, z_{\sigma(k)}\}.$$

As $\sigma(k-1) = \sigma(k) + 1$, we have

$$\{z_{\sigma(k-1)}, z_{\sigma(k)}\} = \partial_F(f_{\sigma(k-1)}(F)).$$

It follows that

$$\partial_F(g_k(F)) = \partial_F(f_{\sigma(k-1)}(F)).$$

By the first implication of Theorem 1.9, $F \setminus g_k(F)$ is an extremal component of $F \setminus \{w_{k-1}, w_k\}$. By the second implication of Theorem 1.9, $F \setminus f_{\sigma(k-1)}(F)$ is the only extremal component of $F \setminus \{w_{k-1}, w_k\}$. Thus

$$F \setminus g_k(F) = F \setminus f_{\sigma(k-1)}(F),$$

which yields

$$g_k(F) = f_{\sigma(k-1)}(F) = f_{\sigma\tau^{-1}(k)}(F),$$

where τ^{-1} is the inverse of τ. Of course $\sigma\tau^{-1} \in \mathcal{G}_n$.

As for the case (1.35), let $\sigma = \tau^{j-2} \in \mathcal{G}_n$ be a permutation of I. By a slightly easier argument, we get

$$\partial_F(g_k(F)) = \partial_F(f_{\sigma(k)}(F)).$$

Using Corollary 1.9 as above, we have $g_k(F) = f_{\sigma(k)}(F)$ for each $k \in I$. The proof is completed.

Proof of Theorem 1.6. Let F and G be two topologically equivalent good necklaces in $\mathbb{R}^d$. For clarity let $\{f_1, f_2, \ldots, f_n\}$ be a NIFS of F and $\{g_1, g_2, \ldots, g_m\}$ be a NIFS of G on $\mathbb{R}^d$. Then, by Lemma 1.3 and Theorem 1.3, we have $n = m$. Denote by $z_1, z_2, \ldots, z_n$ the ordered main nodes of F and by $w_1, w_2, \ldots, w_n$ those of G. We are going to show that every homeomorphism $h \in h(F, G)$ is rigid.

Fix $h \in h(F, G)$. Since $\{\{z_{k-1}, z_k\} : k \in I\}$ is the family of extremal 2-cuts of F and $\{\{w_{k-1}, w_k\} : k \in I\}$ is that of G by Theorem 1.3, we have by Lemma 1.3

$$\{\{h(z_{k-1}), h(z_k)\} : k \in I\} = \{\{w_{k-1}, w_k\} : k \in I\}.$$

Now, arguing as we just did in the proof of Theorem 1.5, there is a permutation $\sigma \in \mathcal{G}_n$ such that

$$h(f_k(F)) = g_{\sigma(k)}(G)$$

for every $k \in I$. This shows that h maps every 1-level copy of F onto a 1-level copy of G. Inductively, one has that h maps every m-level copy of F onto an m-level copy of G for every integer $m \geq 1$. Thus h is rigid.

Next we show that $h(F, G)$ is countable. For every integer $m \geq 0$ write

$$M_{F,m} = \bigcup_{i_1 i_2 \ldots i_m \in I^m} f_{i_1 i_2 \ldots i_m}\{z_1, z_2, \ldots, z_n\}$$

and

$$M_{G,m} = \bigcup_{i_1 i_2 \ldots i_m \in I^m} g_{i_1 i_2 \ldots i_m}\{z_1, z_2, \ldots, z_n\}.$$

Then $M_{F,m}$ and $M_{G,m}$ are finite sets. Let

$$M_F = \bigcup_{m=0}^{\infty} M_{F,m} \text{ and } M_G = \bigcup_{m=0}^{\infty} M_{G,m}.$$

Then M_F is dense in F and M_G is dense in G. Let $h \in h(F, G)$. As h was shown to be rigid, the restriction $h|_{M_F}$ of h is a bijection from M_F onto M_G and satisfies $h(M_{F,m}) = M_{G,m}$ for each $m \geq 0$.

Let Φ be the collection of bijections $\phi : M_F \to M_G$ satisfying $\phi(M_{F,m}) = M_{G,m}$ for each $m \geq 0$. Clearly, Φ is countable and

$$\{h|_{M_F} : h \in h(F, G)\} \subseteq \Phi,$$

so $\{h|_{M_F} : h \in h(F, G)\}$ is countable.

To prove that $h(F, G)$ is countable, it suffices to show the correspondence $h \to h|_{M_F}$ from $h(F, G)$ to $\{h|_{M_F} : h \in h(F, G)\}$ is one-to-one. In fact, let $h, \widetilde{h} \in h(F, G)$, $h \neq \widetilde{h}$, then there is a point $z \in F$ such that $h(z) \neq \widetilde{h}(z)$, so $h(A)$ and $\widetilde{h}(A)$ are disjoint for sufficiently small copy A of F with $z \in A$, and so $h|_{M_F} \neq \widetilde{h}|_{M_F}$ due to M_F dense in F. This completes the proof.

Proof of Theorem 1.7. Let F and G be two topologically equivalent good fractal necklaces. Let $h : F \to G$ be a topological embedding. We are going to show that $h(F)$ is a copy of G.

For clarity let $\{f_1, f_2, \ldots, f_n\}$ be a NIFS of F and $\{g_1, g_2, \ldots, g_m\}$ be a NIFS of G. Denote by $z_1, z_2, \ldots, z_n$ the main nodes of F and by $w_1, w_2, \ldots, w_n$ those of G.

Let A be the smallest copy of G such that $h(F) \subseteq A$. Without loss of generality assume $A = G$. We are going to prove $h(F) = G$. Since M_G is dense in G, it suffices to show $h(F) \supset M_G$.

By the assumption, $h(F)$ meets $\text{int}_G B$ for at least two 1-level copies B of G. Since $h(F)$ has no cut points, we see that $h(F)$ actually meets $\text{int}_G B$ for all 1-level copies B of G, so

$$h(F) \supset \{w_1, w_2, \ldots, w_n\}, \text{ i.e. } h(F) \supset M_{G,0}. \tag{1.36}$$

Next we prove $h(F) \supset M_{G,1}$. It suffices to show for each $k \in I$

$$h(F) \supset \{g_k(w_1), g_k(w_2), \ldots, g_k(w_n)\}. \tag{1.37}$$

By Theorem 1.3, $\{\{z_{k-1}, z_k\} : k \in I\}$ is the family of extremal 2-cuts of F, so $\{\{h(z_{k-1}), h(z_k)\} : k \in I\}$ is the family of extremal 2-cuts of $h(F)$ by Lemma 1.3. On the other hand, $\{w_{k-1}, w_k\}$, $k \in I$, are obviously extremal 2-cuts of $h(F)$ by (1.36). It follows that

$$\{\{w_{k-1}, w_k\} : k \in I\} = \{\{h(z_{k-1}), h(z_k)\} : k \in I\},$$

which implies

$$\{h(z_1), h(z_2), \ldots, h(z_n)\} = \{w_1, w_2, \ldots, w_n\} \text{ as sets.} \tag{1.38}$$

Then, arguing as we just did in the proof of Theorem 1.5, there is a permutation $\sigma \in \mathcal{G}_n$ such that for each $k \in I$

$$\partial_{h(F)}(h(f_{\sigma(k)}(F))) = \partial_G(g_k(G)) = \{w_{k-1}, w_k\}. \tag{1.39}$$

Now fix $k \in I$. By (1.39) and the arguments of (1.36), one has

$$h(f_{\sigma(k)}(F)) \subseteq g_k(G) \quad \text{or} \quad h(f_{\sigma(k)}(F)) \supset \{w_1, w_2, \ldots, w_n\},$$

in which the latter case does not occur because it contradicts (1.38). Thus we have

$$h(f_{\sigma(k)}(F)) \subseteq g_k(G).$$

Moreover, since G is good, we see by (1.39) that $g_k(G)$ is the smallest copy containing $h(f_{\sigma(k)}(F))$. Applying (1.36) to the topological embedding $h : f_{\sigma(k)}(F) \to g_k(G)$, we get

$$h(f_{\sigma(k)}(F)) \supset \{g_k(w_1), g_k(w_2), \ldots, g_k(w_n)\},$$

which implies (1.37) and thus $h(F) \supset M_{G,1}$.

Inductively, we have $h(F) \supset M_{G,m}$ for every integer $m \geq 0$, and so $h(F) \supset M_G$. This completes the proof.

2 Dragon Curves

2.1 Convex Hulls of Dragon Curves

2.1.1 Introduction

Let A be a $d \times d$ matrix and $d_i \in \mathbb{R}^d$. We assume that A is contractive. The convex hull of the attractor of iterated function system (IFS for short) $\{f_i \mid i = 1, 2, \dots m\}$ with $f_i = Ax + d_i$ is studied by Strichartz-Wang [30]. They observed an important property of extreme points of the convex hull and deduced that the attractor has a polygonal convex hull if and only if there exists a positive integer s such that A^s is a scalar matrix. Kirat-Kocyigit [20] considered the case that the linear part of f_i may not be identical and proved that, if the attractor has a polygonal convex hull, the vertices must have eventually periodic codings. In contrast with this result, we further get the following theorem.

Theorem 2.1. *Let K be the attractor of an IFS $\{f_i \mid i = 1, 2, \dots m\}$ on the complex plane $\mathbb{C}$ with*

$$f_i(z) = a_i z + b_i,\ \ a_i, b_i \in \mathbb{C},\ \ 0 < |a_i| < 1.$$

Suppose K is not a singleton. If an eventually periodic word $i_1 i_2 \cdots i_l (j_1 \cdots j_k)^\infty$ in $\{1, 2, \dots m\}^{\mathbb{N}}$ is a coding of an extreme point of $co(K)$ then $a_{j_1} a_{j_2} \cdots a_{j_k} > 0$.

For an infinite word $i_1 i_2 \cdots$ in $\{1, 2, \dots m\}^{\mathbb{N}}$ and an integer $k \geq 1$ denote by $f_{i_1 \cdots i_k}$ the composition $f_{i_1} \circ \cdots \circ f_{i_k}$. If $\cap_{k=1}^{\infty} f_{i_1 \cdots i_k}(K) = z$, then $i_1 i_2 \cdots$ is called a coding of the point z in K.

Kirat-Kocyigit [20] also gave a sufficient and necessary condition such that the attractor of a given IFS has a polygonal convex hull. Moreover, they found an algorithm to check their condition, but the termination of the algorithm is not discussed.

The present paper is devoted to studying the convex hulls of dragon curves. Let $\mathbb{C}$ be the complex plane. For $z \in \mathbb{C}$ denote by $\arg z$ the argument of z in $[0, 2\pi)$, by Rez and Imz the real and imaginary part of z, and by $\bar{z}$ the conjugate of z. Let $\eta \in (0, \pi/3)$ and let

$$a := a(\eta) = \frac{e^{-i\eta}}{2\cos\eta}. \tag{2.1}$$

The η-Dragon curve K_η is the attractor of the IFS

$$f_1(z) = az,\ f_2(z) = 1 - \bar{a}z,\ z \in \mathbb{C}.$$

In other words, K_η is an unique nonempty compact subset of $\mathbb{C}$ satisfying

$$K_\eta = f_1(K_\eta) \cup f_2(K_\eta). \tag{2.2}$$

The η-dragon curve has also been obtained as the limit of the renormalized paperfolding curves in the Hausdorff metric as well; see [1, 31]. By using their algorithm, Kirat-Kocyigit [20] verified that the dragon curve $K_{\pi/4}$ has a polygonal convex hull. We will prove that every dragon curve has this property.

Theorem 2.2. *For each $\eta \in (0, \pi/3)$ the convex hull $co(K_\eta)$ is a polygon.*

Actually, we find out a countable subset V of K_η and prove that its convex hull $\mathrm{co}(V)$ is a polygon with $\mathrm{co}(V) = \mathrm{co}(K_\eta)$.

Given $\eta \in (0, \pi/3)$, let a, f_1, f_2, and K_η be defined as above. One has

$$\frac{1}{2} < |a| < 1, \ a + \bar{a} = 1, \ \text{ and } \ 2|a|\cos\eta = 1. \tag{2.3}$$

Let z_0 be the fixed point of the composition f_{2211}. Then one has $z_0 = ca \in K_\eta$ by a simple computation, where

$$c = \frac{1}{1 - |a|^4}. \tag{2.4}$$

For every integer $k \geq 0$ let

$$z_k = f_1^k(z_0), \ w_k = f_2(z_k), \ \text{ and } \ b_k = f_2(w_{k+1}). \tag{2.5}$$

Then

$$z_k = ca^{k+1}, \ w_k = 1 - c|a|^2 a^k, \ \text{ and } \ b_k = a + c|a|^4 a^k. \tag{2.6}$$

We define a countable subset V of K_η by

$$V = \{b_0\} \cup \{z_k : k \geq 0\} \cup \{w_k : k \geq 1\}. \tag{2.7}$$

Since $z_0 \in K_\eta$, one has $V \subset K_\eta$ by (2.5). For every integer $k \geq 1$ let

$$V_k = \{b_0, z_0, z_1, \cdots, z_k, w_1 \cdots, w_k\}. \tag{2.8}$$

We shall show

$$\mathrm{co}(V) = \mathrm{co}(V_k) \tag{2.9}$$

for sufficiently large integer k depending on η. Therefore V has a polygonal convex hull. By the construction of the attractor K_η we may further prove that $\mathrm{co}(K_\eta) = \mathrm{co}(V)$, which gives Theorem 2.2. Detailed proof will be given in Section 3.

We shall see that, in the proof of Theorem 2.2, the vertices of $\mathrm{co}(K_\eta)$ are not determined completely. To answer this question, the first work is to find the smallest integer with the property (2.9). Let

$$\Phi_k(\eta) = (1 - |a|^4)\sin(k-1)\eta - |a|^3 \sin(k-2)\eta + |a|^k \sin\eta. \tag{2.10}$$

We will show that for each integer $k \geq 4$ the function Φ_k has a unique null in the interval $(\pi/k, \pi/(k-1))$. We denote this zero point of Φ_k by η_k. Then the interval $(0, \pi/3)$ has a partition as

$$(0, \pi/3) = [\eta_4, \pi/3) \cup \bigcup_{k=4}^{\infty} [\eta_{k+1}, \eta_k). \tag{2.11}$$

For $\eta \in (0, \eta_4)$ we get the following result.

Theorem 2.3. *Let $k \geq 4$ be an integer and let $\eta \in [\eta_{k+1}, \eta_k)$. Then the vertices of the polygon $co(K_\eta)$ are $b_0, z_0, z_1, \cdots, z_k, w_1, \cdots, w_k$ in clockwise.*

We shall prove Theorem 2.1 in Section 2 and Theorem 2.2 in Section 3. For our purpose, some properties of functions Φ_k will be given in Section 4. Some properties of dragon curves will be given in Section 5. Theorem 2.3 will be proved in Section 6. We give an outline here for the convenience of readers. For $u, v, w \in \mathbb{C}$ denote by $\angle uvw$ the counterclockwise angle of $u - v$ to $w - v$. That is,

$$\angle uvw = \arg \frac{w - v}{u - v}.$$

Then $\angle uvw \in (0, \pi)$ means that v is in the left-hand side of the straight line passing through u and w of direction $(w-u)/|w-u|$. We have the implications:

$$\angle uvw \in (0, \pi) \Leftrightarrow \mathrm{Im}\frac{w - v}{u - v} > 0 \Leftrightarrow \mathrm{Im}(\bar{u} - \bar{v})(w - v) > 0.$$

It is not difficult to get a generic result for all dragon curves as follows:

$$\angle b_0 z_0 z_1 = \angle z_k z_{k+1} z_{k+2} = \angle w_k w_{k+1} w_{k+2} = \pi - \eta$$

for each $\eta \in (0, \pi/3)$ and each integer $k \geq 0$. Moreover, given $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$, we may prove

$$\angle z_{k-1} z_k w_1,\ \angle z_k w_1 w_2,\ \angle w_{k-1} w_k b_0,\ \angle w_k b_0 z_0 \in (0, \pi)$$

and

$$\mathrm{co}(V_k) = \mathrm{co}(V).$$

After that, we infer that for each $\eta \in [\eta_{k+1}, \eta_k)$ the points

$$b_0, z_0, z_1, \cdots, z_k, w_1, w_2, \cdots, w_k$$

are in turn the vertices of the polygon $\mathrm{co}(V_k)$ in clockwise. Once these results are proved, Theorem 2.3 will follow from the proof of Theorem 2.2.

We remark that dragon curves are a class of path-connected self-similar sets in the plane [14], for which some basic geometric questions are subtle. For example, we know very little about when a dragon curve satisfies the open set condition; see [4, 11, 29]. Motivated by a question of Tabachnikov [31], Albers [1], Allouche et al [3], and Kamiya [16] studied self-intersecting and non-intersecting dragon curves, but the study on the question when a dragon curve is an arc is far from conclusive. As for the convex hull of K_η, we shall see that, in the case of $\eta \in [\eta_4, \pi/3)$, the point z_4 is no longer any vertex of $\mathrm{co}(K_\eta)$. Moreover, we shall prove that, if η is near to $\pi/3$, it is not true that

$$b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3$$

are the vertices of the polygon $\mathrm{co}(K_\eta)$ in clockwise. See Remark 2.3 at the end of Section 5. The vertex question of $\mathrm{co}(K_\eta)$ is still open for $\eta \in [\eta_4, \pi/3)$.

2.1.2 The proof of Theorem 2.1

Let K be the attractor of IFS $\{f_n|\ n = 1, 2, \dots, m\}$ on the plane with

$$f_n(z) = a_n z + b_n,\ a_n, b_n \in \mathbb{C},\ 0 < |a_n| < 1.$$

Let $i_1 i_2 \cdots i_l (j_1 \cdots j_k)^\infty$ be a coding of an extreme point of $\mathrm{co}(K)$. We are going to show $a_{j_1} a_{j_2} \cdots a_{j_k} > 0$.

We may write $a_{j_1} \cdots a_{j_k} = re^{i\alpha}$, where $r \in (0, 1)$ and $\alpha \in [0, 2\pi)$ are the modulus and argument of $a_{j_1} \cdots a_{j_k}$. Then

$$a_{j_1} \cdots a_{j_k} > 0 \Longleftrightarrow \alpha = 0.$$

Let w be the unique fixed point of $f_{j_1 \cdots j_k}$. Then $w \in K$, with coding $(j_1 \cdots j_k)^\infty$.

Since K is not a singleton, a point $v \in K$ exists with $v \neq w$. Denote $v_p = f_{(j_1 \cdots j_k)^p}(v)$ for every positive integer p, then $v_p \in K$, $v_p \neq w$ and

$$v_p = (a_{j_1} \cdots a_{j_k})^p (v - w) + w = r^p e^{ip\alpha}(v - w) + w. \tag{2.12}$$

If $\alpha \neq 0$, in view of (2.12), there is an integer $p \geq 2$ such that

$$w \in \mathrm{co}(\{v_1, v_2, \cdots, v_p\}) \text{ and } w \text{ is not a vertex of } \mathrm{co}(\{v_1, v_2, \cdots, v_p\}).$$

We know that $f_{j_1 \cdots j_k}(w)$ is the point of coding $i_1 i_2 \cdots i_l (j_1 \cdots j_k)^\infty$. Then

$$f_{j_1 \cdots j_k}(w) \in \mathrm{co}(\{f_{j_1 \cdots j_k}(v_1), f_{j_1 \cdots j_k}(v_2), \cdots, f_{j_1 \cdots j_k}(v_p)\})$$

and $f_{j_1 \cdots j_k}(w)$ is not a vertex of $\mathrm{co}(\{f_{j_1 \cdots j_k}(v_1), f_{j_1 \cdots j_k}(v_2), \cdots, f_{j_1 \cdots j_k}(v_p)\})$.

Since $v_p \in K$, then $f_{j_1 \cdots j_k}(v_p) \in K$ for every positive integer p. The previous discussion implies that $f_{j_1 \cdots j_k}(w)$ is not an extreme point of $\mathrm{co}(K)$, contradicting the assumption of Theorem 2.1. This proves $\alpha = 0$ and thus completes the proof.

2.1.3 The proof of Theorem 2.2

Given $\eta \in (0, \pi/3)$, let a, f_1, f_2, K_η, c, z_k, w_k, b_k, Φ_k, V and V_k be defined as in Section 1. To prove Theorem 2.2, we only need to show that $\mathrm{co}(V)$ is a polygon and that $\mathrm{co}(K_\eta) = \mathrm{co}(V)$. The definitions of the above parameters and their relationships will be used frequently without mentioning.

Lemma 2.1. *co(V) is a polygon for each $\eta \in (0, \pi/3)$.*

Proof. Let $\eta \in (0, \pi/3)$. By the definition of z_k, we see that there are points of $\{z_k : k \geq 0\}$ in the inner part of every quadrant of the plane. Therefore, 0 is an inner point of the convex hull $\mathrm{co}(V)$. By the definition of w_k, we easily check that 1 is also an inner point of $\mathrm{co}(V)$. Since by (2.6)

$$\lim_{k\to\infty} z_k = 0 \text{ and } \lim_{k\to\infty} w_k = 1,$$

there exists an integer n such that $\{z_k : k > n\} \cup \{w_k : k > n\}$ is in the inner part of co(V). It then follows that

$$\mathrm{co}(V) = \mathrm{co}(V_n),$$

so co(V) is a polygon. □

Remark 2.1. *By the above proof, co(V) is closed and $0, 1 \in$ co(V). Moreover, since $a^k = c^{-1}z_{k-1} = (1 - |a|^4)z_{k-1}$, one has $a^k \in$ co(V) for all integers $k \geq 1$.*

For $z \in \mathbb{C}$ recall that $\arg z$ is the argument of z in $[0, 2\pi)$. For $u, v, w \in \mathbb{C}$ denote by uv the segment of endpoints u and v, and by $\triangle(u, v, w)$ the closed solid triangle of vertices u, v and w. By the definition, $w_0 = 1 - c|a|^2$ is a real number. It can be nonnegative or negative, depending on the choice of η. The next lemma is useful.

Lemma 2.2. *We have $w_0 \in \triangle(0, z_2, z_3) \cap \triangle(0, \bar{z}_2, \bar{z}_3)$, if $w_0 < 0$.*

Proof. Let $\eta \in (0, \pi/3)$ be given such that $w_0 < 0$. Then $1 - |a|^2 - |a|^4 < 0$, which occurs only if $\eta \in (\pi/4, \pi/3)$. Thus,

$$\arg z_2 = 2\pi - 3\eta > \pi \ \text{ and } \ \arg z_3 = 2\pi - 4\eta < \pi,$$

giving $\arg z_3 < \arg w_0 < \arg z_2$. On the other hand, let

$$t = \frac{1}{|a|^2} - 1. \tag{2.13}$$

Then $t \in (0, 1)$ by $|a| < 1$ and $1 - |a|^2 - |a|^4 < 0$. And we have

$$\begin{aligned}
(1-t)z_2 + tz_3 &= ca^3(1 - t + ta) = ca^3(1 - t\bar{a}) \\
&= ca^3(1 - \frac{\bar{a}}{|a|^2} + \bar{a}) = ca^2(a - 1 + |a|^2) \\
&= ca^2(-\bar{a} + |a|^2) = c|a|^2 a(-1 + a) = -c|a|^4.
\end{aligned}$$

Thus the segment $z_2 z_3$ intersects the real axis at $-c|a|^4$. Since

$$1 - c|a|^2 + c|a|^4 = c(1 - |a|^2) > 0,$$

one has $-c|a|^4 < w_0$. The above facts imply $w_0 \in \triangle(0, z_2, z_3)$.

Acting on $w_0 \in \triangle(0, z_2, z_3)$ with $f(z) = \bar{z}$, we get $w_0 \in \triangle(0, \bar{z}_2, \bar{z}_3)$, as desired. □

Lemma 2.3. *$f_1(co(V)) \cup f_2(co(V)) \subset co(V)$ for each $\eta \in (0, \pi/3)$.*

Proof. It suffices to show $f_1(V) \cup f_2(V) \subset$ co(V). Clearly,

$$f_1(V) = \{f_1(b_0)\} \cup \{z_k : k \geq 1\} \cup \{f_1(w_k) : k \geq 1\}$$

and

$$f_2(V) = \{f_2(b_0)\} \cup \{w_k : k \geq 0\} \cup \{b_k : k \geq 0\}.$$

Since

$$f_1(b_0) = a(a + c|a|^4) = a^2 + c|a|^4 a = (1 - |a|^4)z_1 + |a|^4 z_0,$$

we get $f_1(b_0) \in \mathrm{co}(V)$. On the other hand, since

$$b_k = a + c|a|^4 a^k = (1 - |a|^4)z_0 + |a|^4 z_{k-1},$$

we have $b_k \in \mathrm{co}(V)$ for all integers $k \geq 1$. Thus, to complete the proof, we only need to prove $f_1(w_k), f_2(b_0), w_0 \in \mathrm{co}(V)$ for all integers $k \geq 1$.

The proof of $w_0 \in co(V)$. If $w_0 \geq 0$, since $w_0 = 1 - c|a|^2$ and $0, 1 \in \mathrm{co}(V)$, we have $w_0 \in \mathrm{co}(V)$. In the other case, we have $w_0 \in \triangle(0, z_2, z_3)$ by Lemma 2.2, so $w_0 \in \mathrm{co}(V)$.

The proof of $f_2(b_0) \in co(V)$. First, we have

$$f_2(b_0) = 1 - \bar{a}(a + c|a|^4) = 1 - |a|^2 - c|a|^4\bar{a} = 1 - |a|^2 - c|a|^4 + c|a|^4 a.$$

By a long but elementary computation, we get

$$f_2(b_0) = \frac{|a|^4 h + (1 - 2|a|^4 + |a|^6)(1 - |a|^2)}{1 - |a|^4 + |a|^6},$$

where

$$h = (1 - |a|^2 + |a|^4)z_1 + (|a|^2 - |a|^4)z_2.$$

Clearly, $h \in \mathrm{co}(V)$. As was known, $0, 1 \in \mathrm{co}(V)$, which gives $1 - |a|^2 \in \mathrm{co}(V)$. Additionally,

$$1 - 2|a|^4 + |a|^6 = (1 - |a|^2)(1 + |a|^2 - |a|^4) > 0.$$

It then follows that $f_2(b_0) \in \mathrm{co}(V)$.

The proof of $f_1(w_k) \in co(V)$ for all integers $k \geq 1$. It will be done by induction.

We first show $f_1(w_1) \in \mathrm{co}(V)$. Since $a + \bar{a} = 1$ and $f_1(z) = az$, we have

$$\begin{aligned}
& |a|^2(1 - \bar{a}w_0) + (1 - |a|^2)aw_0 \\
= \;& |a|^2 - |a|^2\bar{a}w_0 + aw_0 - |a|^2 aw_0 = |a|^2 + aw_0 - |a|^2 w_0 \\
= \;& |a|^2 + a - c|a|^2 a - |a|^2 + c|a|^4 = a(1 - c|a|^2 + c|a|^2\bar{a}) \\
= \;& a(1 - c|a|^2 a) = aw_1 = f_1(w_1).
\end{aligned}$$

Thus the proof of $f_1(w_1) \in \mathrm{co}(V)$ can be reduced to showing

$$aw_0, 1 - \bar{a}w_0 \in \mathrm{co}(V). \tag{2.14}$$

If $w_0 \geq 0$, one has by $0, 1, a \in \mathrm{co}(V)$

$$aw_0 \in \mathrm{co}(V) \text{ and } 1 - \bar{a}w_0 = (1 - w_0) + w_0 a \in \mathrm{co}(V).$$

If $w_0 < 0$, one has $w_0 \in \triangle(0, z_2, z_3)$ by Lemma 2.2. Acting on it with f_1 and f_2 respectively, we get

$$aw_0 = f_1(w_0) \in \triangle(f_1(0), f_1(z_2), f_1(z_3)) = \triangle(0, z_3, z_4)$$

and

$$1 - \bar{a}w_0 = f_2(w_0) \in \triangle(f_2(0), f_2(z_2), f_2(z_3)) = \triangle(1, w_2, w_3).$$

Therefore $aw_0, 1 - \bar{a}w_0 \in \mathrm{co}(V)$. This proves $f_1(w_1) \in \mathrm{co}(V)$.

Secondly, we show for every integer $k \geq 1$

$$f_1(w_{k+1}) = (1 - |a|^2)f_1(w_k) + |a|^2 b_{k-1}. \tag{2.15}$$

In fact, one has by $a + \bar{a} = 1$

$$-a^{k+1} + a^{k+2} + |a|^2 a^{k+1} + |a|^4 a^{k-1} = 0.$$

Using this equality, we get

$$\begin{aligned} & f_1(w_{k+1}) = a - c|a|^2 a^{k+2} \\ = \;& a - c|a|^2 a^{k+2} + c|a|^2(-a^{k+1} + a^{k+2} + |a|^2 a^{k+1} + |a|^4 a^{k-1}) \\ = \;& a - c|a|^2 a^{k+1} + c|a|^4 a^{k+1} + c|a|^6 a^{k-1} \\ = \;& a - c|a|^2 a^{k+1} - |a|^2 a + c|a|^4 a^{k+1} + |a|^2 a + c|a|^6 a^{k-1} \\ = \;& (1 - |a|^2)(a - c|a|^2 a^{k+1}) + |a|^2(a + c|a|^4 a^{k-1}) \\ = \;& (1 - |a|^2)f_1(w_k) + |a|^2 b_{k-1}. \end{aligned}$$

Finally, since $\{b_k : k \geq 0\} \subset \mathrm{co}(V)$ and $f_1(w_1) \in \mathrm{co}(V)$ have been proved, by the formula (2.15) we get $f_1(w_k) \in \mathrm{co}(V)$ for all integers $k \geq 1$ by induction. □

Proof of Theorem 2.2. Let $\eta \in (0, \pi/3)$. Since $V \subset K_\eta$, one has $\mathrm{co}(V) \subseteq \mathrm{co}(K_\eta)$. On the other hand, since $\mathrm{co}(V)$ is closed, the self-similarity construction of the dragon curve K_η together with Lemma 2.3 implies $K_\eta \subseteq \mathrm{co}(V)$, which yields $\mathrm{co}(K_\eta) \subseteq \mathrm{co}(V)$. Thus, $\mathrm{co}(K_\eta) = \mathrm{co}(V)$. It then follows from Lemma 2.1 that $\mathrm{co}(K_\eta)$ is a polygon.

2.1.4 The properties of functions Φ_k, Ψ_k, and Θ_k

Let

$$\Phi_k(\eta) = (1 - |a|^4)\sin(k-1)\eta - |a|^3 \sin(k-2)\eta + |a|^k \sin\eta,$$

$$\Theta_k(\eta) = (1 - |a|^4)\sin\eta + |a|^{k+1}\sin k\eta$$

and

$$\Psi_k(\eta) = \sin\eta + |a|^{k-2}\sin(k+1)\eta.$$

These functions are closely related to the geometry of dragon curves. In what follows we write $A \asymp B$ if $A = CB$ for some $C > 0$.

Lemma 2.4. *For each $\eta \in (0, \pi/3)$ and for each integer $k \geq 1$ we have*

$$Im((\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) \asymp \Phi_k(\eta) \quad and \quad Im((\bar{z}_k - \bar{w}_1)(w_2 - w_1)) \asymp \Psi_k(\eta).$$

Proof. Let $\eta \in (0, \pi/3)$ and let $k \geq 1$ be an integer. Observing that

$$z_{k-1} - z_k \asymp a^k - a^{k+1} = a^k(1-a) = a^k \bar{a} \asymp a^{k-1},$$

we have

$$\begin{aligned}
& \mathrm{Im}((\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) \\
\asymp\ & \mathrm{Im}(\bar{a}^{k-1}(1 - c|a|^2 a - ca^{k+1})) \\
\asymp\ & \mathrm{Im}(\bar{a}^{k-1}(1 - |a|^4 - |a|^2 a - a^{k+1})) \\
\asymp\ & (1 - |a|^4)\sin(k-1)\eta - |a|^3 \sin(k-2)\eta + |a|^{k+1}\sin 2\eta \\
=\ & (1 - |a|^4)\sin(k-1)\eta - |a|^3 \sin(k-2)\eta + |a|^k \sin\eta,
\end{aligned}$$

On the other hand, since $w_2 - w_1 = c|a|^2(a - a^2) = c|a|^4$, one has

$$\begin{aligned}
& \mathrm{Im}((\bar{z}_k - \bar{w}_1)(w_2 - w_1)) \asymp \mathrm{Im}(\bar{z}_k - \bar{w}_1) \\
=\ & \mathrm{Im}(c\bar{a}^{k+1} - 1 + c|a|^2\bar{a}) = \mathrm{Im}(c\bar{a}^{k+1} + c|a|^2\bar{a}) \\
\asymp\ & |a|^{k-2}\sin(k+1)\eta + \sin\eta,
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *For each integer $k \geq 4$ the function $\Phi_k(\eta)$ has a unique null in the interval $(\pi/k, \pi/(k-1))$.*

Proof. We first show that $\Phi_4(\eta)$ has a unique null in $(\pi/4, \pi/3)$. From the definition

$$\begin{aligned}
\Phi_4(\eta) &= (1 - |a|^4)\sin 3\eta - |a|^3 \sin 2\eta + |a|^4 \sin\eta \\
&\asymp ((1 - |a|^4)(3 - 4\sin^2\eta) - 2|a|^3\cos\eta + |a|^4 \\
&= (1 - |a|^4)(-1 + 4\cos^2\eta) - |a|^2 + |a|^4 \\
&= (1 - |a|^4)(\frac{1}{|a|^2} - 1) - |a|^2 + |a|^4 \\
&\asymp (1 - 2|a|^4)(1 - |a|^2) \asymp 1 - 2|a|^4.
\end{aligned}$$

Since $|a|$ is a strictly increasing function of η, we see from the last relationship that $\Phi_4(\eta)$ has a unique null in $(\pi/4, \pi/3)$. Denote this null of $\Phi_4(\eta)$ by η_4. Then we have for each $\eta \in (0, \eta_4)$

$$1 - 2|a|^4 > 0. \tag{2.16}$$

Given an integer $k \geq 5$, we are going to show that $\Phi_k(\eta)$ has a unique null in $(\pi/k, \pi/(k-1))$. By the definition of Φ_k and a simple triangular formula, we have

$$\Phi_k(\eta) = (1 - |a|^2 - |a|^4)\sin(k-1)\eta + |a|^3 \sin k\eta + |a|^k \sin\eta.$$

Thus

$$\Phi_k(\frac{\pi}{k}) = (1 - |a|^2 - |a|^4 + |a|^k) \sin\frac{\pi}{k} > 0$$

and

$$\Phi_k(\frac{\pi}{k-1}) = (-|a|^3 + |a|^k) \sin\frac{\pi}{k-1} < 0.$$

We only need to show the derivative $\Phi'_k(\eta) < 0$ for each $\eta \in (\pi/k, \pi/(k-1))$.

Since the derivative $d|a|/d\eta = 2|a|^2 \sin\eta$, we have

$$\begin{aligned}\Phi'_k(\eta) \quad = \quad & -8|a|^5 \sin\eta \sin(k-1)\eta + (1-|a|^4)(k-1)\cos(k-1)\eta \\ & -6|a|^4 \sin\eta \sin(k-2)\eta - |a|^3(k-2)\cos(k-2)\eta \\ & +2k|a|^{k+1}\sin^2\eta + |a|^k \cos\eta.\end{aligned}$$

Given $\eta \in (\pi/k, \pi/(k-1))$, since the cosine is decreasing in $(0,\pi)$, we have

$$\cos(k-1)\eta < \cos\frac{(k-1)\pi}{k} = -\cos\frac{\pi}{k} < -\cos\eta$$

and

$$\cos(k-2)\eta > \cos\frac{(k-2)\pi}{k-1} = -\cos\frac{\pi}{k-1} > -\cos\eta.$$

In addition, since $k \geq 5$, we easily get $(k-1)\sin^2\eta \leq 2$ for the given η, in fact, for $k = 5$ we have $(k-1)\sin^2\eta < 4\sin^2(\pi/4) = 2$, and for $k \geq 6$ we have $(k-1)\sin^2\eta \leq (k-1)(\pi/(k-1))^2 \leq \pi^2/5 < 2$. Therefore

$$\begin{aligned} & 2k|a|^{k+1}\sin^2\eta + |a|^k\cos\eta \\ = \quad & 2k|a|^{k+1}\sin^2\eta + 2|a|^{k+1}\cos^2\eta \\ = \quad & 2|a|^{k+1}(1 + (k-1)\sin^2\eta) \\ \leq \quad & 6|a|^{k+1} = 12|a|^{k+2}\cos\eta.\end{aligned}$$

Now, using the above inequalities, we get

$$\frac{\Phi'_k(\eta)}{\cos\eta} < -(k-1)(1-|a|^4) + (k-2)|a|^3 + 12|a|^7.$$

Then, since $|a| < 1/\sqrt{2}$ for the given η, we get

$$\frac{\Phi'_k(\eta)}{\cos\eta} < -\frac{3(k-1)}{4} + \frac{k-2}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \leq -3 + \frac{3}{\sqrt{2}} < 0.$$

This proves that Φ_k has a unique null in $(\pi/k, \pi/(k-1))$. □

For $k \geq 4$ denote by η_k the null of Φ_k in $(\pi/k, \pi/(k-1))$. Clearly,

$$\frac{1}{2} < |a| < \frac{1}{\sqrt{2}} \text{ for each } \eta \in (0, \frac{\pi}{4})$$

and

$$\frac{1}{\sqrt{2}} \leq |a| < \frac{1}{\sqrt[4]{2}} \text{ for each } \eta \in [\frac{\pi}{4}, \eta_4).$$

The latter is due to (2.16). We shall use these two estimates without mentioning them. The properties of functions Φ_k, Θ_k, and Ψ_k, which will be used in the proof of Theorem 2.3, are formulated in the following three lemmas.

Lemma 2.6. *$\Theta_j(\eta) > 0$ for $k \geq 4$, $\eta \in [\eta_{k+1}, \eta_k)$, and $j \geq k$.*

Proof. If $k = 4$ and $\eta \in [\eta_5, \eta_4)$, one has

$$\begin{aligned} \Theta_4(\eta) &\asymp (1-|a|^4) + |a|^5 4\cos\eta\cos 2\eta \\ &= (1-|a|^4) + |a|^4 2\cos 2\eta \\ &> 1 - 2|a|^4 > 0. \end{aligned}$$

If $k = 4$, $j \geq 5$, and $\eta \in [\pi/4, \eta_4)$, one has

$$\begin{aligned} \Theta_j(\eta) &= (1-|a|^4)\sin\eta + |a|^{j+1}\sin j\eta \\ &\geq (1-|a|^4)\sin\frac{\pi}{4} - |a|^6 \\ &\asymp 1 - |a|^4 - \sqrt{2}|a|^6 \\ &> 1 - \frac{1}{2} - \frac{1}{2} = 0. \end{aligned}$$

If $k = 4$, $j \geq 5$, and $\eta \in [\eta_5, \pi/4)$, one has

$$\begin{aligned} \Theta_j(\eta) &= (1-|a|^4)\sin\eta + |a|^{j+1}\sin j\eta \\ &\geq (1-|a|^4)\sin\frac{\pi}{5} - |a|^6 \\ &\geq (1-|a|^4)\frac{2}{5} - |a|^6 \\ &> \frac{3}{4}\cdot\frac{2}{5} - \frac{1}{8} > 0. \end{aligned}$$

If $j \geq k \geq 5$ and $\eta \in [\eta_{k+1}, \eta_k)$, one has

$$\begin{aligned} \Theta_j(\eta) &= (1-|a|^4)\sin\eta + |a|^{j+1}\sin j\eta \\ &> (1-|a|^4)\frac{2\eta}{\pi} - |a|^{k+1} \\ &> (1-|a|^4)\frac{2}{k+1} - |a|^{k+1} \\ &> \frac{3}{2(k+1)} - (\frac{1}{\sqrt{2}})^{k+1} > 0. \end{aligned}$$

This completes the proof. □

Lemma 2.7. *$\Psi_j(\eta) > 0$ for $k \geq 4$, $\eta \in [\eta_{k+1}, \eta_k)$, and $j \geq k$.*

Proof. If $k = 4$, $\eta \in [\pi/4, \eta_4)$ and $j \geq 4$, one has

$$\Psi_j(\eta) = \sin\eta + |a|^{j-2}\sin(j+1)\eta \geq \sin\frac{\pi}{4} - |a|^2 > \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

If $k = 4$ and $\eta \in [\eta_5, \pi/4)$, one has

$$\Psi_4(\eta) > \sin\frac{\pi}{5} + |a|^2 \sin\frac{5\pi}{4} = \sin\frac{\pi}{5} - |a|^2 \sin\frac{\pi}{4} > \frac{2}{5} - \frac{1}{2\sqrt{2}} > 0$$

and for $j \geq 5$

$$\Psi_j(\eta) = \sin\eta + |a|^{j-2}\sin(j+1)\eta > \frac{2\eta}{\pi} - |a|^3 > \frac{2}{5} - \frac{1}{2\sqrt{2}} > 0.$$

If $k = 5$ and $\eta \in [\eta_6, \eta_5)$, one has

$$\Psi_5(\eta) = \sin\eta + |a|^3 \sin 6\eta > \sin\frac{\pi}{6} - |a|^3 > \frac{1}{2} - \frac{1}{2\sqrt{2}} > 0$$

and for $j \geq 6$

$$\Psi_j(\eta) > \frac{2\eta}{\pi} - |a|^4 > \frac{2}{6} - \frac{1}{4} > 0.$$

If $j \geq k \geq 6$ and $\eta \in [\eta_{k+1}, \eta_k)$, one has

$$\Psi_j(\eta) > \frac{2\eta}{\pi} - |a|^{k-2} > \frac{2}{k+1} - \frac{1}{(\sqrt{2})^{k-2}} > 0.$$

This completes the proof. □

Lemma 2.8. *Let $k \geq 4$. We have the following statements.*
(1) $\Phi_k(\eta) > 0$ for $\eta \in [\eta_{k+1}, \eta_k)$.
(2) $\Phi_j(\eta) < 0$ in both the cases C1 and C2, where

C1: $\eta \in [\pi/k, \eta_k)$ and $j \in \{k+1, k+2, \cdots, 2k-3\}$;
C2: $\eta \in [\eta_{k+1}, \pi/k)$ and $j \in \{k+1, k+2, \cdots, 2k-1\}$.

Proof. (1) We have $\Phi_k(\eta) > 0$ for $\eta \in [\pi/k, \eta_k)$ by the proof of Lemma 2.5. In the case $\eta \in [\eta_{k+1}, \pi/k)$, we have

$$1 - |a|^2 - |a|^4 > 0, \ \sin(k-1)\eta > 0, \ \text{ and } \ \sin k\eta > 0,$$

which imply

$$\Phi_k(\eta) = (1 - |a|^2 - |a|^4)\sin(k-1)\eta + |a|^3 \sin k\eta + |a|^k \sin\eta > 0.$$

(2) Let $k \geq 4$. For case C1 one has

$$\pi + \eta \leq (k+1)\eta < (k+2)\eta < \cdots < (2k-3)\eta < 2\pi - \eta.$$

Therefore

$$\sin(j-1)\eta \leq 0 \text{ and } \sin j\eta < -\sin\eta. \tag{2.17}$$

Thus, in the case $1 - |a|^2 - |a|^4 \geq 0$, we immediately get

$$\begin{aligned}\Phi_j(\eta) &= (1 - |a|^2 - |a|^4)\sin(j-1)\eta + |a|^3 \sin j\eta + |a|^j \sin\eta \\ &< -|a|^3 \sin\eta + |a|^j \sin\eta < 0.\end{aligned}$$

The other case $1-|a|^2-|a|^4<0$ occurs only if $k=4$ and $\eta\in[\pi/4,\eta_4)$, for which we get by (2.16) and (2.17)

$$\begin{aligned}\Phi_5(\eta) &< (|a|^4-|a|^2)\sin 4\eta-|a|^3\sin\eta+|a|^5\sin\eta\\ &\asymp -\sin 4\eta-|a|\sin\eta=-\sin 4\eta-|a|^2\sin 2\eta\\ &\asymp -2\cos 2\eta-|a|^2=2-\frac{1}{|a|^2}-|a|^2<0.\end{aligned}$$

For the case C2 one has $1-|a|^2-|a|^4>0$ and

$$\pi+\eta<(k+2)\eta<\cdots<(2k-1)\eta<2\pi-\eta.$$

Then $\Phi_j(\eta)<0$ for $j\in\{k+2,k+3,\cdots,2k-1\}$ by an easier argument than what we just did. In addition, we have $\Phi_{k+1}(\eta)<0$ for $\eta\in[\eta_{k+1},\pi/k)$ by the proof of Lemma 2.5, and thus finish the proof. □

2.1.5 The properties of dragon curves

The following properties on dragon curves are useful.

Lemma 2.9. *For each $\eta\in(0,\eta_4)$ we have $Rez_0, Rew_1, Imw_1\in(0,1)$.*

Proof. Given $\eta\in(0,\eta_4)$, since $1-2|a|^4>0$ by (2.16), one has

$$c|a|^4<1 \text{ and } c<2. \tag{2.18}$$

Then $\mathrm{Re}z_0=c|a|\cos\eta=c/2\in(0,1)$. As for w_1 we have

$$\mathrm{Im}w_1=c|a|^3\sin\eta<|a|^{-1}\sin\eta=\sin 2\eta\le 1$$

and

$$\mathrm{Re}w_1=1-c|a|^3\cos\eta=1-\frac{c|a|^2}{2}>1-|a|^2>0.$$

Then we easily get $\mathrm{Re}w_1,\mathrm{Im}w_1\in(0,1)$. This completes the proof. □

Lemma 2.10. *$\angle 1z_jw_1\in(0,\pi)$ for $k\ge 4$, $\eta\in[\eta_{k+1},\eta_k)$, and $j\ge k$.*

Proof. For $\eta\in(0,\pi/3)$ one has

$$\begin{aligned}&\mathrm{Im}((1-\bar z_j)(w_1-z_j))=\mathrm{Im}((1-\bar z_j)(w_1-1))\\ \asymp\ &\mathrm{Im}((c\bar a^{j+1}-1)a)\asymp\sin\eta+c|a|^{j+1}\sin j\eta\\ \asymp\ &(1-|a|^4)\sin\eta+|a|^{j+1}\sin j\eta=\Theta_j(\eta),\end{aligned}$$

which together with Lemma 2.6 implies the desired result. □

Lemma 2.11. *Let $k\ge 4$. We have in both the cases C1 and C2*

$$z_j\in co(\{0,z_{j-1},w_1,1\}) \text{ and } w_j\in co(\{1,w_{j-1},b_0,a\}).$$

Proof. Remember that $\arg z$ denotes the argument of z in $[0, 2\pi)$. Let $k \geq 4$.

In the case C1, i.e. $\eta \in [\pi/k, \eta_k)$ and $j \in \{k+1, k+2, \cdots, 2k-3\}$, one has

$$\pi + \eta \leq (k+1)\eta < \cdots < (2k-3)\eta < 2\pi - \eta,$$

$$\arg z_k = 2\pi - (k+1)\eta \in (\pi - 2\eta, \pi - \eta],$$
$$\arg z_{2k-3} = 2\pi - (2k-2)\eta \in (0, 2\eta], \text{ and}$$
$$0 < \arg z_{2k-3} \leq \arg z_j < \arg z_{j-1} \leq \arg z_k \leq \pi - \eta.$$

By Lemma 2.10 we have $\angle 1 z_j w_1 \in (0, \pi)$. On the other hand, by Lemma 2.4 and Lemma 2.8(2), we have

$$\mathrm{Im}((\bar{w}_1 - \bar{z}_j)(z_{j-1} - z_j)) = -\mathrm{Im}((\bar{z}_{j-1} - \bar{z}_j)(w_1 - z_j)) \asymp -\Phi_j(\eta) > 0.$$

which implies $\angle w_1 z_j z_{j-1} \in (0, \pi)$. In summary, the point z_j is located in the sector $0 < \arg z < \arg z_{j-1}$, with $\angle 1 z_j w_1, \angle w_1 z_j z_{j-1} \in (0, \pi)$, by which we get

$$z_j \in \mathrm{co}(\{0, z_{j-1}, w_1, 1\}).$$

Now, acting on this relationship with f_2, we get

$$w_j \in \mathrm{co}(\{1, w_{j-1}, b_0, a\}).$$

In case C2 the argument is the same as that for case C1. □

Denote by $D(0, r)$ the closed disk of radius r centered at the origin and by $v_0 v_1 \cdots v_j$ the broken segment formed by segments $v_{k-1} v_k$, $k = 1, \cdots, j$. All η-dragon curves with $\eta \in (0, \eta_4)$ have the following disk property.

Lemma 2.12. *For each $\eta \in (0, \eta_4)$ and each integer $j \geq 1$ we have*

$$\{z_k : k \geq j\} \subset D(0, |z_j|) \quad \text{and} \quad z_0 z_1 \cdots z_{j-1} \subset \mathbb{C} \setminus D(0, |z_j|).$$

Proof. Let $\eta \in (0, \eta_4)$ and $j \geq 1$ be given. For every integer $k \geq 0$ we have

$$|z_{k+1}| < |z_k|,$$

which implies

$$\{z_k : k \geq j\} \subset D(0, |z_j|).$$

On the other hand, one has for each $t \in [0, 1]$

$$\begin{aligned} |1 - t + ta|^2 - |a|^4 &= (1 - t + t|a|\cos\eta)^2 + t^2|a|^2\sin^2\eta - |a|^4 \\ &= (1-t)^2 + 2(1-t)t|a|\cos\eta + t^2|a|^2 - |a|^4 \\ &= (1-t)^2 + (1-t)t + t^2|a|^2 - |a|^4 \\ &= 1 - t + t^2|a|^2 - |a|^4. \end{aligned}$$

If $2|a|^2 < 1$, we have

$$1 - t + t^2|a|^2 - |a|^4 > |a|^2 - |a|^4 > 0.$$

If $2|a|^2 \geq 1$, we have by (2.16)

$$\begin{aligned} 1 - t + t^2|a|^2 - |a|^4 &= 1 - |a|^4 + |a|^2(t - \frac{1}{2|a|^2})^2 - \frac{1}{4|a|^2} \\ &\geq 1 - |a|^4 - \frac{1}{4|a|^2} \asymp 4|a|^2 - 4|a|^6 - 1 \\ &\geq 2|a|^2 - 4|a|^6 = 2|a|^2(1 - 2|a|^4) > 0. \end{aligned}$$

Thus $|1 - t + ta| > |a|^2$ for each $t \in [0, 1]$. Then, given $j \geq 1$, we have

$$|(1 - t)z_{j-1} + tz_j| > |z_{j+1}| \text{ for any } t \in [0, 1].$$

It follows that the broken segment $z_0 z_1 \cdots z_{j-1} \subset \mathbb{C} \setminus D(0, |z_j|)$. □

2.1.6 The Proof of Theorem 2.3

Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. To prove Theorem 2.3, we first prove $\text{co}(V_k) = \text{co}(V)$, which implies $\text{co}(V_k) = \text{co}(K_\eta)$ by the proof of Theorem 2.2. Secondly, we show that the points $b_0, z_0, z_1, \cdots, z_k, w_1, w_2, \cdots, w_k$ are in turn the vertices of the polygon $\text{co}(V_k)$ in clockwise.

The next lemma is generic for all η-dragon curves with $\eta \in (0, \pi/3)$.

Lemma 2.13. *For each $\eta \in (0, \pi/3)$ and each integer $n \geq 0$ we have*

$$\angle b_0 z_0 z_1 = \angle z_n z_{n+1} z_{n+2} = \angle w_n w_{n+1} w_{n+2} = \pi - \eta.$$

Proof. One has

$$\frac{z_1 - z_0}{b_0 - z_0} = \frac{c(a^2 - a)}{a + c|a|^4 - ca} = -\frac{a}{|a|^4}$$

and

$$\frac{z_{n+2} - z_{n+1}}{z_n - z_{n+1}} = \frac{w_{n+2} - w_{n+1}}{w_n - w_{n+1}} = -a.$$

Thus

$$\angle b_0 z_0 z_1 = \angle z_n z_{n+1} z_{n+2} = \angle w_n w_{n+1} w_{n+2} = \arg(-a) = \pi - \eta,$$

as desired. □

Lemma 2.14. *For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have*

$$\angle z_{k-1} z_k w_1,\ \angle z_k w_1 w_2,\ \angle w_{k-1} w_k b_0,\ \angle w_k b_0 z_0 \in (0, \pi).$$

Proof. Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. By Lemma 2.8(1), we have $\Phi_k(\eta) > 0$. By Lemma 2.7, we have $\Psi_k(\eta) > 0$. Then by Lemma 2.4, we get

$$\text{Im}((\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) > 0 \text{ and } \text{Im}((\bar{z}_k - \bar{w}_1)(w_2 - w_1)) > 0,$$

which implies $\angle z_{k-1} z_k w_1, \angle z_k w_1 w_2 \in (0, \pi)$.

Since f_2 preserves angles, one has by the action of f_2

$$\angle z_{k-1}z_kw_1 = \angle w_{k-1}w_kb_0 \text{ and } \angle z_kw_1w_2 = \angle w_kb_0f_2(w_2).$$

As

$$f_2(w_2) = 1 - \bar{a}(1 - c|a|^2a^2) = a + c|a|^4a = ca = z_0,$$

we then get $\angle w_{k-1}w_kb_0$, $\angle w_kb_0z_0 \in (0, \pi)$. This completes the proof. □

Lemma 2.15. *For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have $0, 1 \in co(V_k)$.*

Proof. Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. We consider two cases.

Case 1. $\eta \in [\pi/k, \eta_k)$. We shall prove

$$0 \in \triangle(z_1, z_{k-1}, w_1) \text{ and } 1 \in \triangle(w_1, w_{k-1}, b_0).$$

In this case, one has

$$\arg z_{k-1} = 2\pi - k\eta \in (\pi - \eta, \pi],$$

so z_{k-1} is in the second quadrant of the plane. Since

$$\begin{aligned} \mathrm{Im}(\bar{z}_1w_1) &= \mathrm{Im}(c\bar{a}^2(1 - c|a|^2a)) \asymp \sin 2\eta - c|a|^3 \sin\eta \\ &\asymp 2\cos\eta - c|a|^3 \asymp 1 - c|a|^4 \asymp 1 - 2|a|^4 > 0, \end{aligned}$$

one has $\angle z_10w_1 \in (0, \pi)$. As for the angle $\angle z_{k-1}0z_1$, it is obvious that

$$\angle z_{k-1}0z_1 = (k-2)\eta \in (0, \pi).$$

Since w_1 is in the first quadrant by Lemma 2.9, the above facts imply

$$0 \in \triangle(z_1, z_{k-1}, w_1),$$

which in turn implies $1 \in \triangle(w_1, w_{k-1}, b_0)$ by the action of f_2.

Case 2. $\eta \in [\eta_{k+1}, \pi/k)$. In this case, z_1 is in the fourth quadrant; z_k is in the second quadrant due to $\arg z_k \in (\pi - \eta, \pi)$; $\angle z_k0z_1 = (k-1)\eta \in (0, \pi)$; and w_1 is in the first quadrant. These facts imply $0 \in \triangle(z_1, z_k, w_1)$, which in turn implies $1 \in \triangle(w_1, w_k, b_0)$ by the action of f_2. This completes the proof. □

Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$. One has $\arg z_{2k-3} = 2\pi - (2k-2)\eta \in (0, 2\eta]$.

Lemma 2.16. *Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$. In the case $\arg z_{2k-3} \in (0, \eta]$ we have*

$$z_{2k-2} \in \triangle(0, z_0, 1),\ z_{2k-1} \in \triangle(0, z_1, z_0),$$
$$w_{2k-2} \in \triangle(1, w_0, a), \text{ and } w_{2k-1} \in \triangle(1, w_1, w_0).$$

In the case $\arg z_{2k-3} \in (\eta, 2\eta]$ we have

$$z_{2k-2} \in \triangle(0, 1, z_{2k-3}),\ z_{2k-1} \in \triangle(0, z_0, 1),$$

$$w_{2k-2} \in \triangle(1, a, w_{2k-3}), \text{ and } w_{2k-1} \in \triangle(1, w_0, a).$$

Proof. We only prove the latter. The proof of the former is similar.

Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$ and assume $\arg z_{2k-3} \in (\eta, 2\eta]$. Then

$$\angle z_0 0 z_{2k-1} \leq \eta = \angle z_0 01. \tag{2.19}$$

On the other hand, using the inequality $1 - 2|a|^4 > 0$, we easily get $c < 2$ and

$$|z_{2k-1}| = c|a|^{2k} < c|a|^4 < c/2 < 1.$$

As $\mathrm{Re} z_0 = c|a| \cos \eta = c/2$, we then get

$$\mathrm{Re} z_{2k-1} < \mathrm{Re} z_0 < 1. \tag{2.20}$$

By (2.19) and (2.20) we obtain $z_{2k-1} \in \triangle(0, z_0, 1)$, which yields $w_{2k-1} \in \triangle(1, w_0, a)$ by the action with f_2.

Next we prove $z_{2k-2} \in \triangle(0, 1, z_{2k-3})$. As $\arg z_{2k-3} \in (\eta, 2\eta]$ is assumed, we have

$$0 < \arg z_{2k-2} < \arg z_{2k-3} < \pi.$$

It suffices to show $\angle 1 z_{2k-2} z_{2k-3} \in (0, \pi)$, which can be reduced to showing

$$\mathrm{Im}((1 - \bar{z}_{2k-2})(z_{2k-3} - z_{2k-2})) > 0. \tag{2.21}$$

In fact, if $k = 4$, by using $a + \bar{a} = 1$, $1 - 2|a|^4 > 0$, $c|a|^4 < 1$, $2|a| \cos \eta = 1$, and $|a| > 1/\sqrt{2}$, one has

$$\begin{aligned}
& \mathrm{Im}((1 - \bar{z}_6)(z_5 - z_6)) = \mathrm{Im}((1 - \bar{z}_5)(z_5 - z_6)) \\
\asymp\ & \mathrm{Im}(a^5 - c|a|^{10}\bar{a}) \asymp - \sin 5\eta - c|a|^6 \sin \eta \\
\asymp\ & -4 \cos^2 \eta \cos 2\eta - \cos 4\eta - c|a|^6 \\
>\ & -4 \cos^2 \eta (2 \cos^2 \eta - 1) - 2(2 \cos^2 \eta - 1)^2 + 1 - |a|^2 \\
=\ & -16 \cos^4 \eta + 12 \cos^2 \eta - 1 - |a|^2 \\
\asymp\ & -1 + 3|a|^2 - |a|^4 - |a|^6 \\
=\ & -(1 - |a|^2)^2 + |a|^2(1 - |a|^2)(1 + |a|^2) \\
\asymp\ & -1 + 2|a|^2 + |a|^4 > -1 + 2|a|^2 > 0.
\end{aligned}$$

If $k > 4$, one has $2\pi - (2k - 3)\eta \in (\eta, 3\eta] \subset (\eta, \pi - \eta)$, so

$$\begin{aligned}
& \mathrm{Im}((1 - \bar{z}_{2k-2})(z_{2k-3} - z_{2k-2})) \\
\asymp\ & - \sin(2k - 3)\eta - c|a|^{2k-2} \sin \eta \\
=\ & \sin(2\pi - (2k - 3)\eta) - c|a|^{2k-2} \sin \eta \\
>\ & \sin \eta - c|a|^{2k-2} \sin \eta > 0.
\end{aligned}$$

This proves the inequality (2.21), so we have $z_{2k-2} \in \triangle(0, 1, z_{2k-3})$, which in turn gives $w_{2k-2} \in \triangle(1, a, w_{2k-3})$ by the action of f_2. □

Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \pi/k)$. One has $\arg z_{2k-1} = 2\pi - 2k\eta \in (0, 2\eta]$.

Lemma 2.17. *Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \pi/k)$. For the case* $\arg z_{2k-1} \in (0, \eta]$ *we have*

$$z_{2k} \in \triangle(0, z_0, 1),\ z_{2k+1} \in \triangle(0, z_1, z_0),$$
$$w_{2k} \in \triangle(1, w_0, a),\ \textit{ and }\ w_{2k+1} \in \triangle(1, w_1, w_0).$$

For the case $\arg z_{2k-1} \in (\eta, 2\eta]$ *we have*

$$z_{2k} \in \triangle(0, 1, z_{2k-1}),\ z_{2k+1} \in \triangle(0, z_0, 1),$$
$$w_{2k} \in \triangle(1, a, w_{2k-1}),\ \textit{ and }\ w_{2k+1} \in \triangle(1, w_0, a).$$

Proof. The proof is the same as that of Lemma 2.16. □

Lemma 2.18. *For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have $co(V) = co(V_k)$.*

Proof. Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. One has

$$a, 0, 1, w_0 \in \mathrm{co}(V_k) \tag{2.22}$$

by $a = c^{-1} z_0$, Lemma 2.15, and Lemma 2.2. It suffices to prove

$$\{z_j : j > k\} \cup \{w_j : j > k\} \subset \mathrm{co}(V_k). \tag{2.23}$$

We consider two cases.

Case 1. $\eta \in [\pi/k, \eta_k)$. By using (2.22) and Lemma 2.11, we get

$$z_{k+1}, z_{k+2}, \cdots, z_{2k-3}, w_{k+1}, w_{k+2}, \cdots, w_{2k-3} \in \mathrm{co}(V_k)$$

by induction. Moreover, we have by Lemma 2.16

$$z_{2k-2}, z_{2k-1}, w_{2k-2}, w_{2k-1} \in \mathrm{co}(V_k)$$

and

$$z_0 z_{2k-1} \subset \mathbb{C} \setminus D(0, |z_{2k}|). \tag{2.24}$$

Then, by (2.24) and Lemma 2.12, we get

$$\{z_j : j \geq 2k\} \subset D(0, |z_{2k}|) \subset \mathrm{co}(\{z_0, z_1, \cdots, z_{2k-1}\}).$$

which in turn implies

$$\{w_j : j \geq 2k\} \subset \mathrm{co}(\{w_0, w_1, \cdots, w_{2k-2}, w_{2k-1}\})$$

by the action of f_2. This proves (2.23).

Case 2. $\eta \in [\eta_{k+1}, \pi/k)$. As we just did, by using (2.22), Lemma 2.11, 2.12, and 2.17, we may prove (2.23) by showing step in step

$$z_{k+1}, \cdots, z_{2k-1}, w_{k+1}, \cdots, w_{2k-1} \in \mathrm{co}(V_k),$$
$$z_{2k}, z_{2k+1}, w_{2k}, w_{2k+1} \in \mathrm{co}(V_k),$$

and

$$\{z_j : j > 2k+1\} \cup \{w_j : j > 2k+1\} \subset \mathrm{co}(V_k).$$

This completes the proof. □

The proof of Theorem 2.3. Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. By Lemma 2.18 and the proof of Theorem 2.2, we have

$$\mathrm{co}(V_k) = \mathrm{co}(V) = \mathrm{co}(K_\eta).$$

Since Lemma 2.13 and Lemma 2.14 have been proved, to show that

$$b_0, z_0, z_1, \cdots, z_k, w_1, \cdots, w_k$$

are the vertices of the polygon $\mathrm{co}(K_\eta)$ in clockwise, it suffices to show that the broken segment $b_0 z_0 z_1 \cdots z_k w_1 \cdots w_k b_0$ is a loop.

Clearly, z_0, b_0 are in the lower half-plane and w_1 is in the upper half-plane.

Denote by l_1 the directed straight line passing through 0 of direction $-a/|a|$ and by l_2 the directed straight line passing through 1 of direction $-a/|a|$. Then $z_0 \in l_1$ and $w_1 \in l_2$. Since

$$\mathrm{Im}((\bar{z}_0 - \bar{b}_0)(0 - b_0)) \asymp -\sin\eta < 0$$

and

$$\mathrm{Im}((1 - \bar{b}_0)(w_1 - b_0)) \asymp (1 - 2|a|^4)\sin\eta > 0,$$

we see that b_0 is on the right side of l_1 and the left side of l_2.

Case 1. $\eta \in [\pi/k, \eta_k)$. In this case,

$$\pi \leq k\eta < \pi + \eta.$$

Thus $\mathrm{Im} w_k \asymp \sin k\eta < 0$, implying that w_k is in the lower half-plane.

Since

$$\mathrm{Im}((\bar{z}_0 - \bar{z}_j)(0 - z_j)) \asymp \sin j\eta$$

and

$$\mathrm{Im}((1 - \bar{w}_j)(w_1 - w_j)) \asymp -\sin(j-1)\eta,$$

we easily check that the broken segment $z_0 z_1 \cdots z_{k-1}$ is a simple arc on the left side of l_1 and that $w_1 w_2 \cdots w_k$ is a simple arc on the right side of l_2.

Since

$$\pi - 2\eta < \arg z_k = 2\pi - (k+1)\eta \leq \pi - \eta, \tag{2.25}$$

z_k is in the upper half-plane. Moreover, since

$$\mathrm{Im}((\bar{z}_0 - \bar{z}_k)(0 - z_k)) \asymp \sin k\eta \leq 0$$

and

$$\mathrm{Im}((1 - \bar{z}_k)(w_1 - z_k)) \asymp (1 - |a|^4)\sin\eta - |a|^{k+1}\sin k\eta > (1 - |a|^4)\sin\eta > 0,$$

z_k is on the right side of l_1 and the left side of l_2.

The above facts together imply that the broken segment $b_0 z_0 \cdots z_k w_1 \cdots w_k b_0$ is a loop.

Case 2. $\eta \in [\eta_{k+1}, \pi/k)$. The proof is similar.

Remark 2.2. *The vertices of the polygon* $co(K_\eta)$ *are* $b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3$ *in clockwise when* $\eta = \eta_4$*. The argument of this case is the same as the proof of Theorem 2.3.*

Remark 2.3. *In the case* $\eta \in (\eta_4, \pi/3)$*, we have* $\Phi_4(\eta) < 0$ *by Lemma 2.5. Then, by Lemma 2.4, we have* $\angle z_3 z_4 w_1 \in (\pi, 2\pi)$*. Moreover, we easily see that* z_4 *is in the inner part of* $co(K_\eta)$*. One may ask: Is it true that* $co(K_\eta)$ *is a polygon of vertices* $b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3$ *in clockwise?*

The answer to this question is no. In fact, by simple computation we get

$$Im((\bar{b}_0 - \bar{z}_6)(w_3 - z_6)) \asymp 6 - 9|a|^2 - 6|a|^4 + 16|a|^6 - 7|a|^8.$$

Let $h(x) = 6 - 9x - 6x^2 + 16x^3 - 7x^4$*. Then* $h(1) = 0$ *and* $h'(1) > 0$*, which implies* $Im((\bar{b}_0 - \bar{z}_6)(w_3 - z_6)) < 0$*, provided that* $\eta \in [\eta_4, \pi/3)$ *is sufficiently near to* $\pi/3$*. Moreover, for such* η*, one has* $\angle b_0 z_6 w_3 \in (\pi, 2\pi)$*, so* z_6 *is not in the polygon of vertices* $b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3$ *in clockwise.*

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3 Future plan

- **Fractal necklaces**

1. We want to find the sufficient and necessary condition that fractal necklaces having no cut points.
2. We want to know if every necklace satisfies the OSC in the higher dimensional case.
3. We want to prove that copies and main nodes are actually independent of the choice of NIFSs for every fractal necklace.

- **Dragon curves**

1. We want to know when a dragon curve is an arc.
2. We want to know when a dragon curve satisfies the open set condition.

- **Lagrange spectrum of geometric progressions**

1. We want to give the upper bounds for Hausdorff dimension of 2-Lagrange spectrum of geometric progressions.

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