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Article

Scaling Limits of a Tandem Queue with Two Infinite Orbits †

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Abstract: This paper considers a tandem queueing network with a Poisson arrival process of incoming calls, two servers, and two infinite orbits by the method of asymptotic analysis. The servers provide services for incoming calls for exponentially distributed random times. Blocked customers at each server join the orbit of that server and retry to enter the server again after an exponentially distributed time. Under the condition of low retrial rates, we prove that the joint stationary distribution of scaled numbers of calls in the orbits weakly converges to a two-variable Normal distribution.

Keywords: tandem queueing networks; retrial; asymptotic analysis; two infinite orbits

MSC: 60K25; 60K05; 90B22

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1. Introduction

Retrial queues have been extensively studied due to their applications in service and computer and telecommunication systems. The main feature of retrial queues is that a blocked customer that cannot receive service upon arrival joins a virtual queue called an orbit and retries to enter a server after some random time. The analysis of retrial queues is challenging due to the non-homogeneity of the underlying Markov chain of these systems. As a result, analytical solutions for the stationary queue length of retrial queues are found in only a few special cases with a small number of servers (see some surveys and related references [1–5]).

Analytical solutions are even more challenging for the network of retrial queues as the so-called product-form solutions do not exist [6]. This motivated us to consider scaling limits for these models. In a recent series of work [7–10], we studied tandem queues with one orbit. In these papers, we studied an asymptotic regime where the retrial rate is extremely small, proving that in the transient regime, two scaled versions of the number of customers in orbit converge to a deterministic process and the diffusion process, respectively. Furthermore, in the stationary regime, two scaled versions of the number of customers in orbit converge to the constant and the Normal distribution, respectively.

This paper extends our work to a new framework with multiple orbits. Each orbit corresponds to a buffer of a server in the tandem queue. Our model is formulated by a four-dimensional Markov chain representing the state of the two servers and two orbits. As the size of each orbit is unlimited, the underlying Markov chain has two infinite dimensions. Furthermore, the underlying Markov chain is non-homogeneous because the retrial rate is proportional to the number of customers in orbit. This makes the analytical solution of the joint queue-length distribution very challenging. In order to obtain exact results, we consider a regime where the retrial rates of both orbits are scaled by a scaling factor. In this regime, the numbers of customers in both orbits explode. We, however, prove that two scaled versions of the numbers of customers in orbits converge to a deterministic vector and a vector of two random variables with Normal distribution, respectively. The proof is based on the characteristic function of the joint distribution of the numbers of calls in orbits.

As for closely related work, Avrachenkov and Yechiali [11] studied tandem blocking queues with a common retrial queue (constant retrial rate), while Takahara [12] proposed a fixed point approximation for a queueing network with caller's retrial. In Ref. [12], the joint distribution of the number of customers in orbit(s) was not considered, and all nodes are assumed to be independent. To the best of our knowledge, our work is the first to consider an analytic solution for the joint stationary distribution of the numbers of customers in orbits. Furthermore, some related models were also presented in Refs. [13–21].

The rest of our paper is organized as follows. Section 2 presents the model and the problem statement. Section 3 shows the system of Kolmogorov equations. Section 4 is devoted to the first-order asymptotic where we show that a scaled version of the numbers of customers in orbit converges to a constant vector. In Section 5, we further prove that another version of the numbers of customers in orbits extracting the constant converges to a vector following two-variable Normal distribution. Finally, Section 6 demonstrates some numerical examples showing the applicability of our asymptotic results as approximations.

2. Model Description and Markov Chain

We consider a tandem queueing network with Poisson arrival process of incoming calls with rate λ fed to two servers (see Figure 1). Upon the arrival of a call, if the first server is free, the call occupies it and is served for an exponentially distributed time with mean $1/\mu_1$. If the first server is busy, the call is sent to the orbit of the first server for an exponentially distributed time with mean $1/\sigma_1$ and retries to enter the first server again. Upon service completion from server 1, the call tries to enter to the second server. If the second server is free, the call moves to it for a service with an exponentially distributed random time with mean $1/\mu_2$. In case the second server is busy, the call moves to the orbit of the second server and retries to enter the second server after some exponentially distributed time with mean $1/\sigma_2$. The call leaves the system after receiving a service from the second server.

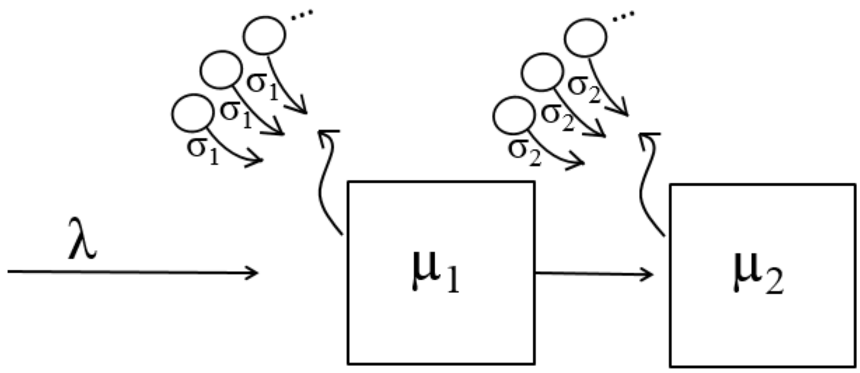

Figure 1. The model.

Let $n_1(t)$ and $n_2(t)$ denote the state of the first server and that of the second server at time t. If server i is busy $n_i(t) = 1$ otherwise $n_i(t) = 0$ for $i = 1, 2$. Furthermore let $i_1(t)$ and $i_2(t)$ denote the numbers of calls in the first and the second orbits at the time t, respectively. The process $X(t) = \{n_1(t), n_2(t), i_1(t), i_2(t)\}$ is a Markov chain on the state space $S = \{0,1\} \times \{0,1\} \times \{0,1,\dots\} \times \{0,1,\dots\}$.

The goal of the study is to obtain an exact asymptotic expression for the two-dimensional stationary probability distribution of the numbers of calls in orbits $\{i_1(t), i_2(t)\}$.

Lemma 1. *The necessary stability condition for X(t) is* $\lambda < \min(\mu_1, \mu_2)$.

Proof. Because the first server and its orbit form an M/M/1/1 retrial queue, this queue's necessary and sufficient stability condition is $\lambda < \mu_1$. It should be noted that the output process of the first queue is not a renewal process [2]. However, under the stability condition of the first queue, i.e., $\lambda < \mu_1$, the departure rate from the first queue is also λ. Under the stability condition of the second queue, the probability that the second server is busy is given by $\lambda/\mu_2 < 1$, which implies the proof. □

3. Balance Equations and Characteristic Functions

Under the steady state, we define the stationary probabilities

$$P_{n_1 n_2}(i_1, i_2) = \lim_{t \to \infty} P\{n_1(t) = n_1, n_2(t) = n_2, i_1(t) = i_1, i_2(t) = i_2\}. \quad (1)$$

We define the partial characteristic functions, denoting $j = \sqrt{-1}$ as follows:

$$H_{n_1 n_2}(u_1, u_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} e^{ju_1 i_1} e^{ju_2 i_2} P_{n_1 n_2}(i_1, i_2). \quad (2)$$

The balance equations for the probabilities in (1) are given as follows.

$$\begin{aligned}
(\lambda + i_1\sigma_1 + i_2\sigma_2)P_{00}(i_1, i_2) &= \mu_2 P_{01}(i_1, i_2), \\
(\lambda + \mu_1 + i_2\sigma_2)P_{10}(i_1, i_2) &= \lambda P_{00}(i_1, i_2) + (i_1 + 1)\sigma_1 P_{00}(i_1 + 1, i_2) \\
&\quad + \mu_2 P_{11}(i_1, i_2), \\
(\lambda + i_1\sigma_1 + \mu_2)P_{01}(i_1, i_2) &= \mu_1 P_{10}(i_1, i_2) + (i_2 + 1)\sigma_2 P_{00}(i_1, i_2 + 1), \\
(\lambda + \mu_1 + \mu_2)P_{11}(i_1, i_2) &= \lambda P_{11}(i_1 - 1, i_2) + (i_1 + 1)\sigma_1 P_{01}(i_1 + 1, i_2) \\
&\quad + (i_2 + 1)\sigma_2 P_{10}(i_1, i_2 + 1),
\end{aligned}$$

with the convention that $P_{n_1 n_2}(i_1, i_2) = 0$ if $i_1 < 0$ or $i_2 < 0$. Multiplying these equations by $e^{ju_1 i_1} e^{ju_2 i_2}$, taking the summation over $i_1, i_2 \in \{0, 1, 2, \dots\}$, and arranging the results we obtain the following system of equations for characteristic functions.

$$\begin{gathered}
-\lambda H_{00}(u_1, u_2) + j\sigma_1 \frac{\partial H_{00}(u_1, u_2)}{\partial u_1} + j\sigma_2 \frac{\partial H_{00}(u_1, u_2)}{\partial u_2} + \mu_2 H_{01}(u_1, u_2) = 0, \\
\lambda H_{00}(u_1, u_2) - j\sigma_1 e^{-ju_1} \frac{\partial H_{00}(u_1, u_2)}{\partial u_1} - (\lambda + \mu_1 - \lambda e^{ju_1}) H_{10}(u_1, u_2) \\
+ j\sigma_2 \frac{\partial H_{10}(u_1, u_2)}{\partial u_2} + \mu_2 H_{11}(u_1, u_2) = 0, \\
-j\sigma_2 e^{-ju_2} \frac{\partial H_{00}(u_1, u_2)}{\partial u_2} + \mu_1 H_{10}(u_1, u_2) - (\lambda + \mu_2) H_{01}(u_1, u_2) \\
+ j\sigma_1 \frac{\partial H_{01}(u_1, u_2)}{\partial u_1} + \mu_1 e^{ju_2} H_{11}(u_1, u_2) = 0, \\
\lambda H_{01}(u_1, u_2) - j\sigma_1 e^{-ju_1} \frac{\partial H_{01}(u_1, u_2)}{\partial u_1} - j\sigma_2 e^{-ju_2} \frac{\partial H_{10}(u_1, u_2)}{\partial u_2} \\
-(\lambda + \mu_1 + \mu_2 - \lambda e^{ju_1}) H_{11}(u_1, u_2) = 0. \quad (3)
\end{gathered}$$

Denote matrices

$$\mathbf{A} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -(\lambda+\mu_1) & \mu_1 & 0 \\ \mu_2 & 0 & -(\lambda+\mu_2) & \lambda \\ 0 & \mu_2 & 0 & -(\lambda+\mu_1+\mu_2) \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \end{bmatrix}, \mathbf{I}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{I}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{I}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

Let us write the system (3) in the matrix form and multiply the resulting system by the identity column vector **e** (with all elements of 1) to obtain the following system.

$$\begin{gathered} \mathbf{H}(u_1,u_2)\Big\{\mathbf{A} + e^{ju_1}\mathbf{B}_1 + e^{ju_2}\mathbf{B}_2\Big\} + j\sigma_1 \frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_1}\Big\{\mathbf{I}_0 - e^{-ju_1}\mathbf{I}_1\Big\} \\ + j\sigma_2 \frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_2}\Big\{\mathbf{I}_2 - e^{-ju_2}\mathbf{I}_3\Big\} = 0, \\ \Big(e^{ju_1} - 1\Big)\bigg\{\mathbf{H}(u_1,u_2)\mathbf{B}_1 + j\sigma_1 e^{-ju_1}\frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_1}\mathbf{I}_0\bigg\}\mathbf{e} \\ +(e^{ju_2} - 1\Big)\bigg\{\mathbf{H}(u_1,u_2)\mathbf{B}_2 + j\sigma_2 e^{-ju_2}\frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_2}\mathbf{I}_2\bigg\}\mathbf{e} = 0, \end{gathered} \tag{5}$$

where $\mathbf{H}(u_1,u_2) = \{H_{00}(u_1,u_2), H_{10}(u_1,u_2), H_{01}(u_1,u_2), H_{11}(u_1,u_2)\}$.

The system of Equation (5) is the basis for further research. We will solve it by the method of asymptotic analysis [22] under the asymptotic condition $\sigma_1 = \sigma\gamma_1, \sigma_2 = \sigma\gamma_2$ where $\sigma \to 0$. The method of asymptotic analysis is carried out in two stages: the first stage $\sigma = \varepsilon$ and the second stage $\sigma = \varepsilon^2$.

4. The First Order Asymptotic

By denoting $\sigma_1 = \sigma\gamma_1, \sigma_2 = \sigma\gamma_2$, we get the following system

$$\begin{gathered} \mathbf{H}(u_1,u_2)\Big\{\mathbf{A} + e^{ju_1}\mathbf{B}_1 + e^{ju_2}\mathbf{B}_2\Big\} + j\sigma\gamma_1 \frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_1}\Big\{\mathbf{I}_0 - e^{-ju_1}\mathbf{I}_1\Big\} \\ + j\sigma\gamma_2 \frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_2}\Big\{\mathbf{I}_2 - e^{-ju_2}\mathbf{I}_3\Big\} = 0, \\ \Big(e^{ju_1} - 1\Big)\bigg\{\mathbf{H}(u_1,u_2)\mathbf{B}_1 + j\sigma\gamma_1 e^{-ju_1}\frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_1}\mathbf{I}_0\bigg\}\mathbf{e} \\ +(e^{ju_2} - 1\Big)\bigg\{\mathbf{H}(u_1,u_2)\mathbf{B}_2 + j\sigma\gamma_2 e^{-ju_2}\frac{\partial \mathbf{H}(u_1,u_2)}{\partial u_2}\mathbf{I}_2\bigg\}\mathbf{e} = 0. \end{gathered} \tag{6}$$

Furthermore, we perform in the system (6) the following substitution

$$\sigma = \varepsilon, u_1 = \varepsilon w_1, u_2 = \varepsilon w_2, \mathbf{H}(u_1,u_2) = \mathbf{F}(w_1,w_2,\varepsilon). \tag{7}$$

With this substitution $\mathbf{F}(w_1,w_2,\varepsilon)$ represents the vector of characteristic functions of $\{\varepsilon i_1(t), \varepsilon i_2(t)\}$.

We derive the following system.

$$
\begin{aligned}
&\mathbf{F}(w_1,w_2,\varepsilon)\Big\{\mathbf{A}+e^{j\varepsilon w_1}\mathbf{B}_1+e^{j\varepsilon w_2}\mathbf{B}_2\Big\}\\
&+j\gamma_1\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_1}\Big\{\mathbf{I}_0-e^{-j\varepsilon w_1}\mathbf{I}_1\Big\}\\
&+j\gamma_2\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_2}\Big\{\mathbf{I}_2-e^{-j\varepsilon w_2}\mathbf{I}_3\Big\}=0,\\
&\Big(e^{j\varepsilon w_1}-1\Big)\Big\{\mathbf{F}(w_1,w_2,\varepsilon)\mathbf{B}_1+j\gamma_1e^{-j\varepsilon w_1}\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_1}\mathbf{I}_0\Big\}\mathbf{e}\\
&+\Big(e^{j\varepsilon w_2}-1\Big)\Big\{\mathbf{F}(w_1,w_2,\varepsilon)\mathbf{B}_2+j\gamma_2e^{-j\varepsilon w_2}\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_2}\mathbf{I}_2\Big\}\mathbf{e}=0,
\end{aligned}
\tag{8}
$$

which we will solve under the following assumptions.

$$
\begin{aligned}
\mathbf{F}(w_1,w_2)&=\lim_{\varepsilon\to0}\mathbf{F}(w_1,w_2,\varepsilon),\\
\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_1}&=\lim_{\varepsilon\to0}\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_1},\\
\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_2}&=\lim_{\varepsilon\to0}\frac{\partial\mathbf{F}(w_1,w_2,\varepsilon)}{\partial w_2}.
\end{aligned}
$$

Theorem 1. *We have*

$$
\lim_{\sigma\to0}Ee^{jw_1\sigma i_1(t)+jw_2\sigma i_2(t)}=e^{jw_1a_1+jw_2a_2}.\tag{9}
$$

The vector $\mathbf{r}$ *is a vector of the states of servers that satisfies the normalization condition* $\mathbf{re}=1$ *and is the solution of the matrix equation*

$$
\mathbf{r}(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2)+\gamma_1a_1\mathbf{r}(\mathbf{I}_1-\mathbf{I}_0)+\gamma_2a_2\mathbf{r}(\mathbf{I}_3-\mathbf{I}_2)=0,\tag{10}
$$

where a_1 *and* a_2 *are solutions of the equations*

$$
\mathbf{r}(\mathbf{B}_1-\gamma_1a_1\mathbf{I}_0)\mathbf{e}=0,\tag{11}
$$

$$
\mathbf{r}(\mathbf{B}_2-\gamma_2a_2\mathbf{I}_2)\mathbf{e}=0,\tag{12}
$$

and where γ_1 *and* γ_2 *are parameters of the asymptotic analysis.*

Remark 1. *It should be noted that*

$$
\mathbf{Q}=(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2)+\gamma_1a_1(\mathbf{I}_1-\mathbf{I}_0)+\gamma_2a_2(\mathbf{I}_3-\mathbf{I}_2)
$$

represents the infinitesimal generator of the Markov chain of the states of the two-server tandem queue without buffers and retrials. In particular, the arrival process to the first server is superposed by a Poisson with rate λ*, and an additional Poisson process with rate* γ_1a_1 *(representing the retrials from the first orbit), while the input to the second server is the output from the first server and a Poisson process with rate* γ_2a_2 *(representing the retrials from orbit 2). Furthermore, we interpret (11) and (12) as the balance equations of the rates coming into and out of orbit 1 and orbit 2, respectively. In fact,* $\mathbf{rB_1e}$ *represents the blocking flow going into the orbit and* $\mathbf{r}\gamma_1a_1\mathbf{I_0e}$ *does the flow successfully going out the orbit (seeing the first server idle). The same interpretation is also applied for (12).*

Proof. Let us take the limit $\varepsilon\to0$ in the system (8) and obtain

$$
\mathbf{F}(w_1,w_2)\{\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2\}+j\gamma_1\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_1}\{\mathbf{I}_0-\mathbf{I}_1\}
$$

$$+j\gamma_2\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_2}\{\mathbf{I}_2-\mathbf{I}_3\}=0,$$

$$jw_1\left\{\mathbf{F}(w_1,w_2)\mathbf{B}_1+j\gamma_1\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_1}\mathbf{I}_0\right\}\mathbf{e}$$

$$+jw_2\left\{\mathbf{F}(w_1,w_2)\mathbf{B}_2+j\gamma_2\frac{\partial\mathbf{F}(w_1,w_2)}{\partial w_2}\mathbf{I}_2\right\}\mathbf{e}=0. \quad (13)$$

It should be noted that the nontrivial ($\neq 0$) solution of (13) is unique because the underlying Markov chain has unique stationary distribution and its characteristic function. We find the solution of this system in the form

$$\mathbf{F}(w_1,w_2)=\mathbf{r}\Phi(w_1,w_2), \quad (14)$$

where row vector $\mathbf{r}=\left[r_{00},r_{10},r_{01},r_{11}\right]$ defines the probability distribution of the states of servers. If we can find the solution in this form, it will be the unique solution of (13).

Substituting the Equation (14) in the system (13), we obtain

$$\mathbf{r}\{\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2\}+j\gamma_1\mathbf{r}\frac{\partial\Phi(w_1,w_2)/\partial w_1}{\Phi(w_1,w_2)}\{\mathbf{I}_0-\mathbf{I}_1\}$$

$$+j\gamma_2\mathbf{r}\frac{\partial\Phi(w_1,w_2)/\partial w_2}{\Phi(w_1,w_2)}\{\mathbf{I}_2-\mathbf{I}_3\}=0,$$

$$jw_1\left\{\mathbf{r}\mathbf{B}_1+j\gamma_1\mathbf{r}\frac{\partial\Phi(w_1,w_2)/\partial w_1}{\Phi(w_1,w_2)}\mathbf{I}_0\right\}\mathbf{e}$$

$$+jw_2\left\{\mathbf{r}\mathbf{B}_2+j\gamma_2\mathbf{r}\frac{\partial\Phi(w_1,w_2)/\partial w_2}{\Phi(w_1,w_2)}\mathbf{I}_2\right\}\mathbf{e}=0. \quad (15)$$

We will find the solution of the system (15) in the following form

$$\Phi(w_1,w_2)=e^{jw_1a_1+jw_2a_2}, \quad (16)$$

then $\frac{\partial\Phi(w_1,w_2)/\partial w_1}{\Phi(w_1,w_2)}=ja_1$ and $\frac{\partial\Phi(w_1,w_2)/\partial w_2}{\Phi(w_1,w_2)}=ja_2$.

So, substituting these expressions into (15), we obtain

$$\mathbf{r}\{\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2\}-\gamma_1a_1\mathbf{r}\{\mathbf{I}_0-\mathbf{I}_1\}-\gamma_2a_2\mathbf{r}\{\mathbf{I}_2-\mathbf{I}_3\}=0,$$

$$jw_1\mathbf{r}\{\mathbf{B}_1-\gamma_1a_1\mathbf{I}_0\}\mathbf{e}=0,$$

$$jw_2\mathbf{r}\{\mathbf{B}_2-\gamma_2a_2\mathbf{I}_2\}\mathbf{e}=0,$$

$$\mathbf{r}\mathbf{e}=1. \quad (17)$$

Because

$$\lim_{\sigma\to0}Ee^{jw_1\sigma i_1(t)+jw_2\sigma i_2(t)}=\lim_{\sigma\to0}\mathbf{F}(w_1,w_2,\sigma)\mathbf{e}$$
$$=\mathbf{F}(w_1,w_2)\mathbf{e}=e^{jw_1a_1+jw_2a_2},$$

the theorem is proved. □

Solving this system, we find the probability distribution of states of servers $\mathbf{r}$, and parameters a_1 and a_2.

Remark 2. *At first look, the system of Equation (17) has 7 equations while the number of unknowns is 6. However, as we mentioned in Remark 1, the matrix* $\mathbf{Q}$ *is the infinitesimal generator of a Markov chain the system of equations* $\mathbf{rQ} = 0$ *has one redundant equation. Thus, in fact, we have 6 equations for 6 unknowns.*

Remark 3. *Theorem 1 implies that* $\{\sigma i_1(t), \sigma i_2(t)\}$ *converges in distribution to* $\{a_1, a_2\}$ *as* $\sigma \to \infty$*. Thus, we can have the approximation*

$$\{i_1(t), i_2(t)\} \approx \left\{\frac{a_1}{\sigma}, \frac{a_2}{\sigma}\right\}, \quad \sigma \to 0. \tag{18}$$

Equation (18) represents the deterministic part of the number of calls in the orbits. In order to see the stochastic part, we consider the second-order asymptotic in Section 5.

5. The Second Order Asymptotic

We subtract the deterministic part in (18) to investigate the stochastic part. To this end, we define

$$\mathbf{H}(u_1, u_2) = exp\left\{ju_1\frac{a_1}{\sigma} + ju_2\frac{a_2}{\sigma}\right\}\mathbf{H}^{(2)}(u_1, u_2).$$

$\mathbf{H}^{(2)}(u_1, u_2)$ represents the characteristic function of $\{i_1(t) - a_1/\sigma, i_2(t) - a_2/\sigma\}$. Substituting the following in the system (5)

$$\mathbf{H}^{(2)}(u_1, u_2) = exp\left\{-j\frac{u_1}{\sigma}a_1 - ju_2\frac{u_2}{\sigma}a_2\right\}\mathbf{H}(u_1, u_2), \tag{19}$$

we obtain

$$\begin{gathered}
\mathbf{H}^{(2)}(u_1, u_2)\Big\{\mathbf{A} + e^{ju_1}\mathbf{B}_1 + e^{ju_2}\mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - e^{-ju_1}\mathbf{I}_1) \\
-\gamma_2 a_2(\mathbf{I}_2 - e^{-ju_2}\mathbf{I}_3)\Big\} + j\sigma\gamma_1\frac{\partial\mathbf{H}^{(2)}(u_1, u_2)}{\partial u_1}\Big\{\mathbf{I}_0 - e^{-ju_1}\mathbf{I}_1\Big\} \\
+j\sigma\gamma_2\frac{\partial\mathbf{H}^{(2)}(u_1, u_2)}{\partial u_2}\Big\{\mathbf{I}_2 - e^{-ju_2}\mathbf{I}_3\Big\} = 0, \\
\Big(e^{ju_1} - 1\Big)\Big\{\mathbf{H}^{(2)}(u_1, u_2)(\mathbf{B}_1 - e^{-ju_1}a_1\gamma_1\mathbf{I}_0) \\
+j\sigma\gamma_1 e^{-ju_1}\frac{\partial\mathbf{H}^{(2)}(u_1, u_2)}{\partial u_1}\mathbf{I}_0\Big\}\mathbf{e} \\
+(e^{ju_2} - 1\Big)\Big\{\mathbf{H}^{(2)}(u_1, u_2)(\mathbf{B}_2 - e^{-ju_2}a_2\gamma_2\mathbf{I}_2) \\
+j\sigma\gamma_2 e^{-ju_2}\frac{\partial\mathbf{H}^{(2)}(u_1, u_2)}{\partial u_2}\mathbf{I}_2\Big\}\mathbf{e} = 0.
\end{gathered} \tag{20}$$

In the system (20), we make substitutions

$$\sigma = \varepsilon^2, u_1 = \varepsilon w_1, u_2 = \varepsilon w_2, \mathbf{H}^{(2)}(u_1, u_2) = \mathbf{F}^{(2)}(w_1, w_2, \varepsilon). \tag{21}$$

With this substitution, $\mathbf{F}^{(2)}(w_1, w_2, \varepsilon)$ represents the characteristic function of $\left\{\sqrt{\sigma}\left(i_1(t) - \frac{a_1}{\sigma}\right), \sqrt{\sigma}\left(i_2(t) - \frac{a_2}{\sigma}\right)\right\} = \frac{1}{\sqrt{\sigma}}\{\sigma i_1(t) - a_1, \sigma i_2(t) - a_2\}$.
We then rewrite the system in the following form

$$\begin{gathered}
\mathbf{F}^{(2)}(w_1, w_2, \varepsilon)\Big\{\mathbf{A} + e^{j\varepsilon w_1}\mathbf{B}_1 + e^{j\varepsilon w_2}\mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - e^{-j\varepsilon w_1}\mathbf{I}_1) \\
-\gamma_2 a_2(\mathbf{I}_2 - e^{-j\varepsilon w_2}\mathbf{I}_3)\Big\} + j\varepsilon\gamma_1\frac{\partial\mathbf{F}^{(2)}(w_1, w_2, \varepsilon)}{\partial w_1}\Big\{\mathbf{I}_0 - e^{-j\varepsilon w_1}\mathbf{I}_1\Big\}
\end{gathered}$$

$$+j\varepsilon\gamma_2\frac{\partial\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)}{\partial w_2}\left\{\mathbf{I}_2-e^{-j\varepsilon w_2}\mathbf{I}_3\right\}=0,$$

$$\left(e^{j\varepsilon w_1}-1\right)\left\{\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)(\mathbf{B}_1-e^{-j\varepsilon w_1}a_1\gamma_1\mathbf{I}_0)\right.$$
$$\left.+j\varepsilon\gamma_1 e^{-j\varepsilon w_1}\frac{\partial\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)}{\partial w_1}\mathbf{I}_0\right\}\mathbf{e}$$
$$+\left(e^{j\varepsilon w_2}-1\right)\left\{\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)(\mathbf{B}_2-e^{-j\varepsilon w_2}a_2\gamma_2\mathbf{I}_2)\right.$$
$$\left.+j\varepsilon\gamma_2 e^{-j\varepsilon w_2}\frac{\partial\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)}{\partial w_2}\mathbf{I}_2\right\}\mathbf{e}=0. \tag{22}$$

For the system (22), we will solve under the assumption that $\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)$ and its derivatives have limits as $\varepsilon\to 0$.

Theorem 2. *In the context of Theorem 1, we have*

$$\lim_{\sigma\to 0} Ee^{jw_1\sqrt{\sigma}(i_1(t)-\frac{a_1}{\sigma})+jw_2\sqrt{\sigma}(i_2(t)-\frac{a_2}{\sigma})}$$
$$=e^{\frac{(jw_1)^2}{2}K_{11}+\frac{(jw_2)^2}{2}K_{22}+jw_1jw_2K_{12}}, \tag{23}$$

where K_{11}, K_{22}, and K_{12} are the second-order central moments, defined as

$$K_{11}=\frac{(\mathbf{f}_1(\mathbf{B}_1-a_1\gamma_1\mathbf{I}_0)+a_1\gamma_1\mathbf{r}\mathbf{I}_0)\mathbf{e}}{(\gamma_1\mathbf{r}\mathbf{I}_0)\mathbf{e}},\ K_{22}=\frac{(\mathbf{f}_2(\mathbf{B}_2-a_2\gamma_2\mathbf{I}_2)+a_2\gamma_2\mathbf{r}\mathbf{I}_2)\mathbf{e}}{(\gamma_2\mathbf{r}\mathbf{I}_2)\mathbf{e}},$$
$$K_{12}=\frac{(\mathbf{f}_2(\mathbf{B}_1-a_1\gamma_1\mathbf{I}_1)+\mathbf{f}_1(\mathbf{B}_2-a_2\gamma_2\mathbf{I}_3))\mathbf{e}}{(\mathbf{r}(\gamma_1\mathbf{I}_1+\gamma_2\mathbf{I}_3))\mathbf{e}}. \tag{24}$$

Furthermore, $\mathbf{f}_1$ and $\mathbf{f}_2$ are given in the following form:

$$\mathbf{f}_1=C\mathbf{r}+K_{11}\mathbf{g}_{11}+K_{12}\mathbf{g}_{12}-\mathbf{z}_1,$$

$$\mathbf{f}_2=C\mathbf{r}+K_{12}\mathbf{g}_{21}+K_{22}\mathbf{g}_{22}-\mathbf{z}_2, \tag{25}$$

where C is an arbitrary constant, $\mathbf{g}_{11}$, $\mathbf{g}_{12}$, $\mathbf{g}_{21}$, $\mathbf{g}_{22}$, $\mathbf{z}_1$, and $\mathbf{z}_2$ are solutions of the following systems

$$\mathbf{g}_{11}(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\gamma_1\mathbf{r}(\mathbf{I}_0-\mathbf{I}_1),$$
$$\mathbf{g}_{12}(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\gamma_2\mathbf{r}(\mathbf{I}_2-\mathbf{I}_3),$$
$$\mathbf{z}_1(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\mathbf{r}(\mathbf{B}_1-a_1\gamma_1\mathbf{I}_0),$$
$$\mathbf{g}_{11}\mathbf{e}=0,\ \mathbf{g}_{12}\mathbf{e}=0,\ \mathbf{z}_1\mathbf{e}=0,$$
$$\mathbf{g}_{21}(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\gamma_1\mathbf{r}(\mathbf{I}_0-\mathbf{I}_1),$$
$$\mathbf{g}_{22}(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\gamma_2\mathbf{r}(\mathbf{I}_2-\mathbf{I}_3),$$
$$\mathbf{z}_2(\mathbf{A}+\mathbf{B}_1+\mathbf{B}_2-a_1\gamma_1(\mathbf{I}_0-\mathbf{I}_1)-a_2\gamma_2(\mathbf{I}_2-\mathbf{I}_3))=\mathbf{r}(\mathbf{B}_2-a_2\gamma_2\mathbf{I}_2),$$

$$\mathbf{g}_{21}\mathbf{e}=0,\ \mathbf{g}_{22}\mathbf{e}=0,\ \mathbf{z}_2\mathbf{e}=0. \tag{26}$$

Proof. Let us substitute the following expansion into the system (22)

$$\mathbf{F}^{(2)}(w_1,w_2,\varepsilon)=\Phi_2(w_1,w_2)(\mathbf{r}+j\varepsilon w_1\mathbf{f}_1+j\varepsilon w_2\mathbf{f}_2)+O(\varepsilon^2). \tag{27}$$

Taking a series expansion of the exponent, we obtain

$$(\mathbf{r} + j\varepsilon w_1 \mathbf{f}_1 + j\varepsilon w_2 \mathbf{f}_2)\{\mathbf{A} + (1 + j\varepsilon w_1)\mathbf{B}_1 + (1 + j\varepsilon w_2)\mathbf{B}_2$$
$$-\gamma_1 a_1(\mathbf{I}_0 - (1 - j\varepsilon w_1)\mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - (1 - j\varepsilon w_2)\mathbf{I}_3)\}$$
$$+j\varepsilon\gamma_1 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}\{\mathbf{I}_0 - (1 - j\varepsilon w_1)\mathbf{I}_1\}$$
$$+j\varepsilon\gamma_2 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}\{\mathbf{I}_2 - (1 - j\varepsilon w_2)\mathbf{I}_3\} = O(\varepsilon^2),$$
$$\left(j\varepsilon w_1 + \frac{(j\varepsilon w_1)^2}{2}\right)\{(\mathbf{r} + j\varepsilon w_1 \mathbf{f}_1 + j\varepsilon w_2 \mathbf{f}_2)(\mathbf{B}_1 - (1 - j\varepsilon w_1)a_1\gamma_1 \mathbf{I}_0)$$
$$+j\varepsilon\gamma_1(1 - j\varepsilon w_1)\mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}\mathbf{I}_0\bigg\}\mathbf{e}$$
$$+\left(j\varepsilon w_2 + \frac{(j\varepsilon w_2)^2}{2}\right)\{(\mathbf{r} + j\varepsilon w_1 \mathbf{f}_1 + j\varepsilon w_2 \mathbf{f}_2)(\mathbf{B}_2 - (1 - j\varepsilon w_2)a_2\gamma_2 \mathbf{I}_2)$$
$$+j\varepsilon\gamma_2(1 - j\varepsilon w_2)\mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}\mathbf{I}_2\bigg\}\mathbf{e} = O(\varepsilon^3). \tag{28}$$

Let us rewrite the system (28) in the following form:

$$j\varepsilon(w_1 \mathbf{f}_1 + w_2 \mathbf{f}_2)\{\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3)\}$$
$$+j\varepsilon\mathbf{r}\{w_1 \mathbf{B}_1 + w_2 \mathbf{B}_2 - \gamma_1 a_1 w_1 \mathbf{I}_1 - \gamma_2 a_2 w_2 \mathbf{I}_3\}$$
$$+j\varepsilon\gamma_1 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}(\mathbf{I}_0 - \mathbf{I}_1) + j\varepsilon\gamma_2 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}(\mathbf{I}_2 - \mathbf{I}_3) = O(\varepsilon^2),$$
$$j\varepsilon w_1\bigg\{j\varepsilon(w_1 \mathbf{f}_1 + w_2 \mathbf{f}_2)(\mathbf{B}_1 - a_1\gamma_1 \mathbf{I}_0) + \frac{1}{2}j\varepsilon w_1 \mathbf{r}(\mathbf{B}_1 + a_1\gamma_1 \mathbf{I}_0)$$
$$+j\varepsilon\gamma_1 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}\mathbf{I}_0\bigg\}\mathbf{e}$$
$$+j\varepsilon w_2\bigg\{j\varepsilon(w_1 \mathbf{f}_1 + w_2 \mathbf{f}_2)(\mathbf{B}_2 - a_2\gamma_2 \mathbf{I}_2) + \frac{1}{2}j\varepsilon w_2 \mathbf{r}(\mathbf{B}_2 + a_2\gamma_2 \mathbf{I}_2)$$
$$+j\varepsilon\gamma_2 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}\mathbf{I}_2\bigg\}\mathbf{e} = O(\varepsilon^3). \tag{29}$$

Furthermore, let us divide the first equation by $j\varepsilon$, the second by $j^2\varepsilon^2$ and take the limit $\varepsilon \to 0$ to obtain

$$w_1\{\mathbf{f}_1(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3)) + \mathbf{r}(\mathbf{B}_1 - \gamma_1 a_1 \mathbf{I}_1)\}$$
$$+w_2\{\mathbf{f}_2(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3)) + \mathbf{r}(\mathbf{B}_2 - \gamma_2 a_2 \mathbf{I}_3)\}$$
$$+\gamma_1 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}(\mathbf{I}_0 - \mathbf{I}_1) + \gamma_2 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}(\mathbf{I}_2 - \mathbf{I}_3) = 0,$$
$$w_1\left\{(w_1 \mathbf{f}_1 + w_2 \mathbf{f}_2)(\mathbf{B}_1 - a_1\gamma_1 \mathbf{I}_0) + w_1 a_1\gamma_1 \mathbf{r}\mathbf{I}_0 + \gamma_1 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_1}{\Phi_2(w_1,w_2)}\mathbf{I}_0\right\}\mathbf{e}$$
$$+w_2\{(w_1 \mathbf{f}_1 + w_2 \mathbf{f}_2)(\mathbf{B}_2 - a_2\gamma_2 \mathbf{I}_2) + w_2 a_2\gamma_2 \mathbf{r}\mathbf{I}_2$$
$$+\gamma_2 \mathbf{r}\frac{\partial\Phi_2(w_1,w_2)/\partial w_2}{\Phi_2(w_1,w_2)}\mathbf{I}_2\bigg\}\mathbf{e} = 0. \tag{30}$$

We will find the solution in the following form

$$\Phi_2(w_1, w_2) = e^{\left\{\frac{(jw_1)^2}{2}K_{11} + \frac{(jw_2)^2}{2}K_{22} + jw_1 jw_2 K_{12}\right\}}, \tag{31}$$

and then

$$\frac{\partial\Phi_2(w_1, w_2)/\partial w_1}{\Phi_2(w_1, w_2)} = -(w_1 K_{11} + w_2 K_{12}),$$
$$\frac{\partial\Phi_2(w_1, w_2)/\partial w_2}{\Phi_2(w_1, w_2)} = -(w_2 K_{22} + w_1 K_{12}).$$

We substitute these two equations into the system (30) to obtain

$$w_1\{\mathbf{f}_1(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3)) + \mathbf{r}(\mathbf{B}_1 - \gamma_1 a_1 \mathbf{I}_1)$$
$$-\gamma_1 \mathbf{r} K_{11}(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 \mathbf{r} K_{12}(\mathbf{I}_2 - \mathbf{I}_3)\}$$
$$+w_2\{\mathbf{f}_2(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3))$$
$$+\mathbf{r}(\mathbf{B}_2 - \gamma_2 a_2 \mathbf{I}_3) - \gamma_2 \mathbf{r} K_{22}(\mathbf{I}_2 - \mathbf{I}_3) - \gamma_1 \mathbf{r} K_{12}(\mathbf{I}_0 - \mathbf{I}_1)\} = 0,$$
$$w_1^2\left\{\mathbf{f}_1(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0) + \frac{1}{2}\mathbf{r}(\mathbf{B}_1 + a_1\gamma_1\mathbf{I}_0) - \gamma_1 \mathbf{r} K_{11}\mathbf{I}_0\right\}\mathbf{e}$$
$$+w_2^2\left\{\mathbf{f}_2(\mathbf{B}_2 - a_2\gamma_2\mathbf{I}_2) + \frac{1}{2}\mathbf{r}(\mathbf{B}_2 + a_2\gamma_2\mathbf{I}_2) - \gamma_2 \mathbf{r} K_{22}\mathbf{I}_2\right\}\mathbf{e}$$
$$+w_1 w_2\{\mathbf{f}_2(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0) + \mathbf{f}_1(\mathbf{B}_2 - a_2\gamma_2\mathbf{I}_2) - \mathbf{r}K_{12}(\gamma_1\mathbf{I}_0 + \gamma_2\mathbf{I}_2)\}\mathbf{e} = 0. \tag{32}$$

Let us consider equations of the system (32) separately, i.e., the coefficients of w_1 and w_2 are 0.

$$\mathbf{f}_1(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3))$$
$$= -\mathbf{r}(\mathbf{B}_1 - \gamma_1 a_1 \mathbf{I}_1) + \gamma_1 \mathbf{r} K_{11}(\mathbf{I}_0 - \mathbf{I}_1) + \gamma_2 \mathbf{r} K_{12}(\mathbf{I}_2 - \mathbf{I}_3),$$
$$\mathbf{f}_2(\mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3))$$
$$= -\mathbf{r}(\mathbf{B}_2 - \gamma_2 a_2 \mathbf{I}_3) + \gamma_2 \mathbf{r} K_{22}(\mathbf{I}_2 - \mathbf{I}_3) + \gamma_1 \mathbf{r} K_{12}(\mathbf{I}_0 - \mathbf{I}_1). \tag{33}$$

Recall that

$$\mathbf{Q} = \mathbf{A} + \mathbf{B}_1 + \mathbf{B}_2 - \gamma_1 a_1(\mathbf{I}_0 - \mathbf{I}_1) - \gamma_2 a_2(\mathbf{I}_2 - \mathbf{I}_3).$$

The system (33) is an inhomogeneous system of linear algebraic equations for $\mathbf{f}_1$ and $\mathbf{f}_2$. Since the matrix of the coefficients $\mathbf{Q}$ (as explained in Remark 1) is an infinitesimal generator, and the rank of the extended matrix is equal to the rank of $\mathbf{Q}$, the system has many solutions.

Let us consider the inhomogeneous system of Equation (33) and the homogeneous system of Equation (17). If we compare them, we can see that system (17) is the homogeneous system for system (33). In this case, we can write the solution to the system (33) in the form

$$\mathbf{f}_1 = C\mathbf{r} + K_{11}\mathbf{g}_{11} + K_{12}\mathbf{g}_{12} - \mathbf{z}_1,$$

$$\mathbf{f}_2 = C\mathbf{r} + K_{12}\mathbf{g}_{21} + K_{22}\mathbf{g}_{22} - \mathbf{z}_2, \tag{34}$$

where C is a constant, and each of $\mathbf{g}_{11}$, $\mathbf{g}_{12}$,$\mathbf{g}_{21}$, $\mathbf{g}_{22}$, $\mathbf{z}_1$, and $\mathbf{z}_2$ is a particular solution of the inhomogeneous system (33).

By substituting the expression (34) in the system (33), we obtain

$$(K_{11}\mathbf{g}_{11} + K_{12}\mathbf{g}_{12} - \mathbf{z}_1)\mathbf{Q}$$
$$= -\mathbf{r}(\mathbf{B}_1 - \gamma_1 a_1 \mathbf{I}_1) + \gamma_1 \mathbf{r} K_{11}(\mathbf{I}_0 - \mathbf{I}_1) + \gamma_2 \mathbf{r} K_{12}(\mathbf{I}_2 - \mathbf{I}_3),$$
$$(K_{12}\mathbf{g}_{21} + K_{22}\mathbf{g}_{22} - \mathbf{z}_2)\mathbf{Q}$$
$$= -\mathbf{r}(\mathbf{B}_2 - \gamma_2 a_2 \mathbf{I}_3) + \gamma_2 \mathbf{r} K_{22}(\mathbf{I}_2 - \mathbf{I}_3) + \gamma_1 \mathbf{r} K_{12}(\mathbf{I}_0 - \mathbf{I}_1) \quad (35)$$

and get the system of equations to find a particular solution of the inhomogeneous system (33) as follows.

$$\mathbf{g}_{11}\mathbf{Q} = \gamma_1 \mathbf{r}(\mathbf{I}_0 - \mathbf{I}_1), \mathbf{g}_{12}\mathbf{Q} = \gamma_2 \mathbf{r}(\mathbf{I}_2 - \mathbf{I}_3), \mathbf{z}_1\mathbf{Q} = \mathbf{r}(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_1),$$
$$\mathbf{g}_{11}\mathbf{e} = 0, \mathbf{g}_{12}\mathbf{e} = 0, \mathbf{z}_1\mathbf{e} = 0,$$
$$\mathbf{g}_{21}\mathbf{Q} = \gamma_1 \mathbf{r}(\mathbf{I}_0 - \mathbf{I}_1), \mathbf{g}_{22}\mathbf{Q} = \gamma_2 \mathbf{r}(\mathbf{I}_2 - \mathbf{I}_3), \mathbf{z}_2\mathbf{Q} = \mathbf{r}(\mathbf{B}_2 - a_2\gamma_2\mathbf{I}_3),$$
$$\mathbf{g}_{21}\mathbf{e} = 0, \mathbf{g}_{22}\mathbf{e} = 0, \mathbf{z}_2\mathbf{e} = 0. \quad (36)$$

All these systems of linear equations are feasible because $\mathbf{Q}$ is the infinitesimal generator of a Markov chain, as explained in Remark 1.

From the system (36) we can find $\mathbf{g}_{11}$, $\mathbf{g}_{12}$, $\mathbf{g}_{21}$, $\mathbf{g}_{22}$, and $\mathbf{z}_1$, $\mathbf{z}_2$ and substitute to (34) to find $\mathbf{f}_1$, $\mathbf{f}_2$. In order to find K_{11}, K_{22}, and K_{12}, we solve the system of equations such that the coefficients of $w_1^2, w_2^2, w_1 w_2$ in (32) are zeros.

$$K_{11} = \frac{(\mathbf{f}_1(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0) + \frac{1}{2}\mathbf{r}(\mathbf{B}_1 + a_1\gamma_1\mathbf{I}_0))\mathbf{e}}{\gamma_1 \mathbf{r}\mathbf{I}_0\mathbf{e}},$$
$$K_{22} = \frac{(\mathbf{f}_2(\mathbf{B}_2 - a_2\gamma_2\mathbf{I}_2) + \frac{1}{2}\mathbf{r}(\mathbf{B}_2 + a_2\gamma_2\mathbf{I}_2))\mathbf{e}}{\gamma_2 \mathbf{r}\mathbf{I}_2\mathbf{e}},$$
$$K_{12} = \frac{(\mathbf{f}_2(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0) + \mathbf{f}_1(\mathbf{B}_2 - a_2\gamma_2\mathbf{I}_2))\mathbf{e}}{(\mathbf{r}(\gamma_1\mathbf{I}_0 + \gamma_2\mathbf{I}_2))\mathbf{e}}. \quad (37)$$

The theorem is proved. □

So, the second order asymptotic shows that the asymptotic probability distribution of the number of calls in the orbit is a two-dimensional Gaussian with asymptotic means a_1/σ and a_2/σ, dispersions K_{11}/σ and K_{22}/σ, covariance K_{12}/σ.

Remark 4. *In the above procedure, we can see that C is an arbitrary parameter. At the first look, the values of $\mathbf{f}_1$ and $\mathbf{f}_2$ are not unique. As a result, K_{11}, K_{22} and K_{12} may not be unique. However, it turns out that these values are unique because we can prove that C disappears from (37). In fact, we look at*

$$\mathbf{f}_1(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0)\mathbf{e},$$

which includes C due to $\mathbf{f}_1$. Substituting $\mathbf{f}_1$ in (34) into this quantity, we see that the coefficient of C is given by $\mathbf{r}(\mathbf{B}_1 - a_1\gamma_1\mathbf{I}_0)\mathbf{e} = 0$ due to (11). Similarly, we also have that the coefficients of C in the expressions of K_{22} and K_{12} are also zero. These imply that K_{11}, K_{22} and K_{12} do not depend on C.

6. Numerical Examples

We consider $\lambda = 2$, $\mu_1 = 3$, $\mu_2 = 2.5$, $\sigma_1 = 1$, $\sigma_2 = 0.9$, $\sigma = 0.1$, $\gamma_1 = 10$, $\gamma_2 = 9$. Figure 2 presents the approximation of the probability distribution of the number of calls in orbits.

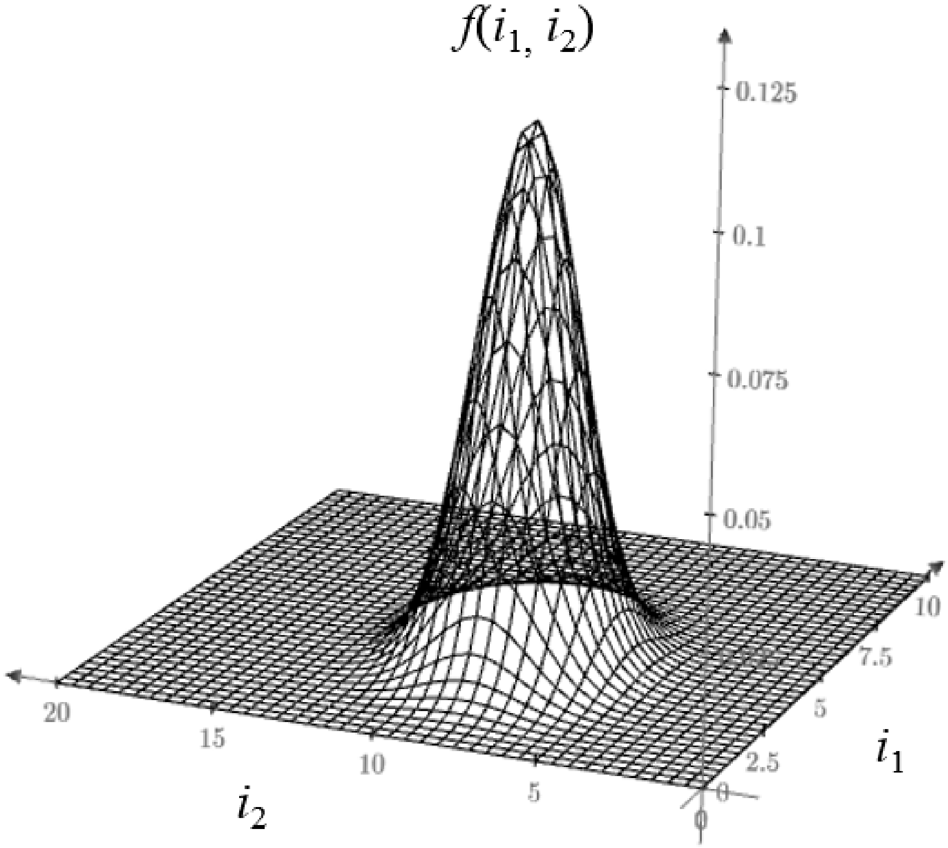

Figure 2. The joint probability distribution $f(i_1, i_2)$ of the numbers of calls in orbits.

In this case, covariance $K_{12}/\sigma = 2.92$, asymptotic means $a_1/\sigma = 4$, $a_2/\sigma = 8.63$, and dispersions $K_{11}/\sigma = 12$, $K_{22}/\sigma = 41.93$ of the number of calls in the orbits and probability distribution of states of servers $\mathbf{r} = [0.051, 0.149, 0.282, 0.518]$.

The characteristic function $H(u)$ of the number of calls in the first orbit has the form [2]

$$H(u) = (1 + \rho_1 - \rho_1 e^{ju}) \left(\frac{1 - \rho_1}{1 - \rho_1 e^{ju}} \right)^{\frac{\lambda}{\sigma_1}+1}, \rho_1 = \frac{\lambda}{\mu_1}. \tag{38}$$

We apply the inverse Fourier transform of the pre-limit characteristic function (38) and write the pre-limit density of the probability distribution of the number of calls in the first orbit in the form:

$$p(i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jui} H(u) du. \tag{39}$$

We can also explicitly express $p(i)$ as in Ref. [2]. We obtain the Gaussian limiting probability distribution of the number of calls in the first orbit p_ass_i with asymptotic mean a_1/σ and dispersions K_{11}/σ. We consider a numerical example for $\lambda = 0.5$, $\mu_1 = 2$, $\sigma = 0.1$, $\sigma_1 = 0.01$, $\gamma_1 = 0.1$, $\mu_2 = 2.5$, $\sigma_2 = 0.9$, $\gamma_2 = 9$ and show the comparison of the asymptotic density p_ass_i and the pre-limit probability density p_i of the number of calls in the first orbit in Figure 3, where the Kolmogorov distance is equal to 0.017, which is acceptable.

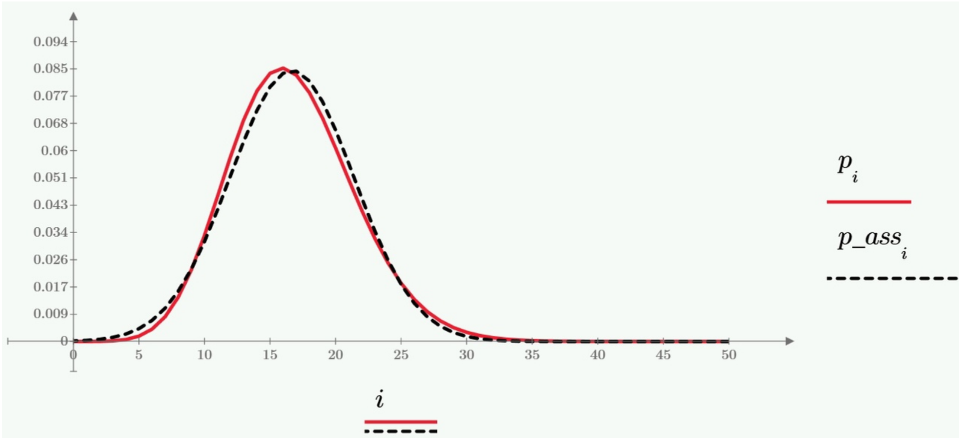

Figure 3. Comparison of the asymptotic and prelimit densities probability distribution of the number of calls in the first orbit.

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