a review of MR4535663 Rogers, Morgan Toposes of topological monoid actions

Hirokazu Nishimura
University of Tsukuba

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Toposes of topological monoid actions. (English summary)

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The author ["Toposes of discrete monoid actions", preprint, arXiv:1905.10277] investigated properties of presheaf toposes of the form $\mathrm{PSh}(M)$ for a monoid M. For a topological group (G,τ), the category $\mathrm{Cont}(G,\tau)$ of continuous G-actions on discrete topological spaces is a Grothendieck topos. This paper studies the category of continuous actions of topological monoids.

The synopsis of the paper goes as follows:

§1 exhibits the necessary data to establish that the forgetful functor from the category $\mathrm{Cont}(M,\tau)$ of continuous actions of a monoid with respect to an arbitrary topology τ to the topos $\mathrm{PSh}(M)$ is left exact and comonadic (Proposition 1.4). The adjoint is to be expressed using either clopen subsets of (M,τ) or open relations. The existence of the adjunction guarantees that $\mathrm{Cont}(M,\tau)$ is an elementary topos, so that the forgetful functor is the inverse image of a hyperconnected geometric morphism. Results from [M. Rogers, Theory Appl. Categ. **37** (2021), Paper No. 32, 1017–1079; MR4326108] on supercompactly generated toposes are recalled in §1.2, and applied in §1.3 to conclude that any topos of the form $\mathrm{Cont}(M,\tau)$ is moreover a supercompactly generated Grothendieck topos, which gives rise to an intuitive Morita-equivalence result in terms of the category of continuous M-sets (Corollary 1.22). Finally, it is shown in §1.4 that the above characterization of toposes of the form $\mathrm{Cont}(M,\tau)$ is not yet complete.

§2 examines the question of how much is recoverable about a topology τ on a monoid M from the hyperconnected morphism

$$\mathrm{PSh}(M)\to\mathrm{Cont}(M,\tau).$$

§3 considers the canonical surjective point of $\mathrm{Cont}(M,\tau)$, which is the composite of the canonical essential surjective point of $\mathrm{PSh}(M)$ and the hyperconnected morphism obtained in §1. The author aims to characterize this class of toposes in terms of the existence of a point of this form, just as O. Caramello [Adv. Math. **291** (2016), 646–695; MR3459027] did for topological groups.

§4 shows that continuous semigroup homomorphisms between topological monoids induce geometric morphisms between the corresponding toposes of continuous actions, while it was shown in [M. Rogers, op. cit., arXiv:1905.10277] that semigroup homomorphisms correspond to essentially geometric morphisms between toposes of discrete monoid actions. When a geometric morphism g is induced by a continuous semigroup homomorphism ϕ between complete monoids, it is shown that the surjection-inclusion factorization of g is canonically represented by the factorization of ϕ into a monoid homomorphism followed by an inclusion of a subsemigroup (Theorem 4.15). It is also shown that the hyperconnected-localic factorization of g can be identified with the dense-closed factorization of ϕ (Theorem 4.19).

§5 summarizes the unresolved problems the author has encountered along the way, suggesting some future directions in which this research might proceed.

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Hirokazu Nishimura