Colored d-complete Posets Associated to the Weyl Group Orbits through Certain Weights for Multiply-laced Simple Lie Algebras, and Multiple Hook Removing Game Related to them

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1 Introduction

The notion of a d-complete poset was introduced by Robert A. Proctor ([15, 16]). A d-complete poset is a finite poset which satisfies some local conditions described in terms of double-tailed diamonds (see Section 2.3), and can be regarded as extensions of Young diagrams and shifted Young diagrams, having similar properties to the hook length property ([17]) and the jeu de taquin property ([18]) for Young diagrams. So, it is natural to expect that d-complete posets play important roles in the combinatorial representation theory as Young diagrams and shifted Young diagrams do.

We recall the fundamental relation between d-complete posets and finite-dimensional simple Lie algebras (for the details, see Section 3.2). Let $\mathfrak{g}$ be a *simply-laced* finite-dimensional simple Lie algebra, with I the index set of simple roots. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group, where s_i is the simple reflection corresponding to $i \in I$. Let λ be a dominant integral weight of $\mathfrak{g}$, and set $W_\lambda := \{w \in W \mid w\lambda = \lambda\}$. We know that each coset in W/W_λ has a unique element whose length is minimal among the elements in the coset; we regard W/W_λ as a subset of W by taking the complete system of these "minimal-length coset representatives" for the cosets in W/W_λ. Let $\leq_s$ (resp., $\leq_w$) be the partial order on $W\lambda$ corresponding to the Bruhat order (resp., weak Bruhat order) on $W/W_\lambda \subset W$ under the canonical map $W\lambda \xrightarrow{\sim} W/W_\lambda \subset W$. If λ is minuscule (in this case, $\leq_s$ is identical to $\leq_w$), then there exists a connected self-dual d-complete poset $(P_\lambda, \leq)$ such that $(W\lambda, \leq_s) = (W\lambda, \leq_w)$ and $(\mathcal{F}(P_\lambda), \subseteq)$ are isomorphic as posets, where $\mathcal{F}(P_\lambda)$ is the set of order filters of P_λ ([15, Section 14]). Furthermore, using a unique map $\kappa : P_\lambda \to I$ called the coloring, we construct an I-colored d-complete poset $(P_\lambda, \leq, \kappa, I)$. Then, there exists a unique order isomorphism $f : W\lambda \to \mathcal{F}(P_\lambda)$ satisfying the condition that $\mu \to s_i\mu$ is a cover relation in $W\lambda$ if and only if $f(s_i\mu) \setminus f(\mu)$ consists of one element x with $\kappa(x) = i$ ([16, Proposition 9.1]). There are some important applications of these results. For example, the problem counting the λ-minuscule elements in W is reduced to the combinatorial problem counting the "standard tableaux" for the corresponding d-complete posets ([22, Theorem 3.5]). Also, the "colored hook formula" for d-complete posets obtained in [13] is a generalization of the famous hook length formula for Young diagrams in terms of the reflections in the positive roots of $\mathfrak{g}$.

In this thesis, we study the relation between the Weyl group orbit through a dominant integral weight for a *multiply-laced* finite-dimensional simple Lie algebra and the set of order filters in a d-complete poset. To do this, we use the "folding" technique (see Section 4.1). Assume that $\mathfrak{g}$ is of type A_n, D_n, E_6, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\sigma : I \to I$ be a non-trivial automorphism of the Dynkin diagram of $\mathfrak{g}$; note that σ canonically induces a Lie algebra automorphism of $\mathfrak{g}$ such that $\sigma(\mathfrak{h}) = \mathfrak{h}$, and a linear automorphism on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Then the fixed point subalgebra $\mathfrak{g}(0) := \{x \in \mathfrak{g} \mid \sigma(x) = x\}$ is isomorphic to a multiply-laced finite-dimensional simple Lie algebra with the set J of σ-orbits in I its index set of simple roots and $\mathfrak{h}(0) := \{h \in \mathfrak{h} \mid \sigma(h) = h\}$ its Cartan subalgebra. Let $\tilde{W} = \langle \tilde{s}_p \mid p \in J \rangle \subset GL(\mathfrak{h}(0)^*)$ be the Weyl group of $\mathfrak{g}(0)$. We know that the subgroup $\hat{W} := \{w \in W \mid \sigma w \sigma^{-1} = w\}$ of W is isomorphic to $\tilde{W}$. Let $\mathrm{res} : \mathfrak{h}^* \to \mathfrak{h}(0)^*$ be the restriction map. The map $\mathrm{res}|_{\hat{W}\lambda}$ gives a bijection $\hat{W}\lambda$ onto $\tilde{W}\mathrm{res}(\lambda)$ for a dominant integral weight λ of $\mathfrak{g}$.

Now, let λ be a minuscule dominant integral weight of $\mathfrak{g}$, and $(P_\lambda, \leq)$ the corresponding d-complete poset mentioned above; recall the order isomorphism $f : W\lambda \to \mathcal{F}(P_\lambda)$. We define $\tilde{f} : \tilde{W}\mathrm{res}(\lambda) \to \mathcal{F}(P_\lambda)$ by $\tilde{f} \circ \mathrm{res} = f$, and set $\tilde{\mathcal{F}}(P_\lambda) := \mathrm{Im}(\tilde{f}) \subset \mathcal{F}(P_\lambda)$. The following is the first main theorem of this thesis.

Theorem 1.1 (= Theorem 4.11)**.**

(1) The poset $(\tilde{W}\mathrm{res}(\lambda), \leq_w)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \tilde{\trianglelefteq})$, where $\tilde{\trianglelefteq}$ is a partial order on $\mathcal{F}(P_\lambda)$ defined in terms of an involution $\tilde{S}_p$ $(p \in J)$ on $\mathcal{F}(P_\lambda)$.

(2) The poset $(\tilde{W}\mathrm{res}(\lambda), \leq_s)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \subseteq)$.

In addition, in the case that $\mathfrak{g}$ is of type A_n, we give an explicit description of $\tilde{\mathcal{F}}(P_\lambda)$ (see Theorem 5.4).

In order to prove Theorem 1.1, we introduce a J-colored d-complete poset $(P_\lambda, \leq, \tilde{\kappa}, J)$, where $\tilde{\kappa}$ is a new coloring naturally induced by the coloring κ for $(P_\lambda, \leq)$ and the Dynkin diagram automorphism $\sigma : I \to I$. Based on this coloring $\tilde{\kappa}$, we define a new impartial combinatorial game, named "Multiple Hook Removing Game" (MHRG for short) which is a variation of Hook Removing Game (HRG for short); HRG is an impartial combinatorial game whose game positions are (shifted) Young diagrams, and in which each of two players (called A and B) alternately removes one hook from the (shifted) Young diagram (given as a game position in his/her turn). HRG was introduced by Mikio Sato (see [19] and [20]), who also gave a formula for the $\mathcal{G}$-values (or the Sprague-Grundy values). Our MHRG is an impartial combinatorial game whose rules are as follows (see also Example 1.2 below):

(M1) The starting position is a Young diagram Y^s with a numbering $\alpha : Y^s \to \mathbb{Z}_{>0}$. All game positions are Young diagrams Y contained in Y with a numbering $\alpha|_Y$.

(M2) Given a Young diagram Y with the numbering $\alpha|_Y$, each player chooses a box in Y and removes the hook h corresponding to the box on his/her turn. Let $\mathcal{A}_\alpha(h)$ be the multiset of the numbers (in boxes) in the hook h, and let Y' be the Young diagram obtained by removing h from Y, with the numbering $\alpha|_{Y'}$.

(M2a) If there does not exist any box in Y' whose corresponding hook h' satisfies $\mathcal{A}_\alpha(h') = \mathcal{A}_\alpha(h)$ as multisets, then the player's turn is over, and the next player is given Y'.

(M2b) If there exists a box in Y' whose corresponding hook h' satisfies $\mathcal{A}_\alpha(h') = \mathcal{A}_\alpha(h)$, then the player must choose one such boxes, and remove the hook h' corresponding to the box. Let Y'' be the Young diagram obtained by removing h' from Y', with the numbering $\alpha|_{Y''}$.

(M2c) Do the same operation as (M2a) and (M2b), with Y' replaced by Y''. As long as such a box exists, repeat this operation.

(M3) The winner is the player who removes the last remaining hook in the diagram.

In this thesis, we mainly treat MHRG(m, n) for $m, n \in \mathbb{Z}_{>0}$, which is MHRG whose starting position Y^s is the rectangular Young diagram $Y_{m,n}$ of size $m \times n$ with the "unimodal numbering" $\alpha_{m,n}$ (see Section 6.2); this numbering $\alpha_{m,n}$ is derived from the coloring $\tilde{\kappa}$ for the case that $\mathfrak{g}$ is of type A (and hence $\mathfrak{g}(0)$ is of type B or C).

Example 1.2. At the beginning of MHRG$(3, 5)$ played by A and B, the following Young diagram $Y = Y_{3,5}$ with the numbering $\alpha_{3,5}$ is given to the player, say A, having the first move, as the starting position.

$$Y = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&3&2\\\hline 1&2&3&4&3\\\hline\end{array}$$

If the player A removes the hook h corresponding to the box $(2,4)$ from Y, then A obtains Y' (with $\alpha_{3,5}|_{Y'}$) below:

$$Y = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&\colorbox{gray}{3}&\colorbox{gray}{2}\\\hline 1&2&3&\colorbox{gray}{4}&3\\\hline\end{array} \quad \to \quad Y' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3&4&3\\\cline{1-4} 1&2&3\\\cline{1-3}\end{array}$$

Note that $\mathcal{A}_{\alpha_{3,5}}(h) = [2,3,4]$ (as multisets). Since there does not exist a box in Y' whose corresponding hook h' satisfies $\mathcal{A}_{\alpha_{3,5}}(h') = \mathcal{A}_{\alpha_{3,5}}(h) = [2,3,4]$, the player A's turn is over. If the player B removes the hook h' corresponding to the box $(2,1)$ from Y', then B obtains Y'' (with $\alpha_{3,5}|_{Y''}$) below:

$$Y' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline \colorbox{gray}{2}&\colorbox{gray}{3}&\colorbox{gray}{4}&\colorbox{gray}{3}\\\cline{1-4} \colorbox{gray}{1}&2&3\\\cline{1-3}\end{array} \quad \to \quad Y'' = \begin{array}{|c|c|c|c|c|}\hline 3&4&3&2&1\\\hline 2&3\\\cline{1-2}\end{array}$$

Note that $\mathcal{A}_{\alpha_{3,5}}(h') = [3,4,3,2,1]$ (as multisets). Notice that the box $(1,2)$ in Y'' is a unique box in Y'' whose corresponding hook h'' satisfies $\mathcal{A}_{\alpha_{3,5}}(h'') = \mathcal{A}_{\alpha_{3,5}}(h') = [3,4,3,2,1]$. Because of (M4b), B must remove the hook h'' from Y'', and obtains Y''' (with $\alpha_{3,5}|_{Y'''}$) below:

$$Y'' = \begin{array}{|c|c|c|c|c|}\hline 3&\colorbox{gray}{4}&\colorbox{gray}{3}&\colorbox{gray}{2}&\colorbox{gray}{1}\\\hline 2&\colorbox{gray}{3}\\\cline{1-2}\end{array} \quad \to \quad Y''' = \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}$$

If the player A removes the hook h''' corresponding to the box $(1,1)$ from Y''', then A obtains the empty Young diagram $\emptyset$:

$$Y''' = \begin{array}{|c|}\hline \colorbox{gray}{3}\\\hline \colorbox{gray}{2}\\\hline\end{array} \quad \to \quad \emptyset$$

In this case, the winner is the player A.

We remark that the Young diagram Y'' above is not appear as a position of MHRG$(3,5)$. In general, not every Young diagram contained in $Y_{m,n}$ is a position of MHRG(m,n). Motegi [12] gave a characterization of the set of all game positions in MHRG(m,n); our proof of Theorem 5.4 (describing $\tilde{\mathcal{F}}(P_\lambda)$ in the case of type A) essentially came from Motegi's proof for this result.

Now, by computer, we obtain the following table on the $\mathcal{G}$-value of the starting position $Y^s = Y_{m,n}$ in MHRG(m,n) for each $1 \le m,n \le 9$. By Table 1, we make the following conjectures (1)-(4) on the $\mathcal{G}$-value of the starting position $Y_{m,n}$:

(1) If $m \le n$ and $m+n$ is even, then the $\mathcal{G}$-value of the starting position $Y_{m,n}$ in MHRG(m,n) is equal to the $\mathcal{G}$-value of the starting position $Y_{m,n+1}$ in MHRG$(m,n+1)$.

(2) The sequence $\{\mathcal{G}(Y_{1,n})\}_{n\ge 1}$ of the $\mathcal{G}$-values of the starting positions $Y_{1,n}$ in

$m\backslash n$	1	2	3	4	5	6	7	8	9
1	1	1	3	3	5	5	7	7	9
2	1	3	3	1	1	1	1	1	1
3	3	3	0	0	0	0	3	3	10
4	3	1	0	4	4	2	2	5	5
5	5	1	0	4	1	1	14	14	18
6	5	1	0	2	1	7	7	0	0
7	7	1	3	2	14	7	0	0	10
8	7	1	3	5	14	0	0	8	8
9	9	1	10	5	18	0	10	8	1

Table 1 $\mathcal{G}$-value of the starting position $Y_{m,n}$ in MHRG(m,n) for $1 \le m,n \le 9$.

MHRG$(1,n)$ for $n \ge 1$ is arithmetic periodic.

(3) The sequence $\{\mathcal{G}(Y_{2,n})\}_{n\ge 2}$ of the $\mathcal{G}$-values of the starting positions $Y_{2,n}$ in MHRG$(2,n)$ for $n \ge 2$ is periodic.

(4) The $\mathcal{G}$-value of the starting position in MHRG(n,n) and MHRG$(n,n+1)$ is equal to $\bigoplus_{1\le k\le n} k$, where $\bigoplus_i a_i$ denotes the nim-sum (the addition of numbers in binary form without carry) of all a_i's.

In this thesis, we prove the following four theorems which show that our conjectures above are true. Let $\mathcal{T}(Y_{m,n})$ be the subset of $\mathcal{F}(Y_{m,n})$ consisting of all positions in MHRG(m,n).

Theorem 1.3 (= Theorem 8.7)**.** Let $m,n \in \mathbb{Z}_{>0}$ be such that $m \le n$ and $m+n$ is even. There exists an isomorphism E from MHRG(m,n) to MHRG$(m,n+1)$. Therefore, it holds that $\mathcal{G}(Y) = \mathcal{G}(E(Y))$ for every $Y \in \mathcal{T}(Y_{m,n})$, and hence $\mathcal{G}(Y_{m,n})$ in MHRG(m,n) is equal to $\mathcal{G}(Y_{m,n+1})$ in MHRG$(m,n+1)$.

Theorem 1.4 (= Theorem 9.1)**.** Let $m = 1$ and $n \in \mathbb{Z}_{>0}$. In MHRG$(1,n)$,

$$\mathcal{T}(Y_{1,n}) = \begin{cases} \mathcal{F}(Y_{1,n}) & \text{if } n \text{ is odd,} \\ \mathcal{F}(Y_{1,n}) \setminus \{Y_{1,\frac{n}{2}}\} & \text{if } n \text{ is even.} \end{cases}$$

Moreover, for $0 \le l \le n$ such that $Y_{1,l} \in \mathcal{T}(Y_{1,n})$,

$$\mathcal{G}(Y_{1,l}) = \begin{cases} l & \text{if } n \text{ is odd,} \\ l & \text{if } n \text{ is even and } l < n/2, \\ l-1 & \text{if } n \text{ is even and } n/2 < l. \end{cases}$$

In particular,

$$\mathcal{G}(Y_{1,n}) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 1.5 (see Lemma 9.4, Theorem 9.5, and Corollary 9.6)**.** Let $m = 2$ and $n' \in \mathbb{Z}_{>0}$. In MHRG$(2,2n')$,

$$\mathcal{T}(Y_{2,2n'}) = \mathcal{F}(Y_{2,2n'}) \setminus \{(\boldsymbol{k}'_1, \boldsymbol{k}'_2) \in \mathcal{F}(Y_{2,2n'}) \mid \boldsymbol{k}'_1 + \boldsymbol{k}'_2 = 2n'\}.$$

Moreover, the list of those $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$ whose $\mathcal{G}$-values are 0, 1 or 2 is given by Table 7 (see p.47). In particular, in $\mathrm{MHRG}(2,n)$ for $n \geq 2$, the $\mathcal{G}$-value of the starting position $Y_{2,n}$ is given as follows:

$$\mathcal{G}(Y_{2,n}) = \begin{cases} 3 & \text{if } n = 2, 3, \\ 2 & \text{if } n \neq 2, 3, \text{ and } n \equiv 2, 3 \bmod 8, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 1.6 (see Theorem 10.15 and Corollary 10.16)**.** For $n \in \mathbb{Z}_{>0}$, $\mathrm{MHRG}(n, n+1)$ and $\mathrm{HRG}(SY_n)$ are isomorphic, where $\mathrm{HRG}(SY_n)$ is the Hook Removing Game whose starting position is the triangular shifted Young diagram of size n. In particular, both $\mathcal{G}(Y_{n,n})$ in $\mathrm{MHRG}(n,n)$ and $\mathcal{G}(Y_{n,n+1})$ in $\mathrm{MHRG}(n, n+1)$ are equal to $\mathcal{G}(SY_n) = \bigoplus_{1 \leq k \leq n} k$ in $\mathrm{HRG}(SY_n)$.

This paper is organized as follows. In Section 2, we review Young diagrams and (colored) d-complete posets. Also, we introduce an involution S_c on $\mathcal{F}(P)$ for each color c. In Section 3, we fix our notation for finite-dimensional simple Lie algebras, and review the orders $\leq_s, \leq_w$ on $W\lambda$. Also, we explain the fundamental relation between d-complete posets and simply-laced finite-dimensional simple Lie algebras. In Section 4, we review the "folding" technique for a simply-laced finite-dimensional simple Lie algebra, and then introduce "J-colored" d-complete posets by using it. Also, we prove Theorem 1.1 above. In Section 5, we give an explicit description of $\tilde{\mathcal{F}}(P_\lambda)$ in the case that $\mathfrak{g}$ is of type A_n. In Section 6, we fix our notation for impartial combinatorial games. Also, we review hooks in Young diagrams, and then introduce the impartial combinatorial game $\mathrm{MHRG}(m,n)$. In Section 7, we explain the diagonal expression for a Young diagram. In Section 8 (resp., 9, 10), we prove Theorem 1.3 (resp., Theorems 1.4, 1.5, 1.6) above.

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2 d-complete Posets and Coloring

Denote by $\mathbb{Z}_{>0}$ the set of all positive integers and $\mathbb{Z}_{\geq 0}$ the set of all non-negative integers.

2.1 Young Diagram

A Young diagram is a finite collection of boxes arranged in left-adjusted rows where the row lengths are in non-increasing order. Let $m \in \mathbb{Z}_{>0}$, and let $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m \in \mathbb{Z}_{\geq 0}$ be such that $\boldsymbol{k}_1 \geq \cdots \geq \boldsymbol{k}_m \geq 0$. Then, the set $Y = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m) := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq m, 1 \leq j \leq \boldsymbol{k}_i\}$ is called the Young diagram corresponding to $(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m)$.

An element of a Young diagram is called a "box" and each box is located by a pair (i,j). For example, the Young diagram $(6,6,5,3,3)$ is given as follows:

$Y =$

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	
$(4,1)$	$(4,2)$	$(4,3)$			
$(5,1)$	$(5,2)$	$(5,3)$			

Fig. 1 Young diagram $(6,6,5,3,3)$

For $i \in \mathbb{Z}_{>0}$, the subset $\{(i,j) \mid j \in \mathbb{Z}\} \cap Y$ of Y is called the i-th row of Y. Similarly, for $j \in \mathbb{Z}_{>0}$, the subset $\{(i,j) \mid i \in \mathbb{Z}\} \cap Y$ of Y is called the j-th column of Y.

For a Young diagram Y, let $\mathcal{F}(Y)$ denote the set of all Young diagrams contained in Y. Also, let $\#(Y)$ denote the number of boxes contained in Y. It is obvious that if $Y' \subseteq Y$, then $\#(Y') \leq \#(Y)$.

For fixed $m, n \in \mathbb{Z}_{>0}$, we denote by $Y_{m,n} := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ the rectangular Young diagram.

2.2 Shifted Young Diagram

Shifted Young diagrams are described as follows (see [14] for additional details). Let $m \in \mathbb{Z}_{>0}$, and let $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m \in \mathbb{Z}_{>0}$ be such that $\boldsymbol{k}_1 > \cdots > \boldsymbol{k}_m > 0$. The set $S = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m) := \{(i,j) \in \mathbb{Z}^2 \mid i \leq j,\ 1 \leq i \leq m,\ 1 \leq j \leq \boldsymbol{k}_i\}$ is called the shifted Young diagram corresponding to $(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_m)$. An element of the shifted Young diagram is called a box, and the shifted Young diagram is described in terms of boxes as follows.

$S = (7, 6, 4, 3, 2) =$

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$
	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$
		$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	
			$(4,4)$	$(4,5)$	$(4,6)$	
				$(5,5)$	$(5,6)$	

For $i \in \mathbb{Z}_{>0}$, the subset $\{(i,j) \mid j \in \mathbb{Z}\} \cap S$ of S is called the i-th row of S. Similarly, for $j \in \mathbb{Z}_{>0}$, the subset $\{(i,j) \mid i \in \mathbb{Z}\} \cap S$ of S is called the j-th column of S. We call $h(S) := \max\{i \mid (i,j) \in S\}$ the height of S.

For a shifted Young diagram S, let $\mathcal{F}(S)$ denote the set of all shifted Young diagrams contained in S.

For fixed $n \in \mathbb{Z}_{>0}$, we denote by $SY_n = \{(i,j) \in \mathbb{Z}^2 \mid \ 1 \leq i \leq n, i \leq j \leq n\}$ the triangular shifted Young diagram.

2.3 d-complete Posets

Let $(P, \leq)$ be a poset. When x is covered by y in P, we write $x \to y$. For $x, y \in P$, we set $[x, y] := \{z \in P \mid x \leq z \leq y\}$, which we call an interval. A subset F is called an order filter if every element in P greater than an element in F is always contained in F. Let $\mathcal{F}(P)$ be the set of all order filters in P. Let $(P, \leq)^*$ denote the order dual set of $(P, \leq)$. If $(P, \leq)$ is isomorphic, as a poset, to $(P, \leq)^*$, then $(P, \leq)$ is said to be self-dual. If the Hasse diagram of P is connected, then the P is said to be connected.

Definition 2.1 ([15, Section 2])**.** For $k \geq 3$, we define a poset $d_k(1)$ by the following conditions (1) and (2) (see also Figure 2):

(1) $d_k(1)$ consists of $2k-2$ elements $w_k, w_{k-1}, \cdots, w_3, x, y, z_3, \cdots, z_{k-1}, z_k$.
(2) The partial order on $d_k(1)$ is as follows:

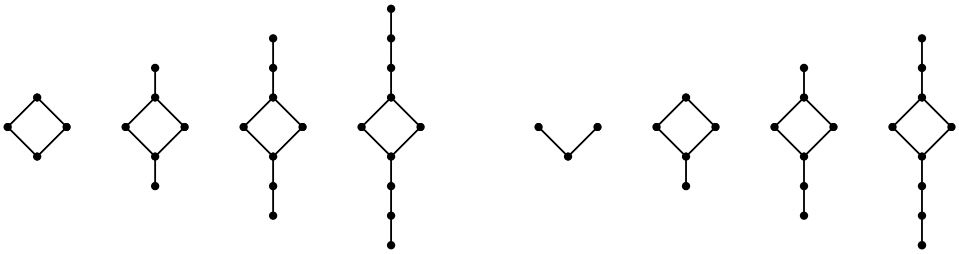
$$w_k < w_{k-1} < \cdots < w_3,\ w_3 < x < z_3,\ w_3 < y < z_3,$$

$$x \not\leq y,\ x \not\geq y,\ z_3 < \cdots < z_{k-1} < z_k.$$

We call $d_k(1)$ the double-tailed diamond. Also, we define $d_k^-(1) := d_k(1) \setminus \{z_k\}$ for $k \geq 3$.

Definition 2.2 ([15, Section 2])**.** Let P be a poset, and $x, y \in P$. For $k \geq 3$ (resp., $k \geq 4$), if the interval $[x, y]$ is isomorphic to $d_k(1)$ (resp., $d_k^-(1)$), then we say that $[x, y]$ is a d_k-interval (resp., d_k^--interval). If $w, x, y \in P$ satisfy $w \to x$ and $w \to y$, then we say that $\{w, x, y\}$ is a d_3^--interval.

Definition 2.3 ([15, Section 3])**.** Let P be a poset. Let $k \geq 4$ (resp., $k = 3$), and let $I = [x, y]$ (resp., $I = \{w, x, y\}$) be a d_k^--interval in P. If $I \cup \{z\}$ is not a d_k-interval for any $z \in P$, then the d_k^--interval I is called an incomplete d_k^--interval. If there is another d_k^--interval $I' = [x', y']$ (resp., $I' = \{w', x', y'\}$) such that $I \setminus \{\min I\} = I' \setminus \{\min I'\}$ and $\min I \neq \min I'$, then the d_k^--interval I is called an overlapping d_k^--interval.

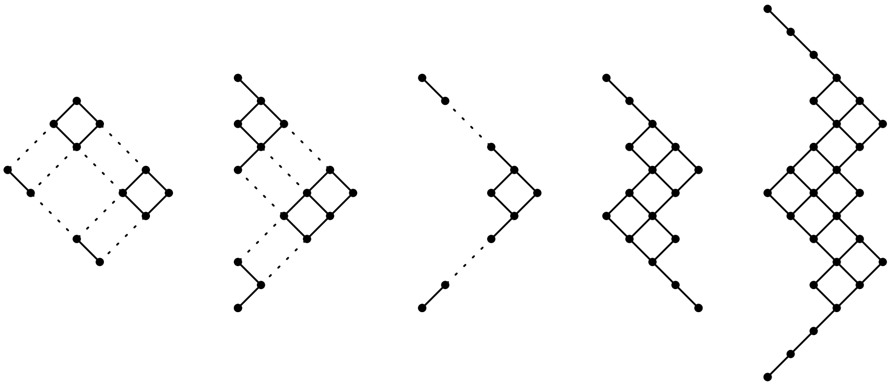


$d_3(1)$ $d_4(1)$ $d_5(1)$ $d_6(1)$ $d_3^-(1)$ $d_4^-(1)$ $d_5^-(1)$ $d_6^-(1)$

Fig. 2 Double-tailed diamonds.

Definition 2.4 ([15, Section 3]). A finite poset P is called a d-complete poset if P satisfies the following conditions (D1)-(D3):

(D1) There is no incomplete d_k^--interval in P for any $k \geq 3$.
(D2) If I is a d_k-interval in P for some $k \geq 3$, then there is no element that is not included in I and is covered by $\max I$.
(D3) There is no overlapping d_k^--interval in P for any $k \geq 3$.



Shape Shifted shape Inset Swivel Bat

Fig. 3 Connected, self-dual d-complete posets.

Definition 2.5 ([15, Section 4]). Let P be a d-complete poset. We define the top tree T_P of P to be the subset of P consisting of all elements $x \in P$ satisfying the condition that

(T) $\#\{z \in P \mid y \to z\} \leq 1$ for every $y \in P$ such that $x \leq y$.

Proposition 2.6 ([15, Sections 3 and 14],[16, Proposition 8.6]). Let P be a d-complete poset.

(1) If P is connected, then P has a unique maximum element.
(2) For each $w \in P \setminus T_P$, there are unique $z \in P$ and $k \geq 3$ such that $[w, z]$ is a d_k-interval.
(3) A connected self-dual d-complete poset is isomorphic, as a poset, to one of those in Figure 3.

Example 2.7. (1) For $m, n \in \mathbb{Z}_{>0}$, we define a partial order on the rectangular Young diagram $Y_{m,n}$ as follows. If $i_1 \geq i_2$ and $j_1 \geq j_2$, then $(i_1, j_1) \leq (i_2, j_2)$. Then the poset $(Y_{m,n}, \leq)$ is a d-complete poset of Shape class in Figure 3. The top tree $T_{Y_{m,n}}$ of $Y_{m,n}$ is identical to the set of those boxes in the first row or in the first column; see the right diagram in Figure 4.

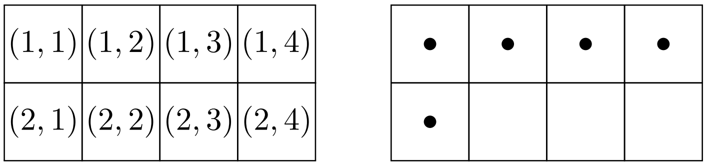


Fig. 4 The Young diagram corresponding to $Y_{2,4}$ and its top tree.

(2) For $n \in \mathbb{Z}_{>0}$, we define a partial order on the triangular shifted Young diagram SY_n as that on $Y_{m,n}$. Then the poset $(SY_n, \leq)$ is a d-complete poset of Shifted Shape class in Figure 3. The top tree T_{SY_n} of SY_n is identical to the set of those boxes in the first row or in the second column; see the right diagram in Figure 5.

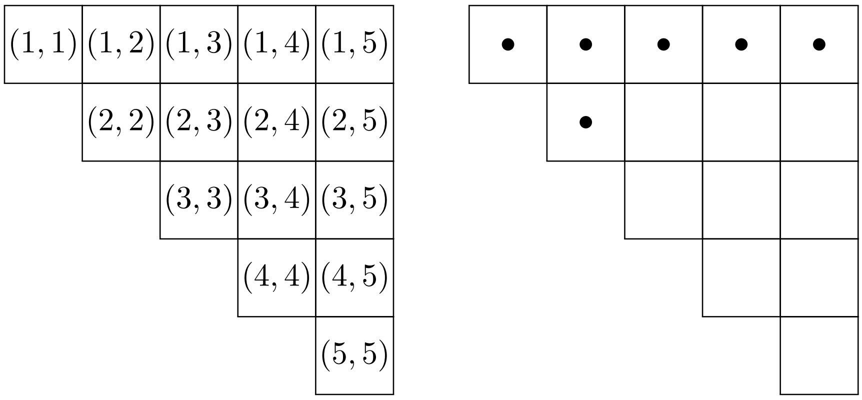


Fig. 5 The shifted Young diagram corresponding to SY_5 and its top tree.

In what follows, we use Young diagrams and shifted Young diagrams for d-complete posets of Shape and Shifted Shape classes. For a given subset X in these d-complete posets P, we indicate an element in X (resp., in $P \setminus X$) by a white box (resp., gray box). For example, the left diagram in Figure 6 indicates the subset $\{(1,1), (1,2), (1,3), (2,1)\}$ of $Y_{2,4}$, which is in fact an order filter of $Y_{2,4}$. The right diagram in Figure 6 indicates the subset $\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (3,3)\}$ of SY_5, which is in fact an order filter of SY_5.

2.4 Colored d-complete Posets and Involutions on $\mathcal{F}(P)$

Let $(P, \leq)$ be a poset, and let C be a set. We call a map $\kappa : P \to C$ a coloring of P with C the set of colors, and the quadruple $(P, \leq, \kappa, C)$ a colored poset.

Proposition 2.8 ([16, Proposition 8.6])**.** Let $(P, \leq)$ be a d-complete poset, and let C be a set such that $\#C = \#T_P$. There exists a coloring $\kappa : P \to C$ of P satisfying the following conditions (a) and (b):

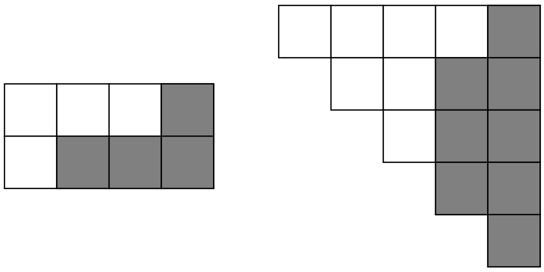


Fig. 6 Examples of order filters of d-complete posets.

(a) The restriction of $\kappa : P \to C$ to the top tree T_P is a bijection from T_P onto C. Namely, each element of T_P has a different color from each other.
(b) If $[w, z]$ is a d_k-interval for some $k \geq 3$, then $\kappa(w) = \kappa(z)$.

Moreover, this coloring of P with C the set of colors is unique, up to the coloring of the top tree T_P in (a). In this case, we call the quadruple $(P, \leq, \kappa, C)$ a colored d-complete poset.

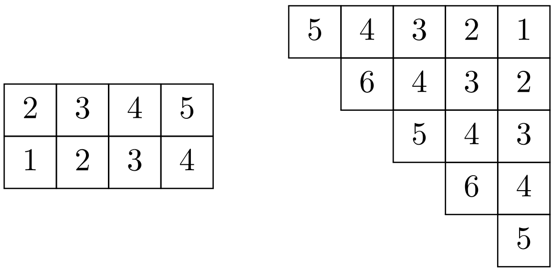


Fig. 7 Colored d-complete posets.

Proposition 2.9 ([16, Section 3])**.** Let $(P, \leq, \kappa, C)$ be a colored d-complete poset.

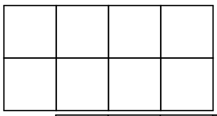
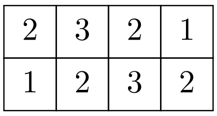
(1) Let $x, y \in P$. If there is the covering relation between x and y, or if x and y are incomparable, then $\kappa(x) \neq \kappa(y)$, that is, x and y have distinct colors.
(2) Let I be an interval of P. If I is a totally order set, then $\kappa(x) \neq \kappa(y)$ for all elements $x, y \in I$ with $x \neq y$, that is, each element in I has a distinct color from each other.
(3) For each $c \in C$, the subset $\kappa^{-1}(\{c\})$ consisting of elements in P having the color c is a totally order set.

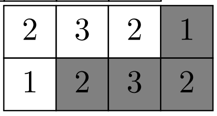
Definition 2.10. Let $(P, \leq, \kappa, C)$ be a finite colored poset. For each $c \in C$, we define maps $A_c, R_c, S_c : \mathcal{F}(P) \to \mathcal{F}(P)$ as follows. For each $F \in \mathcal{F}(P)$,

$$A_c(F) := \bigcup_{\substack{F' \in \mathcal{F}(P) \\ F' \setminus F \subseteq \kappa^{-1}(\{c\})}} F', \qquad R_c(F) := \bigcap_{\substack{F' \in \mathcal{F}(P) \\ F \setminus F' \subseteq \kappa^{-1}(\{c\})}} F',$$

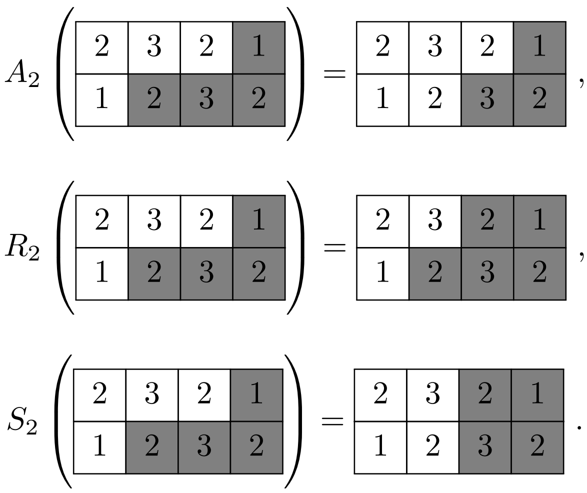
$$S_c(F) := \begin{cases} (A_c(F) \setminus F) \cup R_c(F) & \text{if } (A_c(F) \setminus F) \cup R_c(F) \in \mathcal{F}(P), \\ F & \text{otherwise.} \end{cases}$$

Remark 2.11. It is obvious by the definition that $A_c(F) \supseteq F \supseteq R_c(F)$. If F satisfies $R_c(F) = F$ (resp., $A_c(F) = F$), then $S_c(F) = A_c(F)$ (resp., $S_c(F) = R_c(F)$). Also, it can be easily verified that $A_c(F) \supseteq S_c(F) \supseteq R_c(F)$.

Example 2.12. Let $P = Y_{2,4} =$  , and define a coloring $\kappa : P \to \{1, 2, 3\}$ for P by 

. Let $F =$ 

; notice that F is an order filter of P. Then, $A_2(F), R_2(F), S_2(F)$ are as follows:



Lemma 2.13. Let $(P, \leq, \kappa, C)$ be a colored poset. For every $F \in \mathcal{F}(P)$ and $c \in C$, the following hold.

(1) $A_c(S_c(F)) = A_c(F)$.
(2) $R_c(S_c(F)) = R_c(F)$.
(3) $S_c(S_c(F)) = F$. Namely, the map $S_c : \mathcal{F}(P) \to \mathcal{F}(P)$ is an involution on $\mathcal{F}(P)$.

Proof. By the definition of $S_c(F)$, it suffices to consider the case that $(A_c(F) \setminus F) \cup R_c(F)$ is an order filter of P.

(1) Since all elements of $A_c(F) \setminus R_c(F)$ have the color c and since $S_c(F) \supseteq R_c(F)$, all elements of $A_c(F) \setminus S_c(F)$ also have the color c. Hence, $A_c(F) \in \{F' \in \mathcal{F}(P) \mid F' \setminus S_c(F) \subseteq \kappa^{-1}(\{c\})\}$, and hence $A_c(S_c(F)) \supseteq A_c(F)$ by the definition of A_c. This inclusion relation also implies that all elements in $A_c(S_c(F)) \setminus A_c(F)$ have the color c. By the definition of A_c, all elements in $A_c(F) \setminus F$ have the color c. Hence, $A_c(S_c(F)) \in \{F' \in \mathcal{F}(P) \mid F' \setminus F \subseteq \kappa^{-1}(\{c\})\}$. By the definition of A_c, we obtain $A_c(S_c(F)) \subseteq A_c(F)$. Therefore, $A_c(S_c(F)) = A_c(F)$.

(2) Similar to Part (1).

(3) We compute

$$\begin{aligned}
&(A_c(S_c(F)) \setminus S_c(F)) \cup R_c(S_c(F)) \\
&= (A_c(F) \setminus ((A_c(F) \setminus F) \cup R_c(F))) \cup R_c(F) \\
&= ((A_c(F) \setminus (A_c(F) \setminus F)) \cap (A_c(F) \setminus R_c(F))) \cup R_c(F) \\
&= (F \cup R_c(F)) \cap ((A_c(F) \setminus R_c(F)) \cup R_c(F)) \\
&= F \cap A_c(F) \\
&= F
\end{aligned}$$

Therefore, $(A_c(S_c(F)) \setminus S_c(F)) \cup R_c(S_c(F))$ is an order filter of P, and $S_c(S_c(F)) = F$. ☐

Definition 2.14. Let $(P, \leq, \kappa, C)$ be a colored poset. We define an order $\trianglelefteq$ on $\mathcal{F}(P)$ as follows. For $F, F' \in \mathcal{F}(P)$, $F \trianglelefteq F'$ if there exists a sequence of order filters $F = F_0, F_1, \ldots, F_{n-1}, F_n = F'$ such that for all $i \in \{0, 1, \ldots, n-1\}$, there exist $c_i \in C$ such that $S_{c_i}(F_i) = F_{i+1} \supset F_i$.

Lemma 2.15. Let $(P, \leq, \kappa, C)$ be a colored d-complete poset. For an order filter F of P and a color $c \in C$, the symmetric difference of F and $S_c(F)$ has at most one element.

Proof. Suppose, for a contradiction, that the symmetric difference of F and $S_c(F)$ has more than one element. Let x, y be the elements of the symmetric difference, with $x \neq y$. Because both x and y have the color c, it follows from Proposition 2.9(3) that either $x < y$ or $x > y$ holds; we may assume that $x < y$. Because both F and $S_c(F)$ are order filters, we deduce that either $x, y \in F \setminus S_c(F)$ or $x, y \in S_c(F) \setminus F$ holds. Assume that $x, y \in F \setminus S_c(F)$. Since $x < y$, there exists an element $z \in P$ such that $x \to z$ and $z \leq y$. Because $x \in F$, and F is an order filter, we see that $z \in F$. Similarly, because $y \notin S_c(F)$, and $S_c(F)$ is an order filter, we see that $z \notin S_c(F)$. Thus we get $z \in F \setminus S_c(F)$; in particular, z has the color c. However, this contradicts Proposition 2.9(1); recall that $x \to z$, and x has the color c. A proof for the case that $x, y \in S_c(F) \setminus F$ is similar. ☐

Remark 2.16. Let $(P, \leq, \kappa, C)$ be a colored d-complete poset. By Lemma 2.15, it is clear that for $F, F' \in \mathcal{F}(P)$, $F \subseteq F'$ if and only if $F \trianglelefteq F'$. In particular, $(\mathcal{F}(P), \subseteq)$ and $(\mathcal{F}(P), \trianglelefteq)$ are order isomorphic.

3 Weyl Groups and d-complete Posets

3.1 Finite-dimensional Simple Lie Algebras

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, with $A = (a_{ij})_{i,j \in I}$ the Cartan matrix. Denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathfrak{h}$ the set of simple coroots, $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ the set of simple roots, $\Delta_+ \subset \mathfrak{h}^*$ the set of positive roots, $\Delta_- \subset \mathfrak{h}^*$ the set of negative roots, $\Lambda_i \in \mathfrak{h}^* (i \in I)$ the fundamental weights, and $e_i, f_i \in \mathfrak{g} (i \in I)$ the Chevalley generators. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, where s_i is the simple reflection in α_i for $i \in I$. For $\beta \in \Delta_+$, $\beta^\vee \in \mathfrak{h}$ denotes the dual root of β, and $s_\beta \in W$ denotes the reflection in β; recall that if $\beta = w(\beta')$ for $\beta' \in \Delta_+$ and $w \in W$, then $s_\beta = s_{w(\beta')} = w s_{\beta'} w^{-1}$.

Definition 3.1. Let λ be a dominant integral weight of $\mathfrak{g}$. We define the order $\leq_s$

on the Weyl group orbit $W\lambda$ through λ as follows. For $\mu, \mu' \in W\lambda$, $\mu \leq_s \mu'$ if there exists a finite sequence $\mu = \mu_0, \mu_1, \ldots, \mu_{k-1}, \mu_k = \mu'$ of elements in $W\lambda$ and a finite sequence $\beta_0, \ldots, \beta_{k-1}$ of elements in Δ_+ such that $s_{\beta_i}(\mu_i) = \mu_{i+1}$ and $\mu_i(\beta_i^\vee) > 0$ for each $i \in \{0, 1, \ldots k-1\}$.

Lemma 3.2. Let μ be an integral weight of $\mathfrak{g}$, and $\beta \in \Delta_+$. For $w \in W$, if $\mu <_s s_\beta(\mu)$ and $w(\beta) \in \Delta_+$, then $w(\mu) <_s ws_\beta(\mu)$.

Proof. Since $s_{w(\beta)}(w\mu) = ws_\beta w^{-1}(w\mu) = ws_\beta(\mu)$, and since $w(\mu) \neq ws_\beta(\mu)$, either $w(\mu) <_s ws_\beta(\mu)$ or $w(\mu) >_s ws_\beta(\mu)$ holds. By the definition of $\leq_s$, there exists $n \in \mathbb{Z}_{>0}$ such that $s_\beta(\mu) = \mu - n\beta$. Thus, $ws_\beta(\mu) = w(\mu - n\beta) = w(\mu) - nw(\beta)$. Because $w(\beta) \in \Delta_+$, we obtain $w(\mu) <_s ws_\beta(\mu)$, as desired. □

Proposition 3.3 ([11, Lemma 4.1])**.** Let $\mu_1, \mu_2 \in W\lambda$, and $i \in I$.

(1) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \leq 0$, then $\mu_1 \leq_s s_i(\mu_2)$.
(2) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \leq 0$, then $s_i(\mu_1) \leq_s \mu_2$.
(3) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \leq 0$ and $\mu_2(h_i) \leq 0$, then $s_i(\mu_1) \leq_s s_i(\mu_2)$.
(4) If $\mu_1 \leq_s \mu_2$, $\mu_1(h_i) \geq 0$ and $\mu_2(h_i) \geq 0$, then $s_i(\mu_1) \leq_s s_i(\mu_2)$.

Definition 3.4. Let λ be a dominant integral weight of $\mathfrak{g}$. We define the order $\leq_w$ on $W\lambda$ as follows. For $\mu, \mu' \in W\lambda$, $\mu \leq_w \mu'$ if there exists a finite sequence $\mu = \mu_0, \mu_1, \ldots, \mu_{k-1}, \mu_k = \mu'$ of elements in $W\lambda$ and a finite sequence $j_0, \ldots, j_{k-1}$ of elements in I such that $s_{j_i}(\mu_i) = \mu_{i+1}$ and $\mu_i(h_j) > 0$ for each $i \in \{0, 1, \ldots, k-1\}$.

Remark 3.5 (see, e.g., [8, Section 4.3] and [4, Section 2.4])**.** Let λ be a dominant integral weight, and $W_\lambda := \{w \in W \mid w\lambda = \lambda\}$ the stabilizer of λ; we have the canonical bijection $W/W_\lambda \to W\lambda, wW_\lambda \mapsto w\lambda$. It is known that W_λ is the subgroup of W generated by s_i for $i \in I$ such that $\lambda(h_i) = 0$, and each coset in W/W_λ has a unique element whose length is minimal among the element in the coset; we regard W/W_λ as a subset of W by taking the minimal-length coset representative from each coset in W/W_λ. The poset W/W_λ in the restriction of the Bruhat order (resp., the weak Bruhat order) on W is order isomorphic to $(W\lambda, \leq_s)$ (resp., $(W\lambda, \leq_w)$) under the canonical map $W/W_\lambda \to W\lambda$ above.

3.2 Order Isomorphism between $W\lambda$ and $\mathcal{F}(P_\lambda)$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$.

Definition 3.6. Let λ be a dominant integral weight of $\mathfrak{g}$. We call λ a minuscule weight if λ satisfies $(w\lambda)(h_i) \in \{-1, 0, 1\}$ for all $w \in W$ and $i \in I$.

Table 2 below is the list of minuscule weights of simply-laced finite-dimensional simple Lie algebras; the vertices of the Dynkin diagram are numbered as Figure 8.

Remark 3.7 ([8, Lemma 11.1.18] and Remark 3.5)**.** Assume that λ is minuscule. For $\mu, \mu' \in W\lambda$, $\mu \leq_s \mu'$ if and only if $\mu \leq_w \mu'$. Therefore, $(W\lambda, \leq_s)$ and $(W\lambda, \leq_w)$ are order isomorphic.

Proposition 3.8 ([15, Section 14])**.** Assume that $\mathfrak{g}$ is simply-laced. Let λ be a minuscule weight of $\mathfrak{g}$. There exists a connected, self-dual d-complete poset P_λ such that $(W\lambda, \leq_s)$

$\mathfrak{g}$	minuscule weight λ
A_n	$\Lambda_1, \dots, \Lambda_n$
D_n	$\Lambda_1, \Lambda_{n-1}, \Lambda_n$
E_6	Λ_1, Λ_5
E_7	Λ_6
E_8	none

Table 2 Minuscule weights; simply-laced case.

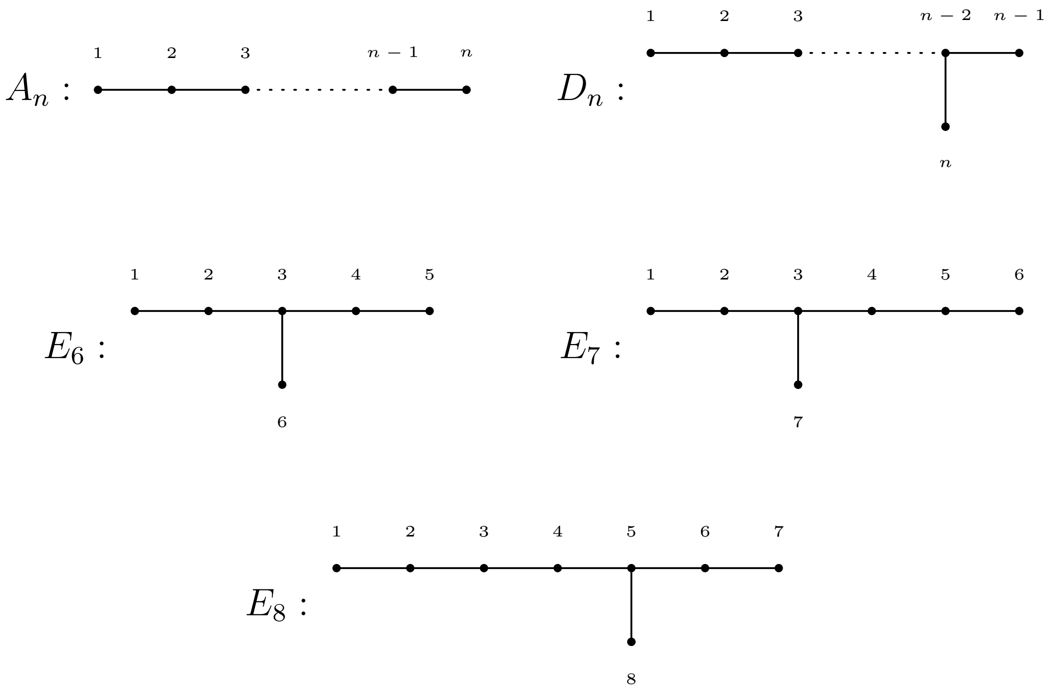


Fig. 8 Simply-laced Dynkin diagrams.

and $(\mathcal{F}(P_\lambda), \subseteq)$ are isomorphic, as posets (see also Table 3).

$\mathfrak{g}$	minuscule weight λ	corresponding d-complete poset P_λ
A_n	$\Lambda_i (1 \leq i \leq n)$	$Y_{i,n-i+1}$ (Shape)
D_n	Λ_1	SY_{n-1} (Shifted Shape)
D_n	Λ_{n-1}, Λ_n	$d_n(1)$ (Inset)
E_6	Λ_1, Λ_5	Swivel
E_7	Λ_6	Bat

Table 3 The d-complete posets P_λ corresponding to minuscule weights λ.

Keep the setting in Proposition 3.8, with $\lambda = \Lambda_i$ for some $i \in I$ such that Λ_i is minuscule. We know from [16, Proposition 8.6] that the graph obtained from the Hasse diagram of the top tree T_{P_λ} of P_λ by replacing each allow by an edge is identical to the Dynkin diagram of $\mathfrak{g}$; in particular, $\#I = \#T_{P_\lambda}$. By Proposition 2.8, we can obtain the colored poset $(P_\lambda, \leq, \kappa, I)$ such that $\kappa|_{T_{P_\lambda}} : T_{P_\lambda} \to I$ is the graph isomorphism and the maximum element of P_λ (notice that it is contained in T_{P_λ}) is sent to the i under the map κ. We call $(P_\lambda, \leq, \kappa, I)$ the I-colored d-complete poset for the minuscule weight λ.

Proposition 3.9 ([16, Proposition 9.1])**.** Keep the notation and setting in Proposition 3.8. Let $(P_\lambda, \leq, \kappa, I)$ be the I-colored d-complete poset. There exists a unique order isomorphism $f : (W\lambda, \leq_s) \xrightarrow{\sim} (\mathcal{F}(P_\lambda), \subseteq)$ such that for $\mu \in W\lambda$ and $i \in I$, there exists the cover relation $\mu \to s_i\mu$ in $W\lambda$ if and only if $f(s_i(\mu)) \setminus f(\mu)$ consists of one element having the color i.

Example 3.10. Let $\mathfrak{g}$ be of type A_5, and $\lambda = \Lambda_2$; in this case, the corresponding (connected, self-dual) d-complete poset P_{Λ_2} is $Y_{2,4}$. Let $(P_{\Lambda_2}, \leq, \kappa, I)$ be the I-colored d-complete poset, with the coloring κ as in Figure 7. The Hasse diagrams of $(W\Lambda_2, \leq_s)$ and $(\mathcal{F}(P_{\Lambda_2}), \subseteq)$ are given in Figure 9 below:

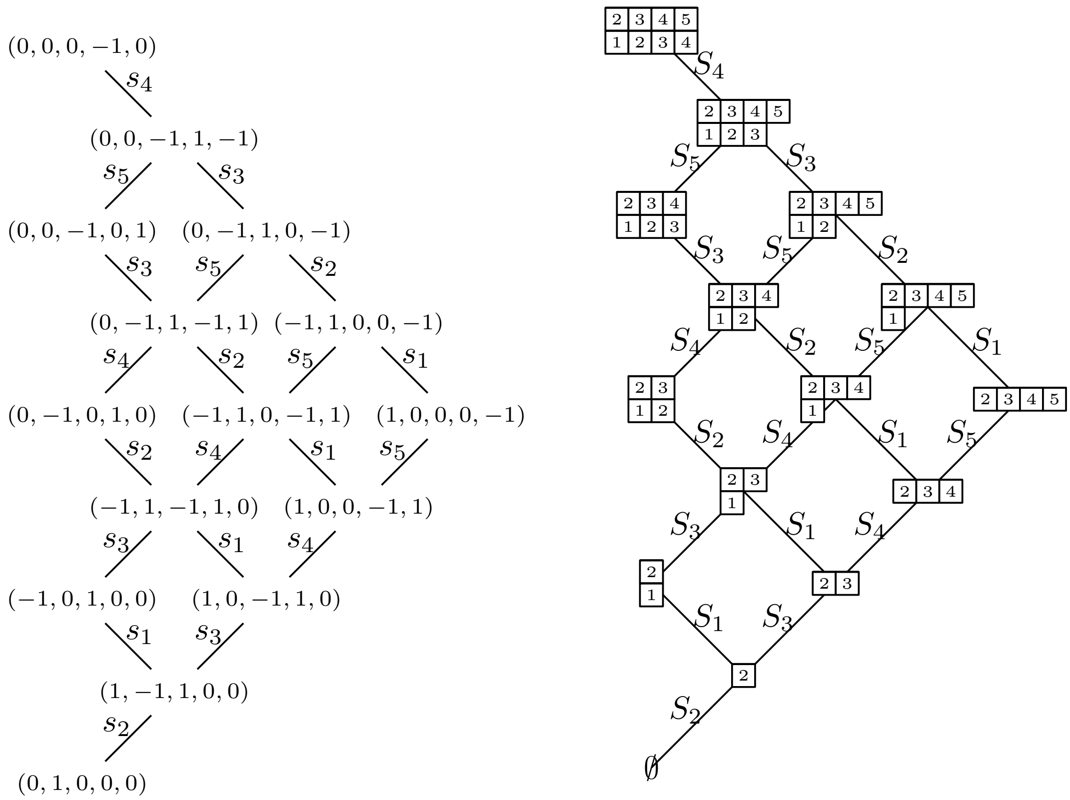


Fig. 9 $(W\Lambda_2, \leq_s)$ and $(\mathcal{F}(P_{\Lambda_2}), \subseteq)$ of type A_5

The next corollary follows from Remark 2.16 and Proposition 3.9.

Corollary 3.11. Assume that $\mathfrak{g}$ is simply-laced. Let λ be a minuscule weight of $\mathfrak{g}$, and let $(P_\lambda, \leq)$ be the d-complete poset such that $(\mathcal{F}(P_\lambda), \subseteq)$ is isomorphic to $(W\lambda, \leq_s)$ (see Proposition 3.8). Let $(P_\lambda, \leq, \kappa, I)$ be the I-colored d-complete poset, and let $f : (W\lambda, \leq_s) \xrightarrow{\sim} (\mathcal{F}(P_\lambda), \subseteq)$ be the order isomorphism in Proposition 3.9. For $\mu \in W\lambda$ and $i \in I$,

$$f(s_i(\mu)) = S_i(f(\mu)).$$

For $F \in \mathcal{F}(P_\lambda)$ and $i \in I$, we define $c_i(F) := \#\{x \in F \mid \kappa(x) = i\}$. Because λ is minuscule, we see that if there exists the cover relation $\mu \to s_i(\mu)$ in $W\lambda$, then $\mu(h_i) = 1$ and $s_i(\mu) = \mu - \alpha_i$. Hence we have the next corollary.

Corollary 3.12. For $\mu \in W\lambda$ and $F = f(\mu)$,

$$\mu = \sum_{i \in I} (\#(S_i(F)) - \#(F))\Lambda_i = \lambda - \sum_{i \in I} c_i(F)\alpha_i.$$

For $F \in \mathcal{F}(P_\lambda)$, we define

$$g(F) := \sum_{i \in I} (\#(S_i(F)) - \#(F))\Lambda_i = \lambda - \sum_{i \in I} c_i(F)\alpha_i.$$

By Corollary 3.12, $g : (\mathcal{F}(P_\lambda), \subseteq) \xrightarrow{\sim} (W\lambda, \leq_s)$ is the inverse of f.

We will use the following proposition later.

Proposition 3.13 ([16, Proposition 8.6])**.** Keep the notation and setting in Proposition 3.8. Let $(P_\lambda, \leq, \kappa, I)$ be the I-colored d-complete poset. If there exists the covering relation between $x, y \in P_\lambda$, then the color $\kappa(x)$ of x is adjacent to the color $\kappa(y)$ of y in the Dynkin diagram of $\mathfrak{g}$.

4 Order Isomorphism between $\tilde{W}\mathrm{res}(\lambda)$ and $\tilde{\mathcal{F}}(P_\lambda)$

4.1 Folding of a Lie Algebra

We review the "folding" of a simply-laced finite-dimensional simple Lie algebra; for the details, see [9, Sections 7.9 and 7.10] and [5, Section 9.5] in example.

Let $\mathfrak{g}$ be the finite-dimensional simple Lie algebra of type A_n, D_n or E_6; we use the notation in Section 3.1. Let σ be a non-trivial graph automorphism of the Dynkin diagram of $\mathfrak{g}$. Denote by $\langle\sigma\rangle$ the cyclic group generated by σ (in the group of permutations on I), and J the set of $\langle\sigma\rangle$-orbits on I. We say that $p \in J$ satisfies the orthogonality condition if $a_{ij} = a_{ji} = 0$ for all $i, j \in p$ with $i \neq j$; notice that $p \in J$ does not satisfy the orthogonality condition if and only if $\mathfrak{g}$ is of type A_{2n} and $p = \{n, n+1\}$. It is known that the graph automorphism σ induces a (unique) Lie algebra automorphism of $\mathfrak{g}$ such that $\sigma(e_i) = e_{\sigma(i)}, \sigma(f_i) = f_{\sigma(i)}, \sigma(h_i) = h_{\sigma(i)}$ for $i \in I$; we set $\mathfrak{g}(0) := \{x \in \mathfrak{g} \mid \sigma(x) = x\}$. For each $p \in J$, we define $H_p, E_p, F_p \in \mathfrak{g}(0)$ as follows:

(1) If p satisfies the orthogonality condition, then

$$H_p := \sum_{i \in p} h_i, \qquad E_p := \sum_{i \in p} e_i, \qquad F_p := \sum_{i \in p} f_i.$$

(2) If p does not satisfy the orthogonality condition, then

$$H_p := 2\sum_{i \in p} h_i, \qquad E_p := \sum_{i \in p} e_i, \qquad F_p := 2\sum_{i \in p} f_i.$$

Proposition 4.1 (see, e.g., [9, Sections 7.9 and 7.10])**.** The fixed point subalgebra $\mathfrak{g}(0)$

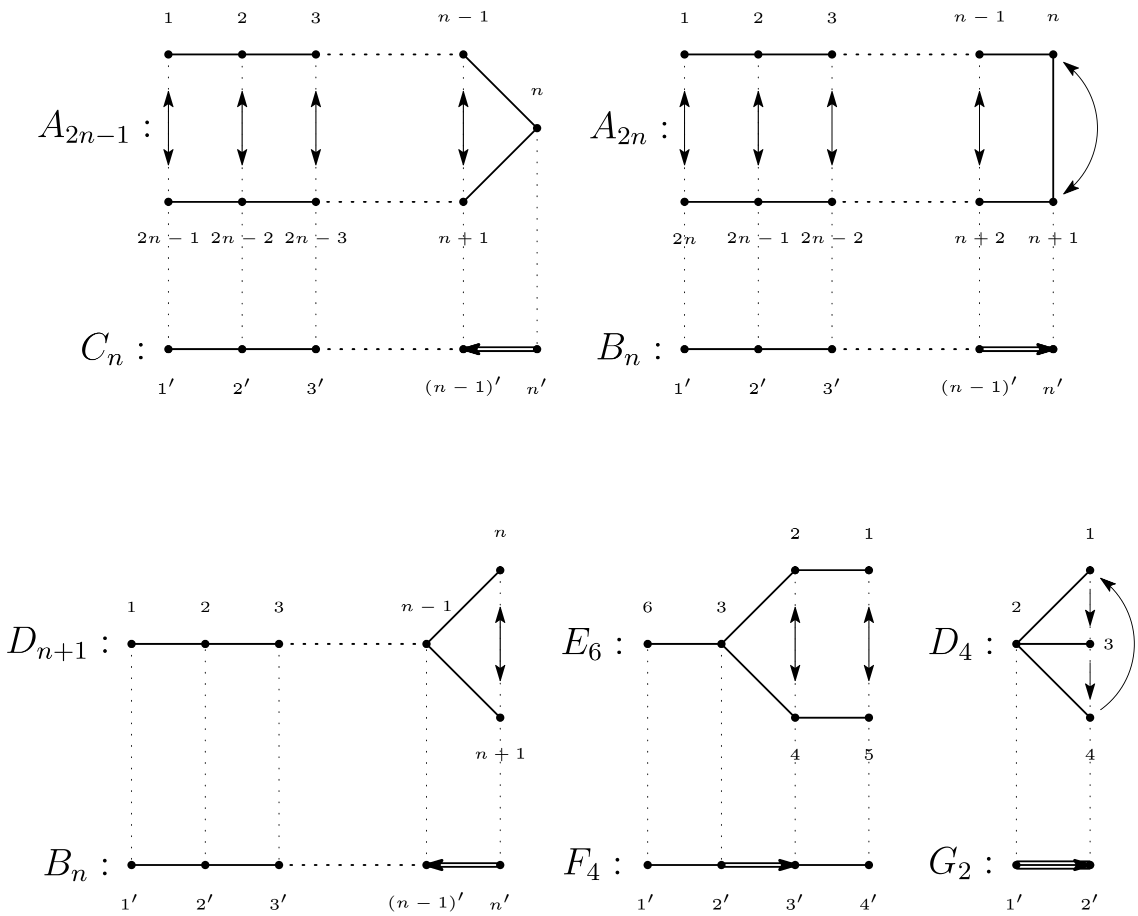


Fig. 10 The Dynkin diagram of $\mathfrak{g}$, its (non-trivial) graph automorphism $\sigma : I \to I$, and the Dynkin diagram of the fixed point subalgebra $\mathfrak{g}(0)$.

is generated by $\{H_p, E_p, F_p\}_{p \in J}$, and is isomorphic to a multiply-laced finite-dimensional simple Lie algebra; see Figure 10 and Table 4.

type of $\mathfrak{g}$	A_{2n}	A_{2n-1}	D_{n+1}	E_6	D_4
order of σ	2	2	2	2	3
type of $\mathfrak{g}(0)$	B_n	C_n	B_n	F_4	G_2

Table 4 $\mathfrak{g}$, σ, and $\mathfrak{g}(0)$. The vertices of the Dynkin diagram of $\mathfrak{g}(0)$ are "numbered" as Figure 10.

Let $\mathfrak{h}(0)$ be the subspace of $\mathfrak{h}$ spanned by $\{H_p\}_{p \in J}$, which is a Cartan subalgebra of $\mathfrak{g}(0)$. Denote by $\mathrm{res} : \mathfrak{h}^* \to \mathfrak{h}(0)^*, \mu \mapsto \mu|_{\mathfrak{h}(0)}$, the restriction map, and set $\beta_p := \mathrm{res}(\alpha_i) \in \mathfrak{h}(0)^*$ for $p \in J$, where i is an arbitrary element in the $\langle\sigma\rangle$-orbit p; note that β_p is independent of the choice of $i \in p$. The set of simple coroots and the set of simple roots of $\mathfrak{g}(0)$ are given by $\{H_p\}_{p \in J}$ and $\{\beta_p\}_{p \in J}$, respectively. Denote by $\tilde{\Delta}_+ \subset \mathfrak{h}(0)^*$ the set of positive roots of $\mathfrak{g}(0)$, and $\tilde{\Delta}_- \subset \mathfrak{h}(0)^*$ the set of negative roots of $\mathfrak{g}(0)$. For $p \in J$, we define $\tilde{s}_p(\nu) := \nu - \nu(H_p)\beta_p$ for $\nu \in \mathfrak{h}(0)^*$. Then, $\widetilde{W} := \langle \tilde{s}_p \mid p \in J \rangle$ is the Weyl group of $\mathfrak{g}(0)$.

For each $p \in J$, we define $\hat{s}_p \in W$ as follows:

(1) If p satisfies the orthogonality condition, then

$$\hat{s}_p := \prod_{k \in p} s_k.$$

(2) If p does not satisfy the orthogonality condition, that is, if $\mathfrak{g}$ is of type A_{2n} and $p = \{n, n+1\}$ (see also page 17), then

$$\hat{s}_p := s_n s_{n+1} s_n = s_{n+1} s_n s_{n+1}.$$

Lemma 4.2. For $p \in J$, $\tilde{s}_p(\mathrm{res}(\mu)) = \mathrm{res}(\hat{s}_p(\mu))$ for all $\mu \in \mathfrak{h}^*$.

Proof. If p satisfies the orthogonality condition, then we compute

$$\mathrm{res}(\hat{s}_p(\mu)) = \mathrm{res}\left(\mu - \sum_{i \in p} \mu(h_i)\alpha_i\right) = \mathrm{res}(\mu) - \mathrm{res}(\mu)(H_p)\beta_p = \tilde{s}_p(\mathrm{res}(\mu)).$$

If p does not satisfy the orthogonality condition, then we compute

$$\begin{aligned}\mathrm{res}(\hat{s}_p(\mu)) &= \mathrm{res}(\mu - \mu(h_n + h_{n+1})(\alpha_n + \alpha_{n+1})) \\ &= \mathrm{res}(\mu) - \mathrm{res}(\mu)(H_p)\beta_p = \tilde{s}_p(\mathrm{res}(\mu)).\end{aligned}$$

□

Since σ acts on $\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C}h_i$, σ naturally acts also on $\mathfrak{h}^*$ by $(\sigma(\mu))(h) = \mu(\sigma^{-1}(h))$ for $\mu \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$; we see that $\sigma(\Lambda_i) = \Lambda_{\sigma(i)}, \sigma(\alpha_i) = \alpha_{\sigma(i)}$ for $i \in I$. Notice that $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$ for $i \in I$ in $GL(\mathfrak{h}^*)$. Hence, $\sigma W \sigma^{-1} \subseteq W$.

Proposition 4.3 ([5, Proposition 9.17])**.** Set $\hat{W} := \{w \in W \mid \sigma w \sigma^{-1} = w\}$. There is a group isomorphism from $\hat{W}$ onto $\tilde{W}$ such that $\hat{s}_p \mapsto \tilde{s}_p$. Therefore $\hat{W}$ is the subgroup of W generated by $\{\hat{s}_p\}_{p \in J}$.

Remark 4.4. Because $\tilde{W}$ and $\hat{W}$ are generated by $\{\tilde{s}_p\}_{p \in J}$ and $\{\hat{s}_p\}_{p \in J}$, we see by Lemma 4.2 that $\mathrm{res}(\hat{W}\lambda) = \tilde{W}\mathrm{res}(\lambda)$ for every (dominant) integral weight λ.

Let $\tilde{\Lambda}_p \in \mathfrak{h}(0)^* (p \in J)$ be the fundamental weights of $\mathfrak{g}(0)$. We can easily show the following lemma.

Lemma 4.5. Let $p \in J$, and $i \in p$.

(1) If p satisfies the orthogonality condition, then $\mathrm{res}(\Lambda_i) = \tilde{\Lambda}_p$.
(2) If p does not satisfy the orthogonality condition, then $\mathrm{res}(\Lambda_i) = 2\tilde{\Lambda}_p$.

Lemma 4.6. Let λ be a dominant integral weight of $\mathfrak{g}$, and let $\mu_1, \mu_2 \in \hat{W}\lambda$. If $\mathrm{res}(\mu_1) = \mathrm{res}(\mu_2)$, then $\mu_1 = \mu_2$. Therefore the map $\mathrm{res}|_{\hat{W}\lambda} : \hat{W}\lambda \to \tilde{W}\mathrm{res}(\lambda)$ is bijective (see Remark 4.4).

Proof. For each $i = 1, 2$, let $\hat{w}_i \in \hat{W}$ be such that $\mu_i = \hat{w}_i\lambda$, and let $\tilde{w}_i \in \tilde{W}$ be such that $\mathrm{res} \circ \hat{w}_i = \tilde{w}_i \circ \mathrm{res}$ (see Lemma 4.2). We have $\tilde{w}_1\mathrm{res}(\lambda) = \mathrm{res}(\hat{w}_1\lambda) = \mathrm{res}(\mu_1) = \mathrm{res}(\mu_2) = \mathrm{res}(\hat{w}_2\lambda) = \tilde{w}_2\mathrm{res}(\lambda)$. Since $\mathrm{res}(\lambda)$ is a dominant integral weight for $\mathfrak{g}(0)$ by

Lemma 4.5, it follows that $\tilde{w_1}^{-1}\tilde{w_2} \in \langle \tilde{s}_p \mid (\mathrm{res}(\lambda))(H_p) = 0\rangle$, and hence $\hat{w_1}^{-1}\hat{w_2} \in \langle \hat{s}_p \mid (\mathrm{res}(\lambda))(H_p) = 0\rangle$. Observe that $(\mathrm{res}(\lambda))(H_p) = 0$ if and only if $\lambda(h_i) = 0$ for all $i \in p$. Thus we obtain $\hat{w_1}^{-1}\hat{w_2}(\lambda) = \lambda$, and hence $\mu_1 = \hat{w}_1\lambda = \hat{w}_2\lambda = \mu_2$, as desired. □

Notice that σ preserves Δ and Δ_+, Δ_-.

Lemma 4.7. Let λ be a dominant integral weight of $\mathfrak{g}$.

(1) For each $\mu \in \hat{W}\lambda$ and $p \in J$, either $\mu(h_i) \geq 0$ for all $i \in p$ or $\mu(h_i) \leq 0$ for all $i \in p$.
(2) For each $\mu \in \hat{W}\lambda$ and $p \in J$, if $\mu(h_i) > 0$ (resp., $\mu(h_i) < 0$) for some $i \in p$, then $\mu <_w \hat{s}_p(\mu)$ (resp., $\mu >_w \hat{s}_p(\mu)$).

Proof. (1) Let $w \in \hat{W}$ be such that $\mu = w\lambda$. Because $\mu(h_i) = (w\lambda)(h_i) = \lambda(w^{-1}h_i) = \lambda((w^{-1}\alpha_i)^\vee)$, and because λ is a dominant integral weight, it suffices to show that either $w^{-1}\alpha_i \in \Delta_+$ for all $i \in p$ or $w^{-1}\alpha_i \in \Delta_-$ for all $i \in p$. If $w^{-1}\alpha_i \in \Delta_+$ (resp., $w^{-1}\alpha_i \in \Delta_-$) for some $i \in p$, then $w^{-1}\alpha_{\sigma(i)} = w^{-1}\sigma\alpha_i = \sigma w^{-1}\alpha_i \in \Delta_+$ (resp., $\in w^{-1}\alpha_i \in \Delta_-$). Since p is a $\langle\sigma\rangle$-orbit, the assertion above follows.

(2) We give a proof only for the case that $\mu(h_i) > 0$ for some $i \in p$, and $\#p = 2$; the proofs for the other cases are similar. Since $\mu(h_i) > 0$, it follows that $\mu <_w s_i(\mu)$. If $p = \{i, j\}$, then we see by part (1) that $\mu(h_j) \geq 0$. Assume that p satisfies the orthogonality condition. Then,

$$\begin{aligned} s_js_i(\mu) &= s_j(\mu - \mu(h_i)\alpha_i) = s_j(\mu) - \mu(h_i)s_j(\alpha_i) \\ &= \mu - \mu(h_j)\alpha_j - \mu(h_i)\alpha_i = s_i(\mu) - \mu(h_j)\alpha_j \geq_w s_i(\mu). \end{aligned}$$

Thus we obtain $\mu <_w s_i(\mu) \leq_w s_js_i(\mu) = \hat{s}_p(\mu)$, as desired. Assume that p does not satisfy the orthogonality condition. Then,

$$\begin{aligned} s_js_i(\mu) &= s_j(\mu - \mu(h_i)\alpha_i) = s_j(\mu) - \mu(h_i)s_j(\alpha_i) \\ &= \mu - \mu(h_j)\alpha_j - \mu(h_i)(\alpha_i + \alpha_j) = \mu - \mu(h_i)\alpha_i - \mu(h_i)\alpha_j - \mu(h_j)\alpha_j \\ &= s_i(\mu) - \mu(h_i)\alpha_j - \mu(h_j)\alpha_j >_w s_i(\mu), \end{aligned}$$

$$\begin{aligned} s_is_js_i(\mu) &= s_i(s_i(\mu) - \mu(h_i + h_j)\alpha_j) = \mu - s_i(\mu(h_i + h_j)\alpha_j) \\ &= \mu - \mu(h_i + h_j)(\alpha_i + \alpha_j) = \mu - \mu(h_i)\alpha_i - \mu(h_j)\alpha_i - \mu(h_i + h_j)\alpha_j \\ &\geq_w \mu - \mu(h_i)\alpha_i - \mu(h_i + h_j)\alpha_j = s_js_i(\mu). \end{aligned}$$

Thus we obtain $\mu <_w s_i(\mu) <_w s_js_i(\mu) \leq_w s_is_js_i(\mu) = \hat{s}_p(\mu)$, as desired. □

Definition 4.8. We set $Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\nu = \sum_{i\in I} m_i\alpha_i \in Q_+$, we define the height $\mathrm{ht}(\nu)$ of ν by $\mathrm{ht}(\nu) := \sum_{i\in I} m_i$. Similarly, we set $\tilde{Q}_+ := \sum_{p\in J} \mathbb{Z}_{\geq 0}\beta_p$. For $\xi = \sum_{p\in J} n_p\beta_p \in \tilde{Q}_+$, we define the height $\mathrm{ht}(\xi)$ of ξ by $\mathrm{ht}(\xi) := \sum_{p\in J} n_p$.

Lemma 4.9. Let λ be a dominant integral weight, and $\mu_1, \mu_2 \in \hat{W}\lambda$. Then, $\mu_1 \leq_s \mu_2$ if and only if $\mathrm{res}(\mu_1) \leq_s \mathrm{res}(\mu_2)$.

Proof. First, we show the "if" part. We see that $\mathrm{res}(\lambda) - \mathrm{res}(\mu_2) \in \tilde{Q}_+$ since $\mathrm{res}(\lambda)$ is dominant and $\mathrm{res}(\mu_2) \in \tilde{W}\mathrm{res}(\lambda)$. We show the assertion by induction on $\tilde{h} := \mathrm{ht}(\mathrm{res}(\lambda) - \mathrm{res}(\mu_2))$. If $\tilde{h} = 0$, then $\mathrm{res}(\mu_2) = \mathrm{res}(\lambda)$. Because $\mathrm{res}(\lambda) - \mathrm{res}(\mu_1) \in \tilde{Q}_+$, and because

$\mathrm{res}(\mu_1) - \mathrm{res}(\lambda) = \mathrm{res}(\mu_1) - \mathrm{res}(\mu_2) \in \tilde{Q}_+$ by the definition of $\leq_s$ on $\tilde{W}\mathrm{res}(\lambda)$, we get $\mathrm{res}(\mu_1) = \mathrm{res}(\lambda)$. Now, for $i = 1, 2$, we see that $\lambda - \mu_i \in Q_+$. Since $\mathrm{res}(\lambda - \mu_i) = \mathrm{res}(\lambda) - \mathrm{res}(\mu_i) = \mathrm{res}(\lambda) - \mathrm{res}(\lambda) = 0$, we deduce that $\lambda = \mu_i$. Thus, we obtain $\mu_1 = \lambda \leq_s \lambda = \mu_2$.

Assume that $\tilde{h} > 0$. In this case, there exists $p \in J$ such that $\mathrm{res}(\mu_2)(H_p) < 0$, because $\mathrm{res}(\lambda)$ is a unique dominant integral weight in $\tilde{W}\mathrm{res}(\lambda)$; note that $\mathrm{ht}(\mathrm{res}(\lambda) - \tilde{s}_p\mathrm{res}(\mu_2)) < \tilde{h}$. Here, we give a proof only for the case that $p = \{i, j\}$ with $i \neq j$, and p satisfies the orthogonality condition; the proofs for the other cases are similar. If $\mathrm{res}(\mu_1)(H_p) \geq 0$, then we get $\mathrm{res}(\mu_1) \leq_s \tilde{s}_p\mathrm{res}(\mu_2)$ by Proposition 3.3 (1). By the induction hypothesis, it follows that $\mu_1 \leq_s \hat{s}_p(\mu_2)$. Because $\mu_2(h_i + h_j) = \mathrm{res}(\mu_2)(H_p) < 0$, we see by Lemma 4.7 that $\hat{s}_p(\mu_2) = s_j s_i(\mu_2) \leq_s s_i(\mu_2) \leq_s \mu_2$. Thus we obtain $\mu_1 \leq_s \mu_2$. If $\mathrm{res}(\mu_1)(H_p) \leq 0$, then we get $\tilde{s}_p\mathrm{res}(\mu_1) \leq_s \tilde{s}_p\mathrm{res}(\mu_2)$ by Proposition 3.3 (3). By the induction hypothesis, it follows that $\hat{s}_p(\mu_1) \leq_s \hat{s}_p(\mu_2)$. Similarly to the case above, we deduce that $\hat{s}_p(\mu_k)(h_i) \leq 0$ and $s_i\hat{s}_p(\mu_k)(h_j) \leq 0$ for $k = 1, 2$. By Proposition 3.3 (4), we obtain $s_i\hat{s}_p(\mu_1) \leq_s s_i\hat{s}_p(\mu_2)$, and then $\mu_1 = s_j s_i \hat{s}_p(\mu_1) \leq_s s_j s_i \hat{s}_p(\mu_2) = \mu_2$, as desired.

Next, we show the "only if" part by the induction on $h := \mathrm{ht}(\lambda - \mu_2)$. If $h = 0$, then we see by the same argument as above that $\mu_1 = \mu_2 = \lambda$. Hence, $\mathrm{res}(\mu_1) \leq_s \mathrm{res}(\mu_2)$. Assume that $h > 0$. Then there exists $i \in I$ such that $\mu_2(h_i) < 0$. Let $p \in J$ be such that $i \in p$. Here, we give a proof only for the case that $p = \{i, j\}$ with $i \neq j$, and p satisfies the orthogonality condition; the proofs for the other cases are similar. By Lemma 4.7, $\mu_2(h_j) \leq 0$ and $\hat{s}_p(\mu_2) = s_j s_i(\mu_2) \leq_s s_i(\mu_2) <_s \mu_2$; note that $\mathrm{ht}(\lambda - \hat{s}_p(\mu_2)) < h$. Assume that $\mu_1(h_i) \geq 0$. It follows from Proposition 3.3 (1) that $\mu_1 \leq_s s_i(\mu_2)$. Also, we see by Lemma 4.7 (1) that $\mu_1(h_j) \geq 0$. By Proposition 3.3 (1), we get $\mu_1 \leq_s s_j s_i(\mu_2) = \hat{s}_p(\mu_2)$. By the induction hypothesis, it follows that $\mathrm{res}(\mu_1) \leq_s \tilde{s}_p\mathrm{res}(\mu_2)$. Because $\mathrm{res}(\mu_2)(H_p) = \mu_2(h_i + h_j) < 0$, we have $\tilde{s}_p\mathrm{res}(\mu_2) \leq_s \mathrm{res}(\mu_2)$, and hence $\mathrm{res}(\mu_1) \leq_s \mathrm{res}(\mu_2)$. Assume that $\mu_1(h_i) \leq 0$. It follows from Proposition 3.3 (3) that $s_i(\mu_1) \leq_s s_i(\mu_2)$. Also, we see by Lemma 4.7 (2) that $(s_i(\mu_1))(h_j) \leq 0$. By Proposition 3.3 (3), we get $\hat{s}_p(\mu_1) = s_j s_i(\mu_1) \leq_s s_j s_i(\mu_2) = \hat{s}_p(\mu_2)$. By the induction hypothesis, it follows that $\tilde{s}_p\mathrm{res}(\mu_1) \leq_s \tilde{s}_p\mathrm{res}(\mu_2)$. Because $\mathrm{res}(\mu_1)(H_p) \leq 0$ and $\mathrm{res}(\mu_2)(H_p) \leq 0$, we obtain $\mathrm{res}(\mu_1) \leq_s \mathrm{res}(\mu_2)$ by Proposition 3.3 (4), as desired. □

4.2 J-colored d-complete Poset

Let $\mathfrak{g}$ be a simply-laced finite-dimensional Lie algebra, and let σ be a non-trivial graph automorphsim of the Dynkin diagram of $\mathfrak{g}$ (see Figure 10). Let λ be a minuscule weight of $\mathfrak{g}$. Recall from Proposition 3.8 that there exists a connected self-dual d-complete poset $(P_\lambda, \leq)$ such that $(W\lambda, \leq_s)$ and $(\mathcal{F}(P_\lambda), \subseteq)$ are isomorphic. Let $(P_\lambda, \leq, \kappa, I)$ be the I-colored d-complete poset (see the comment after Proposition 3.8). By Proposition 3.9 and Corollary 3.11, there exists a unique order isomorphism $f : (W\lambda, \leq_s) \xrightarrow{\sim} (\mathcal{F}(P_\lambda), \subseteq)$ such that $f(s_i(\mu)) = S_i(f(\mu))$ for all $\mu \in W\lambda$ and $i \in I$. Because the map $\mathrm{res}|_{\hat{W}\lambda} : \hat{W}\lambda \to \tilde{W}\mathrm{res}(\lambda)$ is bijective (see Lemma 4.6), we can define a map $\tilde{f} : \tilde{W}\mathrm{res}(\lambda) \to \mathcal{F}(P_\lambda)$ by the following commutative diagram (4.1):

$$\begin{array}{ccc} W\lambda & \overset{f}{\searrow} & \\ \rotatebox{90}{$\supseteq$} & & \\ \hat{W}\lambda & \xrightarrow{f|_{\hat{W}\lambda}} & (\mathcal{F}(P_\lambda), \subseteq) \\ \mathrm{res}\downarrow & \circlearrowleft & \underset{\tilde{f}}{\nearrow} \\ \tilde{W}\mathrm{res}(\lambda) & & \end{array} \tag{4.1}$$

We define $\tilde{\mathcal{F}}(P_\lambda) := \mathrm{Im}(\tilde{f}) \subseteq \mathcal{F}(P_\lambda)$ (see also (4.3) below).

Definition 4.10. Keep the setting above. We define a map $\tilde{\kappa} : P_\lambda \to J$ to be the composition of $\kappa : P_\lambda \to I$ and the canonical projection $I \twoheadrightarrow J$. We call the colored poset $(P_\lambda, \leq, \tilde{\kappa}, J)$ the J-colored d-complete poset corresponding to $\mathfrak{g}(0)$ and $\mathrm{res}(\lambda)$.

For $F \in \mathcal{F}(P_\lambda)$ and $p \in J$, we define $\tilde{c}_p(F) := \#\{x \in F \mid \tilde{\kappa}(x) = p\}$. By Corollary 3.12, it follows that for $\mu \in \hat{W}\lambda$ and $F = f(\mu)$,

$$\mathrm{res}(\mu) = \mathrm{res}(\lambda) - \sum_{p \in J} \left(\sum_{i \in p} c_i(F) \right) \beta_p = \mathrm{res}(\lambda) - \sum_{p \in J} \tilde{c}_p(F) \beta_p.$$

We define $\tilde{g} : \tilde{\mathcal{F}}(P_\lambda) \to \tilde{W}\mathrm{res}(\lambda)$ by

$$\tilde{g}(F) := \mathrm{res}(\lambda) - \sum_{p \in J} \tilde{c}_p(F) \beta_p$$

for $F \in \tilde{\mathcal{F}}(P_\lambda)$. It can be easily checked that $\tilde{g}$ is the inverse of $\tilde{f}$.

Denote by $\tilde{A}_p, \tilde{R}_p, \tilde{S}_p : \mathcal{F}(P_\lambda) \to \mathcal{F}(P_\lambda)$ $(p \in J)$ the maps in Definition 2.10 for the J-colored d-complete poset $(P_\lambda, \leq, \tilde{\kappa}, J)$. Also, we define the order $\tilde{\trianglelefteq}$ on $\mathcal{F}(P_\lambda)$ in exactly the same way as Definition 2.14. Namely, for $F, F' \in \mathcal{F}(P_\lambda)$, $F \mathrel{\tilde{\trianglelefteq}} F'$ if there exists a sequence of order filters $F = F_0, F_1, \ldots, F_{n-1}, F_n = F'$ in $\mathcal{F}(P_\lambda)$ such that for each $i \in \{0, 1, \ldots, n-1\}$, there exists $p_i \in J$ such that $\tilde{S}_{p_i}(F_i) = F_{i+1} \supset F_i$.

Theorem 4.11. Keep the notation and setting above.

1. (1) The poset $(\tilde{W}\mathrm{res}(\lambda), \leq_w)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \tilde{\trianglelefteq})$ under the map $\tilde{f} : \tilde{W}\mathrm{res}(\lambda) \to \tilde{\mathcal{F}}(P_\lambda)$.
2. (2) The poset $(\tilde{W}\mathrm{res}(\lambda), \leq_s)$ is isomorphic to the poset $(\tilde{\mathcal{F}}(P_\lambda), \subseteq)$ under the map $\tilde{f} : \tilde{W}\mathrm{res}(\lambda) \to \tilde{\mathcal{F}}(P_\lambda)$.

Example 4.12. Let $\mathfrak{g}$ be of type A_5, and $\lambda = \Lambda_2$. Recall from Example 3.10 that the corresponding (connected, self-dual) d-complete poset P_{Λ_2} is $Y_{2,4}$, and the I-colored d-complete poset $(P_{\Lambda_2}, \leq, \kappa, I)$ is the left diagram in Figure 7. In this case, $\mathfrak{g}(0)$ is of type C_3, and $\mathrm{res}(\Lambda_2) = \tilde{\Lambda}_{2'}$. The J-colored d-complete poset $(P_{\Lambda_2}, \leq, \tilde{\kappa}, J)$ is below. The Hasse diagrams of $(\tilde{W}\tilde{\Lambda}_{2'}, \leq_w)$ and $(\tilde{\mathcal{F}}(P_{\Lambda_2}), \tilde{\trianglelefteq})$ (resp., $(\tilde{W}\tilde{\Lambda}_{2'}, \leq_s)$ and $(\tilde{\mathcal{F}}(P_{\Lambda_2}), \subseteq)$) are given in Figure 11 (resp., Figure 12).

$\mathfrak{g}$	λ	$\mathfrak{g}(0)$	$\mathrm{res}(\lambda)$	P_λ
A_{2n-1}	$\Lambda_1, \ldots, \Lambda_n$	C_n	$\tilde{\Lambda}_{1'}, \ldots, \tilde{\Lambda}_{n'}$	Shape
A_{2n}	$\Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_n$	B_n	$\tilde{\Lambda}_{1'}, \ldots, \tilde{\Lambda}_{(n-1)'}, 2\tilde{\Lambda}_{n'}$	Shape
D_{n+1}	Λ_1	B_n	$\tilde{\Lambda}_{1'}$	Inset
D_{n+1}	Λ_n	B_n	$\tilde{\Lambda}_{n'}$	Shifted Shape
E_6	Λ_1	F_4	$\tilde{\Lambda}_{4'}$	Swivel
D_4	Λ_1	G_2	$\tilde{\Lambda}_{2'}$	Shifted Shape

Table 5 Correspondence between $\mathfrak{g}$, λ, $\mathfrak{g}(0)$, $\mathrm{res}(\lambda)$, and P

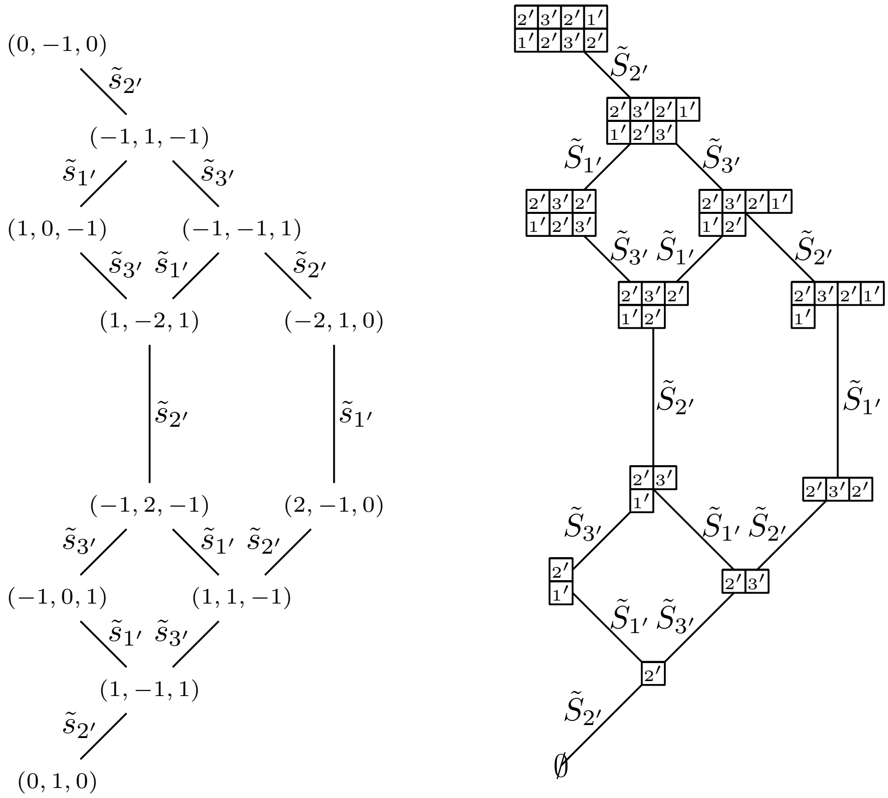


Fig. 11 $(\tilde{W}\tilde{\Lambda}_{2'}, \leq_w)$ and $(\tilde{\mathcal{F}}(P_{\Lambda_2}), \tilde{\trianglelefteq})$ of type C_3

.

4.3 Proof of Theorem 4.11

Keep the notation and setting in the previous section.

Definition 4.13. For $p \in J$, we define $\hat{S}_p : \mathcal{F}(P_\lambda) \to \mathcal{F}(P_\lambda)$ as follows:

(1) If p satisfies the orthogonality condition, then

$$\hat{S}_p := \prod_{k \in p} S_k;$$

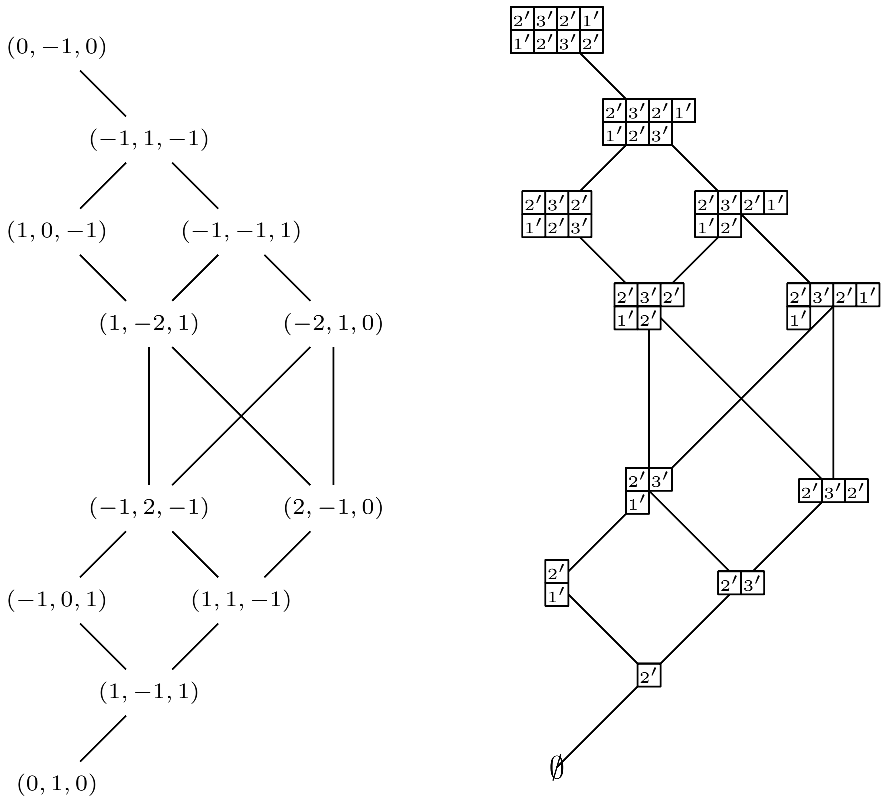


Fig. 12 $(\tilde{W}\tilde{\Lambda}_{2'}, \leq_s)$ and $(\tilde{\mathcal{F}}(P_{\Lambda_2}), \subseteq)$ of type C_3

.

we see by Lemma 3.11 that $\hat{S}_p$ does not depend on the order of the product of S_k's.

(2) If p does not satisfy the orthogonality condition, that is, if $\mathfrak{g}$ is of type A_{2n} and $p = \{n, n+1\}$ (see page 17), then

$$\hat{S}_p := S_n S_{n+1} S_n = S_{n+1} S_n S_{n+1};$$

the second equality follows from Lemma 3.11, together with $s_n s_{n+1} s_n = s_{n+1} s_n s_{n+1}$.

We need the following fact to prove Lemma 4.15 below.

Proposition 4.14 ([7, page 23]). Let P be an arbitrary poset, and let $F \in \mathcal{F}(P)$ be an order filter of P.

(1) For $x \in F$, x is a minimal element of F if and only if $F \setminus \{x\}$ is an order filter.
(2) For $x \in F$, x is a maximal element of $P \setminus F$ if and only if $F \cup \{x\}$ is an order filter.

Lemma 4.15. Let $\mu \in W\lambda$, and set $F := f(\mu) \in \mathcal{F}(P_\lambda)$. It holds that

$$\tilde{S}_p(F) = \hat{S}_p(F) \text{ for all } p \in J. \tag{4.2}$$

Proof. First, we assume that $p \in J$ satisfies the orthogonality condition. The case that

$\#p = 1$ is easy. Assume that $\#p = 2$ (the proof for the case that $\#p = 3$ is similar). Let we write p as: $p = \{i, j\}$, with $i, j \in I, i \neq j$. We deduce by Lemma 2.15 that for each $k \in p = \{i, j\}$, $S_k(F)$ satisfies one of the following:

(i) $S_k(F) = A_k(F) = F \sqcup \{x_k\}$ for some $x_k \in P_\lambda \setminus F$; in this case, $R_k(F) = F$.
(ii) $S_k(F) = R_k(F) = F \setminus \{x_k\}$ for some $x_k \in F$; in this case, $A_k(F) = F$.
(iii) $S_k(F) = A_k(F) = R_k(F) = F$.

Here, we give a proof only for the case that both $S_i(F)$ and $S_j(F)$ satisfy (i); the proofs for the other cases are similar. In this case, there exist $x_i, x_j \in P_\lambda \setminus F$ such that $S_i(F) = F \sqcup \{x_i\}$ and $S_j(F) = F \sqcup \{x_j\}$; note that $\kappa(x_i) = i$ and $\kappa(x_j) = j$. By Definition 4.13, we have $S_j(F \sqcup \{x_i\}) = S_jS_i(F) = S_iS_j(F) = S_i(F \sqcup \{x_j\})$. Since $\kappa(x_i) = i$ and $i \neq j$, we see from the definition of S_j that when we apply S_j to $F \sqcup \{x_i\}$, x_i is not removed. Hence, $x_i \in S_j(F \sqcup \{x_i\}) = S_jS_i(F)$. Similarly, $x_j \in S_jS_i(F)$. Since the symmetric difference of $S_jS_i(F)$ and F has at most two element by Lemma 2.15, we see that $S_jS_i(F) = F \sqcup \{x_i\} \sqcup \{x_j\}$. Therefore, it suffices to show that $\tilde{S}_p(F) = F \sqcup \{x_i\} \sqcup \{x_j\}$.

Suppose, for a contradiction, that $F \supsetneq \tilde{R}_p(F)$. Let y be a minimal element of $F \setminus \tilde{R}_p(F)$. Because $\tilde{R}_p(F)$ is an order filter by the definition of $\tilde{R}_p$, we deduce that y is a minimal element of F. Hence, by Proposition 4.14 (1), $F \setminus \{y\}$ is an order filter of P_λ. Note that $\tilde{\kappa}(y) = p$, and recall that $\tilde{\kappa}(y) = p$ if and only if $\kappa(y) = i$ or $\kappa(y) = j$. Assume that $\kappa(y) = i$. Since $F \setminus \{y\}$ is an order filter of P_λ satisfying $F \setminus (F \setminus \{y\}) = \{y\} \subseteq \kappa^{-1}(\{i\})$, we see by the definition of R_i that $R_i(F) \neq F$. Similarly, if $\kappa(y) = j$, then $R_j(F) \neq F$. Thus we conclude that $R_i(F) \neq F$ or $R_j(F) \neq F$. However, this contradicts the assumption that both $S_i(F)$ and $S_j(F)$ satisfy (i). Therefore, $\tilde{R}_p(F) = F$, and hence $\tilde{S}_p(F) = \tilde{A}_p(F)$. Since $F \sqcup \{x_i\} \sqcup \{x_j\} = S_jS_i(F)$ is an order filter of P_λ, we see by the definition of $\tilde{A}_p(F)$ that $F \sqcup \{x_i\} \sqcup \{x_j\} \subseteq \tilde{A}_p(F) = \tilde{S}_p(F)$.

Suppose, for a contradiction, that $\tilde{S}_p(F) \supsetneq F \sqcup \{x_i\} \sqcup \{x_j\}$. Since $F \sqcup \{x_j\}, F \sqcup \{x_i\} \in \mathcal{F}(P_\lambda)$, it follows from Proposition 4.14 (1) that x_i and x_j are minimal elements of $F \sqcup \{x_i\} \sqcup \{x_j\}$. Let z be a maximal element of $\tilde{S}_p(F) \setminus (F \sqcup \{x_i\} \sqcup \{x_j\})$; note that $\tilde{\kappa}(z) = p$, which implies that $\kappa(z) \in p = \{i, j\}$. If z and x_i are comparable, then $z \to x_i$ because $F \sqcup \{x_i\} \sqcup \{x_j\}$ is an order filter, and x_i is a minimal element of $F \sqcup \{x_i\} \sqcup \{x_j\}$ as seen above. By Proposition 3.13, $\kappa(z) \in \{i, j\}$ and $\kappa(x_i) = i$ are adjacent in the Dynkin diagram of $\mathfrak{g}$. However, this contradicts that p satisfies the orthogonality condition. Thus, z and x_i are incomparable. Similarly, we can show that z and x_j are incomparable. Thus, z is a maximal element of $\tilde{S}_p(F) \setminus F$. Since $\tilde{S}_p(F)$ is an order filter of P_λ, we see that z is a maximal element of $P_\lambda \setminus F$. Hence, by Proposition 4.14 (2), $F \sqcup \{z\}$ is an order filter of P_λ. Since $\kappa(z) \in p = \{i, j\}$, we see by the definitions of A_i and A_j that z is contained in either $A_i(F)$ or $A_j(F)$. However, this contradicts the assumption that both $S_i(F)$ and $S_j(F)$ satisfy (i). Therefore, we obtain $F \sqcup \{x_i\} \sqcup \{x_j\} = \tilde{S}_p(F)$, as desired.

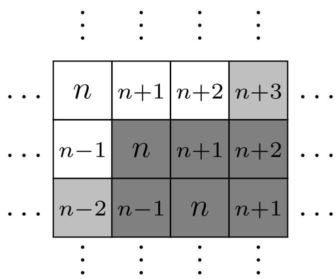
Next, we assume that p does not satisfy the orthogonality condition, that is, $\mathfrak{g}$ is of type A_{2n} and $p = \{n, n+1\}$. Let $\lambda = \Lambda_i$. In this case, the corresponding d-complete poset P_λ is $Y_{i,2n-i+1}$ (see Example 2.7 (1)), and its I-coloring $\kappa : P_\lambda \to I$ is given as follows (see also Figure 7):

$$(P_\lambda, \leq, \kappa, I) = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
i & \cdots & * & * & * & * & \cdots & 2n \\
\hline
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\hline
* & \cdots & n & n{+}1 & n{+}2 & n{+}3 & \cdots & * \\
\hline
* & \cdots & n{-}1 & n & n{+}1 & n{+}2 & \cdots & * \\
\hline
* & \cdots & n{-}2 & n{-}1 & n & n{+}1 & \cdots & * \\
\hline
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\hline
1 & \cdots & * & * & * & * & \cdots & 2n{-}i{+}1 \\
\hline
\end{array}\,.$$

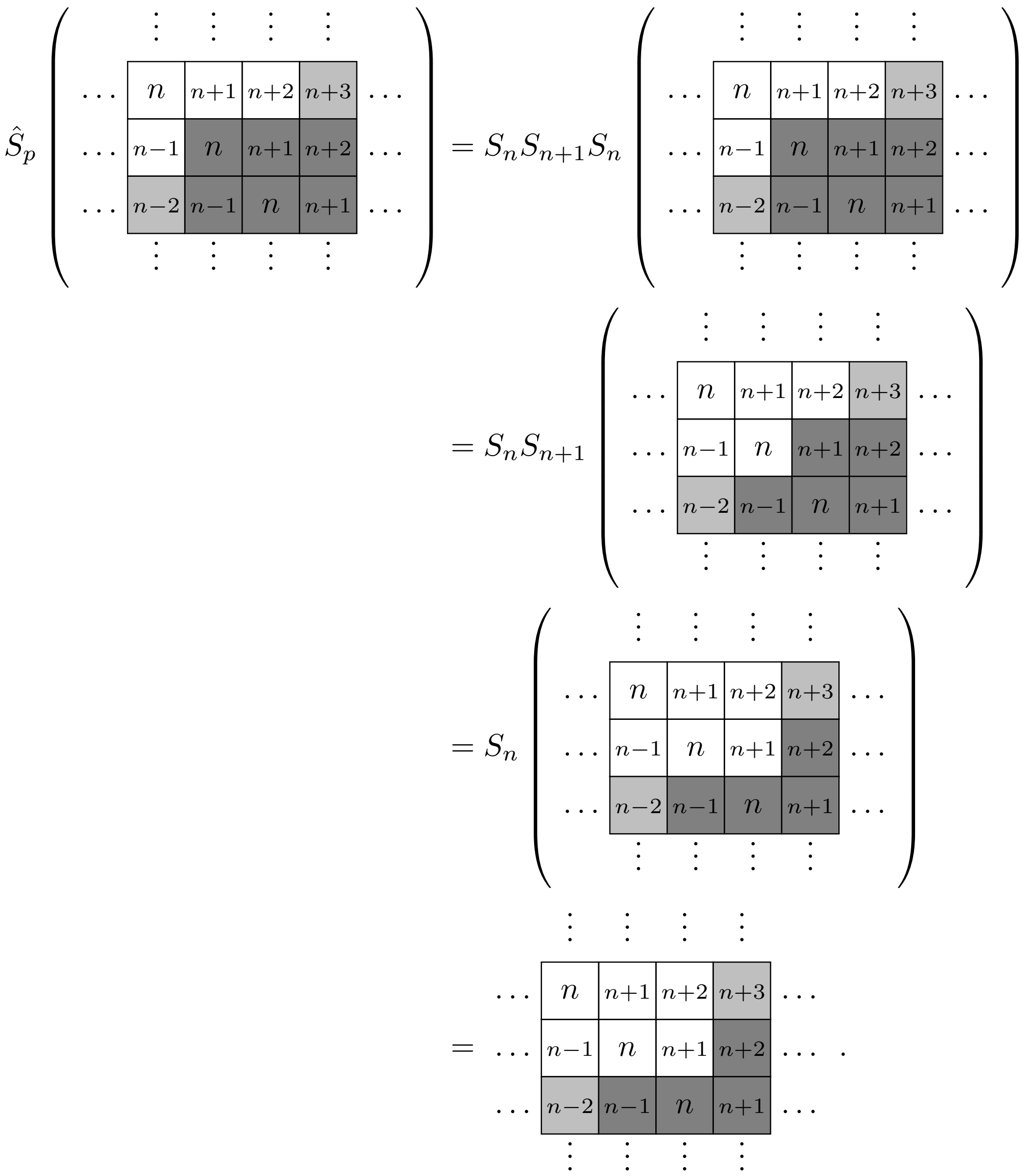
In this proof, the boxes having the color n or $n+1$ are important; if $1 \leq i \leq n$ (resp., $n{+}1 \leq i \leq 2n$), then $\kappa^{-1}(\{n\}) = \{(1, n{-}i{+}1), (2, n{-}i{+}2), \dots, (i, n)\}$ and $\kappa^{-1}(\{n{+}1\}) = \{(1, n-i+2), (2, n-i+3), \dots, (i, n+1)\}$ (resp., $\kappa^{-1}(\{n\}) = \{(i-n+1, 1), (i-n+2, 2), \dots, (n{+}1, 2n{-}i{+}1)\}$ and $\kappa^{-1}(\{n{+}1\}) = \{(i{-}n, 1), (i{-}n{+}1, 2), \dots, (n, 2n{-}i{+}1)\}$). Notice that the subset $\kappa^{-1}(\{n, n+1\}) \subset P_\lambda$ is a totally order set. Similarly to the case that p satisfies the orthogonality condition, each of $S_n(F)$ and $S_{n+1}(F)$ satisfies one of (i),(ii),(iii). Suppose, for a contradiction, that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (i). Then, there exist x_n, x_{n+1} such that both x_n and x_{n+1} are maximal elements of $P_\lambda \setminus F$, and $\kappa(x_n) = n, \kappa(x_{n+1}) = n+1$. However, this contradict the fact that $\kappa^{-1}(\{n, n+1\})$ is a totally order set. Therefore, the case that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (i) does not happen. Similarly, we deduce that the case that both $S_n(F)$ and $S_{n+1}(F)$ satisfy (ii) does not happen. So, it suffices to consider the other 7 cases.

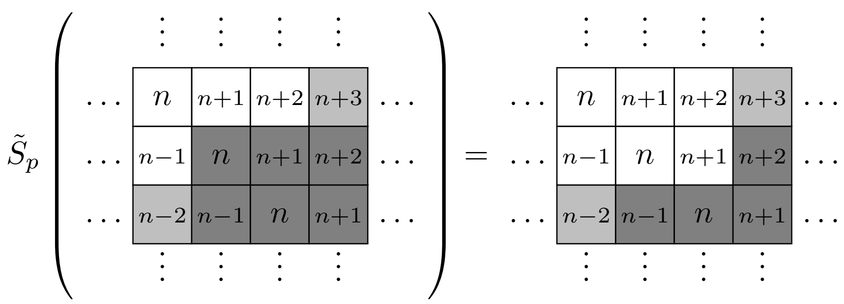
Now, we give a proof only for the case that $S_n(F)$ satisfies (i), and $S_{n+1}(F)$ satisfies (iii); the proofs for the other cases are similar. Then, under the description mentioned at the end of Section 2, F has a “block” of the following form:

$F =$



Here, each element corresponding to the right-gray box (with the color $n+3$ or $n-2$) is not necessarily an element of F. Then, $\hat{S}_p(F)$ and $\tilde{S}_p(F)$ are as follows:





Thus we obtain $\tilde{S}_p(F) = \hat{S}_p(F)$, as desired. $\square$

Lemma 4.16. For $\mu \in \hat{W}\lambda$ and $p \in J$,

$$\tilde{f}(\tilde{s}_p(\mathrm{res}(\mu))) = \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu))).$$

In particular,

$$\tilde{\mathcal{F}}(P_\lambda) = \{\tilde{S}_{p_n} \cdots \tilde{S}_{p_2}\tilde{S}_{p_1}(f(\lambda)) \,|\, n \geq 0, p_k \in J (1 \leq k \leq n)\}. \tag{4.3}$$

Proof. We compute that

$$\begin{aligned}
\tilde{f}(\tilde{s}_p(\mathrm{res}(\mu))) &= \tilde{f}(\mathrm{res}(\hat{s}_p(\mu))) && \text{(by Lemma 4.2)}\\
&= f(\hat{s}_p(\mu)) && \text{(by the definition of } \tilde{f})\\
&= \hat{S}_p(f(\mu)) && \text{(by Corollary 3.11)}\\
&= \hat{S}_p(\tilde{f}(\mathrm{res}(\mu))) && \text{(by the definition of } \tilde{f})\\
&= \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu))) && \text{(by Lemma 4.15).}
\end{aligned}$$

□

Proof of Theorem 4.11. (1) By the definitions of $\leq_w$ and $\tilde{\trianglelefteq}$, it suffices to show that for $\mu \in \hat{W}\lambda$ and $p \in J$, $\mathrm{res}(\mu) <_w \tilde{s}_p(\mathrm{res}(\mu))$ if and only if $\tilde{f}(\mathrm{res}(\mu)) \tilde{\triangleleft} \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu)))$. First, we assume that $\mathrm{res}(\mu) <_w \tilde{s}_p(\mathrm{res}(\mu)) = \mathrm{res}(\hat{s}_p(\mu))$. Because $\mathrm{res}(\mu)(H_p) > 0$, there exists $i \in p$ such that $\mu(h_i) > 0$. Then we deduce by Lemma 4.7(2) that $\mu <_w \hat{s}_p(\mu)$ in $(W\lambda, \leq_w)$. By the definition of $<_w$ and $<_s$, we have $\mu <_s \hat{s}_p(\mu)$ in $(W\lambda, \leq_s)$. So we compute

$$\begin{aligned}
f(\mu) &\subsetneq f(\hat{s}_p(\mu)) && \text{(by Proposition 3.8)}\\
&= \hat{S}_p(f(\mu)) && \text{(by Corollary 3.11)}\\
&= \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu))) && \text{(by the definition of } \tilde{f} \text{ and Lemma 4.15).}
\end{aligned}$$

Therefore, we obtain $\tilde{f}(\mathrm{res}(\mu)) \tilde{\triangleleft} \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu)))$, as desire.

Next, we assume that $\tilde{f}(\mathrm{res}(\mu)) \tilde{\triangleleft} \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu)))$. Then we have $\tilde{f}(\mathrm{res}(\mu)) \subsetneq \tilde{S}_p(\tilde{f}(\mathrm{res}(\mu)))$. Since $\tilde{f}(\mathrm{res}(\mu)) = f(\mu)$ and $\tilde{S}_p(\tilde{f}(\mathrm{res}(\mu))) = f(\hat{s}_p(\mu))$ as seen above, we get $f(\mu) \subsetneq f(\hat{s}_p(\mu))$. Hence, by Proposition 3.8, $\mu <_s \hat{s}_p(\mu)$ in $(W\lambda, \leq_s)$. Write $\hat{s}_p(\mu)$ as: $\hat{s}_p(\mu) = \mu - \sum_{i \in p} m_i\alpha_i$; since $\mu <_s \hat{s}_p(\mu)$, we see that $m_i \geq 0$ for all $i \in I$, and $m := \sum_{i\in p} m_i > 0$. Because $\tilde{s}_p(\mathrm{res}(\mu)) = \mathrm{res}(\mu) - m\beta_p$, we obtain $\mathrm{res}(\mu) <_w \tilde{s}_p(\mathrm{res}(\mu))$, as desire.

(2) For $\mu_1, \mu_2 \in \hat{W}\lambda$, We deduce

$$\begin{aligned}
&\mathrm{res}(\mu_1) <_s \mathrm{res}(\mu_2)\\
\Leftrightarrow\ &\mu_1 <_s \mu_2 && \text{(by Lemma 4.9)}\\
\Leftrightarrow\ &f(\mu_1) \subset f(\mu_2) && \text{(by Proposition 3.8)}\\
\Leftrightarrow\ &\tilde{f}(\mathrm{res}(\mu_1)) \subset \tilde{f}(\mathrm{res}(\mu_2)) && \text{(by the definition of } \tilde{f}).
\end{aligned}$$

□

5 Explicit Description of $\tilde{\mathcal{F}}(P_\lambda)$

Keep the notation and setting in Section 4.2. We give an explicit description of $\tilde{\mathcal{F}}(P_\lambda)$ in the case that $\mathfrak{g}$ is of type A_n; in fact, our description, Theorem 5.4 below, and its proof are essentially restatements of [12, Theorem 1.1 and its proof]; however, we give a proof (in terms of our notation) for the convenience of the readers.

Assume that $\mathfrak{g}$ is of type A_n, and $\lambda = \Lambda_m$ with $1 \leq m \leq (n+1)/2$. We regard P_λ as a rectangular Young diagram $Y_{m,n-m+1}$ (see Example 2.7). Note that $\kappa((i,j)) = j-i+m$ and $\tilde{\kappa}((i,j)) = (\min\{j-i+m, i-j+n-m+1\})'$.

For $i,j,p \in \mathbb{Z}$, we set $[i,j] := \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ and $\binom{[i,j]}{p} := \{I \subseteq [i,j] \mid \#I = p\}$.

Definition 5.1. Let $Y = (\boldsymbol{k}_1, \dots, \boldsymbol{k}_m) \in \mathcal{F}(Y_{m,n-m+1})$. We set $\mathcal{I}(Y) := \{\boldsymbol{k}_i + m + 1 - i \mid i \in [1,m]\}, \overline{\mathcal{I}(Y)} := \{n+2-i \mid i \in \mathcal{I}(Y)\}$. Observe that the map $\mathcal{I} : \mathcal{F}(Y_{m,n-m+1}) \to \binom{[1,n+1]}{m}, Y \mapsto \mathcal{I}(Y)$, is a bijection.

Lemma 5.2. Let $Y \in \mathcal{F}(Y_{m,n-m+1})$, and $k \in [1,n]$. Then,

(1) $k \in \mathcal{I}(Y)$ and $k+1 \notin \mathcal{I}(Y)$ if and only if $S_k(Y) \supset Y$;
(2) $k \notin \mathcal{I}(Y)$ and $k+1 \in \mathcal{I}(Y)$ if and only if $S_k(Y) \subset Y$;
(3) $k, k+1 \in \mathcal{I}(Y)$ or $k, k+1 \notin \mathcal{I}(Y)$ if and only if $S_k(Y) = Y$.

Proof. Notice that for $Y \in \mathcal{F}(Y_{m,n-m+1})$ and $k \in [1,n]$, $S_k(Y)$ satisfies one of the following (see Lemma 2.15 and Corollary 3.11):

(i) $S_k(Y) = A_k(Y) = Y \sqcup \{(i,j)\}$ for some $(i,j) \in Y_{m,n-m+1} \setminus Y$.
(ii) $S_k(Y) = R_k(Y) = Y \setminus \{(i,j)\}$ for some $(i,j) \in Y$.
(iii) $S_k(Y) = A_k(Y) = R_k(Y) = Y$.

(1) First, we show the "only if" part. Because $k \in \mathcal{I}(Y)$, there exists $i \in [1,m]$ such that $k = \boldsymbol{k}_i + m + 1 - i$. Then we see that $(i, \boldsymbol{k}_i) = (i, k-m-1+i) \in Y$ or $\boldsymbol{k}_i = 0$. In both cases, we get $(i, k-m+i) \notin Y$. Note that $\kappa(i, k-m+i) = k$. If $i = 1$, then we get $Y \sqcup \{(i, k-m+i)\} \in \mathcal{F}(Y_{m,n-m+1})$ and $S_k(Y) = Y \sqcup \{(i,k-m+i)\} \supset Y$. If $i > 1$, then $\boldsymbol{k}_{i-1} + m + 1 - (i-1) > k+1$ by $k+1 \notin \mathcal{I}(Y)$. Thus we have $k - m + i \leq \boldsymbol{k}_{i-1}$ and $(i-1, k-m+i) \in Y$. Hence we get $Y \sqcup \{(i, k-m+i)\} \in \mathcal{F}(Y_{m,n-m+1})$ and $S_k(Y) = Y \sqcup \{(i, k-m+i)\} \supset Y$.

Next, we show the "if" part. Because $S_k(Y) \supset Y$, there exists $(i,j) \in S_k(Y)$ such that $S_k(Y) = Y \sqcup \{(i,j)\}$ and $\kappa(i,j) = j - i + m = k$. Then we see that $(i, j-1) \in Y$ or $j-1 = 0$. In both cases, we get $\boldsymbol{k}_i = j-1 = i-m+k-1$. Hence, $k = \boldsymbol{k}_i + m + 1 - i \in \mathcal{I}(Y)$. If $i = 1$, then $\max(\mathcal{I}(Y)) = k$, and hence $k+1 \notin \mathcal{I}(Y)$. If $i > 1$, then $(i-1, j) \in Y$ and $\boldsymbol{k}_{i-1} \geq j$. Thus we obtain $\boldsymbol{k}_{i-1} + m + 1 - (i-1) \geq j + m + 1 - i + 1 = k+2$, which implies that $k+1 \notin \mathcal{I}(Y)$.

(2) Similar to part (1).

(3) Since $S_k(Y)$ satisfies one of (i)-(iii), the assertion is obvious from parts (1) and (2). □

Remark 5.3. By Lemma 5.2, if $k \in \mathcal{I}(Y)$ and $k+1 \notin \mathcal{I}(Y)$, then $k \notin \mathcal{I}(S_k(Y))$ and $k+1 \in \mathcal{I}(S_k(Y))$. Moreover, either $k' \in \mathcal{I}(Y), k' \in \mathcal{I}(S_k(Y))$ or $k' \notin \mathcal{I}(Y), k' \notin \mathcal{I}(S_k(Y))$ for $k' \in [1, n+1]$ with $k' \neq k, k+1$.

For $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0}$ such that $1 \leq m \leq (n+1)/2$, we set $\mathcal{SS}(Y_{m,n-m+1}) :=$

$\{Y \in \mathcal{F}(Y_{m,n-m+1}) \mid \mathcal{I}(Y) \cap \overline{\mathcal{I}(Y)} = \emptyset\}$.

Theorem 5.4 (cf. [12, Theorem 1.1])**.** It holds that $\tilde{\mathcal{F}}(Y_{m,n-m+1}) = \mathcal{SS}(Y_{m,n-m+1})$.

Proof. We will show that $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$ if and only if $Y \in \mathcal{SS}(Y_{m,n-m+1})$ by induction on $\#Y$. If $\#Y = 0$, then $Y = \emptyset$. It is obvious that $\emptyset \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. Also, because $I(\emptyset) = \{1, 2, \dots, m\}$ and $I(\emptyset) = \{n+1, n, \dots, n+2-m\}$, with $m < n+2-m$, it follows that $\mathcal{I}(\emptyset) \cap \overline{\mathcal{I}(\emptyset)} = \emptyset$, and hence $\emptyset \in \mathcal{SS}(Y_{m,n-m+1})$.

Assume that $\#Y > 0$. First, we will show the "only if" part. Because $Y \neq \emptyset$, there exists $p \in J$ such that $\tilde{S}_p(Y) \subset Y$. Since $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$, we have $\tilde{S}_p(Y) \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. By the induction hypothesis, it follows that $\tilde{S}_p(Y) \in \mathcal{SS}(Y_{m,n-m+1})$. Here, we give a proof only for the case that $\#p = 2$; the proof for the case that $\#p = 1$ is similar (and simpler). Assume that p satisfies the orthogonality condition. We write p as: $p = \{i, n+1-i\}$ with $i \neq n+1-i$. By Lemma 4.7, $\tilde{S}_p(Y)$ satisfies one of the following:

(i) $\tilde{S}_p(Y) \subset S_i\tilde{S}_p(Y), \tilde{S}_p(Y) = S_{n+1-i}\tilde{S}_p(Y)$.
(ii) $\tilde{S}_p(Y) = S_i\tilde{S}_p(Y), \tilde{S}_p(Y) \subset S_{n+1-i}\tilde{S}_p(Y)$.
(iii) $\tilde{S}_p(Y) \subset S_i\tilde{S}_p(Y), \tilde{S}_p(Y) \subset S_{n+1-i}\tilde{S}_p(Y)$.

We see by Lemma 5.2 that (i) (resp., (ii), (iii)) holds if and only if the following (i)' (resp., (ii)', (iii)') holds:

(i)' $i \in \mathcal{I}(\tilde{S}_p(Y)), i+1 \notin \mathcal{I}(\tilde{S}_p(Y)), n+1-i \notin \mathcal{I}(\tilde{S}_p(Y)), n+2-i \notin \mathcal{I}(\tilde{S}_p(Y))$.
(ii)' $i \notin \mathcal{I}(\tilde{S}_p(Y)), i+1 \notin \mathcal{I}(\tilde{S}_p(Y)), n+1-i \in \mathcal{I}(\tilde{S}_p(Y)), n+2-i \notin \mathcal{I}(\tilde{S}_p(Y))$.
(iii)' $i \in \mathcal{I}(\tilde{S}_p(Y)), i+1 \notin \mathcal{I}(\tilde{S}_p(Y)), n+1-i \in \mathcal{I}(\tilde{S}_p(Y)), n+2-i \notin \mathcal{I}(\tilde{S}_p(Y))$.

Moreover, it can be easily checked that (i)' (resp., (ii)', (iii)') holds if and only if the following (i)" (resp., (ii)", (iii)") holds:

(i)" $i \notin \mathcal{I}(Y), i+1 \in \mathcal{I}(Y), n+1-i \notin \mathcal{I}(Y), n+2-i \notin \mathcal{I}(Y)$,
(ii)" $i \notin \mathcal{I}(Y), i+1 \notin \mathcal{I}(Y), n+1-i \notin \mathcal{I}(Y), n+2-i \in \mathcal{I}(Y)$,
(iii)" $i \notin \mathcal{I}(Y), i+1 \in \mathcal{I}(Y), n+1-i \notin \mathcal{I}(Y), n+2-i \in \mathcal{I}(Y)$.

By Remark 5.3, we obtain $Y \in \mathcal{SS}(Y_{m,n-m+1})$ for any cases. Assume that p does not satisfy the orthogonality condition; in this case, n is even, and $p = \{i, i+1\}$ with $i = n/2$. By Lemmas 4.7 and 5.2, $\tilde{S}_p(Y)$ satisfies $i \in \mathcal{I}(\tilde{S}_p(Y)), i+1 \notin \mathcal{I}(\tilde{S}_p(Y))$, and $i+2 \notin \mathcal{I}(\tilde{S}_p(Y))$. Also, Y satisfies $i \notin \mathcal{I}(Y), i+1 \notin \mathcal{I}(Y)$, and $i+2 \in \mathcal{I}(Y)$. Thus we obtain $Y \in \mathcal{SS}(Y_{m,n-m+1})$, as desired.

Next, we will show the "if" part. Because $Y \neq \emptyset$, there exists $k \in [1, n]$ such that $k \notin \mathcal{I}(Y)$ and $k+1 \in \mathcal{I}(Y)$; we set $p := \{k, n+1-k\} \in J$. Let $Y' \in \mathcal{F}(Y_{m,n-m+1})$ be such that $\mathcal{I}(Y') = \mathcal{I}(Y) \sqcup \{k\} \setminus \{k+1\}$; note that $\#Y' = \#Y - 1$. Assume that $n+2-k \notin \mathcal{I}(Y')$. By Remark 5.3, we have $Y' \in \mathcal{SS}(Y_{m,n-m+1})$. By the induction hypothesis, it follows that $Y' \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. Notice that $n/2+1 \notin \mathcal{I}(Y)$, because $n+2-(n/2+1) = n/2+1$. Because $k+1 \in \mathcal{I}(Y)$, we have $n+1-k \neq k+1$. Also, because $k \in \mathcal{I}(Y')$ and $n+2-k \notin \mathcal{I}(Y')$, we have $k \neq n+2-k$, and hence $n+1-k \neq k-1$. Thus, p satisfies the orthogonality condition. If $\#p = 1$, then $p = \{k\}$, and $\tilde{S}_p(Y') = S_k(Y') = Y$ by Lemma 5.2. If $\#p = 2$, then $k \neq n+1-k$ and $\{k, k+1\} \cap \{n+1-k, n+2-k\} = \emptyset$, which implies that $n+2-k \notin \mathcal{I}(Y)$ by Remark 5.3, and $n+1-k \notin \mathcal{I}(Y)$ by Lemmas 4.7 and 5.2. Hence we have $\tilde{S}_p(Y') = S_{n+1-k}S_k(Y') = S_{n+1-k}(Y) = Y$. In both cases, we obtain $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. Assume that $n+2-k \in \mathcal{I}(Y')$. Let $Y'' \in \mathcal{F}(Y_{m,n-m+1})$ be such that $\mathcal{I}(Y'') = \mathcal{I}(Y') \sqcup \{n+1-k\} \setminus \{n+2-k\}$; note that $\#Y'' = \#Y' - 1$. Because $n+2-(n+1-k) = k+1 \notin \mathcal{I}(Y)$, we have $Y'' \in \mathcal{SS}(Y_{m,n-m+1})$. By the induction hypothesis, it follows that $Y'' \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$. Because $\#Y'' = \#Y - 2$, we have $\#p = 2$. We see by Lemmas 4.7 and 5.2 that if p satisfies the orthogonality condition, then $\tilde{S}_p(Y'') = S_k S_{n+1-k}(Y'') = S_k(Y') = Y$. If p does not satisfy the orthogonality condition, then $n+1-k = k-1$. Thus we obtain $k-1 \in \mathcal{I}(Y'')$, $k, k+1 \notin \mathcal{I}(Y'')$, and hence $\tilde{S}_p(Y'') = S_{k-1}S_kS_{k-1}(Y'') = S_{k-1}Y = Y$. In both cases, we obtain $Y \in \tilde{\mathcal{F}}(Y_{m,n-m+1})$, as desired. □

6 The Rules of MHRG(m,n)

6.1 Impartial Combinatorial Games

Combinatorial games satisfy the requirements stated below. One should consult with Berlekamp, Conway, and Guy [3] for the classical introduction to such games. See Conway [6] and Siegel [21] for more advanced treatments.

- A combinatorial game is played by two players (we will call them "A" and "B").
- Two players alternate in making a move.
- There are no chance elements (no moves are determined by rolling dice, etc.).
- No position can appear more than once during a game. And, in particular, combinatorial games are "short games"-they always end following a finite number of moves.

In addition, if both players have the same set of options in each position, then the game is an impartial combinatorial game. As previously mentioned, MHRG(m,n) is such a game.

Given an impartial combinatorial game G, a game position is called an $\mathcal{N}$-position (resp., $\mathcal{P}$-position) if the next (resp., previous) player has a winning strategy, and each game position is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. Additionally, if G is an $\mathcal{N}$-position, then there exists a move from G to a $\mathcal{P}$-position. If G is a $\mathcal{P}$-position, then there exists no move from G to a $\mathcal{P}$-position (see [3], [21]).

Let G be an impartial game and set

$$\mathcal{C}(G) = \{G' \mid G' \text{ is a game position of } G\} \text{ (of course } G \in \mathcal{C}(G)).$$

If G' is an option of G, then we write $G \to G'$, and we set

$$\mathcal{O}(G) = \{G' \mid G \to G'\} \ (\mathcal{O}(G) \subset \mathcal{C}(G)).$$

A transition from G to G' is, by definition, a sequence $G = G_0, G_1, \ldots, G_k = G', k \in \mathbb{Z}_{\geq 0}$, of game positions in $\mathcal{C}(G)$ such that

$$G = G_0 \to G_1 \to \cdots \to G_k = G'.$$

Definition 6.1. Let G and H be impartial combinatorial games. If there exists a bijection $f : \mathcal{C}(G) \to \mathcal{C}(H)$ such that $f(\mathcal{O}(G')) = \mathcal{O}(f(G'))$ for all $G' \in \mathcal{C}(G)$, then we say that G is isomorphic to H, and we call f an isomorphism from G to H. In other words, G is isomorphic to H if G and H have identical game trees [2].

Definition 6.2. For any proper subset T of $\mathbb{Z}_{\geq 0}$, we define the minimal excluded number $\mathrm{mex}(T)$ as follows:

$$\mathrm{mex}(T) = \min(\mathbb{Z}_{\geq 0} \setminus T).$$

We recall the $\mathcal{G}$-value (or Sprague-Grundy value) of a position in an impartial combinatorial game.

Definition 6.3. Let G be a game position. We define $\mathcal{G}(G) \in \mathbb{Z}_{\geq 0}$, called the $\mathcal{G}$-value (or Sprague-Grundy value) of G, by

$$\mathcal{G}(G) := \mathrm{mex}\{\mathcal{G}(G') \mid G \to G'\}.$$

The following theorem is well-known.

Proposition 6.4 ([21, Chapter IV])**.** For a game position G, $\mathcal{G}(G) = 0$ if and only if G is a $\mathcal{P}$-position.

The following proposition can be easily shown.

Proposition 6.5. Let G and H be impartial combinatorial games. If there exists a bijection $f : \mathcal{C}(G) \to \mathcal{C}(H)$, then $\mathcal{G}(G') = \mathcal{G}(f(G'))$ for all $G' \in \mathcal{C}(G)$.

6.2 Unimodal Numbering of a Rectangular Young Diagram

For a Young diagram Y, a map $\alpha : Y \to \mathbb{Z}_{>0}$ is called a numbering of Y. For a box $(i, j) \in Y$, if $\alpha(i, j) = x$, then we say that the box (i, j) has the number x. Let Y be a Young diagram with a numbering α. For a subset X of Y, we set $\mathcal{A}_\alpha(X) = [\alpha(i, j) \mid (i, j) \in X]$, where $[x_1, \ldots, x_N]$ denotes the multiset consisting of $x_1, \ldots, x_N$.

Let $m, n \in \mathbb{Z}_{>0}$. For $Y \in \mathcal{F}(Y_{m,n})$, we define a special numbering $\alpha_{m,n} : Y \to \mathbb{Z}_{>0}$, called the unimodal numbering of Y, as follows: For $(i, j) \in Y$, we set $\alpha_{m,n}(i, j) := \min\{j - i + m, i - j + n\} \in \mathbb{Z}_{>0}$. In what follows, the boxes in $Y \in \mathcal{F}(Y_{m,n})$ are always numbered by the unimodal numbering $\alpha_{m,n}$.

3	3	2	1
2	3	3	2
1	2	3	3

3	4	3	2	1
2	3	4	3	
1	2	3		

Fig. 13 unimodal numberings

Remark 6.6. Let $Y \in \mathcal{F}(Y_{m,n})$. By the definition of unimodal numbering $\alpha_{m,n}$, we can easily check the following.

(1) If Y contains the box $(m, 1)$, then it has the number 1. If Y contains the box $(1, n)$, then it has the number 1.
(2) The boxes (i, j) and $(i + 1, j + 1)$ have the same number (if they exist in Y).
(3) The maximum value of $\alpha_{m,n} : Y_{m,n} \to \mathbb{Z}_{>0}$ is equal to $\hat{\alpha}_{m,n} := \lfloor (n+m)/2 \rfloor$, where $\lfloor x \rfloor := \max\{y \in \mathbb{Z} \mid y \leq x\}$ for $x \in \mathbb{R}$.

Remark 6.7. Assume that $\mathfrak{g}$ is of type A_n, and $\lambda = \Lambda_m$ with $1 \le m \le (n+1)/2$. As mentioned in Section 5, we regard P_λ as a rectangular Young diagram $Y_{m,n-m+1}$. Then it holds that $\tilde{\kappa}(i,j) = (\min\{j-i+m, i-j+n-m+1\})'$ for $(i,j) \in Y_{m,n-m+1}$. Thus we have $\tilde{\kappa}(i,j) = (\alpha_{m,n-m+1}(i,j))'$. Thus we can regard the J-colored d-complete poset $(P_\lambda, \le, \tilde{\kappa}, J)$ as the rectangular Young diagram $Y_{m,n-m+1}$ with the unimodal numbering $\alpha_{m,n-m+1}$.

6.3 Rules of the Multiple Hook Removing Game

In this subsection, we explain the rules of MHRG(m,n).

Definition 6.8. For a box (i,j) of a Young diagram Y,

$$h(i,j) = h_Y(i,j) := \{(i,j)\} \sqcup \{(i',j) \in Y \mid i' > i\} \sqcup \{(i,j') \in Y \mid j' > j\}$$

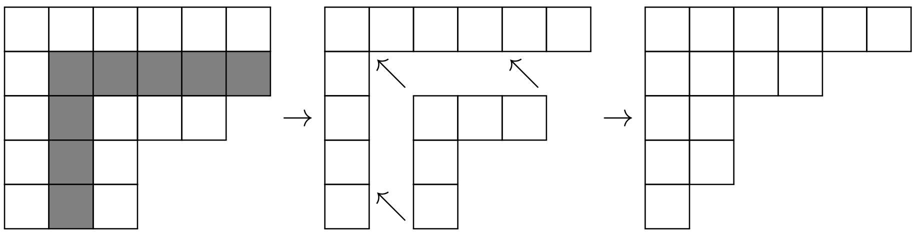
is called the hook (in Y) corresponding to the box (i,j).

Definition 6.9. For a box (i,j) of a Young diagram Y, we remove the hook $h_Y(i,j)$ corresponding to the box (i,j) as follows:

1. Remove each box in the hook $h_Y(i,j)$.
2. Move each box (i',j') satisfying $i' > i$ and $j' > j$ to $(i'-1, j'-1)$.

We denote by $Y \setminus h_Y(i,j)$ the Young diagram obtained by removing the hook $h_Y(i,j)$ corresponding to the box (i,j) from Y.

Example 6.10. If we remove the hook corresponding to the box $(2,2)$ from the Young diagram $Y = (6,6,5,3,3)$, then we get $Y' = Y \setminus h_Y(2,2) = (6,4,2,2,1)$.



Definition 6.11. Let $m, n \in \mathbb{Z}_{>0}$. MHRG(m,n) is an impartial combinatorial game whose rules are as follows:

(M1) The starting position is a rectangular Young diagram $Y_{m,n}$ with the unimodal numbering $\alpha_{m,n}$. All game positions are Young diagrams Y contained in Y with a numbering $\alpha_{m,n}|_Y$.

(M2) Given a Young diagram Y with the numbering $\alpha_{m,n}|_Y$, each player chooses a box in Y and removes the hook h corresponding to the box on his/her turn. Let $\mathcal{A}_{\alpha_{m,n}}(h)$ be the multiset of the numbers (in boxes) in the hook h, and let Y' be the Young diagram obtained by removing h from Y, with the numbering $\alpha_{m,n}|_{Y'}$.

  (M2a) If there does not exist any box in Y' whose corresponding hook h' satisfies $\mathcal{A}_{\alpha_{m,n}}(h') = \mathcal{A}_{\alpha_{m,n}}(h)$ as multisets, then the player's turn is over, and the next player is given Y'.

  (M2b) If there exists a box in Y' whose corresponding hook h' satisfies $\mathcal{A}_{\alpha_{m,n}}(h') = \mathcal{A}_{\alpha_{m,n}}(h)$, then the player must choose one such boxes, and remove the hook h'

corresponding to the box. Let Y'' be the Young diagram obtained by removing h' from Y', with the numbering $\alpha_{m,n}|_{Y''}$.

(M2c) Do the same operation as (M2a) and (M2b), with Y' replaced by Y''. As long as such a box exists, repeat this operation.

(M3) The winner is the player who removes the last remaining hook in the diagram.

For an example, see Example 1.2 in Introduction.

7 Diagonal Expressions for Young Diagrams and Hooks

The diagonal expression for $Y \in \mathcal{F}(Y_{m,n})$ is now defined in terms of the following elements.

Let $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{m+n+1}$ be given by $\boldsymbol{a} = (a_{-m}, a_{-m+1}, \ldots, a_n)$, where we call a_k the k-th component of $\boldsymbol{a}$ for $-m \leq k \leq n$. For $-m < i \leq 0$ (resp., $0 < i \leq n$), we say that the pair (a_{i-1}, a_i) satisfies the adjacency requirement if $0 \leq a_i - a_{i-1} \leq 1$ (resp., $0 \leq a_{i-1} - a_i \leq 1$). Additionally, we say that $\boldsymbol{a}$ satisfies the adjacency requirement if (a_{i-1}, a_i) satisfies the adjacency requirement for all $-m < i \leq n$.

For $m, n \in \mathbb{Z}_{>0}$, let $\mathbb{D}_{m,n} \subset \mathbb{Z}_{\geq 0}^{m+n+1}$ denote the set of all elements $\boldsymbol{a} = (a_{-m}, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{m+n+1}$ with $a_{-m} = a_n = 0$ satisfying the adjacency requirement. Finally, set $d_k(Y) := \#\{(i,j) \in Y \mid j - i = k\}$ for $k \in \mathbb{Z}$. Note that if $k \leq -m$ or $k \geq n$, then $d_k(Y) = 0$.

Remark 7.1. For $i, j \geq 2$, if $(i,j) \in Y$, then $(i-1, j-1) \in Y$. Also, if $(i,j) \notin Y$, then $(i+a, j+a) \notin Y$ for $a \in \mathbb{Z}_{>0}$. Hence we see that $d_k(Y) = \max\{\min\{i,j\} \mid (i,j) \in Y, j - i = k\}$ for $k \in \mathbb{Z}$.

Given the above setting, the following lemma is easily verified.

Lemma 7.2. The following statements hold.

(1) Let $k \geq 0$. Then, $(d_k(Y)+1, d_k(Y)+k+1) \notin Y$. Moreover, if $d_k(Y) > 0$, then $(d_k(Y), d_k(Y)+k) \in Y$.

(2) Let $k < 0$. Then $(d_k(Y)-k+1, d_k(Y)+1) \notin Y$. Moreover, if $d_k(Y) > 0$, then $(d_k(Y)-k, d_k(Y)) \in Y$.

Definition 7.3. For every $Y \in \mathcal{F}(Y_{m,n})$, the diagonal expression for Y is given by

$$\boldsymbol{d}(Y) = \boldsymbol{d}_{m,n}(Y) = (d_{-m}(Y), d_{-m+1}(Y), \ldots, d_n(Y))$$

Lemma 7.4. Let $Y \in \mathcal{F}(Y_{m,n})$; recall that $d_k = d_k(Y) = \#\{(i,j) \in Y \mid j - i = k\}$ for $k \in \mathbb{Z}$.

(1) If $k > 0$, then $0 \leq d_{k-1} - d_k \leq 1$.

(2) If $k \leq 0$, then $0 \leq d_k - d_{k-1} \leq 1$.

Proof. (1) Assume that $d_k = 0$. Then, $(1, k+1) \notin Y$ by Lemma 7.2, which implies that $(2, k+1) \notin Y$. Hence, $d_{k-1} = \max\{\min\{i,j\} \mid (i,j) \in Y, j - i = k-1\}$ is equal to 0 or 1 (see Remark 7.1). Thus we obtain $0 \leq d_{k-1} - d_k = d_{k-1} \leq 1$. Assume $d_k > 0$. By Lemma 7.2, it follows that $(d_k, d_k + k) \in Y$ and $(d_k + 1, d_k + k + 1) \notin Y$. Then we have $(d_k, d_k + k - 1) \in Y$ and $(d_k + 2, d_k + k + 1) \notin Y$. Therefore $d_{k-1} = \max\{\min\{i,j\} \mid (i,j) \in Y, j - i = k-1\}$ is d_k or $d_k + 1$ by Remark 7.1. Thus we obtain $0 \leq d_{k-1} - d_k \leq 1$.

(2) The proof of (2) is similar to that of (1). ◻

Proposition 7.5. The function $\boldsymbol{d}_{m,n}(Y)$ is a bijection of $\mathcal{F}(Y_{m,n}) \to \mathbb{D}_{m,n}$.

Proof. From Lemma 7.4, the pair $(d_{i-1}(Y), d_i(Y))$ satisfies the adjacency requirement for all $-m < i \le n$. Since $d_{-m}(Y) = d_n(Y) = 0$, we have $(d_{-m}(Y), \dots, d_n(Y)) \in \mathbb{D}_{m,n}$. Also, by the definition of $\boldsymbol{d} = \boldsymbol{d}_{m,n}$, it is obvious that $\boldsymbol{d}$ is an injection.

For $\boldsymbol{a} = (a_{-m}, \dots, a_n) \in \mathbb{D}_{m,n}$, we define Y as follows. The box (i,j) is contained in Y if and only if $\min\{i,j\} \le a_{j-i}$. Note that if $j - i \le -m$ or $n \le j - i$, then the box (i,j) is not contained in Y. We claim that Y is a Young diagram. It suffices to show that if the box (i,j) is not contained in Y, then neither the box $(i+1,j)$ nor $(i,j+1)$ is contained in Y. If $-m < j-i < 0$, then $\min\{i,j\} = j > a_{j-i}$. By the definition of $\mathbb{D}_{m,n}$, we have $0 \le a_{j-i} - a_{j-i-1}$ and $a_{j-i+1} - a_{j-i} \le 1$. Then we get $\min\{i+1,j\} = j > a_{j-i-1}$ and $\min\{i,j+1\} = j+1 > a_{j-i+1}$. Hence, by the definition of Y, we obtain $(i+1,j), (i,j+1) \notin Y$. The proofs for the cases that $j - i = 0$ and $0 < j - i < n$ are similar. Thus we have shown that Y is a Young diagram. Further, since $(m+1,1), (1,n+1) \notin Y$, it follows that $Y \in \mathcal{F}(Y_{m,n})$.

By the definition of Y, we have $d_k = a_k$ for $-m < k < n$. Hence we obtain $\boldsymbol{d}(Y) = \boldsymbol{a}$, which shows that $\boldsymbol{d}$ is a surjection. Thus we have proved that $\boldsymbol{d}$ is a bijection. □

Example 7.6. Assume that $m = 3$ and $n = 5$. If $Y \in \mathcal{F}(Y_{3,5})$ is

$$Y = \begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 3 & 2 & 1 \\ \hline 2 & 3 & 4 & 3 & \\ \cline{1-4} 1 & 2 & 3 & & \\ \cline{1-3}\end{array},$$

then $\boldsymbol{d}(Y) = \boldsymbol{d}_{3,5}(Y) = (0, 1, 2, \dot{3}, 2, 2, 1, 1, 0)$ (if necessary, we will accentuate the 0-th component by putting a dot above it as above).

Let $Y \in \mathcal{F}(Y_{m,n}), (i,j) \in Y$ and set $Y' := Y \setminus h_Y(i,j)$. We set $i' := \max\{x \in \mathbb{Z}_{>0} \mid (x,j) \in Y\}$ and $j' := \max\{x \in \mathbb{Z}_{>0} \mid (i,x) \in Y\}$ For $k \in \mathbb{Z}$, it holds that

$$d_k(Y) - d_k(Y') = \begin{cases} 1 & \text{if } j - i' \le k \le j' - i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the diagonal expression of Y' is

$$\begin{aligned} \boldsymbol{d}(Y') = (\dots, d_{j-i'-1}(Y),\, & d_{j-i'}(Y) - 1, d_{j-i'+1}(Y) - 1, \dots, \\ & d_{j'-i-1}(Y) - 1, d_{j'-i}(Y) - 1, d_{j'-i+1}(Y), \dots). \end{aligned} \tag{3.1}$$

Let $\boldsymbol{a} = (a_{-m}, \dots, a_n) \in \mathbb{D}_{m,n}$, $\boldsymbol{a'} = (a'_{-m}, \dots, a'_n) \in \mathbb{Z}_{\ge 0}^{m+n+1}$, and let l, r be such that $-m < l \le r < n$. If $a'_k = a_k - 1$ for $l \le k \le r$, and $a'_k = a_k$ for the other k's, then we write $\boldsymbol{a} \xrightarrow{l,r} \boldsymbol{a'}$ or $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]}$. In this case, the pair (a'_{k-1}, a'_k) satisfies the adjacency requirement for all $-m < k \le n$ but $k = l$ and $r + 1$. Hence, if (a'_{l-1}, a'_l) and (a'_r, a'_{r+1}) satisfy the adjacency requirement, then $\boldsymbol{a'} \in \mathbb{D}_{m,n}$.

Lemma 7.7. Let $Y, Y' \in \mathcal{F}(Y_{m,n})$. The following are equivalent.

(1) There exists a box $(i,j) \in Y$ such that $Y' = Y \setminus h_Y(i,j)$.

(2) There exist $-m < l \le r < n$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$.

In this case, it holds that $l = j - i'$ and $r = j' - i$, where (i',j) is the bottom box in the

j-th column of Y, and (i, j') is the rightmost box in the i-th row of Y.

Example 7.8. Let Y be as in Example 7.6, and let $Y' = Y \setminus h_Y(1,4)$.

$$Y = \begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 3 & 2 & 1 \\ \hline 2 & 3 & 4 & 3 \\ \cline{1-4} 1 & 2 & 3 \\ \cline{1-3} \end{array} \quad \to \quad Y' = \begin{array}{|c|c|c|} \hline 3 & 4 & 3 \\ \hline 2 & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

In the diagonal expression, we see that

$$\boldsymbol{d}(Y) = (0, 1, 2, \dot{3}, 2, 2, 1, 1, 0), \qquad \boldsymbol{d}(Y') = (0, 1, 2, \dot{3}, 2, \underline{1, 0, 0}, 0),$$

and $\boldsymbol{d}(Y) \xrightarrow{2,4} \boldsymbol{d}(Y')$.

Proof of Lemma 7.7. The implication (1)⇒(2) and equalities $l = j - i'$ and $r = j' - i$ follow from (3.1). Let us show (2)⇒(1). A proof is given only for the case that $l \le r \le 0$. Proofs of the cases $l \le 0 \le r$ and $0 \le l \le r$ are similar. Notice that $d_l(Y), d_r(Y) > 0$. By Lemma 7.2 (2), we have both $(d_l(Y) - l, d_l(Y)), (d_r(Y) - r, d_r(Y)) \in Y$. Also, by the adjacency requirement, it follows that $d_l(Y) \le d_r(Y)$. Because $d_l(Y') = d_l(Y) - 1$ and $d_r(Y') = d_r(Y) - 1$, we see by Lemma 7.2 (2) that both $(d_l(Y) - l, d_l(Y)), (d_r(Y) - r, d_r(Y)) \notin Y'$, which implied that $(d_l(Y) - l + 1, d_l(Y)), (d_r(Y) - r, d_r(Y) + 1) \notin Y'$. Since $d_k(Y') = d_k(Y)$ for $k < l$ and $r < k$, we deduce that for i, j such that $j - i < l$ or $r < j - i$, we have $(i, j) \in Y$ if and only if $(i, j) \in Y'$. Hence, both $(d_l(Y) - l + 1, d_l(Y)), (d_r(Y) - r, d_r(Y) + 1) \notin Y$.

Let h be the hook in Y corresponding to the box $(d_r(Y) - r, d_l(Y))$. Since $(d_l(Y) - l, d_l(Y)) \in Y$ and $(d_l(Y) - l + 1, d_l(Y)) \notin Y$, the bottom box in the $d_l(Y)$-th column of Y is $(d_l(Y) - l, d_l(Y))$. Also, since $(d_r(Y) - r, d_r(Y)) \in Y$ and $(d_r(Y) - r, d_r(Y) + 1) \notin Y$, the rightmost box in the $(d_r(Y) - r)$-th row of Y is $(d_r(Y) - r, d_r(Y))$. We see from (3.1) that the diagonal expression of $Y \setminus h$ is

$$(\dots, d_{l-1}(Y), d_l(Y) - 1, d_{l+1}(Y) - 1, \dots, d_{r-1}(Y) - 1, d_r(Y) - 1, d_{r+1}(Y), \dots),$$

which is equal to $\boldsymbol{d}(Y')$. Thus we have proved the lemma. □

The next lemma follows from the proof of Lemma 7.7.

Lemma 7.9. Let $Y \in \mathcal{F}(Y_{m,n})$ and $Y' = Y \setminus h_Y(i,j)$ for $(i,j) \in Y$. Also, let $-m < l \le r < n$ be such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ in the diagonal expression. Then, $\#(h_Y(i,j)) = \#\mathcal{A}_{\alpha_{m,n}}(h_Y(i,j)) = r - l + 1$.

Definition 7.10. Let $\boldsymbol{a} = (a_{-m}, \dots, a_n) \in \mathbb{Z}_{\ge 0}^{m+n+1}$. Assume that (a_{k-1}, a_k) satisfies the adjacency requirement for some $-m < k \le n$. If $(a_{k-1} - 1, a_k)$ (resp., $(a_{k-1}, a_k - 1)$) also satisfies the adjacency requirement, then we say that (a_{k-1}, a_k) is a left (resp., right) bulge, and we write $a_{k-1} \searrow a_k$ (resp., $a_{k-1} \nearrow a_k$).

The following lemma can be easily verified.

Lemma 7.11. Let $\boldsymbol{a} = (a_{-m}, \dots, a_n) \in \mathbb{Z}_{\ge 0}^{m+n+1}$.

(1) If (a_{k-1}, a_k) satisfies the adjacency requirement, then (a_{k-1}, a_k) is either a left bulge or a right bulge.
(2) Assume that (a_{k-1}, a_k) satisfies the adjacency requirement. If (a_{k-1}, a_k) is a left

bulge, then $(a_{k-1}-1, a_k)$ is a right bulge.

(3) Assume that (a_{k-1}, a_k) satisfies the adjacency requirement. If (a_{k-1}, a_k) is a right bulge, then $(a_{k-1}, a_k - 1)$ is a left bulge.

Lemma 7.12. Let $\boldsymbol{a} = (a_{-m}, \dots, a_n), \boldsymbol{a'} = (a'_{-m}, \dots, a'_n) \in \mathbb{D}_{m,n}$. Assume that $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]} \in \mathbb{D}_{m,n}$ for some $-m < l \le r < n$. Then, $a_{l-1} \nearrow a_l, a_r \searrow a_{r+1}$ and $a'_{l-1} \searrow a'_l, a'_r \nearrow a'_{r+1}$. Moreover, for $-m < k \le n$ with $k \neq l, r+1$, if $a_{k-1} \nearrow a_k$ (resp., $a_{k-1} \searrow a_k$), then $a'_{k-1} \nearrow a'_k$ (resp., $a'_{k-1} \searrow a'_k$).

Proof. Since $\boldsymbol{a'} = \boldsymbol{a}_{[l,r]} \in \mathbb{D}_{m,n}$, it follows that $(a_{l-1}, a_l - 1)$ and $(a_r - 1, a_{r+1})$ satisfy the adjacency requirement. Hence, (a_{l-1}, a_l) is a right bulge and (a_r, a_{r+1}) is a left bulge. By Lemma 7.11 (2) and (3), (a'_{l-1}, a'_l) is a left bulge and (a'_r, a'_{r+1}) is a right bulge. By the definition of $\boldsymbol{a}_{[l,r]}$, we have $a_k - a_{k-1} = a'_k - a'_{k-1}$ for $-m < k \le n$ with $k \neq l, r+1$. Hence, both (a_{l-1}, a_l) and (a'_{l-1}, a'_l) are both left bulges or rights bulges. Thus we have proved the lemma. □

Let $Y \in \mathcal{F}(Y_{m,n})$ be a Young diagram with the unimodal numbering $\alpha_{m,n}$. By Remark 6.6 (2), it holds that $\alpha_{m,n}(i', j') = \alpha_{m,n}(i' + a, j' + a)$ for all $(i', j') \in Y$ and $a \in \mathbb{Z}_{>0}$ such that $(i' + a, j' + a) \in Y$. Hence we see that $\mathcal{A}_{\alpha_{m,n}}(Y) = \mathcal{A}_{\alpha_{m,n}}(Y \setminus h_Y(i,j)) \cup \mathcal{A}_{\alpha_{m,n}}(h_Y(i,j))$ for $(i,j) \in Y$.

Lemma 7.13. For $Y \in \mathcal{F}(Y_{m,n})$ and $1 \le k \le \hat{\alpha}_{m,n} = \lfloor (n+m)/2 \rfloor$,

$$\#\{x \in \mathcal{A}_{\alpha_{m,n}}(Y) \mid x = k\} = \begin{cases} d_{-m+k} + d_{n-k} & \text{if } -m+k \neq n-k, \\ d_{-m+k} & \text{if } -m+k = n-k. \end{cases}$$

Proof. Assume that $-m + k \neq n - k$. Then we compute

$$\begin{aligned} &\#\{x \in \mathcal{A}_{\alpha_{m,n}}(Y) \mid x = k\} = \#\{(i,j) \in Y \mid j - i = -m + k \text{ or } n - k\} \\ &= \#\{(i,j) \in Y \mid j - i = -m + k\} + \#\{(i,j) \in Y \mid j - i = n - k\} \\ &= d_{-m+k} + d_{n-k}. \end{aligned}$$

The proof of the case $-m + k = n - k$ is similar. □

Lemma 7.14. Let $Y \in \mathcal{F}(Y_{m,n})$ and $Y' = Y \setminus h_Y(i,j)$ for $(i,j) \in Y$. Let $-m < l \le r < n$ be such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ in the diagonal expression (see (3.1)). Assume that there exists $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{m,n}}(h_{Y'}(i',j')) = \mathcal{A}_{\alpha_{m,n}}(h_Y(i,j))$. Set $Y'' = Y' \setminus h_{Y'}(i',j')$. Then, $\boldsymbol{d}(Y') \xrightarrow{n-m-r, n-m-l} \boldsymbol{d}(Y'')$ in the diagonal expression. Also, there exists no box $(i'', j'') \in Y''$ such that $\mathcal{A}_{\alpha_{m,n}}(h_{Y''}(i'',j'')) = \mathcal{A}_{\alpha_{m,n}}(h_Y(i,j))$.

Example 7.15. Let Y be as in Example 7.6 (note that $m = 3$ and $n = 5$), and set $Y' = Y \setminus h_Y(1,3)$. Then we have $\mathcal{A}_{\alpha_{3,5}}(h_Y(1,3)) = [3,4,3,2,1]$. Notice that $\mathcal{A}_{\alpha_{3,5}}(h_{Y'}(1,1)) = [1,2,3,4,3] = \mathcal{A}_{\alpha_{3,5}}(h_Y(1,3))$. Here we set $Y'' = Y' \setminus h_{Y'}(1,1)$ and it follows that

$Y =$

3	4	3	2	1
2	3	4	3	
1	2	3		

$\rightarrow$ $Y' =$

3	4	3
2	3	
1	2	

$\rightarrow$ $Y'' =$

3
2

In this case,

$$\boldsymbol{d}(Y) = (0, 1, 2, \dot{3}, 2, 2, 1, 1, 0),$$

$$\boldsymbol{d}(Y') = (0,1,2,\underline{\dot{2},1,1,0,0},0),$$

$$\boldsymbol{d}(Y'') = (0,\underline{0,1,\dot{1},0,0},0,0,0),$$

and hence $\boldsymbol{d}(Y) \xrightarrow{0,4} \boldsymbol{d}(Y') \xrightarrow{-2,2} \boldsymbol{d}(Y'')$ where $-2 = 5-3-4$ and $2 = 5-3-0$.

Proof of Lemma 7.14. We set $h := h_Y(i,j)$ and $h' := h_{Y'}(i',j')$. Since $Y'' = Y' \setminus h'$, we see by (3.1) that $\boldsymbol{d}(Y') \xrightarrow{l',r'} \boldsymbol{d}(Y'')$ for some $-m < l' \le r' < n$. Now we show that $l' = m-n-r$ and $r' = n-m-l$. Since $\mathcal{A}_{\alpha_{m,n}}(h') = \mathcal{A}_{\alpha_{m,n}}(h)$ and $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, we have $\#\mathcal{A}_{\alpha_{m,n}}(h') = \#\mathcal{A}_{\alpha_{m,n}}(h) = r-l+1$ by Lemma 7.9. Hence we see that $\boldsymbol{d}(Y') \xrightarrow{a,a+r-l} \boldsymbol{d}(Y'')$ for some $a \in \mathbb{Z}$.

Now it is sufficient to show that $a = n-m-r$. On the contrary, suppose that $a = l$. Note that $\boldsymbol{d}(Y') \xrightarrow{l,r} \boldsymbol{d}(Y'')$. Hence we see by Lemma 7.12 that $d_{l-1}(Y') \nearrow d_l(Y')$. Similarly, since $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, it follows from Lemma 7.12 that $d_{l-1}(Y') \searrow d_l(Y')$. Thus we get $d_{l-1}(Y') \nearrow d_l(Y')$ and $d_{l-1}(Y') \searrow d_l(Y')$, which contradicts Lemma 7.11 (1).

Next, suppose that $a \ne l, n-m-r$. For $k \in \mathbb{Z}$, we define $\mu(k) := \min\{k+m, -k+n\}$. Since $\boldsymbol{d}(Y') \xrightarrow{a,a+r-l} \boldsymbol{d}(Y'')$, we have $\mathcal{A}_{\alpha_{m,n}}(h') = [\alpha_{m,n}(i',j') \mid (i',j') \in h'] = [\min\{j'-i'+m, i'-j'+n\} \mid (i',j') \in h'] = [\mu(k) \mid a \le k \le a+r-l]$. Note that $\min \mathcal{A}_{\alpha_{m,n}}(h) = \min[\min\{j'-i'+m, i'-j'+n\} \mid (i',j') \in h] = \min\{\min\{l+m, l+n\}, \min\{r+m, r+n\}\} = \min\{\mu(l), \mu(r)\}$. We give a proof only for the case that $\mu(l) < \mu(r)$. The proofs for the cases in which $\mu(l) = \mu(r)$ and $\mu(l) > \mu(r)$ are similar. If $l \ge n-m-l$, then

$$\begin{aligned}\mu(l) &= \min\{l+m, -l+n\} = m + \min\{l, -l+n-m\} = n-l \\ &\ge \min\{r+m, (n-l) + \underbrace{(l-r)}_{\le 0}\} = \min\{r+m, -r+n\} = \mu(r),\end{aligned}$$

which is a contradiction. Hence we get $l < n-m-l$ and $\mu(l) = \mu(n-m-l) = l+m$. If $l < a < n-m-r$, then $a+r-l < n-m-l$. Then, we have $\mu(b) = \min\{b+m, -b+n\} = \min\{a+m, -a-r+l+n\} > \min\{l+m, -n+m+l+n\} = l+m = \mu(l)$ for $a \le b \le a+r-l$. Since $\mathcal{A}_{\alpha_{m,n}}(h') = [\mu(k) \mid a \le k \le a+r-l]$, it follows that $\mu(l) \in \mathcal{A}_{\alpha_{m,n}}(h)$ is not contained in $\mathcal{A}_{\alpha_{m,n}}(h')$, which is a contradiction. If $a < l$, then $a+m < l+m < n-m-l+m < -a+n$ and $\mu(a) = \min\{a+m, -a+n\} = a+m < l+m = \mu(l) = \min \mathcal{A}_{\alpha_{m,n}}(h)$. Hence we obtain $\mu(a) \notin \mathcal{A}_{\alpha_{m,n}}(h)$, another contradiction. If $n-m-r < a$, then $a+r-l+m > -l+n > l+m > -a-r+l+n$ and $\mu(a+r-l) = \min\{a+r-l+m, -a-r+l+n\} = -a-r+l+n < l+m = \mu(l) = \min \mathcal{A}_{\alpha_{m,n}}(h)$. Hence we get $\mu(a+r-l) \notin \mathcal{A}_{\alpha_{m,n}}(h)$, yet another contradiction. Thus we obtain $a = n-m-r$, as desired.

Suppose that there exists a box $(i'',j'') \in Y''$ such that $\mathcal{A}_{\alpha_{m,n}}(h'') = \mathcal{A}_{\alpha_{m,n}}(h)$, where $h'' := h_{Y''}(i'',j'')$. Note that $\mathcal{A}_{\alpha_{m,n}}(h'') = \mathcal{A}_{\alpha_{m,n}}(h')$. Since $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$, it follows by the argument above that $\boldsymbol{d}(Y'' \setminus h'')$ is equal to $\boldsymbol{d}(Y'')_{[l,r]}$ or $\boldsymbol{d}(Y'')_{[n-m-r,n-m-l]}$.

If $\boldsymbol{d}(Y'' \setminus h'') = \boldsymbol{d}(Y'')_{[l,r]}$, then we see by Lemma 7.12 that $d_{l-1}(Y'') \nearrow d_l(Y'')$ and $d_r(Y'') \searrow d_{r+1}(Y'')$. Similarly, since $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, it follows from Lemma 7.12 that $d_{l-1}(Y') \searrow d_l(Y')$ and $d_r(Y') \nearrow d_{r+1}(Y')$. Note that $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$. If

$l = n - m - l + 1$, then $r \geq l = n - m - l + 1 \geq n - m - r + 1 > n - m - r - 1$. Thus we see by Lemma 7.12 that $d_{l-1}(Y'') \searrow d_l(Y'')$ or $d_r(Y'') \nearrow d_{r+1}(Y'')$. Thus we have $[d_{l-1}(Y'') \nearrow d_l(Y'')$ and $d_{l-1}(Y'') \searrow d_l(Y'')]$ or $[d_r(Y'') \searrow d_{r+1}(Y'')$ and $d_r(Y'') \nearrow d_{r+1}(Y'')]$, which contradicts Lemma 7.11 (1).

If $\boldsymbol{d}(Y''\backslash h'') = \boldsymbol{d}(Y'')_{[n-m-r,n-m-l]}$, then we see by Lemma 7.12 that $d_{n-m-r-1}(Y'') \nearrow d_{n-m-r}(Y'')$. Similarly, since $\boldsymbol{d}(Y') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y'')$, it follows from Lemma 7.12 that $d_{n-m-r-1}(Y'') \searrow d_{n-m-r}(Y'')$. Thus we get $d_{n-m-r-1}(Y'') \nearrow d_{n-m-r}(Y'')$ and $d_{n-m-r-1}(Y'') \searrow d_{n-m-r}(Y'')$, another contradiction of Lemma 7.11 (1). □

8 An Isomorphism between Rectangular Diagrams

For fixed $m, n \in \mathbb{Z}_{>0}$, it can be easily shown that MHRG(m, n) is isomorphic to MHRG(n, m). In what follows, we assume that $m \leq n$.

Assume that $m + n$ is even. We define $c := (n - m)/2$; note that c is a non-negative integer. Here we will prove that MHRG(m, n) is isomorphic to MHRG$(m, n + 1)$.

Let $\mathcal{T}(Y_{m,n})$ be the subset of $\mathcal{F}(Y_{m,n})$ consisting of all $Y \in \mathcal{F}(Y_{m,n})$ such that there exists a transition from $Y_{m,n}$ to Y, that is, $\mathcal{T}(Y_{m,n}) = \mathcal{C}(\text{MHRG}(m, n))$.

Remark 8.1. We see by Lemma 7.14 that in MHRG(m, n), the operation (M2b) is performed at most once, and the operation (M2c) is never performed.

Let $Y \in \mathcal{T}(Y_{m,n})$ and $Y' \in \mathcal{O}(Y)$. By Lemmas 7.7 and 7.14, there exists $-m < l \leq r < n$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$$

or there exist $-m < l \leq r < n$ and $Y'' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y'') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y').$$

Definition 8.2. We define the map $E : \mathbb{Z}_{\geq 0}^{m+n+1} \to \mathbb{Z}_{\geq 0}^{m+n+2}$ as follows. If $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{m+n+1}$ is

$$\boldsymbol{a} = (a_{-m}, \ldots, a_{c-1}, \underbrace{a_c}_{c\text{-th}}, a_{c+1}, \ldots, a_n),$$

then

$$E(\boldsymbol{a}) := (a_{-m}, \ldots, a_{c-1}, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, a_{c+1}, \ldots, a_n).$$

It can be easily verified that

$$\boldsymbol{a} \in \mathbb{D}_{m,n} \text{ if and only if } E(\boldsymbol{a}) \in \mathbb{D}_{m,n+1}. \tag{8.1}$$

Hence the map $E : \mathbb{Z}_{\geq 0}^{m+n+1} \to \mathbb{Z}_{\geq 0}^{m+n+2}$ induces the map $E : \mathcal{F}(Y_{m,n}) \to \mathcal{F}(Y_{m,n+1})$ as follows. For $Y \in \mathcal{F}(Y_{m,n})$, we define $E(Y)$ to be the unique element of $\mathcal{F}(Y_{m,n+1})$ whose diagonal expression is

$$E(\boldsymbol{d}(Y)) = (d_{-m}(Y), \ldots, d_{c-1}(Y), d_c(Y), d_c(Y), d_{c+1}(Y), \ldots, d_n(Y)).$$

Note that $\boldsymbol{d}(E(Y)) = E(\boldsymbol{d}(Y))$. Notice, also, that $E : \mathcal{F}(Y_{m,n}) \to \mathcal{F}(Y_{m,n+1})$ is an injection. For $l, r \in \mathbb{Z}$, we define $e_l, e_r : \mathbb{Z} \to \mathbb{Z}$ by

$$e_l(k) := \begin{cases} k & \text{if } k \leq c, \\ k+1 & \text{if } k > c, \end{cases} \qquad e_r(k) := \begin{cases} k & \text{if } k < c, \\ k+1 & \text{if } k \geq c. \end{cases}$$

In particular, note that $e_l(k) \neq c+1$ and $e_r(k) \neq c$. The following lemma can be shown easily.

Lemma 8.3. Let $l, r \in \mathbb{Z}$. It holds that $e_l(n-m-k) = n-m+1-e_r(k)$ for $k \in \mathbb{Z}$.

Lemma 8.4. For $l, r \in \mathbb{Z}$ and $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{m+n+1}$, it holds that $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[e_l(l), e_r(r)]}$. Therefore, for $Y \in \mathcal{F}(Y_{m,n})$, it holds that $\boldsymbol{d}(Y)_{[l,r]} \in \mathbb{D}_{m,n}$ if and only if $\boldsymbol{d}(E(Y))_{[e_l(l), e_r(r)]} \in \mathbb{D}_{m,n+1}$.

Proof. If $c < l \leq r$, then $l+1 = e_l(l), r+1 = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, \ldots, a_{l-1}, \underbrace{a_l - 1}_{(l+1)\text{-th}}, \ldots, \underbrace{a_r - 1}_{(r+1)\text{-th}}, a_{r+1}, \ldots).$$

Thus we obtain $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l+1,r+1]} = E(\boldsymbol{a})_{[e_l(l), e_r(r)]}$.

If $l \leq c \leq r$, then $l = e_l(l), r+1 = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, a_{l-1}, \underbrace{a_l - 1}_{l\text{-th}}, \ldots, \underbrace{a_c - 1}_{c\text{-th}}, \underbrace{a_c - 1}_{(c+1)\text{-th}}, \ldots, \underbrace{a_r - 1}_{(r+1)\text{-th}}, a_{r+1}, \ldots).$$

This implies $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l,r+1]} = E(\boldsymbol{a})_{[e_l(l), e_r(r)]}$.

If $l \leq r < c$, then $l = e_l(l), r = e_r(r)$ and

$$E(\boldsymbol{a}_{[l,r]}) = (\ldots, a_{l-1}, \underbrace{a_l - 1}_{l\text{-th}}, \ldots, \underbrace{a_r - 1}_{r\text{-th}}, a_{r+1}, \ldots, \underbrace{a_c}_{c\text{-th}}, \underbrace{a_c}_{(c+1)\text{-th}}, \ldots).$$

And, hence, we obtain $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[l,r]} = E(\boldsymbol{a})_{[e_l(l), e_r(r)]}$.

In all cases above, we have $E(\boldsymbol{a}_{[l,r]}) = E(\boldsymbol{a})_{[e_l(l), e_r(r)]}$ for $-m < l \leq r < n$. Hence, by $\boldsymbol{d}(E(Y)) = E(\boldsymbol{d}(Y))$ and (8.1), we obtain

$$\begin{aligned} \boldsymbol{d}(Y)_{[l,r]} \in \mathbb{D}_{m,n} &\overset{(8.1)}{\Leftrightarrow} E(\boldsymbol{d}(Y)_{[l,r]}) \in \mathbb{D}_{m,n+1} \\ &\Leftrightarrow E(\boldsymbol{d}(Y))_{[e_l(l), e_r(r)]} \in \mathbb{D}_{m,n+1} \\ &\Leftrightarrow \boldsymbol{d}(E(Y))_{[e_l(l), e_r(r)]} \in \mathbb{D}_{m,n+1}, \end{aligned}$$

as desired. □

Lemma 8.5. Let $Y, Y' \in \mathcal{T}(Y_{m,n})$. Assume that $Y \to Y'$ and $E(Y) \in \mathcal{T}(Y_{m,n+1})$. Then, $E(Y') \in \mathcal{T}(Y_{m,n+1})$ and $E(Y) \to E(Y')$.

Proof. Since $Y \to Y'$, it follows from definition that

(a) there exists $-m < l \leq r < n$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$ or

(b) there exist $-m < l \leq r < n$ and $Y'' \in \mathcal{F}(Y_{m,n})$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r}$

$$\boldsymbol{d}(Y'') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(Y').$$

We give a proof only for the case (b); the proof for the case (a) is easier and entirely similar.

By Lemma 8.3, we have $e_l(n-m-r) = n-m+1-e_r(r)$ and $e_r(n-m-l) = n-m+1-e_l(l)$. Thus we have

$$\boldsymbol{d}(E(Y)) \xrightarrow{e_l(l),e_r(r)} \boldsymbol{d}(E(Y'')) \xrightarrow{e_l(n-m-r),e_r(n-m-l)} \boldsymbol{d}(E(Y'))$$

by Lemma 8.4, which implies that $E(Y) \to E(Y')$. Thus we have proved the lemma. □

Let $Y' \in \mathcal{T}(Y_{m,n})$, and let $Y_{m,n} = Y_0 \to Y_1 \to \cdots \to Y_k = Y'$ be a transition from $Y_{m,n}$ to Y' in MHRG(m,n). Note that $E(Y_0) = E(Y_{m,n}) = Y_{m,n+1} \in \mathcal{T}(Y_{m,n+1})$. Also, we see by Lemma 8.5 that for $0 \le p < k$, if $E(Y_p) \in \mathcal{T}(Y_{m,n+1})$, then $E(Y_{p+1}) \in \mathcal{T}(Y_{m,n+1})$. Thus we obtain $E(Y') \in \mathcal{T}(Y_{m,n+1})$ by inductive argument. Therefore, we obtain

$$E(\mathcal{T}(Y_{m,n})) \subset \mathcal{T}(Y_{m,n+1}). \tag{8.2}$$

Moreover, it is obvious from Lemma 8.5 that

$$E(\mathcal{O}(Y)) \subseteq \mathcal{O}(E(Y)) \tag{8.3}$$

for $Y \in \mathcal{T}(Y_{m,n+1})$.

Lemma 8.6. It follows that $d_c(Y) = d_{c+1}(Y)$ for all $Y \in \mathcal{T}(Y_{m,n+1})$.

Proof. Suppose, for a contradiction, that there exists $Y \in \mathcal{T}(Y_{m,n+1})$ such that $d_c(Y) \neq d_{c+1}(Y)$. Let $\mathcal{V} \subset \mathcal{T}(Y_{m,n+1})$ be the subset of $\mathcal{T}(Y_{m,n+1})$ consisting of elements $Y \in \mathcal{T}(Y_{m,n+1})$ such that $d_c(Y) \neq d_{c+1}(Y)$; also, let $Y_0 \in \mathcal{V}$ be such that $\#(Y_0) \geq \#(Y)$ for all $Y \in \mathcal{V}$. Since $c \geq 0$ and $(d_c(Y_0), d_{c+1}(Y_0))$ satisfies the adjacency requirement, we have $d_c(Y_0) = d_{c+1}(Y_0) + 1$ and $d_c(Y_0) \searrow d_{c+1}(Y_0)$.

Since $Y_0 \neq Y_{m,n+1}$, there exists $Y_1 \in \mathcal{T}(Y_{m,n+1})$ such that $Y_1 \to Y_0$. Note that $\#(Y_1) \geq \#(Y_0)$, which implies that $Y_1 \notin \mathcal{V}$ by the maximality of Y_0. Thus we have $d_c(Y_1) = d_{c+1}(Y_1)$ and $d_c(Y_1) \nearrow d_{c+1}(Y_1)$. By Lemma 7.13, we set that for $p = 0, 1$, the number t_p of boxes in Y_p having the number $\hat{\alpha}_{m,n} = (m+n)/2$ is equal to $d_c(Y_p) + d_{c+1}(Y_p)$. Thus, $t_1 - t_0$ is odd. If two hooks are removed in $Y_1 \to Y_0$, then the two hooks have the same multiset of numbers. Thus $t_1 - t_0$ is even, but this contradict the fact that $t_1 - t_0$ is odd. Consequently, one hook is removed in $Y_1 \to Y_0$. Hence $d_c(Y_0) = d_c(Y_1)$ and $d_{c+1}(Y_0) = d_{c+1}(Y_1) - 1$ by $0 \leq d_k(Y_1) - d_k(Y_0) \leq 1$ for $-m \leq k \leq n$. Also, there exists $c+1 \leq k = k(Y_1) < n+1$ such that $\boldsymbol{d}(Y_1) \xrightarrow{c+1,k} \boldsymbol{d}(Y_0)$ and $\boldsymbol{d}(Y_0)_{[n+1-m-k,c]} \notin \mathbb{D}_{m,n+1}$. Note that $n+1-m-(c+1) = c$. By Lemma 7.12, we have $d_{n-m-k}(Y_1) \searrow d_{n+1-m-k}(Y_1)$, $d_c(Y_1) \nearrow d_{c+1}(Y_1)$, and $d_k(Y_1) \searrow d_{k+1}(Y_1)$. Now we choose Y_1 such that $k = k(Y_1)$ is maximum.

Suppose that $Y_1 = Y_{m,n+1}$. In this case, we have $d_p(Y_1) \nearrow d_{p+1}(Y_1)$ for $-m \leq p < n-m$, and $d_p(Y_1) \searrow d_{p+1}(Y_1)$ for $n-m \leq p \leq n$. Since $c \leq n-m$, we have $d_{n-m-k}(Y_1) \nearrow d_{n+1-m-k}(Y_1)$ and $d_{n-m-k}(Y_0) \nearrow d_{n+1-m-k}(Y_0)$ by Lemma 7.12. Thus we have $\boldsymbol{d}(Y_0)_{[n+1-m-k,c]} \in \mathbb{D}_{m,n+1}$ by Lemma 7.12, which is a contradiction. Hence we obtain $Y_1 \neq Y_{m,n+1}$. Then, there exists $Y_2 \in \mathcal{T}(Y_{m,n+1})$ such that $Y_2 \to Y_1$. Note that $d_c(Y_2) = d_{c+1}(Y_2)$ and $d_c(Y_2) \nearrow d_{c+1}(Y_2)$.

Suppose that $d_{n-m-k}(Y_2) \searrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \searrow d_{k+1}(Y_2)$. By Lemma

7.12, we have $\boldsymbol{d}(Y_2)_{[c+1,k]} \in \mathbb{D}_{m,n+1}$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,k]}$; also, notice that $d_c(Y_1') \neq d_{c+1}(Y_1')$ and $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[n+1-m-k,c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$ and hence $Y_1' \in \mathcal{V}$. Since $\#(Y_1) - \#(Y_0) = \#(Y_2) - \#(Y_1')$, we have $\#(Y_1') = \#(Y_2) - \#(Y_1) + \#(Y_0) > \#(Y_0)$ which contradicts the maximality of Y_0.

Suppose that $d_{n-m-k}(Y_2) \searrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \nearrow d_{k+1}(Y_2)$. If two hooks are removed in $Y_2 \to Y_1$, then there exist $-m < l \leq r < n$ and $Y' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n+1-m-l} \boldsymbol{d}(Y_1).$$

Since $d_k(Y_2) \nearrow d_{k+1}(Y_2)$ and $d_k(Y_1) \searrow d_{k+1}(Y_1)$, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n-m-k} \boldsymbol{d}(Y_1)$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{l,n-m-k} \boldsymbol{d}(Y') \xrightarrow{k+1,n+1-m-l} \boldsymbol{d}(Y_1).$$

Thus we have $d_{n-m-k}(Y_1) \nearrow d_{n+1-m-k}(Y_1)$, another contradiction. Hence one hook is removed in $Y_2 \to Y_1$. Then there exist $p \geq k+1$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,p} \boldsymbol{d}(Y_1).$$

Note that $\boldsymbol{d}(Y_1)_{[n+1-m-p,n-m-k]} \notin \mathbb{D}_{m,n+1}$, also, that $d_{n-m-p}(Y_2) \searrow d_{n+1-m-p}(Y_2)$ and $d_p(Y_2) \searrow d_{p+1}(Y_2)$. By Lemma 7.12, we have $\boldsymbol{d}(Y_2)_{[c+1,p]} \in \mathbb{D}_{m,n+1}$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,p]}$; notice that $d_c(Y_1') \neq d_{c+1}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[n+1-m-p,c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$. Hence $Y_1' = \boldsymbol{d}(Y_2)_{[c+1,p]} = (\boldsymbol{d}(Y_2)_{[k+1,p]})_{[c+1,k]} = Y_0$ which contradicts the maximality of k.

Suppose that $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \searrow d_{k+1}(Y_2)$. If two hooks are removed in $Y_2 \to Y_1$, then there exist $-m < l \leq r < n$ and $Y' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,n+1-m-l} \boldsymbol{d}(Y_1).$$

Since $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_{n-m-k}(Y_1) \searrow d_{n+1-m-k}(Y_1)$, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,r} \boldsymbol{d}(Y') \xrightarrow{n+1-m-r,k} \boldsymbol{d}(Y_1)$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{l,k} \boldsymbol{d}(Y') \xrightarrow{n+1-m-k,n+1-m-l} \boldsymbol{d}(Y_1).$$

Thus we have $d_k(Y_1) \nearrow d_{k+1}(Y_1)$, another contradiction, hence one hook is removed in $Y_2 \to Y_1$. Consequently, there exist $p \geq n+1-m-k$ such that

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,p} \boldsymbol{d}(Y_1).$$

Note that $\boldsymbol{d}(Y_1)_{[n+1-m-p,k]} \notin \mathbb{D}_{m,n+1}$ and $d_{n-m-p}(Y_2) \searrow d_{n+1-m-p}(Y_2)$, $d_p(Y_2) \searrow d_{p+1}(Y_2)$. Since $d_c(Y_2) \nearrow d_{c+1}(Y_2)$, we have $p \neq c$. By Lemma 7.12, we have $\boldsymbol{d}(Y_2)_{[c+1,\max(p,n+1-m-p)]} \in \mathbb{D}_{m,n+1}$. Also, notice that $c+1 \leq \max(p, n+1-m-p)$ since $p \neq c$. Let $Y_1' \in \mathcal{F}(Y_{m,n+1})$ be the Young diagram whose diagonal expression is equal to $\boldsymbol{d}(Y_2)_{[c+1,\max(p,n+1-m-p)]}$; note that $d_c(Y_1') \neq d_{c+1}(Y_1')$. Since $\boldsymbol{d}(Y_1')_{[\min\{p,n+1-m-p\},c]} \notin \mathbb{D}_{m,n+1}$, it follows that $Y_1' \in \mathcal{O}(Y_2)$ and hence $Y_1' \in \mathcal{V}$. If $\max(p, n+1-m-p) \leq k$, then $\#(Y_1) - \#(Y_0) \geq \#(Y_2) - \#(Y_1')$ and hence $\#(Y_1') \geq \#(Y_2) - \#(Y_1) + \#(Y_0) > \#(Y_0)$. If $\max(p, n+1-m-p) > k$, then $\#(Y_2) - \#(Y_1) > \#(Y_2) - \#(Y_1')$ and, hence, $\#(Y_1') > \#(Y_1) > \#(Y_0)$. In any case, we obtain $\#(Y_1') > \#(Y_0)$, which contradicts the maximality of Y_0.

Suppose that $d_{n-m-k}(Y_2) \nearrow d_{n+1-m-k}(Y_2)$ and $d_k(Y_2) \nearrow d_{k+1}(Y_2)$. Let $Y_1' \in \mathcal{O}(Y_2)$. If $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$ and $d_k(Y_1') \searrow d_{k+1}(Y_1')$, then by Lemma 7.12, we have

$$\boldsymbol{d}(Y_2) \xrightarrow{n+1-m-k,n-m-k} \boldsymbol{d}(Y') \xrightarrow{k+1,k} \boldsymbol{d}(Y_1')$$

or

$$\boldsymbol{d}(Y_2) \xrightarrow{k+1,k} \boldsymbol{d}(Y') \xrightarrow{n+1-m-k,n-m-k} \boldsymbol{d}(Y_1')$$

for $Y' \in \mathcal{F}(Y_{m,n})$, which is a contradiction. Thus there exists no option $Y_1' \in \mathcal{O}(Y_2)$ such that $d_{n-m-k}(Y_1') \searrow d_{n+1-m-k}(Y_1')$ and $d_k(Y_1') \searrow d_{k+1}(Y_1')$, which contradicts $Y_2 \to Y_1$. Thus we have proved Lemma 8.6. $\square$

Theorem 8.7. Let $m, n \in \mathbb{Z}_{>0}$ be such that $m \leq n$ and $m+n$ is even. Then the map E gives an isomorphism from $\mathrm{MHRG}(m,n)$ to $\mathrm{MHRG}(m,n+1)$. Therefore, for each $Y \in \mathcal{T}(Y_{m,n})$, it holds that $\mathcal{G}(Y) = \mathcal{G}(E(Y))$. In particular, $\mathcal{G}(Y_{m,n})$ in $\mathrm{MHRG}(m,n)$ is equal to $\mathcal{G}(Y_{m,n+1})$ in $\mathrm{MHRG}(m,n+1)$.

Proof. We have shown that the map $E : \mathcal{T}(Y_{m,n}) \to \mathcal{T}(Y_{m,n+1})$ is injective (see (8.2)) and $E(\mathcal{O}(Y)) \subseteq \mathcal{O}(E(Y))$ for $Y \in \mathcal{T}(Y_{m,n})$ (see (8.3)). Hence it remains to show that $E(\mathcal{O}(Y)) \supseteq \mathcal{O}(E(Y))$ for $Y \in \mathcal{T}(Y_{m,n})$ and $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$.

We first show that $E(\mathcal{O}(Y)) \supseteq \mathcal{O}(E(Y))$. Let $Y \in \mathcal{T}(Y_{m,n})$, and let $X \in \mathcal{O}(E(Y))$. There exists $-m < l \leq r < n$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(X) \tag{a}$$

or there exist $-m < l \leq r < n$ and $X' \in \mathcal{F}(Y_{m,n})$ such that

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(X') \xrightarrow{n-m-r,n-m-l} \boldsymbol{d}(X). \tag{b}$$

By Lemma 8.6, we have $d_c(E(Y)) \nearrow d_{c+1}(E(Y))$ and $r \neq c$.

In the first case (a), we get $\boldsymbol{d}(X)_{[n+1-m-r,n+1-m-l]} \notin \mathbb{D}_{m,n+1}$. If $l = c+1$, then $d_c(X) \searrow d_{c+1}(X)$ and $d_c(X) > d_{c+1}(X)$, which contradicts Lemma 8.6. If $l \neq c+1$, then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = l, e_r(r_0) = r$. By Lemma 8.4, we have $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$ and $(\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \notin \mathbb{D}_{m,n}$. Thus the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$ whose diagonal expression is equal to $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$ is an option of Y. By the proof of Lemma 8.5, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$.

Consider the second case (b). If $l = c+1$, then $\boldsymbol{d}(X) = (\boldsymbol{d}(E(Y))_{[c+1,r]})_{[n+1-m-r,c]} = \boldsymbol{d}(E(Y))_{[n+1-m-r,r]}$. Then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = n+1-m-r, e_r(r_0) = r$. By Lemma 8.3, we have $e_l(n-m-r_0) = e_l(n-m-r_0) + e_r(r_0) - r = n-m+1-r = e_l(l_0)$ and hence $l_0 = n-m-r_0$. By Lemma 8.4, we have $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$ and $(\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} = \boldsymbol{d}(Y)_{[l_0,r_0]})_{[l_0,r_0]} \notin \mathbb{D}_{m,n}$. Thus the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$ whose diagonal expression is equal to $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$ is an option of Y. By the proof of Lemma 8.5, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$. If $l \neq c+1$, then there exist $-m < l_0 \leq r_0 < n$ such that $e_l(l_0) = l, e_r(r_0) = r$. By Lemma 8.4, we have $\boldsymbol{d}(Y)_{[l_0,r_0]} \in \mathbb{D}_{m,n}$ and $(\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \in \mathbb{D}_{m,n}$. Thus the Young diagram $Y' \in \mathcal{T}(Y_{m,n})$ whose diagonal expression is equal to $(\boldsymbol{d}(Y)_{[l_0,r_0]})_{[n-m-r_0,n-m-l_0]} \in \mathbb{D}_{m,n}$ is an option of Y. By the proof of Lemma 8.5, we obtain $X = E(Y') \in E(\mathcal{O}(Y))$. In any case, we obtain $X \in E(\mathcal{O}(Y))$, as desired.

We next show that $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$. Let $X' \in \mathcal{T}(Y_{m,n+1})$, and let $Y_{m,n+1} = X_0 \to X_1 \to \cdots \to X_k = X'$ be a transition from $Y_{m,n+1}$ to X' in MHRG$(m, n+1)$. We show by induction on k that $X' \in E(\mathcal{T}(Y_{m,n}))$. If $k = 0$, then $X' = Y_{m,n+1} = E(Y_{m,n}) \in E(\mathcal{T}(Y_{m,n}))$. Assume that $k > 0$; note that $X_{k-1} \in E(\mathcal{T}(Y_{m,n}))$ by the induction hypothesis. Let $X'_{k-1} \in \mathcal{T}(Y_{m,n})$ be such that $X_{k-1} = E(X'_{k-1})$. Since $E(\mathcal{O}(X'_{k-1})) = \mathcal{O}(E(X'_{k-1})) = \mathcal{O}(X_{k-1})$ as shown above, we get $X' = X_k \in \mathcal{O}(X_{k-1}) = E(\mathcal{O}(X'_{k-1})) \subset E(\mathcal{T}(Y_{m,n}))$, as desired. Therefore, we conclude that $E(\mathcal{T}(Y_{m,n})) \supset \mathcal{T}(Y_{m,n+1})$ and hence $E(\mathcal{T}(Y_{m,n})) = \mathcal{T}(Y_{m,n+1})$. This completes the proof of Theorem 8.7. □

9 Sprague-Grundy Values of the Starting Position of MHRG(m, n) with $m = 1$ or 2

9.1 Case of MHRG$(1, n)$

Theorem 9.1. Let $m = 1$ and $n \in \mathbb{Z}_{>0}$. In MHRG$(1, n)$,

$$\mathcal{T}(Y_{1,n}) = \begin{cases} \mathcal{F}(Y_{1,n}) & \text{if } n \text{ is odd,} \\ \mathcal{F}(Y_{1,n}) \setminus \{Y_{1,\frac{n}{2}}\} & \text{if } n \text{ is even.} \end{cases}$$

Moreover, for $0 \leq l \leq n$ such that $Y_{1,l} \in \mathcal{T}(Y_{1,n})$,

$$\mathcal{G}(Y_{1,l}) = \begin{cases} l & \text{if } n \text{ is odd,} \\ l & \text{if } n \text{ is even and } l < n/2, \\ l-1 & \text{if } n \text{ is even and } n/2 < l. \end{cases}$$

In particular,

$$\mathcal{G}(Y_{1,n}) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Theorem 8.7, it suffices to show the assertion for the case that n is odd.

We set $k = (n+1)/2 \in \mathbb{Z}_{>0}$. We see that for $0 \leq l \leq n$, the unimodal numbering of $Y_{1,l} \in \mathcal{F}(Y_{1,n})$ is as follows:

1	2	$\cdots$	$l-1$	l

if $0 \leq l \leq k$,

1	2	$\cdots$	$k-1$	k	$k-1$	$\cdots$	$n+2-l$	$n+1-l$

if $k < l \leq n$.

By this fact, we deduce that in MHRG$(1, n)$ (with odd n), the operation removing two hooks never takes place. Hence, we obtain $\mathcal{O}(Y_{1,l}) = \{Y_{1,i} \mid 0 \leq i < l\}$ for all $0 \leq l \leq n$. The assertion of the theorem follows immediately from the latter and the definition of the $\mathcal{G}$-value. □

9.2 Case of MHRG$(2, n)$

Let $m = 2$ and $n \geq 2$. Recall that $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2)$ denotes the Young diagram having $\boldsymbol{k}_1$ boxes in the 1st row and $\boldsymbol{k}_2$ boxes in the 2nd row. If n is even, then MHRG$(2, n)$ is isomorphic to MHRG$(2, n+1)$ (see Theorem 8.7). Thus it suffices to study the case in which n is even; we set $n' := n/2 \in \mathbb{Z}_{>0}$.

Lemma 9.2. Let $(\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ and $\boldsymbol{k}'_1, \boldsymbol{k}'_2 \in \mathbb{Z}_{\geq 0}$ with $2n' \geq \boldsymbol{k}'_1 \geq \boldsymbol{k}'_2 \geq 0$. Then $(\boldsymbol{k}_1, \boldsymbol{k}_2) = (\boldsymbol{k}'_1, \boldsymbol{k}'_2)$ if and only if $d_{\boldsymbol{k}'_1-1}(Y) \searrow d_{\boldsymbol{k}'_1}(Y)$, $d_{\boldsymbol{k}'_2-2}(Y) \searrow d_{\boldsymbol{k}'_2-1}(Y)$, and $d_{k-1}(Y) \nearrow d_k(Y)$ for $-2 < k \leq 2n' = n$ with $k \neq \boldsymbol{k}'_1, \boldsymbol{k}'_2 - 1$.

Proof. If $\boldsymbol{k}_2 = 0$, then $(\boldsymbol{k}_1, 0) = (\boldsymbol{k}'_1, \boldsymbol{k}'_2)$ if and only if $\boldsymbol{k}'_2 = 0$, $d_{-1}(Y) = 0$, $d_k(Y) = 1$ for $0 \leq k < \boldsymbol{k}'_1$, and $d_k(Y) = 0$ for $\boldsymbol{k}'_1 \leq k < 2n'$. The latter is equivalent to $d_{\boldsymbol{k}'_1-1}(Y) \searrow d_{\boldsymbol{k}'_1}(Y)$, $d_{\boldsymbol{k}'_2-2}(Y) \searrow d_{\boldsymbol{k}'_2-1}(Y)$, and $d_{k-1}(Y) \nearrow d_k(Y)$ for $-2 < k \leq 2n' = n$ with $k \neq \boldsymbol{k}'_1, \boldsymbol{k}'_2 - 1$.

If $\boldsymbol{k}_2 > 0$, then $(\boldsymbol{k}_1, \boldsymbol{k}_2) = (\boldsymbol{k}'_1, \boldsymbol{k}'_2)$ if and only if $d_{-1}(Y) = 1$, $d_k(Y) = 2$ for $0 \leq k < \boldsymbol{k}'_2 - 1$, $d_k(Y) = 1$ for $\boldsymbol{k}'_2 - 1 \leq k < \boldsymbol{k}'_1$, and $d_k(Y) = 0$ for $\boldsymbol{k}'_1 \leq k < 2n'$. The latter is equivalent to $d_{\boldsymbol{k}'_1-1}(Y) \searrow d_{\boldsymbol{k}'_1}(Y)$, $d_{\boldsymbol{k}'_2-2}(Y) \searrow d_{\boldsymbol{k}'_2-1}(Y)$, and $d_{k-1}(Y) \nearrow d_k(Y)$ for $-2 < k \leq 2n' = n$ with $k \neq \boldsymbol{k}'_1, \boldsymbol{k}'_2 - 1$.

Thus we have proved the lemma. □

Lemma 9.3. Let $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ and $(i, j) \in Y$. Also, set $Y' = (\boldsymbol{k}'_1, \boldsymbol{k}'_2) = Y \setminus h_Y(i, j)$. Then, $\boldsymbol{k}'_1 + \boldsymbol{k}'_2 = 2n'$ if and only if there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i, j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i', j'))$. In this case, $Y'' := Y' \setminus h'_Y(i', j')$ is equal to $(2n' - \boldsymbol{k}_2, 2n' - \boldsymbol{k}_1)$.

Proof. We first show the "if" part. By Lemma 7.7, there exist $-2 < l, r < 2n'$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$. If there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i, j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i', j'))$, then it follows from Lemma 7.14 that $\boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-l]} \in \mathbb{D}_{2,2n'}$. Note that $2n' - 2 - l + 1 \neq l$. By Lemmas 7.12 and 9.2, the pair $(r, 2n' - 2 - l)$ is equal to $(\boldsymbol{k}_1 - 1, \boldsymbol{k}_2 - 2)$ or $(\boldsymbol{k}_2 - 2, \boldsymbol{k}_1 - 1)$, and hence $\boldsymbol{k}'_1 + \boldsymbol{k}'_2 = (\boldsymbol{k}_1 + \boldsymbol{k}_2) - (r - l + 1) = (\boldsymbol{k}_1 + \boldsymbol{k}_2) - (2 - 2n' + \boldsymbol{k}_1 - 1 + \boldsymbol{k}_2 - 2 + 1) = (\boldsymbol{k}_1 + \boldsymbol{k}_2) - (2n' + \boldsymbol{k}_1 + \boldsymbol{k}_2) = 2n'$.

We next show the "only if" part. As above, assume that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$. If $\boldsymbol{k}'_1 + \boldsymbol{k}'_2 = 2n'$, then $d_{\boldsymbol{k}'_1-1}(Y') \searrow d_{\boldsymbol{k}'_1}(Y'), d_{2n'-\boldsymbol{k}'_1-2}(Y') \searrow d_{2n'-\boldsymbol{k}_1-1}(Y')$, and $d_{k-1}(Y') \nearrow d_k(Y')$ for $-2 < k \leq 2n'$ with $k \neq \boldsymbol{k}'_1, 2n' - \boldsymbol{k}'_1 - 1$. By Lemma 7.12, we have $l = \boldsymbol{k}'_1$ or $l = 2n' - \boldsymbol{k}'_1 - 1$. If $l = \boldsymbol{k}'_1$, then $r \neq 2n' - \boldsymbol{k}'_1 - 2$ and hence $2n' - 2 - r \neq \boldsymbol{k}'_1$. Thus, $\boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-l]} = \boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-\boldsymbol{k}'_1]} \in \mathbb{D}_{2,2n'}$. If $l = 2n' - \boldsymbol{k}'_1 - 1$, then $r \neq \boldsymbol{k}'_1 - 1$ and hence $2n' - 2 - r \neq 2n' - 1 - \boldsymbol{k}'_1$. Thus, $\boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-l]} = \boldsymbol{d}(Y')_{[2n'-2-r, \boldsymbol{k}'_1-1]} \in \mathbb{D}_{2,2n'}$. In both cases, we have $\boldsymbol{d}(Y')_{[2n'-2-r, 2n'-2-l]} \in \mathbb{D}_{2,2n'}$, which implies that there exists a box $(i', j') \in Y'$ such that $\mathcal{A}_{\alpha_{2,n}}(h_Y(i, j)) = \mathcal{A}_{\alpha_{2,n}}(h_{Y'}(i', j'))$ (see Lemma 7.14).

Finally, let us show that $Y'' := Y' \setminus h_{Y'}(i', j')$ is equal to $(2n' - \boldsymbol{k}_2, 2n' - \boldsymbol{k}_1)$. By Lemma 7.12, we have

$$\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y') \xrightarrow{2n'-2-r, 2n'-2-l} \boldsymbol{d}(Y''),$$

and $d_{l-1}(Y'') \searrow d_l(Y''), d_{2n'-2-r-1}(Y'') \searrow d_{2n'-2-r}(Y'')$, and $d_k(Y'') \nearrow d_k(Y'')$ for

$-2 < k \leq 2n'$ with $k \neq l, 2n'-2-r$. As seen above, the pair $(r, 2n'-2-l)$ is equal to $(\boldsymbol{k}_1-1, \boldsymbol{k}_2-2)$ or $(\boldsymbol{k}_2-2, \boldsymbol{k}_1-1)$. If $(r, 2n'-2-l) = (\boldsymbol{k}_1-1, \boldsymbol{k}_2-2)$, then $l = 2n'-\boldsymbol{k}_2 > 2n'-1-\boldsymbol{k}_1 = 2n'-2-l$. Otherwise, if $(r, 2n'-2-l) = (\boldsymbol{k}_2-2, \boldsymbol{k}_1-1)$, then $l = 2n'-1-\boldsymbol{k}_1 < 2n'-\boldsymbol{k}_2 = 2n'-2-l$. In both cases, we get $d_{2n'-2-\boldsymbol{k}_1}(Y'') \searrow d_{2n'-1-\boldsymbol{k}_1}(Y''), d_{2n'-\boldsymbol{k}_2-1}(Y'') \searrow d_{2n'-\boldsymbol{k}_2}(Y'')$, and $d_k(Y'') \nearrow d_k(Y'')$ for $-2 < k \leq 2n'$ with $k \neq 2n'-1-\boldsymbol{k}_1$ and $k \neq 2n'-\boldsymbol{k}_2$. Hence we obtain $Y'' = (2n'-\boldsymbol{k}_2, 2n'-\boldsymbol{k}_1)$ by Lemma 9.2, as desired. $\square$

For $Y \in \mathcal{F}(Y_{2,2n'})$, we set $OH(Y) := \{Y \setminus h_Y(i,j) \mid (i,j) \in Y\}$. If $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2)$, then

$$OH(Y) = \{(\boldsymbol{k}_1', \boldsymbol{k}_2) \mid \boldsymbol{k}_2 \leq \boldsymbol{k}_1' < \boldsymbol{k}_1\} \cup \{(\boldsymbol{k}_1, \boldsymbol{k}_2') \mid 0 \leq \boldsymbol{k}_2' < \boldsymbol{k}_2\} \\ \cup \{(\boldsymbol{k}_2-1, \boldsymbol{k}_1') \mid 0 \leq \boldsymbol{k}_1' < \boldsymbol{k}_2\}.$$

By Lemma 9.3, we can easily show the following lemma.

Lemma 9.4. In MHRG$(2, 2n')$,

$$\mathcal{T}(Y_{2,2n'}) = \mathcal{F}(Y_{2,2n'}) \setminus \{(\boldsymbol{k}_1', \boldsymbol{k}_2') \in \mathcal{F}(Y_{2,2n'}) \mid \boldsymbol{k}_1' + \boldsymbol{k}_2' = 2n'\}.$$

Moreover, for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$,

(1) if $\boldsymbol{k}_1 + \boldsymbol{k}_2 < 2n'$, then $\mathcal{O}(Y) = OH(Y)$;
(2) if $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$, then $\mathcal{O}(Y) = OH(Y) \setminus \{(\boldsymbol{k}_1', \boldsymbol{k}_2') \in \mathcal{F}(Y_{2,2n'}) \mid \boldsymbol{k}_1' + \boldsymbol{k}_2' = 2n'\} \cup \{(2n'-\boldsymbol{k}_2, 2n'-\boldsymbol{k}_1)\}$.

By Lemma 9.4, the $\mathcal{G}$-value of $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 < n = 2n'$ is equal to the $\mathcal{G}$-value of the game position corresponding to Y in Sato-Welter game (see, e.g., [10, Theorem 2]). For later use, we list those $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 < 2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

$\mathcal{G}(Y) = 0$	$\mathcal{G}(Y) = 1$	$\mathcal{G}(Y) = 2$
$(2i, 2i)$	$(1+4i, 4i)$ $(2+4i, 1+4i)$	$(2+4i, 4i)$ $(1+4i, 1+4i)$

Table 6 $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 < 2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

Theorem 9.5. As above, assume that n is even, and set $n' = n/2$. In MHRG$(2, 2n')$, the list of those $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$ whose $\mathcal{G}$-values are 0,1 or 2 is given by Table 7.

Proof. We give a proof only for the case of $n' = 4n''$ for $n'' \in \mathbb{Z}_{>0}$; the proofs of the cases $n' = 4n''+1, 4n''+2, 4n''+3$ for $n'' \in \mathbb{Z}_{\geq 0}$ are similar. We set $G_k := \{(\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'}) \mid \boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n', \mathcal{G}((\boldsymbol{k}_1, \boldsymbol{k}_2)) = k\}$ for $k \in \mathbb{Z}_{\geq 0}$.

First, we determine G_0. Let $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$. If $\boldsymbol{k}_2 < n'$ and $\boldsymbol{k}_2$ is even (resp., odd), then we deduce that $Y' = (\boldsymbol{k}_2, \boldsymbol{k}_2)$ (resp., $Y' = (\boldsymbol{k}_2-1, \boldsymbol{k}_2-1)$) is contained in $\mathcal{O}(Y)$. Since $\mathcal{G}(Y') = 0$ by Table 6, we obtain $Y \notin G_0$.

n'	$\mathcal{G}(Y)=0$	$\mathcal{G}(Y)=1$	$\mathcal{G}(Y)=2$
$4n''$	$(n'+1+4i, n'+4i)$ $(n'+2+4i, n'+1+4i)$	$(n'+2, n')$ $(n'+1, n'+1)$ $(n'+4+2i, n'+4+2i)$	$(n'+2, n'+2)$ $(n'+3, n')$ $(n'+4, n'+1)$ $(n'+7+4i, n'+6+4i)$ $(n'+8+4i, n'+7+4i)$
$4n''+1$	$(n'+2+4i, n'+1+4i)$ $(n'+3+4i, n'+2+4i)$	$(n'+2+2i, n'+2i)$	$(n'+1, n')$ $(n'+2, n'-1)$ $(n'+3, n'+1)$ $(n'+5+2i, n'+5+2i)$
$4n''+2$	$(n'+1+4i, n'+4i)$ $(n'+2+4i, n'+1+4i)$	$(n'+2+2i, n'+2+2i)$	$(n'+3+4i, n'+2+4i)$ $(n'+4+4i, n'+3+4i)$
$4n''+3$	$(n'+2+4i, n'+1+4i)$ $(n'+3+4i, n'+2+4i)$	$(n'+1+2i, n'+1+2i)$	$(n'+4+8i, n'+1+8i)$ $(n'+5+8i, n'+2+8i)$ $(n'+6+8i, n'+3+8i)$ $(n'+7+8i, n'+4+8i)$

Table 7 $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$ whose $\mathcal{G}$-values are 0, 1, or 2.

Now, we see by Lemma 9.4 that

$$\begin{aligned}
\mathcal{O}((n'+1, n')) &= \Big(\{(n', n')\} \cup \{(n'+1, \boldsymbol{k}'_2) \mid 0 \le \boldsymbol{k}'_2 < n'\} \\
&\quad \cup \{(n'-1, \boldsymbol{k}'_1) \mid 0 \le \boldsymbol{k}'_1 < n'\}\Big) \\
&\quad \setminus \{(\boldsymbol{k}'_1, \boldsymbol{k}'_2) \mid \boldsymbol{k}'_1 + \boldsymbol{k}'_2 = 2n'\} \cup \{(n', n'-1)\} \\
&= \{(n'+1, \boldsymbol{k}'_2) \mid 0 \le \boldsymbol{k}'_2 < n'-1\} \\
&\quad \cup \{(n'-1, \boldsymbol{k}'_1) \mid 0 \le \boldsymbol{k}'_1 < n'\} \cup \{(n', n'-1)\}.
\end{aligned}$$

Note that $n' = 4n''$ is even. By Table 6 and the argument above, it can be seen that $\mathcal{O}((n'+1, n'))$ has no position whose $\mathcal{G}$-value is 0. Thus we get $\mathcal{G}((n'+1, n')) = 0$. If $Y \in \{(n'+1, n'+1)\} \cup \{(\boldsymbol{k}'_1, n') \mid n'+2 \le \boldsymbol{k}'_1 \le 2n'\} \cup \{(\boldsymbol{k}'_1, n'+2) \mid n'+2 \le \boldsymbol{k}'_1 \le 2n'\}$, then $(n'+1, n') \in \mathcal{O}(Y)$, which implies that $Y \notin G_0$.

Similarly, we see by Lemma 9.4 that

$$\begin{aligned}
\mathcal{O}((n'+2, n'+1)) &= \Big(\{(n'+1, n'+1)\} \cup \{(n'+2, \boldsymbol{k}'_2) \mid 0 \le \boldsymbol{k}'_2 < n'+1\} \\
&\quad \cup \{(n', \boldsymbol{k}'_1) \mid 0 \le \boldsymbol{k}'_1 < n'+1\}\Big) \\
&\quad \setminus \{(\boldsymbol{k}'_1, \boldsymbol{k}'_2) \mid \boldsymbol{k}'_1 + \boldsymbol{k}'_2 = 2n'\} \cup \{(n'-1, n'-2)\} \\
&= \{(n'+1, n'+1)\} \\
&\quad \cup \Big(\{(n'+2, \boldsymbol{k}'_2) \mid 0 \le \boldsymbol{k}'_2 < n'+1\} \setminus \{(n'+2, n'-2)\}\Big) \\
&\quad \cup \{(n', \boldsymbol{k}'_1) \mid 0 \le \boldsymbol{k}'_1 < n'\} \cup \{(n'-1, n'-2)\}.
\end{aligned}$$

By Table 6 and the argument above, we deduce that $\mathcal{O}((n'+2, n'+1))$ has no position whose $\mathcal{G}$-value is 0. Thus we obtain $\mathcal{G}((n'+2, n'+1)) = 0$. If $Y \in \{(n'+2, n'+$

$2)\} \cup \{(\boldsymbol{k}_1', n'+1) \mid n'+3 \le \boldsymbol{k}_1' \le 2n'\} \cup \{(\boldsymbol{k}_1', n'+3) \mid n'+3 \le \boldsymbol{k}_1' \le 2n'\}$, then $(n'+2, n'+1) \in \mathcal{O}(Y)$, which implies that $Y \notin G_0$. Therefore, for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $n' \le \boldsymbol{k}_2 \le n'+3$ and $\boldsymbol{k}_2 \le \boldsymbol{k}_1 \le 2n'$,

$$Y \in G_0 \text{ if and only if } Y = [(n'+1, n'), (n'+2, n'+1)]. \tag{9.1}$$

Let $i \in \mathbb{Z}_{>0}$ with $n'+4+4i \le 2n'$. By Lemma 9.4, $(n'+4+4i, n'+4+4i) \to (n'-4-4i, n'-4-4i)$. Since $\mathcal{G}((n'-4-4i, n'-4-4i)) = \mathcal{G}((4n''-4-4i, 4n''-4-4i)) = 0$ by Table 6, we obtain $\mathcal{G}((n'+4+4i, n'+4+4i)) \neq 0$. Furthermore, in the same way that (9.1) was obtained, it can be verified that for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4i \le \boldsymbol{k}_2 \le n'+3+4i$ and $\boldsymbol{k}_2 \le \boldsymbol{k}_1 \le 2n'$, $Y \in G_0$ if and only if $Y = [(n'+1+4i, n'+4i), (n'+2+4i, n'+1+4i)]$. Therefore, we obtain

$$\begin{aligned} G_0 = \Big(&\{(n'+1+4i, n'+4i) \mid i \ge 0\} \cup \{(n'+2+4i, n'+1+4i) \mid i \ge 0\}\Big) \\ &\cap \mathcal{F}(Y_{2,2n'}), \end{aligned}$$

as desired.

Next, we determine G_1. Let $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$. Similar to the determination of G_0, if $\boldsymbol{k}_2 < n'$, then $Y \notin G_1$. By Table 6 and $\mathcal{G}((n'+1, n')) = 0$, we deduce that $\mathcal{O}((n'+2, n'))$ and $\mathcal{O}((n'+1, n'+1))$ have no position whose $\mathcal{G}$-value is 1, but we have a position $(n'+1, n')$, whose $\mathcal{G}$-value is 0. Thus we get $\mathcal{G}((n'+2, n')) = \mathcal{G}((n'+1, n'+1)) = 1$. If $Y \in \{(\boldsymbol{k}_1', n') \mid n'+2 \le \boldsymbol{k}_1' \le 2n'\} \cup \{(\boldsymbol{k}_1', n'+1) \mid n'+1 \le \boldsymbol{k}_1' \le 2n'\} \cup \{(\boldsymbol{k}_1', n'+2) \mid n'+1 \le \boldsymbol{k}_1' \le 2n'\} \cup \{(\boldsymbol{k}_1', n'+3) \mid n'+2 \le \boldsymbol{k}_1' \le 2n'\}$, then $(n'+2, n') \in \mathcal{O}(Y)$ or $(n'+1, n'+1) \in \mathcal{O}(Y)$, which implies that $Y \notin G_1$. Therefore, for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $n' \le \boldsymbol{k}_2 \le n'+3$ and $\boldsymbol{k}_2 \le \boldsymbol{k}_1 \le 2n'$, $Y \in G_1$ if and only if $Y = (n'+2, n'), (n'+1, n'+1)$.

We see by Lemma 9.4 that

$$\begin{aligned} \mathcal{O}((n'+4, n'+4)) = \Big(&\{(n'+4, \boldsymbol{k}_2') \mid 0 \le \boldsymbol{k}_2' < n'+3\} \\ &\cup \{(n'+3, \boldsymbol{k}_1') \mid 0 \le \boldsymbol{k}_1' < n'+3\}\Big) \\ &\setminus \{(\boldsymbol{k}_1', \boldsymbol{k}_2') \mid \boldsymbol{k}_1' + \boldsymbol{k}_2' = 2n'\} \cup \{(n'-4, n'-4)\}. \end{aligned}$$

By Table 6 and the argument above, we deduce that $\mathcal{O}((n'+2, n'+1))$ has no position whose $\mathcal{G}$-value is 1, but we have a position $(n'-4, n'-4)$, whose $\mathcal{G}$-value is 0. Thus we get $\mathcal{G}((n'+4, n'+4)) = 1$. If $Y \in \{(\boldsymbol{k}_1', n'+4) \mid n'+5 \le \boldsymbol{k}_1' \le 2n'\} \cup \{(\boldsymbol{k}_1', n'+5) \mid n'+5 \le \boldsymbol{k}_1' \le 2n'\}$, then $(n'+4, n'+4) \in \mathcal{O}(Y)$, which implies that $Y \notin G_1$. Therefore, for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4 \le \boldsymbol{k}_2 \le n'+5$ and $\boldsymbol{k}_2 \le \boldsymbol{k}_1 \le 2n'$, $Y \in G_1$ if and only if $Y = (n'+4, n'+4)$. Similarly, for each $i \in \mathbb{Z}_{>0}$ (with $n'+4+2i \le 2n'$), it can be verified that for $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{F}(Y_{2,2n'})$ with $n'+4+2i \le \boldsymbol{k}_2 \le n'+5+2i$ and $\boldsymbol{k}_2 \le \boldsymbol{k}_1 \le 2n'$, $Y \in G_1$ if and only if $Y = (n'+4+2i, n'+4+2i)$. Therefore, we obtain

$$\begin{aligned} G_1 = \Big(&\{(n'+2, n'), (n'+1, n'+1)\} \cup \{(n'+4+2i, n'+4+2i) \mid i \ge 0\}\Big) \\ &\cap \mathcal{F}(Y_{2,2n'}) \end{aligned}$$

as desired.

Finally, we determine G_2. Let $Y = (\boldsymbol{k}_1, \boldsymbol{k}_2) \in \mathcal{T}(Y_{2,2n'})$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 > 2n'$. Similar to G_0 and G_1, we determine G_2 as follows.

- If $\boldsymbol{k}_2 < n'$, then $Y \notin G_2$.
- If $n' \leq \boldsymbol{k}_2 \leq n' + 5$ and $\boldsymbol{k}_2 \leq \boldsymbol{k}_1 \leq 2n'$, then $Y \in G_1$ if and only if $Y = (n'+2, n'+2), (n'+3, n'), (n'+4, n'+1)$.
- For each $i \in \mathbb{Z}_{\geq 0}$ (with $n' + 6 + 4i \leq 2n'$), if $n' + 6 + 4i \leq \boldsymbol{k}_2 \leq n' + 9 + 4i$ and $\boldsymbol{k}_2 \leq \boldsymbol{k}_1 \leq 2n'$, then $Y \in G_0$ if and only if $Y = (n'+7+4i, n'+6+4i), (n'+8+4i, n'+7+4i)$.

Therefore, we obtain

$$\begin{aligned} G_2 = \Big(&\{(n'+2, n'+2), (n'+3, n'), (n'+4, n'+1)\} \\ &\cup \{(n'+7+4i, n'+6+4i) \mid i \geq 0\} \\ &\cup \{(n'+8+4i, n'+7+4i) \mid i \geq 0\}\Big) \cap \mathcal{F}(Y_{2,2n'}), \end{aligned}$$

as desired. This complete the proof of Theorem 9.5. □

The following is an immediate consequence of Theorem 9.5, together with Theorem 8.7.

Corollary 9.6. Let $n \geq 2$. In MHRG$(2, n)$, the $\mathcal{G}$-value of the starting position $Y_{2,n}$ is given as follows:

$$\mathcal{G}(Y_{2,n}) = \begin{cases} 3 & \text{if } n = 2, 3, \\ 2 & \text{if } n \neq 2, 3, \text{ and } n \equiv 2, 3 \bmod 8, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We can easily calculate the $\mathcal{G}$-value of the starting position in the cases that $n = 2, 3$. In the other case, we can prove the equality by Theorem 9.5 and Theorem 8.7. □

10 Relation between MHRG and HRG in terms of Shifted Young Diagrams

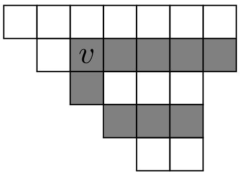
10.1 Hooks of a Shifted Young Diagram

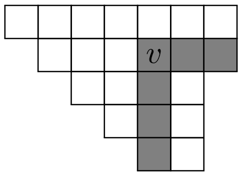
Definition 10.1. For a box (i, j) of a shifted Young diagram S, we define

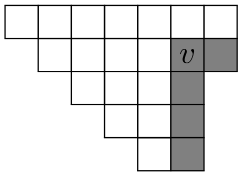
$$\begin{aligned} \mathrm{arm}_S(i,j) &:= \{(i',j') \in S \mid i = i', j < j'\}, \\ \mathrm{leg}_S(i,j) &:= \{(i',j') \in S \mid i < i', j = j'\}, \\ \mathrm{tail}_S(i,j) &:= \{(i',j') \in S \mid j+1 = i', j < j'\}, \\ h_S(i,j) &:= \{(i,j)\} \sqcup \mathrm{arm}_S(i,j) \sqcup \mathrm{leg}_S(i,j) \sqcup \mathrm{tail}_S(i,j). \end{aligned}$$

The set $h_S(i, j)$ is called the hook corresponding to the box (i, j).

Example 10.2. In the figures below, the shadowed boxes form the hook corresponding to the box $v = (i, j)$.

(a): 

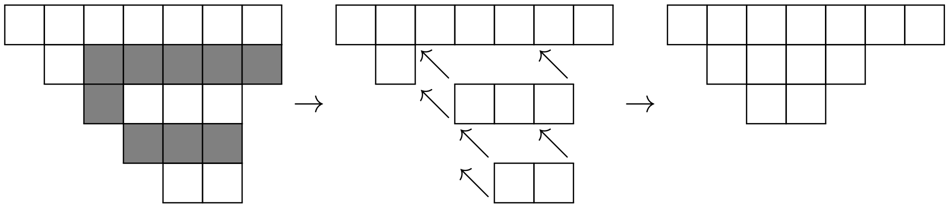
(b): 

(c): 

Definition 10.3. For a box (i, j) of a shifted Young diagram S, we remove the hook $h_S(i, j)$ corresponding to the box (i, j) as follows:

1. Remove all boxes in the hook $h_S(i, j)$.
2. Move each box (i', j') satisfying $j + 1 > i' > i$ and $j' > j$ to $(i' - 1, j' - 1)$.
3. Move each box (i', j') satisfying $i' > j + 1$ to $(i' - 2, j' - 2)$.

Example 10.4. If we remove the hook corresponding to the box $(2, 3)$ from the shifted Young diagram $S = (7, 6, 4, 3, 2)$, then we get $S' = (7, 4, 2)$.



Definition 10.5. A Hook Removing Game (HRG for short) in terms of shifted Young diagrams is an impartial combinatorial game. The rules of this game are as follows:

(HS1) Given a shifted Young diagram S, each player chooses a box $(i, j) \in S$, and remove the hook $h_S(i, j)$ corresponding to the box (i, j) from S on his/her turn.
(HS2) The player who makes the empty shifted Young diagram $\emptyset$ wins.

We denote HRG (in terms of shifted Young diagrams) whose starting position is a shifted Young diagram S by HRG(S). It is clear from the definition of HRG(S) that $\mathcal{F}(S)$ is identical to the set of all positions in HRG(S).

Proposition 10.6. Let $S = (\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_n)$ be a shifted Young diagram, and let T be a shifted Young diagram containing S. The $\mathcal{G}$-value of S in HRG(T) is equal to

$$\mathcal{G}(S) = \bigoplus_{1 \le i \le n} \boldsymbol{k}_i,$$

where $\bigoplus_i a_i$ denotes the nim-sum (the addition of numbers in binary form without carry) of all a_i's.

While this formula is (apparently) well-known by experts, will deduce it from the results of [10], or by the fact that HRG(S) is isomorphic to Turning Turtles (for Turning Turtles, see, e.g., [21, page 182]).

10.2 Diagonal Expression of a Shifted Young Diagram

We now describe the diagonal expression for shifted Young diagrams. Fix $n \in \mathbb{Z}_{>0}$. An element $\boldsymbol{b} \in \mathbb{Z}_{\ge 0}^{n+1}$ is written as $\boldsymbol{b} = [b_0, \ldots, b_n]$. Also, we denote by $\mathbb{SD}_n \subset \mathbb{Z}_{\ge 0}^{n+1}$ the set of all elements $\boldsymbol{b} = [b_0, \ldots, b_n] \in \mathbb{Z}_{\ge 0}^{n+1}$ with $b_n = 0$ satisfying $0 \le b_k - b_{k+1} \le 1$ for $0 \le k < n$.

Let $S \in \mathcal{F}(SY_n)$; recall that $SY_n = \{(i,j) \in \mathbb{Z}^2_{>0} \mid 1 \le i \le n, i \le j \le n\}$. We set $d_k = d_k(S) := \#\{(i,j) \in S \mid j - i = k\}$ for $k \in \mathbb{Z}$. Note that if $k \ge n$, then $d_k = 0$. As in Proposition 7.5, we deduce that $\boldsymbol{sd}(S) = \boldsymbol{sd}_n(S) := [d_0(S), \dots, d_n(S)] \in \mathbb{SD}_n$ for $S \in \mathcal{F}(SY_n)$ and the fact that the map $\boldsymbol{sd} = \boldsymbol{sd}_n : \mathcal{F}(SY_n) \to \mathbb{SD}_n$, $S \mapsto \boldsymbol{sd}(S)$ is bijective.

Definition 10.7. We call $\boldsymbol{sd}(S) = \boldsymbol{sd}_n(S)$ the diagonal expression of $S \in \mathcal{F}(SY_n)$.

Let $\boldsymbol{b} = [b_0, \dots, b_n] \in \mathbb{SD}_n$, $\boldsymbol{b'} = [b'_0, \dots, b'_n] \in \mathbb{Z}^{n+1}_{\ge 0}$, and $0 \le l \le r < n$. If

$$b'_k = \begin{cases} b_k - 1 & \text{if } l \le k \le r, \\ b_k & \text{otherwise,} \end{cases}$$

then we write $\boldsymbol{b} \xrightarrow{l,r} \boldsymbol{b'}$. If

$$b'_k = \begin{cases} b_k - 2 & \text{if } 0 \le k \le r' < r, \\ b_k - 1 & \text{if } r' < k \le r, \\ b_k & \text{otherwise,} \end{cases}$$

then we write $\boldsymbol{b} \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{b'}$ (or $\boldsymbol{b} \xrightarrow{0,r'}\xrightarrow{0,r} \boldsymbol{b'}$). Otherwise, if

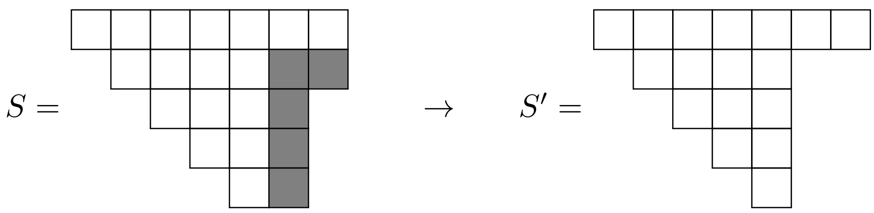
$$b'_k = \begin{cases} b_k - 2 & \text{if } 0 \le k \le r' < r, \\ b_k - 1 & \text{if } r' < k \le r, \\ b_k & \text{otherwise,} \end{cases}$$

then $\boldsymbol{b'} \in \mathbb{SD}_n$.

Lemma 10.8. Let $S, S' \in \mathcal{F}(SY_n)$. The following are equivalent.

(1) There exists a box $(i,j) \in S$ such that $S' = S \setminus h_S(i,j)$.
(2) There exists $0 \le l \le r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$ or we have $0 \le r' < r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{sd}(S')$.

Let us explain the key point of a proof of the lemma by using some examples. Let $S \in \mathcal{F}(SY_n)$, and write $\boldsymbol{sd}(S)$ as $\boldsymbol{sd}(S) = [d_0, \dots, d_n]$ for $S \in \mathcal{F}(SY_n)$. Let us consider (1) $\Longrightarrow$ (2). If $h(S) \le j$, then the removed hook $h_s(i,j)$ is of the form either (b) or (c) in Example 10.2. Thus, there exist $0 \le l \le r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$. For example, let S be as in Example 10.2, and let $S' = S \setminus h_S(2,6)$. Note that the right-half of S is an (ordinary) Young diagram. Removing the hook $h_s(i,j)$ of this form from S naturally corresponds to removing a hook from the Young diagram (see [14, Chapter 4]).

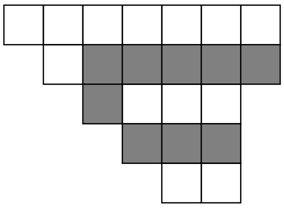
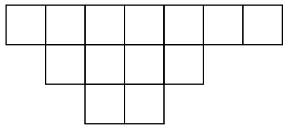


In the diagonal expression, we see that

$$\boldsymbol{sd}(S) = [5,5,4,3,2,2,1,0], \qquad \boldsymbol{sd}(S') = [5,4,3,2,1,1,1,0],$$

and hence $\boldsymbol{sd}(S) \xrightarrow{1,5} \boldsymbol{sd}(S')$.

If $j < h(S)$, then the removed hook is of the form (a) in Example 10.2. In this case, we deduce that $\boldsymbol{sd}(S) \xrightarrow{0,r}\xrightarrow{0,r'} \boldsymbol{sd}(S')$ for some $0 \le r' < r < n$. For example, let S be as in Example 10.2, and let $S' = S \setminus h_S(2,3)$.

$S =$  $\rightarrow$ $S' =$ 

In the diagonal expression, we see that

$$\boldsymbol{sd}(S) = [5,5,4,3,2,2,1,0], \qquad \boldsymbol{sd}(S') = [3,3,2,2,1,1,1,0],$$

and hence $\boldsymbol{sd}(S) \xrightarrow{0,5}\xrightarrow{0,2} \boldsymbol{sd}(S')$.

The implication (2) $\Longrightarrow$ (1) can be verified as in Lemma 7.7.

Definition 10.9. A sequence $(a_{-m}, \dots, a_n) \in \mathbb{D}_{m,n}$ is said to be symmetric if $a_i = a_{n-m-i}$ for all $-m \le i \le n$.

Lemma 10.10.

(1) Let $Y \in \mathcal{F}(Y_{n,n})$. The sequence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric if and only if $Y \in \mathcal{T}(Y_{n,n})$.

(2) Let $Y \in \mathcal{F}(Y_{n,n+1})$. The sequence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n+1}$ is symmetric if and only if $Y \in \mathcal{T}(Y_{n,n+1})$.

Proof. By Theorem 8.7, we need only to show part (1) since it is clear that for $Y \in \mathcal{T}(Y_{n,n})$, $\boldsymbol{d}(Y)$ is symmetric if and only if $\boldsymbol{d}(E(Y))$ is symmetric. We show by induction on $\#Y$ that if $Y \in \mathcal{T}(Y_{n,n})$, then $\boldsymbol{d}(Y) = (d_{-n}(Y), \dots, d_n(Y)) \in \mathbb{D}_{n,n}$ is symmetric. If $Y = Y_{n,n}$, then $\boldsymbol{d}(Y_{n,n}) = (0,1,\dots,n-1,n,n-1,\dots,1,0)$ is symmetric. Assume that $Y \ne Y_{n,n}$. Then there exists $\hat{Y} \in \mathcal{T}(Y_{n,n})$ such that $\hat{Y} \to Y$. Note that $d(\hat{Y}) = (d_{-n}(\hat{Y}), \dots, d_n(\hat{Y}))$ is symmetric by the induction hypothesis, and $d_{k-1}(\hat{Y}) \nearrow d_k(\hat{Y})$ if and only if $d_{-k}(\hat{Y}) \searrow d_{-k+1}(\hat{Y})$ for $-n < k \le n$. Then,

(i) there exist $-n < l \le r < n$ such that $\boldsymbol{d}(\hat{Y}) \xrightarrow{l,r} \boldsymbol{d}(Y)$, or

(ii) there exist $-n < l \le r < n$ such that $\boldsymbol{d}(\hat{Y}) \xrightarrow{l,r} \boldsymbol{d}(\hat{Y}') \xrightarrow{l'=-r,r'=-l} \boldsymbol{d}(Y)$.

Let us consider case (i). Suppose that $l \ne -r$. Note that $d_{l-1}(\hat{Y}) \nearrow d_l(\hat{Y})$, $d_{-l}(\hat{Y}) \searrow d_{-l+1}(\hat{Y})$, $d_r(\hat{Y}) \searrow d_{r+1}(\hat{Y})$, and $d_{-r-1}(\hat{Y}) \nearrow d_{-r}(\hat{Y})$. By Lemma 7.12, we have $d_{-r-1}(Y) \nearrow d_{-r}(Y)$ and $d_{-l}(Y) \searrow d_{-l+1}(Y)$. Thus $\boldsymbol{d}(Y)_{[-r,-l]} \in \mathbb{D}_{n,n}$ by Lemma 7.14, but this is a contradiction. Hence we deduce that $l = -r$. Then we have $\boldsymbol{d}(\hat{Y}) \xrightarrow{-r,r} \boldsymbol{d}(Y)$. In this case, it is obvious that $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric.

Let us consider case (ii). We will show that $d_k(Y) = d_{-k}(Y)$ for any $-n < k < n$. Assume that $l \le k \le r$ and $-r \le k \le -l$. In this case, we have $d_k(\hat{Y}) = d_k(\hat{Y}') + 1 =$

$d_k(Y)+2$. Since $l \le -k \le r$ and $-r \le -k \le -l$, we have $d_{-k}(\hat{Y}) = d_{-k}(\hat{Y}') + 1 = d_{-k}(Y) + 2$. Thus we have $d_k(Y) = d_k(\hat{Y}) - 2 = d_{-k}(\hat{Y}) - 2 = d_{-k}(Y)$. The proofs for the other cases are similar. Hence $\boldsymbol{d}(Y) \in \mathbb{D}_{n,n}$ is symmetric.

Next, we show that if $\boldsymbol{d}(Y) = (d_{-n}(Y), \dots, d_n(Y)) \in \mathbb{D}_{n,n}$ is symmetric, then $Y \in \mathcal{T}(Y_{n,n})$. Let $\mathbb{A} := \{0 \le i \le n-1 \mid d_i = d_{i+1} + 1\}$ and write it as $\mathbb{A} = \{i_1, i_2, \dots, i_k\}$. Then there exists a transition $Y_{n,n} = Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_{k-1} \to Y_k = Y$ such that $\boldsymbol{d}(Y_{l-1}) \xrightarrow{-i_l, i_l} \boldsymbol{d}(Y_l)$ for $1 \le l \le k$. Thus we obtain $Y \in \mathcal{T}(Y_{n,n})$, as desired. □

Let $\boldsymbol{a} = (a_{-n}, a_{n-1}, \dots, a_{-1}, \dot{a_0}, a_1, \dots, a_n, a_{n+1}) \in \mathbb{D}_{n,n+1}$. Assume that

$$\hat{\boldsymbol{a}} := [a_1, a_2, \dots, a_n, a_{n+1}] \in \mathbb{Z}_{\ge 0}^{n+1}.$$

By the definition of $\mathbb{D}_{n,n+1}$, we thus have $\hat{\boldsymbol{a}} \in \mathbb{SD}_n$.

Definition 10.11. The map $A : \mathcal{T}(Y_{n,n+1}) \to \mathcal{F}(SY_n)$ is defined as follows. If the diagonal expression of $Y \in \mathcal{T}(Y_{n,n+1})$ is

$$\boldsymbol{d}(Y) = (a_{-n}, a_{n-1}, \dots, a_{-1}, \dot{a_0}, a_1, \dots, a_n, a_{n+1}),$$

then we define $A(Y) \in \mathcal{F}(SY_n)$ to be the shifted Young diagram in $\mathcal{F}(SY_n)$ whose diagonal expression is equal to

$$\boldsymbol{sd}(A(Y)) = [a_1, a_2, \dots, a_n, a_{n+1}].$$

Lemma 10.12. Let $Y \in \mathcal{T}(Y_{n,n+1})$, and let $Y' \in \mathcal{O}(Y)$. Also, set $S := A(Y) \in \mathcal{F}(SY_n)$. Then there exists $S' \in \mathcal{O}(S)$ such that $A(Y') = S'$.

Proof. Since $Y' \in \mathcal{O}(Y)$, we see that

(i) there exist $-n < l \le r < n+1$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y')$, or
(ii) there exist $-n < l \le r < n+1$ and $Y'' \in \mathcal{F}(Y_{n,n+1})$ such that $\boldsymbol{d}(Y) \xrightarrow{l,r} \boldsymbol{d}(Y'') \xrightarrow{-r+1,-l+1} \boldsymbol{d}(Y')$.

First, we consider case (i). By the proof of Lemma 10.10, we see that $l = -r+1$ and hence $\boldsymbol{d}(Y) \xrightarrow{-r+1,r} \boldsymbol{d}(Y')$. In this case, we have $d_{r-1}(S) = d_r(S) + 1$. Let $S' \in \mathcal{F}(SY_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{0,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Next, we consider case (ii). By the proof of Lemma 10.10, we see that $l \ne -r+1$ and hence $\boldsymbol{d}(Y)_{[l,r]}, (\boldsymbol{d}(Y)_{[l,r]})_{[-r+1,-l+1]} \in \mathbb{D}_{n,n+1}$.

Assume that $0 \le l \le r$. In this case, we have $d_{l-2}(S) = d_{l-1}(S)$ and $d_{r-1}(S) = d_r(S) + 1$. Let $S' \in \mathcal{F}(SY_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{l-1,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Assume that $l \le r \le 0$. In this case, we have $d_{-r-1}(S) = d_{-r}(S)$ and $d_{-l}(S) = d_{-l+1}(S) + 1$. Let $S' \in \mathcal{F}(SY_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{-r,-l} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Assume that $l \le 0 < r$. In this case, we have $d_{r-1}(S) = d_r(S) + 1$ and $d_{-l}(S) = d_{-l+1}(S) + 1$. Let $S' \in \mathcal{F}(SY_n)$ be such that $\boldsymbol{sd}(S) \xrightarrow{0,-l} \xrightarrow{0,r-1} \boldsymbol{sd}(S')$. Then we deduce that $A(Y') = S'$.

Thus we have proved the lemma. □

Let $\boldsymbol{b} = [b_0, b_1, \ldots, b_{n-1}, b_n] \in \mathbb{SD}_n$. Assume that

$$\hat{\boldsymbol{b}} := (b_{-n}, b_{n-1}, \ldots, b_{-1}, \dot{b_0}, b_0, b_1, \ldots, b_{n-1}, b_n) \in \mathbb{Z}_{\geq 0}^{2n+2}.$$

By the definition of $\mathbb{SD}_n$, we have $\hat{\boldsymbol{b}} \in \mathbb{D}_{n,n+1}$.

Definition 10.13. The map $B : \mathcal{F}(SY_n) \to \mathcal{T}(Y_{n,n+1})$ is defined as follows. If the diagonal expression of $Y \in \mathcal{F}(SY_n)$ is

$$\boldsymbol{sd}(S) = [a_0, a_1, \ldots, a_{n-1}, a_n].$$

then we define $B(S) \in \mathcal{T}(Y_{n,n+1})$ to be the rectangular Young diagram in $\mathcal{T}(Y_{n,n+1})$ whose diagonal expression is equal to

$$\boldsymbol{d}(B(S)) = (a_n, a_{n-1}, \ldots, \dot{a_0}, \underbrace{a_0}_{\text{1st}}, a_1, \ldots, a_{n-1}, \underbrace{a_n}_{(n+1)\text{-th}}).$$

Lemma 10.14. Let $S \in \mathcal{F}(SY_n)$, and let $S' \in \mathcal{O}(S)$. Also, set $Y := B(S) \in \mathcal{T}(Y_{n,n+1})$. Then there exists $Y' \in \mathcal{O}(Y)$ such that $B(S') = Y'$.

Proof. Since $S' \in \mathcal{O}(S)$, we see that

(i) there exist $0 \leq l \leq r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{l,r} \boldsymbol{sd}(S')$, or

(ii) there exist $0 \leq r' < r < n$ such that $\boldsymbol{sd}(S) \xrightarrow{0,r} \xrightarrow{0,r'} \boldsymbol{sd}(S')$.

First, we consider case (i). Assume that $l = 0$. In this case, $d_r(S) = d_{r+1}(S) + 1$. Then, we have $d_{r+1}(B(S)) = d_{r+2}(B(S)) + 1$, $d_{-r-1}(B(S)) + 1 = d_{-r}(B(S))$, and hence $\boldsymbol{d}(B(S))_{[-r,r+1]} \in \mathbb{D}_{n,n+1}$ by Lemma 7.12. Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{-r,r+1} \boldsymbol{d}(Y')$. Then we deduce that $B(S') = Y'$. Assume that $0 < l \leq r$. In this case, $d_{l-1}(S) = d_l(S)$ and $d_r(S) = d_{r+1}(S) + 1$. Then, we have $d_l(B(S)) = d_{l+1}(B(S))$, $d_{-l}(B(S)) = d_{-l+1}(B(S))$, $d_{r+1}(B(S)) = d_{r+2}(B(S))+1$, $d_{-r-1}(B(S))+1 = d_{-r}(B(S))$, and hence $\boldsymbol{d}(B(S))_{[l+1,r+1]}$, $(\boldsymbol{d}(B(S))_{[l+1,r+1]})_{[-r,-l]} \in \mathbb{D}_{n,n+1}$ by Lemma 7.12. Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{l+1,r+1} \boldsymbol{d}(Y'') \xrightarrow{-r,-l} \boldsymbol{d}(Y')$. Then we deduce that $B(S') = Y'$.

Next, we consider case (ii). In this case, $d_r(S) = d_{r+1}(S) + 1$ and $d_{r'}(S) = d_{r'+1}(S) + 1$. Then, we have $d_{r+1}(B(S)) = d_{r+2}(B(S)) + 1$, $d_{-r-1}(B(S)) + 1 = d_{-r}(B(S))$, $d_{r'+1}(B(S)) = d_{r'+2}(B(S)) + 1$, $d_{-r'-1}(B(S)) + 1 = d_{-r'}(B(S))$, and hence, by Lemma 7.12, we have $\boldsymbol{d}(B(S))_{[-r',r+1]}, (\boldsymbol{d}(B(S))_{[-r',r+1]})_{[-r,r'+1]} \in \mathbb{D}_{n,n+1}$. Let $Y' \in \mathcal{O}(Y)$ be such that $\boldsymbol{d}(Y) \xrightarrow{-r',r+1} \boldsymbol{d}(Y'') \xrightarrow{-r,r'+1} \boldsymbol{d}(Y')$. This implies that $B(S') = Y'$.

Thus we have proved the lemma. □

The next theorem follows from Lemmas 10.12 and 10.14.

Theorem 10.15. For $n \in \mathbb{Z}_{>0}$, MHRG$(n, n + 1)$ and HRG(SY_n) are isomorphic. In particular, $\mathcal{G}(Y_{n,n+1})$ in MHRG$(n, n + 1)$ is equal to $\mathcal{G}(SY_n)$ in HRG(SY_n).

Combining Proposition 10.6, Theorems 8.7, and 10.15, we obtain the following corollary.

Corollary 10.16. In MHRG(n, n) (resp., MHRG$(n, n + 1)$), the $\mathcal{G}$-value of the starting position $Y_{n,n}$ (resp., $Y_{n,n+1}$) is equal to

$$\mathcal{G}(Y_{n,n}) = \mathcal{G}(Y_{n,n+1}) = \bigoplus_{1 \leq k \leq n} k.$$

Example 10.17. Assume that $n = 3$. The $\mathcal{G}$-value of $Y_{3,4} = \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 1 \\ \hline 2 & 3 & 3 & 2 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array}$ is equal to $1 \oplus 2 \oplus 3 = 0$.

References

[1] T. Abuku and M. Tada, *Multiple Hook Removing Game Whose Starting Position is a Rectangular Young Diagram with the Unimodal Numbering*, arXiv:2112.12963 (2021), to appear in Integers.

[2] H. M. Albert, J. R. Nowakowski, and D. Wolfe, *Lessons in play: An introduction to combinatorial game theory*, A K Peters, 2007.

[3] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, Vol 1-4, A K Peters, 2001-2004.

[4] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics Book 231, Springer, 2005.

[5] R. Carter, *Lie Algebra of Finite and Affine Type*, Cambridge Studies in Advanced Mathematics 96, Cambridge University Press, 2005.

[6] J. H. Conway, *On Numbers and Games*, 2nd edition, A K Peters, 2001.

[7] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990.

[8] R. M. Green, *Combinatorics of Minuscule Representations*, Cambridge Tracts in Mathematics 199, Cambridge University Press, 2013.

[9] V. G. Kac, *Infinite Dimensional Lie Algebras*, Third edition, Cambridge University Press, 1990.

[10] N. Kawanaka, *Sato-Welter game and Kac-Moody Lie algebras, Topics in combinatorial representation theory* (Kyoto, 2000), Sūrikaisekikenkyūsho Kōkyūroku No. 1190 (2001), 95-106. (in Japanese)

[11] P. Littelmann, *Paths and Root Operators in Representation Theory*, Annals of Mathematics Second Series 142 (1995), 499-525.

[12] Y. Motegi, *Game positions of Multiple Hook Removing Game*, arXiv:2112.14200 (2021).

[13] K. Nakada, *Colored Hook Formula for a Generalized Young Diagram*, Osaka Journal of Mathematics 45 (2008), 1085-1120.

[14] J. Olsson, *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1994.

[15] Robert A. Proctor, *Dynkin Diagram Classification of λ-Minuscule Bruhat Lattices and of d-Complete Posets*, Journal of Algebraic Combinatorics 9 (1999), 61-94.

[16] Robert A. Proctor, *Minuscule Elements of Weyl Groups, the Numbers Game, and d-Complete Posets*, Journal of Algebra 213 (1999), 272-303.

[17] Robert A. Proctor, *d-Complete Posets Generalize Young Diagrams for the Hook Product Formula : Partial Presentation of Proof*, RIMS Kôkyûroku 1913 (2014), 120-140.

[18] Robert A. Proctor, *d-Complete Posets Generalize Young Diagrams for the Jeu de Taquin Property*, arXiv:0905.3716 (2009).

[19] M. Sato, (Notes by H. Enomoto), *On Maya game*, Suugaku-no-Ayumi 15-1 (Special Issue : Mikio Sato) (1970), 73-84. (in Japanese)

[20] M. Sato (Notes by T. Umeda), *Lectures (on Soliton Theory) at Kyoto University*, 1984-85 , RIMS Kyoto University, 1989. (in Japanese)

[21] A. N. Siegel, *Combinatorial Game Theory*, Graduate Studies in Mathematics, 146, American Mathematical Society, 2013.

[22] J. R. Stembridge, *Quasi-Minuscule Quotients and Reduced Words for Reflections*, Journal of Algebraic Combinatorics 13 (2001), 275-293.

[23] M. Tada, *Relation between the Weyl group orbits of fundamental weights for multiply-laced finite dimensional simple Lie algebras and d-complete posets*, arXiv:2209.09945 (2022), submitted.