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A model structure for weakly horizontally invariant double categories. (English)
Zbl 07706509
Algebr. Geom. Topol. 23, No. 4, 1725-1786 (2023)

This paper aims to study and compare the homotopy theories of two related types of 2-dimensional categories, namely, 2-*categories* and *double categories*. A -category $\mathcal{A}$ can always be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ with only trivial vertical morphisms, giving rise a full embedding of 2-categories into double categories

$$\mathbb{H} : 2\mathrm{Cat} \to \mathrm{DblCat}$$

The category 2Cat of 2-categories and 2-functors admits a model structure constructed by *S. Lack* [*K*-Theory 26, No. 2, 171–205 (2002; Zbl 1017.18005); *K*-Theory 33, No. 3, 185–197 (2004; Zbl 1069.18008)], where the weak equivalences are the biequivalences, the trivial fibrations are the 2-functors which are surjective on objects, full on morphisms, and fully faithful on 2-morphisms, and all 2-categories are fibrant, while Lack gives a characterization of the cofibrant objects as the 2-categories whose underlying category is free. The authors of this paper are interested in the question whether there is a homotopy theory for double categories containing that of 2-categories. This paper gives an affirmative answer to this question nicely.

Several model structures for double categories were first constructed in [*T. M. Fiore* and *S. Paoli*, Algebr. Geom. Topol. 10, No. 4, 1933–2008 (2010; Zbl 1203.18014); *T. M. Fiore* et al., Algebr. Geom. Topol. 8, No. 4, 1855–1959 (2008; Zbl 1159.18302)], but the homotopy theory of 2-categories does not embed in any of these homotopy theories for double categories. The first positive answer to this question was given by the authors in [*L. Moser* et al., Cah. Topol. Géom. Différ. Catég. 63, No. 2, 184–236 (2022; Zbl 1498.18031)], whose model structure is right-induced from two copies of Lack's model struccture on 2Cat, This model structure is very well behaved with respect to the horizontal embedding $\mathbb{H}$, which is both left and right Quillen, and both left- and right-induces Lack's model structure. As is expected, this model structure is of horizontal bias, being not well with respect to the vertical direction and, particularly, being deterred from being monoidal with respect to the Gray tensor product for double categories defined by *G. Böhm* [Appl. Categ. Struct. 28, No. 3, 477–515 (2020; Zbl 1475.18030)].

This paper aims to provide a new model structure on DblCat, whose trivial fibrations behave symmetrically with respect to the horizontal and vertical directions, and whose fibrant objects are the weakly horizontally invariant double categories.

The synopsis of the paper goes as follows.

- §2 recalls some notations and definitions of double category theory introduced in [*L. Moser* et al., Cah. Topol. Géom. Différ. Catég. 63, No. 2, 184–236 (2022; Zbl 1498.18031)], introducing weakly horizontally invariant double categories and the homotopical horizontal embedding functor

$$\mathbb{H}^{\simeq}; 2\mathrm{Cat} \to \mathrm{DblCat}$$

- §3 gives the main features of the model structure on DblCat.
- §4 presents several technical results to establish the existence of the above model structure.
- §5 completes the proof of the existence of the above model structure.
- §6 studies its relation with the model structure on DblCat in [*L. Moser* et al., Cah. Topol. Géom. Différ. Catég. 63, No. 2, 184–236 (2022; Zbl 1498.18031)] and with Lack's one on 2Cat, showing in Theorem 6.1 that the identity functor from the authors' new model structure on DblCat to the one of [*L. Moser* et al., Cah. Topol. Géom. Différ. Catég. 63, No. 2, 184–236 (2022; Zbl 1498.18031)] is right Quillen and homotopically fully faithful. It is shown in Theorem 6.5 that the inclusion

$$\mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$$

is a weak equivalence, exhibiting $\mathbb{H}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{H}\mathcal{A}$ in the model structure for weakly horizontally invariant double categories. It is established in Theorem 6.6 that $\mathbb{H}^{\simeq}$ embeds the homotopy theory of into that of weakly horizontally invariant double categories in a reflective way. It is demonstrated in Theorem 6.8 that $\mathbb{H}^{\simeq}$ not only preseves, but also reflects weak equivalences and fibrations.

§7 establishes that the model structure is monoidal with respect to the Gray tensor product for double categories (Theorem 7.8).

§8 is devoted to the proof of the Whitehead theorem for double categories, which implies that the weak equivalences between fibrant objects in the model structure for weakly horizontally invariant double categories are akin to the biequivalences between 2-categories.

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MSC:

18D20 Enriched categories (over closed or monoidal categories)
18N10 2-categories, bicategories, double categories
18N40 Homotopical algebra, Quillen model categories, derivators

Keywords:

model structure; double categories; 2-categories; monoidal model structure; Whitehead theorem

Full Text: DOI arXiv

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