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A Moment Approach for a Conditional Central Limit Theorem of Infinite-Server Queue: A Case of M/M^X/∞ Queue

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Abstract: Several studies have been conducted on scaling limits for Markov-modulated infinite-server queues. To the best of our knowledge, most of these studies adopt an approach to prove the convergence of the moment-generating function (or characteristic function) of the random variable that represents a scaled version of the number of busy servers and show the weak law of large numbers and the central limit theorem (CLT). In these studies, an essential assumption is the finiteness of the phase process and, in most of them, the CLT for the number of busy servers conditional on the phase (or the joint states) has not been considered. This paper proposes a new method called the *moment approach* to address these two limitations in an infinite-server batch service queue, which is called the M/M^X/∞ queue. We derive the conditional weak law of large numbers and a recursive formula that suggests the conditional CLT. We derive series expansion of the conditional raw moments, which are used to confirm the conditional CLT by a symbolic algorithm.

Keywords: queueing model; infinite server; asymptotic analysis; weak law of large numbers; central limit theorem; moment approach; raw moment; factorial moment; stirling number; symbolic algorithm

MSC: 60K25; 60K05; 90B22

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1. Introduction

Infinite-server queues have a wide range of applications as an approximation for queues with a sufficient number of servers and have been studied for a long time. In particular, studies have been conducted on the scaling models, where the arrival and transition rates of the phase are scaled by a factor. For example, Blom et al. [1] considered an infinite-server queue with arrival and service rates depending on the state of the Markovian background process. They derived the weak law of large numbers and the central limit theorem (CLT) for the number of busy servers under heavy traffic. To the best of our knowledge, most of these studies have adopted an approach to prove the convergence of the moment-generating function (or characteristic function) of the random variable that represents the number of busy servers [1–5]. The limitations of this approach are: (i) the state space of the phase (e.g., Markovian background state in [1]) is finite, and (ii) the number of busy servers conditional on the phase has not been considered (it should be noted that only [5] derives the CLT in terms of a matrix representation for the two-dimensional joint states of the M/G/∞ queue in a random environment, by using the characteristic function in Kolmogorov differential equations of the joint state probabilities).

This paper proposes a new method called the *moment approach* for the analysis of scaling models of a special class of infinite-server queues. In a nutshell, the idea is to prove the convergence of an arbitrary order moment of the random variable for the scaled number of the number of busy servers *conditional* on the phase to the corresponding moments of the targeting distribution. The proposed approach may have the following advantages:

1. Models whose phases have *infinite* state space can be considered, as long as their joint moments (of the phase and number of busy servers) are explicitly obtained.
2. Scaling limits for the number of busy servers *conditional* on the phase can be considered.

As the first to investigate this approach, we consider an infinite-server batch service queue, which is called the $M/M^X/\infty$ queue. The batch size distribution X is an *infinite* support discrete distribution. The $M/M^X/\infty$ queue has several applications where customers, products, or data are served in a group (whose size is determined by an arbitrary distribution), for example, transportation systems such as shuttle buses and ride-sharing, logistic systems such as home delivery service, and batch-processing in data centers [6,7]. The detailed settings of the $M/M^X/\infty$ queue are described in the next section. Note that most related studies (e.g., [6,8,9]) assume a finite support distribution, although many discrete distributions have infinite support, for example, geometric and Poisson distributions. Regarding the scaling model, our previous research [7] derived the weak law of large numbers and the CLT of the number of busy servers for the $M/M^X/\infty$ queue. However, this research imposed the constraint that X follows a finite support distribution in order to adapt to [1]. In this study, we consider the scaling limits for the number of busy servers conditional on the number of waiting customers and the batch size of the $M/M^X/\infty$ queue with an arbitrary support batch-size distribution. In particular, we prove the conditional weak law of large numbers and a recursive formula suggesting the conditional CLT. Furthermore, we derive series expansions of the conditional moments and propose a symbolic computation algorithm to confirm the CLT.

The reminder of this paper is organized as follows. First, we describe the $M/M^X/\infty$ queue in detail in Section 2. In Section 3, we summarize the results of the previous research and prepare some lemmas. Based on Section 3, we present the main results in Section 4. Furthermore, some symbolic computation results for the proof of CLT are shown in Section 5. Finally, concluding remarks are presented in Section 6.

2. $M/M^X/\infty$ Queue and Preliminary Results

This section summarizes the existing results of the $M/M^X/\infty$ queue that were obtained in [7]. The arrival process of customers is a Poisson process with rate λ, and the service time follows an exponential distribution with rate μ. The batch size distribution and its probability generating function (PGF) are defined as $q_c = \mathbf{P}(X = c)$ $(c \geqq 1)$ and $\Psi(z) = \sum_{c=1}^{\infty} z^c q_c$, respectively. We define the j-th factorial moment B_j as

$$B_j = \mathbf{E}[X(X-1)\dots(X-1+j)],$$

and we assume that $B_j < \infty$ for any j. Once the number of waiting customers reaches X, all these customers are served by one server for an exponentially distributed time with mean $1/\mu$, and a new batch size is newly determined by X. Let $I(t)$, $S(t)$, and $U(t)$ denote the number of busy servers, number of waiting customers in the system, and size of the batch collecting customers, respectively, at time t. In addition, defining

$$\mathcal{V}_1 = \{(0,1)\},$$

$$\mathcal{V}_a = \{(1,a),(2,a),\dots(a-1,a)\}\ (a \geqq 2),$$

$$\mathcal{V} = \{\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\dots\}$$

$$\mathcal{U} = \mathbb{Z} \times \mathcal{V},$$

it is clear that $\{(I(t),S(t),U(t)) \mid t \geqq 0\}$ becomes an irreducible continuous-time Markov chain with the state space $\mathcal{U}$. Moreover, let I, S, and U denote $I(t), S(t)$, and $U(t)$ in the steady state, respectively.

We can also denote the stationary distribution and PGF as

$$\pi_{i,k,c} = \lim_{t\to\infty} \mathbf{P}(I(t) = i, S(t) = k, U(t) = c),$$

$$\Pi_{k,c}(z) = \sum_{i=0}^{\infty} \pi_{i,k,c} z^i, \qquad 0 \leqq k \leqq c-1, \quad 1 \leqq c \quad z \leqq 1,$$

respectively. The stationary distribution $\pi_{i,k,c}$ and PGF $\Pi_{k,c}(z)$ are given by Lemma 1.

Lemma 1 (Theorem 1 in [7])**.** *$\pi_{i,k,c}$ and $\Pi_{k,c}(z)$ are given as follows:*

$$\pi_{i,k,c} = \frac{1}{i!} \sum_{n=0}^{\infty} f_{n+i}^{(k,c)} \frac{(-1)^n}{n!}, \qquad \Pi_{k,c}(z) = \mathcal{F}_{k,c}(z-1), \tag{1}$$

where

$$f_n^{(0,1)} = \frac{n!\left(\dfrac{\lambda}{\mu n + \lambda}\right)}{1 - \Psi\left(\dfrac{\lambda}{\mu n + \lambda}\right)} \prod_{l=1}^{n-1} \left\{ \frac{\Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)}{1 - \Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)} \right\} \times \frac{1}{\mathbf{E}[X]},$$

$$f_n^{(k,c)} = \frac{n!\left(\dfrac{\lambda}{\mu n + \lambda}\right)^{k+1}}{1 - \Psi\left(\dfrac{\lambda}{\mu n + \lambda}\right)} \prod_{l=1}^{n-1} \left\{ \frac{\Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)}{1 - \Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)} \right\} \times \frac{q_c}{\mathbf{E}[X]},$$
$$1 \leqq k \leqq c-1, \quad 2 \leqq c.$$

with the convention that $\prod_{l=1}^{0} = 1$, and

$$\mathcal{F}_{k,c}(z) = \sum_{n=0}^{\infty} f_n^{(k,c)} \frac{z^n}{n!}.$$

Proof. The proof is given by Appendix A. □

Based on Lemma 1, Lemma 2 can be shown immediately.

Lemma 2. *Let $I_{(k,c)}$ denote the conditional random variable for the number of busy servers when $S = k, U = c$, that is,*

$$\mathbf{P}(I_{(k,c)} = i) = \frac{\mathbf{P}(I = i, S = k, U = c)}{\mathbf{P}(S = k, U = c)},$$

under the steady state. The n-th factorial moment is defined as

$$f_{(k,c,n)} = \mathbf{E}[I_{(k,c)}(I_{(k,c)} - 1)(I_{(k,c)} - 2) \ldots (I_{(k,c)} - n + 1)],$$

and this can be calculated as follows:

$$f_{(k,c,n)} = \frac{n!\left(\dfrac{\lambda}{\mu n + \lambda}\right)^{k+1}}{1 - \Psi\left(\dfrac{\lambda}{\mu n + \lambda}\right)} \prod_{l=1}^{n-1} \left\{ \frac{\Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)}{1 - \Psi\left(\dfrac{\lambda}{\mu l + \lambda}\right)} \right\}.$$

Proof. The stationary probability that the number of waiting customers is k and the size of the batch collecting customers in progress is c at the moment a customer arrives at the system is given by

$$\mathbf{P}(S = k, U = c) = \frac{q_c}{\mathbf{E}[X]}.$$

By substituting this result for Lemma 1, we obtain Lemma 2. □

Here, let $\hat{\mathcal{F}}_{k,c}(z)$ denote the factorial moment-generating function (FMGF) of the conditional number of busy servers given by

$$\hat{\mathcal{F}}_{k,c}(z) = \sum_{n=0}^{\infty} f_{(k,c,n)} \frac{z^n}{n!}.$$

Then Lemma 3 can be naturally shown.

Lemma 3. *The steady state probabilities and PGF, i.e.,*

$$\hat{\Pi}_{k,c}(z) := \sum_{i=0}^{\infty} \mathbf{P}(I_{(k,c)} = i) z^i,$$

for the conditional number of busy servers are given as follows:

$$\mathbf{P}(I_{(k,c)} = i) = \frac{1}{i!} \sum_{n=0}^{\infty} f_{(k,c,n+i)} \frac{(-1)^n}{n!} \qquad \text{and} \qquad \hat{\Pi}_{k,c}(z) = \hat{\mathcal{F}}_{k,c}(z-1). \tag{2}$$

Proof. Lemma 2 and the discussion of Lemma 1 yield the conclusion. □

Remark 1. *In what follows, let the symbol with the superscript (N) denote the corresponding one where λ is replaced by $N\lambda$. For example, $f^{(N)}_{(k,c,n)}$ represents $f_{(k,c,n)}$ where λ is replaced by $N\lambda$ as follows:*

$$f^{(N)}_{(k,c,n)} = \frac{n!\left(\frac{N\lambda}{\mu n + \lambda}\right)^{k+1}}{1 - \Psi\left(\frac{N\lambda}{\mu n + N\lambda}\right)} \prod_{l=1}^{n-1} \left\{ \frac{\Psi\left(\frac{N\lambda}{\mu l + N\lambda}\right)}{1 - \Psi\left(\frac{N\lambda}{\mu l + N\lambda}\right)} \right\}.$$

Some asymptotic results for the $M/M^X/\infty$ queue are also proven in [7]. In [7], a weak law of large numbers and CLT for the number of busy servers under heavy traffic regime and the constraint that X follows a finite support distribution were proven (see Lemmas 4 and 5). The method of the proof complies with [1]. Due to the assumption for X, $M/M^X/\infty$ can be considered a special case of the model in [1]. To prove Lemmas 4 and 5, it is enough to prove

$$\lim_{N\to\infty} \frac{\mathbf{E}[I^{(N)}]}{N} = \frac{\lambda}{\mathbf{E}[X]\mu}, \tag{3}$$

and

$$\lim_{N\to\infty} \frac{\mathbf{E}[(I^{(N)} - \mathbf{E}[I^{(N)}])^2]}{N} = \frac{\lambda}{\mu} \frac{\mathbf{E}[X]^2 + \mathbf{E}[X] + \mathbf{V}[X]}{2\mathbf{E}[X]^3}. \tag{4}$$

Equations (3) and (4) can be proven by using the closed-form expressions of the factorial moments in Lemma 1. We omit the detail of the proof in this paper (see [7]). Note that (3) and (4) are true for any distributions of X. Assuming that X has finite support, Lemmas 4 and 5 are established due to [1].

Lemma 4 (Theorem 5 in [7])**.** *Under the constraint that X follows a finite support distribution,*

$$N^{-1} I^{(N)} \text{ converges in the distribution to } \frac{\lambda}{\mathbf{E}[X]\mu} \ (:= g) \text{ as } N \to \infty.$$

Lemma 5 (Theorem 6 in [7])**.** *Under the constraint that X follows a finite support distribution,*

$$N^{1/2}\left(\frac{I^{(N)}}{N} - g\right)$$

converges to a random variable with normal distribution with a zero mean and variance σ^2, where

$$\sigma^2 := \frac{\lambda}{\mu}\frac{\mathbf{E}[X]^2 + \mathbf{E}[X] + \mathbf{V}[X]}{2\mathbf{E}[X]^3},$$

as $N \to \infty$.

3. Preliminaries for the Moment Approach

This section presents the preliminaries for the moment approach in Section 4.

Lemma 6 (Theorem 1 in [10])**.** *Let $\{F_n(x)\}$ be a sequence of distribution functions for which the moments*

$$M_r(n) = \int_{-\infty}^{\infty} x^r dF_n(x)$$

exist for all $r = 0, 1, 2, \ldots$. Moreover, let $F(x)$ be a distribution function in which the moments

$$M_r = \int_{-\infty}^{\infty} x^r dF(x),$$

exist for all $r = 0, 1, 2, \ldots$. If

$$\lim_{n\to\infty} M_r(n) = M_r,$$

for all $r = 0, 1, 2, \ldots$, and if $F(x)$ is uniquely determined by the sequence of moments M_0, $M_1, M_2, \cdots$, then

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for each continuity point of $F(x)$.

Lemma 7 (Theorem 1 in [11])**.** *For a random variable X with $m_k = \mathbf{E}[X^k]$ and $f_k = \mathbf{E}[X(X-1)\ldots(X-k+1)]$,*

$$m_n = \sum_{k=0}^{n} S(n,k) f_k,$$

for $n \geqq 1$, where $S(n,k)$ is the Stirling number of the second kind,

$$S(n+1,k) = kS(n,k) + S(n,k-1),$$

with the initial conditions $S(0,0) = 1$ and $S(n,0) = S(0,n) = 0$.

Lemma 8. *The following fact is well known:*
Letting X denote the random variable that follows the normal distribution with $m_k = \mathbf{E}[X^k]$ and variance σ^2, we obtain

$$m_{n+1} = m_1 m_n + n\sigma^2 m_{n-1}.$$

Lemma 9. *For an arbitrary number a, the following relationship holds:*

$$\Psi\left(\frac{1}{1+ax}\right) = 1 + \sum_{i=1}^{\infty}\sum_{j=1}^{i} \frac{B_j}{j!}(-ax)^i.$$

Proof. We define the function $h(x)$ as

$$h(x) = \Psi\left(\frac{1}{1+ax}\right) = \sum_{i=0}^{\infty} \frac{h^{(i)}(0)}{i!} x^i.$$

We can easily calculate the derivative $h^{(i)}(x)$ and prove this lemma. □

Lemma 10. *For an arbitrary natural number c, the following relationship holds:*

$$\left(\frac{1}{1+cx}\right)^{k+1} = \sum_{i=0}^{\infty} {}_{k+i}C_i(-cx)^i,$$

where ${}_kC_i = k!/\{i!(k-i)!\}$.

Proof. The following transformation is obtained:

$$(1+cx)^{-(k+1)} = \sum_{i=0}^{\infty} \frac{\prod_{j=0}^{i-1}(k+1+j)}{i!}(-cx)^i = \sum_{i=0}^{\infty} \frac{(k+i)!}{k!i!}(-cx)^i = \sum_{i=0}^{\infty} {}_{k+i}C_i(-cx)^i.$$

The above immediately yields this lemma. □

4. Main Results

This section presents the main results of this study. In this section, we consider the weak law of large numbers and CLT for the number of busy servers conditional on the number of waiting customers and the batch size of the $M/M^X/\infty$ queue with an infinite support batch-size distribution.

It should be noted that we cannot use the framework of [1]. This is because elements 1 and 2 in Section 1 were not considered in [1]. Bolm et al. [1] derived differential equations for the moment-generating functions of the random variables which are scaled versions of the number of busy servers, i.e., $N^{-1}I^{(N)}$ and $N^{1/2}\left(I^{(N)}/N - g\right)$ in Lemmas 4 and 5, and then proved the pointwise convergence to the moment-generating functions of degenerate distribution and normal distribution. However, in our settings, it seems to be difficult to even derive the differential equations that the conditional number of busy servers, i.e., $I_{(k,c)}$, satisfies from (A5)–(A7) in Lemma 1. Although the infinite series expression for the PGF is shown in Lemma 3, it is hard to discuss the limit of this representation directly.

Therefore, as we introduced in Section 1, we tackle this challenging analysis using the moment approach. Based on Lemma 6, we consider the limit for the explicit expression of the conditional factorial moments in Lemma 2 and discuss the weak law of large numbers and CLT.

First, Theorem 1 shows the weak law of large numbers $N^{-1}I_{(k,c)}^{(N)}$.

Theorem 1. $N^{-1}I_{(k,c)}^{(N)}$ *converges in the distribution to g as $N \to \infty$.*

Proof. We can rewrite the n-th factorial moment $f_{(k,c,n)}^{(N)}$ for $n \geqq 1$ as follows,

$$f_{(k,c,n)}^{(N)} = \xi_{(k,c,n)}^{(N)} f_{(k,c,n-1)}^{(N)},$$

where

$$\xi_{(k,c,n)}^{(N)} = \frac{\Psi\left(\dfrac{N\lambda}{\mu(n-1)+N\lambda}\right)}{\left(\dfrac{N\lambda}{\mu(n-1)+N\lambda}\right)^{k+1}} \frac{n\left(\dfrac{N\lambda}{n\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\dfrac{N\lambda}{n\mu+N\lambda}\right)},$$

with the initial condition $f_{(k,c,0)}^{(N)} = 1$. By using Lemma 9 as $x = \mu/(N\lambda)$, we obtain

$$\lim_{N\to\infty} \frac{\xi_{(k,c,n)}^{(N)}}{N} = \frac{\lambda}{B_1\mu} = \frac{\lambda}{\mathbf{E}[X]\mu} = g \qquad \Longrightarrow \qquad \lim_{N\to\infty} \frac{f_{(k,c,n)}^{(N)}}{N^n} = \left(\frac{\lambda}{\mathbf{E}[X]\mu}\right)^n = g^n$$

for $n \geqq 0$. From Lemma 7, the n-th raw moment of $I_{(k,c)}$ is given by

$$m_{(k,c,n)}^{(N)} = \sum_{k=0}^{n} S(n,k) f_{(k,c,n)}^{(N)}.$$

Therefore, we obtain

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{I_{(k,c)}^{(N)}}{N}\right)^n\right] = \lim_{N\to\infty} \frac{m_{(k,c,n)}^{(N)}}{N^n} = \lim_{N\to\infty} \frac{\sum_{k=0}^{n} S(n,k) f_{(k,c,n)}^{(N)}}{N^n} = \left(\frac{\lambda}{\mathbf{E}[X]\mu}\right)^n = g^n$$

for $n \geqq 0$. By applying Lemma 6, we can prove the lemma. □

Next, we move on to the discussion of CLT. We define $\kappa_{(k,c,n)}^{(N)}$ as

$$\kappa_{(k,c,n)}^{(N)} = \frac{m_{(k,c,n)}^{(N)}}{N^n} = \frac{\sum_{k=0}^{n} S(n,k) f_{(k,c,n)}^{(N)}}{N^n} \tag{5}$$

and obtain the following lemma and theorem.

Lemma 11. *The following infinite series expressions hold:*

$$\frac{\xi_{(k,c,n)}^{(N)}}{N} = \sum_{i=0}^{\infty} \frac{\xi_{(k,c,n,i)}}{N^i} = g + \sum_{i=1}^{\infty} \frac{\xi_{(k,c,n,i)}}{N^i}, \qquad \text{and} \qquad \frac{f_{(k,c,n)}^{(N)}}{N^r} = \sum_{i=0}^{\infty} \frac{f_{(k,c,n,i)}}{N^i} = g^n + \sum_{i=1}^{\infty} \frac{f_{(k,c,n,i)}}{N^i},$$

where

$$\xi_{(k,c,n,i)} = \frac{1}{l_0}\left(r_i - \sum_{m=0}^{i-1} \xi_{(k,c,n,i)} l_{i-m}\right), \tag{6}$$

$$l_i = \frac{\mu}{\lambda} {}_{k+i}C_i \left\{\frac{-(n-1)\mu}{\lambda}\right\}^i B_1 + \frac{\mu}{\lambda} \sum_{m=1}^{i} {}_{k+i}C_{i-m} \left\{\frac{-(n-1)\mu}{\lambda}\right\}^{i-m} \sum_{j=1}^{i+1} \frac{B_j}{j!} \left\{\frac{-n\mu}{\lambda}\right\}^m, \tag{7}$$

$$r_i = {}_{k+i}C_i \left(\frac{-n\mu}{\lambda}\right)^i + \sum_{m=1}^{i} \sum_{j=1}^{i} \frac{B_j}{j!} \left\{\frac{-(n-1)\mu}{\lambda}\right\}^m {}_{k+i}C_{i-m} \left(\frac{-n\mu}{\lambda}\right)^{i-m}, \tag{8}$$

$$f_{(k,c,n,i)} = \sum_{h=0}^{i} \xi_{(k,c,n,h)} f_{(k,c,n-1,i-h)}, \tag{9}$$

under initial conditions

$$\xi_{(k,c,n,0)} = g, \qquad f_{(k,c,0,0)} = 1, \qquad f_{(k,c,0,i)} = 0 \ \ (i \geqq 1).$$

Proof. Evidently, the following holds:

$$\frac{f_{(k,c,n)}^{(N)}}{N^n} = \frac{f_{(k,c,n-1)}^{(N)}}{N^{n-1}} \frac{\xi_{(k,c,n)}^{(N)}}{N}.$$

Therefore, we obtain the following relationship for $n \geqq 1$:

$$f_{(k,c,n,i)} = \sum_{h=0}^{i} \xi_{(k,c,n,h)} f_{(k,c,n-1,i-h)} \tag{10}$$

under initial conditions

$$f_{(k,c,0,0)} = 1, \qquad f_{(k,c,0,i)} = 0 \ \ (i \geqq 1).$$

Here, we obtain the following transformation for $\xi^{(N)}_{(k,c,n)}/N$ by applying Lemmas 9 and 10 (we define $x = \mu/(N\lambda)$ as the second equality).

$$\begin{aligned}\frac{\xi^{(N)}_{(k,c,n)}}{N} &= \frac{1}{N} \times \frac{\Psi\left(\dfrac{N\lambda}{\mu(n-1)+N\lambda}\right)}{\left(\dfrac{N\lambda}{\mu(n-1)+N\lambda}\right)^{k+1}} \frac{n\left(\dfrac{N\lambda}{n\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\dfrac{N\lambda}{n\mu+N\lambda}\right)} \\ &= \frac{1}{N} \times \frac{1+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\dfrac{B_j}{j!}(-(n-1)x)^i}{\sum_{i=0}^{\infty} {}_{k+i}C_i(-(n-1)x)^i} \times \frac{n\sum_{i=0}^{\infty} {}_{k+i}C_i(-nx)^i}{-\sum_{i=1}^{\infty}\sum_{j=1}^{i}\dfrac{B_j}{j!}(-nx)^i}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\left(\sum_{i=0}^{\infty}\frac{\xi_{(k,c,n,i)}}{N^i}\right)\left(\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(-\sum_{i=1}^{\infty}\sum_{j=1}^{i}N\frac{B_j}{j!}\left\{\frac{-n\mu}{N\lambda}\right\}^i\right) \\ &\qquad= \left(1+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{B_j}{j!}\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(r\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-n\mu}{N\lambda}\right\}^i\right) \\ \Longrightarrow\quad &\left(\sum_{i=0}^{\infty}\frac{\xi_{(k,c,n,i)}}{N^i}\right)\left(\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(\frac{n\mu}{\lambda}\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{B_j}{j!}\left\{\frac{-n\mu}{N\lambda}\right\}^{i-1}\right) \\ &\qquad= \left(1+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{B_j}{j!}\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(r\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-n\mu}{N\lambda}\right\}^i\right) \\ \Longrightarrow\quad &\frac{\mu}{\lambda}\left(\sum_{i=0}^{\infty}\frac{\xi_{(k,c,n,i)}}{N^i}\right)\left(\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(B_1+\sum_{i=1}^{\infty}\sum_{j=1}^{i+1}\frac{B_j}{j!}\left\{\frac{-n\mu}{N\lambda}\right\}^i\right) \\ &\qquad= \left(1+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{B_j}{j!}\left\{\frac{-(n-1)\mu}{N\lambda}\right\}^i\right)\left(\sum_{i=0}^{\infty} {}_{k+i}C_i\left\{\frac{-n\mu}{N\lambda}\right\}^i\right).\end{aligned} \tag{11}$$

Then, we can rewrite (11) as

$$\sum_{i=0}^{\infty}\frac{\xi_{(k,c,n,i)}}{N^i}\sum_{i=0}^{\infty}\frac{l_i}{N^i} = \sum_{i=0}^{\infty}\frac{r_i}{N^i}, \tag{12}$$

where l_i and r_i are given in (7) and (8), respectively. Therefore, considering (7), (8), and (12), we obtain

$$\begin{aligned}&\sum_{m=0}^{i}\xi_{(k,c,n,i)}l_{i-m} = r_i \\ &\Longrightarrow \xi_{(k,c,n,i)} = \frac{1}{l_0}\left(r_i - \sum_{m=0}^{i-1}\xi_{(k,c,n,i)}l_{i-m}\right).\end{aligned}$$

This yields the lemma. □

Theorem 2. *The following recursive formula holds for $n \geqq 1$. Hereafter, we define the notations $o(x)$ and $O(x)$, which satisfy $\lim_{x\to 0} o(x)/x = 0$ and $\lim_{x\to 0} O(x)/x = K$, respectively, where K is a non-zero constant.*

$$\kappa^{(N)}_{(k,c,n+1)} = \kappa^{(N)}_{(k,c,1)}\kappa^{(N)}_{(k,c,n)} + n\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-1)} + O(N^{-2})$$

Proof. By comparing the coefficients of $1/N$ in (11) (note that $\xi_{(k,c,n,0)} = g$), we obtain

$$\xi_{(k,c,n,1)} = \frac{-2(n-1)B_1^2 + 2(n-k-1)B_1 + nB_2}{2B_1^2}. \tag{13}$$

Next, we define the following. The coefficient of $1/N$ of the right-hand side in (14) is $0\ (= g^{n+1} - g \times g^n)$.

$$N\kappa^{(N)}_{(k,c,n+1)} - N\kappa^{(N)}_{(k,c,1)}\kappa^{(N)}_{(k,c,n)} - n\sigma^2\kappa^{(N)}_{(k,c,n-1)} = \sum_{i=0}^{\infty}\frac{a_{(k,c,n,i)}}{N^i}, \tag{14}$$

$$\kappa^{(N)}_{(k,c,n)} = \sum_{i=0}^{\infty}\frac{b_{(k,c,n,i)}}{N^i}, \tag{15}$$

where

$$a_{(k,c,n,i)} = b_{(k,c,n+1,i+1)} - \sum_{j=0}^{i+1} b_{(k,c,1,j)}b_{(k,c,n,i+1-j)} - n\sigma^2 b_{(k,c,n-1,i)}, \tag{16}$$

$$b_{(k,c,n,i)} = \begin{cases} \sum_{r=n-i}^{n} S(n,r) f_{(k,c,r,r-n+i)}, & i \leqq n, \\ \sum_{r=0}^{n} S(n,r) f_{(k,c,r,r-n+i)}, & i \geqq n, \end{cases} \tag{17}$$

by using (5). Then, we can find

$$\begin{aligned} a_{(k,c,n,0)} &= S(n+1,n)g^n + f_{(k,c,n+1,1)} - gf_{(k,c,n,1)} - S(n,n-1)g^n - b_{(k,c,1,1)}g^n - n\sigma^2 g^{n-1} \\ &= f_{(k,c,n+1,1)} - gf_{(k,c,n,1)} - \xi_{(k,c,1,1)}g^n + ng^n - ng^n\frac{B_2+2B_1}{2(B_1)^2} \\ &= \xi_{(k,c,n+1,1)}g^n - \xi_{(k,c,1,1)}g^n + ng^n - ng^n\frac{B_2+2B_1}{2(B_1)^2} \\ &= g^n\frac{-2nB_1^2 + 2nB_1 + nB_2}{2B_1^2} + ng^n - ng^n\frac{B_2+2B_1}{2(B_1)^2} \\ &= 0. \end{aligned} \tag{18}$$

Note that we used some previously derived results for the transformation in (18), namely, (13),

$$b_{(k,c,1,1)} = \xi_{(k,c,1,1)}, \qquad \sigma^2 = g\frac{B_2+2B_1}{2(B_1)^2},$$

$$S(n+1,n) - S(n,n-1) = nS(n,n) = n \qquad (\because \text{Lemma } 7).$$

We can easily confirm $S(n,n) = 1$ for $n \geqq 0$ due to the definition of Lemma 7, and

$$f_{(k,c,n+1,1)} - gf_{(k,c,n,1)} = \xi_{(k,c,n+1,1)}g^n \qquad (\because (9)).$$

This concludes the proof. □

Based on Theorem 2 and Lemma 8, we have the normal approximation as follows:

Remark 2. *We have the following normal approximation for $N \gg 1$:*

$$\frac{I^{(N)}_{(k,c)}}{N} \approx Normal\left(g, \frac{\sigma^2}{N}\right).$$

Based on the above, we have the following conjecture.

Conjecture 1. *The following CLT holds:*

$$N^{1/2}\left(\frac{I_{(k,c)}^{(N)}}{N}-g\right)$$

converges to a random variable following a normal distribution with a zero mean and variance σ^2 *as* $N \to \infty$.

Remark 3. *Assuming that the CLT in Conjecture 1 holds, we obtain the following insights for the* $M/M^X/\infty$ *queue under heavy traffic:*

- *The larger the variance of the batch-size distribution, the larger the variance of the normal distribution, i.e., the variance of the CLT takes the minimum value for the constant batch size.*
- *The CLT is equivalent to Lemma 4, i.e., the number of busy servers is independent of the number of waiting customers and batch size under heavy traffic, although the steady state probabilities in Lemma 3 show the dependency of these random variables.*

In the following, we present a symbolic computation algorithm to confirm CLT in Conjecture 1. To this end, we prepare the following lemma:

Lemma 12. *Provided that*

$$\lim_{N\to\infty} N^{n/2}\mathbf{E}\left[\left(\frac{I_{(k,c)}^{(N)}}{N}-g\right)^n\right]=\theta_n \tag{19}$$

exists, the necessary and sufficient condition for the CLT in Conjecture 1 is

$$\theta_n = \begin{cases} 0 & (n \text{ is odd}) \\ (n-1)!!\sigma^n & (n \text{ is even}) \end{cases} \tag{20}$$

where $n!! = \prod_{k=0}^{\lceil n/2\rceil -1}(n-2k)$, *or*

$$\theta_{n+1} = n\sigma^2\theta_{n-1}, \tag{21}$$

for $n \geqq 1$.

Proof. Lemmas 6 and 8 yield the lemma immediately. □

Regarding Lemma 12, the following theorem can be proved:

Theorem 3. *(19) and (20) in Lemma 12 holds true for* $1 \leqq n \leqq 4$.

Proof. We prove that (19) and (20) in Lemma 12 hold for $1 \leqq n \leqq 4$ in a straightforward manner by using Lemmas 9 and 10. We show the proof for $n = 1, 2$. We omit the case $n = 3, 4$ (the simple outline of the proof for $n = 3$ is shown in Appendix B).

The proof for the case $n = 1$ is shown as follows:

$$\begin{aligned}
\lim_{N\to\infty} N^{1/2}\left(\frac{\mathbf{E}\left[I_{(k,c)}^{(N)}\right]}{N} - g\right) &= \lim_{N\to\infty} N^{1/2}\left(\frac{1}{N}\frac{\left(\frac{N\lambda}{\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)} - g\right) \\
&= \lim_{N\to\infty} N^{1/2}\left(\frac{1}{N}\frac{\left(\frac{1}{\mu/(N\lambda)+1}\right)^{k+1}}{1-\Psi\left(\frac{1}{1+\mu/(N\lambda)}\right)} - g\right) \\
&= \lim_{N\to\infty} N^{1/2}\left(\frac{1}{N}\frac{1}{B_1\mu/(N\lambda)+o(N^{-2})} - g\right) \\
&= 0.
\end{aligned} \tag{22}$$

With regard to the case $n = 2$, we have

$$\begin{aligned}
\mathbf{E}\left[(I_{(k,c)}^{(N)} - Ng)^2\right] &= -{m_{(k,c,1)}^{(N)}}^2 + f_{(k,c,2)}^{(N)} + f_{(k,c,1)}^{(N)} \\
&= -\frac{\left(\frac{N\lambda}{\mu+N\lambda}\right)^{2k+2}}{\left\{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)\right\}^2} + \frac{2\left(\frac{N\lambda}{2\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\frac{N\lambda}{2\mu+N\lambda}\right)}\frac{\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)}{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)} \\
&\quad + \frac{\left(\frac{N\lambda}{\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)} \\
&= \frac{\text{(numerator)}}{\text{(denominator)}} + \frac{\left(\frac{N\lambda}{\mu+N\lambda}\right)^{k+1}}{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)},
\end{aligned}$$

where

$$\begin{aligned}
\text{(numerator)} =& 2\left(\frac{N\lambda}{2\mu+N\lambda}\right)^{k+1}\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)\left\{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)\right\} \\
&- \left(\frac{N\lambda}{\mu+N\lambda}\right)^{2k+2}\left\{1-\Psi\left(\frac{N\lambda}{2\mu+N\lambda}\right)\right\},
\end{aligned}$$

and

$$\text{(denominator)} = \left\{1-\Psi\left(\frac{N\lambda}{2\mu+N\lambda}\right)\right\}\left\{1-\Psi\left(\frac{N\lambda}{\mu+N\lambda}\right)\right\}^2.$$

Considering Lemmas 9 and 10 and letting x denote $\mu/(N\lambda)$, the numerator and the denominator can be transformed as follows:

$$\begin{aligned}
(\text{numerator}) =& 2\left\{1-2x+(2x)^2+o(N^{-3})\right\}^{k+1}\left\{1-B_1x+\left(B_1+\frac{B_2}{2}\right)x^2+o(N^{-3})\right\}\times \\
&\left\{B_1x-\left(B_1+\frac{B_2}{2}\right)x^2+o(N^{-3})\right\} \\
&-\left\{1-x+x^2+o(N^{-3})\right\}^{2k+2}\left\{B_1x-\left(B_1+\frac{B_2}{2}\right)x^2+o(N^{-3})\right\} \\
=& 2B_1x+2(k+1)(-2x)B_1x-2B_1^2x^2-2\left(B_1+\frac{B_2}{2}\right)x^2 \\
&-2B_1x-(2k+2)(-x)B_1(2x)+\left(B_1+\frac{B_2}{2}\right)(2x)^2+o(N^{-3}) \\
=& -2B_1^2x^2+2B_1x^2+B_2x^2+o(N^{-3}) \\
=& \left\{-2B_1^2+2B_1+B_2\right\}\left(\frac{\mu}{N\lambda}\right)^2+o(N^{-3}) \\
=& \left\{-2\mathbf{E}[X]^2+2\mathbf{E}[X]+\mathbf{E}[X(X-1)]\right\}\left(\frac{\mu}{N\lambda}\right)^2+o(N^{-3}),
\end{aligned}$$

$$\begin{aligned}
(\text{denominator}) =& 2B_1^3x^3+o(N^{-4}) \\
=& 2\mathbf{E}[X]^3\left(\frac{\mu}{N\lambda}\right)^3+o(N^{-4}).
\end{aligned}$$

Therefore, considering (22), we obtain

$$\begin{aligned}
\lim_{N\to\infty}\frac{\mathbf{E}\left[(I_{(k,c)}^{(N)}-Ng)^2\right]}{N} &= \lim_{N\to\infty}\frac{1}{N}\frac{\{-2\mathbf{E}[X]^2+2\mathbf{E}[X]+\mathbf{E}[X(X-1)]\}\left(\frac{\mu}{N\lambda}\right)^2+o(N^{-3})}{2\mathbf{E}[X]^3\left(\frac{\mu}{N\lambda}\right)^3+o(N^{-4})}+g \\
&= \frac{\lambda}{\mu}\frac{2\mathbf{E}[X]+\mathbf{E}[X(X-1)]}{2\mathbf{E}[X]^3} = \frac{\lambda}{\mu}\frac{\mathbf{E}[X]^2+\mathbf{E}[X]+\mathbf{V}[X]}{2\mathbf{E}[X]^3} = \sigma^2.
\end{aligned}$$

This concludes the proof. □

The case of $n = 3, 4$ can be proved after tedious calculations. The main idea is to group the terms in the expansion of the central moments, i.e., $(I_{(k,c)}^{(N)}/N-g)^n$. At the first glance, this quantity seems to have the order of a constant. However, we must prove that it has at most the order of $N^{-\lceil n/2\rceil}$. When n is large, however, the number of terms in the binomial expansion is also large, and it is difficult to find the rule for grouping these terms to evaluate the order. As a result, the proof (21) in Lemma 12 for any n is extremely complex (we demonstrate the difficulty in the proof for the case $n = 3$ in Appendix B). Therefore, for the general case of n, we propose a symbolic computation procedure for the proof as in Lemma 13:

Lemma 13. *The recurrence formula (21) holds if*

$$N^{(n+1)/2}\sum_{i=1}^{\infty}\frac{\sum_{l=0}^{n-1}{}_nC_l a_{(k,c,n-l,i)}(-g)^l}{N^{i+1}} = o(1) \tag{23}$$

holds.

Proof. First, we obtain the following transformation of the central moment using the binomial theorem and (14) (note that $a_{(k,c,n-l,0)} = 0$ was proved in (18)):

$$
\begin{aligned}
&N^{n/2}\sum_{l=0}^{n}{}_nC_l\kappa^{(N)}_{(k,c,n-l)}(-g)^l \qquad \left(= N^{n/2}\mathbf{E}\left[\left(\frac{I^{(N)}_{(k,c)}}{N}-\frac{\lambda}{\mathbf{E}[X]\mu}\right)^n\right]\right)\\
&= N^{n/2}\sum_{l=1}^{n-1}{}_nC_l\left(\kappa^{(N)}_{(k,c,n-l+1)}-(n-l)\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-l-1)}-\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\right)\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad + N^{n/2}\kappa^{(N)}_{(k,c,n)} + N^{n/2}(-g)^n\\
&= N^{n/2}\sum_{l=1}^{n}{}_nC_l\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=1}^{n-1}{}_nC_l(n-l)\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-l-1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad - N^{n/2}\sum_{l=1}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} + N^{n/2}\kappa^{(N)}_{(k,c,n)}\\
&= N^{n/2}\sum_{l=1}^{n}({}_{n+1}C_l - {}_nC_{l-1})\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=1}^{n-1}{}_{n-1}C_l n\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-l-1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad - N^{n/2}\sum_{l=1}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} + N^{n/2}\kappa^{(N)}_{(k,c,n)}\\
&= N^{n/2}\sum_{l=1}^{n}({}_{n+1}C_l - {}_nC_{l-1})\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=0}^{n-1}{}_{n-1}C_l n\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-l-1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad - N^{n/2}\sum_{l=1}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} + N^{n/2}\kappa^{(N)}_{(k,c,n)} + N^{n/2}n\frac{\sigma^2}{N}\kappa^{(N)}_{(k,c,n-1)}\frac{1}{\kappa^{(N)}_{(k,c,1)}}\\
&= N^{n/2}\sum_{l=1}^{n+1}{}_{n+1}C_l\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\frac{(-g)^{n+1}}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=1}^{n}{}_nC_{l-1}\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad - N^{-1/2}n\frac{\sigma^2}{\kappa^{(N)}_{(k,c,1)}}N^{(n-1)/2}\sum_{l=0}^{n-1}{}_{n-1}C_l\kappa^{(N)}_{(k,c,n-1-l)}(-g)^l\\
&\qquad + N^{n/2}\frac{\kappa^{(N)}_{(k,c,n+1)}}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=1}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa_{(k,c,1)}1^{(N)}} - N^{n/2}\frac{1}{\kappa^{(N)}_{(k,c,1)}}\sum_{i=1}^{\infty}\frac{a_{(k,c,n,i)}}{N^{i+1}}\\
&= N^{n/2}\sum_{l=0}^{n+1}{}_{n+1}C_l\kappa^{(N)}_{(k,c,n-l+1)}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=0}^{n}{}_nC_l\kappa^{(N)}_{(k,c,n-l)}\frac{(-g)^{l+1}}{\kappa^{(N)}_{(k,c,1)}}\\
&\qquad - N^{-1/2}n\frac{\sigma^2}{\kappa^{(N)}_{(k,c,1)}}N^{(n-1)/2}\sum_{l=0}^{n-1}{}_{n-1}C_l\kappa^{(N)}_{(k,c,n-1-l)}(-g)^l - N^{n/2}\sum_{l=0}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}\\
&= N^{-1/2}\frac{N^{(n+1)/2}}{\kappa^{(N)}_{(k,c,1)}}\sum_{l=0}^{n+1}{}_{n+1}C_l\kappa^{(N)}_{(k,c,n+1-l)}(-g)^l\\
&\qquad - N^{-1/2}n\frac{\sigma^2}{\kappa^{(N)}_{(k,c,1)}}N^{(n-1)/2}\sum_{l=0}^{n-1}{}_{n-1}C_l\kappa^{(N)}_{(k,c,n+1-l)}(-g)^l\\
&\qquad - N^{n/2}\sum_{l=0}^{n}{}_nC_l\kappa^{(N)}_{(k,c,n-l)}\frac{(-g)^{l+1}}{\kappa^{(N)}_{(k,c,1)}} - N^{n/2}\sum_{l=0}^{n-1}{}_nC_l\sum_{i=1}^{\infty}\frac{a_{(k,c,n-l,i)}}{N^{i+1}}\frac{(-g)^l}{\kappa^{(N)}_{(k,c,1)}}.
\end{aligned}
$$

Therefore, we obtain

$$N^{(n+1)/2}\sum_{l=0}^{n+1} {}_{n+1}C_l\kappa^{(N)}_{(k,c,n+1-l)}(-g)^l - n\sigma^2 N^{(n-1)/2}\sum_{l=0}^{n-1} {}_{n-1}C_l\kappa^{(N)}_{(k,c,n+1-l)}(-g)^l$$
$$= N^{1/2}N^{n/2}\sum_{l=0}^{n} {}_nC_l\kappa^{(N)}_{(k,c,n-l)}(-g)^l\left(\kappa^{(N)}_{(k,c,1)} - g\right) + N^{(n+1)/2}\sum_{i=1}^{\infty}\sum_{l=0}^{n-1} {}_nC_l\frac{a_{(k,c,n-l,i)}}{N^{i+1}}(-g)^l, \quad (24)$$

and from (15), we find that

$$N^{1/2}N^{n/2}\sum_{l=0}^{n} {}_nC_l\kappa^{(N)}_{(k,c,n-l)}(-g)^l\left(\kappa^{(N)}_{(k,c,1)} - g\right) = N^{n/2}\sum_{l=0}^{n} {}_nC_l\kappa^{(N)}_{(k,c,n-l)}(-g)^l \times N^{1/2}\sum_{i=1}^{\infty}\frac{b_{(k,c,1,i)}}{N^i}$$
$$= o(1). \quad (25)$$

Consequently, to prove (21), we must prove (23). □

Remark 4. *If (23) holds, it follows from Theorem 3, (24), and (25) that (19) exists for all n. Thus, we have to confirm only (23) and take the limits as $N \to \infty$ in (24) to obtain (21).*

To confirm (23), we conduct the procedure in Algorithm 1. We start the procedure from $n = 3$ since (23) holds for $n = 1, 2$ clearly.

Algorithm 1 Procedure to confirm (23).

Step 1 Calculate $\xi_{(k,c,n,i)}$ for $0 \leqq l \leqq n+1$, $0 \leqq i \leqq \lceil n/2 \rceil$, by (6)–(8).

Step 2 Calculate $f_{(k,c,l,i)}$ for $0 \leqq l \leqq n+1$, $0 \leqq i \leqq \lceil n/2 \rceil$, by (9).

Step 3 Calculate $b_{(k,c,l,i)}$ for $1 \leqq l \leqq n+1$, $0 \leqq i \leqq \lceil n/2 \rceil$, by (17).

Step 4 Calculate $a_{(k,c,l,i)}$ for $1 \leqq l \leqq n$, $1 \leqq i \leqq \lceil n/2 \rceil - 1$, by (16).

Step 5 Calculate $\sum_{l=0}^{n-1} {}_nC_l a_{(k,c,n-l,i)}(-g)^l$ $(:= \chi(n,i))$ for $1 \leqq i \leqq \lceil n/2 \rceil - 1$, and if $\chi(n,i) = 0$ for $1 \leqq i \leqq \lceil n/2 \rceil - 1$, then we can guarantee (23).

5. Examples of Symbolic Computation

In this section, we show some results of the symbolic computation proposed in Lemma 5. The experiments were conducted by SymPy, which is a Python library for symbolic mathematics. Table 1 shows the results of $\chi(n,i)$ for $3 \le n \le 10$. Here, '#' denotes a non-zero constant. We can confirm $\chi(n,i) = 0$ for $1 \leqq i \leqq \lceil n/2 \rceil - 1$ within $3 \le n \le 10$. That is, it is guaranteed that (21) holds for $3 \le n \le 10$ at least. These results support the establishment of Conjecture 1.

As examples, we show the detailed results of $\chi(3,1), \chi(4,1), \chi(5,1)$, and $\chi(5,2)$ in Appendix C.

Table 1. Results for symbolic computation of $\chi(n,i)$ ($3 \le n \le 10$).

$\chi(n,i)$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$n = 3$	0	#	#	#	#
$n = 4$	0	#	#	#	#
$n = 5$	0	0	#	#	#
$n = 6$	0	0	#	#	#
$n = 7$	0	0	0	#	#
$n = 8$	0	0	0	#	#
$n = 9$	0	0	0	0	#
$n = 10$	0	0	0	0	#

6. Conclusions

In this paper, we proposed the *moment approach* to analyze scaling models for infinite-server queues. This method considers the scaling limits for the number of busy servers *conditional* on a phase with *infinite* state space. We applied this method to the $M/M^X/\infty$ queue and presented the following results: the weak law of large numbers (Theorem 1), a recursive formula for the necessary condition of CLT (Theorem 2) that can be utilized for a normal approximation of the system (Remark 2), and a symbolic computation algorithm (Lemma 13) to confirm the CLT (Conjecture 1). Some results of the symbolic computation that support Conjecture 1 were shown in Table 1.

As a future issue, it would be challenging but significant to rigorously prove Conjecture 1, i.e., to prove (23) explicitly. Another interesting direction to explore is to apply the method in this study to any other infinite-server queueing system, especially one whose phase has an infinite state space.

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Appendix A

The poof of Lemma 1 is given by Appendix A. The balance equations of the Markov chain $(I(t), S(t), U(t))$ are given by (A1)–(A4).

$$\lambda\pi_{0,0,1} = \mu\pi_{1,0,1}, \qquad i = 0, k = 0, c = 1, \tag{A1}$$

$$(\lambda + i\mu)\pi_{i,0,1} = \sum_{c=2}^{\infty} \lambda\pi_{i-1,c-1,c} + q_1\lambda\pi_{i-1,0,1} + (i+1)\mu\pi_{i+1,0,1}, \qquad i \geqq 1, \quad k = 0, \quad c = 1, \tag{A2}$$

$$(\lambda + i\mu)\pi_{i,1,c} = q_c\lambda\pi_{i,0,1} + (i+1)\mu\pi_{i+1,1,c}, \qquad i \geqq 0, \quad k = 1, \quad c \geqq 2, \tag{A3}$$

$$(\lambda + i\mu)\pi_{i,k,c} = \lambda\pi_{i,k-1,c} + (i+1)\mu\pi_{i+1,k,c}, \qquad i \geqq 0, \quad 2 \leqq k \leqq c-1, \quad 3 \leqq c. \tag{A4}$$

Multiplying (A1)–(A4) by z^i, taking the sum over $i \in \mathbb{Z}$, and rearranging the result, we obtain the following system of differential equations:

$$(\mu z - \mu)\Pi'_{0,1}(z) = -\lambda\Pi_{0,1}(z) + \sum_{c=2}^{\infty} \lambda z\Pi_{c-1,c}(z) + q_1\lambda z\Pi_{0,1}(z), \qquad k = 0, \quad c = 1, \tag{A5}$$

$$(\mu z - \mu)\Pi'_{1,c}(z) = -\lambda\Pi_{1,c}(z) + q_c\lambda\Pi_{0,1}(z), \qquad k = 1, \quad 2 \leqq c, \tag{A6}$$

$$(\mu z-\mu)\Pi'_{k,c}(z)=-\lambda\Pi_{k,c}(z)+\lambda\Pi_{k-1,c}(z),\qquad 2\leqq k\leqq c-1,\quad 3\leqq c. \tag{A7}$$

However, it is not easy to obtain the solution for the system of differential equations because the coefficients are not constant. Hence, we use the factorial moment-generating function method [12]. Letting I, S, and U denote $I(t)$, $S(t)$, and $U(t)$ in the steady state, respectively, we define the factorial moments as follows:

$$f_0^{(k,c)}=\sum_{i=0}^{\infty}\pi_{i,k,c},$$

$$f_n^{(k,c)}=\mathbf{E}\left[I(I-1)(I-2)\dots(I-n+1)\mathbb{1}_{\{S=k,\ U=c\}}\right],\qquad n\geqq 1.$$

Moreover, we define the FMGF as:

$$\mathcal{F}_{k,c}(z)=\sum_{n=0}^{\infty}f_n^{(k,c)}\frac{z^n}{n!}=\Pi_{k,c}(z+1). \tag{A8}$$

Using FMGF, we can rearrange (A5)–(A7) as the following system of differential equations:

$$\mu z\mathcal{F}'_{0,1}(z)=-\lambda\mathcal{F}_{0,1}(z)+\sum_{c=2}^{\infty}\lambda(z+1)\mathcal{F}_{c-1,c}(z)+q_1\lambda(z+1)\mathcal{F}_{0,1},\qquad k=0,\quad c=1, \tag{A9}$$

$$\mu z\mathcal{F}'_{1,c}(z)=-\lambda\mathcal{F}_{1,c}(z)+q_c\lambda\mathcal{F}_{0,1}(z),\qquad k=1,\quad 2\leqq c, \tag{A10}$$

$$\mu z\mathcal{F}'_{k,c}(z)=-\lambda\mathcal{F}_{k,c}(z)+\lambda\mathcal{F}_{k-1,c}(z),\qquad 2\leqq k\leqq c-1,\quad 3\leqq c. \tag{A11}$$

Equating the coefficients of z^n on both sides of (A9)–(A11) yields

$$(\mu n+\lambda)f_n^{(0,1)}=\sum_{c=2}^{\infty}\lambda f_n^{(c-1,c)}+q_1\lambda f_n^{(0,1)}+\sum_{c=2}^{\infty}\lambda n f_{n-1}^{(c-1,c)}+q_1\lambda n f_{n-1}^{(0,1)},\qquad k=0,\quad c=1, \tag{A12}$$

$$(\mu n+\lambda)f_n^{(1,c)}=q_c\lambda f_n^{(0,1)},\qquad k=1,\quad 2\leqq c, \tag{A13}$$

$$(\mu n+\lambda)f_n^{(k,c)}=\lambda f_n^{(k-1,c)},\qquad 2\leqq k\leqq c-1,\quad 3\leqq c. \tag{A14}$$

Furthermore, it is clear by the definition that

$$f_0^{(0,1)}=\sum_{i=0}^{\infty}\pi_{i,0,1}=\Pi_{0,1}(1),\qquad k=0,\quad c=1,$$

$$f_0^{(k,c)}=\sum_{i=0}^{\infty}\pi_{i,k,c}=\Pi_{k,c}(1),\qquad 1\leqq k\leqq c-1,\quad 2\leqq c,$$

and thus, by substituting $z=1$ into (A5)–(A7), we obtain

$$0=-\lambda\Pi_{0,1}(1)+\sum_{c=2}^{\infty}\lambda\Pi_{c-1,c}(1)+q_1\lambda\Pi_{0,1}(1),\qquad k=0,\quad c=1,$$

$$0=-\lambda\Pi_{1,c}(1)+q_c\lambda\Pi_{0,1}(1),\qquad k=1,\quad 2\leqq c, \tag{A15}$$

$$0=-\lambda\Pi_{k,c}(1)+\lambda\Pi_{k-1,c}(1),\qquad 2\leqq k\leqq c-1,\quad 3\leqq c. \tag{A16}$$

From (A15), (A16), and

$$1 = \Pi_{0,1}(1) + \sum_{c=2}^{\infty}\sum_{k=1}^{c-1}\Pi_{k,c}(1)$$
$$= \sum_{c=1}^{\infty} c q_c \Pi_{0,1}(1),$$

the following holds:

$$\Pi_{0,1}(1) = f_0^{(0,1)} = \frac{1}{\mathbf{E}[X]}, \qquad k = 0, \quad c = 1,$$

$$\Pi_{k,c}(1) = f_0^{(k,c)} = \frac{q_c}{\mathbf{E}[X]}, \qquad 1 \leqq k \leqq c-1, \quad 2 \leqq c.$$

Solving (A12)–(A14) using these initial conditions, we obtain

$$f_n^{(0,1)} = \prod_{l=1}^{n} \frac{\sum_{c=1}^{\infty}\left(\frac{\lambda l}{\mu l + \lambda}\right)\left(\frac{\lambda}{\mu(l-1)+\lambda}\right)^{c-1} q_c}{1 - \sum_{c=1}^{\infty}\left(\frac{\lambda}{\mu l + \lambda}\right)^{c} q_c} \times \frac{1}{\mathbf{E}[X]}, \tag{A17}$$

and

$$f_n^{(0,1)} = \frac{n!\left(\frac{\lambda}{\mu n + \lambda}\right)}{1 - \Psi\left(\frac{\lambda}{\mu n + \lambda}\right)} \prod_{l=1}^{n-1}\left\{\frac{\Psi\left(\frac{\lambda}{\mu l + \lambda}\right)}{1 - \Psi\left(\frac{\lambda}{\mu l + \lambda}\right)}\right\} \times \frac{1}{\mathbf{E}[X]},$$

$$f_n^{(k,c)} = \frac{n!\left(\frac{\lambda}{\mu n + \lambda}\right)^{k+1}}{1 - \Psi\left(\frac{\lambda}{\mu n + \lambda}\right)} \prod_{l=1}^{n-1}\left\{\frac{\Psi\left(\frac{\lambda}{\mu l + \lambda}\right)}{1 - \Psi\left(\frac{\lambda}{\mu l + \lambda}\right)}\right\} \times \frac{q_c}{\mathbf{E}[X]},$$
$$1 \leqq k \leqq c-1, \quad 2 \leqq c,$$

which yields Lemma 1.

Appendix B

We show the outline of the poof of Theorem 3 for $n = 3$. First, we obtain the following transformation.

$$\begin{aligned}\mathbf{E}\left[(I_{(k,c)}^{(N)} - Ng)^3\right] &= -{m_{(k,c,1)}^{(N)}}^3 + (3{m_{(k,c,1)}^{(N)}}^2 - 3m_{(k,c,1)}^{(N)} + 1)f_{(k,c,1)}^{(N)} + 3(1 - m_{(k,c,1)}^{(N)})f_{(k,c,2)}^{(N)} \\ &\quad + f_{(k,c,3)}^{(N)} \\ &= (-{m_{(k,c,1)}^{(N)}}^3 + 3{m_{(k,c,1)}^{(N)}}^2 f_{(k,c,1)}^{(N)} - 3m_{(k,c,1)}^{(N)} f_{(k,c,2)}^{(N)} + f_{(k,c,3)}^{(N)}) \\ &\quad + 3(-m_{(k,c,1)}^{(N)} f_{(k,c,1)}^{(N)} + f_{(k,c,2)}^{(N)}) + f_{(k,c,1)}^{(N)}.\end{aligned}$$

After some algebraic manipulations, we obtain

$$\lim_{N\to\infty} \frac{-{m_{(k,c,1)}^{(N)}}^3 + 3{m_{(k,c,1)}^{(N)}}^2 f_{(k,c,1)}^{(N)} - 3m_{(k,c,1)}^{(N)} f_{(k,c,2)}^{(N)} + f_{(k,c,3)}^{(N)}}{N^{3/2}} = 0,$$

$$\lim_{N\to\infty} \frac{3(-m^{(N)}_{(k,c,1)} f^{(N)}_{(k,c,1)} + f^{(N)}_{(k,c,2)})}{N^{3/2}} = 0,$$

and

$$\lim_{N\to\infty} \frac{f^{(N)}_{(k,c,1)}}{N^{3/2}} = 0.$$

Therefore, it is clear that the following holds:

$$\lim_{N\to\infty} \frac{\mathbf{E}\left[(I^{(N)}_{(k,c)} - Ng)^3\right]}{N^{3/2}} = 0.$$

Appendix C

The detailed expressions for $\chi(3,1)$, $\chi(4,1)$, $\chi(5,1)$, and $\chi(5,2)$ are given as follows:

$$\begin{aligned}
\chi(3,1) =& \frac{3\lambda^2\left(B_1^3k - B_1^2B_2/2 - B_1^2k - B_1B_2k + 3B_1B_2/2 - B_1B_3/2 + B_2^2\right)}{B_1^6\mu^2} \\
&+ \frac{\lambda^2\left(6B_1^3k - 6B_1^3 - 6B_1^2B_2 - 6B_1^2k + 12B_1^2 - 6B_1B_2k + 27B_1B_2 - 5B_1B_3 + 12B_2^2\right)}{2B_1^6\mu^2} \\
&- \frac{\lambda^2\left(12B_1^3k - 6B_1^3 - 9B_1^2B_2 - 12B_1^2k + 12B_1^2 - 12B_1B_2k + 36B_1B_2 - 8B_1B_3 + 18B_2^2\right)}{2B_1^6\mu^2} \\
=& 0,
\end{aligned}$$

$$\begin{aligned}
\chi(4,1) =& -\frac{4\lambda^3\left(B_1^3k - B_1^2B_2/2 - B_1^2k - B_1B_2k + 3B_1B_2/2 - B_1B_3/2 + B_2^2\right)}{B_1^7\mu^3} \\
&+ \frac{\lambda^3\left(4B_1^3k - 6B_1^3 - 5B_1^2B_2 - 4B_1^2k + 12B_1^2 - 4B_1B_2k + 24B_1B_2 - 4B_1B_3 + 10B_2^2\right)}{B_1^7\mu^3} \\
&- \frac{2\lambda^3\left(6B_1^3k - 6B_1^3 - 6B_1^2B_2 - 6B_1^2k + 12B_1^2 - 6B_1B_2k + 27B_1B_2 - 5B_1B_3 + 12B_2^2\right)}{B_1^7\mu^3} \\
&+ \frac{\lambda^3\left(12B_1^3k - 6B_1^3 - 9B_1^2B_2 - 12B_1^2k + 12B_1^2 - 12B_1B_2k + 36B_1B_2 - 8B_1B_3 + 18B_2^2\right)}{B_1^7\mu^3} \\
=& 0,
\end{aligned}$$

$$\begin{aligned}
\chi(5,1) =& \frac{5\lambda^4\left(B_1^3k - B_1^2B_2/2 - B_1^2k - B_1B_2k + 3B_1B_2/2 - B_1B_3/2 + B_2^2\right)}{B_1^8\mu^4} \\
&- \frac{5\lambda^4\left(4B_1^3k - 6B_1^3 - 5B_1^2B_2 - 4B_1^2k + 12B_1^2 - 4B_1B_2k + 24B_1B_2 - 4B_1B_3 + 10B_2^2\right)}{B_1^8\mu^4} \\
&+ \frac{5\lambda^4\left(6B_1^3k - 12B_1^3 - 9B_1^2B_2 - 6B_1^2k + 24B_1^2 - 6B_1B_2k + 45B_1B_2 - 7B_1B_3 + 18B_2^2\right)}{6B_1^8\mu^4} \\
&+ \frac{5\lambda^4\left(6B_1^3k - 6B_1^3 - 6B_1^2B_2 - 6B_1^2k + 12B_1^2 - 6B_1B_2k + 27B_1B_2 - 5B_1B_3 + 12B_2^2\right)}{B_1^8\mu^4} \\
&- \frac{5\lambda^4\left(12B_1^3k - 6B_1^3 - 9B_1^2B_2 - 12B_1^2k + 12B_1^2 - 12B_1B_2k + 36B_1B_2 - 8B_1B_3 + 18B_2^2\right)}{3B_1^8\mu^4} \\
=& 0,
\end{aligned}$$

$$
\begin{aligned}
\chi(5,2) = & -\frac{5\lambda^3}{24B_1^9\mu^3}\big(36B_1^4k^2 + 36B_1^4k - 36B_1^3B_2k - 12B_1^3B_3 + 18B_1^2B_2^2 \\
& - 24B_1^2B_2k^2 + 24B_1^2B_2k - 24B_1^2B_3k + 20B_1^2B_3 - 7B_1^2B_4 + 48B_1B_2^2k \\
& - 42B_1B_2^2 + 36B_1B_2B_3 - 33B_2^3\big) \\
& -\frac{5\lambda^3}{6B_1^9\mu^3}\big(-12B_1^5k + 6B_1^4B_2 - 60B_1^4k^2 + 96B_1^3B_2k - 54B_1^3B_2 + 30B_1^3B_3 + 24B_1^3k^2 \\
& - 24B_1^3k - 60B_1^2B_2^2 + 48B_1^2B_2k^2 - 144B_1^2B_2k + 72B_1^2B_2 + 48B_1^2B_3k - 68B_1^2B_3 \\
& + 13B_1^2B_4 - 120B_1B_2^2k + 186B_1B_2^2 - 84B_1B_2B_3 + 93B_2^3\big) \\
& +\frac{5\lambda^3}{24B_1^9\mu^3}\big(-48B_1^5k + 48B_1^5 + 48B_1^4B_2 - 132B_1^4k^2 + 396B_1^4k - 432B_1^4 + 420B_1^3B_2k \\
& - 936B_1^3B_2 + 132B_1^3B_3 + 96B_1^3k^2 - 528B_1^3k + 576B_1^3 - 438B_1^2B_2^2 + 120B_1^2B_2k^2 \\
& - 1152B_1^2B_2k + 2088B_1^2B_2 + 168B_1^2B_3k - 524B_1^2B_3 + 43B_1^2B_4 - 522B_1B_2^2k \\
& + 2166B_1B_2^2 - 432B_1B_2B_3 + 669B_2^3\big) \\
& -\frac{5\lambda^3}{6B_1^9\mu^3}\big(-36B_1^5k + 24B_1^5 + 30B_1^4B_2 - 108B_1^4k^2 + 216B_1^4k - 180B_1^4 + 288B_1^3B_2k \\
& - 486B_1^3B_2 + 90B_1^3B_3 + 72B_1^3k^2 - 288B_1^3k + 216B_1^3 - 261B_1^2B_2^2 + 96B_1^2B_2k^2 \\
& - 708B_1^2B_2k + 972B_1^2B_2 + 120B_1^2B_3k - 308B_1^2B_3 + 31B_1^2B_4 - 372B_1B_2^2k \\
& + 1158B_1B_2^2 - 276B_1B_2B_3 + 393B_2^3\big) \\
& +\frac{5\lambda^3}{4B_1^9\mu^3}\big(-24B_1^5k + 8B_1^5 + 16B_1^4B_2 - 84B_1^4k^2 + 84B_1^4k - 48B_1^4 + 180B_1^3B_2k \\
& - 204B_1^3B_2 + 56B_1^3B_3 + 48B_1^3k^2 - 120B_1^3k + 48B_1^3 - 138B_1^2B_2^2 + 72B_1^2B_2k^2 \\
& - 372B_1^2B_2k + 348B_1^2B_2 + 80B_1^2B_3k - 160B_1^2B_3 + 21B_1^2B_4 - 228B_1B_2^2k \\
& + 528B_1B_2^2 - 162B_1B_2B_3 + 207B_2^3\big) \\
= & \; 0.
\end{aligned}
$$

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