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The over-topos at a model. (English) Zbl 07692245
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This paper aims to construct a site of definition for the topos classifying model homomorphisms towards the internalization of a fixed model of a geometric theory in a given Grothendieck topos, explicitly constructing a site of definition for the over-topos from model-theoretic data and describing both the logical and fibrational aspects of the construction.

The synopsis of the paper goes as follows.

§1 recalls the well-known notion of totally connected topos and the construction of the over-topos as a finite bilimits of Grothendieck toposes.

§2 focuses on the case of a set-based model, introducing the *antecedent topology* on a category of elements associated to the model in order to obtain a site of definition of the over-topos.

§3 introduces a number of stacks for the generalization of the construction to an arbitrary topos, providing a fully explicit description of the *lifted topology* on a category of the form $(1_{\mathcal{F}} \downarrow f^*)$, where $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, as the smallest topology making both projection functors to $\mathcal{E}$ and $\mathcal{F}$ comorphisms to the associated sites on $\mathcal{E}$ and $\mathcal{F}$.

§4 generalizes the construction of the over-topos to a model in an arbitrary Grothendieck topos, for which the authors construct a canonical stack associated with the model, applying Giraud's general construction of the classifying topos of a stack to establish the desired universal property. Particularly, an explicit generalization of the antecedent topology is provided and recovered as a restriction of the lifted topology. It should be mentioned that a construction of the topology also appeared in [*A. Abbes* and *M. Gros*, "Topos co-évanescents et généralisations", Preprint, `arXiv:1107.2380`, §4.1], where site-theoretic description of general comma toposes are provided. However, the construction there works only in the case of small sites with finite limits and under the hypothesis that the relavant goemetric morphisms are induced by morphism of sites.

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MSC:

18F10 Grothendieck topologies and Grothendieck topoi
18C10 Theories (e.g., algebraic theories), structure, and semantics

Keywords:

over-topos; totally connected topos; Giraud topology; colocalization

Full Text: arXiv Link

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