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The Integration Theory of Selling Problem and Buying Problem Based on the Concepts of Symmetry and Analogy

(ver.001)

by

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May 15, 2023

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Alice's Adventures in Wonderland*

The Integration Theory of Selling Problem and Buying Problem Based on the Concepts of Symmetry and Analogy

— `ver.001` —

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Keywords

selling problem, buying problem, optimal stopping problem, search, symmetry, analogy, quitting penalty, market restriction, recognizing time, staring time, initiating time, null time zone, deadline falling

Abstract

A trading problem can be classified into the following four kinds: a selling problem and a buying problem, each of which, moreover, falls into a problem with reservation price mechanism (opponent trader proposes the trading price) and a problem with posted price mechanism (leading trader proposes the trading price). Let us refer to a group of the four problems as the *quadruple-asset-trading-problems*. The main objective of the paper is twofold: the construction of a general theory which integrates the quadruple-asset-trading-problems and the analyses of some basic models of these problems by using the theory. To attain the objective, some novel ideas are introduced in the present paper: symmetry, analogy, initiating time, quitting penalty, market restriction, etc. which will lead us to quite a new horizon that has not been perceived at all by any researcher. The most notable findings obtained from the analyses of these basic models are the following two. One is the drastic collapse of symmetry between the selling problem and the buying problem, the other is the existence of null-time-zone, at every point in time on which any decision-making activity makes no sense at all. Especially, the latter finding could confront us with the all-around review of the whole discussion that has been made thus far for a trading problem as a decision process. Furthermore interestingly, when the zone is over all points in time on the planning horizon, excluding the *deadline*, it follows that all activities of the decision-making that are scheduled for over the entire planning horizon are swallowed up into the *deadline* as if all substances, even light, fall into a black hole. Finally, we propose the enormous number of models for asset trading problems which have not yet been posed so far and wind up this study with insisting that the treatment of these problems are almost impossible without the integration theory.

As far as the laws of mathematics refer to reality,
they are not certain,
and as far as they are certain,
they do not refer to reality.

Albert Einstein

— *This study starting with this apothegm ends with this apothegm* (see C27.4(p.267)) —

This is the first substantial version of "*An integration of the optimal stopping problem and the optimal pricing problem*"

`https://commons.sk.tsukuba.ac.jp/discussion/page/27` (No.1084/2004)

The integration theory posed in this present version is exemplified by use of some generalized models of the asset selling problem in [20, Ee & Ikuta (2006)]. Let us refer to the above original paper as `ver.000` and to the present paper as `ver.001`. According to the progress of our study, this will be updated as `ver.002`, `ver.003`, $\cdots$

*Readers will be bewildered by different uncommon notions, concepts, definitions, events, results, findings, etc. appearing in the present paper (see Alice's 1(p.9), 2(p.36), 3(p.36), 4(p.36), 5(p.37), and 6(p.53)).

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Contents

Abbreviations

ATP	Asset Trading Problem
ASP	Asset Selling Problem
ABP	Asset Buying Problem
ATM	Asset Trading Model
ASM	Asset Selling Model
ABM	Asset Buying Model
Tom	Lemma in Total market ($\mathscr{F}$)
Pom	Lemma in Positive market ($\mathscr{F}^+$)
Mim	Lemma in Mixed market ($\mathscr{F}^\pm$)
Nem	Lemma in Negative market ($\mathscr{F}^-$)
A	Assertion
$\mathscr{A}$	$\mathscr{A}$ssertion system
$\mathbb{R}$	$\mathbb{R}$eservation price mechanism
$\mathbb{P}$	$\mathbb{P}$osted price mechanism
M:x[$\mathbb{X}$][X]	Model of asset selling problem ($x = 1, 2, 3, \mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
M̃:x[$\mathbb{X}$][X]	M̃odel of asset buying problem ($x = 1, 2, 3, \mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
$\mathcal{Q}$(Models)	quadruple-asset-trading-models
$\mathcal{S}$(Models)	structured unit of models
SOE	System of Optimality Equations [soé]
OIT	Optimal-Initiating-Time [òuít]
dOITs	degenerate OIT to the starting time
dOITd	degenerate OIT to the deadline
ndOIT	nondegenerate OIT
pSkip	posterior-Skip of search
Nul-TZ	Null-Time-Zone [nʌ́ltí:zí:]
a-E-case	attack-Enforced-case
a-A-case	attack-Allowed-case
s-E-case	search-Enforced-case
s-A-case	search-Allowed-case
a-E-model	attack-Enforced-model
a-A-model	attack-Allowed-model
s-E-model	search-Enforced-model
s-A-model	search-Allowed-model
ii-E-case	immediate-initiation-Enforced-case
ii-A-case	immediate-initiation-Allowed-case
ii-E-model	immediate-initiation-Enforced-model
ii-A-model	immediate-initiation-Allowed-model
$\doteqdot$	"not always equal" ("$\neq$" is "equal")

Part 1

Prologue

This part provides all of what will be needed prior to the construction of the integration theory in Part 2 (p.38).

Chapter 1

Introduction

1.1 Two Examples

To begin with, let us here give two examples which convey a flavor of problems treated throughout the present paper.

Example **1.1.1 (selling problem)** *Suppose that you have to sell your car up to a specified date (deadline) for a compelling reason (for instance, you must suddenly return to your mother country by order of the head office) and that a buyer who wants to buy it for a certain price has just appeared. If the price is high enough, you will sell the car to the buyer, but if not so, you will hesitate to sell. Then, if you sell, you might incur the risk that a buyer proposing a higher price may appear in the future. Conversely, if you do not sell, you might incur the risk that any buyer proposing a higher price may not appear in the future; in the worst case, no buyer could appear up to the deadline at all; if so, you must sell it for a giveaway price or dispose of it by paying some cost. Taking such risks into consideration, you have to determine whether or not to sell it to each buyer who will successively appear. The above scenario implies that you must sell the car for a lower and lower price as getting closer and closer to the deadline, or equivalently you must set the minimum permissible selling price (reservation price) so as to become lower and lower as getting closer and closer to the deadline.*

We can also consider a buying problem which is the inverse of the above selling problem, described as below.

Example **1.1.2 (buying problem)** *Suppose that you have to buy a car up to a specified date (deadline) and that a seller who wants to sell a car for a certain price has just appeared. If the price is low enough, you will buy the car from the seller, but if not so, you will hesitate to buy. Then, if you buy, you might incur the risk that a seller proposing a lower price may appear in the future. Conversely, if you do not buy, you might incur the risk that any seller proposing a lower price may not appear in the future; in the worst case, no seller could appear up to the deadline at all; if so, you must buy a car for a very high price in a black market. Taking such risks into consideration, you have to determine whether or not to buy a car from each seller who will successively appear. The above scenario implies that you must buy a car for a higher and higher price as getting closer and closer to the deadline, or equivalently you must set the maximum permissible buying price (reservation price) so as to become higher and higher as getting closer and closer to the deadline.*

The above two problems have been already investigated so far under the name "*optimal stopping problem*", implying "*when to stop the behavior of "not selling the asset" in the former problem or "not buying the asset" in the letter problem*". The earliest papers for these problems can be traced back to 1960's as long as we know [43,1961][0005] [30,1962][0521] [10,1971][0014] [34,1973][0525].

Now, the above two problems would seem to be special cases of different decision-making problems that have been treated so far in the fields of operational research and economics. In the present paper, however, we will try to thoroughly and in detail clarify all aspects of the two problems by introducing some novel ideas: symmetry, analogy, initiating time, quitting penalty, market restriction, etc. which have not been perceived by any researcher at all, including the authors in the past. After having finished reading the paper, it will be known that the above two problems which seem to be symmetrical at a glance are not always so and, moreover, that this study in the present paper will throw a *new light* and open up a *genuinely new horizon* not only for the above two problems but also for a more generalized decision processes (see Section A 5(p.291)).

1.2 Quadruple-Asset-Trading-Problems

Before proceeding with our study, below let us review a general framework of a selling problem and a buying problem. First, let us have an eye to the fact that an economic behaviour is basically constituted by different types of transactions, in each of which different models of trading assets (house, car, a lot of land, etc.), commodities (wheat, copper, gasoline, etc.), and goods (fruit, fish, clothes, etc.) have been posed and examined thus far. As general terms, let us refer to an item traded there as the *asset* and to a problem of trading an asset as the *asset trading problem*, `ATP` for short. In addition, let us refer to persons involved in `ATP` as *traders*, consisting of a seller and a buyer, who face an *asset selling problem*† and an *asset buying problem* respectively,‡

† [30,1962][0521], [32,1963][0303], [3,1977][0080], [38,1983][0138], [37,1983][0136], [40,1990][0390], [7,1991][0506], [33,1993][0480], [44,1993][0494], [36,1995][0356], [27,1995][0374], [45,1995][0500], [4,1995][0491], [47,1997][0282], [9,1997][0393], [12,1998][0389], [21,1999][0308], [1,1999][0371], [13,2001][0387], [35,2002][0388], [11,2002][0319], [15,2004][0413], [19,2005][0507], [16,2005][0517]

‡ [9,1998][0393], [11,2002][0319].

denoted by ASP and ABP respectively for short. Now, a *leading trader* and *its opponent trader* in ASP (ABP) are a seller (buyer) and a buyer (seller) respectively, and each of ASP and ABP is furthermore separated into the two cases, depending on which of leading trader and opponent trader proposes a trading price. Accordingly, it follows that ATP consists of the following four kinds of problems:

⟨1⟩ ASP in which a buyer as an opponent trader proposes a buying price,
⟨2⟩ ABP in which a seller as an opponent trader proposes a selling price,
⟨3⟩ ASP in which a seller as a leading trader proposes a selling price,
⟨4⟩ ABP in which a buyer as a leading trader proposes a buying price.

In ATP in which an opponent trader proposes a trading price ⟨1/2⟩, a leading trader faces the problem of determining whether or not to accept the trading price proposed by the opponent trader (buying price/selling price). Then, the leading trader must set the *optimal reservation price* as a threshold based on which it is judged whether or not to accept it. It is called the problem with *reservation price mechanism*,[§] $\mathbb{R}$-mechanism for short; let us represent the problem as ATP[$\mathbb{R}$] for short. In ASP[$\mathbb{R}$], the reservation price is the *minimum permissible selling price*; a seller (leading trader) is willing to sell the asset if and only if the proposed price is greater than or equal to it. In ABP[$\mathbb{R}$], the reservation price is the *maximum permissible buying price*; a buyer (leading trader) is willing to buy the asset if and only if the proposed price is less than or equal to it. On the other hand, in ATP in which a leading trader proposes a trading price ⟨3/4⟩, the leading trader faces the problem of determining the trading price to propose (selling price/buying price). It is called the problem with *posted price mechanism*,[¶] $\mathbb{P}$-mechanism for short; let us represent the problem by ATP[$\mathbb{P}$]. From the above, one sees that ATP consists of the four kinds of problems, ASP[$\mathbb{R}$], ABP[$\mathbb{R}$], ASP[$\mathbb{P}$], and ABP[$\mathbb{P}$] respectively, called the *quadruple-asset-trading-problems.* Furthermore, let us represent models for the above four problems by ASM[$\mathbb{R}$], ABM[$\mathbb{R}$], ASM[$\mathbb{P}$], and ABM[$\mathbb{P}$] respectively, called the *quadruple-asset-trading-models.*

1.3 Two Motives

The writing of the present paper was strongly urged by the motives of trying to answer to the two questions below:

Motive 1 **Is a buying problem always symmetrical to a selling problem ?**

From long before this study started, we had been continuing to conceive a naïve question "*Does a buying problem and a selling problem always become symmetrical each other in the sense that once a property of the latter problem is known, its corresponding property of the former problem can be immediately and easily known by merely changing the signs of variables, parameters, constants, etc. appearing within the description of the property of the latter problem ? and vice versa ?*" Our final answer to the above naïve question to which almost all researchers, including even the authors in the past, had called no unambivalent attention is "nay !".

Motive 2 **Can the theory integrating quadruple-asset-trading-problems exist ?**

Before starting to write this paper, we had widely read many papers related to these problems and spontaneously obtained a *rude expectation* that there might exist "a common denominator" at the base of all discussions developed there. This feeling led us to the insight that the common denominator is closely involved with a function called the T-function (see Section 5.1.1(p.17)). Urged by this insight, we had before long a faint anticipation that there could exist a theory integrating the quadruple-asset-trading-problems, and finally we were led to a ray of hope that the construction of the theory can first become possible by introducing the concepts of *symmetry* and *analogy*, and in the final stage, fortunately we succeeded in its construction.

1.4 Four Novel Factors

In the present paper we introduce the following four novel factors:

Factor 1 **Search skip** (see Concept 2c(p.10))

A leading trader has an option whether to conduct the search for opponent trader or to skip it.

Factor 2 **Quitting penalty** (see A6(p.7))

This is a penalty by accepting which a leading trader can quit the process.

Factor 3 **Market restrictions** (see A7(p.8) and Chapter 16(p.99))

It will be shown later that the successful construction of the integration theory can become possible under the basic premise that a price ξ, whether reservation price or posted price, is defined on $(-\infty, \infty)$. However, since the price is positive in a usual transaction, i.e., $\xi \in (0, \infty)$, the above premise must be said to be irrational a little bit from a practical viewpoint. To remove the irrationality the paper employs the methodology of restricting results obtained on $(-\infty, \infty)$ to ones on $(0, \infty)$, called the market restriction (see Chapter 16(p.99)).

§ [4,1995][0491], [6,2001][0490]
¶ [5,1998][0492], [6,2001][0490], [22,1994][0283], [44,1993][0494], [45,1995][0500]

Factor 4 **Optimal initiating time** (see Section 7.2.4(p.34))

When encountering a decision problem, normally we lean to at once trying to *initiate* its treatment; however, from a managerial and/or economical viewpoints it might become better to postpone its initiation, which inevitably leads us to the notion of the optimal-initiating-time (see Section 7.2.4(p.34)). Quite strangely enough, thus far we had not been able to find any literature at all in which this rational thinking is taken into consideration in the study of decision problems.

It will be known later that the introduction of the above four factors will lead us to quite a *new horizon* for the whole study of asset trading problems.

Chapter 2

Assumptions and Concepts

2.1 Basic Assumptions

The section provides the basic assumptions that will be employed in models of asset trading problems dealt with in the present paper.

A1 Points in time

The process of trading proceeds intermittently on points in time equally spaced on a finite length of time axis as shown in Figure 2.1.1 below. Let us refer to each point in time as the *time*, which is numbered backward from the final point in time, time 0 (deadline), as $0, 1, \cdots$. Accordingly, if the *present* point in time is time t, then the two adjacent times $t+1$ and $t-1$ are the *previous* point in time and the *next* point in time respectively.

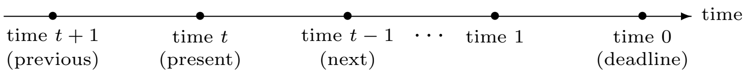

Figure 2.1.1: Points in time

A2 Absolutely necessary condition

In ASP (ABP), the seller (buyer) who is a leading trader must necessarily sell out his asset on hand (buy an asset from an appearing seller) up to time 0. In other words, the seller (buyer) is not absolutely allowed to quit the selling process (buying process) without selling (buying) the asset.

A3 Stop of process

In ATP[$\mathbb{R}$] the process stops when the leading trader accepts a price proposed by an opponent trader, and in ATP[$\mathbb{P}$] the process stops when an opponent trader accepts a price proposed by the leading trader. In the present paper we use distinctively the terms “stop” and “terminate”; the former is used when the process stops at a point in time *before the deadline* and the latter is used when it terminates *at the deadline*.

A4 Search cost

A cost $s \geq 0$, called the *search cost*, must be paid to search for opponent traders.

A5 Opponent trader appearing probability λ

If the search is conducted at a certain point in time, then an opponent trader appears at the *next* point in time with a probability λ $(0 < \lambda \leq 1)$.

A6 Quitting penalty

Assume $0 < \lambda < 1$ (i.e. $\lambda \neq 1$). Then, it is possible that no opponent trader appears at the next point in time even if the search is conducted. Accordingly, this assumption, as long as leaving all other conditions intact, inevitably leads us to the possibility that no opponent trader appears at all points in time over the entire planning horizon, including the deadline. If so, it follows that a leading trader must be faced with the situation of having to quit the process at the *terminal* point in time of the process (deadline) without trading the asset, which contradicts the assumption A2. One of the most general requirements that can be taken when encountering such a situation is the introduction of a *penalty* ρ [20,Ee & Ikuta (2006)][0514], implying that the leading trader can quit the process in exchange for the penalty; in this sense, let us call the penalty the *terminal quitting penalty.* Now, we can consider also the case that such a quitting penalty ρ is available also at every point in time besides the terminal point in time of the process; let us call the ρ the *intervening quitting penalty.* Note here that there exists a fine difference in its implication between a selling problem and a buying problem as stated below.

a. Selling Problem: When confronting such a situation in a selling problem, the seller (leading trader) will endeavor to devise means of selling out the asset, for example, by proposing a giveaway price ρ (a very low price) to a buyer whom he can then contact.

b. Buying Problem: When confronting such a situation in a buying problem, the buyer (leading trader) will endeavor to buy the asset by proposing a very high price ρ to a seller whom he can then contact.

Here note that in the above description the quitting price ρ is implicitly assumed to be positive (i.e., $\rho > 0$); however, in order to generalize the discussions that follows, we define it on $-\infty < \rho < \infty$ (i.e. $(-\infty, \infty)$). See Section 16.5(p.100) for further implication of this convenient extension.

A7 Region of price

In ATP[ℝ] (ATP[ℙ]), the price ξ proposed by an appearing opponent trader (the reservation price ξ of an appearing opponent trader) is positive in a normal market of the real world (i.e., $\xi > 0$) or equivalently the region of the price ξ is the interval $(0, \infty)$. However, in order to generalize the discussions that follows, we define it on $-\infty < \xi < \infty$ (i.e. $(-\infty, \infty)$). See Section 16.1(p.99) for the reason of the practical validity of this convenient extension.

A8 Distribution function

In ATP[ℝ] (ATP[ℙ]) let us assume that the prices proposed by successive opponent traders (the reservation prices of successive opponent traders), $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\cdots$, are independent identically distributed random variables having a *continuous* distribution function

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} \tag{2.1.1}$$

with a finite expectation μ where

$$\begin{aligned} F(\xi) &= 0 & \cdots (1) \quad & \xi \le a, \\ 0 < F(\xi) &< 1 & \cdots (2) \quad & a < \xi < b, \\ F(\xi) &= 1 & \cdots (3) \quad & b \le \xi, \end{aligned} \tag{2.1.2}$$

for given constants a and b such that

$$-\infty < a < \mu < b < \infty. \tag{2.1.3}$$

Furthermore, for its probability density function $f(\xi)$ let us assume

$$\begin{aligned} f(\xi) &= 0 & \cdots (1) \quad & \xi < a, \\ 0 < f(\xi) &< 1 & \cdots (2) \quad & a \le \xi \le b, \\ f(\xi) &= 0 & \cdots (3) \quad & b < \xi. \end{aligned} \tag{2.1.4}$$

Let us represent the set of all possible distribution functions defined above by

$$\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}, \tag{2.1.5}$$

called the *distribution function space.*

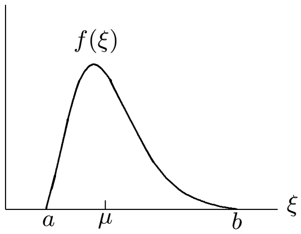

Figure 2.1.2: Probability density function $f(\xi)$.

A9 Recallability of once rejected opponent trader

Whether in the model with ℝ-mechanism or in the model with ℙ-mechanism, if an once rejected opponent trader can be *recalled* later and accepted, then it is called the *recall-model* or *model-with-recall*, conversely, if such recallability is impossible, then it is called the *no-recall-model*, *model-with-no-recall*, or *model-without-recall.*

A10 Discount factor β

In an asset selling problem, a seller (leading trader) can invest the profit ξ obtained by selling his asset at a given rate of interest $r > 0$; as a result, the profit ξ obtained at the present point time increases to $(1 + r)\xi$ a period hence (the next point in time); accordingly, a profit ξ a period hence is equivalent to $(1 + r)^{-1}\xi$ at the present point in time (see [39,Ross, 1961][0535]).† Usually $(1 + r)^{-1}$ is denoted by $\beta = (1 + r)^{-1}$ $(0 < \beta \le 1)$, called the *discount factor*, hence $(1 + r)^{-1}\xi = \beta\xi$. Now, let the today's profit (present point in time) be x and the tomorrow's profit (next point in time) be y. Then, the tomorrow's profit y is evaluated as βy at the point in time of today; accordingly, the comparison between the today's profit x and the tomorrow's profit y should be made by the inequality "$x > (= (<))\ \beta y$". Thus, it follows that the comparison between the today's profit x and the profit y obtained after n days should be made by the inequality "$x > (= (<))\ \beta^n y$". On the other hand, in an asset buying problem, a buying price paid by the buyer (leading trader) is a *cost*, which is not a currency on hand; in other words, it is what has been already paid, so it can not be invested. Hence, it follows that the introduction of discount factor β to the cost *seems* to be beside the question *at a glance.*

† $(1 + r)(1 + r)^{-1}\xi = \xi$

Alice 1 **(discount factor for cost)** *Herein, Alice came to grips with the following question. "But, but —, if this is true, it follows that the concept of the discount factor cannot be introduced not only to the asset buying problem but also to any other decision process with the objective of cost minimization !; Is this true ?. Then, Dr. Rabbit clad in the waistcoat-pocket suddenly appeared in front of her and told to her. The pay of a cost y can be regarded as the occurrence of a loss in the sense of losing the currency y on hand that should be avoided if it were not paid; in other words, "the today's cost y" can be regarded as a synonym of " the loss of the currency y on hand"; such a cost is usually called the opportunity loss. This scenario further proceeds as below. Since the today's currency y as an opportunity loss must increased, if it was not paid, to $(1+r)y = \beta^{-1}y$ tomorrow, the tomorrow's currency y is evaluated as the currency βy today. Accordingly, since the comparison between the today's cost x and the cost y after n days can be replaced by the comparison between the today's opportunity loss x and the opportunity loss y after n days, it eventually follows that the former comparison should be made by the inequality "$x > (= (<))\ \beta^n y$. From the above scenario it follows that the discount factor β can and should be introduced to all kinds of costs including the search cost s. And then, taking a watch out of its waistcoat-pocket and murmuring "Oh dear! Oh dear !, I shall be too late for the faculty meeting", Dr. Rabbit disappeared down the hole.* □

2.2 Two Novel Concepts

The two novel concepts below are introduced over the whole discussion of the present paper.

Concept 1 Recognizing time, starting time, initiating time, and null time zone (see Section 7.1(p.33))

Since a decision-making is basically a behaviour of human-beings, it first starts only when the existence of the decision-making problem has been recognized; in other words, it can never start without recognizing its existence. Now, suppose that the existence of an asset trading problem (ATP) has been recognized at a certain point in time. Then, let us refer to this point in time as the ***recognizing time*** of the process. Next, for different reasons, say making budgets, arranging staffs, etc., some amount of time will be needed to start actually tackling the problem. Let us refer to the point in time when all of such preparations finish as the ***starting time*** τ of the process. Now, reaching the starting time, *implicitly* or *unconsciously* one will immediately try to initiate the search for an opponent trader and to make the decision whether or not to make a deal with opponent traders that will appear after that. However, from a managerial and economic viewpoint, this is not always profitable since there exists the possibility that it can become better to postpone the initiation of search and decision until a point in time in the future. Let us refer to such a point in time as the ***initiating time*** and to the best of all possible initiating times as the *optimal initiating -time* $t^*_\tau \leq \tau$, OIT for short (see Figure 2.2.1 below). Let us call the time interval between the starting time τ and the optimal initiating time t^*_τ the *null-time-zone*, denoted by Nul-TZ (see Sections 7.2.4.6(p.36)). Quite oddly enough, however, we were not able to find any works, as long as we know, in which the above time concepts are taken into account.

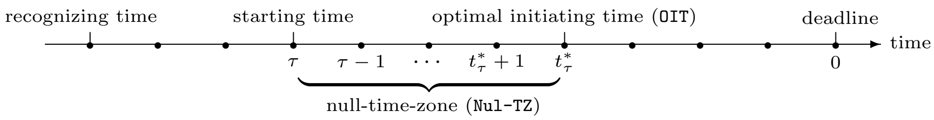

Figure 2.2.1: Null-Time-Zone (Nul-TZ)

Concept 2 Enforced-case and allowed-case

As reading the paper more, it will be known that all the concepts below are imperative in making the framework of each model treated in this paper more clear.

a. Attack-enforced-case and attack-allowed-case

When the existence of a decision-making problem has been recognized, what should be first questioned is whether it is enforced to attack the decision problem or not.

i. a-E-case: Let us refer to the former as the *attack-enforced-case*. In this case, even if it is known to yield no profit, a decision-maker must be resigned to the red ink.

ii. a-A-case: Let us refer to the latter as the *attack-allowed* (*not-enforced*)*-case* in a sense that a leading trader has the option whether to attack or not, hence, in this case, if it is known to yield no profit, it suffices to quit the decision-making itself without taking any action.

b. Immediate-initiation-enforced-case and immediate-initiation-allowed-case

Whether a-E-case or a-A-case, the following two cases can be considered, provided that in a-A-case it has been determined to attack the decision-making problem.

i. ii-E-case: The case that it is enforced to immediately initiate the attack, called the *immediate-initiation-enforced-case*.

ii. ii-A-case: The case that it is not enforced to immediately initiate the attack; in other words, the immediate initiation is allowed as one option, so it is possible to postpone its initiation. For this reason, we call the case the *immediate-initiation-allowed* (*not-enforced*)*-case*.

c. Search-Enforced-case and search-Allowed-case

i. s-E-case: The case that once the process has initiated, it is *enforced* to conduct the search at every point in time after that, called the *search-enforced-case.* In this case, as seen in Figure 2.2.2 below, a leading trader must continue to conduct the search until the process stops.

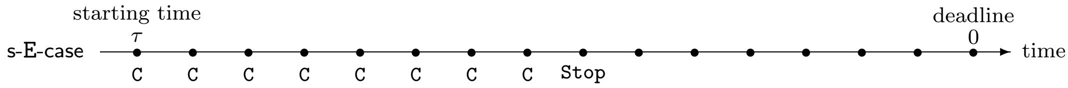

Figure 2.2.2: Flow of Search-Conducts in s-E-case

ii. s-A-case: The case that even if once the process has initiated, it is *allowed* to skip the search at every point in time after that, called the *search-allowed* (*not-enforced*)*-case.* In other words, a leading trader has an option whether to conduct the search or to skip as long as the process does not stop. In this case, we can consider different types of the flows of search-Conduct and search-Skip as shown in Figure 2.2.3 below where "⇝" represents the switch from search-Skip to search-Conduct or from search-Conduct to search-Skip.

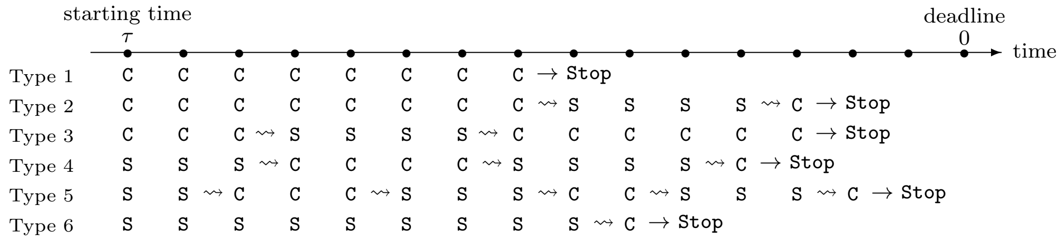

Figure 2.2.3: Different flows of search-Conducts and search-Skips in s-A-case

Definition 2.2.1 (search-Enforced-model and search-Allowed-model) Let us refer to the model of s-E-case and the model of s-A-case as search-**E**nforced-model and search-**A**llowed-model respectively, represented by s-E-model and s-A-model for short. □

Chapter 3

Tables of Models

3.1 Three Kinds of Models

In the present paper, we consider the following three kinds of models (see A6(p.7)):

- Model 1 in which the quitting penalty ρ (whether "*terminal*" or "*intervening*") is not available.
- Model 2 in which only the *terminal* quitting penalty ρ is available.
- Model 3 in which the *terminal* quitting penalty ρ and the *intervening* quitting penalty ρ are both available.

3.2 Simplified Symbols

For expressional simplicity, let us employ the following simplified symbols:

$$\mathbb{R}\text{-mech-model} \rightarrow \mathbb{R}\text{-model}, \quad \text{search-Allowed-model} \rightarrow \text{s-A-model}, \quad \text{selling-model} \rightarrow \text{S-model}, \tag{3.2.1}$$

$$\mathbb{P}\text{-mech-model} \rightarrow \mathbb{P}\text{-model}, \quad \text{search-Enforced-model} \rightarrow \text{s-E-model}, \quad \text{buying-model} \rightarrow \text{B-model}. \tag{3.2.2}$$

3.3 Tables of Models

Models treated in the present paper are classified into the no-recall-model and the recall-model (see A9(p.8)), each of which is moreover classified into s-A-model and s-E-model (see Def. 2.2.1(p.10)).

3.3.1 No-Recall-Model

Let us designate s-A-model with no recall (see Concept 2cii(p.10) (s-A-case)) by

$$\mathsf{M}{:}x[\mathbb{X}][\mathtt{A}] \quad (\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{A}]) \quad x = 1, 2, 3, \quad \mathbb{X} = \mathbb{R}, \mathbb{P}, {}^{\ddagger}$$

and s-E-model with no recall (see Concept 2ci(p.10) (s-E-case)) by

$$\mathsf{M}{:}x[\mathbb{X}][\mathtt{E}] \quad (\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}]) \quad x = 1, 2, 3, \quad \mathbb{X} = \mathbb{R}, \mathbb{P}.$$

Furthermore, let us define the set

$$\mathcal{Q}\langle \mathsf{M} : x[\mathtt{X}] \rangle = \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}], \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]\}, \quad x = 1, 2, 3, \ \mathtt{X} = \mathtt{A}, \mathtt{E}, \tag{3.3.1}$$

called the *quadruple-asset-trading-models*. Table 3.3.1 below provides the bird's-eye view of $\mathcal{Q}\langle \mathsf{M} : x[\mathtt{X}] \rangle$.

Table 3.3.1: Twenty Four No-recall-Models

		ASP[$\mathbb{R}$]	ABP[$\mathbb{R}$]	ASP[$\mathbb{P}$]	ABP[$\mathbb{P}$]	
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{A}]\}$	= {	$\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$	}
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{E}]\}$	= {	$\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$	}
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\}$	= {	$\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$	}
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{E}]\}$	= {	$\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]$	}
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\}$	= {	$\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]$	}
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{E}]\}$	= {	$\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]$	}

‡Throughout the paper, the model of the asset *buying* problem (ABP) is represented by the symbol upon which the tilde "˜" is capped like $\tilde{\mathsf{M}}$.

3.3.2 Recall-Model

Let us designate s-A-model with recall by

$$\mathsf{rM}{:}x[\mathbb{X}][\mathtt{A}] \quad (\mathrm{r}\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{A}]) \quad x = 1,2,3, \quad \mathbb{X} = \mathbb{R},\mathbb{P},$$

and s-E-model with recall by

$$\mathsf{rM}{:}x[\mathbb{X}][\mathtt{E}] \quad (\mathrm{r}\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}]) \quad x = 1,2,3, \quad \mathbb{X} = \mathbb{R},\mathbb{P}.$$

Furthermore, let us define the set

$$\mathcal{Q}\langle \mathsf{rM}:x[\mathtt{X}]\rangle = \{\mathsf{rM}{:}x[\mathbb{R}][\mathtt{X}], \mathrm{r}\tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}], \mathsf{rM}{:}x[\mathbb{P}][\mathtt{X}], \mathrm{r}\tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]\}, \quad x = 1,2,3, \quad \mathtt{X} = \mathtt{A},\mathtt{E}, \tag{3.3.2}$$

called the *quadruple-asset-trading-models-with-recall* as below. Table 3.3.2 below provides the bird's-eye view of $\mathcal{Q}\langle \mathsf{rM}:x[\mathtt{X}]\rangle$.

Table 3.3.2: Twenty Four Recall-Models

	ASP[ℝ]	ABP[ℝ]	ASP[ℙ]	ABP[ℙ]
$\mathcal{Q}\{\mathrm{rM}{:}1[\mathtt{A}]\}$ = {	rM:1[ℝ][A],	r$\tilde{\mathsf{M}}$:1[ℝ][A],	rM:1[ℙ][A],	r$\tilde{\mathsf{M}}$:1[ℙ][A] }
$\mathcal{Q}\{\mathrm{rM}{:}1[\mathtt{E}]\}$ = {	rM:1[ℝ][E],	r$\tilde{\mathsf{M}}$:1[ℝ][E],	rM:1[ℙ][E],	r$\tilde{\mathsf{M}}$:1[ℙ][E] }
$\mathcal{Q}\{\mathrm{rM}{:}2[\mathtt{A}]\}$ = {	rM:2[ℝ][A],	r$\tilde{\mathsf{M}}$:2[ℝ][A],	rM:2[ℙ][A],	r$\tilde{\mathsf{M}}$:2[ℙ][A] }
$\mathcal{Q}\{\mathrm{rM}{:}2[\mathtt{E}]\}$ = {	rM:2[ℝ][E],	r$\tilde{\mathsf{M}}$:2[ℝ][E],	rM:2[ℙ][E],	r$\tilde{\mathsf{M}}$:2[ℙ][E] }
$\mathcal{Q}\{\mathrm{rM}{:}3[\mathtt{A}]\}$ = {	rM:3[ℝ][A],	r$\tilde{\mathsf{M}}$:3[ℝ][A],	rM:3[ℙ][A],	r$\tilde{\mathsf{M}}$:3[ℙ][A] }
$\mathcal{Q}\{\mathrm{rM}{:}3[\mathtt{E}]\}$ = {	rM:3[ℝ][E],	r$\tilde{\mathsf{M}}$:3[ℝ][E],	rM:3[ℙ][E],	r$\tilde{\mathsf{M}}$:3[ℙ][E] }

3.4 Structured-Unit-of-Models

Let us refer to the whole of 24 models defined in each of the above two tables as the *structured-unit-of-models*, denoted by $\mathcal{S}(\mathsf{M})$ and $\mathcal{S}(\mathrm{r}\mathsf{M})$ respectively, i.e.,

$$\mathcal{S}(\mathsf{M}) = \{\mathcal{Q}\{\mathsf{M}:1[\mathtt{A}]\}, \mathcal{Q}\{\mathsf{M}:1[\mathtt{E}]\}, \mathcal{Q}\{\mathsf{M}:2[\mathtt{A}]\}, \mathcal{Q}\{\mathsf{M}:2[\mathtt{E}]\}, \mathcal{Q}\{\mathsf{M}:3[\mathtt{A}]\}, \mathcal{Q}\{\mathsf{M}:3[\mathtt{E}]\}, \tag{3.4.1}$$

$$\mathcal{S}(\mathrm{r}\mathsf{M}) = \{\mathcal{Q}\{\mathrm{r}\mathsf{M}:1[\mathtt{A}]\}, \mathcal{Q}\{\mathrm{r}\mathsf{M}:1[\mathtt{E}]\}, \mathcal{Q}\{\mathrm{r}\mathsf{M}:2[\mathtt{A}]\}, \mathcal{Q}\{\mathrm{r}\mathsf{M}:2[\mathtt{E}]\}, \mathcal{Q}\{\mathrm{r}\mathsf{M}:3[\mathtt{A}]\}, \mathcal{Q}\{\mathrm{r}\mathsf{M}:3[\mathtt{E}]\}. \tag{3.4.2}$$

3.5 Decisions

What a leading trader should determine in each of the models defined in Tables 3.3.1 and 3.3.2 are as follows:

1. Whether or not to accept the price proposed by an opponent trader (only for ℝ-model),
2. What price to propose (only for ℙ-model),
3. Whether or not to conduct the search for opponent traders (only for s-A-model),
4. When to initiate the process (for all models).

Chapter 4

No-Recall-Models

In this chapter we provide the strict definitions of no-recall-models in Table 3.3.1(p.11), by use of which the integration theory will be constructed in Part 2(p.38).

4.1 Model 1

4.1.1 Search-Enforced-Model: $\mathcal{Q}\langle \mathsf{M}{:}1[\mathsf{E}]\rangle = \{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}] : \mathsf{M}{:}1[\mathbb{P}][\mathsf{E}],\ \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$

4.1.1.1 $\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]$ and $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$

The two are the most basic models of the asset selling problem [8,Ber1995,p.158-162][0044] [46,You1998][0054] †, which are defined by the following assumptions:

*A*1. Once the process initiates, at every point in time after that it is enforced to conduct the search for buyers (see Concept 2ci(p.10) (s-E-case), hence the search cost $s \geq 0$ is paid at every point in time (see A4(p.7)).

*A*2. After the search has been conducted at a point in time $t > 0$, a buyer (opponent trader) certainly appears at time $t-1$ (next point in time), i.e., the buyer appearing probability $\lambda = 1$ (see A5(p.7)).

*A*3. Prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ which successively appearing buyers in $\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]$ propose and reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ which successively appearing buyers in $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$ have been both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$ (see A8(p.8)).

*A*4. Both terminal quitting penalty ρ and intervening quitting penalty ρ are not available (see A6(p.7)).

*A*5. The selling process stops at the point in time when the asset is sold to an appearing buyer (see A3(p.7)).

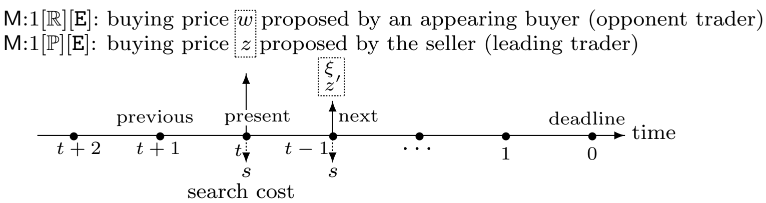

Figure 4.1.1: $\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]$ and $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$

The objective is to maximize the total expected present discounted *profit*, i.e., the expected present discounted value of the price for which the asset is sold, *minus* the total expected present discounted value of the search costs which will be paid until the process stops with selling the asset.

Remark 4.1.1

(a) The starting time τ must be greater than 0, i.e., $\tau > 0$ for the following reason. If $\tau = 0$, there exists no buyer at time 0, hence the process must stop without selling the asset, which contradicts A2(p.7).

(b) Suppose the process has proceeded up to time 1. Then, since the search is conducted at that time due to *A*1(p.13), a buyer certainly appears at time 0 due to *A*2(p.13). Then, at time 0 (deadline):

1. In $\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]$, due to A2(p.7) the seller must sell the asset to the buyer however small the price proposed by the buyer may be.
2. In $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$, the seller must propose a price a to the buyer where a is the lower bound of the distribution function F for the reservation price $\boldsymbol{\xi}$ of the buyer (see Figure 2.1.2(p.8)). Then, the buyer certainly buys the asset. □

†The case with $n = 1$.

4.1.1.2 $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] and $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]

The two are the models of the asset *buying* problem, each of which is the inverse of the asset *selling* problem in Section 4.1.1.1, defined by the following assumptions:

$A1$. Once the process initiates, at every point in time after that it is enforced to conduct the search for sellers, hence the search cost $s \geq 0$ is paid at every point in time.

$A2$. After the search has been conducted at a point in time $t > 0$, a seller (opponent trader) certainly appears at time $t-1$ (next point in time), i.e., the seller appearing probability $\lambda = 1$.

$A3$. Prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ which successively appearing sellers in $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] propose and reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ which successively appearing sellers in $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E] have been both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$.

$A4$. Both terminal quitting penalty ρ and intervening quitting penalty ρ are not available.

$A5$. The buying process stops at the point in time when the asset is bought by an appearing seller.

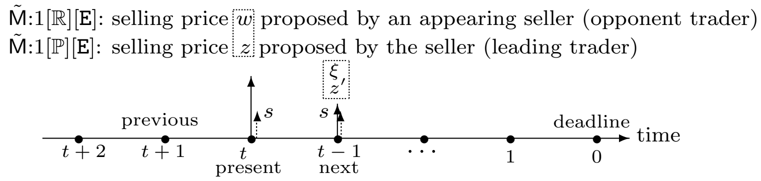

Figure 4.1.2: $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] and $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]

The objective is to minimize the total expected present discounted *cost*, i.e., the expected present discounted value of the price for which the asset is bought, *plus* the total expected present discounted value of the search costs which will be paid until the process stops with buying the asset.

Remark 4.1.2 Herein it should be noted that the direction of the vector representing a trading price ξ or z' and the direction of the vector representing a search cost s are <u>converse</u> in Figures 4.1.1 (selling model) and <u>identical</u> in 4.1.2 (buying model). □

4.1.2 Search-Allowed-Model 1: $\mathcal{Q}\langle$M:1[A]$\rangle$ = {M:1[$\mathbb{R}$][A], M:1[$\mathbb{P}$][A], $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A], $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]}

4.1.2.1 M:1[$\mathbb{R}$][A] and M:1[$\mathbb{P}$][A]

The two are the same as M:1[$\mathbb{R}$][E] and M:1[$\mathbb{P}$][E] in Section 4.1.1.1 only except that $A1$(p.13) is changed into as follows:

$A1$. At every point in time $t > 0$, it is allowed to skip the search (see Concept 2cii(p.10) (s-A-case)); in other words, the seller has an option whether to conduct the search or to skip.

Remark 4.1.3

(a) The starting time τ must be greater than 0, i.e., $\tau > 0$ for the same reason as in Remark 4.1.1(a).

(b) Suppose the process has proceeded up to time $t = 1$. Then, if the search is skipped at that time, no buyer appears at time $t = 0$, hence the seller is faced with the situation of having to quit the process without selling the asset, which contradicts A2(p.7). Accordingly, *the search must be necessarily conducted at time* $t = 1$; as a result, a buyer certainly appears at time 0 due to the assumption $A2$. □

4.1.2.2 $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] and $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]

The two are the same as $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] and $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E] in Section 4.1.1.2 only except that after the process has initiated, it is allowed to skip the search.

4.2 Model 2

4.2.1 Search-Enforced-Model 2: $\mathcal{Q}\langle \mathsf{M}:2[\mathtt{E}]\rangle = \{\mathsf{M}:2[\mathbb{R}][\mathtt{E}],\ \mathsf{M}:2[\mathbb{P}][\mathtt{E}],\ \tilde{\mathsf{M}}:2[\mathbb{R}][\mathtt{E}],\ \tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{E}]\}$

The quadruple models are the same as in Section 4.1.1.1 only except that the assumptions $A2$(p.13) and $A4$(p.13) are changed into as follows:

$A2$. After the search has been conducted at time $t > 0$, a buyer appears at the next point in time with a probability $\lambda \leq 1$.

$A4$. The terminal quitting penalty ρ is available.

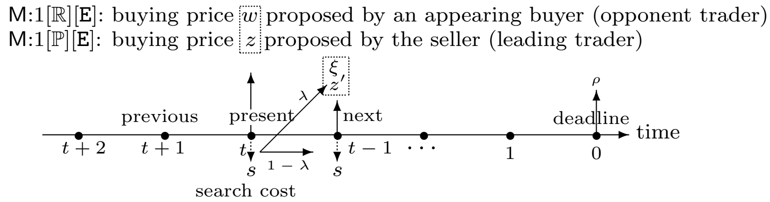

Figure 4.2.1: $\mathsf{M}:2[\mathbb{R}][\mathtt{E}], \mathsf{M}:2[\mathbb{P}][\mathtt{E}], \tilde{\mathsf{M}}:2[\mathbb{R}][\mathtt{E}],$ and $\tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{E}]$

Remark 4.2.1 In this model it is possible to stop the process by accepting the terminal quitting penalty ρ at time 0, hence the starting time $\tau = 0$ is permitted since the leading trader can quit the process with accepting the ρ at time 0 even if no opponent trader exists at time 0. Accordingly, in these models it follows that the starting time τ is greater than or equal to 0, i.e., $\tau \geq 0$. □

4.2.2 Search-Allowed-Model 2: $\mathcal{Q}\langle \mathsf{M}:2[\mathtt{A}]\rangle = \{\mathsf{M}:2[\mathbb{R}][\mathtt{A}],\ \mathsf{M}:2[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}:2[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{A}]\}$

The quadruple models are the same as in Section 4.2.1 only except that $A1$(p.13) is changed as follows:

$A1$. After the process has initiated, it is allowed to skip the search at every point in time $t > 0$.

4.3 Model 3

4.3.1 Search-Enforced-Model 3: $\mathcal{Q}\langle \mathsf{M}:3[\mathtt{E}]\rangle = \{\mathsf{M}:3[\mathbb{R}][\mathtt{E}],\ \mathsf{M}:3[\mathbb{P}][\mathtt{E}],\ \tilde{\mathsf{M}}:3[\mathbb{R}][\mathtt{E}],\ \tilde{\mathsf{M}}:3[\mathbb{P}][\mathtt{E}]\}$

The quadruple models are the same as in Section 4.2.1 only except that the assumption $A4$(p.15) is changed as follows:

$A4$. In addition to the terminal quitting penalty ρ, the intervening quitting penalty ρ is also available.

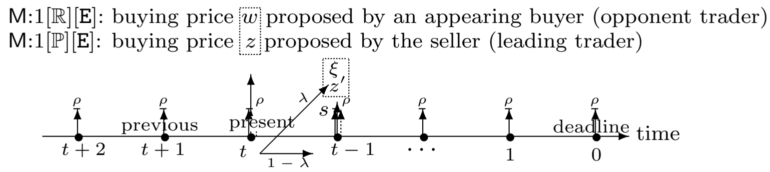

Figure 4.3.1: $\mathsf{M}:3[\mathbb{R}][\mathtt{E}], \mathsf{M}:3[\mathbb{P}][\mathtt{E}], \tilde{\mathsf{M}}:3[\mathbb{R}][\mathtt{E}]$, and $\tilde{\mathsf{M}}:3[\mathbb{P}][\mathtt{E}]$

4.3.2 Search-Allowed-Model 3: $\mathcal{Q}\langle \mathsf{M}:3[\mathtt{A}]\rangle = \mathsf{M}:3[\mathbb{R}][\mathtt{A}],\ \mathsf{M}:3[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}:3[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}:3[\mathbb{P}][\mathtt{A}]$

The quadruple models are the same as in Section 4.3.1 only except that after the process has initiated, it is allowed to skip the search.

4.4 Parameter and Parameter Space

Let us refer to $\lambda \in (0, 1]$, $\beta \in (0, 1]$, $s \in [0, \infty)$, and $\rho \in (-\infty, \infty)$ as the *parameter* of models, all of which are independent of the distribution function F. Then, let $\boldsymbol{p} = (\lambda, \beta, s)$ for Model 1 and $\boldsymbol{p} = (\lambda, \beta, s, \rho)$ for Models 2,3, called the *parameter vector*, and let us represent the set of all possible $\boldsymbol{p}$'s by

$$\mathscr{P} = \{\boldsymbol{p} \mid \lambda = 1,\ 0 < \beta \leq 1,\ 0 \leq s\} \quad \text{for Model 1,} \tag{4.4.1}$$
$$\mathscr{P} = \{\boldsymbol{p} \mid 0 < \lambda \leq 1,\ 0 < \beta \leq 1,\ 0 \leq s,\ -\infty < \rho < \infty\} \quad \text{for Models 2,3,} \tag{4.4.2}$$

called the *parameter space*.

‡Cartesian product.

Definition 4.4.1 (total-space) Let us consider the following product (Cartesian product):

$$\mathscr{P} \times \mathscr{F} = \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}, F \in \mathscr{F}\}, \tag{4.4.3}$$

called the *total-space*, depicted as the deformative circle in Figure 4.4.1 below. □

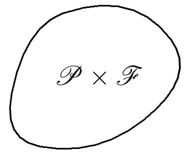

Figure 4.4.1: Total-space $\mathscr{P} \times \mathscr{F}$

4.5 Different Variations

Different variations can be considered for the models that were defined in the previous sections.

■ **No-Recall-Model** In Section 4.1.1.1 we provided the definition of the most textbookish and basic no-recall-model. Below let us show its some variations.

⟨1⟩ *Limited search budget* [25,Iku1992][0236] † This is the model in which the total amount of budget that can be invested in the search activities is limited. Then, how to allocate the limited budget among search activities in all points in time over the entire planning horizon ?

⟨2⟩ *Price mechanism switching* [17,Ee2006][0033] [15,Ee2004][0413] * This is the model in which the price mechanism can be switched between $\mathbb{R}$-mechanism and $\mathbb{P}$-mechanism at every point in time over the planning horizon.

⟨3⟩ *Several search areas* [26,Iku1995][0261] ‡ For example, let Tokyo, Kyoto, and Osaka be areas in which the leading trader can search for opponent traders. Suppose he is in Tokyo today. Then, tomorrow, to stay in Tokyo, to move to Kyoto, or to move to Osaka ?

⟨4⟩ *Uncertain deadline* [18,Ee2009][0018] In Example 1.1.1(p.3), let the date of the return home be not yet definite at present; it may be right now, one week after, or maybe the return directive itself might be countermanded.

■ **Recall-Model** In Section 22.3.1.1.1(p.222) we will provide the three definitions of the most basic recall-models defined in [43, Sak1961][0005]. Below let us show the three variations of them:

⟨5⟩ *Uncertain recall* [31,Kar1977][0078] [2,Aki2014][0009] [24,Iku1988][0180] § This is the model in which the recall of opponent traders once rejected is uncertain.

⟨6⟩ *Costly recall* [28,Kan1999][0276],[29,Kan2005][0034] This is the model in which some cost must be paid to recall opponent traders once rejected.

⟨7⟩ *Reserved recall* [41,Sai1998][0275],[42,Sai1999][0188] This is the model in which the availability of recall can be reserved by paying some deposit

Models in the above references are all Model 1, in which the quitting penalty ρ is not available, the search skip is not permitted, and the initiating time is not introduced. The analyses for models with the quitting penalty ρ, the search skip, and the initiating time are left as future subjects (see Section 28.3(p.269)).

■ **Others** In addition to the above variations, we will have other variations in the future which are not yet posed by any researchers. For example:

⟨8⟩ *Multiple assets model* This is the model in which multiple assets are traded. In the model, the optimal decision rule depends on the number of assets that remain without being traded.

⟨9⟩ *Lasting effect of search activity* This is the model in which the effect of the search activity lasts for a while. The simplest case of the variation is that its effect disappears with a given probability p at the next point in time; hence it lasts with the probability $1 - p$.

⟨10⟩ *Search activity impossibility probability* For example, as it suddenly rains, you cannot go out to search for buyers.

⋮

†https://www.orsj.or.jp/~archive/pdf/e_mag/Vol.35_02_172.pdf
*https://commons.sk.tsukuba.ac.jp/discussion/page/27 No.1098 (2004)
‡https://www.orsj.or.jp/~archive/pdf/e_mag/Vol.38_01_089.pdf
§https://www.orsj.or.jp/~archive/pdf/e_mag/Vol.31_02_145.pdf

Chapter 5

Underlying Functions

All the functions defined in this chapter are used to derive the systems of optimality equations (see Chapter 6(p.21)) of the twenty four models in Table 3.3.1(p.11) .

5.1 Definition

5.1.1 T, L, K, and $\mathcal{L}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$T(x) = \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}] \tag{5.1.1}$$
$$= \int_{-\infty}^{\infty} \max\{\xi - x, 0\} f(\xi) d\xi,^{\dagger\ddagger} \tag{5.1.2}$$

and then define

$$L(x) = \lambda\beta T(x) - s, \tag{5.1.3}$$
$$K(x) = \lambda\beta T(x) - (1-\beta)x - s,^{\S} \tag{5.1.4}$$
$$\mathcal{L}(s) = L(\lambda\beta\mu - s), \tag{5.1.5}$$
$$\kappa = \lambda\beta T(0) - s \tag{5.1.6}$$
$$= L(0) = K(0) = \lambda\beta T(0) - s \tag{5.1.7}$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{R}$ and to κ as the *κ-value* of Type $\mathbb{R}$. The formula below will be sometimes used in the rest of the paper.

$$K(x) + (1-\beta)x = L(x), \tag{5.1.8}$$
$$K(x) + x = L(x) + \beta x, \tag{5.1.9}$$
$$\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x\}] + (1-\lambda)\beta x - s = K(x) + x \tag{5.1.10}$$

5.1.2 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$\tilde{T}(x) = \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}] \tag{5.1.11}$$
$$= \int_{-\infty}^{\infty} \min\{\xi - x, 0\} f(\xi) d\xi, \tag{5.1.12}$$

and then define

$$\tilde{L}(x) = \lambda\beta\tilde{T}(x) + s, \tag{5.1.13}$$
$$\tilde{K}(x) = \lambda\beta\tilde{T}(x) - (1-\beta)x + s, \tag{5.1.14}$$
$$\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s), \tag{5.1.15}$$
$$\tilde{\kappa} = \lambda\beta\tilde{T}(0) + s \tag{5.1.16}$$
$$= \tilde{L}(0) = \tilde{K}(0). \tag{5.1.17}$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{R}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{R}$.

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† See [14,DeGroot70][0540] .
‡ See Figure A 7.3(p.296) (I) ,
§ See Figure A 7.3(p.296) (II) ,

5.1.3 T, L, K, and $\mathcal{L}$ of Type $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$p(z) = \Pr\{z \le \boldsymbol{\xi}\}, \tag{5.1.18}$$
$$T(x) = \max_z p(z)(z-x)^\dagger \tag{5.1.19}$$

and then define

$$L(x) = \lambda\beta T(x) - s, \tag{5.1.20}$$
$$K(x) = \lambda\beta T(x) - (1-\beta)x - s, \tag{5.1.21}$$
$$\mathcal{L}(s) = L(\lambda\beta a - s), \tag{5.1.22}$$
$$\kappa = \lambda\beta T(0) - s \tag{5.1.23}$$
$$= L(0) = K(0) \tag{5.1.24}$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{P}$ and to κ as the *κ-value* of Type $\mathbb{P}$. Let us denote z maximizing $p(z)(z-x)$ by $z(x)$ if it exists, i.e.,

$$T(x) = p(z(x))(z(x)-x). \tag{5.1.25}$$

Definition 5.1.1 If there exist multiple $z(x)$, let us define the *smallest* of them as $z(x)$. □

Furthermore, for convenience of later discussions, let us define

$$a^\star = \inf\{x \mid T(x) + x > a\} = \inf\{x \mid T(x) > a - x\}, \tag{5.1.26}$$
$$x^\star = \inf\{x \mid z(x) > a\}. \tag{5.1.27}$$

Noting that (5.1.18) can be rewritten as $p(z) = 1 - \Pr\{\boldsymbol{\xi} < z\} = 1 - \Pr\{\boldsymbol{\xi} \le z\}$ due to the assumption of F being continuous (see A8(p.8), we have $p(z) = 1 - F(z)$. Accordingly, it can be immediately seen that

$$p(z) \begin{cases} = 1, & z \le a \quad \cdots (1) \quad \text{due to } (2.1.2\,(1)\,(\text{p.8})), \\ < 1, & a < z \quad \cdots (2) \quad \text{due to } (2.1.2\,(2,3)), \end{cases} \tag{5.1.28}$$

$$p(z) \begin{cases} > 0, & z < b \quad \cdots (1), \quad \text{due to } (2.1.2\,(1,2)), \\ = 0, & b \le z \quad \cdots (2), \quad \text{due to } (2.1.2)3. \end{cases} \tag{5.1.29}$$

Example **5.1.1** $p(z)(z-x)$ can be depicted as below.

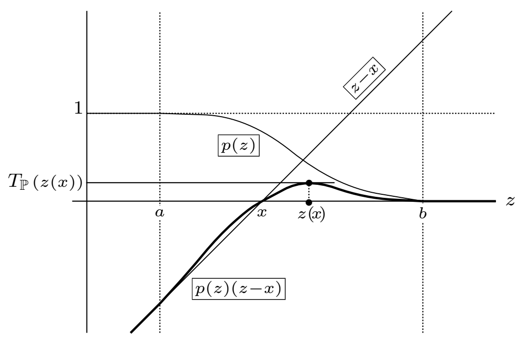

Figure 5.1.1: Graph of $p(z)(z-x)$

When F is the uniform distribution function, we have

$$a^\star = 2a - b \quad (\text{see } (\text{A } 7.7\,(1)\,(\text{p.297}))\,). \quad \square \tag{5.1.30}$$

5.1.4 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of Type $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \le z\}, \tag{5.1.31}$$
$$\tilde{T}(x) = \min_z \tilde{p}(z)(z-x), \tag{5.1.32}$$

† See Figure A 7.4(p.296) .

and then define

$$\begin{aligned}
\tilde{L}(x) &= \lambda\beta\tilde{T}(x) + s, && (5.1.33)\\
\tilde{K}(x) &= \lambda\beta\tilde{T}(x) - (1-\beta)x + s, && (5.1.34)\\
\tilde{\mathcal{L}}(s) &= \tilde{L}(\lambda\beta b + s), && (5.1.35)\\
\tilde{\kappa} &= \lambda\beta\tilde{T}(0) + s && (5.1.36)\\
&= \tilde{L}(0) = \tilde{K}(0). && (5.1.37)
\end{aligned}$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{P}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{P}$.

Definition 5.1.2 Let us denote z minimizing $\tilde{p}(z)(z-x)$ by $\tilde{z}(x)$ if it exists, i.e.,

$$\tilde{T}(x) = \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x). \tag{5.1.38}$$

If there exist multiple $\tilde{z}(x)$, let us define the *largest* of them as $\tilde{z}(x)$. □

Furthermore, for convenience of later discussions, let us define

$$\begin{aligned}
b^\star &= \sup\{x \mid \tilde{T}(x) + x < b\} = \sup\{x \mid \tilde{T}(x) < b - x\}, && (5.1.39)\\
\tilde{x}^\star &= \sup\{x \mid \tilde{z}(x) < b\}. && (5.1.40)
\end{aligned}$$

Noting that (5.1.31) can be rewritten as $\tilde{p}(z) = F(z)$, we can immediately see that

$$\tilde{p}(z)\begin{cases} = 0, & z \le a \quad \cdots (1) \quad \text{due to } (2.1.2\,(1)\,\text{(p.8)}),\\ > 0, & a < z \quad \cdots (2) \quad \text{due to } (2.1.2\,(2.3)),\end{cases} \tag{5.1.41}$$

$$\tilde{p}(z)\begin{cases} < 1, & z < b \quad \cdots (1) \quad \text{due to } (2.1.2\,(1,2)),\\ = 1, & b \le z \quad \cdots (2) \quad \text{due to } (2.1.2\,(3)).\end{cases} \tag{5.1.42}$$

5.2 Solutions

(a) Let us define the solutions of the equations $L(x) = 0$, $K(x) = 0$, and $\mathcal{L}(s) = 0$, whether Type $\mathbb{R}$ or Type $\mathbb{P}$, by x_L, x_K, and $s_{\mathcal{L}}$ respectively if they exist, i.e.,

$$L(x_L) = 0 \cdots (1), \qquad K(x_K) = 0 \cdots (2), \qquad \mathcal{L}(s_{\mathcal{L}}) = 0 \cdots (1). \tag{5.2.1}$$

If multiple solutions exist for each of the above three equations, we employ the *smallest* as its solution.

(b) Let us define the solutions of the equations $\tilde{L}(x) = 0$, $\tilde{K}(x) = 0$, and $\tilde{\mathcal{L}}(s) = 0$, whether $\tilde{\text{T}}$ype $\mathbb{R}$ or $\tilde{\text{T}}$ype $\mathbb{P}$, by $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$ respectively if they exist.

$$\tilde{L}(x_{\tilde{L}}) = 0 \cdots (1), \qquad \tilde{K}(x_{\tilde{K}}) = 0 \cdots (2), \qquad \tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0 \cdots (1). \tag{5.2.2}$$

If multiple solutions exist for each of the above three equations, we employ the *largest* as its solution.

5.3 Primitive Underlying Functions and Derivative Underlying Functions

Sometimes let us refer to each of T- and $\tilde{T}$-functions as the *primitive underlying function* and to each of L-, K-, $\mathcal{L}$-, $\tilde{L}$-, $\tilde{L}$-, and $\tilde{\mathcal{L}}$-functions as the *derivative underlying function*, each of which is defined by use of the primitive underlying function T.

5.4 Identical Representation and Explicit Representation

In the rest of the paper, when we need to distinguish

$$T,\ L,\ K,\ \mathcal{L},\ \kappa,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ \tilde{\kappa},\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}} \tag{5.4.1}$$

between Type $\mathbb{R}$ and Type $\mathbb{P}$ and between $\tilde{\text{T}}$ype $\mathbb{R}$ and $\tilde{\text{T}}$ype $\mathbb{P}$, let us denote them by

$$\begin{aligned}
&T_{\mathbb{R}},\ L_{\mathbb{R}},\ K_{\mathbb{R}},\ \mathcal{L}_{\mathbb{R}},\ \kappa_{\mathbb{R}},\ x_{L_{\mathbb{R}}},\ x_{K_{\mathbb{R}}},\ s_{\mathcal{L}_{\mathbb{R}}},\ \tilde{T}_{\mathbb{R}},\ \tilde{L}_{\mathbb{R}},\ \tilde{K}_{\mathbb{R}},\ \tilde{\mathcal{L}}_{\mathbb{R}},\tilde{\kappa}_{\mathbb{R}},\ x_{\tilde{L}_{\mathbb{R}}},\ x_{\tilde{K}_{\mathbb{R}}},\ s_{\tilde{\mathcal{L}}_{\mathbb{R}}}, && (5.4.2)\\
&T_{\mathbb{P}},\ L_{\mathbb{P}},\ K_{\mathbb{P}},\ \mathcal{L}_{\mathbb{P}},\ \kappa_{\mathbb{P}},\ x_{L_{\mathbb{P}}},\ x_{K_{\mathbb{P}}},\ s_{\mathcal{L}_{\mathbb{P}}},\ \tilde{T}_{\mathbb{P}},\ \tilde{L}_{\mathbb{P}},\ \tilde{K}_{\mathbb{P}},\ \tilde{\mathcal{L}}_{\mathbb{P}},\tilde{\kappa}_{\mathbb{P}},\ x_{\tilde{L}_{\mathbb{P}}},\ x_{\tilde{K}_{\mathbb{P}}},\ s_{\tilde{\mathcal{L}}_{\mathbb{P}}}. && (5.4.3)
\end{aligned}$$

Let us refer to (5.4.1) as the *identical representation* and to (5.4.2) and (5.4.3) as the *explicit representation*.

5.5 Characteristic Vector and Characteristic Element

Let us here define the two vectors, $\boldsymbol{C}_{\mathbb{R}}$ consisting of (5.1.3(p.17))-(5.1.6) and $\tilde{\boldsymbol{C}}_{\mathbb{R}}$ consisting of (5.1.13(p.17))-(5.1.16), represented as respectively

$$\boldsymbol{C}_{\mathbb{R}} = (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{R}} = (\tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}). \tag{5.5.1}$$

Likewise, let us define the two vectors, $\boldsymbol{C}_{\mathbb{P}}$ consisting of (5.1.20(p.18))-(5.1.23) and $\tilde{\boldsymbol{C}}_{\mathbb{P}}$ consisting of (5.1.33(p.19))-(5.1.36), i.e.,

$$\boldsymbol{C}_{\mathbb{P}} = (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{P}} = (\tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}). \tag{5.5.2}$$

Furthermore, adding T- and $\tilde{T}$-functions to the above vectors, let us define

$$\boldsymbol{C}^T_{\mathbb{R}} = (T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \quad \tilde{\boldsymbol{C}}^T_{\mathbb{R}} = (\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}), \tag{5.5.3}$$

$$\boldsymbol{C}^T_{\mathbb{P}} = (T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \quad \tilde{\boldsymbol{C}}^T_{\mathbb{P}} = (\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}). \tag{5.5.4}$$

Let us call each of the vectors defined above the *characteristic vector* and its element the *characteristic element*. In the identical representation, the above vectors are all represented by $\boldsymbol{C}$, $\tilde{\boldsymbol{C}}$, $\boldsymbol{C}^T$, and $\tilde{\boldsymbol{C}}^T$ respectively.

Chapter 6

Systems of Optimality Equations

6.1 Preliminary

In this chapter we derive the system of optimality equations, denoted by SOE for short, for each of the twenty-four models in Table 3.3.1(p.11) .

Definition 6.1.1 (Conduvt/Skip) For expressional simplicity, below, by $\texttt{Conduct}_t$ ($\texttt{Skip}_t$) let us represent "Conduct the search at time t" ("Skip the search at time t"). □

Remark 6.1.1 (relationship between SOE and assertion) In general, a model M of a decision process, whether in this paper or not, has the system of optimality equations, denoted by SOE{M}, which should be said to be a mirror exhaustively reflecting the entire aspect of the model M. In other words, SOE{M} involves the exhaustive information of the model M as if a gene has the exhaustive information of a life. This implies that any assertion which is characterized by the sequence $\{V_t\}$ generated from SOE{M} can be regarded as an assertion on the model M; conversely, an assertion which is not characterized by the sequence $\{V_t\}$ cannot be said to be an assertion on the M. □

Below let us represent "buyer (seller) proposing a price w" by "buyer (seller) w" for short.

6.2 Search-Allowed-Model

6.2.1 Model 1

Let us note here that $\lambda = 1$ is assumed in this model.

6.2.1.1 M:1[ℝ][A]

By $v_t(w)$ ($t \geq 0$) and V_t ($t > 0$) let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively. Then, we have

$$v_0(w) = w, \tag{6.2.1}$$
$$v_t(w) = \max\{w, V_t\}, \quad t > 0, \tag{6.2.2}$$

where V_t is the maximum of the total expected present discounted profit from rejecting the proposed price w. Then, we have

$$V_1 = \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] - s = \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s, \tag{6.2.3}$$
$$V_t = \max\{\mathtt{C} : \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s,\ \mathtt{S} : \beta V_{t-1}\}, \quad t > 1, \tag{6.2.4}$$

where C and S represent the actions of Conducting the search and Skipping the search respectively. Then, since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = T(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.1(p.17))), hence (6.2.4) can be written as

$$\begin{aligned} V_t &= \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4(p.17)) with } \lambda = 1) \qquad (6.2.5) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.8(p.17)))}. \qquad (6.2.6) \end{aligned}$$

□ SOE{M:1[ℝ][A]} is given by the set of (6.2.1) – (6.2.4). However, since the sequence $\{V_t\}$ is generated from the two expressions (6.2.3) and (6.2.5), due to Remark 6.1.1 it can be reduced to only the two in Table 6.5.1(p.31) (I). □

Now, let us here define

$$\mathbb{S}_t = \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 1. \tag{6.2.7}$$

Then, (6.2.4) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{6.2.8}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.9}$$

From (6.2.2) we can rearrange (6.2.7) as $\mathbb{S}_t = \beta(\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s$. Accordingly, from (5.1.1(p.17)) and (5.1.3) with $\lambda = 1$ we have

$$\begin{aligned} \mathbb{S}_t &= \beta T(V_{t-1}) - s && (6.2.10)\\ &= L(V_{t-1}), \quad t > 1. && (6.2.11) \end{aligned}$$

6.2.1.2 $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller w and with no seller respectively. Then, we have

$$v_0(w) = w, \tag{6.2.12}$$

$$v_t(w) = \min\{w, V_t\}, \quad t > 0, \tag{6.2.13}$$

where V_t is the minimum of the total expected present discounted cost from rejecting the proposed price w. Then, we have

$$\begin{aligned} V_1 &= \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] + s = \beta\,\mathbf{E}[\boldsymbol{\xi}] + s = \beta\mu + s, && (6.2.14)\\ V_t &= \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s,\ \beta V_{t-1}\}, \quad t > 1. && (6.2.15) \end{aligned}$$

Then, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.11(p.17))), hence (6.2.15) can be written as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\}\\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14) with } \lambda = 1) && (6.2.16)\\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}\\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.14) and (5.1.13) with } \lambda = 1). && (6.2.17) \end{aligned}$$

□ SOE{$\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A]} can be reduced to (6.2.14) and (6.2.16), listed in Table 6.5.1(p.31) (II). ■

Remark 6.2.1 Note here that the same notations $v_t(w)$ and V_t are used for both M:1[$\mathbb{R}$][A] and $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A]. For explanatory convenience, later on we sometimes represent $v_t(w)$ and V_t for $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] by $\tilde{v}_t(w)$ and $\tilde{V}_t$ respectively. Then (6.2.12)-(6.2.15) are written as $\tilde{v}_0(w) = w$, $\tilde{v}_t(w) = \min\{w, \tilde{V}_t\}$, $\tilde{V}_1 = \beta\mu + s$, and $\tilde{V}_t = \min\{\beta\,\mathbf{E}[\tilde{v}_{t-1}(\boldsymbol{\xi})] + s, \beta\tilde{V}_{t-1}\}$ respectively. ■

Now, let us here define

$$\tilde{\mathbb{S}}_t = \beta(\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 1. \tag{6.2.18}$$

Then, (6.2.15) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{6.2.19}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.20}$$

From (6.2.13) we can rearrange (6.2.18) as $\tilde{\mathbb{S}}_t = \beta(\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s$. Accordingly, from (5.1.11(p.17)) and (5.1.13(p.17)) with $\lambda = 1$ we have

$$\begin{aligned} \tilde{\mathbb{S}}_t &= \beta\tilde{T}(V_{t-1}) + s && (6.2.21)\\ &= \tilde{L}(V_{t-1}), \quad t > 1. && (6.2.22) \end{aligned}$$

6.2.1.3 M:1[$\mathbb{P}$][A]

By v_t $(t \geq 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. In this model, since the search must be necessarily conducted at time 1 (see Remark 4.1.3(p.14) (b)), there exists a buyer at time 0. Suppose the process has proceeded up to time 0. Then, since the seller must necessarily sell the asset at that time, he must propose the price $a^\dagger$ to a buyer appearing at that time (see Remark 4.1.1(p.13) (b2)), thus we have

$$z_0 = a. \tag{6.2.23}$$

Hence, the profit that the seller obtains at time 0 becomes a, i.e.,

$$v_0 = a. \tag{6.2.24}$$

† The lower bound of the distribution function for the reservation price (maximum permissible buying price) of the buyer.

Now, since the search is conducted at time $t = 1$, we have

$$V_1 = \beta v_0 - s = \beta a - s. \tag{6.2.25}$$

In addition, we have

$$V_t \;=\; \max\{\beta v_{t-1} - s,\, \beta V_{t-1}\}, \quad t > 1. \tag{6.2.26}$$

If the seller proposes a price z, the probability of a buyer buying the asset is given by $p(z) = \Pr\{z \le \boldsymbol{\xi}\}$ (see (5.1.18(p.18))), hence we have

$$v_t \;=\; \max_z\{p(z)z + (1 - p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.27}$$

due to (5.1.19(p.18)), implying that the optimal price z_t which the seller should propose is given by

$$z_t = z(V_t), \quad t > 0, \tag{6.2.28}$$

due to (5.1.25(p.18)). Now, since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (6.2.27)), we can rearrange (6.2.26) as follows

$$\begin{aligned} V_t &= \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.18)) with } \lambda = 1) \qquad (6.2.29) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \quad \text{(see (5.1.21) and (5.1.20) with } \lambda = 1) \qquad (6.2.30) \end{aligned}$$

□ SOE{M:1[ℙ][A]} can be reduced to (6.2.25) and (6.2.29), listed in Table 6.5.1(p.31) (III). ∎

Now, let us here define

$$\mathbb{S}_t \;=\; \beta(v_{t-1} - V_{t-1}) - s, \quad t > 1. \tag{6.2.31}$$

Then, (6.2.26) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{6.2.32}$$

implying that

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.33}$$

From (6.2.27) and (5.1.20(p.18)) we can rewrite (6.2.31(p.23)) as

$$\begin{aligned} \mathbb{S}_t &= \beta T(V_{t-1}) - s \qquad (6.2.34) \\ &= L(V_{t-1}), \quad t > 1. \qquad (6.2.35) \end{aligned}$$

6.2.1.4 M̃:1[ℙ][A]

By v_t $(t \ge 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \ge 0$ by z_t. In this model, since the search must be necessarily conducted at time 1 (see Remark 4.1.3(p.14) (b)), there exists a seller at time 0. Suppose the process has proceeded up to time 0. Then, since the buyer must necessarily buy the asset at that time, he must propose the price b[†] to a seller appearing at that time (see Remark 4.1.1(p.13) (b2)), thus we have

$$z_0 = b. \tag{6.2.36}$$

Hence, the cost that the buyer pays at time 0 becomes b, i.e.,

$$v_0 = b. \tag{6.2.37}$$

Now, since the search is conducted at time $t = 1$, we have

$$V_1 = \beta v_0 + s = \beta b + s. \tag{6.2.38}$$

In addition, we have

$$V_t \;=\; \min\{\beta v_{t-1} + s,\, \beta V_{t-1}\}, \quad t > 1. \tag{6.2.39}$$

If the buyer proposes a price z, the probability of a seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \le z\}$ (see (5.1.31(p.18))), hence we have

$$v_t \;=\; \min_z\{\tilde{p}(z)z + (1 - \tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.40}$$

due to (5.1.32(p.18)), implying that the optimal price z_t which the buyer should propose is given by

$$z_t = \tilde{z}(V_t), \quad t > 0, \tag{6.2.41}$$

[†] The upper bound of the distribution function for the reservation price (minimum permissible selling price) of the seller

due to (5.1.38(p.19)). Now, since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (6.2.40)), we can rearrange (6.2.39) as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.19)) with } \lambda = 1) \qquad (6.2.42) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad \text{(see (5.1.34) and (5.1.33) with } \lambda = 1) \qquad (6.2.43) \end{aligned}$$

□ SOE{$\tilde{\text{M}}$:1[ℙ][A]} can be reduced to (6.2.38) and (6.2.42), listed in Table 6.5.1(p.31) (IV). ∎

Now, let us here define

$$\tilde{\mathbb{S}}_t \;=\; \beta(v_{t-1} - V_{t-1}) + s, \quad t > 1. \qquad (6.2.44)$$

Then, (6.2.39) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \qquad (6.2.45)$$

implying that

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \qquad (6.2.46)$$

From (6.2.40) and (5.1.33(p.19)) we can rewrite (6.2.44(p.24)) as

$$\begin{aligned} \mathbb{S}_t &= \beta\tilde{T}(V_{t-1}) + s \qquad (6.2.47) \\ &= \tilde{L}(V_{t-1}), \quad t > 1. \qquad (6.2.48) \end{aligned}$$

6.2.2 Model 2

6.2.2.1 M:2[ℝ][A]

By $v_t(w)$ $(t \ge 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively. Then we have

$$\begin{aligned} v_0(w) &= \max\{w, \rho\}, \qquad (6.2.49) \\ v_t(w) &= \max\{w, V_t\}, \quad t > 0, \qquad (6.2.50) \end{aligned}$$

where

$$V_t \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \qquad (6.2.51)$$

Let us here define

$$V_0 = \rho. \qquad (6.2.52)$$

Then (6.2.50) holds for $t \ge 0$ instead of $t > 0$, i.e.,

$$v_t(w) \;=\; \max\{w, V_t\}, \quad t \ge 0. \qquad (6.2.53)$$

Since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.1(p.17))), from (6.2.51) we have

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4(p.17)))} \qquad (6.2.54) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.8))}. \qquad (6.2.55) \end{aligned}$$

□ SOE{M:2[ℝ][A]} can be reduced to (6.2.52) and (6.2.54), listed in Table 6.5.3(p.31) (I). ∎

Let us here define

$$\mathbb{S}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \qquad (6.2.56)$$

Then, (6.2.51) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \qquad (6.2.57)$$

implying that

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0. \qquad (6.2.58)$$

From (6.2.56) and (6.2.50) we have $\mathbb{S}_t = \beta(\,\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s$. Accordingly, from (5.1.1(p.17)) and (5.1.3(p.17)) we have

$$\begin{aligned} \mathbb{S}_t &= \beta T(V_{t-1}) - s \qquad (6.2.59) \\ &= L(V_{t-1}), \quad t > 0. \qquad (6.2.60) \end{aligned}$$

6.2.2.2 $\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]

By $v_t(w)$ ($t \geq 0$) and V_t ($t > 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller w and with no seller respectively. Then, we have

$$v_0(w) = \min\{w, \rho\}, \tag{6.2.61}$$

$$v_t(w) = \min\{w, V_t\}, \quad t > 0, \tag{6.2.62}$$

where

$$V_t = \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.63}$$

Let us here define

$$V_0 = \rho. \tag{6.2.64}$$

Then (6.2.62) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(w) = \min\{w, V_t\}, \quad t \geq 0. \tag{6.2.65}$$

Since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.11(p.17))), from (6.2.63) we have

$$\begin{aligned} V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.17)))} \qquad (6.2.66) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0. \quad \text{(see (5.1.14) and (5.1.13))} \qquad (6.2.67) \end{aligned}$$

□ SOE{$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]} can be reduced to (6.2.64) and (6.2.66), listed in Table 6.5.3(p.31) (II). ■

Let us here define

$$\tilde{\mathbb{S}}_t = \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{6.2.68}$$

Then, (6.2.63) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{6.2.69}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.70}$$

From (6.2.68) and (6.2.62) we have $\tilde{\mathbb{S}}_t = \beta(\,\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s$. Accordingly, from (5.1.11(p.17)) and (5.1.13(p.17)) we have

$$\tilde{\mathbb{S}}_t = \beta\tilde{T}(V_{t-1}) + s \tag{6.2.71}$$

$$= \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.72}$$

6.2.2.3 M:2[$\mathbb{P}$][A]

By v_t ($t \geq 0$) and V_t ($t \geq 0$) let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, it must be determined whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. If the ρ is accepted, the profit which the seller can obtain is ρ. On the other hand, since the seller must necessarily sell the asset to the buyer due to A2(p.7), the price $a^{\dagger}$ must be proposed to the buyer; in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \tag{6.2.73}$$

hence the profit which the seller obtains at that time is a. Hence, the profit that the seller obtain at time 0 becomes

$$v_0 = \max\{\rho, a\}. \tag{6.2.74}$$

Suppose there exists a buyer at a time $t > 0$. Then, since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \leq \boldsymbol{\xi}\}$ (see (5.1.18(p.18))). Hence we have

$$v_t = \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.75}$$

due to (5.1.19), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.76}$$

due to (5.1.25). Finally V_t can be expressed as follows.

---

† The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer.

$$V_0 = \rho, \tag{6.2.77}$$

$$V_t = \max\{\lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.78}$$

For $t = 1$ we have

$$\begin{aligned} V_1 &= \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &= \max\{\lambda\beta \max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &= \max\{\lambda\beta \max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \end{aligned} \tag{6.2.79}$$

Since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.75), we can rearrange (6.2.78) as follows.

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.18)))} \qquad (6.2.80) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.21) and (5.1.20))}. \qquad (6.2.81) \end{aligned}$$

□ SOE{M:1[ℙ][A]} can be reduced to (6.2.77), (6.2.79), and (6.2.80), listed in Table 6.5.3(p.31) (III). ■

Now, let us here define

$$\mathbb{S}_t = \lambda\beta(v_{t-1} - V_{t-1}) - s, \quad t > 0. \tag{6.2.82}$$

Then, (6.2.78) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{6.2.83}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.2.84}$$

From (6.2.75) and (5.1.20(p.18)) we have

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.85}$$

$$= L(V_{t-1}), \quad t > 0. \tag{6.2.86}$$

6.2.2.4 $\tilde{\mathsf{M}}$:2[ℙ][A]

By v_t ($t \geq 0$) and V_t ($t \geq 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, it must be determined whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. If the ρ is accepted, the cost which the buyer pays is ρ. On the other hand, since the buyer must necessarily buy the asset from the seller due to A2(p.7), the price b[†] must be proposed to the seller; in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = b, \tag{6.2.87}$$

hence the cost which the buyer pays at that time is b. Hence the cost that the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\}. \tag{6.2.88}$$

Suppose there exists a seller at a time $t > 0$. Then, since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes a price z, the probability of the seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.18))). Hence we have

$$v_t = \min_z\{\tilde{p}(z)z + (1 - p(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.89}$$

due to (5.1.32), implying that the optimal buying price z_t which the buyer should propose is given by

$$z_t = \tilde{z}(V_t), \qquad t > 0, \tag{6.2.90}$$

due to (5.1.38). Finally V_t can be expressed as follows.

$$V_0 = \rho, \tag{6.2.91}$$

$$V_t = \min\{\lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.92}$$

For $t = 1$ we have

[†] The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

$$
\begin{aligned}
V_1 &= \min\{\lambda\beta v_0 + (1-\lambda)\beta V_0 + s, \beta V_0\} \\
&= \min\{\lambda\beta \min\{\rho, b\} + (1-\lambda)\beta\rho + s, \beta\rho\} \\
&= \min\{\lambda\beta \min\{0, b-\rho\} + \beta\rho + s, \beta\rho\}. \qquad (6.2.93)
\end{aligned}
$$

Since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.89), we can rearrange (6.2.92) as follows.

$$
\begin{aligned}
V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\
&= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\
&= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.34(\text{p.19}))) \qquad (6.2.94) \\
&= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\
&= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad (\text{see } (5.1.34) \text{ and } (5.1.33)) \qquad (6.2.95)
\end{aligned}
$$

□ SOE{$\tilde{\text{M}}$:2[$\mathbb{P}$][A]} can be reduced to (6.2.91), (6.2.93), and (6.2.94), listed in Table 6.5.3(p.31) (IV). ■

Now, let us here define

$$\tilde{\mathbb{S}}_t = \lambda\beta(v_{t-1} - V_{t-1}) + s, \quad t > 0. \qquad (6.2.96)$$

Then, (6.2.92) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \qquad (6.2.97)$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \qquad (6.2.98)$$

From (6.2.89) and (5.1.33(p.19)) we have

$$
\begin{aligned}
\mathbb{S}_t &= \beta\tilde{T}(V_{t-1}) + s \quad t > 0. \qquad (6.2.99) \\
&= \tilde{L}(V_{t-1}), \quad t > 0. \qquad (6.2.100)
\end{aligned}
$$

6.2.3 Model 3

Since it is proven in Chapter 20(p.211) that Model 3 is reduced to Model 2, the discussions for this model becomes redundant. Accordingly, below let us confine only to the derivation of the system of optimality equations.

6.2.3.1 M:3[$\mathbb{R}$][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer w and with no buyer respectively, expressed as

$$
\begin{aligned}
v_0(w) &= \max\{w, \rho\}, \qquad (6.2.101) \\
v_t(w) &= \max\{w, \rho, U_t\}, \quad t > 0, \qquad (6.2.102) \\
V_0 &= \rho, \qquad (6.2.103) \\
V_t &= \max\{\rho, U_t\}, \quad t > 0, \qquad (6.2.104)
\end{aligned}
$$

where U_t is the maximum of the total expected present discounted *profit* from rejecting both the price w and intervening quitting penalty ρ in (6.2.102) and from rejecting the intervening quitting penalty ρ in (6.2.104). Then, U_t can be expressed as

$$U_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \qquad (6.2.105)$$

For convenience, let us here define $U_0 = \rho$, hence from (6.2.103) we have

$$V_0 = U_0 = \rho. \qquad (6.2.106)$$

Then, it follows that both (6.2.102) and (6.2.104) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$
\begin{aligned}
v_t(w) &= \max\{w, \rho, U_t\}, \quad t \geq 0, \qquad (6.2.107) \\
V_t &= \max\{\rho, U_t\}, \quad t \geq 0, \qquad (6.2.108)
\end{aligned}
$$

thus (6.2.107) can be expressed as

$$v_t(w) = \max\{w, V_t\}, \quad t \geq 0. \qquad (6.2.109)$$

Accordingly, since $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] = \mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.1(p.17)), we can rewrite (6.2.105) as

$$
\begin{aligned}
U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\
&= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\
&= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4))} \qquad (6.2.110) \\
&= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\
&= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.8))}. \qquad (6.2.111)
\end{aligned}
$$

□ SOE{M:3[ℝ][A]} can be reduced to (6.2.106), (6.2.108), and (6.2.110), listed in Table 6.5.5(p.31) (I). ∎

6.2.3.2 M̃:3[ℝ][A]

By $v_t(w)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time $t \geq 0$ with a seller w and with no seller respectively, expressed as

$$
\begin{aligned}
v_0(w) &= \min\{w, \rho\}, && (6.2.112) \\
v_t(w) &= \min\{w, \rho, U_t\}, \quad t > 0, && (6.2.113) \\
V_0 &= \rho, && (6.2.114) \\
V_t &= \min\{\rho, U_t\}, \qquad t > 0, && (6.2.115)
\end{aligned}
$$

where U_t is the minimum of the total expected present discounted *cost* from rejecting both the price w and intervening quitting penalty ρ in (6.2.113) and from rejecting the intervening quitting penalty ρ in (6.2.115). Then, U_t can be expressed as

$$U_t = \min\{\mathtt{C}: \lambda\beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\ \mathtt{S}: \beta V_{t-1}\}, \quad t > 0. \qquad (6.2.116)$$

For convenience, let us here define $U_0 = \rho$, hence from (6.2.114) we have

$$V_0 = U_0 = \rho. \qquad (6.2.117)$$

Then, it follows that both (6.2.113) and (6.2.115) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$
\begin{aligned}
v_t(w) &= \min\{w, \rho, U_t\}, \quad t \geq 0, && (6.2.118) \\
V_t &= \min\{\rho, U_t\}, \qquad t \geq 0, && (6.2.119)
\end{aligned}
$$

thus (6.2.113) can be expressed as

$$v_t(w) = \min\{w, V_t\}, \quad t \geq 0. \qquad (6.2.120)$$

Accordingly, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.11(p.17)), we can rewrite (6.2.116) as follows.

$$
\begin{aligned}
U_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\
&= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\
&= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14))} \qquad (6.2.121) \\
&= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\
&= \max\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.14) and (5.1.13))}. \qquad (6.2.122)
\end{aligned}
$$

□ SOE{M̃:3[ℝ][A]} can be reduced to (6.2.117), (6.2.119), and (6.2.121), listed in Table 6.5.5(p.31) (II). ∎

6.2.3.3 M:3[ℙ][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, it must be determined whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. If the ρ is accepted, the profit which the seller can obtain is ρ. On the other hand, since the seller must sell the asset to the buyer due to A2(p.7), the price $a^\dagger$ must be proposed to the buyer, in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \qquad (6.2.123)$$

hence the profit which the seller obtains at that time is a. Hence the profit that the seller obtains at time 0 becomes

$$v_0 = \max\{\rho, a\}. \qquad (6.2.124)$$

Next we have

† The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer

$$v_t = \max\{\rho, H_t\}, \quad t > 0, \tag{6.2.125}$$

$$V_0 = \rho, \tag{6.2.126}$$

$$V_t = \max\{\rho, U_t\}, \quad t > 0, \tag{6.2.127}$$

where H_t and U_t are defined as follows. Firstly H_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since a buyer exists due to the above definition of v_t and since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \leq \boldsymbol{\xi}\}$ (see (5.1.18(p.18))). Hence we have

$$H_t \;=\; \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0 \tag{6.2.128}$$

due to (5.1.19(p.18)), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.129}$$

due to (5.1.25(p.18)). Finally U_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since no buyer exists due to the above definition of V_t, it can be expressed as follows.

$$U_t = \max\{\,\mathtt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s,\ \mathtt{S}\colon \beta V_{t-1}\}, \quad t > 0. \tag{6.2.130}$$

For $t = 1$ we have

$$\begin{aligned} U_1 \;&=\; \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &=\; \max\{\lambda\beta \max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &=\; \max\{\lambda\beta \max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \end{aligned} \tag{6.2.131}$$

Now, from (6.2.128) we have $H_t - V_t = T(V_t)$ for $t > 0$, hence from (6.2.125) we have $v_t - V_t = \max\{\rho - V_t, H_t - V_t\} = \max\{\rho - V_t, T(V_t)\} \cdots (1)$ for $t > 0$. Since $V_t \geq \rho$ for $t > 0$ from (6.2.127), we have $\rho - V_t \leq 0$ for $t > 0$. In addition, since $p(b) = 0$ due to (5.1.29 (2) (p.18)), from (5.1.19) we have $T(V_t) \geq p(b)(b - V_t) = 0$. Therefore, since $\rho - V_t \leq 0 \leq T(V_t)$, from (1) we have $v_t - V_t = T(V_t)$ for $t > 0$, i.e., $v_t = T(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.130) with $t > 1$[‡] can be rearranged as

$$\begin{aligned} U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.18)))} \qquad (6.2.132) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.21) and (5.1.20))}. \qquad (6.2.133) \end{aligned}$$

For convenience, let $U_0 = \rho$. Then, due to (6.2.126) we have

$$V_0 = U_0 = \rho, \tag{6.2.134}$$

hence it follows that (6.2.127) holds true for $t \geq 0$ instead of $t > 0$, i.e.,

$$V_t = \max\{\rho, U_t\}, \quad t \geq 0. \tag{6.2.135}$$

□ SOE{M:3[ℙ][A]} can be reduced to (6.2.134), (6.2.135), (6.2.131), and (6.2.132), listed in Table 6.5.5(p.31) (III). ∎

6.2.3.4 $\tilde{\mathsf{M}}$:3[ℙ][A]

By v_t ($t \geq 0$) and V_t ($t \geq 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, it must be determined whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. If the ρ is accepted, the cost which the buyer pays at time 0 is ρ. On the other hand, since the buyer must buy the asset from the seller due to A2(p.7), the price b[†] must be is proposed to the seller; in other words, the optimal price to propose is given by

$$z_0 = b, \tag{6.2.136}$$

hence the cost which the buyer pays at that time is b. Hence the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\} \tag{6.2.137}$$

[‡]Instead of $t > 0$.

[†]The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

Next we have

$$v_t = \min\{\rho, H_t\}, \quad t > 0. \tag{6.2.138}$$

$$V_0 = \rho, \tag{6.2.139}$$

$$V_t = \min\{\rho, U_t\}, \quad t > 0, \tag{6.2.140}$$

where H_t and U_t are defined as follows. Firstly H_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since a seller exists due to the above definition of v_t and since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes the price z to an appearing seller, the probability of the seller selling the asset for the price z is $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \le z\}$ (see (5.1.31(p.18))). Hence we have

$$H_t \;=\; \min_z\{\tilde{p}(z)z + (1-\tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.141}$$

due to (5.1.32(p.18)), implying that the optimal buying price which the buyer should pay is given by

$$z_t = \tilde{z}(V_t), \qquad t \ge 0, \tag{6.2.142}$$

due to (5.1.38(p.19)). Finally U_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since no seller exists due to the above definition of V_t, it can be expressed as follows.

$$U_t \;=\; \min\{\texttt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} + s,\ \texttt{S}\colon \beta V_{t-1}\}, \quad t > 0. \tag{6.2.143}$$

For $t = 1$ we have

$$\begin{aligned} U_1 \;&=\; \min\{\lambda\beta v_0 + (1-\lambda)\beta V_0 + s, \beta V_0\} \\ &=\; \min\{\lambda\beta \min\{\rho, b\} + (1-\lambda)\beta\rho + s, \beta\rho\} \\ &=\; \min\{\lambda\beta \min\{0, b-\rho\} + \beta\rho + s, \beta\rho\}. \end{aligned} \tag{6.2.144}$$

Now, from (6.2.141) we have $H_t - V_t = \tilde{T}(V_t)$ for $t > 0$, hence from (6.2.138) we have $v_t - V_t = \min\{\rho - V_t, H_t - V_t\} = \min\{\rho - V_t, \tilde{T}(V_t)\} \cdots \mathbf{(2)}$ for $t > 0$. Since $V_t \le \rho$ for $t > 0$ from (6.2.140), we have $\rho - V_t \ge 0$ for $t > 0$. In addition, since $\tilde{p}(a) = 0$ due to (5.1.41 (1) (p.19)), from (5.1.32(p.18)) we have $\tilde{T}(V_t) \le \tilde{p}(a)(a - V_t) = 0$. Therefore, since $\rho - V_t \ge 0 \ge \tilde{T}(V_t)$, from (2) we have $v_t - V_t = \tilde{T}(V_t)$ for $t > 0$, i.e., $v_t = \tilde{T}(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.143) with $t > 1$ can be rearranged as

$$\begin{aligned} U_t \;&=\; \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &=\; \min\{\lambda\beta\tilde{T}(V_{t-1}) + V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &=\; \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34))} && (6.2.145) \\ &=\; \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}, \quad t > 1 \\ &=\; \max\{\tilde{L}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.19)) and (5.1.33(p.19)))} && (6.2.146) \end{aligned}$$

(6.2.147)

For convenience, let $U_0 = \rho$. Then, due to (6.2.139) we have

$$V_0 = U_0 = \rho, \tag{6.2.148}$$

hence it follows that (6.2.140) holds true for $t \ge 0$ instead of $t > 0$, i.e.,

$$V_t = \min\{\rho, U_t\}, \quad t \ge 0. \tag{6.2.149}$$

□ $\texttt{SOE}\{\tilde{\textsf{M}}\text{:}3[\mathbb{R}][\texttt{A}]\}$ can be reduced to (6.2.148), (6.2.149), (6.2.144), and (6.2.145), listed in Table 6.5.5(p.31) (IV). ■

6.3 Search-Enforced-Model

In s-E-model ($\textsf{M}\text{:}x[\mathbb{X}][\texttt{E}]$ and $\tilde{\textsf{M}}\text{:}x[\mathbb{X}][\texttt{E}]$ with $x = 1, 2, 3$ and $\mathbb{X} = \mathbb{R}, \mathbb{P}$) a leading trader needs to make no decision regarding whether or not to conduct the search. This implies that eliminating the terms related to this decision from the systems of optimality equations in s-A-model ($\texttt{SOE}\{\textsf{M}\text{:}x[\mathbb{X}][\texttt{A}]\}$ and $\texttt{SOE}\{\tilde{\textsf{M}}\text{:}x[\mathbb{X}][\texttt{A}]\}$) produces $\texttt{SOE}\{\textsf{M}\text{:}x[\mathbb{X}][\texttt{E}]\}$ and $\texttt{SOE}\{\tilde{\textsf{M}}\text{:}x[\mathbb{X}][\texttt{E}]\}$ respectively. Noting this, from Tables 6.5.1, 6.5.3, and 6.5.5 we can immediately obtain the systems of optimality equations for s-E-model, which are given by Tables 6.5.2, 6.5.4, and 6.5.6.

6.4 Assertion and Assertion System

In general, let us call a description on whether or not a given statement is true the *assertion*, denoted by A, and a set consisting of some assertions the *assertion system*, denoted by $\mathscr{A}$. In addition, let us denote an assertion and an assertion system for a given Model by respectively $A\{\textsf{Model}\}$ and $\mathscr{A}\{\textsf{Model}\}$.

6.5 Summary of the System of Optimality Equations SOE's

Model 1

Table 6.5.1: **Search-Allowed-Model** 1

(I) SOE{M:1[ℝ][A]}

$$V_1 = \beta\mu - s, \tag{6.5.1}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.2}$$

(II) SOE{$\tilde{\mathsf{M}}$:1[ℝ][A]}

$$V_1 = \beta\mu + s, \tag{6.5.3}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.4}$$

(III) SOE{M:1[ℙ][A]}

$$V_1 = \beta a - s, \tag{6.5.5}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.6}$$

(IV) SOE{$\tilde{\mathsf{M}}$:1[ℙ][A]}

$$V_1 = \beta b + s, \tag{6.5.7}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.8}$$

Table 6.5.2: **Search-Enforced-Model** 1

(I) SOE{M:1[ℝ][E]}

$$V_1 = \beta\mu - s, \tag{6.5.9}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.10}$$

(II) SOE{$\tilde{\mathsf{M}}$:1[ℝ][E]}

$$V_1 = \beta\mu + s, \tag{6.5.11}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.12}$$

(III) SOE{M:1[ℙ][E]}

$$V_1 = \beta a - s, \tag{6.5.13}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.14}$$

(IV) SOE{$\tilde{\mathsf{M}}$:1[ℙ][E]}

$$V_1 = \beta b + s, \tag{6.5.15}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.16}$$

Model 2

Table 6.5.3: **Search-Allowed-Model** 2

(I) SOE{M:2[ℝ][A]}

$$V_0 = \rho, \tag{6.5.17}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.18}$$

(II) SOE{$\tilde{\mathsf{M}}$:2[ℝ][A]}

$$V_0 = \rho, \tag{6.5.19}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.20}$$

(III) SOE{M:2[ℙ][A]}

$$V_0 = \rho, \tag{6.5.21}$$
$$V_1 = \max\{\lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.5.22}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.23}$$

(IV) SOE{$\tilde{\mathsf{M}}$:2[ℙ][A]}

$$V_0 = \rho, \tag{6.5.24}$$
$$V_1 = \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.5.25}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.26}$$

Table 6.5.4: **Search-Enforced-Model** 2

(I) SOE{M:2[ℝ][E]}

$$V_0 = \rho, \tag{6.5.27}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.5.28}$$

(II) SOE{$\tilde{\mathsf{M}}$:2[ℝ][E]}

$$V_0 = \rho, \tag{6.5.29}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.5.30}$$

(III) SOE{M:2[ℙ][E]}

$$V_0 = \rho, \tag{6.5.31}$$
$$V_1 = \lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \tag{6.5.32}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.33}$$

(IV) SOE{$\tilde{\mathsf{M}}$:2[ℙ][E]}

$$V_0 = \rho, \tag{6.5.34}$$
$$V_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \tag{6.5.35}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.36}$$

Model 3

Table 6.5.5: **Search-Allowed-Model** 3

(I) SOE{M:3[ℝ][A]}

$$V_0 = U_0 = \rho, \tag{6.5.37}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.38}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.39}$$

(II) SOE{$\tilde{\mathsf{M}}$:3[ℝ][A]}

$$V_0 = U_0 = \rho, \tag{6.5.40}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.41}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.42}$$

(III) SOE{M:3[ℙ][A]}

$$V_0 = U_0 = \rho, \tag{6.5.43}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.44}$$
$$U_1 = \max\{\lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.5.45}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.46}$$

(IV) SOE{$\tilde{\mathsf{M}}$:3[ℙ][A]}

$$V_0 = U_0 = \rho, \tag{6.5.47}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.48}$$
$$U_1 = \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.5.49}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.50}$$

Table 6.5.6: **Search-Enforced-Model** 3

(I) SOE{M:3[ℝ][E]}

$$V_0 = U_0 = \rho, \tag{6.5.51}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.52}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.5.53}$$

(II) SOE{$\tilde{\mathsf{M}}$:3[ℝ][E]}

$$V_0 = U_0 = \rho, \tag{6.5.54}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.55}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.5.56}$$

(III) SOE{M:3[ℙ][E]}

$$V_0 = U_0 = \rho, \tag{6.5.57}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.58}$$
$$U_1 = \lambda\beta \max\{0, a - \rho\} + \beta\rho - s, \tag{6.5.59}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.60}$$

(IV) SOE{$\tilde{\mathsf{M}}$:3[ℙ][E]}

$$V_0 = U_0 = \rho, \tag{6.5.61}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.62}$$
$$U_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \tag{6.5.63}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.64}$$

Chapter 7

Optimal Decision Rules

7.1 Five Kinds of Points in Time

To start with, let us note herein that the optimal decision rule prescribed for each model in Table 3.3.1(p.11) is closely related to the following four kinds of points in time (see Concept 1(p.9)).

1. *Recognizing time* $t_r \geq 0$,
2. *Starting time* $t_s \geq 0$, represented by τ, i.e., $\tau = t_s$,
3. *Initiating time* $\tau \geq t_i \geq 0$, sometimes represented by t,
4. *Deadline* $t_d = 0$, the final point in time of the decision process. Here by t_{qd} let us denote the smallest of all possible initiating times, called *quasi-deadline*, where

$$t_{qd} = 1 \text{ for Model 1 (see Remark 4.1.1(p.13) (a))}, \tag{7.1.1}$$
$$t_{qd} = 0 \text{ for Model 2 (see Remark 4.2.1(p.15))}. \tag{7.1.2}$$

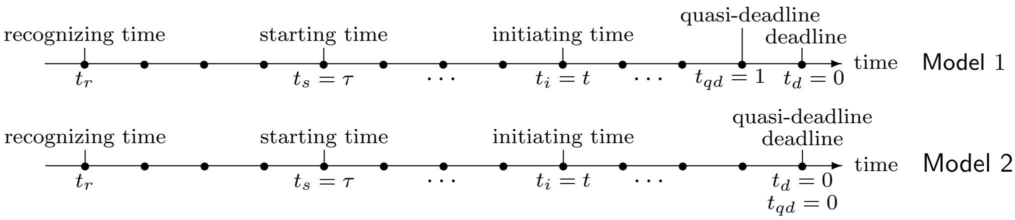

Figure 7.1.1: Four kinds of points in time

Remark 7.1.1 In Chapter 20(p.211) we will show the following two facts:

1. It becomes optimal to accept the intervening quitting penalty ρ at the starting time τ and then stop the process,
2. Model 3 is reduced to Model 2, hence it becomes redundant to discuss any more for Model 3. For this reason, in this chapter we consider only Model 1 and Model 2. □

Remark 7.1.2 (finite planning horizon vs. infinite planning horizon) In the present paper we consider only models with the *finite* planning horizon. Our basic standpoint over the whole of this paper lies in a *grim reality* that a process with the *infinite* planning horizon is a *product of fantasy* created by mathematics, which does not exist in the real world at all; in fact, it is an inanity to consider a model with the planning horizon of more than 135 hundred millions years. However, we can have the two reasons for which it becomes still meaningful to discuss the model with the *infinite* planning horizon. One is that it can become an approximation for the process with an enough long (finite) planning horizon, the other is that results obtained from it can provide a meaningful information for the analyses of models with the *finite* planning horizon (see Section 27.4(p.267)). □

7.2 Four Kinds of Decisions

Here let us recall the four kinds of decision rules that were prescribed in Section 3.5(p.12).

7.2.1 Whether or Not to Accept the Proposed Price

This is the decision only for $\mathbb{R}$-model. In M:1[$\mathbb{R}$][A] and M:2[$\mathbb{R}$][A] (S-model) let $\texttt{Accept}_t\langle w\rangle$ and $\texttt{Reject}_t\langle w\rangle$ denote "Accept a price w at time t" and "Reject a price w at time t" respectively. Here suppose that a buyer appearing at a time t has proposed a buying price w. Then, from (6.2.2(p.21)) and (6.2.50(p.24)) we see

$$w \geq (\leq)\, V_t \Rightarrow \texttt{Accept}_t\langle w\rangle \;\; (\texttt{Reject}_t\langle w\rangle). \tag{7.2.1}$$

Similarly, in $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] and $\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A] (B-model), from (6.2.13(p.22)) and (6.2.62(p.25)) we see

$$w \leq (\geq)\, V_t \Rightarrow \texttt{Accept}_t\langle w\rangle \;\; (\texttt{Reject}_t\langle w\rangle). \tag{7.2.2}$$

From the above we see that the reservation-price for each of the above models is given by

$$V_t. \tag{7.2.3}$$

By $A_t(w)$ and $R_t(w)$ let us represent here the profits (costs) in S-model (B-model) from respectively accepting and rejecting the price w proposed at time t. Then, we have $A_t(w) = w$ and $R_t(w) = V_t$. Furthermore, let us define

$$\Delta_t(w) \stackrel{\text{def}}{=} A_t(w) - B_t(w) = w - V_t. \tag{7.2.4}$$

Accordingly, it follows that

$$\Delta_t(w) \geq (\leq)\ 0 \Leftrightarrow w \geq (\leq)\ V_t \Rightarrow \texttt{Accept}_t\langle w\rangle\ \ (\texttt{Reject}_t\langle w\rangle)\quad (\textsf{S-model}), \tag{7.2.5}$$

$$\Delta_t(w) \leq (\geq)\ 0 \Leftrightarrow w \leq (\geq)\ V_t \Rightarrow \texttt{Accept}_t\langle w\rangle\ \ (\texttt{Reject}_t\langle w\rangle)\quad (\textsf{B-model}). \tag{7.2.6}$$

7.2.2 What Price to Propose

This is the decision only for $\mathbb{P}$-model. In M:1[$\mathbb{P}$][A] (M:2[$\mathbb{P}$][A]) the optimal selling price which a seller who is a leading trader should propose at a time t is given by

$$z_t = z(V_t) \quad (\text{see } (6.2.28(\text{p.23}))\ ((6.2.76(\text{p.25})))). \tag{7.2.7}$$

Similarly, in $\tilde{\text{M}}$:1[$\mathbb{P}$][A] ($\tilde{\text{M}}$:2[$\mathbb{P}$][A]) the optimal buying price which a buyer who is a leading trader should propose at a time t is given by

$$z_t = \tilde{z}(V_t) \quad (\text{see } (6.2.41(\text{p.23}))\ ((6.2.90(\text{p.26})))). \tag{7.2.8}$$

7.2.3 Whether or not to Conduct the Search

This is the decision only for s-A-model (see C2cii(p.10)). Then, the decision rule is given by (6.2.9(p.22)), (6.2.20(p.22)), (6.2.33(p.23)), (6.2.46(p.24)), (6.2.58(p.24)), (6.2.70(p.25)), (6.2.84(p.26)), and (23.2.4(p.242)).

Remark 7.2.1 (posterior-skip-of-search (pSkip)) Figure 7.2.1(I) below sketches the case that once the search-conduct starts at the optimal initiating time t^*_τ, continue it up to the quasi-deadline $t_{qd} = 1$ (Model 1); it will be known that this case occurs everywhere in the paper. Contrary to this, Figure 7.2.1(II) schematizes the case that once the search-conduct starts at the optimal initiating time t^*_τ, continue it for a while and then switches to the search-skip at a certain point in time t'; this is a very rare case that occurs only in Tom's 19.1.4(p.144) (b3iii), 19.1.12(p.154) (b3iii), and 19.1.15(p.155) (b3iii). Let us call the case the *posterior-skip-of-search*, represented by pSkip for short. □

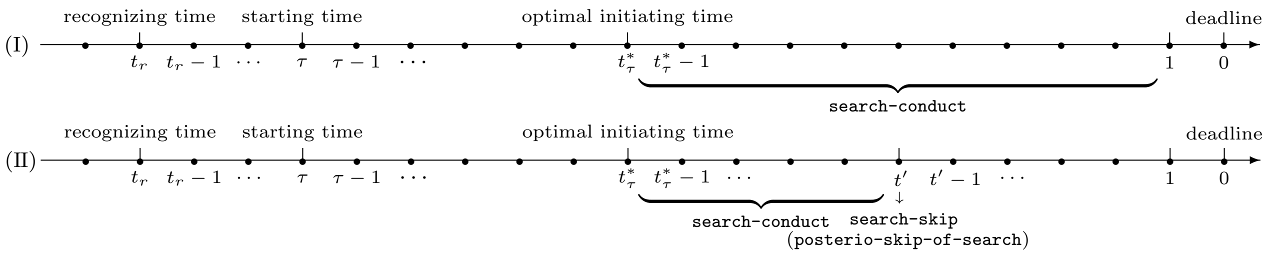

Figure 7.2.1: Posterio-Skip-of-Search (pSkip)

7.2.4 When to Initiate the Process (Optimal Initiating Time)

This is a notion only for ii-A-case (see Concept 2bii(p.9)) defined in both s-E-case and s-A-case (see Concept 2ai,2aii) and in both $\mathbb{R}$-model and $\mathbb{P}$-model.

7.2.4.1 Revolutionary Switch of Conventional Conception

Throughout the whole of the present paper we treat an activity of "decision-making" as "one unit" which is included in a given space; let us call the unit the *decision-making-unit* and the space the *decision-making-universe*, or the *decision-unit* and the *decision-universe* respectively for short. Of course, it can be also permitted that multiple decision-units are included within a given decision-universe. What should not be forgot herein is that all decision-units in an decision-universe can be treated independently each other and that the problem of selecting the best decision-units among them can raise. At a glance, the above scenario seems to be non-descriptive; however, this perspective is what should be said to be a "revolutionary right-about-face" in the sense that the switch of this way of viewing yields the novel "null-time-zone" (see Section 7.2.4.6(p.36)), which furthermore causes the unbelievable event of "deadline-falling" (see Section 7.2.4.7(p.36)).

7.2.4.2 Definition

The definition below is only for S-model with $t_{qd} = 1$ (i.e., Model 1 ($t_{qd} = 0$ for Model 2)). Suppose that its process has started at a starting time τ and that a seller (leading trader) has determined to *initiate* the process at a time t ($\tau \geq t \geq t_{qd} = 1$), i.e., $\tau - t$ periods hence. Then, the total expected present discounted profit at the *starting time* τ is given by (see Section 20(p.266) for the definition of V_t)

$$I^t_\tau \stackrel{\text{def}}{=} \beta^{\tau-t} V_t, \quad \tau \geq t \geq t_{qd}.^\dagger \tag{7.2.9}$$

By t^*_τ let us denote t maximizing I^t_τ on $\tau \ge t \ge t_{qd}$, i.e.,

$$I^{t^*_\tau}_\tau = \max_{\tau \ge t \ge t_{qd}} I^t_\tau \quad \text{or equivalently} \quad I^{t^*_\tau}_\tau \ge I^t_\tau, \quad \tau \ge t \ge t_{qd}. \tag{7.2.10}$$

Let us call the t^*_τ the *optimal initiating time*, denoted by $\mathtt{OIT}_\tau\langle t^*_\tau\rangle_\vartriangle$. If

$$I^{t^*_\tau}_\tau > I^t_\tau \ \text{ for } \ t \ne t^*_\tau, \tag{7.2.11}$$

it is called the *strictly optimal initiating time*, denoted by $\mathtt{OIT}_\tau\langle t^*_\tau\rangle_\blacktriangle$. Throughout the paper, let us employ the following preference rule.

Preference Rule **7.2.1** *Let $I^t_\tau = I^{t-1}_\tau$ for a given t. Then, the seller prefers $t-1$ to t as an initiating time, implying that "Postpone the initiation of the process so long as it is not unprofitable to do so."* □

Remark 7.2.2 (implication of the strict optimality of t^*_τ) Assume that the optimal initiating time t^*_τ is strict in a sense of (7.2.11). Then, since $I^{t^*_\tau}_\tau > I^{t^*_\tau-1}_\tau$, we have $\beta^{\tau-t^*_\tau}V_{t^*_\tau} > \beta^{\tau-t^*_\tau+1}V_{t^*_\tau-1}$, so $V_{t^*_\tau} > \beta V_{t^*_\tau-1}$. Accordingly, since $\max\{\mathbb{S}_{t^*_\tau}, 0\} > 0$ from (6.2.8(p.22)), we have $\mathbb{S}_{t^*_\tau} > 0$, implying that it becomes *strictly* optimal to conduct the search; in other words, it is not allowed to skip the search under the above assumption. □

7.2.4.3 β-adjusted sequence $V_{\beta[\tau]}$

First, let us define the sequence consisting of V_τ, $V_{\tau-1}$, $V_{\tau-2}$, $\cdots$, $V_{t_{qd}}$ by $V_{[\tau]} \overset{\text{def}}{=} \{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{t_{qd}}\}$, called the *original sequence* and let

$$t^{*\prime}_\tau = \arg\max V_{[\tau]} = \arg\max\{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{t_{qd}}\}.$$

Next, let us define the sequence $V_{\beta[\tau]} = \{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^\tau V_{t_{qd}}\} = \{I^\tau_\tau, I^{\tau-1}_\tau, I^{\tau-2}_\tau, \cdots, I^{t_{qd}}_\tau\}$, called the *$\beta$-adjusted sequence* of $V_{[\tau]}$. By definition the optimal initiating time t^*_τ is given by t attaining the maximum of elements within *β-adjusted sequence* $V_{\beta[\tau]}$, i.e.,

$$t^*_\tau = \arg\max V_{\beta[\tau]} = \arg\max\{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^\tau V_{t_{qd}}\}$$

Note herein that the monotonicity of the original sequence $V_{[\tau]}$ is not always inherited to the β-adjusted sequence $V_{\beta[\tau]}$, i.e., $t^*_\tau \ne t^{*\prime}_\tau$ (see Section A 5.2.2(p.292)).

7.2.4.4 Three Possibilities

Below let us show the three types of `OIT` caused by the non-inheritance of monotonicity.

1. **Degeneration to the starting time τ**

   Let $t^*_\tau = \tau$, i.e., it is optimal to initiate the process at the starting time τ, denoted by ⓢ. Then, the optimal initiating time t^*_τ is said to *degenerate* to the *starting time* τ, represented by $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ (ⓢ$_\vartriangle$ for short). If the optimal initiating time t^*_τ is *strict* (see (7.2.11)), it is called the *strictly degenerate* `OIT`, represented by $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ (ⓢ$_\blacktriangle$ for short).

2. **Non-degeneration ($\tau > t^*_\tau > t_{qd}$)**

   Let $\tau > t^*_\tau > t_{qd}$, i.e., the optimal initiating time is between the starting time τ and the quasi-deadline t_{qd}, denoted by ⊛. Then, the optimal initiating time t^*_τ is said to be *non-degenerate* `OIT`, represented by $\boxed{⊛\ \mathtt{ndOIT}_\tau\langle t^*_\tau\rangle}_\vartriangle$ (⊛$_\vartriangle$ for short). If $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^{t^*_\tau}_\tau \ge I^{t_{qd}}_\tau$ as a special case, it is said to be *indifferent* non-degenerate `OIT` (see Preference Rule 7.2.1), represented by $\boxed{⊛\ \mathtt{ndOIT}_\tau\langle t^*_\tau\rangle}_\parallel$ (⊛$_\parallel$ for short). If $I^\tau_\tau < I^{t^*_\tau}_\tau > I^{t_{qd}}_\tau$, it is said to be *strictly* non-degenerate `OIT`, represented by $\boxed{⊛\ \mathtt{ndOIT}_\tau\langle t^*_\tau\rangle}_\blacktriangle$ (⊛$_\blacktriangle$ for short).

3. **Degeneration to the quasi-deadline t_{qd}**

   Let $t^*_\tau = t_{qd} = 1\,(0)$ for Model 1 (Model 2), i.e., the optimal initiating time is the quasi-deadline, denoted by ⓓ. Then, the optimal initiating time t^*_τ is said to *degenerate* to the *quasi-deadline* t_{qd}, represented by $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle t_{qd}\rangle}_\vartriangle$ (ⓓ$_\vartriangle$ for short). If it is *strict*, it is called the *strictly degenerate* `OIT`, represented by $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle t_{qd}\rangle}_\blacktriangle$ (ⓓ$_\blacktriangle$ for short). If $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^{t_{qd}}_\tau$, the degeneration is said to be *indifferent*, represented by $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle t_{qd}\rangle}_\parallel$ (ⓓ$_\parallel$ for short).

7.2.4.5 First Search Conducing Time

For example, consider M:2[ℝ][A] ($t_{qd} = 0$) with the starting time $\tau = 6$ and suppose that $\mathtt{Skip}_{6\vartriangle}$, $\mathtt{Skip}_{5\vartriangle}$, $\mathtt{Skip}_{4\vartriangle}$, $\mathtt{Conduct}_{3\blacktriangle}$, $\mathtt{Conduct}_{2\vartriangle}$, $\mathtt{Conduct}_{1\vartriangle}$. This means that the *first-search-conducting-time* (`f-SCT` for short) is $t^{**}_\tau \overset{\text{def}}{=} 3 \cdots (3)$.[†] In this case, since $\mathbb{S}_6 \le 0$, $\mathbb{S}_5 \le 0$, $\mathbb{S}_4 \le 0$, $\mathbb{S}_3 > 0$, $\mathbb{S}_2 \ge 0$, and $\mathbb{S}_1 \ge 0$ from (6.2.9(p.22)), we have $\max\{\mathbb{S}_6, 0\} = 0$, $\max\{\mathbb{S}_5, 0\} = 0$, $\max\{\mathbb{S}_4, 0\} = 0$, $\max\{\mathbb{S}_3, 0\} > 0$, $\max\{\mathbb{S}_2, 0\} \ge 0$, and $\max\{\mathbb{S}_1, 0\} \ge 0$. Thus, from (6.2.8(p.22)) we have $V_6 = \beta V_5$, $V_5 = \beta V_4$, $V_4 = \beta V_3$, $V_3 > \beta V_2$, $V_2 \ge \beta V_1$, and $V_1 \ge \beta V_0$, so $V_6 = \beta V_5 = \beta^2 V_4 = \beta^3 V_3 > \beta^4 V_2 \ge \beta^5 V_1 \ge \beta^6 V_0$ or equivalently $I^6_6 = I^5_6 = I^4_6 = I^3_6 > I^2_6 \ge I^1_6 \ge I^0_6$ due to (7.2.9), hence we have the *optimal-initiating-time* $t^*_\tau = 3 \cdots (4)$ by definition.

[†]See Section 20(p.266) for the definition of V_t.

[†]If such a time does not exist, let $t^{**}_\tau = \tau\,(=6)$, i.e., the first-search-conducting-time is the starting time τ itself.

Alice **2 (first search conducting time)** *When the story has come up to here, after a moment's reflection, Alice happened to conceive of an idea; "Since $t_\tau^{**} = t_\tau^* = 3$ from (3) and (4), as the optimal initiating time we can employ the first search conducting time $t_\tau^{**} = 3$ instead of t_τ^*!". Then, Dr. Rabbit suddenly appeared and told to her "Surely you are not incorrect, Miss Alice!. However, the profit attained by initiating the process at the first search conducting time t_τ^{**} is the same as the profit attained by initiating the process at the optimal initiating time t_τ^*; in other words, since the former profit does not become greater than the latter profit at all, we have no reason why t_τ^{**} must be used instead of t_τ^*; accordingly, it suffices to employ t_τ^*!! Miss Alice!!!". And then, taking a watch out of the waistcoat-pocket and murmuring "Oh dear! Oh dear! I shall be too late for the faculty meeting", he again disappeared down the hole.* ∎

Alice **3 (jumble of intuition and theory)** *Moreover, Alice was hit by the following question. For example, suppose that $\mathbb{S}_t < 0$ at a time t, meaning that the search-skip becomes strictly optimal at time t. Then, since $\max\{\mathbb{S}_t, 0\} = 0$, we have $V_t = \beta V_{t-1}$ from (6.2.8(p.22)), meaning that initiating the process at the time t becomes indifferent to initiating the process at time $t-1$; but, nevertheless, the search skip becomes strictly optimal! After having mumbled, letting out a strange noise "Is this a little bit funny ?", she gave a shout "Such a laughable affair !". Then, Dr. Rabbit again suddenly appeared and pedantically told to Alice "The above two results are both ones based on a theory of mathematics, but your confusion is one caused by an intuition; there does not exist any logical relationship between the two! Well, your confusion is what is caused by a jumble of intuition and theory!!", and then, he again disappeared down the hole as murmuring "Oh dear! Oh dear! I shall be too late!"* (see `Numerical Examples` 16.8.2(p.109) and 19.1.1(p.174)). ∎

7.2.4.6 Null-Time-Zone

The section describes a *perplexing* situation caused by the optimal initiating time t_τ^*. Herein let $\tau > t_\tau^*$, i.e., the optimal initiating time t_τ^* is not the starting time τ (see Figure 7.2.2 below). This event means that no action of making a decision is taken at every point in time $t = \tau, \tau-1, \cdots, t_\tau^*+1$. Quite strangely enough, however, no researcher, including also the authors in the past, has become aware of the existence of this ***grim reality*** at all thus far which is caused by the introduction of the concept of `OIT`. In other words, it follows that thus far we *unwittingly* or *unconsciously* have been falling into the senselessness of engaging in *unnecessary decision-making activities* over these points in time. Let us refer to each of $\tau, \tau-1, \cdots, t_\tau^*+1$ as the *null point in time* and the whole of these times as the *null-time-zone*, denoted as `Nul-TZ` (see Concept 1(p.9) and Section 7.2.4.1(p.34)), i.e.,

$$\texttt{Nul-TZ} \overset{\text{def}}{=} \langle \tau, \tau-1, \cdots, t_\tau^*+1 \rangle. \tag{7.2.12}$$

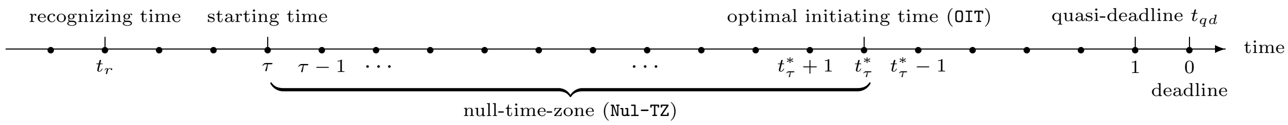

Figure 7.2.2: Null-time-zone in Model 1 with $t_{qd} = 1$ (`Nul-TZ`)

7.2.4.7 Deadline-Falling

Alice **4 (black hole)** *Hereupon, Alice supposed "If the optimal initiating time t_τ^* degenerates to the deadline (time 0), then what will ever happen ?", and screamed out "If so, it follows that don't conduct any decision-making activity up to the deadline !; in other words, the whole of decision-making activities which are scheduled at the starting time τ come to naught as if being sucked and falling into the deadline!". Alice was heavily nonplused and cried "It $\cdots$, it is the same as that black hole into which all physical matters, even light, are squeezed into! If so, $\cdots$, a decision process with an infinite planning horizon vanishes away toward an infinite future!! Oh dear!!! Oh dear !!!! $\cdots$" She hunkered down, and then buried her head in her hands. Then, Dr. Rabbit again suddenly appeared and told to her a little bit ungraciously "This is a theoretical result that cannot be denied!."* ∎

In this paper, let us call the "being sucked and falling into the deadline" the "*deadline-falling*" for short see Section 7.2.4.1(p.34)), symbolically represented by ⓓ-falling. This situation can be depicted as the two figures below.

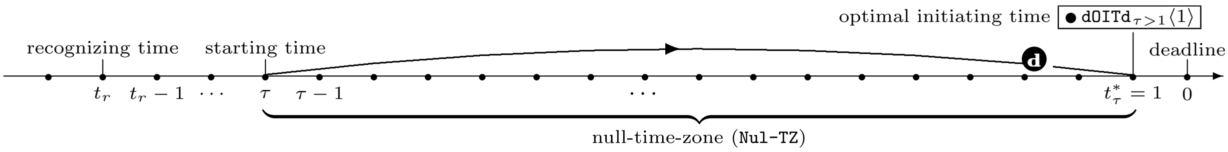

Figure 7.2.3: Deadline-falling (ⓓ) for Model 1

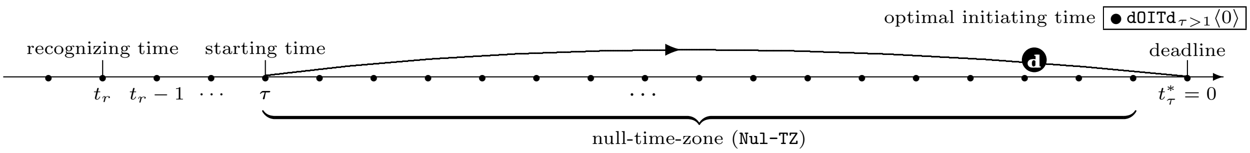

Figure 7.2.4: Deadline-falling (ⓓ) for Model 2

Later on we will see that the ⓓ-falling is not a rare case but a phenomenon which is very often possible. Taking this fact into consideration, we will inevitably be led to a serious re-examination and re-consideration of not only the decision processes dealt with in the present paper but also all of more generalized decision processes, for example, Markovian decision processes [23, Howard,1960][0528] (see Section A 5(p.291)).

7.3 Strong Assertion and Weak Assertion

Alice **5 (strong assertion and weak assertion)** *For example, consider a case such as the inequality $I_\tau^\tau \le I_\tau^{\tau-1} \le \cdots \le I_\tau^0$. In this case, the optimal initiating time is $t_\tau^* = 0$ by definition, i.e.,* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$ (see Preference Rule 7.2.1(p.35)). *However, this inequality includes the equality $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^0$ as a special case. Then, Dr. Rabbit again suddenly appeared and told to Alice a little bit ungraciously "If I told that the optimal initiating time can be said to be also the starting time τ!, i.e., $t_\tau^* = \tau$, then what to do with ?, Ms. Alice $\cdots$!!", and then, murmuring "Oh dear! Oh dear! I shall be too late for faculty meeting", he again disappeared down the rabbit-hole.* □

In the sense of removing a *betwixt and between state* from our discussions, needless to say, $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\blacktriangle$ (strictly optimal) is more desirable than $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$ (merely optimal). For this reason, let us call the former the *strong assertion* and the latter the *weak assertion.* From this viewpoint, throughout the paper, we will make an effort, as much as possible, to show the strictness of optimality.

Part 2

Integration Theory

This part constructs the integration theory.

Chapter 8

Flow of the Construction of Integration Theory

8.1 Bird's-Eye View

This chapter provides a bird's-eye view of the whole flow through which the integration theory will be constructed in the successive chapters that follows.

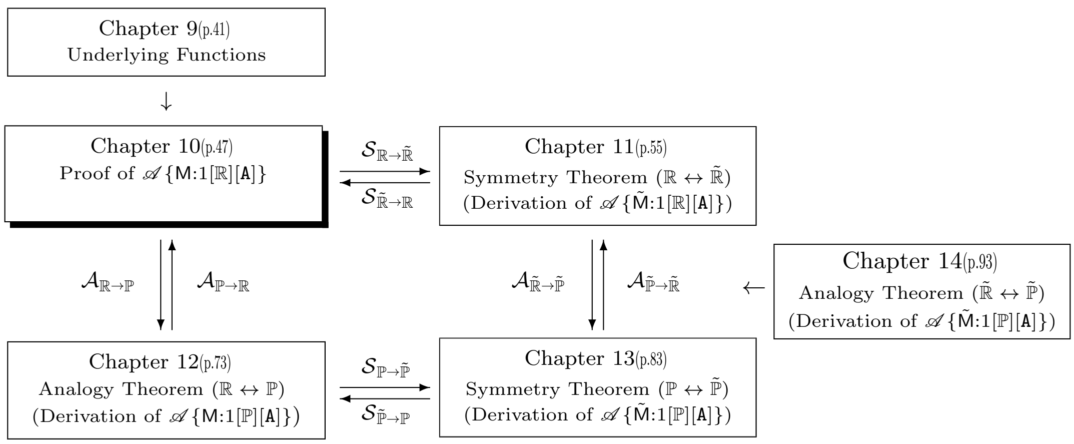

Figure 8.1.1: The flow of the construction of the integration theory

The above figure states the following.

- In Chapter 9(p.41), lemmas and corollaries for underlying functions are proven.
- In Chapter 10(p.47), $\mathscr{A}\{\mathsf{M}:1[\mathbb{R}][\mathtt{A}]\}$ is proven by using the results in Chapter 9.
- In Chapter 11(p.55), the symmetry theorem ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}:1[\mathbb{R}][\mathtt{A}]\}$.
- In Chapter 12(p.73), the analogy theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$) is proven, by which $\mathscr{A}\{\mathsf{M}:1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}:1[\mathbb{R}][\mathtt{A}]\}$.
- In Chapter 13(p.83), the symmetry theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}:1[\mathbb{P}][\mathtt{A}]\}$.
- In Chapter 14(p.93), the analogy theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$) is proven, which gives the relationship between $\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{P}][\mathtt{A}]\}$.

8.2 Connection with Both Directions

We should especially note here that the flow in Figure 8.1.1 above tells us the following:

1. It is only $\boxed{\mathscr{A}\{\mathsf{M}:1[\mathbb{R}][\mathtt{A}]\}}$ that is directly proven.
2. Each of the remaining three $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{R}][\mathtt{A}]\}}$, $\boxed{\mathscr{A}\{\mathsf{M}:1[\mathbb{P}][\mathtt{A}]\}}$, and $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}:1[\mathbb{P}][\mathtt{A}]\}}$ are derived by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\boxed{\mathscr{A}\{\mathsf{M}:1[\mathbb{R}][\mathtt{A}]\}}$.
3. The above four boxes are connected with both directions ($\leftrightarrow \updownarrow$), implying that any given box can be derived from any other box by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$, $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$, $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$.

Chapter 9

Properties of Underlying Functions

This chapter examines the properties of underlying functions $T_{\mathbb{R}}$, $L_{\mathbb{R}}$, $K_{\mathbb{R}}$, and $\mathcal{L}_{\mathbb{R}}$ and $\kappa_{\mathbb{R}}$-value defined by (5.1.1(p.17))-(5.1.6), which are used to clarify the properties of the optimal decision rules for $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Chapter 10(p.47) that follows. Throughout the rest of the paper, in general let us denote an assertion concerning $X_{\mathbb{R}} = T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}$ by $A\{X_{\mathbb{R}}\}$ and an assertion system consisting of some assertions $A\{X_{\mathbb{R}}\}$'s by $\mathscr{A}\{X_{\mathbb{R}}\}$.

9.1 Primitive Underlying Function $T_{\mathbb{R}}$

Lemma 9.1.1 ($\mathscr{A}\{T_{\mathbb{R}}\}$) *For any* $F \in \mathscr{F}$:

(a) $T(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b]$.
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $T(x) + x$ *is strictly increasing on* $[a, \infty)$.
(f) $T(x) = \mu - x$ *on* $(-\infty, a]$ *and* $T(x) > \mu - x$ *on* (a, ∞).
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty)$.
(h) $T(x) \geq \max\{0, \mu - x\}$ *on* $(-\infty, \infty)$.
(i) $T(0) = \mu$ *if* $a > 0$ *and* $T(0) = 0$ *if* $b < 0$.
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x < y$ *and* $a < y$, *then* $T(x) + x < T(y) + y$.
(m) $\lambda\beta T(\lambda\beta\mu - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(n) $a < \mu$.‡ □

Proof Firstly, for any x and y let us prove the following two inequalities:

$$-(x-y)(1-F(y)) \leq T(x) - T(y) \leq -(x-y)(1-F(x)), \tag{9.1.1}$$

$$(x-y)F(y) \leq T(x) + x - T(y) - y \leq (x-y)F(x). \tag{9.1.2}$$

Then, let $T(x,y) \stackrel{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)]$ for any x and y where if a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$. Since $1 \geq I(\boldsymbol{\xi} > y) \geq 0$ for any y by definition and since $\max\{\boldsymbol{\xi} - x, 0\} \geq 0$ and $\max\{\boldsymbol{\xi} - x, 0\} \geq \boldsymbol{\xi} - x$, we have $\max\{\boldsymbol{\xi} - x, 0\} \geq \max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > y) \geq (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)$, hence from (5.1.1(p.17)) we get $T(x) \geq \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] = T(x,y)$. Accordingly, for any x and y we have $T(x) - T(y) \geq T(x,y) - T(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} > y)] = -(x-y)\,\mathbf{E}[I(\boldsymbol{\xi} > y)]$. Since $I(\boldsymbol{\xi} \leq y) + I(\boldsymbol{\xi} > y) = 1$, we have $T(x) - T(y) \geq -(x-y)(\mathbf{E}[1 - I(\boldsymbol{\xi} \leq y)]) = -(x-y)(1 - \mathbf{E}[I(\boldsymbol{\xi} \leq y)])$. Then, since $\mathbf{E}[I(\boldsymbol{\xi} \leq y)] = \int_{-\infty}^{\infty} I(\xi \leq y) f(\xi) d\xi = \int_{-\infty}^{y} 1 \times f(\xi) d\xi = \int_{-\infty}^{y} f(\xi) d\xi = F(y)$, we have $T(x) - T(y) \geq -(x-y)(1 - F(y))$, hence the first inequality in (9.1.1) holds. Multiplying both sides of the inequality by -1 leads to $-T(x) + T(y) \leq (x-y)(1-F(y))$ or equivalently $T(y) - T(x) \leq -(y-x)(1-F(y))$. Then, interchanging the notations x and y yields $T(x) - T(y) \leq -(x-y)(1-F(x))$, hence the second inequality in (9.1.1) holds. (9.1.2) is immediate from adding $x - y$ to (9.1.1). Let us note here that $T(x)$ defined by (5.1.1) can be rewritten as

$$\begin{aligned} T(x) &= \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}I(a \leq \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} < a) \cdots \mathbf{(}1\mathbf{)} \\ &= \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}I(b < \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} \leq b)]. \cdots \mathbf{(}2\mathbf{)} \end{aligned}$$

(a,b) Immediate from the fact that $\max\{\boldsymbol{\xi} - x, 0\}$ is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any given $\boldsymbol{\xi}$.

‡The self-evident assertion is intentionally added here in order to keep the consistency with Lemma 12.2.1(p.77) (n).

(c) Let $y < x < b$, hence $x - y > 0$. Then, since $F(x) < 1$ due to (2.1.2 (1,2) (p.8)), we have $-(x - y)(1 - F(x)) < 0$, so $T(x) < T(y)$ due to (9.1.1), i.e., $T(x)$ is *strictly* decreasing on $(-\infty, b)$. Here note that for any given $x < b$ we have $T(x) \geq T(b)$ due to (b). Let us assume $T(x) = T(b)$ for a given $x < b$. Then, for any sufficiently small $\varepsilon > 0$ such that $b - x > 2\varepsilon$ we have $b > b - \varepsilon > x + \varepsilon > x$, hence $T(b) = T(x) > T(b - \varepsilon) \geq T(b)$ due to the *strict* decreasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, it must be that $T(x) \neq T(b)$ for any given $x < b$, so that we have $T(x) > T(b)$ for any $x < b$, hence it follows that $T(x)$ is strictly decreasing on $(-\infty, b]$ instead of $(-\infty, b)$.

(d) Evident from the fact that $T(x) + x = \mathbf{E}[\max\{\boldsymbol{\xi}, x\}]$ from (5.1.1(p.17)) and that $\max\{\boldsymbol{\xi}, x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $a < y < x$, hence $F(y) > 0$ due to (2.1.2 (2,3) (p.8)). Then, since $(x - y)F(y) > 0$, we have $T(y) + y < T(x) + x$ from (9.1.2), i.e., $T(x) + x$ is *strictly* increasing on (a, ∞). Here note that for any given $x > a$ we have $T(a) + a \leq T(x) + x$ due to (d). Let us assume $T(a) + a = T(x) + x$ for a given $x > a$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > \varepsilon$ we have $a < a + \varepsilon < x$, hence $T(a) + a = T(x) + x > T(a + \varepsilon) + a + \varepsilon \geq T(a) + a$ due to the *strict* increasingness shown above and the nondcreasingness in (d), which is a contradiction. Thus, it must be that $T(x) + x \neq T(a) + a$ for any given $x > a$, so that we have $T(x) + x > T(a) + a$ for any $x > a$, hence it follows that $T(x) + x$ is strictly increasing on $[a, \infty)$ instead of on (a, ∞).

(f) Let $x \leq a$. If $a \leq \boldsymbol{\xi}$, then $x \leq \boldsymbol{\xi}$, hence $\max\{\boldsymbol{\xi} - x, 0\} = \boldsymbol{\xi} - x$, and if $\boldsymbol{\xi} < a$, then $f(\boldsymbol{\xi}) = 0 \cdots \mathbf{(3)}$ due to (2.1.4 (1) (p.8)). Thus, from (1) we have $T(x) = \mathbf{E}[(\boldsymbol{\xi} - x)I(a \leq \boldsymbol{\xi})] + 0$. Then, since $\mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < a)] = \int_{\infty}^{a-}(\xi - x)f(\xi)d\xi = 0$ due to (3), we have $T(x) = \mathbf{E}[(\boldsymbol{\xi} - x)I(a \leq \boldsymbol{\xi})] + \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < a)] = \mathbf{E}[\boldsymbol{\xi} - x] = \mu - x$, hence the former half is true. Then, since $T(a) = \mu - a$ or equivalently $T(a) + a = \mu$, if $a < x$, from (e) we have $T(x) + x > T(a) + a = \mu$, hence $T(x) > \mu - x$, thus the latter half is true.

(g) Let $b \leq x$. If $b < \boldsymbol{\xi}$, then since $f(\boldsymbol{\xi}) = 0$ due to (2.1.4 (3) (p.8)), we have $\mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}I(b < \boldsymbol{\xi})] = \int_{b+}^{\infty}\max\{\xi - x, 0\}f(\xi)d\xi = 0$ and if $\boldsymbol{\xi} \leq b$, then since $\boldsymbol{\xi} \leq x$, we have $\max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} \leq b) = 0$. Accordingly, from (2) we have $T(x) = 0 \cdots \mathbf{(4)}$, so that the latter half is true. Let $x < b$. Then, since $T(x) > T(b)$ from (c) and since $T(b) = 0$ from (4), we have $T(x) > 0$, hence the former half is true.

(h) On $(-\infty, \infty)$ we have $T(x) \geq \mu - x$ from (f) and $T(x) \geq 0$ from (g), hence it follows that $T(x) \geq \max\{0, \mu - x\}$ on $(-\infty, \infty)$.

(i) From (5.1.1(p.17)) and (2.1.4 (1,3) (p.8)) we have $T(0) = \mathbf{E}[\max\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\max\{\boldsymbol{\xi}, 0\}I(a \leq \boldsymbol{\xi} \leq b)]$. Hence, if $a > 0$, then $T(0) = \mathbf{E}[\boldsymbol{\xi}I(a \leq \boldsymbol{\xi} \leq b)] = \mathbf{E}[\boldsymbol{\xi}] = \mu$ and if $b < 0$, then $T(0) = \mathbf{E}[0I(a \leq \boldsymbol{\xi} \leq b)] = 0$.

(j) If $\beta = 1$, then $\beta T(x) + x = T(x) + x$, hence the assertion is true from (d).

(k) Since $\beta T(x) + x = \beta(T(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x < y$ and $a < y$. If $x \leq a$, then $T(x) + x \leq T(a) + a < T(y) + y$ due to (d,e), and if $a < x$, then $a \leq x < y$, hence $K(x) + x < K(y) + y$ due to (e). Thus, whether $x \leq a$ or $a < x$, we have $T(x) + x < T(y) + y$

(m) From (5.1.1(p.17)) we have $\lambda\beta T(\lambda\beta\mu - s) - s = \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - \lambda\beta\mu + s, 0\}] - s = \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu + \lambda\beta s, 0\}] - s = \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu - (1 - \lambda\beta)s, -s\}]$, which is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.

(n) Evident. ∎

9.2 Derivative Underlying Functions

Let us define

$$\delta = 1 - (1 - \lambda)\beta. \tag{9.2.1}$$

Then, due to the assumptions $0 < \beta \leq 1$ and $1 \geq \lambda > 0$ we have

$$\delta \geq 1 - (1 - \lambda) \times 1 = \lambda > 0 \cdots (1), \qquad \delta \leq 1 - (1 - \lambda) \times 0 = 1 \cdots (2). \tag{9.2.2}$$

Now, from (5.1.3(p.17)) and (5.1.4) and from Lemma 9.1.1(f) we obtain

$$L(x) \begin{cases} = \lambda\beta\mu - s - \lambda\beta x \text{ on } (-\infty, a] & \cdots (1), \\ > \lambda\beta\mu - s - \lambda\beta x \text{ on } (a, \infty) & \cdots (2), \end{cases} \tag{9.2.3}$$

$$K(x) \begin{cases} = \lambda\beta\mu - s - \delta x \quad \text{on} \quad (-\infty, a] & \cdots (1), \\ > \lambda\beta\mu - s - \delta x \quad \text{on} \quad (a, \infty) & \cdots (2). \end{cases} \tag{9.2.4}$$

In addition, from (5.1.4(p.17)) and Lemma 9.1.1(g) we have

$$K(x) \begin{cases} > -(1 - \beta)x - s \text{ on } \quad (-\infty, b) & \cdots (1), \\ = -(1 - \beta)x - s \text{ on } \quad [b, \infty) & \cdots (2), \end{cases} \tag{9.2.5}$$

from which we obtain

$$K(x) + x \geq \beta x - s \quad \text{on} \quad (-\infty, \infty). \tag{9.2.6}$$

Then, from (9.2.4 (1)) and (9.2.5 (2)) we get

$$K(x)+x=\begin{cases}\lambda\beta\mu-s+(1-\lambda)\beta x & \text{on} \quad (-\infty,a] \quad \cdots(1),\\ \beta x-s & \text{on} \quad [b,\infty) \quad \cdots(2).\end{cases} \tag{9.2.7}$$

Since $K(x)=L(x)-(1-\beta)x$ and $L(x)=K(x)+(1-\beta)x$ from (5.1.8), if x_L and x_K exist, then

$$K(x_L)=-(1-\beta)x_L \cdots(1), \qquad L(x_K)=(1-\beta)x_K \cdots(2). \tag{9.2.8}$$

Lemma 9.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) *$L(x)$ is continuous.*
(b) *$L(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *Let $s=0$. Then $x_L=b$ where $x_L>(\le)\,x \Leftrightarrow L(x)>(=)\,0 \Rightarrow L(x)>(\le)\,0$.*
(e) *Let $s>0$.*
  1. *x_L uniquely exists with $x_L<b$ where $x_L>(=(<))\,x \Leftrightarrow L(x)>(=(<))\,0$.*
  2. *$(\lambda\beta\mu-s)/\lambda\beta\le(>)\,a \Leftrightarrow x_L=(>)\,(\lambda\beta\mu-s)/\lambda\beta$.* □

Proof (a-c) Immediate from (5.1.3) and Lemma 9.1.1(a-c).

(d) Let $s=0$. Then, since $L(x)=\lambda\beta T(x)$, from Lemma 9.1.1(g) we have $L(x)>0$ for $b>x$ and $L(x)=0$ for $b\le x$, hence $x_L=b$ by the definition of x_L (see Section 5.2(p.19) (a)), thus $x_L>(\le)\,x \Rightarrow L(x)>(=)\,0$. The inverse is true by contraposition. In addition, since $L(x)=0\Rightarrow L(x)\le 0$, we have $L(x)>(=)\,0 \Rightarrow L(x)>(\le)\,0$.

(e) Let $s>0$.

(e1) From (9.2.3 (1)) and the assumptions $\lambda>0$ and $\beta>0$ we have $L(x)>0$ for a sufficiently small $x<0$ such that $x\le a$. In addition, we have $L(b)=\lambda\beta T(b)-s=-s<0$ due to Lemma 9.1.1(g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L<b$ is immediate from $L(b)<0$. The latter half is evident.

(e2) If $(\lambda\beta\mu-s)/\lambda\beta\le(>)\,a$, from (9.2.3) we have $L((\lambda\beta\mu-s)/\lambda\beta)=(>)\,\lambda\beta\mu-s-\lambda\beta(\lambda\beta\mu-s)/\lambda\beta=0$, hence $x_L=(>)\,(\lambda\beta\mu-s)/\lambda\beta$ from (e1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

The corollary below is used when it is not specified whether $s>0$ or $s=0$.

Corollary 9.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) $x_L>(\le)\,x \Leftrightarrow L(x)>(\le)\,0$.
(b) $x_L\ge(\le)\,x \Rightarrow L(x)\ge(\le)\,0$. □

Proof (a) Immediate from Lemma 9.2.1(d,e1).

(b) Since $x_L>(\le)\,x \Rightarrow L(x)>(\le)\,0$ due to (a) and since $L(x)>(\le)\,0 \Rightarrow L(x)\ge(\le)\,0$, we have $x_L>(\le)\,x \Rightarrow L(x)\ge(\le)\,0$. In addition, if $x_L=x$, then $L(x)=L(x_L)=0$ or equivalently $x_L=x\Rightarrow L(x)=0$. Hence it eventually follows that $x_L\ge(\le)\,x \Rightarrow L(x)\ge(\le)\,0$. ∎

Lemma 9.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) *$K(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty,\infty)$ if $\beta<1$.*
(e) *$K(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(f) *$K(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(g) *$K(x)+x$ is strictly increasing on $[a,\infty)$.*
(h) *If $x<y$ and $a<y$, then $K(x)+x<K(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_K=b$ where $x_K>(\le)\,x \Leftrightarrow K(x)>(=)\,0 \Rightarrow K(x)>(\le)\,0$.*
(j) *Let $\beta<1$ or $s>0$.*
  1. *There uniquely exists x_K where $x_K>(=(<))\,x \Leftrightarrow K(x)>(=(<))\,0$.*
  2. *$(\lambda\beta\mu-s)/\delta\le(>)\,a \Leftrightarrow x_K=(>)\,(\lambda\beta\mu-s)/\delta$.*
  3. *Let $\kappa>(=(<))\,0$. Then $x_K>(=(<))\,0$.* □

Proof (a-c) Immediate from (5.1.4(p.17)) and Lemma 9.1.1(a-c).

(d) Immediate from Lemma 9.1.1(b) and (5.1.4).

(e) From (5.1.4) we have $K(x)+x=\lambda\beta T(x)+\beta x-s=\lambda\beta(T(x)+x)+(1-\lambda)\beta x-s\cdots$**(1)**, hence the assertion holds from Lemma 9.1.1(d).

(f) Obvious from (1) and Lemma 9.1.1(d).

(g) Clearly from (1) and Lemma 9.1.1(e).

(h) Let $x < y$ and $a < y$. If $x \le a$, then $K(x) + x \le K(a) + a < K(y) + y$ due to (e,g), and if $a < x$, then $a < x < y$, hence $K(x) + x < K(y) + y$ due to (g). Thus, whether $x \le a$ or $a < x$, we have $K(x) + x < K(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x)$ due to (5.1.4), from Lemma 9.1.1(g) we have $K(x) > 0$ for $x < b$ and $K(x) = 0$ for $b \le x$, hence $x_K = b$ by the definition of x_K (see Section 5.2(p.19) (a)). Thus $x_K > (\le)\ x \Rightarrow K(x) > (=)\ 0$. The inverse holds by contraposition. In addition, since $K(x) = 0 \Rightarrow K(x) \le 0$, we have $K(x) > (=)\ 0 \Rightarrow K(x) > (\le)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) First note (9.2.5 (2)). If $\beta = 1$, then $s > 0$ due to the assumption $\beta < 1$ or $s > 0$, hence $K(x) = -s < 0$ for any $x \ge b$ and if $\beta < 1$, then $K(x) < 0$ for any sufficiently large $x > 0$ such that $x \ge b$. Hence, for any $0 < \beta \le 1$ we have $K(x) < 0$ for any sufficiently large x. Next note (9.2.4 (1)). Then, since $\delta > 0$ from (9.2.2 (1)), for any sufficiently small $x < 0$ such that $x \le a$ we have $K(x) > 0$ for any $0 < \beta \le 1$. Hence, it follows that there exists the solution x_K for any $0 < \beta \le 1$. Let $\beta < 1$. Then, the solution is unique from (d). Let $\beta = 1$. Then since $s > 0$ due to the assumption $\beta < 1$ or $s > 0$, we have $K(b) = -s < 0$ from (9.2.5 (2)), hence $x_K < b$ due to (b), so $K(x)$ is strictly decreasing on the neighbourhood of $x = x_K$ due to (c), thus the solution x_K is unique. Therefore, it follows that the solution x_K is unique for any $0 < \beta \le 1$. From the above we see that the latter half holds.

(j2) Let $(\lambda\beta\mu - s)/\delta \le (>)\ a$. Then, from (9.2.4 (1(2))) we have $K((\lambda\beta\mu - s)/\delta) = (>)\ \lambda\beta\mu - s - \delta(\lambda\beta\mu - s)/\delta = 0$, hence $x_K = (>)\ (\lambda\beta\mu - s)/\delta$ due to (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition.

(j3) If $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$ from (5.1.7(p.17)), hence $x_K > (= (<))\ 0$ from (j1). ∎

The corollary below is used when it is not specified whether $s > 0$ or $s = 0$.

Corollary 9.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) $x_K > (\le)\ x \Leftrightarrow K(x) > (\le)\ 0$.
(b) $x_K \ge (\le)\ x \Rightarrow K(x) \ge (\le)\ 0$. □

Proof (a) Immediate from Lemma 9.2.2(i,j1).

(b) Since $x_K > (\le)\ x \Rightarrow K(x) > (\le)\ 0$ due to (a) and since $K(x) > (\le)\ 0 \Rightarrow K(x) \ge (\le)\ 0$, we have $x_K > (\le)\ x \Rightarrow K(x) \ge (\le)\ 0$. In addition, if $x_K = x$, then $K(x) = K(x_K) = 0$ or equivalently $x_K = x \Rightarrow K(x) = 0$. Hence it eventually follows that $x_K \ge (\le)\ x \Rightarrow K(x) \ge (\le)\ 0$. ∎

Lemma 9.2.3 ($\mathscr{A}\{L_{\mathbb{R}}/K_{\mathbb{R}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_L = x_K = b$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_L = x_K$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $b > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\kappa > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (<))\ 0$.* □

Proof (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma 9.2.1(d) and $x_K = b$ from Lemma 9.2.2(i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (9.2.8 (1)), hence $x_K = x_L$ from Lemma 9.2.2(j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots$ **(1)** from Lemma 9.2.1(d).

1. Suppose $b > 0$. Then, since $x_L > 0$, we have $K(x_L) < 0$ from (9.2.8 (1)), hence $x_L > x_K$ from Lemma 9.2.2(j1). Furthermore, from (5.1.7) we have $K(0) = \lambda\beta T(0) > 0$ due to Lemma 9.1.1(g), hence $x_K > 0 \cdots$ **(2)** from Lemma 9.2.2(j1).
2. Suppose $b = (<)\ 0$. Then, since $x_L = (<)\ 0$ from (1), we have $K(x_L) = (>)\ 0$ from (9.2.8 (1)), thus $x_L = (<)\ x_K$ from Lemma 9.2.2(j1). Thus "⇒" holds and its inverse "⇐" is immediate by contraposition. Furthermore, from (5.1.7) we have $K(0) = \lambda\beta T(0) = 0$ if $b = (<)\ 0$ due to Lemma 9.1.1(g), hence $x_K = 0$ from Lemma 9.2.2(j1) or equivalently $x_K = (=)\ 0$.

(d) Let $\beta < 1$ and $s > 0$. Now, since $\kappa = K(0)$ from (5.1.7), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0 \cdots$ **(3)** from Lemma 9.2.2(j1). Accordingly $L(x_K) > (= (<))\ 0$ from (9.2.8 (2)), hence $x_L > (= (<))\ x_K$ from Lemma 9.2.1(e1). Thus "⇒" holds and its inverse "⇐" is immediate by contraposition. The last "⇒" is the same as (3). ∎

Lemma 9.2.4 ($\mathcal{L}_{\mathbb{R}}$)

(a) *$\mathcal{L}(s)$ is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.*
(b) *Let $\lambda\beta\mu \ge b$.*
   1. *$x_L \le \lambda\beta\mu - s$.*
   2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta\mu - s$.*
(c) *Let $\lambda\beta\mu < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\le)\ s$, then $x_L > (\le)\ \lambda\beta\mu - s$.* □

Proof (a) From (5.1.5(p.17)) and (5.1.3) we have $\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta T(\lambda\beta\mu - s) - s \cdots$ **(1)**, hence the assertion holds from Lemma 9.1.1(m).

(b) Let $\lambda\beta\mu \ge b$. Then, from (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) = 0 \cdots$ **(2)** due to Lemma 9.1.1(g).

(b1) Since $s \geq 0$, from (a) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (2) or equivalently $L(\lambda\beta\mu - s) \leq 0$, hence $x_L \leq \lambda\beta\mu - s$ from Corollary 9.2.1(a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0 \cdots$ **(3)** due to (2) or equivalently $L(\lambda\beta\mu - s) < 0$, hence $x_L < \lambda\beta\mu - s$ from Lemma 9.2.1(e1).

(c) Let $\lambda\beta\mu < b$. From (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) > 0 \cdots$ **(4)** due to Lemma 9.1.1(g). Noting (9.2.3 (1)), for any sufficiently large $s > 0$ such that $\lambda\beta\mu - s \leq a$ and $\lambda\beta\mu - s < 0$ we have $\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s) = (1 - \lambda\beta)(\lambda\beta\mu - s) \leq 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}}$ of $\mathcal{L}(s) = 0$ where $s_{\mathcal{L}} > 0$ due to (4). Then, since $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta\mu - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta\mu - s) \leq 0$ for $s \geq s_{\mathcal{L}}$, from Corollary 9.2.1(a) we get $x_L > \lambda\beta\mu - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta\mu - s$ for $s \geq s_{\mathcal{L}}$. ∎

9.3 $\kappa_{\mathbb{R}}$-value

Lemma 9.3.1 **($\mathscr{A}\{\kappa_{\mathbb{R}}\}$)** *We have*:

(a) $\kappa = \lambda\beta\mu - s$ *if* $a > 0$ *and* $\kappa = -s$ *if* $b < 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$*, Then* $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$. □

Proof (a) Immediate from (5.1.6(p.17)) and Lemma 9.1.1(i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.7(p.17)), hence $x_K > (= (<))\ 0$ from Lemma 9.2.2(j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

Chapter 10

Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

10.1 Preliminary

From (6.2.8(p.22)) and (6.2.11) we have

$$V_t - \beta V_{t-1} = \max\{\mathbb{S}_t, 0\} \tag{10.1.1}$$
$$= \max\{L(V_{t-1}), 0\}, \quad t > 1. \tag{10.1.2}$$

Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence from (5.1.9(p.17)) we have
$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{10.1.3}$$
2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently
$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{10.1.4}$$

Now, from (6.2.5(p.21)) with $t = 2$ we have

$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{10.1.5}$$

Finally, from (6.2.11) and (6.2.9) we have

$$\mathbb{S}_t = L(V_{t-1}) > (<)\, 0 \Rightarrow \ \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle}), \quad t > 1.. \tag{10.1.6}$$

10.2 Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Definition 10.2.1 By $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ let us represent an *assertion* included in each of Tom 10.2.1 and Tom 10.2.2 that follows and by $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ the *assertion system* consisting of all assertions included in the Tom. □

Below note that $\lambda = 1$ is assume in the model.

□ **Tom 10.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, from (5.1.4(p.17)) we have $K(x) = T(x) \geq 0 \cdots \mathbf{(1)}$ for any x due to Lemma 9.1.1(p.41) (g), hence from (6.5.2(p.31)) and (1) we have $V_t = \max\{T(V_{t-1}) + V_{t-1}, V_{t-1}\} = \max\{T(V_{t-1}), 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1} \cdots \mathbf{(2)}$ for $t > 1$.

(a) Since $V_2 = T(V_1) + V_1$, we have $V_2 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 9.1.1(d) we have $V_t \leq T(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Since $V_1 = \mu$ from (6.5.1(p.31)), we have $V_1 < b$. Suppose $V_{t-1} < b$. Then, from (2) we have $V_t < T(b) + b = b$ due to Lemma 9.1.1(l,g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 9.2.1(d); accordingly, $L(V_{t-1}) > 0 \cdots \mathbf{(3)}$ for $\tau \geq t > 1$. Thus, from (10.1.2) we obtain $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly, since $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$, we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\text{ⓢ}\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$, hence we have $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (10.1.6).. ∎

Let us define

$$\mathtt{S}_1\boxed{\text{ⓢ}\blacktriangle}\boxed{\circledast\,\|} = \left\{ \begin{array}{ll} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} & \\ (1) \quad \boxed{\text{ⓢ}\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}, & \\ (2) \quad \boxed{\circledast\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}. & \end{array} \right\}$$

□ Tom **10.2.2** ($\mathscr{A}${M:1[ℝ][A]}) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   i. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $b = 0$ ⦅$\kappa = 0$⦆.*
      1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu - s > a$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $b < 0$ ⦅$\kappa < 0$⦆.*
      1. *Let $\beta\mu - s \leq a$ or $s_{\mathscr{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta\mu - s > a$ and $s < s_{\mathscr{L}}$. Then* $\texttt{S}_1$(p.47)$\boxed{Ⓢ\blacktriangle}\boxed{⊛\parallel}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$. In this model, note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.14) (b)) and that $\delta = 1 \cdots$ **(1)** (see (9.2.1(p.42))) due to the assumption $\lambda = 1 \cdots$ **(2)**.

(a) Since $x_K \geq \beta\mu - s = V_1$ due to Lemma 9.2.2(p.43) (j2) and (6.5.1(p.31)), we have $K(V_1) \geq 0$ due to Lemma 9.2.2(j1), hence $V_2 - V_1 \geq 0$ from (10.1.5), i.e., $V_1 \leq V_2$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.2(p.31)) and Lemma 9.2.2(e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Consider a sufficiently large $M > 0$ with $\beta\mu - s \leq M$ and $b \leq M$, hence $V_1 \leq M$. Suppose $V_{t-1} \leq M$. Then, from (6.5.2(p.31)), Lemma 9.2.2(e), and (9.2.7 (2) (p.43)) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\} \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \leq M$ for $t > 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.2(p.31)) we have $V = \max\{K(V) + V, \beta V\}$, hence $0 = \max\{K(V), -(1-\beta)\beta V\}$. Thus, since $K(V) \leq 0$, we have $V \geq x_K$ from Lemma 9.2.2(j1).

(b) Let $\beta\mu \geq b \cdots$ **(3)**. Then $x_L \leq \beta\mu - s = V_1$ from Lemma 9.2.4(b1) with $\lambda = 1$, hence $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ due to Corollary 9.2.1(a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Hence, from (10.1.4) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^1$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ (see Preference Rule 7.2.1(p.35)).

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots$ **(4)**, hence $s > 0$ due to the assumptions $\beta < 1$ or $s > 0$ in the lemma. Then, from (4), (1), (2) we have $(\lambda\beta\mu - s)/\delta = \mu - s \cdots$ **(5)**. In addition, since $x_L = x_K \cdots$ **(6)** from Lemma 9.2.3(b), we have $K(x_L) = K(x_K) = 0 \cdots$ **(7)**.

(c1i) Let $\mu - s \leq a$. Then $x_L = x_K = \mu - s = V_1$ from (6), Lemma 9.2.2(j2), (5), and (6.5.1(p.31)). Accordingly, since $x_L \leq V_{t-1}$ for $t > 1$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 1$ due to Lemma 9.2.1(e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c1ii) Let $\mu - s > a$. Then $x_L = x_K > \mu - s = V_1 > a$ from (6) and Lemma 9.2.2(j2), hence $a < V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} < x_L$, hence $L(V_{t-1}) > 0$ from Lemma 9.2.1(e1). Then, from (10.1.3), Lemma 9.2.2(g), and (6) we have $V_t < K(x_L) + x_L = K(x_K) + x_L = x_L$. Accordingly, by induction $V_{t-1} < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 9.2.1(p.43) (a). Thus, for the same reason as in the proof of Tom 10.2.1(b) we have $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2) Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.

(c2i) Let $b > 0$ ⦅$\kappa > 0$⦆. Then $x_L > x_K > 0 \cdots$ **(8)** from Lemma 9.2.3(c ⦅d⦆). Now, since $x_K \geq \beta\mu - s$ due to Lemma 9.2.2(j2), (1), and (2), we have $x_K \geq V_1$ from (6.5.1(p.31)). Suppose $x_K \geq V_{t-1}$. Then, from (6.5.2(p.31)) and Lemma 9.2.2(e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to (8). Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (8), thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 9.2.1(a). Hence, for the same reason as in the proof of Tom 10.2.1(b) we have $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2ii) Let $b = 0$ ⦅$\kappa = 0$⦆. Then $x_L = x_K \cdots$ **(9)** from Lemma 9.2.3(c ⦅d⦆).

(c2ii1) Let $\beta\mu - s \leq a$. Then, $x_K = \beta\mu - s = V_1$ from Lemma 9.2.2(j2). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (9), so that $L(V_{t-1}) = L(x_L) = 0$. Then, from (10.1.3) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (9). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (10.1.4), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$ from (6.5.1), we have $V_{t-1} > a$ for $t > 1$ due to (a). In addition, we have $x_K > \beta\mu - s = V_1$ from Lemma 9.2.2(j2). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (9). Then, since $L(V_{t-1}) > 0$ due to Corollary 9.2.1(a), from (10.1.3) and Lemma 9.2.2(g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, so that $x_L > V_{t-1}$ for $t > 1$ due to (9). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 9.2.1(a), for the same reason as in the proof of (c1ii) we have $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2iii) Let $b < 0$ ⦅$\kappa < 0$⦆. Then $x_L < x_K \cdots$ **(10)** from Lemma 9.2.3(c ⦅d⦆).

(c2iii1) Let $\beta\mu - s \leq a$ or $s_{\mathscr{L}} \leq s$. First let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s = V_1$ from Lemma 9.2.2(j2), we have $x_L < V_1$ from (10), hence $x_L \leq V_1$. Next, let $s_{\mathscr{L}} \leq s$. Then, since $x_L \leq \beta\mu - s$ due to Lemma 9.2.4(c), we have $x_L \leq V_1$.

Accordingly, whether $\beta\mu - s \le a$ or $s_{\mathcal{L}} \le s$, we have $x_L \le V_1$, thus $x_L \le V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 9.2.1(a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta\mu - s > a \cdots (11)$ and $s < s_{\mathcal{L}}$. Then, from (10) and Lemma 9.2.4(c) we have $x_K > x_L > \beta\mu - s = V_1 \cdots (12)$, hence $K(V_1) > 0 \cdots (13)$ from Lemma 9.2.2(j1). In addition, since $V_1 > a$ due to (11), we have $V_{t-1} > a$ for $t > 0$ from (a). Now, from (10.1.5) and (13) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from Lemma 9.2.2(g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (12). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \le x_L$ due to (a). Then, from (10) and the fact of $V \ge x_K$ due to (a) we have the contradiction of $V \ge x_K > x_L \ge V$. Hence, it is impossible that $V_{t-1} < x_L$ for all $t > 1$, implying that there exists $t_\tau^\bullet > 1$ such that

$$V_1 < V_2 < \cdots < V_{t_\tau^\bullet - 1} < x_L \le V_{t_\tau^\bullet} < V_{t_\tau^\bullet + 1} < V_{t_\tau^\bullet + 2} < \cdots, \tag{10.2.1}$$

from which

$$V_{t-1} < x_L, \quad t_\tau^\bullet \ge t > 1, \qquad x_L \le V_{t-1}, \quad t > t_\tau^\bullet. \tag{10.2.2}$$

Therefore, from Corollary 9.2.1(a) we have

$$L(V_{t-1}) > 0 \cdots (14), \quad t_\tau^\bullet \ge t > 1, \qquad L(V_{t-1}) \le 0 \cdots (15), \quad t > t_\tau^\bullet.$$

1. Let $t_\tau^\bullet \ge \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots (16)$ for $\tau \ge t > 1$ from (14), for the same reason as in the proof of (c1ii) we have $\boxed{\circledast\ \texttt{dOITs}_{t_\tau^\bullet \ge \tau > 1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{CONDUCT}_{\tau \ge t > 1 \blacktriangle}$. Hence $\texttt{S}_1(1)$ is true.
2. Let $\tau > t_\tau^\bullet$. First let $\tau \ge t > t_\tau^\bullet$. Then, since $L(V_{t-1}) \le 0$ for $\tau \ge t > t_\tau^\bullet$ from (15), we have $V_t = \beta V_{t-1}$ for $\tau \ge t > t_\tau^\bullet$ from (10.1.4), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} \cdots (17).$$

Next let $t_\tau^\bullet \ge t > 1$. Then, from (14) and (10.1.2) we have $V_t - \beta V_{t-1} > 0$ for $t_\tau^\bullet \ge t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t_\tau^\bullet \ge t > 1$, hence

$$V_{t_\tau^\bullet} > \beta V_{t_\tau^\bullet - 1} > \beta^2 V_{t_\tau^\bullet - 2} > \cdots > \beta^{t_\tau^\bullet - 1} V_1 \cdots (18).$$

From (17) and (18) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} > \beta^{\tau - t_\tau^\bullet + 1} V_{t_\tau^\bullet - 1} > \beta^{\tau - t_\tau^\bullet + 2} V_{t_\tau^\bullet - 2} > \cdots > \beta^{\tau - 1} V_1,$$

hence we obtain $t_\tau^* = t_\tau^\bullet$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t_\tau^\bullet}\langle t_\tau^\bullet\rangle}_{\|}$ due to Preference Rule 7.2.1(p.35). In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t_\tau^\bullet \ge t > 1$ due to (14) and (10.1.6). Hence $\texttt{S}_1(2)$ is true. ∎

10.3 Structure of Assertion System $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

In this section let us clarify the structure of the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.

10.3.1 Breakdown and Aggregation

Consider a given set $\mathscr{X}$ and given $k \ge 0$ subsets $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_k \subseteq \mathscr{X}$ where $\mathscr{X}_i \cap \mathscr{X}_j = \emptyset$ for any $i \ne j$ (pairwise disjoint). Here let $\mathscr{X} = \cup_{k=1,2,\cdots,k}\mathscr{X}_i$, and then let us consider the following two operations (see Figure 10.3.1 below ($k = 3$ for example)):

(I) $\mathscr{X}$ is broken down into $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_k$ (breakdown).

(II) $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_k$ are aggregated into $\mathscr{X}$ (aggregation).

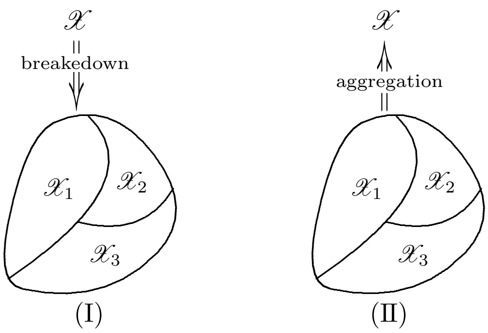

Figure 10.3.1: Breakedown and aggregation

It will be known later on that the fine differences between the above two operations will play an essential role in discussions of Section 10.3.4(p.51) and Step 11.5 (p.63).

10.3.2 Structure of Assertion A

Let us note here that any assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ consists of a *statement* S and a *condition-expression* CE, schematized as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}=\{\mathsf{S} \text{ holds if } \mathsf{CE} \text{ is satisfied}\}. \tag{10.3.1}$$

Example **10.3.1** The assertion of Tom 10.2.2(p.48) (b) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}=\{\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|} \text{ holds if } \beta\mu \ge b \text{ is satisfied}\}$$

where $\mathsf{S}=\{\boxed{\bullet\, \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}\}$ and $\mathsf{CE}=\{\beta\mu \ge b\}$. □

In addition, the condition-expression CE can be regarded as a conditional prescribed as to a parameter vector $\boldsymbol{p}$ and a distribution function F where

$$\boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P}, \tag{10.3.2}$$

$$F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F} \tag{10.3.3}$$

for a given parameter space $\mathscr{P}_A$ and a given distribution function space $\mathscr{F}_{A|\boldsymbol{p}}$ related to a given $\boldsymbol{p} \in \mathscr{P}_A$. Then we can rewrite (10.3.1) as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds for } \boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{10.3.4}$$

Example **10.3.2**

- For the assertion A in Tom 10.2.2(p.48) (c1i) we have

$$\mathscr{P}_A = \{\lambda = 1 \cap \beta = 1 \cap s > 0\},$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{\beta\mu < b \cap \mu - s \le a\}.$$

- For the assertion A in Tom 10.2.2(p.48) (c2iii2) we have

$$\mathscr{P}_A = \{\lambda = 1 \cap \beta < 1 \cap\ s = 0\,(\!(s > 0)\!)\,\},$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{\beta\mu < b \cap\ b < 0\,(\!(\kappa < 0)\!)\ \cap \beta\mu - s > a \cap s <\ s_{\mathcal{L}}\ \}. \ \square$$

Definition 10.3.1 (condition-space $\mathscr{C}\langle A\rangle$) Let us define

$$\mathscr{C}\langle A\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_A \subseteq \mathscr{P}, F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{10.3.5}$$

called the *condition-space* of the assertion $A = A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. □

Then, (10.3.4) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds on } \mathscr{C}\langle A\rangle\ \}. \tag{10.3.6}$$

Throughout the rest of the paper, for explanatory convenience, let us *alternatively* express the whole of (10.3.6) as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A\rangle. \tag{10.3.7}$$

10.3.3 Structure of Tom

In addition to the definition in Section 6.4(p.30) and Def. 10.2.1(p.47), let us here provide the following definition;

Definition 10.3.2 (assertion $A_{\mathtt{Tom}}$) When a given assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is what is included in a given Tom, let us represent it as $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and an assertion system consisting of all $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ included in Tom as $\mathscr{A}_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. □

Then (10.3.4)-(10.3.7) can be rewritten as respectively

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{10.3.8}$$

$$\mathscr{C}\langle A_{\mathtt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P}, F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{10.3.9}$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\ \}, \tag{10.3.10}$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle. \tag{10.3.11}$$

Closely looking into the structure of Tom's 10.2.1(p.47) and 10.2.2, we see that in general a given Tom consists of a *basic-premise* BP and some *assertions* $A_{\mathtt{Tom}_1}$, $A_{\mathtt{Tom}_2}$, $\cdots$, i.e.,

$$\mathtt{Tom} = \{\text{Let BP be true. Then assertions } A_{\mathtt{Tom}_1}, A_{\mathtt{Tom}_2}, \cdots \text{ hold}\} \tag{10.3.12}$$

where the basic-premise BP is given as a *conditional* prescribed as to a parameter vector $\boldsymbol{p}$ and a distribution function F where

$$\boldsymbol{p} \in \mathscr{P}_{\mathtt{Tom}} \subseteq \mathscr{P}, \tag{10.3.13}$$

$$F \in \mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F} \tag{10.3.14}$$

for given subsets $\mathscr{P}_{\mathtt{Tom}}$ and $\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}}$ (see (10.3.2) and (10.3.3)). Then the basic-premise BP can be written as

$$\mathsf{BP} = \{\text{a condition on } \boldsymbol{p} \in \mathscr{P}_{\mathtt{Tom}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{10.3.15}$$

Example **10.3.3** For M:1[ℝ][A] we have

$$(1)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta = 1 \cap s = 0\} \qquad \text{for Tom 10.2.1(p.47)}$$
$$(2)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda = 1 \cap (\beta < 1 \cup s > 0)\} \qquad \text{for Tom 10.2.2(p.48)}$$
$$(1)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \mathscr{F} \qquad \text{for Tom 10.2.1(p.47)}$$
$$(2)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \mathscr{F} \qquad \text{for Tom 10.2.2(p.48)}$$

As the above is too simple, in order to promote a better understanding, below let us provide the example for M:2[ℝ][A].

$$(3)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap \beta = 1 \cap s = 0 \cap -\infty < \rho < \infty\} \qquad \text{for Tom 19.1.1(p.140)}$$
$$(4)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\} \qquad \text{for Tom 19.1.2(p.141)}$$
$$(5)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\} \qquad \text{for Tom 19.1.3(p.143)}$$
$$(6)\ = \mathscr{P}_{\texttt{Tom}} = \{\boldsymbol{p} \mid \lambda \le 1 \cap (\beta < 1 \cup s > 0) \cap -\infty < \rho < \infty\} \qquad \text{for Tom 19.1.4(p.144)}$$
$$(3)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \{F \mid -\infty < a < \mu < b < \infty\} = \mathscr{F} \qquad \text{for Tom 19.1.1(p.140)}$$
$$(4)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho < x_K\} \qquad \text{for Tom 19.1.2(p.141)}$$
$$(5)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho = x_K\} \qquad \text{for Tom 19.1.3(p.143)}$$
$$(6)' = \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} = \{F \mid F \in \mathscr{F} \cap \rho > x_K\} \qquad \text{for Tom 19.1.4(p.144)}$$
□

Definition 10.3.3 (condition-space $\mathscr{C}\langle\texttt{Tom}\rangle$) Let us define

$$\mathscr{C}\langle\texttt{Tom}\rangle \stackrel{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{\texttt{Tom}} \subseteq \mathscr{P}, F \in \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{10.3.16}$$

called the *condition-space* of Tom. □

Then (10.3.15) can be rewritten as

$$\mathsf{BP}\ =\ \{\text{a condition on } \mathscr{C}\langle\texttt{Tom}\rangle\}, \tag{10.3.17}$$

so (10.3.12) can be rewritten as

$$\texttt{Tom}\ =\ \{\text{The assertions } A_{\texttt{Tom}_1},\ A_{\texttt{Tom}_2}.\ \cdots\ \text{hold on } \mathsf{BP}\} \tag{10.3.18}$$

or equivalently

$$\texttt{Tom}\ =\ \{\text{The assertions } A_{\texttt{Tom}_1},\ A_{\texttt{Tom}_2},\ \cdots\ \text{hold on } \mathscr{C}\langle\texttt{Tom}\rangle.\} \tag{10.3.19}$$

10.3.4 Completeness of $\mathscr{C}\langle\texttt{Tom}\rangle$

Breakdown scenario begins from here

As seen in Tom 10.2.2(p.48), in the present paper any given Tom is constructed so that the whole condition-space $\mathscr{C}\langle\texttt{Tom}\rangle$ is *hierarchically*, *encyclopedically*, and *exhaustively* broken down step-by-step from top to bottom; as a result, we obtain a sequence of assertions $A_{\texttt{Tom}_1}$, $A_{\texttt{Tom}_2}$, $\cdots$ with condition-spaces $\mathscr{C}\langle A_{\texttt{Tom}_1}\rangle$, $\mathscr{C}\langle A_{\texttt{Tom}_2}\rangle$, $\cdots$. The above procedure implies that $\mathscr{C}\langle\texttt{Tom}\rangle$ is broken down into $\mathscr{C}\langle A_{\texttt{Tom}_1}\rangle$, $\mathscr{C}\langle A_{\texttt{Tom}_2}\rangle$, $\cdots$ so that the equality

$$\mathscr{C}\langle\texttt{Tom}\rangle = \cup_{A_{\texttt{Tom}} \in \texttt{Tom}} \mathscr{C}\langle A_{\texttt{Tom}}\rangle. \tag{10.3.20}$$

is satisfied. Let us refer to the above equality as the *completeness* of $\mathscr{C}\langle\texttt{Tom}\rangle$ (see Figures 10.3.1(I) and 10.3.2 below). □

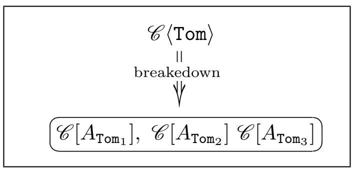

Figure 10.3.2: Exhaustive breakedown of $\mathscr{C}\langle\texttt{Tom}\rangle$ to $\mathscr{C}\langle A_{\texttt{Tom}_1}\rangle$, $\mathscr{C}\langle A_{\texttt{Tom}_2}\rangle$, $\mathscr{C}\langle A_{\texttt{Tom}_3}\rangle$

Remark 10.3.1 It should be noted here that the equality (10.3.20) is not what should be *proven* but what should be *satisfied*. □

Here, consider the list of (10.3.11) over assertions $A_{\texttt{Tom}_1}, A_{\texttt{Tom}_2}, \cdots \in \texttt{Tom}$, i.e.,

$$\text{“ } A_{\texttt{Tom}_1}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\texttt{Tom}_1}\rangle \text{ ”},$$
$$\text{“ } A_{\texttt{Tom}_2}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\texttt{Tom}_2}\rangle \text{ ”}.$$
$$\vdots$$

Then, noting (10.3.20), we see that the whole of the above list can be rewritten as

$$\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\texttt{Tom}\rangle. \tag{10.3.21}$$

10.3.5 Structure of $\mathcal{T}\mathtt{om}$

Let us define

$$\mathcal{T}\mathtt{om} \overset{\text{def}}{=} \{\mathtt{Tom}\} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, , \cdots\}. \tag{10.3.22}$$

Example **10.3.4** For example, we have

$$\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1 = \mathtt{Tom}\ 10.2.1\text{(p.47)}, \mathtt{Tom}_2 = \mathtt{Tom}\ 10.2.2\},$$
$$\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1 = \mathtt{Tom}\ 19.1.1\text{(p.140)}, \mathtt{Tom}_2 = \mathtt{Tom}\ 19.1.2, \mathtt{Tom}_3 = \mathtt{Tom}\ 19.1.3, \mathtt{Tom}_4 = \mathtt{Tom}\ 19.1.4\}. \quad \square$$

Definition 10.3.4 (condition-space $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$) Let us define

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle \overset{\text{def}}{=} \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \mathscr{C}\langle\mathtt{Tom}\rangle, \tag{10.3.23}$$

called the *condition-space* of $\mathcal{T}\mathtt{om}$. $\square$

Using (10.3.20), we can express (10.3.23) as below

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle \overset{\text{def}}{=} \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \mathscr{C}\langle\mathtt{Tom}\rangle = \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}} \mathscr{C}\langle A_{\mathtt{Tom}}\rangle, \tag{10.3.24}$$

schematized as in Figure 10.3.3 below

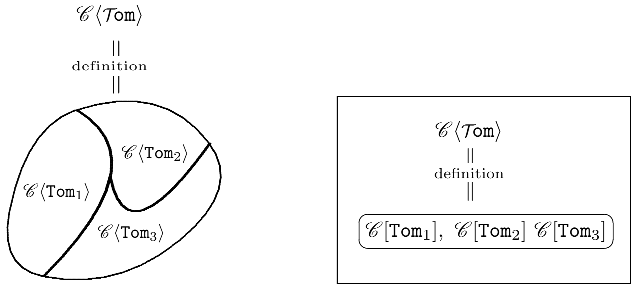

Figure 10.3.3: Definition of $\mathcal{T}\mathtt{om}$

Here, consider the list of (10.3.21) over $\mathtt{Tom}_1, \mathtt{Tom}_2, , \cdots \in \mathcal{T}\mathtt{om}$, i.e.,

$$\text{“}\mathscr{A}_{\mathtt{Tom}_1}\ \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\mathtt{Tom}_1\rangle\ \text{”}.$$
$$\text{“}\mathscr{A}_{\mathtt{Tom}_2}\ \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\mathtt{Tom}_2\rangle\ \text{”}.$$
$$\vdots$$

Then, noting (10.3.23), we see that the whole of the above list can be rewritten as

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle. \tag{10.3.25}$$

Example **10.3.5** Let us consider an example $\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, \mathtt{Tom}_3\}$ where $\mathtt{Tom}_1 = \{A^1_{\mathtt{Tom}_1}, A^2_{\mathtt{Tom}_1}, A^3_{\mathtt{Tom}_1}\}$, $\mathtt{Tom}_2 = \{A^1_{\mathtt{Tom}_2}, A^2_{\mathtt{Tom}_2}, A^3_{\mathtt{Tom}_2}\}$, and $\mathtt{Tom}_3 = \{A^1_{\mathtt{Tom}_3}, A^2_{\mathtt{Tom}_3}, A^3_{\mathtt{Tom}_3}\}$. Then the structure shown by (10.3.24) can be schematized as in Figure 10.3.4 below. $\square$

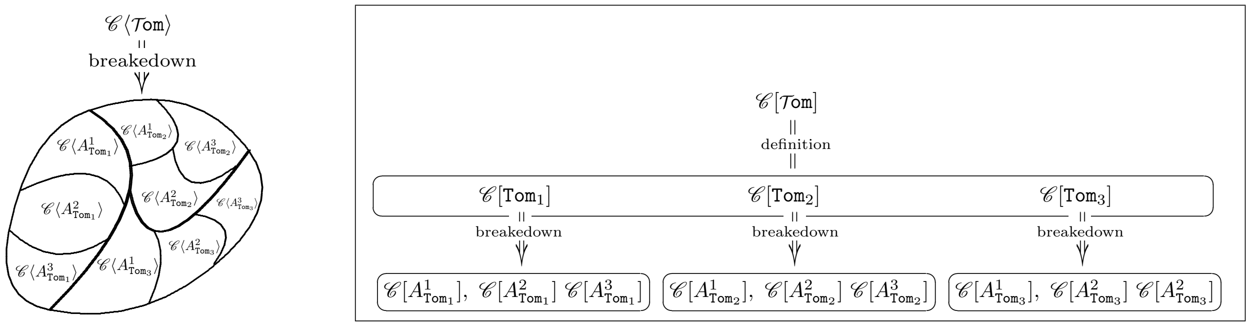

Figure 10.3.4: Breakedown of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i = 1, 2, 3$ and $j = 1, 2, 3$

What is depicted in Figure 10.3.4 above implies that the whole of the statements “$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$” (see (10.3.11)) prescribed on the whole $\mathcal{T}\mathtt{om}$ are aggregated into one statement “$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ over $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$”, schematized as

$$\text{“}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\text{” on } \mathcal{T}\mathtt{om} \Rightarrow \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds over } \mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle. \tag{10.3.26}$$

10.3.6 Completeness of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$

Throughout the paper let the condition-space $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ be constructed so as to become equal to the total-space $\mathscr{P}\times\mathscr{F}$ (see Def. 4.4.1(p.16)), i.e.,

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}. \tag{10.3.27}$$

Here note that (10.3.27) is not what should be proven but what should be satisfied. Then the equality is called the *completeness* of $\mathcal{T}\mathtt{om}$. Due to (10.3.27) we can rewrite Figure 10.3.4 as below.

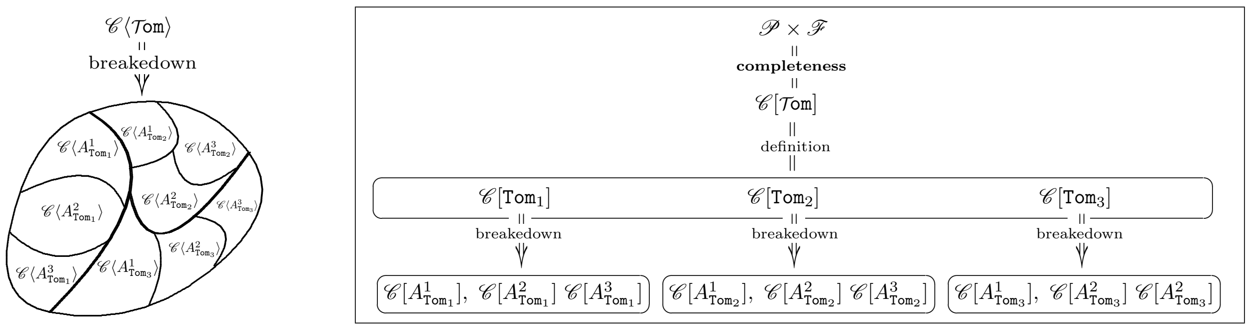

Figure 10.3.5: The completeness of the breakedown of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i = 1, 2, 3$ and $j = 1, 2, 3$

Under the *completeness* of $\mathcal{T}\mathtt{om}$ we can rewrite (10.3.26) as follows:

$$\text{“}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\text{ holds on }\mathscr{C}\langle A_{\mathtt{Tom}}\rangle\text{” on }\mathcal{T}\mathtt{om} \;\Rightarrow\; \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\text{ holds over }\mathscr{P}\times\mathscr{F}. \tag{10.3.28}$$

Let us refer to the relation as the *completeness* of $\mathcal{T}\mathtt{om}$. □

Alice **6 (unknown-box)** *Completely attaining the completeness of a condition-branching may become hard in the sense that it may not be always possible to explicitly specify and describe an assertion in all terminal points of the break-down-process of its condition-branching. Then, Alice faced the question "If so, what should be done ?" and hesitated for a while. Then, Dr. Rabbit appeared again and told to her "Put there the unknown-box* **?** *representing that it is left, as a subject of future study, to clarify and examine what phenomenon or event is hidden there.".* □

Breakdown scenario ends here

Chapter 11

Symmetry Theorem $(\mathbb{R} \leftrightarrow \tilde{\mathbb{R}})$

In this chapter we construct the methodology which derives $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model).

11.1 Two Kinds of Equality

11.1.1 Correspondence Equality

For $\boldsymbol{\xi}$, a, μ, b, $T(x)$, etc., which are all dependent on a given distribution function $F \in \mathscr{F}$, let us define $\hat{\boldsymbol{\xi}} = -\boldsymbol{\xi}$, $\hat{a} = -a$, $\hat{\mu} = -\mu$, $\hat{b} = -b$, $\hat{T}(x) = -T(x)$ respectively, called the *reflection operation* $\mathcal{R}$. For any given distribution function $F \in \mathscr{F}$, i.e.,

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} \subseteq \mathscr{F}, \tag{11.1.1}$$

let us define the distribution function of $\hat{\boldsymbol{\xi}}$ by $\check{F}$, i.e.,

$$\check{F}(\xi) \overset{\text{def}}{=} \Pr\{\hat{\boldsymbol{\xi}} \le \xi\}, \tag{11.1.2}$$

where its probability density function is represented by $\check{f}$ and the set of all possible $\check{F}$'s is denoted by $\check{\mathscr{F}}$, i.e.,

$$\check{\mathscr{F}} \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}\}. \tag{11.1.3}$$

Now, since $\check{\check{F}}(\xi) = \Pr\{\hat{\hat{\boldsymbol{\xi}}} \le \xi\}$ for any ξ due to the definition (11.1.2) and since

$$\hat{\hat{\boldsymbol{\xi}}} = \widehat{-\boldsymbol{\xi}} = -(-\boldsymbol{\xi}) = \boldsymbol{\xi}, \tag{11.1.4}$$

we have $\check{\check{F}}(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = F(\xi)$ for any ξ due to (11.1.1), i.e.,

$$\check{\check{F}} \equiv F. \tag{11.1.5}$$

For any subset $\mathscr{F}' \subseteq \mathscr{F}$ let us define

$$\check{\mathscr{F}}' \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}'\}. \tag{11.1.6}$$

Then, since $\check{\mathscr{F}}' \subseteq \mathscr{F}$, by definition we have

$$\check{\check{\mathscr{F}}}' = \{\check{\check{F}} \mid \check{F} \in \check{\mathscr{F}}'\}. \tag{11.1.7}$$

Here, due to (11.1.5) we get

$$\check{\check{\mathscr{F}}}' = \{F \mid \check{F} \in \check{\mathscr{F}}'\}. \tag{11.1.8}$$

If $F \in \mathscr{F}'$, then $\check{F} \in \check{\mathscr{F}}'$ from (11.1.6), hence $\check{\check{F}} \in \check{\check{\mathscr{F}}}'$ from (11.1.7), so that $F \in \check{\check{\mathscr{F}}}'$ due to (11.1.5); accordingly, we have $\mathscr{F}' \subseteq \check{\check{\mathscr{F}}}' \cdots (*)$. If $F \in \check{\check{\mathscr{F}}}'$, then $\check{F} \in \check{\mathscr{F}}'$ from (11.1.8), hence $F \in \mathscr{F}'$ from (11.1.6), therefore, we have $\check{\check{\mathscr{F}}}' \subseteq \mathscr{F}'$. From this and $(*)$ it follows that

$$\check{\check{\mathscr{F}}}' = \mathscr{F}'. \tag{11.1.9}$$

By $\check{a}$, $\check{\mu}$, and $\check{b}$ let us denote respectively the lower bound, the expectation, and the upper bound of $\check{F} \in \check{\mathscr{F}}$ corresponding to any given $F \in \mathscr{F}$ with the lower bound a, the expectation μ, and the upper bound b. Then, for any ξ we clearly have (see Figure 11.1.1 below)

$$f(\xi) = \check{f}(\hat{\xi}) \tag{11.1.10}$$

where

$$\hat{a} = \check{b}, \quad \hat{\mu} = \check{\mu}, \quad \hat{b} = \check{a}. \tag{11.1.11}$$

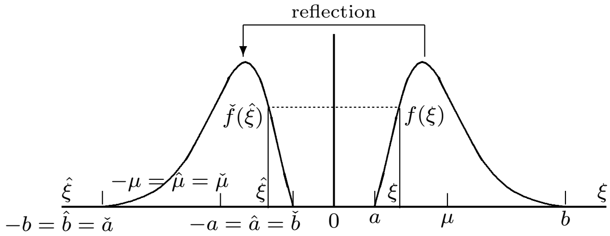

Figure 11.1.1: Relationship between probability density functions f and $\check{f}$

11.1.2 Identity Equality

Lemma 11.1.1

(a) *$\mathscr{F}$ and $\check{\mathscr{F}}$ are one-to-one correspondent where $\mathscr{F} = \check{\mathscr{F}}$.*
(b) *For any $\check{F} \in \check{\mathscr{F}}$ there exists a $F \in \mathscr{F}$ which is identical to the $\check{F}$, i.e., $F \equiv \check{F}$.*[†]
(c) *For any $F \in \mathscr{F}$ there exists a $\check{F} \in \check{\mathscr{F}}$ which is identical to the F, i.e., $\check{F} \equiv F$.*

Proof If $F \in \mathscr{F}$, then $\check{F} \in \check{\mathscr{F}}$ by definition, i.e., $F \in \mathscr{F} \Rightarrow \check{F} \in \check{\mathscr{F}} \cdots$ **(1)**. Conversely, if $\check{F} \in \check{\mathscr{F}}$, then F from which $\check{F} \in \check{\mathscr{F}}$ is defined is clearly an element of $\mathscr{F}$ due to (11.1.3), i.e., $F \in \mathscr{F}$, i.e., $\check{F} \in \check{\mathscr{F}} \Rightarrow F \in \mathscr{F} \cdots$ **(2)**.

(a) First, for any $F \in \mathscr{F}$ and for the $\check{F} \in \check{\mathscr{F}}$ corresponding to the F we have

$$\begin{aligned}\check{F}(\xi) &= \Pr\{\hat{\boldsymbol{\xi}} \le \xi\} = \Pr\{-\hat{\hat{\boldsymbol{\xi}}} \le -\hat{\xi}\} = \Pr\{\hat{\hat{\boldsymbol{\xi}}} \ge \hat{\xi}\} = \Pr\{\boldsymbol{\xi} \ge \hat{\xi}\} \quad \text{(due to (11.1.4))} \\ &= 1 - \Pr\{\boldsymbol{\xi} < \hat{\xi}\} = 1 - \Pr\{\boldsymbol{\xi} \le \hat{\xi}\}^{\ddagger} = 1 - F(\hat{\xi}) \cdots \mathbf{(3)}.\end{aligned}$$

Suppose any $F \in \mathscr{F}$ yields the two different $\check{F}_1 \in \check{\mathscr{F}}$ and $\check{F}_2 \in \check{\mathscr{F}}$, hence there exists at least one ξ' such that $\check{F}_1(\xi') \ne \check{F}_2(\xi')$. Then, since $\check{F}_1(\xi') = 1 - F(\hat{\xi}')$ and $\check{F}_2(\xi') = 1 - F(\hat{\xi}')$ due to (3), we have the contradiction of $\check{F}_1(\xi') = \check{F}_2(\xi')$, hence the $F \in \mathscr{F}$ must correspond to a *unique* $\check{F} \in \check{\mathscr{F}}$. Next, for any $\check{F} \in \check{\mathscr{F}}$ and for $F \in \mathscr{F}$ from which $\check{F} \in \check{\mathscr{F}}$ is defined we have

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = \Pr\{-\hat{\boldsymbol{\xi}} \le -\hat{\xi}\} = \Pr\{\hat{\boldsymbol{\xi}} \ge \hat{\xi}\} = 1 - \Pr\{\hat{\boldsymbol{\xi}} < \hat{\xi}\}^{\ddagger} = 1 - \Pr\{\hat{\boldsymbol{\xi}} \le \hat{\xi}\} = 1 - \check{F}(\hat{\xi}) \cdots \mathbf{(4)}.$$

Suppose any $\check{F} \in \check{\mathscr{F}}$ is yielded by the two different $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}$, hence there exists at least one ξ' such that $F_1(\xi') \ne F_2(\xi')$. Then, since $F_1(\xi') = 1 - \check{F}(\hat{\xi}')$ and $F_2(\xi') = 1 - \check{F}(\hat{\xi}')$ due to (4), we have the contradiction of $F_1(\xi') = F_2(\xi')$, hence the $\check{F} \in \check{\mathscr{F}}$ must correspond to a unique $F \in \mathscr{F}$. Thus, the former half of the assertion is true. The latter half can be proven as follows. First, consider any $F \in \check{\mathscr{F}}$. Then, since $F \in \mathscr{F}$ by definition, we have $\check{\mathscr{F}} \subseteq \mathscr{F} \cdots$ **(5)**. Next, consider any $F \in \mathscr{F}$. Then, since $\check{F} \in \check{\mathscr{F}}$ due to (1), we have $\check{F} \in \mathscr{F}$ due to (5). Hence $\check{\check{F}} \in \check{\mathscr{F}}$ due to (1), so that $F \in \check{\mathscr{F}}$ due to (11.1.5), thus we have $\mathscr{F} \subseteq \check{\mathscr{F}}$. From this and (5) we have $\check{\mathscr{F}} = \mathscr{F} \cdots$ **(6)**.

(b) Consider any $\check{F} \in \check{\mathscr{F}} \cdots$ **(7)**, hence $\check{F} \in \mathscr{F} \cdots$ **(8)** due to (6). Suppose every $F \in \mathscr{F}$ is not identical to the $\check{F}$, i.e., $F \not\equiv \check{F}$, implying that the $\check{F}$ cannot become an element of $\mathscr{F}$, i.e., $\check{F} \notin \mathscr{F}$, which contradicts (8). Hence, it must be that $F \equiv \check{F}$, thus it follows that the assertion holds.

(c) Consider any $F \in \mathscr{F} \cdots$ **(9)**, hence $F \in \check{\mathscr{F}} \cdots$ **(10)** due to (6). Suppose every $\check{F} \in \check{\mathscr{F}}$ is not identical to the F, i.e., $\check{F} \not\equiv F$, implying that the F cannot become an element of $\check{\mathscr{F}}$, i.e., $F \notin \check{\mathscr{F}}$, which contradicts (10). Hence, it must be that $\check{F} \equiv F$, thus it follows that the assertion holds. ∎

From the identity $F \equiv \check{F}$ in Lemma 11.1.1(b,c) we have

$$f(\xi) \equiv \check{f}(\xi), \tag{11.1.12}$$

called the *identity equality.*

11.2 Definitions of Underlying Functions

11.2.1 $\check{T}$, $\check{L}$, $\check{K}$, $\check{\mathcal{L}}$, and $\check{\kappa}$ of Type $\mathbb{R}$

Let us define the underlying functions of Type $\mathbb{R}$ (see Section 5.1.1(p.17)) *for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$* as follows.

$$\check{T}(x) = \check{\mathbf{E}}[\max\{\boldsymbol{\xi} - x, 0\}] = \textstyle\int_{-\infty}^{\infty} \max\{\xi - x, 0\}\check{f}(\xi)d\xi, \tag{11.2.1}$$

$$\check{L}(x) = \lambda\beta\check{T}(x) - s, \tag{11.2.2}$$

$$\check{K}(x) = \lambda\beta\check{T}(x) - (1-\beta)x - s, \tag{11.2.3}$$

$$\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\check{\mu} - s). \tag{11.2.4}$$

Let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ respectively if they exist. If multiple solutions exist for each of $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$, let us employ the *smallest* as its solution (see (a) of Section 5.2(p.19)). Let us define

$$\check{\kappa} = \lambda\beta\check{T}(0) - s. \tag{11.2.5}$$

[†]This means $F(x) = \check{F}(x)$ for all $x \in (-\infty, \infty)$.
[‡]Due to the assumption of F being continuous (see A8)

By $\check{\mathsf{M}}$:1[ℝ][A] let us define M:1[ℝ][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for M:1[ℝ][A] we can express SOE{$\check{\mathsf{M}}$:1[ℝ][A]} as (see Table 6.5.1(p.31) (I))

$$\texttt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} - s,\ V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{11.2.6}$$

11.2.2 $\tilde{\check{T}}$, $\tilde{\check{L}}$, $\tilde{\check{K}}$, $\tilde{\check{\mathcal{L}}}$, and $\tilde{\check{\kappa}}$ of $\tilde{\text{T}}$ype ℝ

Let us define the underlying functions of $\tilde{\text{T}}$ype ℝ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ as follows.

$$\tilde{\check{T}}(x) = \check{\mathbf{E}}[\min\{\boldsymbol{\xi} - x, 0\}] = \textstyle\int_{-\infty}^{\infty} \min\{\xi - x, 0\}\check{f}(\xi)d\xi, \tag{11.2.7}$$

$$\tilde{\check{L}}(x) = \lambda\beta\tilde{\check{T}}(x) + s, \tag{11.2.8}$$

$$\tilde{\check{K}}(x) = \lambda\beta\tilde{\check{T}}(x) - (1-\beta)x + s, \tag{11.2.9}$$

$$\tilde{\check{\mathcal{L}}}(s) = \tilde{\check{L}}(\lambda\beta\check{\mu} + s). \tag{11.2.10}$$

Let the solutions of $\tilde{\check{L}}(x) = 0$, $\tilde{\check{K}}(x) = 0$, and $\tilde{\check{\mathcal{L}}}(s) = 0$ be denoted by $x_{\tilde{\check{L}}}$, $x_{\tilde{\check{K}}}$, and $s_{\tilde{\check{\mathcal{L}}}}$ respectively if they exist. If multiple solutions exist for each of $x_{\tilde{\check{L}}}$, $x_{\tilde{\check{K}}}$, and $s_{\tilde{\check{\mathcal{L}}}}$, let us employ the *largest* as its solution (see (b) of Section 5.2(p.19)). Let us define

$$\tilde{\check{\kappa}} = \lambda\beta\tilde{\check{T}}(0) + s. \tag{11.2.11}$$

By $\tilde{\check{\mathsf{M}}}$:1[ℝ][A] let us define $\tilde{\mathsf{M}}$:1[ℝ][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\tilde{\mathsf{M}}$:1[ℝ][A] we can express SOE{$\tilde{\check{\mathsf{M}}}$:1[ℝ][A]} as (see Table 6.5.1(p.31) (II))

$$\texttt{SOE}\{\tilde{\check{\mathsf{M}}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} + s,\ V_t = \min\{\tilde{\check{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{11.2.12}$$

11.2.3 List of the Underline Functions of Type ℝ and $\tilde{\text{T}}$ype ℝ

So far we have defined the four kinds of underlying functions, which may cause confusions to readers. To give a clearer picture of these functions, we shall rearrange them as in Table 11.2.1.

Table 11.2.1: List of the underlying functions of Type ℝ and $\tilde{\text{T}}$ype ℝ

Type ℝ	$\tilde{\text{T}}$ype ℝ
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \int_a^b \max\{\xi - x, 0\}f(\xi)d\xi$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta\mu - s)$ See Section 5.1.1(p.17)	$\check{T}(x) = \int_a^b \max\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{\mu} - s)$ See Section 11.2.1(p.56)
$\tilde{T}(x) = \int_a^b \min\{\xi - x, 0\}f(\xi)d\xi$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta\mu + s)$ See Section 5.1.2(p.17)	$\tilde{\check{T}}(x) = \int_a^b \min\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\tilde{\check{L}}(x) = \beta\tilde{\check{T}}(x) + s$ $\tilde{\check{K}}(x) = \beta\tilde{\check{T}}(x) - (1-\beta)x + s$ $\tilde{\check{\mathcal{L}}}(x) = \tilde{\check{L}}(\beta\check{\mu} + s)$ See Section 11.2.2(p.57)

11.3 Two Kinds of Replacements

11.3.1 Correspondence Replacement

The lemma below is used in Step 3 of Scenario[ℝ] (p.60).

Lemma 11.3.1 **($\mathcal{C}_{\mathbb{R}}$)** *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}$, $\hat{\mu} = \check{\mu}$, $\hat{b} = \check{a}$.
(c) $\hat{T}(x) = \tilde{\check{T}}(\hat{x})$.
(d) $\hat{L}(x) = \tilde{\check{L}}(\hat{x})$.
(e) $\hat{K}(x) = \tilde{\check{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \tilde{\check{\mathcal{L}}}(s)$.
(g) $\hat{x}_L = x_{\tilde{\check{L}}}$.
(h) $\hat{x}_K = x_{\tilde{\check{K}}}$.
(i) $s_{\mathcal{L}} = s_{\tilde{\check{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \tilde{\check{\kappa}}$. □

Proof (a) The same as (11.1.10).

(b) The same as (11.1.11(p.55)).

(c) The function $T(x)$ for any F (see (5.1.2(p.17))) can be rewritten as

$$\begin{aligned} T(x) &= \int_{-\infty}^{\infty} \max\{-\hat{\xi}+\hat{x},0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi}-\hat{x},0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi}-\hat{x},0\} \check{f}(\hat{\xi}) d\xi \quad \text{due to (a).} \end{aligned}$$

Let $\eta \stackrel{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then, we have

$$\begin{aligned} T(x) &= \int_{\infty}^{-\infty} \min\{\eta-\hat{x},0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\eta-\hat{x},0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\xi-\hat{x},0\} \check{f}(\xi) d\xi \quad \text{(without loss of generality}^{\dagger}\text{)} \\ &= -\check{\tilde{T}}(\hat{x}) \quad \text{(see (11.2.7)),} \end{aligned}$$

hence $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.

(d) From (5.1.3(p.17)) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\tilde{T}}(\hat{x}) - s = -\check{\tilde{L}}(\hat{x})$ from (11.2.8(p.57)), hence $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.

(e) From (5.1.4(p.17)) and (c) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\tilde{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\tilde{K}}(\hat{x})$ from (11.2.9(p.57)), hence $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.

(f) From (5.1.5(p.17)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta\mu - s)$, hence from (d) we obtain $\mathcal{L}(s) = -\check{\tilde{L}}(\widehat{\lambda\beta\mu - s}) = -\check{\tilde{L}}(-\lambda\beta\mu + s) = -\check{\tilde{L}}(\lambda\beta\hat{\mu} + s) = -\check{\tilde{L}}(\lambda\beta\check{\mu} + s)$ due to (b). Accordingly, from (11.2.10(p.57)) we obtain $\mathcal{L}(s) = -\check{\tilde{\mathcal{L}}}(s)$, hence $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $\hat{L}(x_L) = 0$, which can be rewritten as $\check{\tilde{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\tilde{L}}(x) = 0$ has the solution $x_{\check{\tilde{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\hat{K}(x_K) = 0$, which can be rewritten as $\check{\tilde{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\tilde{K}}(x) = 0$ has the solution $x_{\check{\tilde{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, which can be rewritten as $\check{\tilde{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\tilde{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\tilde{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.6(p.17)) we have $\kappa = -\lambda\beta\hat{T}(0) - s$, which can be rewritten as $\kappa = -\lambda\beta\check{\tilde{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\tilde{T}}(0) - s = -\check{\tilde{\kappa}}$ from (11.2.11(p.57)), thus $\hat{\kappa} = \check{\tilde{\kappa}}$. ∎

Definition 11.3.1 (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 11.3.1 by its right-hand the *correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$. □

The lemma below is used in S̃tep 3 of S̃cenario[$\mathbb{R}$] (p.69).

Lemma 11.3.2 ($\tilde{\mathcal{C}}_{\mathbb{R}}$) *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{b} = \check{a}, \quad \hat{\mu} = \check{\mu}, \quad \hat{a} = \check{b}$.
(c) $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.
(d) $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.
(e) $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.
(f) $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.
(g) $\hat{x}_{\tilde{L}} = x_{\check{L}}$.
(h) $\hat{x}_{\tilde{K}} = x_{\check{K}}$.
(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.
(j) $\hat{\tilde{\kappa}} = \check{\kappa}$. □

Proof (a) The same as (11.1.10).

(b) The same as (11.1.11(p.55)).

(c) The function $\tilde{T}(x)$ for any F (see (5.1.12(p.17))) can be rewritten as

$$\begin{aligned} \tilde{T}(x) &= \int_{-\infty}^{\infty} \min\{-\hat{\xi}+\hat{x},0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\} \check{f}(\hat{\xi}) d\xi \quad \text{(due to (a)).} \end{aligned}$$

Let $\eta = \hat{\xi} = -\xi$. Then, since $d\eta = -d\xi$, we have

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†The mere replacement of the symbol η by ξ.

$$\begin{aligned}\tilde{T}(x) &= \int_{\infty}^{-\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta\\ &= -\int_{-\infty}^{\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta\\ &= -\int_{-\infty}^{\infty} \max\{\xi-\hat{x},0\}\check{f}(\xi)d\xi \quad \text{(without loss of generality}^{\dagger}\text{)}\\ &= -\check{T}(\hat{x}) \quad \text{(see (11.2.1))},\end{aligned}$$

hence $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(d) From (5.1.13(p.17)) and (c) we have $\tilde{L}(x) = -\lambda\beta\hat{\tilde{T}}(x) + s = -\lambda\beta\check{T}(\hat{x}) + s = -\check{L}(\hat{x})$ from (11.2.2(p.56)), hence $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) From (5.1.14(p.17)) and (c) we have $\tilde{K}(x) = -\lambda\beta\hat{\tilde{T}}(x) + (1-\beta)\hat{x} + s = -\lambda\beta\check{T}(\hat{x}) + (1-\beta)\hat{x} + s = -\check{K}(\hat{x})$ from (11.2.3(p.56)), hence $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) From (5.1.15(p.17)) we have $\tilde{\mathcal{L}}(s) = -\hat{\tilde{L}}(\lambda\beta\mu + s)$, hence from (d) we obtain $\tilde{\mathcal{L}}(s) = -\check{L}(\widehat{\lambda\beta\mu + s}) = -\check{L}(-\lambda\beta\mu - s) = -\check{L}(\lambda\beta\hat{\mu} - s) = -\check{L}(\lambda\beta\check{\mu} - s)$ due to (b). Accordingly, from (11.2.4(p.56)) we obtain $\tilde{\mathcal{L}}(s) = -\check{\mathcal{L}}(s)$, hence $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, which can be rewritten as $\check{L}(\hat{x}_{\tilde{L}}) = 0$ from (d), implying that $\check{L}(x) = 0$ has the solution $x_{\check{L}} = \hat{x}_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $\hat{\tilde{K}}(x_{\tilde{K}}) = 0$, which can be rewritten as $\check{K}(\hat{x}_{\tilde{K}}) = 0$ from (e), implying that $\check{K}(x) = 0$ has the solution $x_{\check{K}} = \hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $\hat{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, which can be rewritten as $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), implying that $\check{\mathcal{L}}(s) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.16(p.17)) we have $\tilde{\kappa} = -\lambda\beta\hat{\tilde{T}}(0) + s$, which can be rewritten as $\tilde{\kappa} = -\lambda\beta\check{T}(\hat{0}) + s$ from (c), hence $\tilde{\kappa} = -\lambda\beta\check{T}(0) + s = -\check{\kappa}$ from (11.2.5(p.56)), thus $\hat{\tilde{\kappa}} = \check{\kappa}$. ∎

Definition 11.3.2 (correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 11.3.2 by its right-hand the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{R}}$. □

Definition 11.3.3 (reflective element and non-reflective element) It should be noted that the left-hand of each of the equalities in Lemmas 11.3.1(i) and 11.3.2(i) have not the hat symbol " ˆ ". In other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reflection. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reflective element* and to each of all the other elements as the *reflective element*. □

11.3.2 Identity Replacement

The two lemmas are used in Step 4 of Scenario[ℝ] (p.60).

Lemma 11.3.3 ($\mathcal{I}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right-hand side is for $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots [1^*]$.*[†]

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ.
(b) $\check{a} = a,\ \check{\mu} = \mu,\ \check{b} = b$.
(c) $\check{\tilde{T}}(x) = \tilde{T}(x)$.
(d) $\check{\tilde{L}}(x) = \tilde{L}(x)$.
(e) $\check{\tilde{K}}(x) = \tilde{K}(x)$.
(f) $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$.
(g) $x_{\check{\tilde{L}}} = x_{\tilde{L}}$.
(h) $x_{\check{\tilde{K}}} = x_{\tilde{K}}$.
(i) $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$.
(j) $\check{\tilde{\kappa}} = \tilde{\kappa}$. □

Proof (a) Clear from [1*].

(b) Immediate from (a).

(c) Evident from (11.2.7(p.57)), (5.1.12(p.17)), and [3*].

(d) Immediate from (11.2.8(p.57)), (5.1.13(p.17)), and (c).

(e) Immediate from (11.2.9(p.57)), (5.1.14(p.17)), and (c).

(f) Immediate from (11.2.10(p.57)), (5.1.15(p.17)), (d), and $\check{\mu} = \mu$ due to (b).

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $\check{\tilde{L}}(x_{\tilde{L}}) = 0$ from (d), hence $x_{\check{\tilde{L}}} = x_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $\check{\tilde{K}}(x_{\tilde{K}}) = 0$ from (e), hence $x_{\check{\tilde{K}}} = x_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $\check{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), hence $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) Immediate from (11.2.11(p.57)), (5.1.16(p.17)), and (c) with $x = 0$. ∎

[†] The mere replacement of the symbol η by ξ.
[†] See Lemma 11.1.1(b).

Definition 11.3.4 (identity replacement operation $\mathcal{I}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand side of each equality in Lemma 11.3.3 by its right-hand side the *identity replacement operation* $\mathcal{I}_{\mathbb{R}}$. □

The lemma below is used in $\tilde{\text{S}}$tep 4 of $\tilde{\text{S}}$cenario[ℝ] (p.69).

Lemma 11.3.4 ($\tilde{\mathcal{I}}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right-hand side is for $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots [1^*]$.*[†]

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ.
(b) $\check{a} = a,\ \check{\mu} = \mu,\ \check{b} = b$.
(c) $\check{T}(x) = T(x)$.
(d) $\check{L}(x) = L(x)$.
(e) $\check{K}(x) = K(x)$.
(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$.
(g) $x_{\check{L}} = x_L$.
(h) $x_{\check{K}} = x_K$.
(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$.
(j) $\check{\kappa} = \kappa$. □

Proof (a) Clear from $[1^*]$.

(b) Immediate from (a).

(c) Evident from (11.2.1(p.56)), (5.1.2(p.17)), and $[3^*]$.

(d) Immediate from (11.2.2(p.56)), (5.1.3(p.17)), and (c).

(e) Immediate from (11.2.3(p.56)), (5.1.4(p.17)), and (c).

(f) Immediate from (11.2.4(p.56)), (5.1.5(p.17)), (d), and $\check{\mu} = \mu$ due to (b).

(g) Since $L(x_L) = 0$ by definition, we have $\check{L}(x_L) = 0$ from (d), hence $x_{\check{L}} = x_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\check{K}(x_K) = 0$ from (d), hence $x_{\check{K}} = x_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\check{\mathcal{L}}(s_{\mathcal{L}}) = 0$ from (f), hence $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$ by definition.

(j) Immediate from (11.2.5(p.56)), (5.1.6(p.17)), and (c) with $x = 0$. ∎

Definition 11.3.5 (identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 11.3.4 by its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{R}}$. □

11.4 Attribute Vector

Closely looking into the contents of all assertions $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \in \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, we can immediately see that each assertion is stated by using a part of the following twelve kinds of elements;

$$a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t$$

where V_t represents the sequence $\{V_t,\ t = 1, 2, \cdots\}$ generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. Let us call each element the *attribute element* and the vector of them the *attribute vector*, denoted by

$$\boldsymbol{\theta}(A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t). \tag{11.4.1}$$

In addition, also for the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ we can employ the similar definition, denoted by

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t). \tag{11.4.2}$$

11.5 Scenario[ℝ]

In this section we write up a scenario of constructing a methodology which derives an assertion on $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (*buying model*) from a given assertion on $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ (*selling model*). Let us refer to this as the scenario of Type ℝ, denoted by Scenario[ℝ].

■ Step 1 (opening)

∘ The system of optimality equations of $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ is given by Table 6.5.1(p.31) (I), i.e.,

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu - s,\ V_t = \max\{K(V_{t-1}) + V_{t-1},\ \beta V_{t-1}\},\quad t > 1\}. \tag{11.5.1}$$

∘ Let us consider an assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$[†] in $\mathtt{Tom}$'s 10.2.1(p.47) or 10.2.2, which can be written as

$$\begin{aligned} A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} &= \{\mathsf{S} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\} \quad \text{(see (10.3.8(p.50)))} && (11.5.2)\\ &= \{\mathsf{S} \text{ is true on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\} \quad \text{(see (10.3.10(p.50)))}. && (11.5.3)\end{aligned}$$

[†]See Def. 10.3.2(p.50) for the symbol "$\mathtt{Tom}$" in $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.

To facilitate the understanding of the discussion that follows, let us use the following example.‡

$$\mathsf{S} \;=\; \langle V_t - \; s_{\mathcal{L}} \; + \; x_L \; + \kappa + a + \mu + b \geq 0, \;\; t > 0 \,\rangle. \tag{11.5.4}$$

- The attribute vector of the assertion $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (11.4.1), i.e.,

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; (a, \mu, b, \; x_L\,, \, x_K\,, \; s_{\mathcal{L}}\,, \kappa, T, L\,, K\,, \mathcal{L}\,, V_t). \tag{11.5.5}$$

■ Step **2** (reflection operation $\mathcal{R}$)

- Applying the reflection operation $\mathcal{R}$ (see Section 11.1.1(p.55)) to (11.5.1) produces

$$\begin{aligned}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\; -\hat{V}_t = \max\{-\hat{K}\,(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\;\; t > 1\} \\ &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\; -\hat{V}_t = -\min\{\hat{K}\,(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\;\; t > 1\} \\ &= \{\hat{V}_1 = \beta\hat{\mu} + s,\; \hat{V}_t = \min\{\hat{K}\,(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\;\; t > 1\}.\end{aligned} \tag{11.5.6}$$

- Applying $\mathcal{R}$ to (11.5.3) yields to

$$\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{R}[\mathsf{S}] \text{ is true on } \mathscr{C}\langle A_{\texttt{Tom}}\rangle\}. \tag{11.5.7}$$

For our example we have:

$$\begin{aligned}\mathcal{R}[\mathsf{S}] \;&=\; \langle\, -\hat{V}_t - \; s_{\mathcal{L}} \; - \hat{x}_L - \hat{\kappa} - \hat{a} - \hat{\mu} - \hat{b} \geq 0, \;\; t > 0\,\rangle^{\S} \\ &=\; \langle\, \hat{V}_t + \; s_{\mathcal{L}} \; + \; \hat{x}_L \; + \hat{\kappa} + \hat{a} + \hat{\mu} + \hat{b} \leq 0, \;\; t > 0\,\rangle.\end{aligned} \tag{11.5.8}$$

- The attribute vector of the assertion $\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (11.5.5), i.e.,

$$\boldsymbol{\theta}(\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]) \;\overset{\text{def}}{=}\; \mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{11.5.9}$$

$$=\; (\hat{a}, \hat{\mu}, \hat{b}, \hat{x}_L\,, \hat{x}_K\,, \; s_{\mathcal{L}}\,, \hat{\kappa}, \hat{T}, \hat{L}\,, \hat{K}\,, \hat{\mathcal{L}}\,, \hat{V}_t). \tag{11.5.10}$$

■ Step **3** (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$).

- Herein let us consider the application of the correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$. By definition, this means the replacement of the left-hand side of each equality in Lemma 11.3.1(p.57) ,

$$f(\xi), \hat{a}, \; \hat{\mu}, \; \hat{b}, \; \hat{x}_L\,, \; \hat{x}_K\,, \;\; s_{\mathcal{L}}\,, \; \hat{\kappa}, \; \hat{T}(x), \; \hat{L}\,(x), \; \hat{K}\,(x), \; \hat{\mathcal{L}}\,(s)\cdots(1^*),$$

by its right-hand,

$$\check{f}(\check{\xi}), \check{b}, \; \check{\mu}, \; \check{a}, \;\; x_{\check{\check{L}}}, \; x_{\check{\check{K}}}, \;\; s_{\check{\check{\mathcal{L}}}}, \; \check{\check{\kappa}}, \; \check{\check{T}}\,(\hat{x}), \; \check{\check{L}}\,(\hat{x}), \; \check{\check{K}}\,(\hat{x}), \; \check{\check{\mathcal{L}}}\,(s)\cdots(2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ corresponding to the $F \in \mathscr{F}$.

- Applying $\mathcal{C}_{\mathbb{R}}$ to (11.5.6) leads to

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\check{\mu} + s,\; \hat{V}_t = \min\{\check{\check{K}}\,(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\;\; t > 1\}. \tag{11.5.11}$$

- Applying $\mathcal{C}_{\mathbb{R}}$ to $\mathcal{R}[\mathsf{S}]$ in (11.5.8) means the replacement of each attribute element within $\mathcal{R}[\mathsf{S}]$ with its correspondent one in (2^*). For our example we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle\, \hat{V}_t + \; s_{\check{\check{\mathcal{L}}}} \; + \; x_{\check{\check{L}}} + \check{\check{\kappa}} + \check{b} + \check{\mu} + \check{a} \leq 0, \;\; t > 0\,\rangle. \tag{11.5.12}$$

Let us note that the replacement performed by the application of $\mathcal{C}_{\mathbb{R}}$ inevitably changes

the condition "$F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$"

included in $\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ (see (11.5.7)) into

the condition "$\check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}$ *corresponding to* $F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$".

Hence we have

$$\begin{aligned}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \;=\; \{&\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \text{ and} \\ &\check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}} \text{ corresponding to } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}\end{aligned}$$

where

$$\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} = \{\check{F} \mid F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\} \subseteq \{\check{F} \mid F \in \mathscr{F}\} = \check{\mathscr{F}} \qquad \text{(see (11.1.3(p.55)))}. \tag{11.5.13}$$

Now, since the phrase "$\check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}$" is *implicitly* accompanied with the phrase "*corresponding to* $F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$", the latter phrase becomes redundant. Accordingly, $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ can be rewritten as

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \;=\; \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \tag{11.5.14}$$

$$=\; \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\;\} \tag{11.5.15}$$

where

$$\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle \;\overset{\text{def}}{=}\; \{\boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \quad \text{(see Def. 10.3.1(p.50))}. \tag{11.5.16}$$

‡The example is a hypothetical assertion which is not contained in $\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$; It is used merely for explanatory convenience.

§Note Def. 11.3.3(p.59) .

Here note that $\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle$ (see (11.5.16)) is derived from $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$ (see (10.3.9(p.50))); in other words, if $(\boldsymbol{p}, F)$ is included in $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$, then $(\boldsymbol{p}, \check{F})$ is included in $\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle$, schematized as

$$(\boldsymbol{p}, F) \in \mathscr{C}\langle A_{\texttt{Tom}}\rangle \Rightarrow (\boldsymbol{p}, \check{F}) \in \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle. \quad \square \tag{11.5.17}$$

- The attribute vector of $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{C}_{\mathbb{R}}$ to (11.5.9), i.e.,

$$\boldsymbol{\theta}(\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]) = \mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{11.5.18}$$

$$= (\check{b}, \check{\mu}, \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}}, \check{\tilde{\kappa}}, \check{\tilde{T}}, \check{\tilde{L}}, \check{\tilde{K}}, \check{\tilde{\mathcal{L}}}, V_t). \tag{11.5.19}$$

■ Step **4** (identity replacement operation $\mathcal{I}_{\mathbb{R}}$).

- Herein let us consider the application of the identity replacement operation $\mathcal{I}_{\mathbb{R}}$. By definition, this means the replacement of the left-hand side of each equality in Lemma 11.3.3(p.59),

$$\check{f}(\xi),\ \check{a},\ \check{\mu},\ \check{b},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}}(x),\ \check{\tilde{L}}(x),\ \check{\tilde{K}}(x),\ \check{\tilde{\mathcal{L}}}(s) \cdots (1^*),$$

by its right-hand side,

$$f(\xi),\ a,\ \mu,\ b,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa}\ \tilde{T}(x),\ \tilde{L}(x),\ \tilde{K}(x),\ \tilde{\mathcal{L}}(s) \cdots (2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ which is identical to the $F \in \mathscr{F}$, i.e., $\check{F} \equiv F \cdots \mathbf{(1)}$.

- Applying $\mathcal{I}_{\mathbb{R}}$ to (11.5.11) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\mu + s,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}. \tag{11.5.20}$$

Now, we have $\hat{V}_1 = \beta\mu + s = V_1$ from (6.5.3(p.31)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, since $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$ from (6.5.4(p.31)), by induction $\hat{V}_t = V_t$ for $t > 0$. Thus (11.5.20) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}, \tag{11.5.21}$$

which is the same as $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.1(p.31) (II)). Thus we have

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\ \}] \tag{11.5.22}$$

$$= \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1\}. \tag{11.5.23}$$

- Applying $\mathcal{I}_{\mathbb{R}}$ to (11.5.15) yields (note the identity $\check{F} \equiv F$ in $\mathbf{(1)}$)

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \}. \tag{11.5.24}$$

Applying $\mathcal{I}_{\mathbb{R}}$ to (11.5.12) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle V_t +\ s_{\tilde{\mathcal{L}}}\ +\ x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ \ t > 0\rangle. \tag{11.5.25}$$

Now V_t within $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}]$ is generated from $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (11.5.24) can be regarded as the assertion on $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (see Remark 6.1.1(p.21)). Thus, we have

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \tag{11.5.26}$$

$$= \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\}, \tag{11.5.27}$$

- The attribute vector of $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\mathcal{I}_{\mathbb{R}}$ to (11.5.19), i.e.,

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{11.5.28}$$

$$= (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t), \tag{11.5.29}$$

■ Step **5** (symmetry transformation operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$)

Below let us line up the attribute vectors given by the four steps that have been discussed so far:

$$\begin{array}{lllr}
\text{Step 11.5:} & \boldsymbol{\theta}(\,a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t\,) & (\leftarrow\ (11.5.5)) & \\
& \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & & \\
\text{Step 11.5:} & \boldsymbol{\theta}(\,\hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t\,) & (\leftarrow\ (11.5.10)) & \\
& \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & & (11.5.30) \\
\text{Step 11.5:} & \boldsymbol{\theta}(\,\check{b},\ \check{\mu},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ \hat{V}_t\,) & (\leftarrow\ (11.5.19)) & \\
& \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & & \\
\text{Step 11.5:} & \boldsymbol{\theta}(\,b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t\,) & (\leftarrow\ (11.5.29)) &
\end{array}$$

The above flow transforming $\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})$ in Step 11.5 into $\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})$ in Step 11.5 can be eventually reduced to the operation

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \stackrel{\text{def}}{=} \left\{\begin{matrix} a, \mu, b, & x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t \\ \downarrow\ \downarrow\ \downarrow & \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ b, \mu, a, & x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t \end{matrix}\right\}, \tag{11.5.31}$$

called the *symmetry transformation operation*, which can be regarded as the successive application "$\mathcal{I}_{\mathbb{R}} \leftarrow \mathcal{C}_{\mathbb{R}} \leftarrow \mathcal{R}$" of the three operations, so we have

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}. \tag{11.5.32}$$

Then (11.5.26) can be rewritten as

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \tag{11.5.33}$$

$$= \{\tilde{\mathsf{S}} \text{ holds for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \quad (\text{see } (10.3.8(\text{p.50}))) \tag{11.5.34}$$

$$= \{\tilde{\mathsf{S}} \text{ holds on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\} \quad (\text{see } (10.3.10(\text{p.50}))) \tag{11.5.35}$$

where

$$\tilde{\mathsf{S}} \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathsf{S}]. \tag{11.5.36}$$

For our example we have

$$\tilde{\mathsf{S}} = \langle V_t + s_{\tilde{\mathcal{L}}} + x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ t > 0\rangle. \tag{11.5.37}$$

Then (11.5.22) can be rewritten as

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{11.5.38}$$

In addition, (11.5.29) can be rewritten as

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{11.5.39}$$

$$= (b, \mu, a, x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{11.5.40}$$

From all the above we see that Scenario[$\mathbb{R}$] starts with (11.5.3) and ends up with (11.5.35), which can be rewritten as respectively

$$A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\texttt{Tom}}\rangle, \tag{11.5.41}$$

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle. \tag{11.5.42}$$

Accordingly, it follows that Scenario[$\mathbb{R}$] starting with (11.5.41) and ending up with (11.5.42) can be eventually stated as Lemma 11.5.1 below.

Lemma 11.5.1 *Let* $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$*. Then* $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle$ *where*

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \ \square \tag{11.5.43}$$

■ Step 6 (Completeness of $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathrm{om}\rangle$)

Aggregation scenario begins from here

□ Applying Lemma 11.5.1 to each assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model) included in `Tom`'s 10.2.1(p.47) and 10.2.2 produces $A\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model), which will be given by `Tom` 11.7.1(p.69) and `Tom` 11.7.2.

Definition 11.5.1 In order to avoid the confusion that follows we sometimes rewrite `Tom` 11.7.1 and `Tom` 11.7.2 as $\tilde{\texttt{Tom}}_1$ and $\tilde{\texttt{Tom}}_2$ respectively; here let $\tilde{\texttt{Tom}} = \tilde{\texttt{Tom}}_1, \tilde{\texttt{Tom}}_2$; in general, $\tilde{\texttt{Tom}} = \tilde{\texttt{Tom}}_1, \tilde{\texttt{Tom}}_2, \cdots$. □

Definition 11.5.2 (condition-space of $\mathrm{A}_{\tilde{\texttt{Tom}}}$) Let us define

$$\check{\mathscr{C}}\langle A_{\tilde{\texttt{Tom}}}\rangle \stackrel{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\tilde{\texttt{Tom}}}} \subseteq \mathscr{P}, F \in \check{\mathscr{F}}_{A_{\tilde{\texttt{Tom}}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \quad (\text{see } (11.5.16(\text{p.61}))). \tag{11.5.44}$$

called the *condition-space of* $\mathrm{A}_{\tilde{\texttt{Tom}}}$. □

Let us define

$$\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}\rangle \stackrel{\text{def}}{=} \cup_{A\in\tilde{\texttt{Tom}}}\check{\mathscr{C}}\langle A_{\tilde{\texttt{Tom}}}\rangle. \tag{11.5.45}$$

Here let us define

$$\tilde{\mathcal{T}}\mathrm{om} \stackrel{\text{def}}{=} \{\tilde{\texttt{Tom}}\} = \{\tilde{\texttt{Tom}}_1, \tilde{\texttt{Tom}}_2, \cdots\} \quad (\text{see } (10.3.22(\text{p.52}))). \tag{11.5.46}$$

Then let us consider the following example, corresponding to Example 10.3.5(p.52).

Example **11.5.1** Let $\tilde{\mathcal{T}}\mathrm{om} = \{\tilde{\texttt{Tom}}_1, \tilde{\texttt{Tom}}_2, \tilde{\texttt{Tom}}_3\}$ where $\tilde{\texttt{Tom}}_1 = \{A^1_{\tilde{\texttt{Tom}}_1}, A^2_{\tilde{\texttt{Tom}}_1}, A^3_{\tilde{\texttt{Tom}}_1}\}$, $\tilde{\texttt{Tom}}_2 = \{A^1_{\tilde{\texttt{Tom}}_2}, A^2_{\tilde{\texttt{Tom}}_2}, A^3_{\tilde{\texttt{Tom}}_2}\}$, and $\tilde{\texttt{Tom}}_3 = \{A^1_{\tilde{\texttt{Tom}}_3}, A^2_{\tilde{\texttt{Tom}}_3}, A^3_{\tilde{\texttt{Tom}}_3}\}$, so $\tilde{\texttt{Tom}} = \{A^1_{\tilde{\texttt{Tom}}}, A^2_{\tilde{\texttt{Tom}}}, A^3_{\tilde{\texttt{Tom}}}\}$ as a general form. □

Now, the structure of the aggregation given by the equality (11.5.45) can be schematized as in Figure 11.5.1 below:

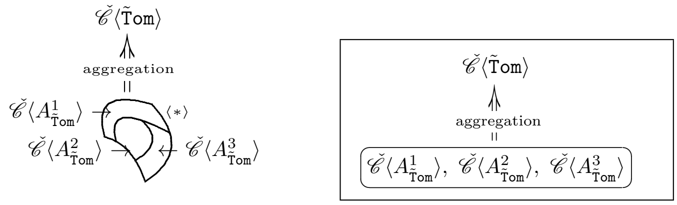

Figure 11.5.1: Aggregation of $\check{\mathscr{C}}\langle A^1_{\tilde{\texttt{Tom}}}\rangle$, $\check{\mathscr{C}}\langle A^2_{\tilde{\texttt{Tom}}}\rangle$,$\check{\mathscr{C}}\langle A^3_{\tilde{\texttt{Tom}}}\rangle$ into $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}\rangle$

Noting that the symbol $\texttt{Tom}$ is replaced by $\tilde{\texttt{Tom}}$ by definition, we see that the symbol $A_{\texttt{Tom}}$ in Lemma 11.5.1 is also replaced by $A_{\tilde{\texttt{Tom}}}$, hence the lemma can be rewritten as Corollary 11.5.1 below.

Corollary 11.5.1 *Let* $A_{\tilde{\texttt{Tom}}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\tilde{\texttt{Tom}}}\rangle$*. Then* $A_{\tilde{\texttt{Tom}}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\tilde{\texttt{Tom}}}\rangle$ *where*

$$A_{\tilde{\texttt{Tom}}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \ = \ \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\tilde{\texttt{Tom}}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}]. \ \square \tag{11.5.47}$$

□ Let us define

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle \ \stackrel{\text{def}}{=} \ \cup_{\tilde{\texttt{Tom}}\in\tilde{\mathcal{T}}\texttt{om}}\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}\rangle \tag{11.5.48}$$

$$= \ \cup_{\tilde{\texttt{Tom}}\in\tilde{\mathcal{T}}\texttt{om}}\cup_{A\in\tilde{\texttt{Tom}}}\check{\mathscr{C}}\langle A_{\tilde{\texttt{Tom}}}\rangle \quad \text{(see (10.3.24(p.52))).} \tag{11.5.49}$$

The structure of the definition given by (11.5.49) can be schematized as in Figure 11.5.2 below (see Figure 10.3.3(p.52)) where the small deformed circle $\langle * \rangle$ in Figure 11.5.1 is what is given by the deformed circle $\langle * \rangle$ in Figure 11.5.2 below. Here the big deformed circle of the left-hand side consist of the three small deformed circles including $\langle * \rangle$.

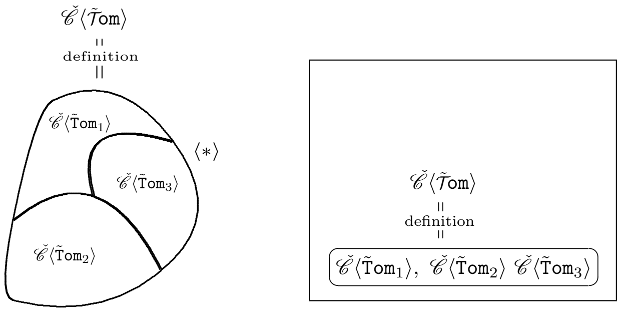

Figure 11.5.2: Definition of $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle$ which consists of $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_1\rangle$, $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_2\rangle$, and $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_3\rangle$

□ Consider here again Example 11.5.1 where $\tilde{\texttt{Tom}}_1 = \{A^1_{\tilde{\texttt{Tom}}_1}, A^2_{\tilde{\texttt{Tom}}_1}, A^3_{\tilde{\texttt{Tom}}_1}\}$, $\tilde{\texttt{Tom}}_2 = \{A^1_{\tilde{\texttt{Tom}}_2}, A^2_{\tilde{\texttt{Tom}}_2}, A^3_{\tilde{\texttt{Tom}}_2}\}$, and $\tilde{\texttt{Tom}}_3 = \{A^1_{\tilde{\texttt{Tom}}_3}, A^2_{\tilde{\texttt{Tom}}_3}, A^3_{\tilde{\texttt{Tom}}_3}\}$. Then, mingling $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_1\rangle$, $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_2\rangle$, and $\check{\mathscr{C}}\langle\tilde{\texttt{Tom}}_3\rangle$ together (see Figure 11.5.1 and the <u>definition</u> (11.5.49)) yields Figure 11.5.3 below.

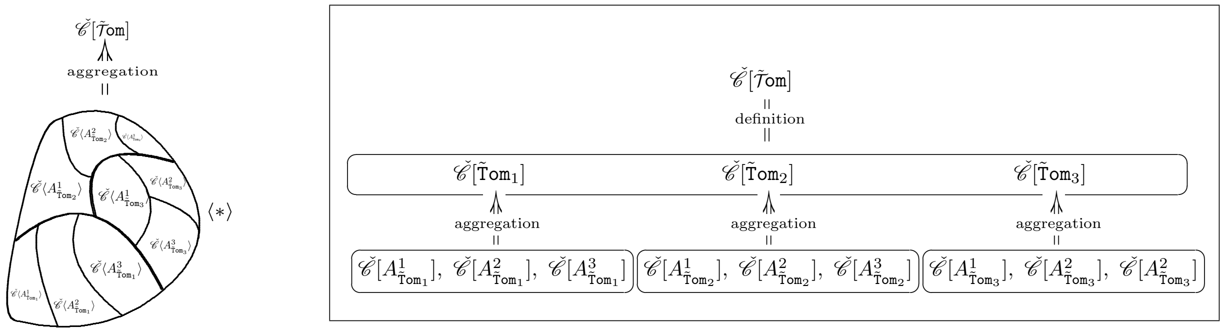

Figure 11.5.3: Aggregation of $\check{\mathscr{C}}\langle A^j_{\tilde{\texttt{Tom}}_i}\rangle$ for $i, j = 1, 2, 3$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle$

Since Figure 11.5.3 above means the aggregation of the nine condition-spaces $\check{\mathscr{C}}\langle A^1_{\tilde{\texttt{Tom}}_i}\rangle$, $\check{\mathscr{C}}\langle A^2_{\tilde{\texttt{Tom}}_i}\rangle$, and $\check{\mathscr{C}}\langle A^3_{\tilde{\texttt{Tom}}_i}\rangle$ for $i = 1, 2, 3$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle$; in other words, aggregating all statements "$A_{\tilde{\texttt{Tom}}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ holds on $\tilde{\mathcal{T}}\texttt{om}$" produces the statement "$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ holds over $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle$", i.e.,

$$\text{“}A_{\tilde{\texttt{Tom}}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\tilde{\texttt{Tom}}}\rangle\text{ ” on } \tilde{\mathcal{T}}\texttt{om} \ \Rightarrow \ \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \text{ holds over } \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle. \tag{11.5.50}$$

From (10.3.26(p.52)) and (11.5.50) it eventually follows that Corollary 11.5.1(p.64) can be aggregated into Corollary 11.5.2 below.

Corollary 11.5.2 (symmetry theorem ($\mathbb{R}\to\tilde{\mathbb{R}}$)) *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle\mathcal{T}\texttt{om}\rangle$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\texttt{om}\rangle$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \ = \ \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}]. \ \square \tag{11.5.51}$$

☐ Quasi-completeness of $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ We have the lemma below:

Lemma 11.5.2 *We have*

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}. \quad \square \tag{11.5.52}$$

Proof For any given $\check{F} \in \check{\mathscr{F}}$ there exists a $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots \mathbf{(1)}$ (see Lemma 11.1.1(p.56) (b)). Similarly, for any given $F \in \mathscr{F}$ there exists a $\check{F} \in \check{\mathscr{F}}$ such that $\check{F} \equiv F \cdots \mathbf{(2)}$ (see Lemma 11.1.1(p.56) (c)). First, from (11.5.44(p.63)) we have $\check{\mathscr{C}}\langle A_{\tilde{\mathtt{T}}\mathtt{om}}\rangle \subseteq \{\boldsymbol{p} \in \mathscr{P}, \check{F} \in \check{\mathscr{F}}\}$, hence due to (2) we get $\check{\mathscr{C}}\langle A_{\tilde{\mathtt{T}}\mathtt{om}}\rangle \subseteq \{\boldsymbol{p} \in \mathscr{P}, F \in \check{\mathscr{F}}\} = \mathscr{P} \times \check{\mathscr{F}}$. Accordingly, from (11.5.49) we obtain $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \subseteq \cup_{\tilde{\mathtt{T}}\mathtt{om}\in\tilde{\mathcal{T}}\mathtt{om}} \cup_{A\in\tilde{\mathtt{T}}\mathtt{om}} \mathscr{P} \times \check{\mathscr{F}} = \mathscr{P} \times \check{\mathscr{F}} \cdots \mathbf{(3)}$. Next, consider any given $(\boldsymbol{p}, F) \in \mathscr{P} \times \check{\mathscr{F}} \cdots \mathbf{(4)}$. Now, since $\mathscr{P} \times \check{\mathscr{F}} \subseteq \mathscr{P} \times \mathscr{F}$, we have $(\boldsymbol{p}, F) \in \mathscr{P} \times \mathscr{F}$ from (4). Then, due to the completeness of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ (see Section 10.3.6(p.53)) we see that $(\boldsymbol{p}, F) \in \mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ for at least one $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$, hence $(\boldsymbol{p}, \check{F}) \in \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ due to (11.5.17(p.62)), so $(\boldsymbol{p}, F) \in \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ due to (1) or equivalently $(\boldsymbol{p}, F) \in \check{\mathscr{C}}\langle A_{\tilde{\mathtt{T}}\mathtt{om}}\rangle$ (see Def. 11.5.1(p.63)), hence $(\boldsymbol{p}, F) \in \check{\mathscr{C}}\langle\tilde{\mathtt{T}}\mathtt{om}\rangle$ from (11.5.45(p.63)), hence $(\boldsymbol{p}, F) \in \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ due to (11.5.49(p.64)). Accordingly, it follows from (4) that $\mathscr{P} \times \check{\mathscr{F}} \subseteq \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \cdots \mathbf{(5)}$. Finally, from (3) and (5) we obtain $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}$. ∎

Let us refer to the equality (11.5.52) as the *quasi-completeness of* $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$. Noting (11.5.52), we can finally rewrite Figure 11.5.3 as Figure 11.5.4 below.

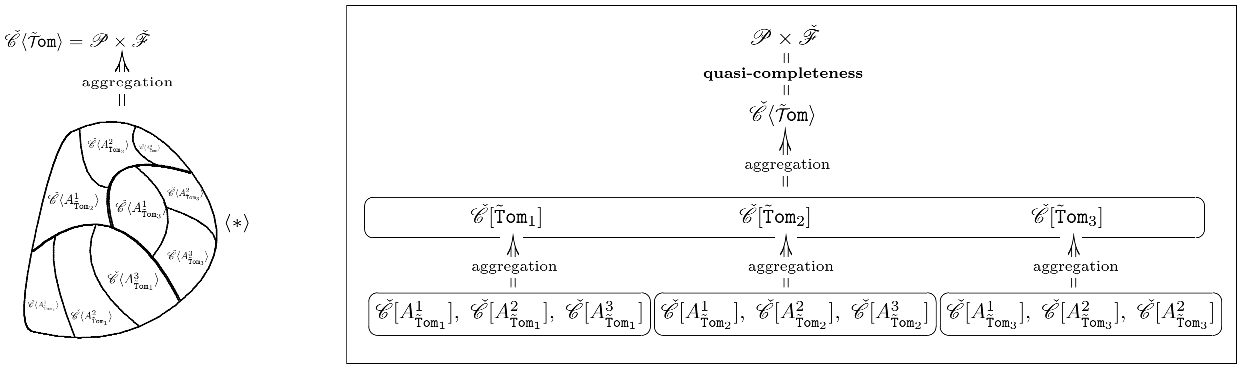

Figure 11.5.4: Aggregation of $\check{\mathscr{C}}\langle A^j_{\tilde{\mathtt{T}}\mathtt{om}_i}\rangle$ for $i, j = 1, 2, 3$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}$

From (10.3.27(p.53)) and (11.5.52) it follows that Corollary 11.5.2(p.64) can be rewritten as Corollary 11.5.1 below.

Corollary 11.5.3 *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \check{\mathscr{F}}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{11.5.53}$$

Aggregation scenario ends here

☐ Completeness of $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$: Moreover, since $\check{\mathscr{F}} = \mathscr{F}$ in Lemma 11.1.1(p.56) (a), we can rewrite (11.5.52(p.65)) as follows.

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}. \tag{11.5.54}$$

Let us refer to the equality (11.5.54) as the *completeness of* $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$. Then, we can rewrite Figure 11.5.4 as Figure 11.5.5 below.

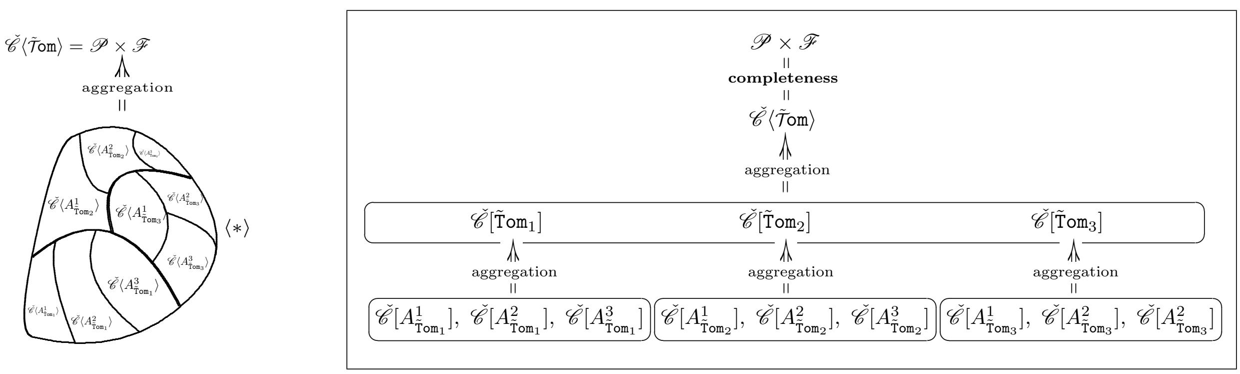

Figure 11.5.5: Aggregation of $\check{\mathscr{C}}\langle A^j_{\tilde{\mathtt{T}}\mathtt{om}_i}\rangle$ for $i, j = 1, 2, 3$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$

■ Step **7** (symmetry theorem ($\mathbb{R} \to \tilde{\mathbb{R}}$))

Finally, due to (11.5.54) we can rewrite Corollary 11.5.3 as Theorem 11.5.1 below.

Theorem 11.5.1 (symmetry theorem ($\mathbb{R} \to \tilde{\mathbb{R}}$)) *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{11.5.55}$$

Then, clearly the attribute vector of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ becomes as follows (see (11.5.39))

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{11.5.56}$$

$$=\; (b, \mu, a,\ x_{\tilde{\mathcal{L}}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \tag{11.5.57}$$

■ Step **8** (**summary of** Scenario[$\mathbb{R}$])

The symmetry transformation operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ is not the operation of *merely* replacing each symbol in the upper row of (11.5.31(p.63)) by its corresponding symbol in the lower row. Since the operation is related to the operation $\mathcal{R}$, each of the plus sign, the minus sign, and the direction of inequality appearing within the description of the original assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is reversed in the assertion $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ derived from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$. Now, at a glance, the operation seems to be rather complicated; however, it can be simply prescribed as follows.

- Firstly, reflect all the *reflective* elements (see Defs 11.3.3(p.59)) appearing within the description of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 10.2.1(p.47) and 10.2.2).
- Next, replace each of all the reflected elements, whether reflective or non-reflective, with the right side of its corresponding equality in Lemma 11.3.1(p.57).
- Then, remove the check sign "ˇ" from all the *replaced* symbols. □

11.6 Derivation of $\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$, and $\tilde{\kappa}_{\mathbb{R}}$

To begin with, let us note here the fact that Scenario[$\mathbb{R}$] is a story for an assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ which is related to the attribute vector $\boldsymbol{\theta}$, and it can be immediately seen that the scenario can be applied also to any other assertions only if it is related to the attribute vector $\boldsymbol{\theta}$. Accordingly, it can be applied also to $T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}$, and $\kappa_{\mathbb{R}}$; in other words, $\tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$ and $\tilde{\kappa}_{\mathbb{R}}$ can be derived by applying the operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to $T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}$, and $\kappa_{\mathbb{R}}$, i.e.,

$$(\tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})]. \tag{11.6.1}$$

Accordingly, we have the following:

Lemma 11.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any* $F \in \mathscr{F}$:

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *strictly increasing on* $(-\infty, b]$.
(f) $\tilde{T}(x) = \mu - x$ *on* $[b, \infty)$ *and* $\tilde{T}(x) < \mu - x$ *on* $(-\infty, b)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $\tilde{T}(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \le \min\{0, \mu - x\}$ *on* $x \in (-\infty, \infty)$.
(i) $\tilde{T}(0) = 0$ *if* $a > 0$ *and* $\tilde{T}(0) = \mu$ *if* $b < 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b > y$*, then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b > \mu$. □

Proof by symmetry The lemma, excluding (a,n), can be easily obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.1.1(p.41) as shown below.

(a) Evident from the fact that $\min\{\boldsymbol{\xi} - x, 0\}$ in (5.1.11(p.17)) is continuous on $(-\infty, \infty)$.

(b) Lemma 9.1.1(p.41) (b) can be rewritten as $A =\{T(x) \ge \tilde{T}(x')$ for $x < x'\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]{=}\{-\hat{T}(x) \ge -\hat{T}(x')$ for $-\hat{x} < -\hat{x}'\}{=}\{\hat{T}(\hat{x}) \le \hat{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] =\{\check{\tilde{T}}(\hat{x}) \le \check{\tilde{T}}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] =\{\tilde{T}(\hat{x}) \le \tilde{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A =\{\tilde{T}(x) \le \tilde{T}(x')$ for $x > x'\}$, meaning that $\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.

(c-e) Almost the same as the proof of (b)

(f) Let the former half of Lemma 9.1.1(f) can by rewritten as $A =\{T(x) = \mu - x$ for $x \le a\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]{=}\{-\hat{T}(x) = -\hat{\mu}{+}\hat{x}$ for $-\hat{x} \le -\hat{a}\}{=}\{\hat{T}(x) = \hat{\mu}{-}\hat{x}$ for $\hat{x} \ge \hat{a}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] =\{\check{\tilde{T}}(\hat{x}) = \check{\mu}{-}\hat{x}$ for $\hat{x} \ge \check{b}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this lead to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] =\{\tilde{T}(\hat{x}) = \mu - \hat{x}$ for $\hat{x} \ge b\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] =\{\tilde{T}(x) = \mu - x$ for $x \ge b\}{=}\{\tilde{T}(x) = \mu - x$ on $[b, \infty)\}$. The proof of the latter half is almost the same as the above.

(g) The former half of Lemma 9.1.1(g) can be rewritten by A ={$T(x) > 0$ for $x < b$}. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]$ ={$-\hat{T}(x) > 0$ for $-\hat{x} < -\hat{b}$}={$\hat{T}(x) < 0$ for $\hat{x} > \hat{b}$}, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\check{\hat{T}}(\hat{x}) < 0$ for $\hat{x} > \check{a}$}. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\tilde{T}(\hat{x}) < 0$ for $\hat{x} > a$}. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\tilde{T}(x) < 0$ for $x > a$}={$\tilde{T}(x) < 0$ on (a,∞)}. The proof of the latter half is almost the same as the above.

(h) Applying $\mathcal{R}$ to Lemma 9.1.1(h) yields $\mathcal{R}[A]$ ={$-\hat{T}(x) \geq \max\{0, -\hat{\mu}+\hat{x}\}$ for $-\infty < -\hat{x} < \infty$}={$\hat{T}(x) \leq \min\{0, \hat{\mu}-\hat{x}\}$ for $\infty > \hat{x} > -\infty$}, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\check{\hat{T}}(\hat{x}) \leq \min\{0, \check{\mu} - \hat{x}\}$ for $\infty > \hat{x} > -\infty$}. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\tilde{T}(\hat{x}) \leq \min\{0, \mu - \hat{x}\}$ for $\infty > \hat{x} > -\infty$}. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\tilde{T}(x) \leq \min\{0, \mu - x\}$ for $\infty > x > -\infty$}={$\tilde{T}(x) \leq \min\{0, \mu - x\}$ on $(-\infty,\infty)$}.

(i) Immediate from $\tilde{T}(0) = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}I(a \leq \boldsymbol{\xi} \leq b)]$ from (5.1.11(p.17)) and (2.1.4(p.8))).

(j,k) Almost the same as the proof of (b and c)

(l) Lemma 9.1.1(l) can be rewritten as A ={If $x < y$ and $a < y$, then $T(x) + x < T(y) + y$}. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]$={If $-\hat{x} < -\hat{y}$ and $-\hat{a} < -\hat{y}$, then $-\hat{T}(x) - \hat{x} < -\hat{T}(y) - \hat{y}$}={If $\hat{x} > \hat{y}$ and $\hat{a} > \hat{y}$, then $\hat{T}(x)\hat{x} > T(y) + \hat{y}$}, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={If $\hat{x} > \hat{y}$ and $\check{b} > \hat{y}$, then $\check{\hat{T}}(\hat{x}) + \hat{x} > \check{\hat{T}}(\hat{y}) + \hat{y}$}={If $x > y$ and $\check{b} > y$, then $\check{\hat{T}}(x) + x > \check{\hat{T}}(y) + y$}. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={If $x > y$ and $b > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$}.

(m) The former half of Lemma 9.1.1(m) can be rewritten as Let A ={$\lambda\beta T(\lambda\beta\mu - s) - s$ is nonincreasing in s}, which can be rewritten as A ={$\lambda\beta T(\lambda\beta\mu - s) - s \geq \lambda\beta T(\lambda\beta\mu - s') - s'$ for $s < s'$}. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A]$ ={$-\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) - s \geq -\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') - s'$ for $s < s'$}={$\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) + s \leq \lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') + s'$ for $s < s'$},[†] and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\lambda\beta\check{\hat{T}}(\widehat{-\lambda\beta\check{\mu} - s}) + s \leq \lambda\beta\check{\hat{T}}(\widehat{-\lambda\beta\check{\mu} - s'}) + s'$ for $s < s'$}={$\lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s) + s \leq \lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s') + s'$ for $s < s'$}. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s \leq \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'$}, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nondecreasing in s. Similarly, the latter half of Lemma 9.1.1(m) can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ ={$\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s < \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'$}, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nonincrecreasing in s.

(n) Clear from (2.1.3(p.8)). ▮

Direct proof See Section A 1.1(p.271) . ▮

We have:

$$\tilde{L}(x) \begin{cases} = \lambda\beta\mu + s - \lambda\beta x \text{ on } [b, -\infty) & \cdots (1), \\ < \lambda\beta\mu + s - \lambda\beta x \text{ on } (-\infty, b) & \cdots (2), \end{cases} \tag{11.6.2}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta\mu + s - \delta x & \text{on} \quad [b, \infty) & \cdots (1), \\ < \lambda\beta\mu + s - \delta x & \text{on} \quad (-\infty, b) & \cdots (2). \end{cases} \tag{11.6.3}$$

$$\tilde{K}(x) \begin{cases} < -(1-\beta)x + s & \text{on} \quad (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s & \text{on} \quad (-\infty, a] & \cdots (2), \end{cases} \tag{11.6.4}$$

$$\tilde{K}(x) + x \leq \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{11.6.5}$$

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta\mu + s + (1-\lambda)\beta x \text{ on} & [b, \infty) & \cdots (1), \\ \beta x + s & \text{on} \quad (-\infty, a] & \cdots (2). \end{cases} \tag{11.6.6}$$

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\, x_{\tilde{L}} \cdots (1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\, x_{\tilde{K}} \cdots (2). \tag{11.6.7}$$

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to (9.2.3(p.42))-(9.2.8). ▮

Direct proof See (A 1.1(p.272))-(A 1.6) . ▮

Lemma 11.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $\tilde{L}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{L}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{L}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) *Let* $s = 0$. *Then* $x_{\tilde{L}} = a$ *where* $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.
(e) *Let* $s > 0$.
   1. $x_{\tilde{L}}$ *uniquely exists with* $x_{\tilde{L}} > a$ *where* $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.
   2. $(\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b$. □

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.2.1(p.43) ▮

Direct proof See Lemma A 1.2(p.272) . ▮

Corollary 11.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.
(b) $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. □

[†]Note Def. 11.3.3(p.59)).

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Corollaries 9.2.1(p.43) ∎

Direct proof See Corollary A 1.1(p.273) . ∎

Lemma 11.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $\tilde{K}(x)$ *is continuous on* $(-\infty,\infty)$.
(b) $\tilde{K}(x)$ *is nonincreasing on* $(-\infty,\infty)$.
(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a,\infty)$.
(d) $\tilde{K}(x)$ *is strictly decreasing on* $(-\infty,\infty)$ *if* $\beta<1$.
(e) $\tilde{K}(x)+x$ *is nondecreasing on* $(-\infty,\infty)$.
(f) $K(x)+x$ *is strictly increasing on* $(-\infty,b]$.
(g) $\tilde{K}(x)+x$ *is strictly increasing on* $(-\infty,\infty)$ *if* $\lambda<1$.
(h) *If* $x>y$ *and* $b>y$, *then* $\tilde{K}(x)+x>\tilde{K}(y)+y$.
(i) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{K}}=a$ *where* $x_{\tilde{K}}<(\geq)\,x \Leftrightarrow \tilde{K}(x)<(=)\,0 \Rightarrow \tilde{K}(x)<(\geq)\,0$.
(j) *Let* $\beta<1$ *or* $s>0$.

1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}}<(=(>))\,x \Leftrightarrow \tilde{K}(x)<(=(>))\,0$.
2. $(\lambda\beta\mu+s)/\delta \geq (<)\, b \Leftrightarrow x_{\tilde{K}}=(<)\,(\lambda\beta\mu+s)/\delta$.
3. *Let* $\tilde{\kappa}<(=(>))\,0$. *Then* $x_{\tilde{K}}<(=(>))\,0$. □

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.2.2(p.43) . ∎

Direct proof See Lemma A 1.3(p.273) . ∎

Corollary 11.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}}<(\geq)\,x \Leftrightarrow \tilde{K}(x)<(\geq)\,0$.
(b) $x_{\tilde{K}}\leq(\geq)\,x \Rightarrow \tilde{K}(x)\leq(\geq)\,0$. □

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Corollaries 9.2.2(p.44) . ∎

Direct proof See Corollary A 1.2(p.274) . ∎

Lemma 11.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}=a$.
(b) *Let* $\beta=1$ *and* $s>0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}$.
(c) *Let* $\beta<1$ *and* $s=0$. *Then* $a<(=(>))\,0 \Leftrightarrow x_{\tilde{L}}<(=(>))\,x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(=))\,0$.
(d) *Let* $\beta<1$ *and* $s>0$. *Then* $\tilde{\kappa}<(=(>))\,0 \Leftrightarrow x_{\tilde{L}}<(=(>))\,x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(>))\,0$. □

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.2.3(p.44) . ∎

Direct proof See Lemma A 1.4(p.274) . ∎

Lemma 11.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta<1$.
(b) *Let* $\lambda\beta\mu\leq a$.

1. $x_{\tilde{L}}\geq\lambda\beta\mu+s$.
2. *Let* $s>0$ *and* $\lambda\beta<1$. *Then* $x_{\tilde{L}}>\lambda\beta\mu+s$.

(c) *Let* $\lambda\beta\mu>a$. *Then, there exists a* $s_{\tilde{\mathcal{L}}}>0$ *such that if* $s_{\tilde{\mathcal{L}}}>(\leq)\,s$, *then* $x_{\tilde{L}}<(\geq)\,\lambda\beta\mu+s$. □

Proof by symmetry Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.2.4(p.44) . ∎

Direct proof See Lemma A 1.5(p.274) . ∎

Lemma 11.6.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have*:

(a) $\tilde{\kappa}=\lambda\beta\mu+s$ *if* $b<0$ *and* $\tilde{\kappa}=s$ *if* $a>0$.
(b) *Let* $\beta<1$ *or* $s>0$. *Then* $\tilde{\kappa}<(=(>))\,0 \Leftrightarrow x_{\tilde{K}}<(=(>))\,0$. □

Proof Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Lemmas 9.3.1(p.45) . ∎

Direct proof See Lemma A 1.6(p.274) . ∎

As the whole of assertions in the above lemmas and corollaries we have the assertion system below.

$$\mathscr{A}\{\{\tilde{L}_{\mathbb{R}},\tilde{K}_{\mathbb{R}},\tilde{\mathcal{L}}_{\mathbb{R}},\tilde{\kappa}_{\mathbb{R}}\}\}=\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\{L_{\mathbb{R}},K_{\mathbb{R}},\mathcal{L}_{\mathbb{R}},\kappa_{\mathbb{R}}\}\}]. \tag{11.6.8}$$

11.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Lemma 11.7.1 ($\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$) *The optimal initiating time t_τ^* (OIT) is not subject to the influence of the symmetry transformation operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$.* □

Proof First, let us represent (7.2.10(p.35)) by $D \stackrel{\text{def}}{=} \{I_\tau^{t_\tau^*} \geq I_\tau^t \text{ for } \tau \geq t \geq t_{qd}\}\cdots(1)$, which can be rewritten as $\{\beta^{\tau-t_\tau^*}V_{t_\tau^*} \geq \beta^{\tau-t}V_t \text{ for } \tau \geq t \geq t_{qd}\}$. Next, applying $\mathcal{R}$ to this yields $\mathcal{R}[D]=\{-\beta^{\tau-t_\tau^*}\hat{V}_{t_\tau^*} \geq -\beta^{\tau-t}\hat{V}_t \text{ for } \tau \geq t \geq t_{qd}\}=\{\beta^{\tau-t_\tau^*}\hat{V}_{t_\tau^*} \leq \beta^{\tau-t}\hat{V}_t$ for $\tau \geq t \geq t_{qd}\}$. Then, even if applying $\mathcal{C}_\mathbb{R}$ to this, no change occurs, i.e., $\mathcal{C}_\mathbb{R}\mathcal{R}[D] = \{\beta^{\tau-t_\tau^*}\hat{V}_{t_\tau^*} \leq \beta^{\tau-t}\hat{V}_t \text{ for } \tau \geq t \geq t_{qd}\}$. Finally, applying $\mathcal{I}_\mathbb{R}$ to this, we have $\mathcal{I}_\mathbb{R}\mathcal{C}_\mathbb{R}\mathcal{R}[A] = \{\beta^{\tau-t_\tau^*}V_{t_\tau^*} \leq \beta^{\tau-t}V_t \text{ for } \tau \geq t \geq t_{qd}\}\cdots(2)$ where $\hat{V}_t$ changes into V_t for the reason stated below (11.5.20(p.62)). The above result means that even if $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ $(= \mathcal{I}_\mathbb{R}\mathcal{C}_\mathbb{R}\mathcal{R})$ is applied, the optimal initiating time t_τ^* in (1) is entirely inherited to t_τ^* in (2) without any change. ∎

□ **Tom 11.7.1** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

Proof by symmetry Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 10.2.1(p.47). ∎

Direct proof See Tom A 4.1(p.284). ∎

□ **Tom 11.7.2** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
(c) *Let $\beta\mu > a$.*
  1. *Let $\beta = 1$.*
     i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
     i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
     ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*
        1. *Let $\beta\mu + s \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
        2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
     iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
        1. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
        2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle}\boxed{⊛\parallel}$ *is true.* □

Proof by symmetry Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 10.2.2(p.48). ∎

Direct proof See Tom A 4.2(p.285). ∎

11.8 $\tilde{\text{S}}$cenario[$\mathbb{R}$]

In this section we write up the inverse of Scenario[$\mathbb{R}$] (p.60) which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 10.2.1(p.47) and 10.2.2) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 11.7.1(p.69) and 11.7.2). Let this scenario be represented as $\tilde{\text{S}}$cenario[$\mathbb{R}$]. For an explanatory simplicity, symbols "$\mathscr{F}$" and "$\check{\mathscr{F}}$" that were used in Scenario[$\mathbb{R}$] are all removed from discussions that will be made below.

■ $\tilde{\text{S}}$tep 1 (opening)

◦ The system of optimality equation of $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ is given by Table 6.5.1(p.31) (II), i.e.,

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\quad t > 1\}. \tag{11.8.1}$$

◦ Let us consider an assertion $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ in Tom's 11.7.1 or 11.7.2, which can be rewritten as

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\tilde{\mathsf{S}} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \text{ and } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \text{ with } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}}\} \tag{11.8.2}$$

$$= \{\tilde{\mathsf{S}} \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\} \quad (\text{see } (10.3.10(\text{p.50}))) \tag{11.8.3}$$

where

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \stackrel{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}}, F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{11.8.4}$$

To facilitate the understanding of the discussion that follows let us use the following example.

$$\tilde{\mathsf{S}} = \langle V_t + s_{\tilde{\mathcal{L}}} + x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \leq 0,\ t > 0\rangle. \tag{11.8.5}$$

◦ The attribute vector of the assertion $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (11.5.40(p.63))), i.e.,

$$\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = (b, \mu, a, x_{\tilde{L}}, x_{\tilde{K}}, s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{11.8.6}$$

■ $\tilde{\rm S}$tep **2** (reflection operation $\mathcal{R}$)

◦ Applying the reflection operation $\mathcal{R}$ to (11.8.1) produces

$$\begin{aligned}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = \min\{-\hat{\tilde{K}}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\ t>1\} \\ &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = -\max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}\} \\ &= \{\hat{V}_1 = \beta\hat{\mu} - s,\ \hat{V}_t = \max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t>1\}. \qquad (11.8.7)\end{aligned}$$

◦ Applying $\mathcal{R}$ to (11.8.3) yields to

$$\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle\ \}. \qquad (11.8.8)$$

For our example we have:

$$\begin{aligned}\mathcal{R}[\tilde{\mathsf{S}}] &= \langle -\hat{V}_t + s_{\tilde{\mathcal{L}}} - \hat{x}_{\tilde{L}} - \hat{\tilde{\kappa}} - \hat{b} - \hat{\mu} - \hat{a} \le 0,\ t>0\,\rangle \\ &= \langle\, \hat{V}_t - s_{\tilde{\mathcal{L}}} + \hat{x}_{\tilde{L}} + \hat{\tilde{\kappa}} + \hat{b} + \hat{\mu} + \hat{a} \ge 0,\ t>0\,\rangle. \qquad (11.8.9)\end{aligned}$$

◦ The attribute vector of the assertion $\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (11.8.6), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &\overset{\text{def}}{=} \mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \qquad (11.8.10) \\ &= (\hat{b}, \hat{\mu}, \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \hat{\kappa}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t). \qquad (11.8.11)\end{aligned}$$

■ $\tilde{\rm S}$tep **3** (correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$).

◦ Herein let us consider the application of the correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$. By definition, this means the replacement of the left-hand side of each equality in Lemma 11.3.2(p.58).

$$\hat{b},\ \hat{\mu},\ \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}}(x),\ \hat{\tilde{L}}(x),\ \hat{\tilde{K}}(x),\ \hat{\tilde{\mathcal{L}}}(s)\cdots(1^*)$$

by its right-hand side

$$\check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(\hat{x}),\ \check{L}(\hat{x}),\ \check{K}(\hat{x}),\ \check{\mathcal{L}}(s)\cdots(2^*)$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ corresponding to the $F \in \mathscr{F}$.

◦ Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (11.8.7) leads to

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\hat{V}_1 = \beta\check{\mu} - s,\ \hat{V}_t = \max\{\check{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t>1\}. \quad (11.8.12)$$

◦ Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to $\mathcal{R}[\tilde{\mathsf{S}}]$ in (11.8.9) means the replacement of each attribute element within $\mathcal{R}[\tilde{\mathsf{S}}]$ with its correspondent one. For our example we have

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] = \langle\, \hat{V}_t + s_{\check{\mathcal{L}}} + \check{x}_L + \check{\kappa} + \check{a} + \check{\mu} + \check{b} \le 0,\ t>0\,\rangle. \qquad (11.8.13)$$

Let us note that the replacement performed by the application of $\tilde{\mathcal{C}}_{\mathbb{R}}$ inevitably changes

the condition "*for* $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$"

included in $\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ (see (11.8.8)) into

the condition "*for* $\check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ *corresponding to any* $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ *with* $\boldsymbol{p} \in \mathscr{P}_A$".

Hence we have

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \text{ and} \qquad (11.8.14) \\ &\qquad \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \text{ corresponding to } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \text{ with } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}}\} \qquad (11.8.15)\end{aligned}$$

where

$$\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} = \{\check{F} \mid F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\} \quad \text{(see (11.1.3(p.55)))}. \qquad (11.8.16)$$

Now, since the phrase "*corresponding to* $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$" in (11.8.15) is what implicitly and inevitably accompanies the phrase "$\check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$", the former phrase becomes redundant. Accordingly $\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ can be rewritten as

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\}, \qquad (11.8.17) \\ &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \} \qquad (11.8.18)\end{aligned}$$

where

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}}, \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \qquad (11.8.19)$$

◦ The attribute vector of $\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (11.8.11), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &= \tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \qquad (11.8.20) \\ &= (\check{a}, \check{\mu}, \check{b},\ \check{x}_L,\ \check{x}_K, s_{\check{\mathcal{L}}}.\check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t). \qquad (11.8.21)\end{aligned}$$

■ $\tilde{\text{S}}$tep **4** (identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$).

∘ Herein let us consider the application of the identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$. By definition, this means the replacement of the left-hand of each equality in Lemma 11.3.4(p.60)

by its right-hand side
$$\check{F},\ \check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(x),\ \check{L}(x),\ \check{K}(x),\ \check{\mathcal{L}}(s)\cdots(1^*)$$
$$F,\ a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T(x),\ L(x),\ K(x),\ \mathcal{L}(s)\cdots(2^*)$$

where (1^*) is for any $F\in\mathscr{F}$ and (2^*) is for $\check{F}\in\check{\mathscr{F}}$ which is identical to the $F\in\mathscr{F}$, i.e., $\check{F}\equiv F\cdots(1)$.

∘ Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (11.8.12) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]=\{\hat{V}_1=\beta\mu-s,\ \hat{V}_t=\max\{K(\hat{V}_{t-1})+\hat{V}_{t-1},\beta\hat{V}_{t-1}\},\ \ t>1\}. \tag{11.8.22}$$

Now, we have $\hat{V}_1=\beta\mu-s=V_1$ from (6.5.5(p.31)). Suppose $\hat{V}_{t-1}=V_{t-1}$. Then, since $\hat{V}_t=\max\{\check{K}(V_{t-1})+V_{t-1},\beta V_{t-1}\}=V_t$ from (6.5.6(p.31)), by induction $\hat{V}_t=V_t$ for $t>0$. Thus we have

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]=\{V_1=\beta\mu-s,\ V_t=\max\{K(V_{t-1})+V_{t-1},\beta V_{t-1}\},\ \ t>1\}, \tag{11.8.23}$$

which is the same as $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.1(p.31) (I)). Thus we have

$$\begin{aligned}\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}&=\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &(11.8.24)\\ &=\{V_1=\beta\mu-s,\ V_t=\max\{K(V_{t-1})+V_{t-1},\beta V_{t-1}\},\ t>1\}. &(11.8.25)\end{aligned}$$

∘ Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (11.8.17) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]\ =\ \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]\text{ is true on }\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. \tag{11.8.26}$$

Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (11.8.13) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]=\langle V_t+\ s_{\mathcal{L}}\ +\ x_L\ +\kappa+a+\mu+b\le 0,\ \ t>0\rangle. \tag{11.8.27}$$

Now V_t within $\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]$ is generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (11.8.26) can be regarded as an assertion as to $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$. Thus, we have

$$\begin{aligned}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\ &=\ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &(11.8.28)\\ &=\ \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]\text{ is true on }\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. &(11.8.29)\end{aligned}$$

∘ The attribute vector of $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (11.8.21), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})\ &=\ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] &(11.8.30)\\ &=\ (a,\mu,b,\ x_L,\ x_K,\ s_{\mathcal{L}},\kappa,T,L,K,\mathcal{L},V_t), &(11.8.31)\end{aligned}$$

■ $\tilde{\text{S}}$tep **5** (symmetry transformation operation $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$)

Below let us line up the attribute vectors given in the four steps that have been discussed so far:

$$\begin{array}{llll}\tilde{\mathsf{S}}\mathsf{tep}\ 1: & \boldsymbol{\theta}(\ b,\ \mu,\ a, & x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},V_t\,) & (\leftarrow (11.8.6))\\ & \ \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\ \tilde{\mathsf{S}}\mathsf{tep}\ 2: & \boldsymbol{\theta}(\ \hat{b},\ \hat{\mu},\ \hat{a}, & \hat{\tilde{x}}_L,\ \hat{\tilde{x}}_K,\ s_{\hat{\tilde{\mathcal{L}}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}},\ \hat{\tilde{L}},\ \hat{\tilde{K}},\ \hat{\tilde{\mathcal{L}}},\hat{V}_t\,) & (\leftarrow (11.8.11))\\ & \ \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\ \tilde{\mathsf{S}}\mathsf{tep}\ 3: & \boldsymbol{\theta}(\ \check{a},\ \check{\mu},\ \check{b}, & x_{\check{L}},x_{\check{K}},\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T},\ \check{L},\ \check{K},\ \check{\mathcal{L}},\hat{V}_t\,) & (\leftarrow (11.8.21))\\ & \ \ \downarrow\ \downarrow\ \downarrow & \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\ \tilde{\mathsf{S}}\mathsf{tep}\ 4: & \boldsymbol{\theta}(\ a,\ \mu,\ b, & x_L,x_K,\ s_{\mathcal{L}},\kappa,\ T,\ L,\ K,\ \mathcal{L},V_t\,) & (\leftarrow (11.8.31))\end{array} \tag{11.8.32}$$

The above flow transforming $\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})$ in $\tilde{\mathsf{S}}$tep 1 into $\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})$ in $\tilde{\mathsf{S}}$tep 4 can be eventually reduced to the operation $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$ depicted below.

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}\overset{\text{def}}{=}\left\{\begin{array}{ll}b,\ \mu,\ a & x_{\tilde{L}},x_{\tilde{K}},s_{\tilde{\mathcal{L}}},\tilde{\kappa},\tilde{T},\tilde{L},\tilde{K},\tilde{\mathcal{L}},\ V_t\\ \downarrow\ \downarrow\ \downarrow & \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\\ a,\mu,\ b & x_L,x_K,s_{\mathcal{L}},\kappa,T,L,K,\mathcal{L},\ V_t\end{array}\right\}, \tag{11.8.33}$$

called the *symmetry transformation operation.* Let us define

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}\ \overset{\text{def}}{=}\ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}. \tag{11.8.34}$$

Then (11.8.28) can be rewritten as

$$\begin{aligned}A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\ &=\ \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &(11.8.35)\\ &=\ \{\mathsf{S}\text{ is true for }\boldsymbol{p}\in\mathscr{P}_{A_{\mathtt{Tom}}}\text{ and }F\in\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\} &(11.8.36)\\ &=\ \{\mathsf{S}\text{ is true on }\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \} &(11.8.37)\end{aligned}$$

where

$$\mathsf{S} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\tilde{\mathsf{S}}]. \tag{11.8.38}$$

For our example we have

$$\mathsf{S} = \langle V_t + \; s_{\mathcal{L}} \; + \; x_L \; + \kappa + a + \mu + b \leq 0, \;\; t > 0\rangle. \tag{11.8.39}$$

Then, (11.8.24) can be rewritten as

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\;\}]. \tag{11.8.40}$$

In addition, (11.5.29) can be rewritten as

$$\begin{aligned}\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] && (11.8.41)\\ &=\; (b, \mu, a, \; x_{\tilde{L}}\,, \, x_{\tilde{K}}\,, \; s_{\tilde{\mathcal{L}}}\,, \tilde{\kappa}, \tilde{T}, \tilde{L}\,, \tilde{K}\,, \tilde{\mathcal{L}}\,, V_t) && (11.8.42)\end{aligned}$$

From all the above we see that S̃cenario[ℝ] starts with (11.8.3) and ends up with (11.8.37), which can be rewritten as respectively

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle, \tag{11.8.43}$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle. \tag{11.8.44}$$

Accordingly, it follows that S̃cenario[ℝ] starting with (11.8.43) and ending up with (11.8.44) can be rewritten as Lemma 11.8.1 below.

Lemma 11.8.1 *Let $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. Then $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ where*

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\; \square \tag{11.8.45}$$

■ S̃tep **6** (aggregation)

We can construct the same procedure as in Step 11.5 (p.63).

■ S̃tep **7** (symmetry theorem $\mathbb{R} \leftarrow \tilde{\mathbb{R}}$)

Through the procedure in S̃tep 11.5 (p.63) we have the following theorem

Theorem 11.8.1 *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\; \square \tag{11.8.46}$$

Proof Immediate for the same reason as in Theorem 11.5.1(p.66). ∎

The attribute vector of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by

$$\begin{aligned}\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] && (11.8.47)\\ &=\; (a, \mu, b, \; x_L\,, \, x_K\,, \; s_{\mathcal{L}}\,, \kappa, T, L\,, K\,, \mathcal{L}\,, V_t) && (11.8.48)\end{aligned}$$

11.9 Definition of Symmetry

Thus far, the term of *symmetry* has been used in the rather intuitive nuance. In order to make our discussions that follows more clear, below let us provide its strict definition.

Definition 11.9.1

(a) Let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be assertions on models M_1 and M_2 respectively. Then, if $A\{\mathsf{M}_2\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A\{\mathsf{M}_1\}]$ and $A\{\mathsf{M}_1\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A\{\mathsf{M}_2\}]$, let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be said to be *symmetrical*, denoted by $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$. Then let us employ the expression of "M_1 and M_2 are symmetrical with respect to A".
(b) For given two assertion systems $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ which are one-to-one correspondent, if $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$ for any pair $(A\{\mathsf{M}_1\}, A\{\mathsf{M}_2\})$ where $A\{\mathsf{M}_1\} \in \mathscr{A}\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\} \in \mathscr{A}\{\mathsf{M}_2\}$, then $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ are said to be *symmetrical*, denoted by $\mathscr{A}\{\mathsf{M}_1\} \sim \mathscr{A}\{\mathsf{M}_2\}$. Then, let us employ the expression of "M_1 and M_2 are symmetrical with respect to $\mathscr{A}$".
(c) Without confusion, let us remove the phrases "with respect to A" and "with respect to $\mathscr{A}$". □

Lemma 11.9.1 *$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ are symmetrical, i.e.,*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \sim \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}. \;\; \square \tag{11.9.1}$$

Proof Immediate from (11.5.55(p.66)) and (11.8.46(p.72)). ∎

11.10 Symmetry-Operation-Free

When no change occurs even if the symmetry operation is applied to a given assertion A, the assertion is said to be *free from* the symmetry operation, called the *symmetry-operation-free assertion.*

Lemma 11.10.1 *Even if the symmetry operation is applied to the symmetry-operation-free assertion, no change occurs.* □

Proof Evident. ∎

Chapter 12

Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

12.1 Preliminary

Lemma 12.1.1 ([46,You][0054])

(a) *Let $x \geq b$. Then $z(x) = b$.*
(b) *Let $x < b$. Then $x < z(x) < b$.*
(c) *$z(x) \geq a$ for any x.* □

Proof (a) Let $x \geq b$. If $z < b$ (I), then $z < x$, hence $p(z)(z-x) < 0$ due to (5.1.29 (1) (p.18)), and if $b \leq z$ (III), then $p(z)(z-x) = 0$ due to (5.1.29 (2)). Hence $z(x)$ can be given by any $z \geq b$, thus $z(x) = b$ due to Def. 5.1.1(p.18).

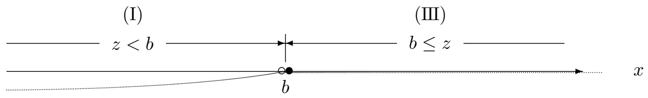

Figure 12.1.1: Case $x \geq b$

(b) Let $x < b$. If $z \leq x$ (I), then $p(z)(z-x) \leq 0$, if $x < z < b$ (II), then $p(z)(z-x) > 0$ due to (5.1.29 (1) (p.18)), and if $b \leq z$ (III), then $p(z)(z-x) = 0$ from (5.1.29 (2)). Hence, $z(x)$ is given by z such that $x < z < b$ or equivalently $x < z(x) < b$.

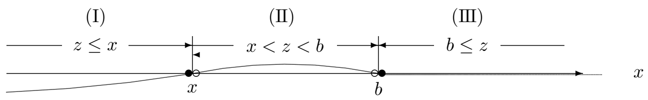

Figure 12.1.2: Case $x < b$

(c) Assume that $z(x) < a$ for a certain x. Then, since $p(z(x)) = 1 = p(a)$ due to (5.1.28 (1)), from (5.1.25(p.18)) we have $T(x) = p(z(x))(z(x)-x) = z(x) - x < a - x = p(a)(a-x) \leq T(x)$, which is a contradiction. Hence, it must be that $z(x) \geq a$ for any x. ∎

Corollary 12.1.1 ([46,You][0054]) *$a \leq z(x) \leq b$ for any x.* □

Proof Immediate from Lemma 12.1.1. ∎

Lemma 12.1.2 ([46,You][0054]) *$p(z)$ is nonincreasing on $(-\infty, \infty)$ and strictly decreasing in $z \in [a, b]$.* □

Proof The former half is immediate from (5.1.18(p.18)). Let $a \leq z' < z \leq b$. Then $p(z') - p(z) = \Pr\{z' \leq \boldsymbol{\xi}\} - \Pr\{z \leq \boldsymbol{\xi}\} = \Pr\{z' \leq \boldsymbol{\xi} < z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.1.4 (2) (p.8))), hence $p(z') > p(z)$, i.e., $p(z)$ is strictly decreasing on $[a, b]$. ∎

Lemma 12.1.3 ([46,You][0054]) *$z(x)$ is nondecreasing on $(-\infty, \infty)$.* □

Proof From (5.1.25(p.18)), for any x and y we have

$$\begin{aligned}
T(x) &= p(z(x))(z(x)-x) \\
&= p(z(x))(z(x)-y) - (x-y)p(z(x)) \\
&\leq T(y) - (x-y)p(z(x)) \\
&= p(z(y))(z(y)-y) - (x-y)p(z(x)) \\
&= p(z(y))\big(z(y) - x + (x-y)\big) - (x-y)p(z(x)) \\
&= p(z(y))(z(y)-x) + (x-y)(p(z(y)) - p(z(x))) \\
&\leq T(x) + (x-y)(p(z(y)) - p(z(x))).
\end{aligned}$$

‡This is the most important property of the function T, which was proven in [?, 0298].

Hence $0 \le (x-y)(p(z(y)) - p(z(x)))$. Let $x > y$. Then $0 \le p(z(y)) - p(z(x))$, so $p(z(x)) \le p(z(y)) \cdots$ **(1)**. Since $a \le z(x) \le b$ and $a \le z(y) \le b$ from Corollary 12.1.1, if $z(x) < z(y)$, then $p(z(x)) > p(z(y))$ from Lemma 12.1.2, which contradicts (1). Hence, it must be that $z(x) \ge z(y)$, i.e., $z(x)$ is nondecreasing in $x \in (-\infty, \infty)$. ▮

Lemma 12.1.4

(a) *$T(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$T(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$T(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$T(x) > 0$ on $(-\infty, b)$ and $T(x) = 0$ on $[b, \infty)$.*
(e) *$T(x) \ge a - x$ on $(-\infty, \infty)$.*
(f) *$T(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(g) *$\beta T(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(h) *$\beta T(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(i) *$T(x) \ge \max\{0, a - x\}$ on $(-\infty, \infty)$.*
(j) *$\lambda\beta T(\lambda\beta a - s) - s$ is nonincreasing in s and is strictly decreasing in s if $\lambda\beta < 1$.* □

Proof (a,b) Immediate from the fact that $p(z)(z-x)$ in (5.1.19(p.18)) is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any z.

(c) Let $x' < x < b$. Then $z(x) < b$ from Lemma 12.1.1(b). Accordingly, since $p(z(x)) > 0$ due to (5.1.29 (1)) and since $z(x) - x < z(x) - x'$, from (5.1.25) we have $T(x) = p(z(x))(z(x) - x) < p(z(x))(z(x) - x') \le T(x')$, implying that $T(x)$ is strictly decreasing on $(-\infty, b) \cdots\cdots$ **(1)**. Assume $T(b) = T(x)$ for a given $x < b$ or equivalently $b - x > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $b - x > 2\varepsilon > 0$ we have $b > b - \varepsilon > x + \varepsilon > x$, hence $T(b) = T(x) > T(b - \varepsilon) \ge T(b)$ due to the *strict* decreasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, since $T(x) \ne T(b)$ for any $x < b$, we have $T(x) > T(b)$ or $T(x) < T(b)$ for any $x < b$. However, the latter is impossible due to (b), hence only the former is possible. From this it must be that the former holds, hence it eventually follows that $T(x)$ is strictly decreasing on $(-\infty, b]$ instead of $(-\infty, b)$.

(d) Let $x \ge b$. Then, since $z(x) = b$ from Lemma 12.1.1(a), we have $p(z(x)) = 0$ due to (5.1.29 (2)), hence $T(x) = p(z(x))(z(x) - x) = 0$ on $[b, \infty)$. Let $x < b$. Then, from (c) we have $T(x) > T(b) = 0$, i.e., $T(x) > 0$ on $(-\infty, b)$.

(e) Since $p(a) = 1$ from (5.1.28 (1)), we have $T(x) \ge p(a)(a - x) = a - x$ for any x on $(-\infty, \infty)$.

(f) Let $x < x'$. Then, we have

$$\begin{aligned} T(x) + x &= p(z(x))(z(x) - x) + x \\ &= p(z(x))z(x) + (1 - p(z(x)))x \\ &\le p(z(x))z(x) + (1 - p(z(x)))x' \\ &= p(z(x))(z(x) - x') + x' \le T(x') + x', \end{aligned}$$

implying that $T(x) + x$ is nondecreasing on $(-\infty, \infty)$.

(g) If $\beta = 1$, then $\beta T(x) + x = T(x) + x$, hence the assertion is true from (f).

(h) Since $\beta T(x) + x = \beta(T(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Immediate from the fact that $T(x) \ge a - x$ on $(-\infty, \infty)$ from (e) and $T(x) \ge 0$ on $(-\infty, \infty)$ from (d).

(j) From (5.1.19(p.18)) we have

$$\lambda\beta T(\lambda\beta a - s) - s = \lambda\beta \max\nolimits_z p(z)(z - \lambda\beta a + s) - s = \max\nolimits_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s) - s.$$

Let $s > s'$. Then, we have

$$\begin{aligned} &\lambda\beta T(\lambda\beta a - s) - s - \lambda\beta T(\lambda\beta a - s') + s' \\ &\quad= \max\nolimits_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s) - \max\nolimits_z p(z)(\lambda\beta z - (\lambda\beta)^2 a + \lambda\beta s') - (s - s') \\ &\quad\le \max\nolimits_z p(z)(s - s')\lambda\beta - (s - s') \\ &\quad\le \max\nolimits_z (s - s')\lambda\beta - (s - s') \quad \text{(due to } p(z) \le 1 \text{ and } s - s' > 0) \\ &\quad= (s - s')\lambda\beta - (s - s') \\ &\quad= -(s - s')(1 - \lambda\beta) \le (<)\ 0 \text{ if } \lambda\beta \le (<)\ 1. \end{aligned}$$

Hence, since $\lambda\beta T(\lambda\beta a - s) - s \le (<)\ \lambda\beta T(\lambda\beta a - s') - s'$ if $\lambda\beta \le (<)\ 1$, it follows that $T(\lambda\beta a - s) - s$ is nonincreasing (strictly decreasing) in s if $\lambda\beta \le (<)\ 1$.

Let us define

$$\begin{aligned} &h(z) = p(z)(z - a)/(1 - p(z)), \quad z > a, \\ &h^\star = \sup\nolimits_{a<z} h(z), \\ &\underline{f} = \min\nolimits_{a \le w \le b} f(w) > 0 \quad \text{due to (2.1.4 (2) (p.8))}. \end{aligned}$$

Below, for a given x let us define the following successive four assertions:

$$\begin{aligned} A_1(x) &= \langle\!\langle\, z(x) > a \,\rangle\!\rangle, \\ A_2(x) &= \langle\!\langle\, T(a,x) < T(z',x,) \text{ for at least one } z' > a \,\rangle\!\rangle, \\ A_3(x) &= \langle\!\langle\, a - h(z') < x \text{ for at least one } z' > a \,\rangle\!\rangle, \\ A_4(x) &= \langle\!\langle\, \inf_{z>a}\{a - h(z)\} < x \,\rangle\!\rangle. \end{aligned}$$

Proposition 12.1.1 For any given x we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. □

Proof Letting $T(z,x) \stackrel{\text{def}}{=} p(z)(z-x)$, we can rewritten (5.1.19(p.18)) as $T(x) = \max_z T(z,x) = T(z(x),x)$.

1. Let $A_1(x)$ be true for any given x. Suppose $T(a,x) \geq T(z',x)$ for all $z' > a$, hence the maximum of $T(z,x)$ for all $z \geq a$ is attained only at $z = a$, i.e., $z(x) = a$ (see Def. 5.1.1(p.18)), which contradicts $A_1(x)$. Hence it must be that $T(a,x) < T(z',x)$ for at least one $z' > a$, thus $A_2(x)$ becomes true. Accordingly, we have $A_1(x) \Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any given x. Then, if $z(x) = a$, we have $T(a,x) < T(z',x) \leq T(x) = T(z(x),x) = T(a,x)$, which is a contradiction, hence it must be that $z(x) > a$ due to Lemma 12.1.1(c). Accordingly, we have $A_2(x) \Rightarrow A_1(x)$. Thus, it follows that $A_1(x) \Leftrightarrow A_2(x)$ for any given x.
2. Since $p(a) = 1$ from (5.1.28 (1)), for $z' > a$ (hence $1 > p(z')$ from (5.1.28 (2))) we have

$$\begin{aligned} &T(a,x) - T(z',x) \\ &\quad = p(a)(a-x) - p(z')(z'-x) \\ &\quad = a - x - p(z')(z'-x) \\ &\quad = a - x - p(z')(a - x + z' - a) \\ &\quad = a - x - p(z')(a-x) - p(z')(z'-a) \\ &\quad = (1-p(z'))(a-x) - p(z')(z'-a) \\ &\quad = (1-p(z'))\big(a - x - p(z')(z'-a)/(1-p(z'))\big) \\ &\quad = (1-p(z'))(a - x - h(z')) \\ &\quad = (1-p(z'))(a - h(z') - x). \end{aligned}$$

   Accordingly, it immediate that $A_2(x) \Leftrightarrow A_3(x)$ for any given x.
3. Let $A_3(x)$ be true for any given x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x) \Rightarrow A_4(x)$. Let $A_4(x)$ be true for any given x. Then evidently $a - \tilde{h}(z' < x$ for at least one $z' > a$, hence $A_3(x)$ is true, so we have $A_4(x) \Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x) \Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. ∎

Lemma 12.1.5

(a) $0 < h^\star < \infty$.
(b) $x^\star = a - h^\star < a$.
(c) $x^\star < (\geq)\ x \Leftrightarrow z(x) > (=)\ a$.
(d) $a^\star < a$. □

Proof (a) For any infinitesimal $\varepsilon > 0$ such that $a < b - \varepsilon < b$ ((II)) we have $0 < p(b-\varepsilon) < 1$ from (5.1.29 (1)) and (5.1.28 (2)). Hence, $h(b-\varepsilon) = p(b-\varepsilon)(b-\varepsilon-a)/(1-p(b-\varepsilon)) > 0 \cdots \mathbf{(1)}$. If $b \leq z$ ((III)), then $p(z) = 0$ due to (5.1.29 (2)), hence $h(z) = 0$ for $z \geq b$. From the above we see that $h^\star > 0 \cdots \mathbf{(2)}$ on $a < z$.

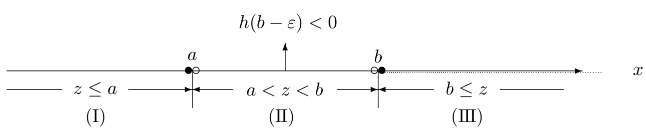

Figure 12.1.3: $h(z) = 0$ for $z \leq a$ and $h(b-\varepsilon) > 0$

Assume that $h^\star = \infty \geq 0$. Then, there exists at least one z' on $a < z' < b$ such that $h(z') \geq N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$ with any $M > 1$, we have $h(z') \geq M/\underline{f}$ or equivalently $p(z')(z'-a)/(1-p(z')) \geq M/\underline{f}$. Hence

$$p(z')(z'-a) \ \geq\ (1-p(z'))M/\underline{f} = (1 - \Pr\{z' \leq \boldsymbol{\xi}\})M/\underline{f} = \Pr\{\boldsymbol{\xi} < z'\}M/\underline{f} \cdots (*)$$

where $\Pr\{\boldsymbol{\xi} < z'\} = \int_a^{z'} f(w)dw \geq \int_a^{z'} dw \times \underline{f} = (z'-a)\underline{f}$. Accordingly, since $p(z')(z'-a) \leq (z'-a)\underline{f}M/\underline{f} = (z'-a)M$, we have $p(z') \geq M > 1$ due to $z' - a > 0$, which is a contradiction. Hence, it must follow that $h^\star < \infty$.

(b) Noting $A_1(x) \Rightarrow A_4(x)$ in Proposition 12.1.1, we can rewritten (5.1.27(p.18)) as

$$
\begin{aligned}
x^\star &= \inf\{x \mid \inf_{z>a}\{a-h(z)\} < x\} \\
&= \inf_{z>a}\{a-h(z)\} \cdots (3) \\
&= a - \sup_{a<z} h(z) = a - h^\star < a \quad \text{(due to (2))},
\end{aligned}
$$

hence (b) holds.

(c) Let $x^\star < x$, hence $\inf_{z>a}\{a-h(z)\} < x$ from (3), so $z(x) > a$ due to $A_4(x) \Rightarrow A_1(x)$. Let $x^\star \geq x$, hence $\inf_{a<z}\{a - h(z)\} \geq x$ from (3), so we have $\inf_{a<z}\{a-h(z)\} \geq x \Rightarrow z(x) \leq a$ as a contraposition of $A_1(x) \Rightarrow A_3(x)$, hence we obtain $z(x) = a$ due to Lemma 12.1.1(c).

(d) First note $T(x) \geq p(z')(z'-x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a+\varepsilon < b$ we have $T(a) \geq p(a+\varepsilon)(a+\varepsilon-a) > 0$, hence, adding a to the inequality yields $T(a)+a > a$. Thus, we have $T(x)+x \geq T(a)+a > a$ for $x \geq a$ due to Lemma 12.1.4(f). Suppose $a^\star \geq a$. Then, since $T(a^\star)+a^\star \geq T(a)+a > a$, from Lemma 12.1.4(a) we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ for any sufficiently small $\varepsilon > 0$ or equivalently $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.18))). Therefore, it must follow that $a^\star > a$. ∎

Lemma 12.1.6

(a) *$T(x)+x$ is strictly increasing on $[a^\star,\infty)$.*
(b) *$T(x) = a-x$ on $(-\infty, a^\star]$ and $T(x) > a-x$ on $(a^\star,\infty)$.*
(c) *$T(0) = a$ if $a^\star > 0$ and $T(0) = 0$ if $b < 0$.*
(d) *If $x < y$ and $a^\star < y$, then $T(x)+x < T(y)+y$.* □

Proof (a) Note here that we have

$$T(x)+x = p(z(x))(z(x)-x)+x = p(z(x))z(x)+(1-p(z(x)))x. \cdots (1)$$

- Let $x^\star < x$. Then $z(x) > a$ from Lemma 12.1.5(c4), hence $p(z(x)) < 1$ due to (5.1.28 (2)) or equivalently $1-p(z(x)) > 0$. If $x < x'$, from (1) we have

  $$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

  i.e., $T(x)+x$ is strictly increasing on $(-\infty,\infty)$, hence understandably so also on $[a^\star,\infty)$.
- Let $x^\star \geq x$. Then $z(x) = a$ from Lemma 12.1.5(c), hence $p(z(x)) = 1$ from (5.1.28 (1)), so $T(x) = p(z(x))(z(x)-x) = a-x \cdots (2)$. Suppose $a^\star < x^\star$. Then, since $a^\star < a^\star+2\varepsilon < x^\star$ for an infinitesimal $\varepsilon > 0$, we have $a^\star < a^\star+\varepsilon < x^\star-\varepsilon < x^\star$ or equivalently $x^\star > a^\star+\varepsilon$; accordingly, due to (2) we obtain $T(a^\star+\varepsilon) = a-(a^\star+\varepsilon) \cdots (3)$. Now, by definition (see (5.1.26(p.18))) we have $T(a^\star+\varepsilon) > a-(a^\star+\varepsilon)$, which contradicts (3). Accordingly, it must be that $x^\star \leq a^\star$. Let $x' > x > a^\star$. Then, since $x^\star < x$, we have $z(x) > a$ Lemma 12.1.5(c4), hence $p(z(x)) < 1$ due to (5.1.28 (2)) or equivalently $1-p(z(x)) > 0$. Thus, from (1) we have

  $$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

  implying that $\tilde{T}(x)+x$ is strictly increasing on $(a^\star,\infty)$, hence so also on $[a^\star,\infty)$ for almost the same reason as in the proof of Lemma 9.1.1(p.41) (c).

Accordingly, whether $x^\star < x$ or $x^\star \geq x$, it follows that $T(x)+x$ is strictly increasing on $[a^\star,\infty)$.

(b) By definition (see (5.1.26(p.18))) we have $T(x) > a-x$ for $x > a^\star$, i.e., $T(x) > a-x$ on $(a^\star,\infty)$. Here note that $T(x) \geq a-x$ on $(-\infty,\infty)$ due to Lemma 12.1.4(e), i.e., $T(x)+x \geq a \cdots (4)$ on $(-\infty,\infty)$. Suppose $T(a^\star)+a^\star > a$. Then, for an infinitesimal $\varepsilon > 0$ we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ due to Lemma 12.1.4(a), i.e., $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.18))). Consequently, we have $T(a^\star)+a^\star = a \cdots (5)$ or equivalently $T(a^\star) = a-a^\star$. Let $x < a^\star$. Then, from Lemma 12.1.4(f) we have $T(x)+x \leq T(a^\star)+a^\star = a$. From the result and (4) we have $T(x)+x = a$, hence $T(x) = a-x$ on $(-\infty,a^\star)$. Thus, from (5) it follows that $T(x) = a-x$ on $(-\infty,a^\star]$.

(c) Let $a^\star > 0$. Then, since $0 \in (-\infty,a^\star])$, we have $T(0) = a$ from the former half of (b). We have $T(0) = \max_z p(z)z$ from (5.1.19(p.18)). Let $b < 0$. Then, if $z \geq b$, we have $p(z)z = 0$ from (5.1.29 (2) (p.18)) and if $z < b\,(< 0)$, then $p(z)z < 0$ from (5.1.29 (1)), hence $T(0) = 0$ due to Def. 5.1.1(p.18).

(d) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $T(x)+x \leq T(a^\star)+a^\star < T(y)+y$ due to Lemma 12.1.4(f) and (a), and if $a^\star < x$, then $a^\star \leq x < y$, hence $T(x)+x < T(y)+y$ due to (a). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $T(x)+x < \tilde{T}(y)+y$. ∎

12.2 Analogy Replacement Operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$

12.2.1 Three Facts

To start with, let us focus on the three results below.

★ Fact 1 First, the following lemma can be obtained.

Lemma 12.2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty) \leftarrow$ ← Lemma 12.1.4(a)
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty) \leftarrow$ ← Lemma 12.1.4(b)
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b] \leftarrow$ ← Lemma 12.1.4(c)
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty) \leftarrow$ ← Lemma 12.1.4(f)
(e) $T(x) + x$ *is strictly increasing on* $[a^\star, \infty) \leftarrow$ ← Lemma 12.1.6(a)
(f) $T(x) = a - x$ *on* $(-\infty, a^\star]$ *and* $T(x) > a - x$ *on* $(a^\star, \infty) \leftarrow$ ← Lemma 12.1.6(b)
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty) \leftarrow$ ← Lemma 12.1.4(d)
(h) $T(x) \geq \max\{0, a - x\}$ *on* $(-\infty, \infty) \leftarrow$ ← Lemma 12.1.4(i)
(i) $T(0) = a$ *if* $a^\star > 0$ *and* $T(0) = 0$ *if* $b < 0 \leftarrow$ ← Lemma 12.1.6(c)
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1 \leftarrow$ ← Lemma 12.1.4(g)
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1 \leftarrow$ ← Lemma 12.1.4(h)
(l) *If* $x < y$ *and* $a^\star < y$, *then* $T(x) + x < T(y) + y \leftarrow$ ← Lemma 12.1.6(d)
(m) $\lambda\beta T(\lambda\beta a - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1 \leftarrow$ ← Lemma 12.1.4(j)
(n) $a^\star < a \leftarrow$ ← Lemma 12.1.5(d)

Herein we shall pay attention to the fact that replacing a and μ in [Lemma 9.1.1($\mathscr{A}\{T_{\mathbb{R}}\}$)(p.41)] by $a^\star$ and a respectively yields [Lemma 12.2.1($\mathscr{A}\{T_{\mathbb{P}}\}$)]. Let us represent this replacement by

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} = \{a \to a^\star,\ \mu \to a\}, \tag{12.2.1}$$

called the *analogy replacement operation*. In other words, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the former lemma leads to the latter lemma, i.e.,

$$\text{Lemma 12.2.1}(\mathscr{A}\{T_{\mathbb{P}}\}) \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{ Lemma 9.1.1}(\mathscr{A}\{T_{\mathbb{R}}\})]. \ \square \tag{12.2.2}$$

Remark 12.2.1 The whole description proving Lemma 9.1.1(p.41) is *quite different* from that proving Lemma 12.2.1(p.77), hence evidently no analogous relation exists at all between both descriptions. Nevertheless, what is amazing here is the fact that the whole description of Lemma 9.1.1(p.41) itself is *completely analogous* to that of Lemma 12.2.1(p.77) itself. $\square$

★ Fact 2 Next, note that replacing μ in [$\mathcal{L}(s) = L(\lambda\beta\mu - s)$] (see (5.1.5(p.17))) by a yields [$\mathcal{L}(s) = L(\lambda\beta a - s)$] (see (5.1.22(p.18))). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the characteristic vector $(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$ (see (5.1.3(p.17))-(5.1.6)) produces $(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$ (see (5.1.20(p.18))-(5.1.23)), i.e.,

$$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})]. \ \square \tag{12.2.3}$$

★ Fact 3 Finally, note that replacing μ in [$V_1 = \beta\mu - s$] (see (6.5.1(p.31))) by a yields [$V_1 = \beta a - s$] (see (6.5.5)). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the system of optimality equations SOE{M:1[ℝ][A]} (see Table 6.5.1(p.31) (I)) leads to SOE{M:1[ℙ][A]} (see Table 6.5.1(p.31) (III)), i.e.,

$$\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\} \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}]. \ \square \tag{12.2.4}$$

12.2.2 Prefiguration

By $F_{(a,\mu,b)}$ let us denote the distribution function with the lower bound a, the expectation μ, and the upper bound b ($a < \mu < b$). For convenience of reference, below let us copy (12.2.2)-(12.2.4):

$$
\begin{array}{rcll}
\text{For } F_{(a,\mu,b)} & & \text{For } F_{(a,\mu,b)} & \\
\text{Lemma 12.2.1}(\mathscr{A}\{T_{\mathbb{P}}\}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \text{Lemma 9.1.1}(\mathscr{A}\{T_{\mathbb{R}}\}) &], \\
(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) &], \\
(1^\star)^l \quad \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}\} & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}\}, \quad (1^\star)^r &].
\end{array}
$$

Closely looking at the flow of discussions in Chapter 10(p.47), we see that $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}$ is what *was* derived *only* from the three items within the right box $(1^\star)^r$ above; let us denote this procedure by Procedure[$\mathbb{R}$], and through almost quite the same reasoning it is easily seen that $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}$ is what *will be* derived from the three items within the left box $(1^\star)^l$ above; let us denote this procedure by Procedure[$\mathbb{P}$]. The flow of the above two discussions can be schematized as below.

$$
\begin{array}{rcll}
\text{For } F_{(a,\mu,b)} & & \text{For } F_{(a,\mu,b)} & \\
\text{Lemma 12.2.1}(\mathscr{A}\{T_{\mathbb{P}}\}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \text{Lemma 9.1.1}(\mathscr{A}\{T_{\mathbb{R}}\}) &], \\
(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) &], \\
(1^\star)^l \quad \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}\} & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}\}, \quad (1^\star)^r &] \\
\downarrow & & \downarrow & \\
\text{Procedure}[\mathbb{P}] & & \text{Procedure}[\mathbb{R}] & \\
\downarrow & & \downarrow & \\
(2^\star)^l \quad \mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\} & & \mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}. \quad (2^\star)^r &
\end{array}
$$

Now, since we have the analogous relation $(1^\star)^l = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(1^\star)^r]$ due to the three Facts, it can be prefigured that this analogous relation will be inherited *also* between Procedure[$\mathbb{P}$] and Procedure[$\mathbb{R}$], i.e. Procedure[$\mathbb{P}$] $= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}$[Procedure[$\mathbb{R}$]], and hence *also* between $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}$, i.e.

$$\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}]. \tag{12.2.5}$$

In other words, $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}$ can be obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}$. Thus, the above figure can be rewritten as below.

$$
\begin{array}{rcll}
\text{For } F_{(a,\mu,b)} & & \text{For } F_{(a,\mu,b)} & \\
\text{Lemma 12.2.1}(\mathscr{A}\{T_{\mathbb{P}}\}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \text{Lemma 9.1.1}(\mathscr{A}\{T_{\mathbb{R}}\}) &], \\
(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) &], \\
(1^\star)^l \quad \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\}\} & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \mathtt{SOE}\{\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}\}, \quad (1^\star)^r &] \\
\downarrow & & \downarrow & \\
\text{Procedure}[\mathbb{P}] & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \text{Procedure}[\mathbb{R}] &] \\
\downarrow & & \downarrow & \\
(2^\star)^l \quad \mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\} & = & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}. \quad (2^\star)^r &]
\end{array}
$$

Remark 12.2.2 (another prefiguration) By Procedure$[\mathbb{R}]_{(a,\mu,b)}$ let us represent Procedure[$\mathbb{R}$] for the distribution function $F_{(a,\mu,b)}$. Now, since $a^\star < a < b$ due to Lemma 12.2.1(n), we can define the distribution function F with the lower bound $a^\star$, the expectation a, and the upper bound b, i.e., $F_{(a^\star,a,b)}$, hence by definition we can properly define also Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, which is *just* what results from replacing a and μ in Procedure$[\mathbb{R}]_{(a,\mu,b)}$ by $a^\star$ and a respectively, i.e.,

$$\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}],$$

Note herein that, in the sense of the symbolic logic[†], Procedure$[\mathbb{P}]_{(a,\mu,b)}$ is *quite* the same as Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, i.e.,

$$\text{Procedure}[\mathbb{P}]_{(a,\mu,b)} \overset{\text{s-logic}}{=} \text{Procedure}[\mathbb{R}]_{(a^\star,a,b)},$$

implying that $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{E}]\}_{(a,\mu,b)}$ derived from Procedure$[\mathbb{P}]_{(a,\mu,b)}$ becomes also identical to $\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{E}]\}_{(a^\star,a,b)}$ derived from Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, i.e.,

$$\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{E}]\}_{(a,\mu,b)} \overset{\text{s-logic}}{=} \mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{E}]\}_{(a^\star,a,b)}. \quad \square$$

[†]A logic is regarded as reducing deduction to the process which transforms the expressions by representing propositions, the concept of logic, and so on with symbols such as $+$, $-$, $>$, $<$, $\vee$, $\wedge$, $\Rightarrow$, and so on (Wikipedia)

12.2.3 Strict Proof

In this section, dividing the *intuitive* prefiguration in Section 12.2.2 into several stages, we strictly prove that (12.2.5) is also *theoretically* true.

☐ More precisely, we can restate Procedure[$\mathbb{R}$], the procedure deriving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Section 10.2(p.47)), as follows. First, by applying $\mathscr{A}\{T_{\mathbb{R}}\}$ (see Lemma 9.1.1(p.41)) to the characteristic vector $(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$ consisting of (5.1.3(p.17))-(5.1.6), we obtain (9.2.3(p.42))-(9.2.8); let us denote these equalities and inequalities by $\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$. Next, by applying $\mathscr{A}\{T_{\mathbb{R}}\}$ to this $\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ we get the assertion system $\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ (see Lemmas 9.2.1(p.43) -9.3.1(p.45)). Finally, by applying the system of optimality equations $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see (I) of Table 6.5.1(p.31)) to $\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$, we get the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see `Tom`'s 10.2.1(p.47) and 10.2.2). The above flow of procedure can be schematized as below.

$$\begin{aligned}\text{Procedure}[\mathbb{R}] \;=\; \langle\!\langle\, & \mathscr{A}\{T_{\mathbb{R}}\} \Rightarrow (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\},\\ & \mathscr{A}\{T_{\mathbb{R}}\} \Rightarrow \{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\} \to \mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\} \to \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}\,\rangle\!\rangle\end{aligned}$$

☐ Applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the above flow leads to

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_{\mathbb{R}}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]\,,\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_{\mathbb{R}}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}],\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned}$$

☐ Due to (12.2.2(p.77))-(12.2.4) we can replace $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_{\mathbb{R}}\}]$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})]$, and $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]$ in the above flow by $\mathscr{A}\{T_{\mathbb{P}}\}$, $(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$, and $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$ respectively, hence we have

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \underline{\mathscr{A}\{T_{\mathbb{P}}\}} \Rightarrow \underline{(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}})} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}],\\ & \underline{\mathscr{A}\{T_{\mathbb{P}}\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}],\\ & \underline{\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned} \tag{12.2.6}$$

☐ Let us here focus our attentions on the terms without underline in the above flow, i.e.,

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]},\\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle\end{aligned} \tag{12.2.7}$$

☐ Then, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the relations $\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ (see (9.2.3(p.42))-(9.2.8)) yields the relations $\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] = \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{12.2.8}$$

In fact, it can be easily shown that the following relation hold:

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x \text{ on } (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \lambda\beta x \text{ on } (a^\star, \infty) & \cdots(2),\end{cases} \tag{12.2.9}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x \quad \text{on} \quad (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \delta x \quad \text{on} \quad (a^\star, \infty) & \cdots(2),\end{cases} \tag{12.2.10}$$

$$K(x)\begin{cases} > -(1-\beta)x - s \text{ on } \quad (-\infty, b) & \cdots(1),\\ = -(1-\beta)x - s \text{ on } \quad [b, \infty) & \cdots(2),\end{cases} \tag{12.2.11}$$

$$K(x) + x \geq \beta x - s \quad \text{on} \quad (-\infty, \infty), \tag{12.2.12}$$

$$K(x) + x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x \text{ on } \quad (-\infty, a^\star] & \cdots(1),\\ \beta x - s \qquad\qquad\qquad\quad \text{on } \quad [b, \infty) & \cdots(2),\end{cases} \tag{12.2.13}$$

$$K(\,x_L\,) = -(1-\beta)\,x_L \cdots(1), \quad L(\,x_K\,) = (1-\beta)\,x_K \cdots(2), \tag{12.2.14}$$

Direct proof See (A 2.1(p.275))-(A 2.6) . ∎

☐ Noting (12.2.8), we can rewrite (12.2.7) as below.

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \langle\!\langle \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},$$
$$\mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]},$$
$$\boxed{\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]}} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle \tag{12.2.15}$$

☐ Now, since $\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]}$ is what is derived by using $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$ (see the relation within the framebox $\boxed{}$ above), due to Remark 6.1.1(p.21) it follows that $\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]}$ can be regarded as the assertion system for $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$, i.e., $\mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$, so we have

$$\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} = \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{12.2.16}$$

Thus, it follows from (12.2.16) that we have the following lemmas and corollaries:

Lemma 12.2.2 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$L(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$L(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *Let $s = 0$. Then $x_L = b$ where $x_L > (\leq)\ x \Leftrightarrow L(x) > (=)\ 0 \Rightarrow L(x) > (\leq)\ 0$.*
(e) *Let $s > 0$.*
  1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\ x \Leftrightarrow L(x) > (= (<))\ 0$.*
  2. *$(\lambda\beta a - s)/\lambda\beta \leq (>)\ a^\star \Leftrightarrow x_L = (>)\ (\lambda\beta a - s)/\lambda\beta$.* □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 9.2.1(p.43). ∎

Direct proof See Lemma A 2.2(p.275). ∎

Corollary 12.2.1 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$x_L > (\leq)\ x \Leftrightarrow L(x) > (\leq)\ 0$.*
(b) *$x_L \geq (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$.* □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 9.2.1(p.43). ∎

Direct proof See Corollary A 2.1(p.276). ∎

Lemma 12.2.3 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$K(x) + x$ is strictly increasing on $[a^\star, \infty)$.*
(g) *$K(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(h) *If $x < y$ and $a^\star < y$, then $K(x) + x < K(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*
  1. *There uniquely exists x_K where $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.*
  2. *$(\lambda\beta a - s)/\delta \leq (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.*
  3. *Let $\kappa > (= (<))\ 0$. Then $x_K > (= (<))\ 0$.* □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 9.2.2(p.43). ∎

Direct proof See Lemma A 2.3(p.276). ∎

Corollary 12.2.2 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$x_K > (\leq)\ x \Leftrightarrow K(x) > (\leq)\ 0$.*
(b) *$x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$.* □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 9.2.2(p.44). ∎

Direct proof See Lemma A 2.2(p.276). ∎

Lemma 12.2.4 ($\mathscr{A}\{L_{\mathbb{P}}/K_{\mathbb{P}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_L = x_K = b$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_L = x_K$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $b > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\kappa > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (<))\ 0$.* □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 9.2.3(p.44). ∎

Direct proof See Lemma A 2.4(p.277). ∎

Lemma 12.2.5 ($\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$)

(a) $\mathcal{L}(s)$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.

(b) *Let* $\lambda\beta a \geq b$.

1. $x_L \leq \lambda\beta a - s$.
2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_L < \lambda\beta a - s$.

(c) *Let* $\lambda\beta a < b$. *Then, there exists a* $s_{\mathcal{L}} > 0$ *such that if* $s_{\mathcal{L}} > (\leq)\ s$, *then* $x_L > (\leq)\ \lambda\beta a - s$. □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 9.2.4(p.44). ∎

Direct proof See Lemma A 2.5(p.277). ∎

Lemma 12.2.6 ($\kappa_{\mathbb{P}}$) *We have*:

(a) $\kappa = \lambda\beta a - s$ *if* $a^\star > 0$ *and* $\kappa = -s$ *if* $b < 0$.

(b) *Let* $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$. □

Proof by analogy Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 9.3.1(p.45). ∎

Direct proof See Lemma A 2.6(p.277). ∎

□ Due to (12.2.16) we can rewrite (12.2.15)ten as below.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; &\langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}]}\,\rangle\!\rangle. \end{aligned} \tag{12.2.17}$$

□ Since the assertion system $\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}}$ in (12.2.17) is what is derived from $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}$, it can be regarded as an assertion system related to the model $\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]$ (see Remark 6.1.1(p.21)), i.e., $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}$, hence we have

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad \text{(the same as (12.2.5(p.78)))}. \tag{12.2.18}$$

Thus (12.2.17) can be rewritten as follows.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; &\langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}\,\rangle\!\rangle \end{aligned} \tag{12.2.19}$$

□ The whole of the r.h.s. of (12.2.19) can be regarded as the procedure which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}$, so let us denote it by Procedure⟨ℙ⟩, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \text{Procedure}[\mathbb{P}]. \tag{12.2.20}$$

Accordingly, finally it follows that we have

$$\begin{aligned}\text{Procedure}[\mathbb{P}] \;=\; &\langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\},\\ &\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}\,\rangle\!\rangle \end{aligned}$$

12.3 Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

Noting the equality (12.2.5(p.78)), we eventually obtain the following theorem.

Theorem 12.3.1 (analogy ($\mathbb{R} \to \mathbb{P}$)**)** *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\square \tag{12.3.1}$$

Then, from the comparison of (I) and (III) of Tables 6.5.1 we also get

$$\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{12.3.2}$$

Moreover, from (11.4.2(p.60)) we obtain the following:

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) \;=\; \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{12.3.3}$$

$$= \;(a^\star, a, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{12.3.4}$$

Since the analogy replacement operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is a mere replacement of the two symbols, $a \to a^\star$ and $\mu \to a$, defining its inverse

$$\mathcal{A}_{\mathbb{P}\to\mathbb{R}} = \{a^\star \to a,\ a \to \mu\}, \tag{12.3.5}$$

we can immediately known that the inverse of the above theorem becomes true, i.e.,

Theorem 12.3.2 (analogy ($\mathbb{P} \leftarrow \mathbb{R}$)) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{12.3.6}$$

In addition, as an inverses of (12.3.2) and (12.3.3) we immediately obtain

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{12.3.7}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{12.3.8}$$

$$= (a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{12.3.9}$$

12.4 Derivation of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

The following two `Tom`'s can be immediately obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to `Tom`'s 10.2.1(p.47) and 10.2.2.

□ **`Tom` 12.4.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) $\boxed{ⓢ\ \mathtt{dOITs}_{\tau}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

Proof by analogy Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to `Tom` 10.2.1(p.47). ∎

Direct proof See `Tom` A 4.3(p.286) . ∎

□ **`Tom` 12.4.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
(c) *Let $\beta a < b$.*
  1. *Let $\beta = 1$.*
     i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     ii. *Let $a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$ ($(s > 0)$).*
     i. *Let $b > 0$ ($(\kappa > 0)$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
     ii. *Let $b = 0$ ($(\kappa = 0)$).*
        1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
        2. *Let $\beta a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
     iii. *Let $b < 0$ ($(\kappa < 0)$).*
        1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
        2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle}\boxed{⊛\parallel}$ *is true.* □

Proof by analogy Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to `Tom` 10.2.2(p.48). ∎

Direct proof See `Tom` A 4.4(p.287) . ∎

12.5 Strict Definition of Analogy

Definition 12.5.1 (analogy)

(a) By $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathfrak{X}]$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathfrak{X}]$) let us denote the assertion defined by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$) to a given $\mathfrak{X}$.
(b) If $A\{\mathfrak{X}_2\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[A\{\mathfrak{X}_1\}]$ and $A\{\mathfrak{X}_1\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[A\{\mathfrak{X}_2\}]$, then $A\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\}$ is said to be *analogous*, denoted by $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$.
(c) For given two assertion systems $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ which are one-to-one correspondent, if $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$ for any pair $(A\{\mathfrak{X}_1\}, A\{\mathfrak{X}_2\})$ where $A\{\mathfrak{X}_1\} \in \mathscr{A}\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\} \in \mathscr{A}\{\mathfrak{X}_2\}$ are correspondent each other, then $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ are said to be *analogous*, denoted by $\mathscr{A}\{\mathfrak{X}_1\} \bowtie \mathscr{A}\{\mathfrak{X}_2\}$. □

12.6 Analogy-Operation-Free

When no change occurs even if the analogy operation is applied to a given assertion A, the assertion is said to be *free from* the analogy operation, called the *analogy-operation-free assertion.*

Lemma 12.6.1 *Even if the analogy operation is applied to the analogy-operation-free assertion, no change occurs.* □

Proof Evident. ∎

12.7 Optimal Price to Propose

Lemma 12.7.1 ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *The optimal price z_t to propose is nondecreasing in $t > 0$.* □

Proof Obvious from (6.2.28(p.23)), `Tom`'s 12.4.1(a) and 12.4.2(a), and
Lemma 12.1.3(p.73) . ∎

Chapter 13

Symmetry Theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$)

This chapter provides the methodology of deriving $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model, see `Tom`'s 12.4.1(p.82) and 12.4.2).

13.1 Functions $\check{T}$, $\check{L}$, $\check{K}$, and $\check{\mathcal{L}}$ of Type $\mathbb{P}$

Let us define the T-function of Type $\mathbb{P}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ (see (5.1.19(p.18)) and (5.1.18(p.18))) by

$$\check{T}(x) = \max_z \check{p}(z)(z-x) \cdots (1), \qquad \check{p}(z) = \Pr\{z \le \hat{\boldsymbol{\xi}}\} \cdots (2). \tag{13.1.1}$$

By $\check{z}(x)$ let us define z maximizing $\check{p}(z)(z-x)$ if it exists, i.e.,

$$\check{T}(x) = \check{p}(\check{z}(x))(\check{z}(x) - x). \tag{13.1.2}$$

Furthermore, let us define

$$\check{L}(x) = \lambda\beta\check{T}(x) - s, \tag{13.1.3}$$
$$\check{K}(x) = \lambda\beta\check{T}(x) - (1-\beta)x - s, \tag{13.1.4}$$
$$\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\check{a} - s), \tag{13.1.5}$$
$$\check{\kappa} = \lambda\beta\check{T}(0) - s. \tag{13.1.6}$$

Then, let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by respectively $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ if they exist; If multiple solutions exist for each of $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$, let us employ the *smallest* as its solution (see Sections 5.2(p.19) (a) and 11.2.1(p.56)). Furthermore, let us define (see Figure 11.1.1(p.56) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{a}^\star = \inf\{x \mid \check{T}(x) > \check{a} - x\} \quad \text{(see (5.1.26(p.18)))}, \tag{13.1.7}$$
$$\check{x}^\star = \inf\{x \mid \check{z}(x) > \check{a}\} \qquad \text{(see (5.1.27(p.18)))}. \tag{13.1.8}$$

By $\check{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ let us define $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (see Table 6.5.1(p.31) (III)) we can obtain

$$\mathtt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \{V_1 = \beta\check{a} - s,\ V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{13.1.9}$$

13.2 Functions $\check{\tilde{T}}$, $\check{\tilde{L}}$, $\check{\tilde{K}}$, and $\check{\tilde{\mathcal{L}}}$ of Type $\mathbb{P}$

Let us define the $\tilde{T}$-function of $\tilde{\text{T}}$ype $\mathbb{P}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ as follows (see (5.1.32(p.18))).

$$\check{\tilde{T}}(x) = \min_z \check{\tilde{p}}(z)(z-x) \cdots (1), \qquad \check{\tilde{p}}(z) = \Pr\{\hat{\boldsymbol{\xi}} \le z\} \cdots (2) \tag{13.2.1}$$

where by $\check{\tilde{z}}(x)$ let us define z minimizing $\check{\tilde{p}}(z)(z-x)$ if it exists, i.e.,

$$\check{\tilde{T}}(x) = \check{\tilde{p}}(\check{\tilde{z}}(x))(\check{\tilde{z}}(x) - x). \tag{13.2.2}$$

Let us define

$$\check{\tilde{L}}(x) = \lambda\beta\check{\tilde{T}}(x) + s, \tag{13.2.3}$$
$$\check{\tilde{K}}(x) = \lambda\beta\check{\tilde{T}}(x) - (1-\beta)x + s, \tag{13.2.4}$$
$$\check{\tilde{\mathcal{L}}}(s) = \check{\tilde{L}}(\lambda\beta\check{b} + s), \tag{13.2.5}$$
$$\check{\tilde{\kappa}} = \lambda\beta\check{\tilde{T}}(0) + s \tag{13.2.6}$$

where let us define the solutions of $\check{\tilde{L}}(x) = 0$, $\check{\tilde{K}}(x) = 0$, and $\check{\tilde{\mathcal{L}}}(x) = 0$ by respectively $x_{\check{\tilde{L}}}$, $x_{\check{\tilde{K}}}$, and $s_{\check{\tilde{\mathcal{L}}}}$; If multiple solutions exist for each of $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$, we shall employ the *largest* as a solution (see Sections 5.2(p.19) (b) and 11.2.2(p.57)). Furthermore let us define (see Figure 11.1.1(p.56) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{b}^\star = \sup\{x \mid \check{\tilde{T}}(x) < \check{b} - x\} \quad \text{(see (5.1.39(p.19)))}, \tag{13.2.7}$$

$$\check{\tilde{x}}^\star = \sup\{x \mid \check{\tilde{z}}(x) < \check{b}\} \quad \text{(see (5.1.40(p.19)))}. \tag{13.2.8}$$

By $\check{\tilde{\mathsf{M}}}$:1[$\mathbb{P}$][A] let us define $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for SOE{$\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]} (see Table 6.5.1(p.31) (IV)) we can obtain

$$\texttt{SOE}\{\check{\tilde{\mathsf{M}}}{:}1[\mathbb{P}][\texttt{A}]\} = \{V_1 = \beta\check{b} + s,\ V_t = \min\{\check{\tilde{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{13.2.9}$$

13.3 List of Underline Functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

The table below is the list of the four kinds of underline functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$ (see Table 11.2.1(p.57)).

Table 13.3.1: List of the underlying functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

Type $\mathbb{P}$	$\tilde{\text{T}}$ype $\mathbb{P}$
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \max_z p(z)(z-x)$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta a - s)$ See Section 5.1.3(p.18)	$\check{T}(x) = \max_z \check{p}(z)(z-x)$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{a} - s)$ See Section 13.1
$\tilde{T}(x) = \min_z \tilde{p}(z)(z-x)$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta b + s)$ See Section 5.1.4(p.18)	$\check{\tilde{T}}(x) = \min_z \check{\tilde{p}}(z)(z-x)$ $\check{\tilde{L}}(x) = \beta\check{\tilde{T}}(x) + s$ $\check{\tilde{K}}(x) = \beta\check{\tilde{T}}(x) - (1-\beta)x + s$ $\check{\tilde{\mathcal{L}}}(x) = \check{\tilde{L}}(\beta\check{b} + s)$ See Section 13.2

13.4 Two Kinds of Replacements

13.4.1 Correspondence Replacement

Lemma 13.4.1 **($\mathcal{C}_{\mathbb{P}}$)** *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F. *Then*:

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}$, $\hat{a}^\star = \check{b}^\star$, $\hat{b} = \check{a}$.
(c) $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.
(d) $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.
(e) $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.
(g) $\hat{x}_L = x_{\check{\tilde{L}}}$.
(h) $\hat{x}_K = x_{\check{\tilde{K}}}$.
(i) $s_{\mathcal{L}} = s_{\check{\tilde{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \check{\tilde{\kappa}}$. □

Proof (a) The same as (11.1.10(p.55)).

(The first and third equalities of (b)) The same as the first and third equalities of (11.1.11(p.55)). The second equality will be proven after the proof of (c).

(c) From (5.1.18(p.18)) we obtain

$$p(z) = \Pr\{-\hat{z} \le -\hat{\boldsymbol{\xi}}\} = \Pr\{\hat{\boldsymbol{\xi}} \le \hat{z}\} = \check{\tilde{p}}(\hat{z}) \tag{13.4.1}$$

due to (13.2.1 (2)), hence from (5.1.19) we have $T(x) = \max_z \check{\tilde{p}}(\hat{z})(-\hat{z} + \hat{x}) = -\min_z \check{\tilde{p}}(\hat{z})(\hat{z} - \hat{x})$. Then, since "$\min_z = \min_{-\infty<z<\infty} = \min_{-\infty<-\hat{z}<\infty} = \min_{\infty>\hat{z}>-\infty} = \min_{-\infty<\hat{z}<\infty} = \min_{\hat{z}}$", the above expression can be rewrite as $T(x) = -\min_{\hat{z}} \check{\tilde{p}}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $T(x) = -\min_z \check{\tilde{p}}(z)(z - \hat{x})$. Accordingly, since $T(x) = -\check{\tilde{T}}(\hat{x})$ from (13.2.1 (1)), we obtain $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.

(The second equality of (b)) From (5.1.26(p.18)) we have $a^\star = \inf\{-\hat{x} \mid -\hat{T}(x) > -\hat{a} + \hat{x}\} = -\sup\{\hat{x} \mid \hat{T}(x) < \hat{a} - \hat{x}\} = -\sup\{\hat{x} \mid \check{\tilde{T}}(\hat{x}) < \check{b} - \hat{x}\}$ due to (c) and (b). Without loss of generality, this can be rewritten as $a^\star = -\sup\{x \mid \check{\tilde{T}}(x) < \check{b} - x\}$, hence $a^\star = -\check{b}^\star$ due to (13.2.7), so that $\hat{a}^\star = \check{b}^\star$.

(d) From (5.1.20) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\check{T}}(\hat{x}) - s = -\check{\check{L}}(\hat{x})$ from (13.2.3), hence $\hat{L}(x) = \check{\check{L}}(\hat{x})$.

(e) From (5.1.21) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\check{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\check{K}}(\hat{x})$ from (13.2.4), hence $\hat{K}(x) = \check{\check{K}}(\hat{x})$.

(f) From (5.1.22(p.18)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta a - s) = -\check{\check{L}}(\widehat{\lambda\beta a - s})$ due to (d). Then, since $\mathcal{L}(s) = -\check{\check{L}}(-\lambda\beta a + s) = -\check{\check{L}}(\lambda\beta\hat{a} + s) = -\check{\check{L}}(\lambda\beta\check{b} + s)$ due to (b), we have $\mathcal{L}(s) = -\check{\check{\mathcal{L}}}(s)$ from (13.2.5), hence $\hat{\mathcal{L}}(s) = \check{\check{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $-\hat{L}(x_L) = 0$, i.e., $\hat{L}(x_L) = 0$, leading to $\check{\check{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\check{L}}(x) = 0$ has the solution $x_{\check{\check{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $-\hat{K}(x_K) = 0$, i.e., $\hat{K}(x_K) = 0$, leading to $\check{\check{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\check{K}}(x) = 0$ has the solution $x_{\check{\check{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $-\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, i.e., $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, leading to $\check{\check{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\check{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\check{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.23) we have $\kappa = -\lambda\beta\hat{T}(0) - s = -\lambda\beta\check{\check{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\check{T}}(0) - s = -\check{\check{\kappa}}$ from (13.2.6), thus $\hat{\kappa} = \check{\check{\kappa}}$. ∎

Definition 13.4.1 (correspondent replacement operation $\mathcal{C}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in the above lemma with its right-hand side the *correspondence replacement operation* $\mathcal{C}_{\mathbb{P}}$. □

Lemma 13.4.2 ($\tilde{\mathcal{C}}_{\mathbb{P}}$) *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to the* F*. Then:*

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}, \quad \hat{b}^\star = \check{a}^\star, \quad \hat{b} = \check{a}$.
(c) $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.
(d) $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.
(e) $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.
(f) $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.
(g) $\hat{x}_{\tilde{L}} = \check{x}_L$.
(h) $\hat{x}_{\tilde{K}} = \check{x}_K$.
(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.
(j) $\hat{\tilde{\kappa}} = \check{\kappa}$. □

Proof (a) The same as (11.1.10(p.55)).

(The first and third equalities of (b)) The same as the first and first equation of (11.1.11(p.55)). The second equality will be proven after the proof of (c).

(c) From (5.1.31(p.18)) we obtain

$$\tilde{p}(z) = \Pr\{-\hat{\boldsymbol{\xi}} \leq -\hat{z}\} = \Pr\{\hat{\boldsymbol{\xi}} \geq \hat{z}\} = \Pr\{\hat{z} \leq \hat{\boldsymbol{\xi}}\} = \check{p}(\hat{z}) \tag{13.4.2}$$

due to (13.1.1 (2)), hence from (5.1.32) we have $\tilde{T}(x) = \min_z \check{p}(\hat{z})(-\hat{z} + \hat{x}) = -\max_z \check{p}(\hat{z})(\hat{z} - \hat{x})$. Then, since "$\max_z = \max_{-\infty<z<\infty} = \max_{-\infty<-\hat{z}<\infty} = \max_{\infty>\hat{z}>-\infty} = \max_{-\infty<\hat{z}<\infty} = \max_{\hat{z}}$", the above expression can be rewritten as $\tilde{T}(x) = -\max_{\hat{z}} \check{p}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $\tilde{T}(x) = -\max_z \check{p}(z)(z - \hat{x})$. Accordingly, since $\tilde{T}(x) = -\check{T}(\hat{x})$ from (13.1.1 (1)), we obtain $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(The second equality of (b)) From (5.1.39(p.19)) we have $b^\star = \sup\{-\hat{x} \mid -\hat{\tilde{T}}(x) < -\hat{b} + \hat{x}\} = \inf\{\hat{x} \mid \hat{\tilde{T}}(x) > \hat{b} - \hat{x}\}$. From (c) and (b) we have $b^\star = \inf\{\hat{x} \mid \check{T}(\hat{x}) > \check{a} - \hat{x}\}$. Then, since $b^\star = \inf\{x \mid \check{T}(x) > \check{a} - x\}$ without loss of generality, we have $b^\star = \check{a}^\star$ due to (13.1.7).

(d) From (5.1.33) and (c) we have $\tilde{L}(x) = -\lambda\beta\hat{\tilde{T}}(x) + s = -\lambda\beta\check{T}(\hat{x}) + s = -\check{L}(\hat{x})$ from (13.1.3), hence $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) From (5.1.34) and (c) we have $\tilde{K}(x) = -\lambda\beta\hat{\tilde{T}}(x) + (1-\beta)\hat{x} + s = -\lambda\beta\check{T}(\hat{x}) + (1-\beta)\hat{x} + s = -\check{K}(\hat{x})$ from (13.1.4), hence $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) From (5.1.35) we have $\tilde{\mathcal{L}}(s) = -\hat{\tilde{L}}(\lambda\beta b + s)$, hence from (d) we obtain $\tilde{\mathcal{L}}(s) = -\check{L}(\widehat{\lambda\beta b + s}) = -\check{L}(-\lambda\beta b - s) = -\check{L}(\lambda\beta\hat{b} - s) = -\check{L}(\lambda\beta\check{a} - s)$ due to (b). Accordingly, from (13.1.5) we obtain $\tilde{\mathcal{L}}(s) = -\check{\mathcal{L}}(s)$, hence $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $-\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, i.e., $\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, leading to $\check{L}(\hat{x}_{\tilde{L}}) = 0$ from (d), implying that $\check{L}(x) = 0$ has the solution $x_{\check{L}} = \hat{x}_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $-\hat{\tilde{K}}(x_{\tilde{K}}) = 0$, i.e., $\hat{\tilde{K}}(x_{\tilde{K}}) = 0$, leading to $\check{K}(\hat{x}_{\tilde{K}}) = 0$ from (e), implying that $\check{K}(x) = 0$ has the solution $x_{\check{K}} = \hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $-\hat{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, i.e., $\hat{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, leading to $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), implying that $\check{\mathcal{L}}(s) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.36) we have $\tilde{\kappa} = -\lambda\beta\hat{\tilde{T}}(0) + s$, leading to $\tilde{\kappa} = -\lambda\beta\check{T}(\hat{0}) + s$ from (c), hence $\tilde{\kappa} = -\lambda\beta\check{T}(0) + s = -\check{\kappa}$ from (13.1.6), thus $\hat{\tilde{\kappa}} = \check{\kappa}$. ∎

Remark 13.4.1 The equality $\hat{\mu} = \check{\mu}$ in Lemmas 11.3.1(p.57) (b) changes into respectively $\hat{a}^\star = \check{b}^\star$ in Lemma 13.4.1(b) and the equality $\hat{\mu} = \check{\mu}$ in (11.1.11(p.55)) changes into $\hat{b}^\star = \check{a}^\star$ in
Lemma 13.4.2(b). □

The definition below is the same as Def. 11.3.3(p.59).

Definition 13.4.2 (reflective element and non-reflective element) It should be noted that the left side of each of the equalities in Lemmas 13.4.1(i) and 13.4.2(i) is respectively $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ without the hat symbol "ˆ"; in other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reflection. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reflective element* and to each of all the other elements as the *reflective element*. □

Definition 13.4.3 (correspondent replacement operation $\tilde{\mathcal{C}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in the above lemma with its right-hand side the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{P}}$. □

13.4.2 Identity Replacement

Lemma 13.4.3 ($\mathcal{I}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $\check{F} \equiv F \cdots [1^*]$.*[†] *Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ,
(b) $\check{a} = a$, $\check{b}^\star = b^\star$, $\check{b} = b$,
(c) $\check{\tilde{T}}(x) = \tilde{T}(x)$,
(d) $\check{\tilde{L}}(x) = \tilde{L}(x)$,
(e) $\check{\tilde{K}}(x) = \tilde{K}(x)$,
(f) $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$,
(g) $x_{\check{\tilde{L}}} = x_{\tilde{L}}$,
(h) $x_{\check{\tilde{K}}} = x_{\tilde{K}}$,
(i) $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$,
(j) $\check{\tilde{\kappa}} = \tilde{\kappa}$. □

Proof (a) Clear from [1*].

(the first and last equalities of (b)) Immediate from (a). The second equality will be proven after the proof of (c).

(c) From (13.2.1 (2)) we have $\check{\tilde{p}}(z) = \Pr\{\check{\boldsymbol{\xi}} \leq z\} = \int_{-\infty}^{z} \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{\tilde{p}}(z) = \int_{-\infty}^{z} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} \leq z\} = \tilde{p}(z)$ from (5.1.31). Hence, we have that $\check{\tilde{T}}(x)$ given by (13.2.1 (1)) becomes $\check{\tilde{T}}(x) = \min_z \tilde{p}(z)(z - x)$, which is identical to $\tilde{T}(x)$ given by (5.1.32), i.e., $\check{\tilde{T}}(x) = \tilde{T}(x)$ for any x.

(the second equality of (b)) From (13.2.7) and (c) we have $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < \check{b} - x\}$, hence from (b) we get $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < b - x\} = b^\star$ due to (5.1.39).

(d,e) Noting (c), from (13.2.3) and (5.1.33) we immediately see that the first equality is true. Similarly, from (13.2.4) and (5.1.34) we immediately see that the second equality is true.

(f) (13.2.5) becomes $\check{\tilde{\mathcal{L}}}(s) = \check{\tilde{L}}(\lambda\beta b + s)$ due to (b). This can be rewritten as $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta b + s)$ due to (d), which is the same as $\tilde{\mathcal{L}}$ (s) given by (5.1.35).

(g-i) Evident from (d-f).

(j) (13.2.6) becomes $\check{\tilde{\kappa}} = \lambda\beta\tilde{T}(0) + s$ due to (c), which is the same as $\tilde{\kappa}$ given by (5.1.36). ∎

Definition 13.4.4 (identity replacement operation $\mathcal{I}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma with its right-hand the *identity replacement operation* $\mathcal{I}_{\mathbb{P}}$. □

Lemma 13.4.4 ($\tilde{\mathcal{I}}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $F \equiv \check{F} \cdots [1^*]$. Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ,
(b) $\check{a} = a$, $\check{a}^\star = a^\star$, $\check{b} = b$,
(c) $\check{T}(x) = T(x)$,
(d) $\check{L}(x) = L(x)$,
(e) $\check{K}(x) = K(x)$,
(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$,
(g) $x_{\check{L}} = x_L$,
(h) $x_{\check{K}} = x_K$,
(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$,
(j) $\check{\kappa} = \kappa$. □

[†]See Lemma 11.1.1(p.56) (b)

Proof (a) Clear from [1*].

(The first and last equalities of b)) Immediate form (a). The second equality will be proven after the proof of (c).

(c) From (13.1.1 (2)) we have $\check{p}(z) = \Pr\{z \le \hat{\boldsymbol{\xi}}\} = \int_z^\infty \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{p}(z) = \int_z^\infty f(\xi)d\xi = \Pr\{z \le \boldsymbol{\xi}\} = p(z)$ from (5.1.18). Hence, we have that $\check{T}(x)$ given by (13.1.1 (1)) becomes $\check{T}(x) = \max_z p(z)(z-x)$, which is identical to $T(x)$ given by (5.1.19), i.e., $\check{T}(x) = T(x)$ for any x.

(the second equality of (b)) From (13.1.7 (1)) and (c) we have $\check{a}^\star = \inf\{x \mid T(x) > \check{a} - x\}$, hence from (b) we get $\check{a}^\star = \inf\{x \mid T(x) > a - x\} = a^\star$ due to (5.1.26). Thus, the second equality of (b) is true.

(d) Noting (c), from (13.1.3) and (5.1.20) we immediately see that the first equality becomes true. Similarly, from (13.1.4) and (5.1.21) we immediately see that the second equality becomes true.

(f) Firstly, (13.1.5) becomes $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta a - s)$ due to (b). This can be rewritten as $\check{\mathcal{L}}(s) = L(\lambda\beta a - s)$ due to (d), which is the same as $\mathcal{L}$ (s) given by (5.1.22).

(g-i) Evident from (d-f).

(j) (13.1.6) becomes $\check{\kappa} = \lambda\beta T(0) - s$ due to (c), which is the same as κ given by (5.1.23). ∎

Definition 13.4.5 (Identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma with its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{P}}$. □

13.5 Scenario of Type $\mathbb{P}$

13.5.1 Scenario[$\mathbb{P}$]

This section provides the scenario that derives $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model), denoted by Scenario[$\mathbb{P}$].

- Before proceeding with the discussion, let us review the process of the transformation of attribute vectors in Scenario[$\mathbb{R}$], summarized as below (the same as (11.5.30(p.62))).

$$\begin{array}{lll}
\textsf{Step } 1[\mathbb{R}]: & & \boldsymbol{\theta}(a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t) \quad (\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 2[\mathbb{R}]: & \to & \boldsymbol{\theta}(\hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 3[\mathbb{R}]: \text{ Lemma 11.3.1(p.57)} & \to & \boldsymbol{\theta}(\check{b},\ \check{\mu},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ \hat{V}_t) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 4[\mathbb{R}]: \text{ Lemma 11.3.3(p.59)} & \to & \boldsymbol{\theta}(b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t) \quad (\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}))
\end{array} \tag{13.5.1}$$

- Prior to entering into Step 1[$\mathbb{P}$] of Scenario[$\mathbb{P}$], first a and μ in Step 1[$\mathbb{R}$] are replace by $a^\star$ and a respectively (application of $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$), and then Step 2[$\mathbb{P}$] follows. Next, in Step 3[$\mathbb{P}$], Lemma 13.4.1(p.84) is used instead of Lemma 11.3.1(p.57) and in Step 4[$\mathbb{P}$], Lemma 13.4.3 is used instead of Lemma 11.3.3(p.59). This flow can be rewritten as follows.

$$\begin{array}{lll}
\textsf{Step } 1[\mathbb{R}]: & & \boldsymbol{\theta}(a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t) \\
\quad\quad\downarrow & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 1[\mathbb{P}]: \text{ Scenario}[\mathbb{P}] & & \boldsymbol{\theta}(a^\star,\ a,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t) \quad (\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 2[\mathbb{P}]: & \to & \boldsymbol{\theta}(\hat{a}^\star,\ \hat{a},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 3[\mathbb{P}]: \text{ Lemma 13.4.1(p.84)} & \to & \boldsymbol{\theta}(\check{b}^\star,\ \check{b},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ \hat{V}_t) \\
 & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 4[\mathbb{P}]: \text{ Lemma 13.4.3(p.86)} & \to & \boldsymbol{\theta}(b^\star,\ b,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t) \quad (\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})
\end{array} \tag{13.5.2}$$

From the above flow of Scenario[$\mathbb{P}$] we see that the operation transforming $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})$ is eventually reduced to the operation transforming the first row into the last row, schematized as

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \left\{\begin{array}{l}
a^\star,\ a,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t \\
\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\
b^\star,\ b,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t
\end{array}\right\}.^\dagger \tag{13.5.3}$$

From (III) and (IV) of Table 6.5.1(p.31) it can be immediately seen that

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{13.5.4}$$

From the above discussion it can be seen that Lemma 11.5.1(p.63) is changed into Lemma 13.5.1 below.

Lemma 13.5.1 *Let* $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. *Then* $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ *where*

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{13.5.5}$$

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† Compare the dash box ⬚ with that in (11.5.31(p.63)).

Finally, for almost the same reason as that for which Theorem 11.5.1(p.66) is derived from Lemma 11.5.1(p.63) we have Theorem 13.5.1 below.

Theorem 13.5.1 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{13.5.6}$$

In addition, we have (see (12.3.4(p.81)))

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) \ \stackrel{\text{def}}{=} \ \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{13.5.7}$$

$$= \ (b^\star, b, a, \ x_{\tilde{L}}, \ s_{\tilde{\mathcal{L}}}, \ x_{\tilde{K}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{13.5.8}$$

13.5.2 $\tilde{\text{S}}$cenario[$\mathbb{P}$]

This section provides the scenario that derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model), denoted by $\tilde{\text{S}}$cenario[$\mathbb{P}$].

- Before proceeding with the discussion, let us review the process of the transformation of attribute vectors in $\tilde{\text{S}}$cenario[$\mathbb{R}$], summarized as below (see (11.8.32(p.71))).

$$\begin{array}{llcll}
\textsf{Step } 1[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}}, V_t) & (\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})) \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 2[\tilde{\mathbb{R}}]: & & \to & \boldsymbol{\theta}(\hat{b},\ \hat{\mu},\ \hat{a},\ \hat{\tilde{x}}_L,\ \hat{\tilde{x}}_K,\ s_{\tilde{\mathcal{L}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}},\ \hat{\tilde{L}},\ \hat{\tilde{K}},\ \hat{\tilde{\mathcal{L}}}, \hat{V}_t) & \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 3[\tilde{\mathbb{R}}]: & \text{Lemma 13.4.1(p.84)} & \to & \boldsymbol{\theta}(\breve{a},\ \breve{\mu},\ \breve{b},\ x_{\check{L}}, x_{\check{K}},\ s_{\check{\mathcal{L}}},\ \breve{\kappa},\ \check{T},\ \check{L},\ \check{K},\ \check{\mathcal{L}}, \hat{V}_t) & \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 4[\tilde{\mathbb{R}}]: & \text{Lemma 13.4.3(p.86)} & \to & \boldsymbol{\theta}(a,\ \mu,\ b,\ x_L, x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L}, V_t) & (\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}))
\end{array} \tag{13.5.9}$$

- Prior to entering into $\textsf{Step } 1[\tilde{\mathbb{P}}]$ of $\tilde{\text{S}}$cenario[$\mathbb{P}$], first b and μ in $\textsf{Step } 1[\tilde{\mathbb{R}}]$ are replace by $b^\star$ and b respectively (application of $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$) and then $\textsf{Step } 2[\tilde{\mathbb{P}}]$ follows. In $\textsf{Step } 3[\tilde{\mathbb{P}}]$, Lemma 13.4.1(p.84) is used instead of Lemma 11.3.1(p.57), and finally in $\textsf{Step } 4[\tilde{\mathbb{P}}]$, Lemma 13.4.3 is used instead of Lemma 11.3.3(p.59). This flow can be rewritten as follows.

$$\begin{array}{llcll}
\textsf{Step } 1[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(\ b,\ \ \mu,\ b,\ x_L, x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L}, V_t) & \\
\hline
 & \quad\downarrow & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 1[\tilde{\mathbb{P}}] & \tilde{\text{S}}\text{cenario}[\mathbb{P}] & & \boldsymbol{\theta}(b^\star,\ b,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}}, V_t) & (\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 2[\tilde{\mathbb{P}}] & & \to & \boldsymbol{\theta}(\hat{b}^\star,\ \hat{b},\ \hat{a},\ \hat{\tilde{x}}_L,\ \hat{\tilde{x}}_K,\ s_{\tilde{\mathcal{L}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}},\ \hat{\tilde{L}},\ \hat{\tilde{K}},\ \hat{\tilde{\mathcal{L}}}, \hat{V}_t) & \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 3[\tilde{\mathbb{P}}] & \text{Lemma 13.4.2(p.85)} & \to & \boldsymbol{\theta}(\breve{a}^\star,\ \breve{a},\ \breve{b},\ x_{\check{L}}, x_{\check{K}},\ s_{\check{\mathcal{L}}},\ \breve{\kappa},\ \check{T},\ \check{L},\ \check{K},\ \check{\mathcal{L}}, \hat{V}_t) & \\
 & & & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \\
\textsf{Step } 4[\tilde{\mathbb{P}}] & \text{Lemma 13.4.4(p.86)} & \to & \boldsymbol{\theta}(a^\star,\ a,\ b,\ x_L, x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L}, V_t) & (\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})
\end{array} \tag{13.5.10}$$

From the above flow of $\tilde{\text{S}}$cenario[$\tilde{\mathbb{P}}$] we see that the operation transforming $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ is eventually reduced to the operation transforming the first row into the last row, schematized as

$$\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} = \left\{\begin{array}{l}
b^\star, b, \ a,\ x_{\tilde{L}}, \tilde{\kappa},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, T, L, K, \mathcal{L},\ V_t \\
\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\
a^\star, a, \ b,\ x_L, \kappa,\ x_K, s_{\mathcal{L}}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}},\ V_t
\end{array}\right\}. \tag{13.5.11}$$

From (III) and (IV) of Table 6.5.1(p.31) it can be immediately confirmed that

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \ \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{13.5.12}$$

From the above discussion it can be seen that Lemma 11.8.1(p.72) is changed into the lemma below.

Lemma 13.5.2 *Let $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. Then $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ where*

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{13.5.13}$$

Finally, for the same reason as the one for which Theorem 11.8.1(p.72) is derived from Lemma 11.8.1(p.72) we have Theorem 13.5.2 below.

Theorem 13.5.2 *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \ \square \tag{13.5.14}$$

From (11.8.47(p.72)) we have

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) \ \stackrel{\text{def}}{=} \ \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{13.5.15}$$

$$= \ (a^\star, a, b, \ x_L, \ s_{\mathcal{L}}, \ x_K, \kappa, T, L, K, \mathcal{L}, V_t). \tag{13.5.16}$$

13.6 Derivation of $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$

For the same reason as in Section 24.1.2(p.249) we see that applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathscr{A}\{T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$ given by Lemmas 12.2.1(p.77) – 12.2.6 yields $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$.

Lemma 13.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, -\infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, b^\star]$.
(f) $\tilde{T}(x) = b - x$ *on* $[b^\star, \infty)$ *and* $\tilde{T}(x) < b - x$ *on* $(-\infty, b^\star)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $T(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \le \min\{0, b - x\}$ *on* $(-\infty, \infty)$.
(i) $\tilde{T}(0) = b$ *if* $b^\star \le 0$ *and* $\tilde{T}(0) = 0$ *if* $a > 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b^\star > y$, *then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b^\star > b$.

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.1(p.77). ∎

Direct proof See Lemma A 3.7(p.281). ∎

Applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (12.2.9(p.79))-(12.2.14), we obtain the relations below:

$$\tilde{L}(x) \begin{cases} = \lambda\beta b + s - \lambda\beta x \text{ on } [b^\star, -\infty) & \cdots (1), \\ < \lambda\beta b + s - \lambda\beta x \text{ on } (-\infty, b^\star) & \cdots (2), \end{cases} \tag{13.6.1}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta b + s - \delta x \text{ on } [b^\star, \infty) & \cdots (1), \\ < \lambda\beta b + s - \delta x \text{ on } (-\infty, b^\star) & \cdots (2). \end{cases} \tag{13.6.2}$$

$$\tilde{K}(x) \begin{cases} < -(1-\beta)x + s \text{ on } (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s \text{ on } (-\infty, a] & \cdots (2), \end{cases} \tag{13.6.3}$$

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{13.6.4}$$

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta b + s + (1-\lambda)\beta x \text{ on } [b^\star, \infty) & \cdots (1), \\ \beta x + s \text{ on } (-\infty, a] & \cdots (2). \end{cases} \tag{13.6.5}$$

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\, x_{\tilde{L}} \cdots (1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\, x_{\tilde{K}} \cdots (2). \tag{13.6.6}$$

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (12.2.9(p.79))-(12.2.14). ∎

Direct proof See (A 3.1(p.281))-(A 3.6). ∎

Lemma 13.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $\tilde{L}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{L}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{L}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) *Let* $s = 0$. *Then* $x_{\tilde{L}} = a$ *where* $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\ge)\ 0$.
(e) *Let* $s > 0$.
   1. $x_{\tilde{L}}$ *uniquely exists with* $x_{\tilde{L}} > a$ *where* $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.
   2. $(\lambda\beta b + s)/\lambda\beta \ge (<)\ b^\star \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta < (\ge)\ b^\star$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.2(p.80). ∎

Direct proof See Lemma A 3.8(p.282). ∎

Corollary 13.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (\ge)\ 0$.
(b) $x_{\tilde{L}} \le (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 12.2.1(p.80). ∎

Direct proof See Corollary A 3.2(p.282). ∎

Lemma 13.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $\tilde{K}(x)$ *is continuous on* $(-\infty,\infty)$.
(b) $\tilde{K}(x)$ *is nonincreasing on* $(-\infty,\infty)$.
(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a,\infty)$.
(d) $\tilde{K}(x)$ *is strictly decreasing on* $(-\infty,\infty)$ *if* $\beta<1$.
(e) $\tilde{K}(x)+x$ *is nondecreasing on* $(-\infty,\infty)$.
(f) $\tilde{K}(x)+x$ *is strictly increasing on* $(-\infty,b^\star]$.
(g) $\tilde{K}(x)+x$ *is strictly increasing on* $(-\infty,\infty)$ *if* $\lambda<1$.
(h) *If* $x>y$ *and* $b^\star>y$, *then* $\tilde{K}(x)+x>\tilde{K}(y)+y$.
(i) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{K}}=a$ *where* $x_{\tilde{K}}<(\geq)\ x \Leftrightarrow \tilde{K}(x)<(=)\ 0 \Rightarrow \tilde{K}(x)<(\geq)\ 0$.
(j) *Let* $\beta<1$ *or* $s>0$.
   1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}}<(=(>))\ x \Leftrightarrow \tilde{K}(x)<(=>))\ 0$.
   2. $(\lambda\beta b+s)/\delta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{K}}=(<)\ (\lambda\beta b+s)/\delta$.
   3. *Let* $\tilde{\kappa}<(=(>))\ 0$. *Then* $x_{\tilde{K}}<(=(>))\ 0$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.3(p.80). ∎

Direct proof See Lemma A 3.9(p.282) . ∎

Corollary 13.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $x_{\tilde{K}}<(\geq)\ x \Leftrightarrow \tilde{K}(x)<(\geq)\ 0$.
(b) $x_{\tilde{K}}\leq(\geq)\ x \Rightarrow \tilde{K}(x)\leq(\geq)\ 0$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 12.2.2(p.80). ∎

Direct proof See Corollary A 3.3(p.283) . ∎

Lemma 13.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}/\tilde{K}_{\mathbb{P}}\}$)

(a) *Let* $\beta=1$ *and* $s=0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}=a$.
(b) *Let* $\beta=1$ *and* $s>0$. *Then* $x_{\tilde{L}}=x_{\tilde{K}}$.
(c) *Let* $\beta<1$ *and* $s=0$. *Then* $a<(=(>))\ 0 \Leftrightarrow x_{\tilde{L}}<(=(>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(=))\ 0$.
(d) *Let* $\beta<1$ *and* $s>0$. *Then* $\tilde{\kappa}<(=(>))\ 0 \Leftrightarrow x_{\tilde{L}}<(=(>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}}<(=(>))\ 0$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.4(p.80). ∎

Direct proof See Lemma A 3.10(p.283) . ∎

Lemma 13.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta<1$.
(b) *Let* $\lambda\beta b\leq a$.
   1. $x_{\tilde{L}}\geq\lambda\beta b+s$.
   2. *Let* $s>0$ *and* $\lambda\beta<1$. *Then* $x_{\tilde{L}}>\lambda\beta b+s$.
(c) *Let* $\lambda\beta b>a$. *Then there exists a* $s_{\tilde{\mathcal{L}}}>0$ *such that if* $s_{\tilde{\mathcal{L}}}>(\leq)\ s$, *then* $x_{\tilde{L}}<(\geq)\ \lambda\beta b+s$.

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.5(p.81). ∎

Direct proof See Lemma A 3.11(p.283) . ∎

Lemma 13.6.6 ($\tilde{\kappa}_{\mathbb{P}}$) *We have*:

(a) $\tilde{\kappa}=\lambda\beta b+s$ *if* $b^\star<0$ *and* $\tilde{\kappa}=s$ *if* $a>0$.
(b) *Let* $\beta<1$ *or* $s>0$. *Then* $\tilde{\kappa}<(=(>))\ 0$. *Then* $x_{\tilde{K}}<(=(>))\ 0$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 12.2.6(p.81). ∎

Direct proof See Lemma A 3.12(p.284) . ∎

13.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

□ `Tom` **13.7.1** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta=1$ *and* $s=0$.

(a) V_t *is nonincreasing in* $t>0$.
(b) Ⓢ `dOITs`$_\tau\langle\tau\rangle$$_\blacktriangle$ *where* `CONDUCT`$_{\tau\geq t>1\blacktriangle}$. □

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 12.4.1(p.82). ∎

Direct proof See `Tom` A 4.5(p.289) . ∎

□ **Tom 13.7.2** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 0$ or $s > 0$. Then, for a given starting time $\tau > 1$:*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \leq a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\|}$.
(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   i. *Let $b + s \geq b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\|}$.
   ii. *Let $b + s < b^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆.*
   i. *Let $a < 0$ ⟅$\tilde{\kappa} < 0$⟆. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $a = 0$ ⟅$\tilde{\kappa} = 0$⟆.*
      1. *Let $\beta b + s \geq b^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\|}$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let $a > 0$ ⟅$\tilde{\kappa} > 0$⟆.*
      1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{\|}$.
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.47) $\boxed{Ⓢ\blacktriangle}\boxed{⊛\|}$ *is true.*

Proof by analogy Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 12.4.2(p.82). ∎

Direct proof See Tom A 4.6(p.289) . ∎

13.8 Optimal Price to Propose

Lemma 13.8.1 ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *The optimal price to propose z_t is nonincreasing in $t > 0$.* □

Proof Obvious from Tom's 13.7.1(a) and 13.7.2(a) and from (6.2.41(p.23)) and Lemma A 3.3(p.278) . ∎

Chapter 14

Analogy Theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)

14.1 Relationship between $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$

14.1.1 Assertion system $\mathscr{A}$

First note the three relations below:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (11.5.55\text{(p.66)})), \tag{14.1.1}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.3.1\text{(p.81)})), \tag{14.1.2}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.5.6\text{(p.88)})). \tag{14.1.3}$$

Then the inverses of the above relations were:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (11.8.46\text{(p.72)})), \tag{14.1.4}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (12.3.6\text{(p.82)})), \tag{14.1.5}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.5.14\text{(p.88)})). \tag{14.1.6}$$

From (14.1.3), (14.1.2), and (14.1.4) we obtain the relation below:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{14.1.7}$$

As an inverse of this, from (14.1.1), (14.1.5), and (14.1.6) we obtain the relation below:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{14.1.8}$$

14.1.2 System of Optimality Equations SOE

First note the three relations below:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \;(\leftarrow (11.5.38\text{(p.63)})), \tag{14.1.9}$$

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \;(\leftarrow (12.3.2\text{(p.81)})), \tag{14.1.10}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.5.4\text{(p.87)})), \tag{14.1.11}$$

The inverses of the above relations were:

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \;(\leftarrow (11.8.40\text{(p.72)})), \tag{14.1.12}$$

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \;(\leftarrow (12.3.7\text{(p.82)})), \tag{14.1.13}$$

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.5.12\text{(p.88)})), \tag{14.1.14}$$

From (14.1.11), (14.1.10), and (14.1.12) we obtain the relation below:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{14.1.15}$$

As an inverse of this, from (14.1.9), (14.1.13), and (14.1.14) we obtain the relation below:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{14.1.16}$$

14.1.3 Attribute Vector $\boldsymbol{\theta}$

First note the three relations below:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (11.5.56(\text{p.66}))), \tag{14.1.17}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (12.3.3(\text{p.81}))), \tag{14.1.18}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (13.5.7(\text{p.88}))), \tag{14.1.19}$$

Then the inverses of the above relations were:

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (11.8.47(\text{p.72}))), \tag{14.1.20}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (12.3.8(\text{p.82}))), \tag{14.1.21}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (13.5.15(\text{p.88}))), \tag{14.1.22}$$

From (14.1.19), (14.1.18), and (14.1.20) we obtain the relation below:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})], \tag{14.1.23}$$

As an inverse of this, from (14.1.17), (14.1.21), and (14.1.22) we obtain the relation below:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})]. \tag{14.1.24}$$

14.2 Analogy Theorem

Let us note

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (11.5.57(\text{p.66}))), \tag{14.2.1}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = (b^\star, b, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (13.5.8(\text{p.88}))). \tag{14.2.2}$$

Herein let us define

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \overset{\text{def}}{\equiv} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \cdots (1), \qquad \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} \overset{\text{def}}{\equiv} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \cdots (2). \tag{14.2.3}$$

Then, from (14.1.7) and (14.1.8), from (14.1.15) and (14.1.16), and from (14.1.23) and (14.1.24) we have respectively

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}\,[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \cdots (1) \qquad \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}\,[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \cdots (2), \tag{14.2.4}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}\,[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \cdots (1) \qquad \mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}\,[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \cdots (2) \tag{14.2.5}$$

$$\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}\,[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}])] \cdots (1) \qquad \boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}\,[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}])] \cdots (2). \tag{14.2.6}$$

From (14.2.4 (1)) we immediately obtain the following theorem.

Theorem 14.2.1 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \overset{\text{def}}{\equiv} \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{14.2.7}$$

From (14.2.4 (2)) we immediately obtain the following theorem.

Theorem 14.2.2 (analogy $[\tilde{\mathbb{P}} \to \tilde{\mathbb{R}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{14.2.8}$$

14.3 Analogy Replacement Operation $[\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{R}}]$

Here note that $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$ in the right hand of (14.1.7) means that the three operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}} \to \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$. Then, putting the flow in vertically, we have

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \overset{\text{def}}{\equiv} \left\{\begin{array}{llllllllllll} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l} \cdots(1) \\ \\ \cdots(2) \end{array} \quad (\leftarrow\ (11.8.33(\text{p.71})))$$

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} \overset{\text{def}}{\equiv} \left\{\begin{array}{ll} a, & \mu \\ \downarrow & \downarrow \\ a^\star, & a \end{array}\right\} \begin{array}{l} \cdots(3) \\ \\ \cdots(4) \end{array} \quad (\leftarrow\ (12.2.1(\text{p.77})))$$

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \overset{\text{def}}{\equiv} \left\{\begin{array}{llllllllllll} a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l} \cdots(5) \\ \\ \cdots(6) \end{array} \quad (\leftarrow\ (13.5.3(\text{p.87})))$$

The above flow means the following:

- First, let us focus attention on elements *outside* the dashbox $\square$. Then, we see that first (1)-row changes into (2)-row, next (2)-row is identical to (5)-row, and finally (5)-row changes into (6)-row, which is identical to the original (1)-row. In other words, it follows that (1)-row remains unchanged *outside* the dashbox even if these operations are applied.
- Next, let us focus attention on elements *inside* the dashbox $\square$. Then, we see that first (1)-row changes into (2)-row, next (2)-row changes into (4)-row via (3)-row, and finally (4)-row changes into (6)-row via (5)-row. In other words, b and μ in (1)-row change into respectively $b^\star$ and b in (6)-row through the application of these operations.

From the above we see that the triple operations $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$ can be eventually reduced to the single operation

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \stackrel{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\}$$

Removing the unchanged elements from the above $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, we can rewrite this as

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \{b \to b^\star,\ \mu \to b\}. \tag{14.3.1}$$

Similarly, $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$ in the right hand of (14.1.8) means that the operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \to \mathcal{A}_{\mathbb{P}\to\mathbb{R}} \to \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$. Then, putting the flow in vertically, we have

$$\begin{array}{l}
\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \stackrel{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l}\cdots(1)\\ \\ \cdots(2)\end{array} \quad (\leftarrow\ (13.5.11(\text{p.88})))\\[2ex]
\mathcal{A}_{\mathbb{P}\to\mathbb{R}} \stackrel{\text{def}}{=} \left\{\begin{array}{cc} a^\star, & a \\ \downarrow & \downarrow \\ a, & \mu \end{array}\right\} \begin{array}{l}\cdots(3)\\ \\ \cdots(4)\end{array} \quad (\leftarrow\ (12.3.5(\text{p.81})))\\[2ex]
\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \stackrel{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l}\cdots(5)\\ \\ \cdots(5)\end{array} \quad (\leftarrow\ (11.5.31(\text{p.63})))
\end{array}$$

Accordingly, it follows that the above flow can be eventually reduced to as follows.

$$\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \{b^\star \to b,\ b \to \mu\}. \tag{14.3.2}$$

Chapter 15

Integration Theory

15.1 Flow of the Whole Discussion

The complete picture of the integration theory can be summarized as follows:

⟨1⟩ $\mathscr{A}\{T_{\mathbb{R}}\}$ is *proven* (Lemma 9.1.1(p.41)).

⟨2⟩ $\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ is *proven* (Lemmas 9.2.1(p.43) – 9.3.1(p.45)).

⟨3⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is *proven* (Tom's 10.2.1(p.47) and 10.2.2).

⟨4⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is *derived* (Tom's 11.7.1(p.69) and 11.7.2).

⟨5⟩ $\mathscr{A}\{T_{\mathbb{P}}\}$ is *proven* (Lemma 12.2.1(p.77)).

⟨6⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ is *derived* (Tom's 12.4.1(p.82) and 12.4.2).

⟨7⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is *derived* (Tom's 13.7.1(p.90) and 13.7.2).

⟨8⟩ The analogous relation between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is shown (Theorems 14.2.1(p.94) and 14.2.2).

⟨9⟩ The diagonal symmetry is proven (Theorems 17.1.1(p.113)-17.1.6).

The above flow over ⟨1⟩ – ⟨9⟩ can be schematized as in Figure 15.1.1 below where the inside of the three shadow-boxes are directly *proven* and the inside of the remaining frame-boxes excluding ⟨9⟩ are *derived* by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$.

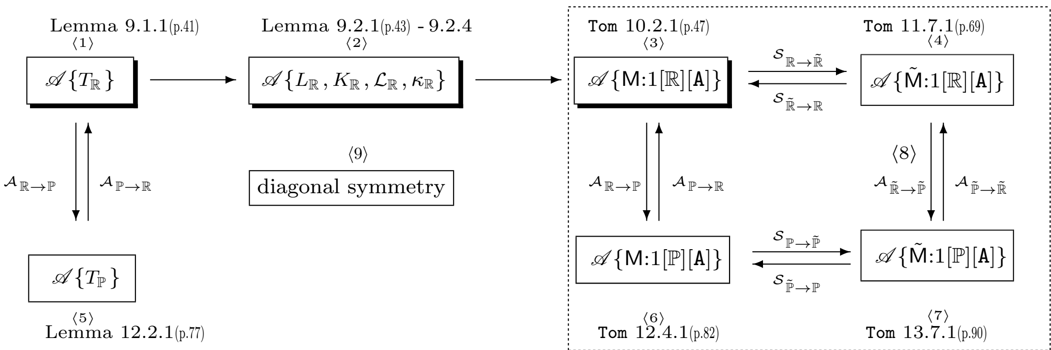


Figure 15.1.1: The flow of the whole discussion of the integration theory

All operations $\mathcal{S}_{\mathbb{X}\to\mathbb{Y}}$ and $\mathcal{A}_{\mathbb{X}\to\mathbb{Y}}$ within the flow depicted by Figure 15.1.1 are ones used in the eight theorems below:

- Theorems 11.5.1(p.66) and 11.8.1(p.72) (symmetry).
- Theorems 12.3.1(p.81) and 12.3.2(p.82) (analogy).
- Theorems 13.5.1(p.88) and 13.5.2(p.88) (symmetry).
- Theorems 14.2.1(p.94) and 14.2.2(p.94) (analogy).

15.2 Overview of Integration Theory

The interrelationship among the quadruple assertion systems within the dashbox ⬚ of Figure 15.1.1 implies the following. First, an assertion system of the model M:1[$\mathbb{R}$][A] selected as a *seed* within the quadruple-asset-trading-models $\mathcal{Q}\langle$M:1[A]$\rangle$ is *defined* and *proven*. Next, the assertion system for each of the remaining three models is *derived* by sequentially applying the symmetry transformation operation and the analogy replacement operation, $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, and $\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$, to results obtained for the seed assertion system. Since it is proven that any of these operations are reversible, even if any other assertion system within $\mathcal{Q}\langle$M:1[A]$\rangle$ is selected as a seed, the same flow as the above can holds. Let us refer to the methodology which integrates the quadruple assertion systems in such a fashion as stated above as the *integration theory.* In the conventional methodology all within the quadruple assertion systems must be separately defined and *one by one* proven. On the other hand, in our methodology based on the integration theory, the number of assertion systems which are defined and proven is *only one* as a seed. In Part 3 that follows we try to apply the integration theory to all of the remaining five quadruple-asset-trading-models in Table 3.3.1(p.11) except for $\mathcal{Q}\langle$M:1[A]$\rangle$ the analysis of which was already ended. After having finished reading Part 3, it will be realized that the integration theory provides a strong tool for the treatment of asset trading problems.

15.3 Summary of Operations

For convenience of reference, below let us copy (11.5.31(p.63)), (13.5.3(p.87)), (12.2.1(p.77)), and (14.3.1(p.95)), which are used as the symmetry transformation operation and the analogy replacement operation in the correlation diagram of Figure 15.1.1 above.

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}=\left\{\begin{array}{llllllllllll} a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\}. \tag{15.3.1}$$

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}=\left\{\begin{array}{llllllllllll} a^\star, & a, & b, & x_L, & x_K, & \kappa, & s_{\mathcal{L}}, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & \tilde{\kappa}, & s_{\tilde{\mathcal{L}}}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\}. \tag{15.3.2}$$

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}=\{a\to a^\star,\ \mu\to a\}. \tag{15.3.3}$$

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}=\{b\to b^\star,\ \mu\to b\}. \tag{15.3.4}$$

Chapter 16

Inheritance and Collapse

16.1 Another Aspect of Trading Problems

First, let us note the following:

1. In a selling problem, a seller (leading trader) *delivers* the asset to a buyer (opponent trader); in other words, the buyer *receives* it from the seller.
2. In a buying problem, a buyer (leading trader) *receives* the asset from a buyer (opponent trader); in other words, the seller *delivers* it to the buyer.

The above can be schematized as below.

$$\begin{array}{lllcll} & \text{leading trader} & & & & \text{opponent trader} \\ & \downarrow & & & & \downarrow \\ \text{selling problem:} & \text{seller} & \text{(delivering-side)} & \rightarrow & \text{(recieving-side)} & \text{buyer} \end{array} \tag{16.1.1}$$

$$\begin{array}{lllcll} \text{buying problem:} & \text{buyer} & \text{(recieving-side)} & \leftarrow & \text{(delivering-side)} & \text{seller} \end{array} \tag{16.1.2}$$

16.2 Trading Problem with Negative Price

Next, let us note that a price w may become negative on the *total market* $\mathscr{F}$ (see A7(p.8)); however, the price is usually positive in the real world; in other words, it cannot become zero or negative. Now, let us consider the case that an asset traded there is, for example, such industrial wastes as surplus soil, concrete blocks, etc. which are disposed of when a building is broken up. Then, in whether a selling problem or a buying problem, the buyer as a receiving-side rightly requires an amount of money as disposal cost instead of paying an amount of money. This implies that the problem of dealing such an item can be regarded as a trading problem with a negative price.

16.3 Three Kinds of Markets

Let us call $\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}$ (see (2.1.5(p.8))), called the *total market*, and let us define the following three kinds of markets:

$$\mathscr{F}^+ \overset{\text{def}}{=} \{F \mid 0 < a < b\} \quad (\textit{positive market}), \tag{16.3.1}$$

$$\mathscr{F}^\pm \overset{\text{def}}{=} \{F \mid a \le 0 \le b\} \quad (\textit{mixed market}), \tag{16.3.2}$$

$$\mathscr{F}^- \overset{\text{def}}{=} \{F \mid a < b < 0\} \quad (\textit{negative market}). \tag{16.3.3}$$

Each of the above three markets implies the following:

(a) Positive market $\mathscr{F}^+$ In an asset trading problem in the real world, the price is usually positive, i.e., the problem is defined on this market.

(b) Negative market $\mathscr{F}^-$ The trading problem in Section 16.2 is defined on this market; this is also called *junk market.*

(c) Mixed market $\mathscr{F}^\pm$ For example, suppose you must waste a piece of well-worn furniture, say a book cabinet, sofa bed, etc. For such a good, normally there exist the two kinds of receiving-sides: One who pays some money on the ulterior motive that some profit might be obtained by reselling it, the other who requires some money for the reason that some cost may be incurred for its disposal. This market can be regarded as a market in which the positive market and the negative market are mixed; this is also called *secondhand market.*

For the above reason, in order to discuss an asset trading problem more generally and comprehensively it is better to extend the region of price to the total market $\mathscr{F} = (-\infty, \infty)$.

Remark 16.3.1 (life of durable goods) A new durable good (automobile, house furnishings, TV, etc.) is first placed on the positive market $\mathscr{F}^+$, gradually deteriorates year after year, then is drove to the mixed market $\mathscr{F}^\pm$ before long, and finally is junked in the negative market $\mathscr{F}^-$. This deterioration flow implies that in order to complete a theory of trading it will become necessary to make discussions over all of the three markets $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$. □

16.4 Market Restriction

Let us refer to the restriction of the total market $\mathscr{F}$ to a given subset $\mathscr{F}' \subset \mathscr{F}$ as the *market restriction* of $\mathscr{F}$ to $\mathscr{F}'$, called the *restricted-total-market.* In the present paper, we consider the three kinds of market restrictions defined in Section 16.3. Let us denote the *operations* of restricting $\mathscr{F}$ to the above three restricted markets by the same symbols $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$, called the *positive market restriction*, the *mixed market restriction*, and the *negative market restriction* respectively. Throughout the rest of this paper, by Model^+, $\mathsf{Model}^\pm$, and Model^- let us denote the models defined on the restricted markets $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ respectively, called the *market restricted models.* For explanatory convenience, for $x = 1, 2, 3$ and $\mathtt{X} = \mathtt{A}, \mathtt{E}$ let us define:

$$\mathcal{Q}\langle \mathsf{M}:x[\mathtt{X}]^+\rangle = \{\mathsf{M}:x[\mathbb{R}][\mathtt{X}]^+, \tilde{\mathsf{M}}:x[\mathbb{R}][\mathtt{X}]^+, \mathsf{M}:x[\mathbb{P}][\mathtt{X}]^+, \tilde{\mathsf{M}}:x[\mathbb{P}][\mathtt{X}]^+\}, \tag{16.4.1}$$

$$\mathcal{Q}\langle \mathsf{M}:x[\mathtt{X}]^\pm\rangle = \{\mathsf{M}:x[\mathbb{R}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}:x[\mathbb{R}][\mathtt{X}]^\pm, \mathsf{M}:x[\mathbb{P}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}:x[\mathbb{P}][\mathtt{X}]^\pm\}, \tag{16.4.2}$$

$$\mathcal{Q}\langle \mathsf{M}:x[\mathtt{X}]^-\rangle = \{\mathsf{M}:x[\mathbb{R}][\mathtt{X}]^-, \tilde{\mathsf{M}}:x[\mathbb{R}][\mathtt{X}]^-, \mathsf{M}:x[\mathbb{P}][\mathtt{X}]^-, \tilde{\mathsf{M}}:x[\mathbb{P}][\mathtt{X}]^-\}. \tag{16.4.3}$$

Remark 16.4.1 (inheritance and collapse) Herein recall that the integration theory consisting of the symmetry theorem and the analogy theorem is what can be constructed under the basic premise that the price, whether reservation price or posted price, is defined on the total market $\mathscr{F} = (-\infty, \infty)$ (see Chapter 16(p.99)). Accordingly, if the total market $\mathscr{F}$ is restricted to a subset $\mathscr{F}' \subset \mathscr{F}$, then it must be examined whether the symmetrical relation and the analogous relation given by the two theorems are inherited or collapses. □

When no change occurs even if a market restriction is applied to a given assertion A, the assertion is said to be *free from* the market restriction, called the *market-restriction-free assertion.*

Lemma 16.4.1 (market-restriction-free) *Even if a market restriction is applied to a market-restriction-free assertion, no change occurs.* □

Proof Evident. ∎

16.5 Market Restriction for Quitting Penalty ρ

- Selling Problem to dispose of it by delivering it to a junk dealer on payment of some <u>cost</u> $\rho' > 0$. Since "paying some cost $\rho' > 0$ to the junk dealer who is a buyer" implies "receiving the negative selling price $\rho = -\rho' < 0$ from the buyer". This implies that the selling problem can be regarded as a selling problem with the negative selling price $\rho < 0$. Not only for the above reason but also to discuss the problem more generally, in this paper we dare to define the ρ on $(-\infty, \infty)$. If there exists no junk dealer who wants to receive it even if proposing however large the cost $\rho' > 0$, the seller must dump it by himself. If it is not illegal, the seller will dump it at his own expense $\rho' > 0$; however, if it is illegal, the seller will be someday punished with a fine $\rho' > 0$. Finally, if the seller is not willing to illegally dump it, then such an `ASP` does not take shape as a real problem to attack; Such problem must be said to be unworthy to discuss.
- Buying Problem Now, since the action of *buying* an asset from a seller can be regarded as that of *receiving* it from the seller, if it is a good such as an industrial waste, the buyer as a *receiving-side* (leading trader) naturally requires some money $\rho' > 0$ to the seller as a *delivering-side* (opponent trader); this problem can be said to be an asset buying problem with the negative buying price $\rho = -\rho' < 0$. Now, herein let us consider a question "Whether or not there can exist an asset buying problem (`ABP`) with a *negative buying price* $\rho' < 0$?" Temporarily, suppose that such an asset buying problem exists. Then, it is an asset buying problem where the buyer pays the negative price $\rho' < 0$. Since "payment of the negative buying price $\rho' < 0$" implies "acquisition of the positive buying price $\rho = -\rho' > 0$ nevertheless it is a buying problem"; needless to say, such an absurd story is not possible at all from a real viewpoint. For this reason, in order to discuss the problem more generally the buying price $\rho' < 0$ must be defined on $(-\infty, \infty)$ similarly to in `ABP`. If there exists no seller who wants to sell it however large the price may be, then such an `ABP` does not take shape as a real problem to tackle, hence it must be said to be unworthy to discuss.

16.6 Constrains of Inequality Conditions by Market Restriction

The lemma below will be used later on when performing the operation of market restrictions.

Lemma 16.6.1 (positive market $\mathscr{F}^+$) *Suppose* $0 < a$.

[1] [ref.8078] $0 < a < \mu < b$. Proof: Evident from (2.1.3(p.8)).

[2] [ref.9343] $\beta b \le b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < 1 \times b \le b$.

[3] [ref.7865] $\beta\mu < b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < 1 \times \mu < b$.

[4] [ref.8369] $\beta a < b$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < 1 \times a < b$.

[5] [ref.9483] $a < \beta\mu$ *and* $\beta\mu \le a$ *are both possible.* Proof: Since $0 < a < 1 \times \mu$, the former is possible for $0 < \beta \le 1$ sufficiently close to 1 and the latter is possible for any sufficiently small $\beta > 0$.

[6] [ref.6867] $a < \beta b$ *and* $\beta b \le a$ *are both possible.* Proof: Since $0 < a < 1 \times b$, the former is possible for $0 < \beta \le 1$ sufficiently close to 1 and the latter is possible for any sufficiently small $\beta > 0$.

[7] [ref.6296] $\beta b < b^\star$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 < 1 \times b < b^\star$ due to Lemma 13.6.1(p.89) (n). □

Lemma 16.6.2 (mixed market $\mathscr{F}^\pm$) *Suppose* $a \le 0 \le b$.

[8] [ref.8062] $a < \beta\mu < b$ *for* $0 < \beta \le 1$. Proof: Let $\mu = 0$. Then $a < \mu = \beta\mu = 0 < b$ for $0 < \beta \le 1$. Let $\mu \ne 0$. If $a < \mu < 0$, then $a < 1\times\mu < 0 \le b$, hence $a < \beta\mu < 0 \le b$ for $0 < \beta \le 1$ and if $0 < \mu < b$, then $a \le 0 < 1\times\mu < b$, hence $a \le 0 < \beta\mu < b$ for $0 < \beta \le 1$. Accordingly, whether $a < \mu < 0$ or $0 < \mu < b$, we have $a < \beta\mu < b$ for any β. Thus, whether $\mu = 0$ or $\mu \ne 0$, it follows that $a < \beta\mu < b$ for $0 < \beta \le 1$.

[9] [ref.6907] $\beta a < b$ *for* $0 < \beta \le 1$. Proof: Since $1 \times a \le 0 \le b$, we have $\beta a \le 0 \le b$ for $0 < \beta \le 1$.

[10] [ref.6892] $a < \beta b$ *for* $0 < \beta \le 1$. Proof: If $b > 0$, then $a \le 0 < b = 1 \times b$, hence $a \le 0 < \beta b$ for any β. If $b = 0$, then $a < b = 0 = \beta \times 0 = \beta b$ for $0 < \beta \le 1$. Therefore, whether $b > 0$ or $b = 0$, we have $a < \beta b$ for $0 < \beta \le 1$.

[11] [ref.6896] $a^\star < \beta a$ *for* $0 < \beta \le 1$. Proof: Immediate from $a^\star < 1 \times a \le 0$ due to Lemma 12.2.1(p.77) (n).

[12] [ref.6298] $\beta b < b^\star$ *for* $0 < \beta \le 1$. Proof: Immediate from $0 \le 1 \times b < b^\star$ due to Lemma 13.6.1(p.89) (n). □

Lemma 16.6.3 (negative market $\mathscr{F}^-$) *Suppose* $b < 0$.

[13] [ref.7486] $a < \mu < b < 0$. Proof: Evident from (2.1.3(p.8)).

[14] [ref.6118] $a \le \beta a$ $0 < \beta \le 1$. Proof: Immediate from $a \le 1 \times a < 0$.

[15] [ref.8068] $a < \beta\mu$ *for* $0 < \beta \le 1$. Proof: Immediate from $a < 1 \times \mu < 0$.

[16] [ref.7482] $a < \beta b$ *for* $0 < \beta \le 1$. Proof: Immediate from $a < 1 \times b < 0$.

[17] [ref.7478] $\beta\mu < b$ *and* $b \le \beta\mu$ *are both possible.* Proof: Since $1 \times \mu < b < 0$, the former is true for $0 < \beta \le 1$ sufficiently close to 1 and the latter is true for any sufficiently small $\beta > 0$.

[18] [ref.8296] $\beta a < b$ *and* $b \le \beta a$ *are both possible.* Proof: Since $1 \times a < b < 0$, the former is possible for $0 < \beta \le 1$ sufficiently close to 1 and the latter is possible for any sufficiently small $\beta > 0$.

[19] [ref.6919] $a^\star < \beta a$ *for* $0 < \beta \le 1$. Proof: Immediate from $a^\star < 1 \times a < 0$ due to Lemma 12.2.1(p.77) (n). □

16.7 Market Restriction of Assertion Systems

16.7.1 $\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^{+,\pm,-}\}$

16.7.1.1 Positive Restriction

□ **Pom 16.7.1** ($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.
(b) ⓢ $\text{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\ge t>1\blacktriangle}$. □ → → ⓢ

Proof The same as Tom 10.2.1 due to Lemma 16.4.1(p.100). ∎

□ **Pom 16.7.2** ($\mathscr{A}\{$M:1[$\mathbb{R}$][A]$^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
(b) *Let* $\beta\mu \ge b$ (impossible). ———
(c) *Let* $\beta\mu < b$ (always holds).
  1. *Let* $\beta = 1$.
    i. *Let* $\mu - s \le a$. *Then* ● $\text{dOITd}_{\tau>1}\langle 1\rangle_\parallel$. → → ⓓ
    ii. *Let* $\mu - s > a$. *Then* ⓢ $\text{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\ge t>1\blacktriangle}$ → → ⓢ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* ⓢ $\text{dOITs}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\text{CONDUCT}_{\tau\ge t>1\blacktriangle}$ → → ⓢ

3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ (see **Numerical Example** 16.8.1(p.108)) → → ⓢ
   ii. *Let $s \geq \beta\mu$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{||}$ (see **Numerical Example** 16.8.2(p.109)) → → ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots$ **(1)**. Let $\beta < 1$ or $s > 0$. Then $\kappa = \beta\mu - s \cdots$ **(2)** from Lemma 9.3.1(p.45) (a) with $\lambda = 1$.

(a) The same as Tom 10.2.2(p.48) (a).

(b,c) Always $\beta\mu < b$ due to [3(p.101)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as Tom 10.2.2(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 10.2.2.

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$. Then, since $\kappa > 0$ due to (2), it suffices to consider only (c2i) of Tom 10.2.2.

(c3ii) Let $\beta\mu \leq s$. Then, since $\kappa \leq 0$ due to (2) and since $\beta\mu - s \leq 0 < a$, it suffices to consider only (c2ii1,c2iii1) of Tom 10.2.2. ∎

16.7.1.2 Mixed Restriction

☐ **Mim 16.7.1** ($\mathscr{A}\{$M:1[ℝ][A]$^\pm\}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. ▯ → → ⓢ

Proof The same as Tom 10.2.1 due to Lemma 16.4.1(p.100). ∎

☐ **Mim 16.7.2** ($\mathscr{A}\{$M:1[ℝ][A]$^\pm\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$* (impossible). ———
(c) *Let $\beta\mu < b$* (always holds).
1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{||}$ → → ⓓ
   ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → ⓢ
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → ⓢ
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{||}$ → → ⓓ
      2. *Let $\beta\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → ⓢ
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{||}$ → → ⓓ
      2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\texttt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle \mid ⊛||}$ *is true* → → ⓢ / ⊛

Proof Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.

(a) The same as Tom 10.2.2(a).

(b,c) Always $\beta\mu < b$ due to [8(p.101)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as Tom 10.2.2(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of Tom 10.2.2 and if $b = 0$, then since always $\beta\mu - s = \beta\mu > a$ due to [8], it suffices to consider only (c2ii2) of Tom 10.2.2. Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from Tom 10.2.2(c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.7(p.17)) with $\lambda = 1$. ∎

16.7.1.3 Negative Restriction

☐ **Nem 16.7.1** ($\mathscr{A}\{$M:1[ℝ][A]$^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → ⓢ

Proof The same as Tom 10.2.1 due to Lemma 16.4.1(p.100). ∎

□ Nem **16.7.2** ($\mathscr{A}\{$M:1[ℝ][A]$^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
   ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\to$ $\to$ ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_1$(p.47) ⓢ▲ ⊛‖ *is true* $\to$ $\to$ ⓢ / ⊛
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
   ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\texttt{S}_1$(p.47) ⓢ▲ ⊛‖ *is true* $\to$ $\to$ ⓢ / ⊛

Proof Suppose $b < 0 \cdots$ **(**1**)**. Let $\beta < 1$ or $s > 0$. Then, we have $\kappa = -s \cdots$ **(**2**)** from Lemma 9.3.1(p.45) (a). Moreover, in this case, both $\beta\mu \geq b$ and $\beta\mu < b$ are possible due to [17(p.101)].

(a,b) The same as `Tom` 10.2.2(a,b).

(c) Let $\beta\mu < b$. Then $s_{\mathcal{L}} > 0 \cdots$ **(**3**)** from Lemma 9.2.4(p.44) (c).

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as `Tom` 10.2.2(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii1,c2iii2) of `Tom` 10.2.2. Since $\beta\mu - s = \beta\mu > a$ due to [15(p.101)] and since $s = 0 < s_L$ due to (3), we have `Tom` 10.2.2(c2iii2).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of `Tom` 10.2.2. ∎

16.7.2 $\mathscr{A}\{\tilde{\text{M}}$:1[ℝ][A]$^{+,\pm,-}\}$

16.7.2.1 Positive Restriction

□ Pom **16.7.3** ($\mathscr{A}\{\tilde{\text{M}}$:1[ℝ][A]$^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. $\to$ ⓢ

Proof The same as `Tom` 11.7.1(p.69) due to Lemma 16.4.1(p.100). ∎

□ Pom **16.7.4** ($\mathscr{A}\{\tilde{\text{M}}$:1[ℝ][A]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
   ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\to$ $\to$ ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_1$(p.47) ⓢ▲ ⊛‖ *is true* $\to$ $\to$ ⓢ / ⊛
3. *Let $\beta < 1$ and $s > 0$.*†
   i. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau}\langle 1\rangle}_{\|}$ $\to$ $\to$ ⓓ
   ii. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\texttt{S}_1$(p.47) ⓢ▲ ⊛‖ *is true* (see `Numerical Example` 16.8.3(p.110)) $\to$ $\to$ ⓢ / ⊛

Proof Suppose $a > 0 \cdots$ **(**1**)**, hence $\tilde{\kappa} = s \cdots$ **(**2**)** from Lemma 11.6.6(p.68) (a). Here note that $\mu\beta \leq a$ and $\mu\beta > a$ are both possible due to [5(p.101)].

(a,b) The same as `Tom` 11.7.2(a,b).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(**3**)** due to Lemma 11.6.5(c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have `Tom` 11.7.2(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.101)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of `Tom` 11.7.2.

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of `Tom` 11.7.2. ∎

16.7.2.2 Mixed Restriction

☐ `Mim` **16.7.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ

Proof The same as `Tom` 11.7.1(p.69) due to Lemma 16.4.1. ∎

☐ `Mim` **16.7.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$. → ⓓ
    ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
      2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
      2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle\,|\,⊛\parallel}$ *is true* → → ⓢ / ⊛

Proof Suppose $a \le 0 \le b$.

(a) The same as `Tom` 11.7.2(a).

(b,c) Always $\beta\mu > a$ due to [8(p.101)], hence $\beta\mu \le a$ is impossible. Hence $s_{\tilde{\mathcal{L}}} > 0 \cdots (1)$ due to Lemma 11.6.5(p.68) (c).

(c1-c1ii) The same as `Tom` 11.7.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of `Tom` 11.7.2. Let $a = 0$. Now, in this case, since $\beta\mu + s = \beta\mu < b$ due to [8(p.101)], it suffices to consider only (c2ii2) of `Tom` 11.7.2. Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions become true from `Tom` 11.7.2(c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.17)). ∎

16.7.2.3 Negative Restriction

☐ `Nem` **16.7.3** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ

Proof The same as `Tom` 11.7.1(p.69) due to Lemma 16.4.1. ∎

☐ `Nem` **16.7.4** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $\mu + s \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
    ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $\beta\mu < -s$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    ii. *Let $\beta\mu \ge -s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\tilde{\kappa} = \beta\mu + s \cdots (3)$ due to Lemma 11.6.6(a).

(a) The same as `Tom` 11.7.2(p.69) (a).

(b,c) Always $a < \beta\mu$ due to [15(p.101)], hence $\beta\mu \le a$ is impossible.

(c1-c1ii) The same as the proof of `Tom` 11.7.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c2i) of `Tom` 11.7.2.

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu < -s$, hence $\beta\mu + s < 0$. Hence, since $\tilde{\kappa} < 0$ due to (3), it suffices to consider only (c2i) of Tom 11.7.2.

(c3ii) Let $\beta\mu \geq -s$, hence $\beta\mu + s \geq 0$. Let $\beta\mu + s = 0$. Then, since $\tilde{\kappa} = 0$ due to (3) and $\beta\mu + s > b$ due to (2), it suffices to consider only (c2iii1) of Tom 11.7.2. Let $\beta\mu + s > 0$. Then, since $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c2iii) of Tom 11.7.2. Then, since $\beta\mu + s > 0 > b$ due to (1), it suffices to consider only (c2ii1) and (c2iii1) of Tom 11.7.2. Accordingly, whether $\beta\mu + s = 0$ or $\beta\mu + s > 0$, we have the same result. ∎

16.7.3 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{+,\pm,-}\}$

16.7.3.1 Positive Restriction

☐ Pom **16.7.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ

Proof The same as Tom 12.4.1 due to Lemma 16.4.1(p.100). ∎

☐ Pom **16.7.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$* (impossible). ______
(c) *Let $\beta a < b$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
    ii. *Let $a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle}\boxed{⊛\parallel}$ $\rightarrow$ → ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$.

(a) The same as Tom 12.4.2(a).

(b,c) Always $\beta a < b$ due to [4(p.101)], hence $\beta a \geq b$ is impossible.

(c1-c1ii) The same as Tom 12.4.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 12.4.2.

(c3-c3iii2) Immediate from Tom 12.4.2(c2-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.18)) with $\lambda = 1$. ∎

16.7.3.2 Mixed Restriction

☐ Mim **16.7.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ

Proof The same as Tom 12.4.1 due to Lemma 16.4.1(p.100). ∎

☐ Mim **16.7.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$* (impossible). ______
(c) *Let $\beta a < b$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
    ii. *Let $a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓢ
    iii. *Let $s > \beta T(0)$.*

1. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\texttt{S}_1$(p.47) $\boxed{Ⓢ\blacktriangle \,|\, ⊛\parallel}$ $\rightarrow$ $\rightarrow$ Ⓢ/⊛

Proof Suppose $a \le 0 \le b$.

(a) The same as Tom 12.4.2(a).

(b,c) Always $\beta a < b$ due to [9(p.101)], hence $\beta a \ge b$ is impossible. .

(c1-c1ii) The same as Tom 12.4.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from Tom 12.4.2(c2i) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.101)], hence the assertion become true from Tom 12.4.2(c2ii2). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as Tom 12.4.2(c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.18))) with $\lambda = 1$. ∎

16.7.3.3 Negative Restriction

☐ **Nem 16.7.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.
(b) *We have* $\boxed{Ⓢ\, \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

Proof Immediate from Tom 12.4.1 due to Lemma 16.4.1(p.100). ∎

☐ **Nem 16.7.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $\ge x_K$ *as* $t \to \infty$.
(b) *Let* $\beta a \ge b$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
(c) *Let* $\beta a < b$.
  1. *Let* $\beta = 1$.
    i. *Let* $a - s \le a^\star$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
    ii. *Let* $a - s > a^\star$. *Then* $\boxed{Ⓢ\, \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\texttt{S}_1$(p.47) $\boxed{Ⓢ\blacktriangle \,|\, ⊛\parallel}$ $\rightarrow$ $\rightarrow$ Ⓢ/⊛
  3. *Let* $\beta < 1$ *and* $s > 0$.
    i. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
    ii. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\texttt{S}_1$(p.47) $\boxed{Ⓢ\blacktriangle \,|\, ⊛\parallel}$ $\rightarrow$ $\rightarrow$ Ⓢ/⊛

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = \kappa_{\mathbb{P}} = -s \cdots (2)$ from Lemma 12.2.6(a). Then, both $\beta a \ge b$ and $\beta a < b$ are possible due to [18(p.101)]. If $\beta a < b$, then $s_{\mathcal{L}} > 0 \cdots (3)$ due to Lemma 12.2.5(p.81) (c) with $\lambda = 1$.

(a) The same as Tom 12.4.2(a).

(b) Let $\beta a \ge b$. Then, the assertion is true Tom 12.4.2(b).

(c) Let $\beta a < b$.

(c1-c1ii) The same as Tom 12.4.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of Tom 12.4.2. In addition, since $\beta a - s = \beta a > a^\star$ due to [19(p.101)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of Tom 12.4.2.

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ from (2), it suffices to consider only (c2iii) of Tom 12.4.2. ∎

16.7.4 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{+,\pm,-}\}$

16.7.4.1 Positive Restriction

☐ **Pom 16.7.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* $\boxed{Ⓢ\, \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

Proof The same as Tom 13.7.1 due to Lemma 16.4.1(p.100). ∎

☐ **Pom 16.7.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \ge x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta b \le a$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
(c) *Let* $\beta b > a$.
  1. *Let* $\beta = 1$.
    i. *Let* $b + s \ge b^\star$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau}\langle 1\rangle}_{\parallel}$ $\rightarrow$
    ii. *Let* $b + s < b^\star$. *Then* $\boxed{Ⓢ\, \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\ge t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\texttt{S}_1$(p.47) $\boxed{Ⓢ\blacktriangle \,|\, ⊛\parallel}$ $\rightarrow$ $\rightarrow$ Ⓢ/⊛
  3. *Let* $\beta < 1$ *and* $s > 0$.
    i. *Let* $\beta b + s \ge b^\star$ *or* $s_{\tilde{\mathcal{L}}} \le s$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau}\langle 1\rangle}_{\parallel}$. $\rightarrow$ ⓓ

ii. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ_\blacktriangle \mid ⊛_{||}}$ $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Suppose $a > 0 \cdots (1)$. Then, $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(a). In this case, $\beta b \leq a$ and $\beta b > a$ are both possible due to [6(p.101)], and if $\beta b > a$, then $s_{\tilde{\mathcal{L}}} > 0 \cdots (3)$ due to Lemma 13.6.5(p.90) (c) with $\lambda = 1$. In addition, we have

(a,b) The same as `Tom` 13.7.2(a,b).

(c) Let $\beta b > a$.

(c1-c1ii)

The same as `Tom` 13.7.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of `Tom` 13.7.2. In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.101)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of `Tom` 13.7.2.

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii-c2iii2) of `Tom` 13.7.2. ∎

16.7.4.2 Mixed Restriction

☐ **`Mim` 16.7.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ

Proof The same as `Tom` 13.7.1 due to Lemma 13.7.1. ∎

☐ **`Mim` 16.7.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \rightarrow \infty$.

(b) *Let* $\beta b \leq a$ (impossible). ———

(c) *Let* $\beta b > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{||}$. $\rightarrow$ ⓓ
   ii. *Let* $b + s < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < -\beta\tilde{T}(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   ii. *Let* $s = -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{||}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   iii. *Let* $s > -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{||}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\mathtt{S}_1$(p.47) $\boxed{ⓢ_\blacktriangle \mid ⊛_{||}}$ $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Let $b \geq 0 \geq a \cdots (1)$.

(a) The same as `Tom` 13.7.2(p.91) (a).

(b,c) Always $\beta b > a$ due to [10(p.101)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as `Tom` 13.7.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of `Tom` 13.7.2. Let $a < 0$. Then, the assertion is true from `Tom` 13.7.2(c2i). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.101)], it suffices to consider only (c2ii2) of `Tom` 13.7.2. Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from `Tom` 13.7.2(c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.19)) with $\lambda = 1$. ∎

16.7.4.3 Negative Restriction

☐ **`Nem` 16.7.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ

Proof The same as `Tom` 13.7.1 due to Lemma 16.4.1(p.100). ∎

☐ **`Nem` 16.7.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \rightarrow \infty$.

(b) *Let* $\beta b \leq a$ (impossible). ———

(c) *Let* $\beta b > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_{||}$ $\rightarrow$ $\rightarrow$ ⓓ
   ii. *Let* $b + s < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ

2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < -\beta\tilde{T}(0)$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   ii. *Let* $s = -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$. *Then* $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   iii. *Let* $-\beta\tilde{T}(0) < s$.
      1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\texttt{S}_1$(p.47) $\boxed{ⓢ\blacktriangle}\boxed{⊛\parallel}$ $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Let $b < 0$, hence $a < b < 0 \cdots$ **(1)**.

(a) The same as Tom 13.7.2(p.91) (a).

(b,c) Always $\beta b > a$ due to [16(p.101)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as Tom 13.7.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 13.7.2.

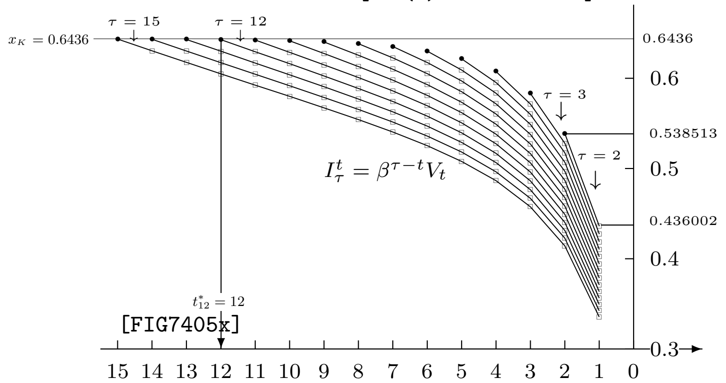
(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 13.7.2(c2-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.19)) with $\lambda = 1$. ∎

16.8 Numerical Example

■ Numerical Example 16.8.1 ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^+$)

This is the example for $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ in Pom 16.7.2(p.101) (c3i) with parameters $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.* Then, we have $x_K = 0.6436$. Figure 16.8.1 below is the graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$ (see (7.2.9(p.34))). For example, the two points on the line of $\tau = 2$ are given by $V_2 = 0.538513\,(\bullet)$ and $\beta V_1 = 0.98 \times 0.444900 = 0.436002\,(\square)$, hence $V_2 > \beta V_1$. Similarly, the three points on the polygonal curve of $\tau = 3$ are given by $V_3 = 0.583152\ (\bullet)$, $\beta V_2 = 0.98 \times 0.538513 = 0.52774274\ (\square)$, and $\beta^2 V_1 = 0.98^2 \times 0.4449 = 0.42728196\ (\square)$, hence $V_3 > \beta V_2 > \beta^2 V_1$. Then, the value of t on the horizontal line corresponding to the bullet • provides the optimal initiating time t^*_τ for each of $\tau = 2, 3, \cdots, 15$, i.e., $\texttt{OIT}_\tau\langle t^*_\tau\rangle$, so we have $t^*_2 = 2$, $t^*_3 = 3$, $\cdots$, $t^*_{15} = 15$ (see t^*_τ-column of the table below). This result means $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ for $\tau = 2, 3, \cdots, 15$. Since $V_t - \beta V_t > 0$ for $t = 2, 3, \cdots, 15$ (see values of $V_t - \beta V_t$-column in the table below), we have $L(V_{t-1}) > 0$ from (10.1.2(p.47)), meaning $\texttt{Conduct}_{15\geq t>1\blacktriangle}$ from (10.1.6(p.47)), i.e., it is strictly optimal to conduct the search on $15 \geq t > 1$.

[006(1)LatexEdit.TEX]



$V_t - \beta V_{t-1}$, t^*_τ (OIT)[006(1)Data.DAT]

t	V_τ	$V_t - \beta V_{t-1}$	t^*_τ	decision
0				
1	0.444900			
2	0.538513	+0.102511	2	Conduct▲
3	0.583152	+0.055409	3	Conduct▲
4	0.607492	+0.036003	4	Conduct▲
5	0.621595	+0.026252	5	Conduct▲
6	0.630035	+0.020871	6	Conduct▲
7	0.635180	+0.017745	7	Conduct▲
8	0.638351	+0.015874	8	Conduct▲
9	0.640318	+0.014734	9	Conduct▲
10	0.641544	+0.014032	10	Conduct▲
11	0.642309	+0.013596	11	Conduct▲
12	0.642788	+0.013325	12	Conduct▲
13	0.643088	+0.013155	13	Conduct▲
14	0.643276	+0.013049	14	Conduct▲
15	0.643393	+0.012983	15	Conduct▲

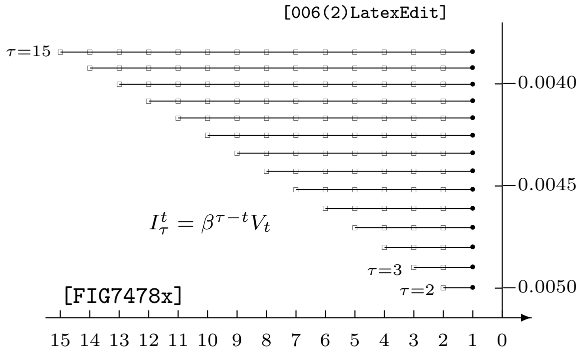
[TAB7473x]

Figure 16.8.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$ where • represents OIT

*Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 > 0.05 = s$. Thus, the condition of this assertion is satisfied.

■ **Numerical Example 16.8.2** ($\mathscr{A}$\{M:1[ℝ][A]\}$^+$)

This is the example for $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ in Pom 16.7.2(p.101) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.50$.† The bullet • in each of the 14 straight lines in Figure 16.8.2 below shows that the optimal initiating time t^*_τ degenerates to time 1 (i.e., $t^*_\tau = 1$ for $\tau = 2, 3, \cdots, 15$) under Preference Rule 7.2.1(p.35), i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau=2,3,\cdots,15}\langle 1\rangle}_{\parallel}$. The result comes from the fact of $V_t - \beta V_t = 0$ for $t = 2, 3, \cdots, 15$ with $t = 2, 3, \cdots, 15$ (see $V_t - \beta V_{t-1}$ - column in the table below), leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$ for $\tau = 2, 3, \cdots, 15$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$ for $\tau = 2, 3, \cdots, 15$.



$V_t - \beta V_{t-1}$ [006(2)Data.DAT]

t	V_t	$V_t - \beta V_{t-1}$	t^*_τ	$\mathbb{S}_t$
0				
1	−0.005100			
2	−0.004998	0.000000	1	−0.00010200
3	−0.004898	0.000000	1	−0.00021960
4	−0.004800	0.000000	1	−0.00029996
5	−0.004704	0.000000	1	−0.00039600
6	−0.004610	0.000000	1	−0.00049008
7	−0.004517	0.000000	1	−0.00058220
8	−0.004427	0.000000	1	−0.00067334
9	−0.004338	0.000000	1	−0.00076154
10	−0.004252	0.000000	1	−0.00084876
11	−0.004167	0.000000	1	−0.00093304
12	−0.004083	0.000000	1	−0.00101634
13	−0.004002	0.000000	1	−0.00109866
14	−0.003922	0.000000	1	−0.00117804
15	−0.003843	0.000000	1	−0.00125644

[TAB7407x]

Figure 16.8.2: Graphs of $I^t_\tau = \beta^{\tau-t} V_t$ ($15 \geq \tau \geq 2, \tau \geq t \geq 1$) where • represents OIT

Note here that numbers in V_t-column are all negative, i.e., red ink, meaning that attacking the asset selling problem makes no profits. Accordingly, if this is of a-E-case (see Concept 2ai(p.9)), you must resign to the red ink and if it is of a-A-case (see Concept 2aii(p.9)), it suffices to pass over the problem without attacking the selling problem itself.

Since $0.5 \times (a + b) = 0.505$ and since $V_t < 0 < 0.01 = a$ for $t = 1, 2, \cdots, 15$ (see V_t-column of the above table), from (A 7.6 (1) (p.296)) we have $T(V_t) = 0.505 - V_t$ for $t = 1, 2, \cdots, 15$, hence we have:

$T(V_1) = 0.505 - (-0.005100) = 0.510100$, $T(V_6) = 0.505 - (-0.004610) = 0.509610$, $T(V_{11}) = 0.505 - (-0.004167) = 0.509167$,
$T(V_2) = 0.505 - (-0.004998) = 0.509998$, $T(V_7) = 0.505 - (-0.004517) = 0.509517$, $T(V_{12}) = 0.505 - (-0.004083) = 0.509083$,
$T(V_3) = 0.505 - (-0.004898) = 0.509898$, $T(V_8) = 0.505 - (-0.004427) = 0.509427$, $T(V_{13}) = 0.505 - (-0.004002) = 0.509002$,
$T(V_4) = 0.505 - (-0.004800) = 0.509800$, $T(V_9) = 0.505 - (-0.004338) = 0.509338$, $T(V_{14}) = 0.505 - (-0.003922) = 0.508922$,
$T(V_5) = 0.505 - (-0.004704) = 0.509704$, $T(V_{10}) = 0.505 - (-0.004252) = 0.509252$, $T(V_{15}) = 0.505 - (-0.003843) = 0.508843$.

Since $\mathbb{S}_t = 0.98 \times T(V_{t-1}) - 0.5$ from (6.2.10(p.22)), we get

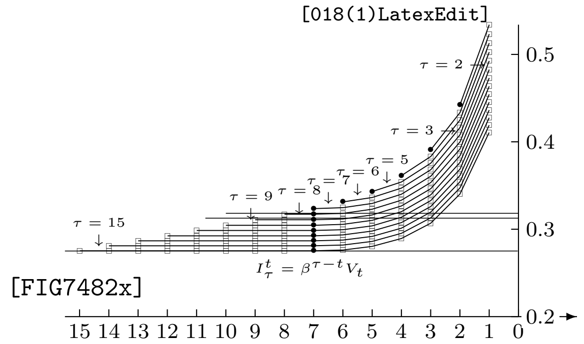
$\mathbb{S}_2 = 0.98 \times 0.510100 - 0.5 = -0.00010200$, $\mathbb{S}_7 = 0.98 \times 0.509610 - 0.5 = -0.00058220$, $\mathbb{S}_{12} = 0.98 \times 0.509167 - 0.5 = -0.00101634$,
$\mathbb{S}_3 = 0.98 \times 0.509998 - 0.5 = -0.00021960$, $\mathbb{S}_8 = 0.98 \times 0.509517 - 0.5 = -0.00067334$, $\mathbb{S}_{13} = 0.98 \times 0.509083 - 0.5 = -0.00109866$,
$\mathbb{S}_4 = 0.98 \times 0.509898 - 0.5 = -0.00029996$, $\mathbb{S}_9 = 0.98 \times 0.509427 - 0.5 = -0.00076154$, $\mathbb{S}_{14} = 0.98 \times 0.509002 - 0.5 = -0.00117804$,
$\mathbb{S}_5 = 0.98 \times 0.509800 - 0.5 = -0.00039600$, $\mathbb{S}_{10} = 0.98 \times 0.509338 - 0.5 = -0.00084876$, $\mathbb{S}_{15} = 0.98 \times 0.508922 - 0.5 = -0.00125644$,
$\mathbb{S}_6 = 0.98 \times 0.509704 - 0.5 = -0.00049008$, $\mathbb{S}_{11} = 0.98 \times 0.509252 - 0.5 = -0.00093304$.

From the results of the above numerical calculation we have $\mathbb{S}_t < 0$ for $15 \geq t > 1$, hence it is *strictly optimal* to skip the search over $15 \geq t > 1$ due to (6.2.9(p.22)), i.e., $\texttt{Skip}_\blacktriangle$. However, since $V_t - \beta V_{t-1} = 0$ for $15 \geq t > 1$ (see $(V_t - \beta V_{t-1})$-column in the above table), we have $V_{15} = \beta V_{14} = \cdots = \beta^{14} V_1$, i.e., the profit attained are indifferent over $15 \geq t > 0$. This is not a contradiction, which is a false feeling caused by confusion from the jumble of intuition and theory (see Alice 3(p.36)).

†Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.50 > 0$. In addition, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 < 0.50 = s$. Thus, the condition of the assertion is satisfied.

■ **Numerical Example 16.8.3** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$ **(buying model)**

This is the numerical example for $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\Vert}$ in $\mathtt{S}_1$(p.47) $\boxed{\circledS_\blacktriangle \mid \circledast_\Vert}$ of Pom 16.7.4(p.103) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.[†] Then, we have $s_{\tilde{\mathcal{L}}} = 0.323274$. Hence, the optimal initiating time t^*_τ is given by t attaining $\min_{\tau\ge t>0} I^t_\tau$ (see (7.2.10(p.35))).[‡] The bullet • in Figure 16.8.3 below shows the optimal initiating time for each of $\tau = 2, 3, \cdots, 15$ (see t^*_τ - column in the table below). From the figure and table we see that $t^*_\tau = \tau$ for $\tau = 2, 3, \cdots, 7$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{7\ge\tau>1}\langle\tau\rangle}_{\Vert}$ (see (1) of $\mathtt{S}_1$(p.47)) and that $t^*_\tau = 7$ for $\tau = 8, 9, \cdots, 15$, i.e., $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>7}\langle 7\rangle}_{\Vert}$ (see (2) of $\mathtt{S}_1$(p.47)). In the numerical example note the fact that $\tilde{\mathbb{S}} = \tilde{L}\,(V_{\tau-1})$ are all negative (< 0 (−), i.e., $\mathtt{Skip}_\blacktriangle$) for $t = 2, 3, \cdots, 7$ and positive (> 0 (+), i.e., $\mathtt{Conduct}_\blacktriangle$) for $t = 8, 9, \cdots, 15$. Moreover, note that we have $V_t - \beta V_{t-1} = 0$ or equivalently $V_t = \beta V_{t-1}$ for $t = 8, 9, \cdots, 15$ and $V_t - \beta V_{t-1} < 0$ or equivalently $V_t < \beta V_{t-1}$ for $t = 2, 3, \cdots, 7$ (see $V_t - \beta V_{t-1}$-column), hence $V_{15} = \beta V_{14} = \beta^2 V_{13} = \cdots = \beta^8 V_7 < \beta^9 V_6 < \beta^{10} V_5 < \cdots < \beta^{14} V_1$ (see $\beta^{15-t}V_t$-column), so we have $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>7}\langle 7\rangle}_{\Vert}$.



t	β^{15-t}	V_t	$V_t - \beta V_{t-1}$	$\beta^{15-t}V_t$	t^*_τ	$\tilde{\mathbb{S}}_t = \tilde{L}\,(V_{t-1})$	decision
0							
1	0.753641	0.544900		0.410658			
2	0.769022	0.442388	−0.091614	0.340206	2	−0.091614	$\mathtt{Conduct}_\blacktriangle$
3	0.784716	0.391004	−0.042535	0.306827	3	−0.042535	$\mathtt{Conduct}_\blacktriangle$
4	0.800731	0.361335	−0.021849	0.289332	4	−0.021849	$\mathtt{Conduct}_\blacktriangle$
5	0.817072	0.343013	−0.011094	0.280266	5	−0.011094	$\mathtt{Conduct}_\blacktriangle$
6	0.833747	0.331264	−0.004889	0.276190	6	−0.004889	$\mathtt{Conduct}_\blacktriangle$
7	0.850763	0.323555	−0.001084	0.275268	7	−0.001084	$\mathtt{Conduct}_\blacktriangle$
8	0.868125	0.317084	0.000000	0.275268	7	+0.001338	$\mathtt{Skip}_\blacktriangle$
9	0.885842	0.310742	0.000000	0.275268	7	+0.003326	$\mathtt{Skip}_\blacktriangle$
10	0.903920	0.304527	0.000000	0.275268	7	+0.005233	$\mathtt{Skip}_\blacktriangle$
11	0.922368	0.298437	0.000000	0.275268	7	+0.007064	$\mathtt{Skip}_\blacktriangle$
12	0.941192	0.292468	0.000000	0.275268	7	+0.008822	$\mathtt{Skip}_\blacktriangle$
13	0.960400	0.286618	0.000000	0.275267	7	+0.010508	$\mathtt{Skip}_\blacktriangle$
14	0.980000	0.280886	0.000000	0.275268	7	+0.012127	$\mathtt{Skip}_\blacktriangle$
15	1.000000	0.275268	0.000000	0.275268	7	+0.013680	$\mathtt{Skip}_\blacktriangle$

[TAB7474x]

Figure 16.8.3: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \ge \tau \ge 2, \tau \ge t \ge 1)$

[†] Note that $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01+1.00)/2 = 0.505$, we have $\beta\mu = 0.98\times 0.505 = 0.4949$, hence $\beta\mu + s = 0.4949 + 0.05 = 0.5449 < 1.00 = b$. In addition, $s_{\tilde{\mathcal{L}}} = 0.323274 > 0.05 = s$. Thus, the conditions for the assertions are satisfied.

[‡] Note that this is a selling model with cost minimization.

Chapter 17

Diagonal Symmetry

17.1 Model with $\mathbb{R}$-mechanism

17.1.1 Identicalness of Condition-Spaces $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ and $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

First, note that $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$ (see (10.3.27(p.53))) and $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}$ (see (11.5.52(p.65))). Then, since $\mathscr{F} = \check{\mathscr{F}}$ (see Lemma 11.1.1(p.56) (a)), we have $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$, hence we have

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F} = \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle. \qquad (17.1.1)$$

In other words, the shapes of the two condition-spaces $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ and $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ are given by the *identical* deformed circle (see Figure 17.1.1 below).

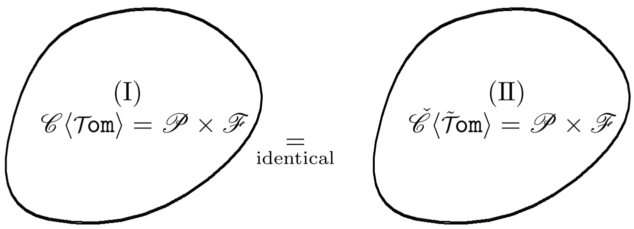


Figure 17.1.1: Identicalness of Condition-Spaces $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ and $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

17.1.2 Non-identicalness of Restricted Condition-Spaces $\mathscr{C}'\langle\mathcal{T}\mathtt{om}\rangle$ and $\check{\mathscr{C}}'\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

17.1.2.1 Restriction of Total-Market

Herein let us again consider the *market restriction* of $\mathscr{F}$ to $\mathscr{F}'$ (see Section 16.3(p.99)) where

$$\mathscr{F}' \subset \mathscr{F}. \qquad (17.1.2)$$

Then, let us define

$$\mathscr{C}'\langle\mathtt{Tom}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{\mathtt{Tom}}, F \in \mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}'\} \quad \text{(see (10.3.16(p.51))),} \qquad (17.1.3)$$

$$\mathscr{C}'\langle A_{\mathtt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}}, F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}'\} \quad \text{(see (10.3.9(p.50))),} \qquad (17.1.4)$$

$$\mathscr{C}'\langle\mathcal{T}\mathtt{om}\rangle \overset{\text{def}}{=} \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}}\mathscr{C}'\langle\mathtt{Tom}\rangle = \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}} \cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}} \mathscr{C}'\langle A_{\mathtt{Tom}}\rangle \quad \text{(see (10.3.24(p.52))).} \qquad (17.1.5)$$

In addition, let us define

$$\check{\mathscr{F}}' \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}'\} \qquad \text{(see (11.1.3(p.55)))} \qquad (17.1.6)$$

where clearly

$$\check{\mathscr{F}}' \subseteq \check{\mathscr{F}}.^{\dagger} \qquad (17.1.7)$$

Then, let us define

$$\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{\tilde{\mathtt{T}}\mathtt{om}}, F \in \mathscr{F}_{\tilde{\mathtt{T}}\mathtt{om}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}'\} \quad \text{(see (10.3.16(p.51))),} \qquad (17.1.8)$$

$$\check{\mathscr{C}}'\langle A_{\tilde{\mathtt{T}}\mathtt{om}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\tilde{\mathtt{T}}\mathtt{om}}}, F \in \mathscr{F}_{A_{\tilde{\mathtt{T}}\mathtt{om}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}'\} \quad \text{(see (11.5.44(p.63))),} \qquad (17.1.9)$$

$$\check{\mathscr{C}}'\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \overset{\text{def}}{=} \cup_{\tilde{\mathtt{T}}\mathtt{om}\in\tilde{\mathcal{T}}\mathtt{om}}\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \cup_{\tilde{\mathtt{T}}\mathtt{om}\in\tilde{\mathcal{T}}\mathtt{om}} \cup_{A_{\tilde{\mathtt{T}}\mathtt{om}}\in\mathtt{Tomtil}} \check{\mathscr{C}}'\langle A_{\tilde{\mathtt{T}}\mathtt{om}}\rangle \quad \text{(see (11.5.49(p.64))).} \qquad (17.1.10)$$

[†] Due to (17.1.2) we have $\check{\mathscr{F}}' = \{\check{F} \mid F \in \mathscr{F}'\} \subseteq \{\check{F} \mid F \in \mathscr{F}\} = \check{\mathscr{F}}$.

Hereupon, let us replace $\mathscr{F}$ and $\mathscr{C}\langle\mathtt{Tom}\rangle$ by $\mathscr{F}'$ and $\mathscr{C}'\langle\mathtt{Tom}\rangle$ respectively. Then, through quite the same reasoning as in Section 10.3(p.49), it can be easily seen that we have

$$\mathscr{C}'\langle\mathtt{Tom}\rangle = \mathscr{P} \times \mathscr{F}' \quad \text{(see (10.3.27(p.53)))}. \tag{17.1.11}$$

Similarly, let us replace $\check{\mathscr{F}}$ and $\check{\mathscr{C}}\langle\tilde{\mathtt{T}}\mathtt{om}\rangle$ by $\check{\mathscr{F}}'$ and $\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle$ respectively. Then, through quite the same reasoning as in Step 11.5 (p.63), it can be easily seen that we have

$$\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}' \quad \text{(see (11.5.52(p.65)))} \tag{17.1.12}$$

Here, note that it cannot be always guaranteed that $\mathscr{F}'$ and $\check{\mathscr{F}}'$ becomes equal (i.e., $\mathscr{F}' \not\doteq \check{\mathscr{F}}'$),‡ hence, since $\mathscr{P} \times \mathscr{F}' \not\doteq \mathscr{P} \times \check{\mathscr{F}}'$, we have

$$\mathscr{C}'\langle\mathtt{Tom}\rangle = \mathscr{P} \times \mathscr{F}' \not\doteq \mathscr{P} \times \check{\mathscr{F}}' = \check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle, \tag{17.1.13}$$

schematized as below

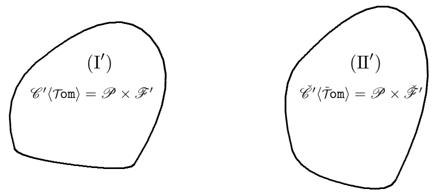


Figure 17.1.2: Non-identicalness of $\mathscr{C}'\langle\mathtt{Tom}\rangle = \mathscr{P} \times \mathscr{F}'$ and $\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}'$

Now, due to (17.1.2) and (17.1.7) we have

$$\mathscr{P} \times \mathscr{F}' \subseteq \mathscr{P} \times \mathscr{F} \quad \text{and} \quad \mathscr{P} \times \check{\mathscr{F}}' \subseteq \mathscr{P} \times \check{\mathscr{F}} = \mathscr{P} \times \mathscr{F} \quad \text{(see Lemma 11.1.1(p.56) (a))}, \tag{17.1.14}$$

hence, noting (17.1.1(p.111)), we have

$$\mathscr{C}'\langle\mathtt{Tom}\rangle = \mathscr{P} \times \mathscr{F}' \subseteq \mathscr{P} \times \mathscr{F} = \mathscr{C}\langle\mathtt{Tom}\rangle \quad \text{and} \quad \check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \mathscr{P} \times \check{\mathscr{F}}' \subseteq \mathscr{P} \times \check{\mathscr{F}} \subseteq \mathscr{P} \times \mathscr{F} = \check{\mathscr{C}}\langle\tilde{\mathtt{T}}\mathtt{om}\rangle. \tag{17.1.15}$$

Accordingly, superimposing Figures 17.1.1 onto 17.1.2 yields Figure 17.1.3 below.

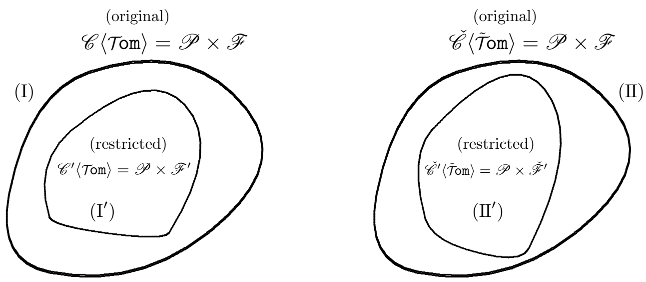


Figure 17.1.3: Superposition of Figures 17.1.1 onto 17.1.2

The inclusion relations depicted in Figure 17.1.3 implies that what holds on $\mathscr{C}\langle\mathtt{Tom}\rangle \cdots$ (I) holds also on $\mathscr{C}'\langle\mathtt{Tom}\rangle \cdots$ (I′) and that what holds on $\check{\mathscr{C}}\langle\tilde{\mathtt{T}}\mathtt{om}\rangle \cdots$ (II) holds also on $\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle \cdots$ (II′). Accordingly it follows that the validity of Corollary 11.5.3(p.65), which holds on $\mathscr{C}\langle\mathtt{Tom}\rangle = \mathscr{F} \times \mathscr{F}$ and $\check{\mathscr{C}}\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \mathscr{F} \times \mathscr{F}$, is in its entirety inherited to $\mathscr{C}'\langle\mathtt{Tom}\rangle = \mathscr{F} \times \mathscr{F}'$ and $\check{\mathscr{C}}'\langle\tilde{\mathtt{T}}\mathtt{om}\rangle = \mathscr{F} \times \check{\mathscr{F}}'$. This fact implies that the corollary can be rewritten as Corollary 17.1.16 below:

Corollary 17.1.1 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}'$. Then, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \check{\mathscr{F}}'$ where*

$$A\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \; = \; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \; \square \tag{17.1.16}$$

17.1.3 Diagonal Symmetry

As the generalized-restricted-total-market $\mathscr{F}'$ let us consider the following three cases:

$$\mathscr{F}' = \mathscr{F}^+ \cdots (1), \quad \mathscr{F}' = \mathscr{F}^\pm \cdots (2), \quad \mathscr{F}' = \mathscr{F}^- \cdots (3). \tag{17.1.17}$$

In addition, as one corresponding to each case above we can consider (see (11.1.3(p.55)))

$$\check{\mathscr{F}}' = \check{\mathscr{F}}^+ = \{\check{F} \mid F \in \mathscr{F}^+\} \cdots (1), \quad \check{\mathscr{F}}' = \check{\mathscr{F}}^\pm = \{\check{F} \mid F \in \mathscr{F}^\pm\} \cdots (2), \quad \check{\mathscr{F}}' = \check{\mathscr{F}}^- = \{\check{F} \mid F \in \mathscr{F}^-\} \cdots (3) \tag{17.1.18}$$

Then, we have the following lemma:

‡ "$\not\doteq$" represents "not always equal" ("$\neq$" is "equal")

Lemma 17.1.1 *We have:*

$$\breve{\mathscr{F}}^{+} = \mathscr{F}^{-} \cdots (1), \quad \breve{\mathscr{F}}^{\pm} = \mathscr{F}^{\pm} \cdots (2), \quad \breve{\mathscr{F}}^{-} = \mathscr{F}^{+} \cdots (1). \ \square \tag{17.1.19}$$

Proof of (1) Consider any $\breve{F} \in \breve{\mathscr{F}}^{+} = \{\breve{F} \mid F \in \mathscr{F}^{+}\}$. Then, since $F \in \mathscr{F}^{+}$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $0 < a < \boldsymbol{\xi} < b$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $0 > \hat{a} > \hat{\boldsymbol{\xi}} > \hat{b}$, we have $\breve{F} \in \mathscr{F}^{-}$, so $\breve{\mathscr{F}}^{+} \subseteq \mathscr{F}^{-} \cdots (1^*)$. Consider any $\breve{F} \in \mathscr{F}^{-}$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $a < \hat{\boldsymbol{\xi}} < b < 0$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $\hat{a} > \boldsymbol{\xi} = \hat{\hat{\boldsymbol{\xi}}} > \hat{b} > 0$, so that $F \in \mathscr{F}^{+}$, hence $\breve{F} \in \breve{\mathscr{F}}^{+}$. Thus $\mathscr{F}^{-} \subseteq \breve{\mathscr{F}}^{+}$. From this and (1^*) we have $\breve{\mathscr{F}}^{+} = \mathscr{F}^{-}$.

Proof of (2) Consider any $\breve{F} \in \breve{\mathscr{F}}^{\pm} = \{\breve{F} \mid F \in \mathscr{F}^{\pm}\}$. Then, since $F \in \mathscr{F}^{\pm}$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $a \le 0 \le b$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $\hat{a} \ge 0 \ge \hat{b}$, we have $\breve{F} \in \mathscr{F}^{\pm}$, so $\breve{\mathscr{F}}^{\pm} \subseteq \mathscr{F}^{\pm} \cdots (2^*)$. Consider any $\breve{F} \in \mathscr{F}^{\pm}$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $a \le 0 \le b$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $\hat{a} \ge 0 \ge \hat{b}$, so that $F \in \mathscr{F}^{\pm}$, hence $\breve{F} \in \breve{\mathscr{F}}^{\pm}$. Thus $\mathscr{F}^{\pm} \subseteq \breve{\mathscr{F}}^{\pm}$. From this and (2^*) we have $\breve{\mathscr{F}}^{\pm} = \mathscr{F}^{\pm}$.

Proof of (3) Consider any $\breve{F} \in \breve{\mathscr{F}}^{-} = \{\breve{F} \mid F \in \mathscr{F}^{-}\}$. Then, since $F \in \mathscr{F}^{-}$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $a < \boldsymbol{\xi} < b < 0$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $\hat{a} > \hat{\boldsymbol{\xi}} > \hat{b} > 0$, we have $\breve{F} \in \mathscr{F}^{+}$, so $\breve{\mathscr{F}}^{-} \subseteq \mathscr{F}^{+} \cdots (3^*)$. Consider any $\breve{F} \in \mathscr{F}^{+}$. Then, since $\breve{F}(\xi) = \Pr\{\breve{\boldsymbol{\xi}} \le \xi\}$ with $0 < a < \hat{\boldsymbol{\xi}} < b$, we have $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with $0 > \hat{a} > \boldsymbol{\xi} = \hat{\hat{\boldsymbol{\xi}}} > \hat{b}$, so that $F \in \mathscr{F}^{-}$, hence $\breve{F} \in \breve{\mathscr{F}}^{-}$. Thus $\mathscr{F}^{+} \subseteq \breve{\mathscr{F}}^{-}$. From this and (3^*) we have $\breve{\mathscr{F}}^{-} = \mathscr{F}^{+}$. ∎

Theorem 17.1.1 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$ holds on $\mathscr{P} \times \mathscr{F}^{+}$. Then, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$ holds on $\mathscr{P} \times \mathscr{F}^{-}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}]. \ \square \tag{17.1.20}$$

Proof Let $\mathscr{F}' = \mathscr{F}^{+}$, hence $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$ (see (16.4.1(p.100))). Then, since $\breve{\mathscr{F}}' = \breve{\mathscr{F}}^{+}$ from (17.1.18 (1)) and $\breve{\mathscr{F}}^{+} = \mathscr{F}^{-}$ from (17.1.19 (1)), we have $\breve{\mathscr{F}}' = \mathscr{F}^{-}$, hence $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}$. Accordingly, Corollary 17.1.1 can be rewritten like the theorem. ∎

Theorem 17.1.2 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$ holds on $\mathscr{P} \times \mathscr{F}^{\pm}$. Then, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$ holds on $\mathscr{P} \times \mathscr{F}^{\pm}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}]. \ \square \tag{17.1.21}$$

Proof Let $\mathscr{F}' = \mathscr{F}^{\pm}$, hence $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}$ (see (16.4.2(p.100))). Then, since $\breve{\mathscr{F}}' = \breve{\mathscr{F}}^{\pm}$ from (17.1.18 (2)) and $\breve{\mathscr{F}}^{\pm} = \mathscr{F}^{\pm}$ from (17.1.19 (2)), we have $\breve{\mathscr{F}}' = \mathscr{F}^{\pm}$, hence $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}$. Accordingly, Corollary 17.1.1 can be rewritten like the theorem. ∎

Theorem 17.1.3 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$ holds on $\mathscr{P} \times \mathscr{F}^{-}$. Then, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$ holds on $\mathscr{P} \times \mathscr{F}^{+}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}]. \ \square \tag{17.1.22}$$

Proof Let $\mathscr{F}' = \mathscr{F}^{-}$, hence $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}$ (see (16.4.3(p.100))). Then, since $\breve{\mathscr{F}}' = \breve{\mathscr{F}}^{-}$ from (17.1.18 (3)) and $\breve{\mathscr{F}}^{-} = \mathscr{F}^{+}$ from (17.1.19 (3)), we have $\breve{\mathscr{F}}' = \mathscr{F}^{+}$, hence $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ in Corollary 17.1.1 can be rewritten as $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}$. Accordingly, Corollary 17.1.1 can be rewritten like the theorem. ∎

It can be easily seen that the inverses of the above three theorems can be given as below:

Theorem 17.1.4 *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$ holds on $\mathscr{P} \times \mathscr{F}^{-}$. Then, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$ holds on $\mathscr{P} \times \mathscr{F}^{+}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}]. \ \square \tag{17.1.23}$$

Theorem 17.1.5 *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$ holds on $\mathscr{P} \times \mathscr{F}^{\pm}$. Then, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}$ holds on $\mathscr{P} \times \mathscr{F}^{\pm}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}]. \ \square \tag{17.1.24}$$

Theorem 17.1.6 *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$ holds on $\mathscr{P} \times \mathscr{F}^{+}$. Then, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}$ holds on $\mathscr{P} \times \mathscr{F}^{-}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}]. \ \square \tag{17.1.25}$$

The relationships showed by (17.1.20)-(17.1.22) and (17.1.23)-(17.1.25) can be schematized as in Figure 17.1.4 below.

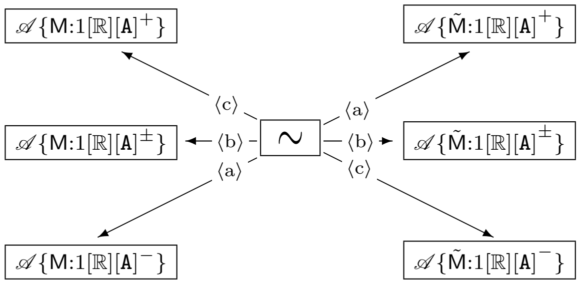


Figure 17.1.4: Symmetrical Relations

Definition 17.1.1 (diagonal-symmetry) Let us refer to the relationships depicted by Figure 17.1.4 above as the *diagonal-symmetry*, denoted by $\text{D-}\sim$. Here let us represent the diagonal-symmetry as below:

$$\langle a\rangle \quad \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\} \ \text{D-}\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\}, \tag{17.1.26}$$

$$\langle b\rangle \quad \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\} \ \text{D-}\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\}, \tag{17.1.27}$$

$$\langle c\rangle \quad \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \ \text{D-}\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}. \ \square \tag{17.1.28}$$

Exercise 17.1.1 **(diagonal symmetry)**

⟨a⟩ *Confirm by yourself that* (17.1.26) *holds in fact by comparing* `Nem` 16.7.2(p.103) *and* `Pom` 16.7.4(p.103) .

⟨b⟩ *Confirm by yourself that* (17.1.27) *holds in fact by comparing* `Mim` 16.7.2(p.102) *and* `Mim` 16.7.4(p.104) .

⟨c⟩ *Confirm by yourself that* (17.1.28) *holds in fact by comparing* `Pom` 16.7.2(p.103) *and* `Nem` 16.7.4(p.104) . □

Here let us represent $\boxed{\sim}$ in Figure 17.1.4 by $\boxed{\text{D-}\sim}$. Then the figure can be rewritten as below.

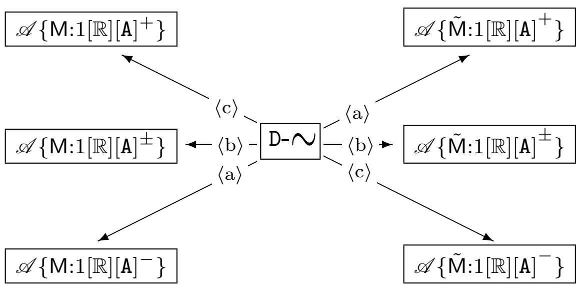


Figure 17.1.5: Diagonal Symmetry

17.2 Simplification of Discussions

17.2.1 Conventional Operations

In the conventional methodology, analyses are separately and one-by-one performed for each of 12 shadow-boxes in Figure 17.2.1 below.

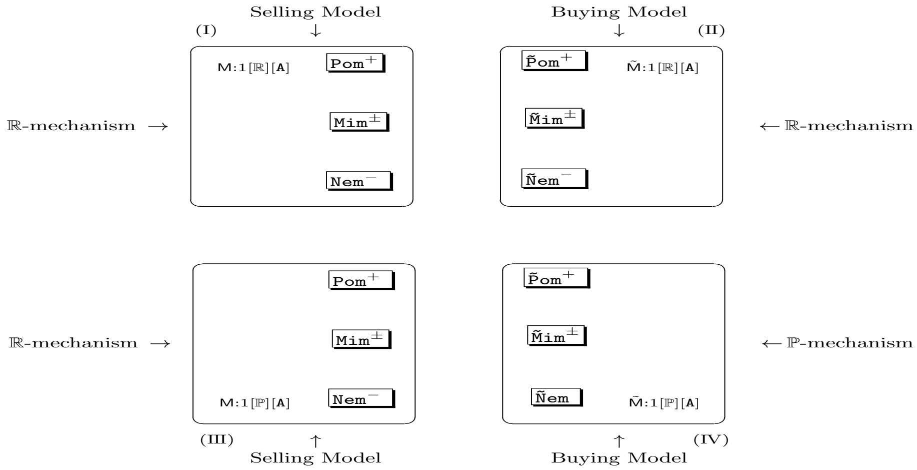


Figure 17.2.1: Conventional method

17.2.2 Operations Based on the Integration Theory

The figure below shows the flow of analyses based on the integration theory where $\mathcal{S}$ in (5^*), (1^*), (2^*), and (6^*) is the symmetry transformation operation (see (11.5.32(p.63)) and (13.5.3(p.87))) and $\mathcal{A}$ in (3^*) and (4^*) is the analogy replacement operation (see (12.2.1(p.77)) and (14.2.3 (1) (p.94))). In the figure, analyses are actually performed only for the 4 shadow-boxes, and the remaining 12 frame-boxes are all derived from applying the market restriction operations $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ to the above 4 shadow-boxes.

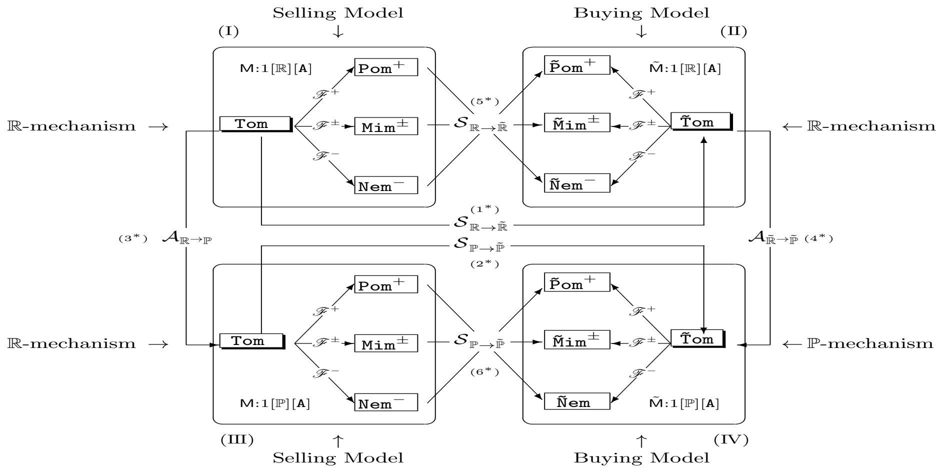


Figure 17.2.2: Correlation diagram

17.2.3 Elimination of Redundant Relations

Here let us recall that what we truly wish to know in the present paper is the existence or non-existence of symmetrical relations between only the two boxes $\boxed{\text{Pom}^+}$ and $\boxed{\tilde{\text{P}}\text{om}^+}$ (see Motive1(p.4)). Now, note that the two boxes can be derived by use of the relations $\boxed{\text{Pom}^+} = \mathscr{F}^+[\boxed{\text{Tom}}]$ by the market restriction (see Chapter 16(p.99)) and $\boxed{\tilde{\text{P}}\text{om}^+} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boxed{\text{Nm}^-}]$ by the diagonal symmetry (see (17.1.22(p.113))). Carefully and in detail looking at the structure of the diagrams in Figure 17.2.2 with noting the above two facts, we immediately see, as shown in Figure 17.2.3 below, that there exist the two methods, **Method A** and **Method B**, which obtain $\boxed{\text{Pom}^+}$ and $\boxed{\tilde{\text{P}}\text{om}^+}$. Accordingly, removing redundant relations within Figure 17.2.2 produces Figure 17.2.3 below; needless to say, **Method A** should be recommended in the sense that it is simpler than **Method B**.

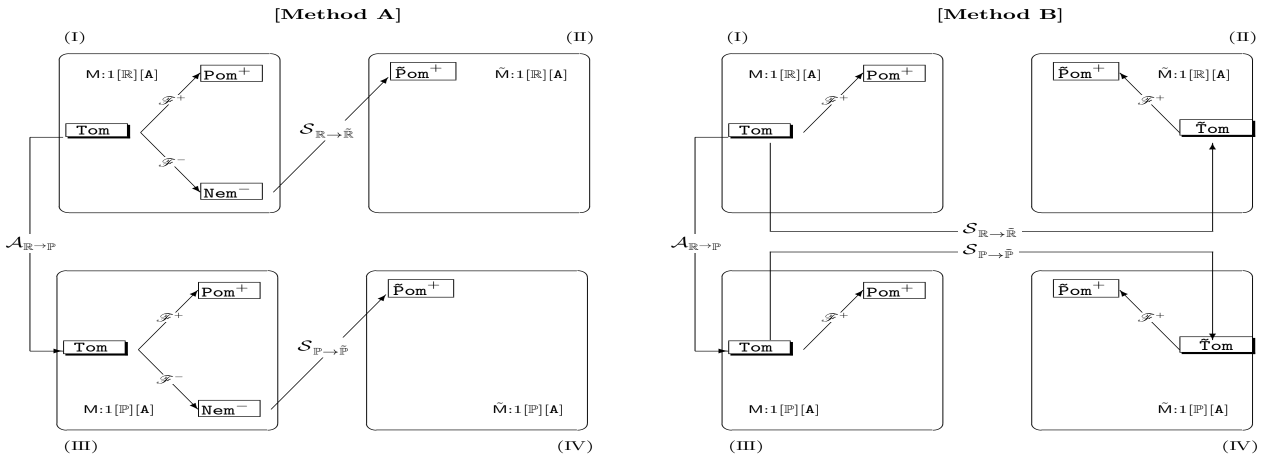


Figure 17.2.3: Correlation diagram

17.3 Model with ℙ-mechanism

Closely looking at the reasoning in discussions for ℝ-mechanism that was made in Section 17.1, one immediately see that it is not directly related to the price mechanism employed there. This fact implies that it holds also for ℙ-mechanism, hence it follows that all of Theorems 17.1.1-17.1.6 hold also for ℙ-mechanism. In other words, the diagonal symmetry holds also between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$. Thus, we see that (17.1.20(p.113))-(17.1.22(p.113)) and (17.1.23(p.113))-(17.1.25(p.113)) hold also for ℙ-mechanism, hence we have

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\}], \tag{17.3.1}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\}], \tag{17.3.2}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}]. \tag{17.3.3}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^+ = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^-\}], \tag{17.3.4}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^\pm\}], \tag{17.3.5}$$

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^- = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}]. \tag{17.3.6}$$

Therefore, it follows that (17.1.26(p.114))-(17.1.28(p.114)) can be written as follows:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}, \tag{17.3.7}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\}, \tag{17.3.8}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\}, \quad \square \tag{17.3.9}$$

*Exercise **17.3.1*** **(diagonal symmetry)**

⟨a⟩ *Confirm by yourself that* (17.3.7) *holds in fact by comparing* `Nem` 16.7.6(p.106) *and* `Pom` 16.7.8(p.106) .

⟨b⟩ *Confirm by yourself that* (17.3.8) *holds in fact by comparing* `Mim` 16.7.6(p.105) *and* `Mim` 16.7.8(p.107) .

⟨c⟩ *Confirm by yourself that* (17.3.9) *holds in fact by comparing* `Pom` 16.7.6(p.105) *and* `Nem` 16.7.8(p.107) . □

Then, as one corresponding to Figure 17.1.5(p.114) we have Figure 17.3.1 below.

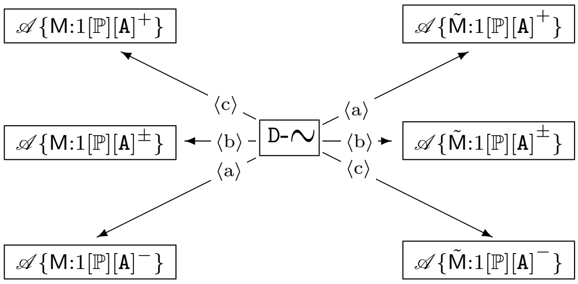


Figure 17.3.1: Diagonal Symmetry

Part 3

No-Recall-Model

Part 3 attempts to analyze all of the basic models with no-recall listed in Table 3.3.1(p.11) by use of the integration theory, excluding $\mathcal{Q}\{\mathtt{M}{:}1[\mathtt{A}]\}$ the analysis of which has already completed in Part 2.

Chapter 18

Model 1

18.1 Search-Allowed-Model 1: $\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{A}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

All analyses for the search-Allowed-model 1 have already completed in Part 2(p.38). Below let us summarize all conclusions obtained there:

Conclusion 1

C1 Monotonicity

a. The optimal reservation price V_t in $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ is nondecreasing in t (see Tom's 10.2.1(p.47) (a) and 10.2.2(a)).
b. The optimal reservation price V_t in $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ is nonincreasing in t (see Tom's 11.7.1(p.69) (a) and 11.7.2(a)).
c. The optimal price z_t to propose in $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ is nondecreasing in t (see Lemmas 12.7.1(p.82)).
d. The optimal price z_t to propose in $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ is nonincreasing in t (see Lemmas 13.8.1(p.91)).

C2 Inheritance and Collapse

a. On the positive-market $\mathscr{F}^+$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.3(p.103) and 16.7.1(p.101)),} \tag{18.1.1}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.7(p.106) and 16.7.5(p.105)),} \tag{18.1.2}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.4(p.103) and 16.7.2(p.101)),} \tag{18.1.3}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.8(p.106) and 16.7.6(p.105)),} \tag{18.1.4}$$

2. Analogy

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.1(p.101) and 16.7.5(p.105)),} \tag{18.1.5}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.3(p.103) and 16.7.7(p.106)).} \tag{18.1.6}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^+\} \not\bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.2(p.101) and 16.7.6(p.105)),} \tag{18.1.7}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\} \quad \text{(see Pom's 16.7.4(p.103) and 16.7.8(p.106)).} \tag{18.1.8}$$

b. On the mixed-market $\mathscr{F}^\pm$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^\pm\} \quad \text{(see Mim's 16.7.3(p.104) and 16.7.1(p.102)),} \tag{18.1.9}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\} \quad \text{(see Mim's 16.7.7(p.107) and 16.7.5(p.105)),} \tag{18.1.10}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^\pm\} \quad \text{(see Mim's 16.7.4(p.104) and 16.7.2(p.102)),} \tag{18.1.11}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\} \quad \text{(see Mim's 16.7.8(p.107) and 16.7.6(p.105)),} \tag{18.1.12}$$

2. Analogy

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\} \quad \text{(see } \mathtt{Mim}\text{'s 16.7.1(p.102) and 16.7.5(p.105)),} \tag{18.1.13}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\} \quad \text{(see } \mathtt{Mim}\text{'s 16.7.3(p.104) and 16.7.8(p.107)).} \tag{18.1.14}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{\pm}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\} \quad \text{(see } \mathtt{Mim}\text{'s 16.7.2(p.102) and 16.7.6(p.105)),} \tag{18.1.15}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{\pm}\} \quad \text{(see } \mathtt{Mim}\text{'s 16.7.4(p.104) and 16.7.8(p.107)).} \tag{18.1.16}$$

c. On the negative-market $\mathscr{F}^{-}$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.3(p.104) and 16.7.1(p.102)),} \tag{18.1.17}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.7(p.107) and 16.7.5(p.106)),} \tag{18.1.18}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.4(p.104) and 16.7.2(p.103)),} \tag{18.1.19}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.8(p.107) and 16.7.6(p.106)),} \tag{18.1.20}$$

2. Analogy

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.1(p.102) and 16.7.5(p.106)),} \tag{18.1.21}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.3(p.104) and 16.7.7(p.107)).} \tag{18.1.22}$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{-}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.2(p.103) and 16.7.6(p.106)),} \tag{18.1.23}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^{-} \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^{-}\} \quad \text{(see } \mathtt{Nem}\text{'s 16.7.4(p.104) and 16.7.8(p.107)).} \tag{18.1.24}$$

C3 Occurrence of Ⓢ, ⊛, and ●d

The symbol "○" in the table below represents "`possible`".

1. Let $\beta = 1$ and $s = 0$. Then, from
   `Pom` 16.7.1(p.101), `Mim` 16.7.1(p.102), `Nem` 16.7.1(p.102),
   `Pom` 16.7.3(p.103), `Mim` 16.7.3(p.104), `Nem` 16.7.3(p.104),
   `Pom` 16.7.5(p.105), `Mim` 16.7.5(p.105), `Nem` 16.7.5(p.106),
   `Pom` 16.7.7(p.106), `Mim` 16.7.7(p.107), `Nem` 16.7.7(p.107)

   we obtain the following table:

Table 18.1.1: Ⓢ, ⊛, and ●d on $\mathscr{F}^{+}$, $\mathscr{F}^{\pm}$, and $\mathscr{F}^{-}$ ($\beta = 1$ and $s = 0$)

		$\mathscr{F}^{+}$	$\mathscr{F}^{\pm}$	$\mathscr{F}^{-}$
[ⓢ $\mathtt{dOITs}_{\tau}\langle\tau\rangle$]$_{\Vert}$	Ⓢ$_{\Vert}$			
[ⓢ $\mathtt{dOITs}_{\tau}\langle\tau\rangle$]$_{\triangle}$	Ⓢ$_{\triangle}$			
[ⓢ $\mathtt{dOITs}_{\tau}\langle\tau\rangle$]$_{\blacktriangle}$	Ⓢ$_{\blacktriangle}$	○	○	○
[⊛ $\mathtt{ndOIT}_{\tau}\langle t_{\tau}^{\bullet}\rangle$]$_{\Vert}$	⊛$_{\Vert}$			
[⊛ $\mathtt{ndOIT}_{\tau}\langle t_{\tau}^{\bullet}\rangle$]$_{\triangle}$	⊛$_{\triangle}$			
[⊛ $\mathtt{ndOIT}_{\tau}\langle t_{\tau}^{\bullet}\rangle$]$_{\blacktriangle}$	⊛$_{\blacktriangle}$			
[● $\mathtt{dOITd}_{\tau}\langle 0\rangle$]$_{\Vert}$	●d$_{\Vert}$			
[● $\mathtt{dOITd}_{\tau}\langle 0\rangle$]$_{\triangle}$	●d$_{\triangle}$			
[● $\mathtt{dOITd}_{\tau}\langle 0\rangle$]$_{\blacktriangle}$	●d$_{\blacktriangle}$			

2. Let $\beta < 1$ or $s > 0$. Then, from
   Pom 16.7.2(p.101), Mim 16.7.2(p.102), Nem 16.7.2(p.103),
   Pom 16.7.4(p.103), Mim 16.7.4(p.104), Nem 16.7.4(p.104),
   Pom 16.7.6(p.105), Mim 16.7.6(p.105), Nem 16.7.6(p.106),
   Pom 16.7.8(p.106), Mim 16.7.8(p.107), Nem 16.7.8(p.107)

   we obtain the following table:

Table 18.1.2: ⓢ, ⊛, and 🅓 on $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ ($\beta < 1$ or $s > 0$)

		$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\parallel}$	$ⓢ_{\parallel}$			
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\vartriangle}$	$ⓢ_{\vartriangle}$			
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\blacktriangle}$	$ⓢ_{\blacktriangle}$	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\parallel}$	$⊛_{\parallel}$	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\vartriangle}$	$⊛_{\vartriangle}$			
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	$⊛_{\blacktriangle}$			
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\parallel}$	$🅓_{\parallel}$	○	○	○
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\vartriangle}$	$🅓_{\vartriangle}$			
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\blacktriangle}$	$🅓_{\blacktriangle}$			

3. The table below is the list of the percents (frequencies) of ⓢ, ⊛, and 🅓 appearing in Section 16.7(p.101).

Table 18.1.3: Percents (frequencies) of ⓢ, ⊛, and 🅓

% (total)	ⓢ	⊛	🅓
100% (102)	55% (56)	14% (14)	31% (32)

C4 Diagonal symmetry

See Figures 17.1.5(p.114) and 17.3.1(p.117).

18.2 Search-Enforced-Model 1: $\mathcal{Q}\{\mathsf{M}{:}1[\mathsf{E}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}], \mathsf{M}{:}1[\mathbb{P}][\mathsf{E}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$

18.2.1 Preliminary

First, let us again note the following three theorems

Theorem 11.5.1(p.66) (symmetry theorem with $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$),
Theorem 12.3.1(p.81) (analogy theorem with $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$),
Theorem 13.5.1(p.88) (symmetry theorem with $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$).

Then, let us recall the fact that in the process of proving the above three theorems it was imperative for the following three relations to hold:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (11.5.38\text{(p.63)})), \tag{18.2.1}$$
$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.2.4\text{(p.77)})), \tag{18.2.2}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.5.4\text{(p.87)})). \tag{18.2.3}$$

In fact, we can immediately reconfirm from Table 6.5.1(p.31) (I,II,III) that the above three relations hold. After the above were clarified, we proved that

Theorem 14.2.1(p.94) (analogy theorem ($\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$))

holds. In addition, from Table 6.5.1(II,III) it can be seen that the following relation holds.

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{18.2.4}$$

The above discussions are all for the search-**A**llowed-model. Also for the search-**E**nforced-model which we will discuss in this section we can immediately confirm from Table 6.5.2(p.31) (I-IV) that the following four relations hold:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}], \tag{18.2.5}$$
$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}], \tag{18.2.6}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}], \tag{18.2.7}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{18.2.8}$$

Accordingly, it can be easily seen that the whole discussions in Part 2(p.38) can be *literally* applied to $\mathcal{Q}\{\mathsf{M}{:}1[\mathsf{E}]\}$; as a results, we have the following four theorems:

Theorem 18.2.1 (symmetry[$\mathbb{R} \to \mathbb{R}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then the equality below holds on $\mathscr{P} \times \mathscr{F}$.*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{18.2.9}$$

Theorem 18.2.2 (analogy[$\mathbb{R} \to \mathbb{P}$)]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then the equality below holds on $\mathscr{P} \times \mathscr{F}$.*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{18.2.10}$$

Theorem 18.2.3 (symmetry[$\mathbb{P} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then the equality below holds on $\mathscr{P} \times \mathscr{F}$.*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}]. \quad \square \tag{18.2.11}$$

Theorem 18.2.4 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then the equality below holds on $\mathscr{P} \times \mathscr{F}$.*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square$$

18.2.2 M:1[$\mathbb{R}$][E]

18.2.2.1 Analysis

To begin with, let us note that

$$\lambda = 1 \tag{18.2.12}$$

is assumed in the model (see $A2$(p.13)), hence from (9.2.1(p.42)) we have

$$\delta = 1 \tag{18.2.13}$$

□ **Tom 18.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, from (5.1.4) we have $K(x) = T(x) \ge 0 \cdots (1)$ for any x due to Lemma 9.1.1(p.41) (g).

(a) From (6.5.10(p.31)) with $t = 2$ we have $V_2 = K(V_1) + V_1 \ge V_1$ due to (1). Suppose $V_{t-1} \le V_t$. Then, from Lemma 9.2.2(p.43) (e) we have $V_t \le K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \le V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) From (6.5.9) we have $V_1 = \mu < b \cdots (2)$. Suppose $V_{t-1} < b$. Then, from (6.5.10) and Lemma 9.2.2(h) we have $V_t < K(b) + b = T(b) + b = b$ due to (1) and Lemma 9.1.1(g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 9.2.1(d), thus $L(V_{t-1}) > 0$ for $\tau \ge t > 1$. Then, from (6.5.10) and from (5.1.8) we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) > 0$ for $\tau \ge t > 1$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$. Hence, since $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, thus $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. ∎

For explanatory simplicity, let us define the statement below:

$$\mathsf{S}_2\boxed{ⓢ^{\blacktriangle}|⊛^{\|}|⊛^{\triangle}|⊛^{\blacktriangle}} = \left\{\begin{array}{l}\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that}\\ (1)\ \boxed{ⓢ\ \mathtt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_{\blacktriangle},\\ (2)\ \boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\triangle},\\ (3)\ \boxed{⊛\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\|}\ (\!(\boxed{⊛\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\blacktriangle})\!).^\dagger\end{array}\right\}$$

□ **Tom 18.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$.
(c) *Let $\beta\mu < b$.*
   1. *Let $\beta = 1$.*
      i. *Let $\mu - s \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
      ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   2. *Let $\beta < 1$ and $s = 0$ ($\!(s > 0)\!$).*
      i. *Let $b > 0$ ($\!(\kappa > 0)\!$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
      ii. *Let $b = 0$ ($\!(\kappa = 0)\!$).*
         1. *Let $\beta\mu - s \le a$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\|}$.
         2. *Let $\beta\mu - s > a$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

† The outer side of ( ) is for $s = 0$ and the inner side is for $s > 0$.

iii. *Let $b < 0$ ($\kappa < 0$).*

1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\texttt{S}_2$ $\boxed{Ⓢ^{\blacktriangle} | ⊛^{\parallel} | ⊛^{\vartriangle} | ⊛^{\blacktriangle}}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$. From (6.5.10(p.31)) and (5.1.8), we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) \cdots$ **(1)** for $t > 1$. From (6.5.10) with $t = 2$ we have $V_2 - V_1 = K(V_1) \cdots$ **(2)**.

(a) Note that $V_1 = \beta\mu - s$ from (6.5.9). Then, from Lemma 9.2.2(j2) we have $x_K \geq \beta\mu - s$ due to (18.2.12) and (18.2.13), hence $x_K \geq V_1 \cdots$ **(3)**. Accordingly, $K(V_1) \geq 0$ due to Lemma 9.2.2(j1), so that $V_1 \leq V_2$ from (2). Suppose $V_{t-1} \leq V_t$. Then, from (6.5.10) and Lemma 9.2.2(e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Note (3). Suppose $V_{t-1} \leq x_K$. Then, from (6.5.10) and Lemma 9.2.2(e) we have $V_t \leq K(x_K) + x_K = x_K$. Hence, by induction $V_t \leq x_K$ for $t > 0$, i.e., V_t is upper bounded in t, thus V_t converges to a finite V as $t \to \infty$. Accordingly, from (6.5.10) we have $V = K(V) + V$, hence $K(V) = 0$, thus $V = x_K$ due to Lemma 9.2.2(j1).

(b) Let $\beta\mu \geq b \cdots$ **(4)**. Then $x_L \leq \beta\mu - s$ from Lemma 9.2.4(p.44)(b1), hence $x_L \leq V_1$, thus $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 9.2.1(a), hence $L(V_{t-1}) \leq 0 \cdots$ **(5)** for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau \leq \beta V_{\tau-1}$, $V_{\tau-1} \leq \beta V_{\tau-2}$, $\cdots$, $V_2 \leq \beta V_1$, leading to $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1}V_1$, hence it follows that $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots$ **(6)**, hence $s > 0$ due to the assumption of $\beta < 1$ and $s > 0$ in the lemma. Then $x_L = x_K \cdots$ **(7)** due to Lemma 9.2.3(b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(8)**.

(c1i) Let $\mu - s \leq a$. Then, noting (6), (18.2.12), and (18.2.13), we have $x_K = \mu - s \cdots$ **(9)** from Lemma 9.2.2(j2), hence $x_K = V_1$ from (6.5.9). Let $V_{t-1} = x_K$. Then, from (6.5.10) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ from (7). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, thus $L(V_{t-1}) = 0$ for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau = \beta V_{\tau-1}$, $V_{\tau-1} = \beta V_{\tau-2}$, $\cdots$, $V_2 = \beta V_1$, leading to $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1}V_1$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ (see Preference-Rule 7.2.1(p.35)).

(c1ii) Let $\mu - s > a$. Then, since $V_1 > a$, we have $V_{t-1} > a$ for $t > 1$ from (a). From (7) and Lemma 9.2.2(j2) we have $x_L = x_K > \mu - s = V_1$. Let $V_{t-1} < x_L$. Then, from (6.5.10) and Lemma 9.2.2(g) we have $V_t < K(x_L) + x_L = x_L$ due to (8), hence by induction $V_{t-1} < x_L$ for $t > 1$. Thus, since $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 9.2.1(e1), for the same reason as in the proof of Tom 18.2.1(b) we obtain $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2) Let $\beta < 1$ and $s = 0$ ($s > 0$).

(c2i) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K \cdots$ **(10)** from Lemma 9.2.3(c (d)). Now, since $x_K \geq \beta\mu - s$ due to Lemma 9.2.2(j2), we have $x_K \geq V_1$. Suppose $x_K \geq V_{t-1}$. Then, from (6.5.10) and Lemma 9.2.2(e) we have $V_t \leq K(x_K) + x_K = x_K$. Thus, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (10). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 9.2.1(a), for the same reason as in the proof of Tom 18.2.1(b) we obtain $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2ii) Let $b = 0$ ($\kappa = 0$). Then $x_L = x_K \cdots$ **(11)** from Lemma 9.2.3(c (d)), hence $K(x_L) = K(x_K) = 0 \cdots$ **(12)**.

(c2ii1) Let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s \cdots$ **(13)** from Lemma 9.2.2(j2), we have $x_K = V_1$. Let $V_{t-1} = x_K$. Then, from (6.5.10) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (11). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, for the same reason as in the proof of (c1i) we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$, we have $V_{t-1} > a$ for $t > 1$ from (a). From (11) and Lemma 9.2.2(j2) we have $x_L = x_K > \beta\mu - s = V_1$. Let $V_{t-1} < x_L$. Then, from (6.5.10) and Lemma 9.2.2(g) we have $V_t < K(x_L) + x_L = x_L$ due to (12), hence, by induction $V_{t-1} < x_L$ for $t > 1$. Consequently, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 9.2.1(a), for the same reason as in the proof of Tom 18.2.1(b) we obtain $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2iii) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots$ **(14)** from Lemma 9.2.3(c (d)).

(c2iii1) If $\beta\mu - s \leq a$, then $x_L < x_K = \beta\mu - s = V_1$ from Lemma 9.2.2(j2) and (6.5.9). If $s_{\mathcal{L}} \leq s$, then $x_L \leq \beta\mu - s$ due to Lemma 9.2.4(c), hence $x_L \leq V_1$. Therefore, whether $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, hence $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 9.2.1(a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

(c2iii2) Suppose $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Hence, since $V_1 > a$, we have $V_{t-1} > a$ for $t > 0$ from (a). Then, since $x_K > x_L > \beta\mu - s = V_1 \cdots$ **(15)** from Lemma 9.2.4(c) and (6.5.9), we have $K(V_1) > 0$ from Lemma 9.2.2(j1), hence $V_2 > V_1$ from (2). Suppose $V_{t-1} < V_t$. Then, from (6.5.10) and Lemma 9.2.2(g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (15). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \leq x_L \cdots$ **(16)** from (a). Then, since $V = x_K$ due to (a), we have the contradiction of $V = x_K > x_L \geq V$ due to (14) and (16). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots \quad \cdots (17),$$

from which we have

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t^\bullet_\tau}, \qquad x_L < V_{t-1}, \quad t > t^\bullet_\tau + 1. \tag{18.2.14}$$

Hence, we have

$$\begin{aligned} &L(V_{t-1}) > 0, && \cdots (18)\ t^\bullet_\tau \geq t > 1 && (\leftarrow \text{Corollary 9.2.1(a)}) \\ &L(V_{t^\bullet_\tau}) \leq 0, && \cdots (19) && (\leftarrow \text{Corollary 9.2.1(a)}) \\ &L(V_{t-1}) = (\!(< 0)\!),^\dagger && \cdots (20)\ t > t^\bullet_\tau + 1 && (\leftarrow \text{Lemma 9.2.1(d(\!(e1)\!))}) \end{aligned}$$

- Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) > 0 \cdots (21)$ for $\tau \geq t > 1$ from (18). Since $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (1) and (21), we have $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$. Therefore, since $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, we obtain $t^*_\tau = \tau$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ⓢ $\texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle \tau \rangle_\blacktriangle$, thus $\texttt{S}_2(1)$ is true. Let us note here that when $\tau = t^\bullet_\tau$, we have $V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (22)$.
- Let $\tau = t^\bullet_\tau + 1$. From (1) with $t = t^\bullet_\tau + 1$ and (19) we have $V_{t^\bullet_\tau+1} - \beta V_{t^\bullet_\tau} \leq 0$, hence $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$. Accordingly, from (22) we have

$$V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau} V_1 \cdots (23),$$

  thus $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$, i.e., ⊛ $\texttt{ndOIT}_{t^*_{t^\bullet_\tau+1}}\langle t^\bullet_\tau \rangle_\vartriangle$, thus $\texttt{S}_2(2)$ is true.
- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau+1}) = (\!(<)\!)\ 0$ from (20) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1}$ from (1), hence from (23) we have

$$V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_1$$

  Similarly we have

$$V_{t^\bullet_\tau+3} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^2 V_{t^\bullet_\tau+1} \leq \beta^3 V_{t^\bullet_\tau} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+2} V_1.$$

  By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain

$$V_\tau = (\!(<)\!)\ \beta V_{\tau-1} = (\!(<)\!)\ \cdots = (\!(<)\!)\ \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^{\tau-1} V_1. \cdots (24)$$

  - Let $s = 0$. Then (24) can be written as

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^{\tau-1} V_1,$$

    hence we have $t^*_\tau = t^\bullet_\tau$, i.e., ⊛ $\texttt{ndOIT}_{\tau > t^\bullet_\tau + 1}\langle t^\bullet_\tau \rangle_\parallel$ (see Preference Rule 7.2.1(p.35)), hence $\texttt{S}_2(3)$ is true.
  - Let $s > 0$. Then (24) can be written as

$$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} < \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^{\tau-1} V_1, \tag{18.2.15}$$

    hence we have $t^*_\tau = t^\bullet_\tau$, i.e., ⊛ $\texttt{ndOIT}_{\tau > t^\bullet_\tau + 1}\langle t^\bullet_\tau \rangle_\blacktriangle$, hence $\texttt{S}_2(3)$ is true. ▮

18.2.2.2 Market Restriction

18.2.2.2.1 Positive Restriction

☐ **Pom 18.2.1** ($\mathscr{A}\{$M:1[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* ⓢ $\texttt{dOITs}_{\tau > 1}\langle \tau \rangle_\blacktriangle$. → ⓢ

Proof The same as Tom 18.2.1 due to Lemma 16.4.1(p.100). ▮

☐ **Pom 18.2.2** ($\mathscr{A}\{$M:1[ℝ][E]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$* (impossible).
(c) *Let $\beta\mu < b$* (always holds).
   1. *Let $\beta = 1$.*
      i. *Let $\mu - s \leq a$. Then* ● $\texttt{dOITd}_{\tau > 1}\langle 1 \rangle_\parallel$ → → ⓓ

† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

3. *Let $\beta < 1$ and $s > 0$.*

   i. *Let $\beta\mu > s$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

   ii. *Let $\beta\mu \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$ $\rightarrow$ $\rightarrow$ ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a) with $\lambda = 1$.

(a) The same as `Tom` 18.2.2(a).

(b,c) Always $\beta\mu < b$ from [3(p.101)], hence $\beta\mu \geq b$ is impossible.

(c1-c1ii) The same as `Tom` 18.2.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of `Tom` 18.2.2.

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$, hence $\kappa > 0$ due to (2). Hence it suffices to consider only (c2i) of `Tom` 18.2.2.

(c3ii) Let $\beta\mu \leq s$, hence $\kappa \leq 0$ due to (2). Then, since $\beta\mu - s \leq 0 < a$, it suffices to consider only (c2iii1) of `Tom` 18.2.2. ∎

18.2.2.2.2 Mixed Restriction

☐ `Mim` **18.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^{\pm}\}$) *Suppose $a \leq 0 \leq 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. $\rightarrow$ ⓢ

Proof The same as `Tom` 18.2.1 due to Lemma 16.4.1(p.100). ∎

☐ `Mim` **18.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*

(b) *Let $\beta\mu \geq b$* (impossible). ———

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*

   i. *Let $\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

   ii. *Let $\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

3. *Let $\beta < 1$ and $s > 0$.*

   i. *Let $s < \beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

   ii. *Let $s = \beta T(0)$.*

      1. *Let $\beta\mu - s \leq a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

      2. *Let $\beta\mu - s > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ $\rightarrow$ ⓢ

   iii. *Let $s > \beta T(0)$.*

      1. *Let $\beta\mu - s \leq a$ or $s_{\mathscr{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$ $\rightarrow$ $\rightarrow$ ⓓ

      2. *Let $\beta\mu - s > a$ and $s_{\mathscr{L}} > s$. Then* $\texttt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^\blacktriangle}$ *is true* $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.

(a) The same as `Tom` 18.2.2(a).

(b,c) Always $\beta\mu < b$ due to [8(p.101)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as `Tom` 18.2.2(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of `Tom` 18.2.2 and if $b = 0$, then since always $\beta\mu - s = \beta\mu > a$ due to [8], it suffices to consider only (c2ii2) of `Tom` 18.2.2. Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from `Tom` 18.2.2(c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.7(p.17)) with $\lambda = 1$. ∎

18.2.2.2.3 Negative Restriction

☐ `Nem` **18.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. $\rightarrow$ ⓢ

Proof The same as `Tom` 18.2.1 due to Lemma 16.4.1(p.100). ∎

☐ Nem **18.2.2** ($\mathscr{A}${M:1[ℝ][E]$^-$}) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\triangle$ → → ⓓ
(c) *Let $\beta\mu < b$.*
  1. *Let $\beta = 1$.*
    i. *Let $\mu - s \leq a$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$ → → ⓓ
    ii. *Let $\mu - s > a$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ → → ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* S$_2$ [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true* → → ⓢ / ⊛
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\triangle$ → → ⓓ
    ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* S$_2$ [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true* → → ⓢ / ⊛

Proof Suppose $b < 0$, hence $a < \mu < b < 0 \cdots (1)$. Hence $\kappa = -s \cdots (2)$ from Lemma 9.3.1(p.45) (a) with $\lambda = 1$. In addition, in this case, $\beta\mu \geq b$ and $\beta\mu < b$ are both possible due to [17(p.101)].

(a,b) The same as Tom 18.2.2(a,b).

(c) Let $\beta\mu < b$.

(c1-c1ii) The same as Tom 18.2.2(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $b < 0$ due to (1), it suffices to consider only (c2iii) of Tom 18.2.2. In this case, since $\beta\mu - s = \beta\mu > \beta a > a$ due to (1) and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 9.2.4(p.44) (c), it suffices to consider only (c2iii2) of Tom 18.2.2.

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (2). Thus, it suffices to consider only (c2iii1-c2iii2) of Tom 18.2.2. ∎

18.2.3 M̃:1[ℝ][E]

18.2.3.1 Analysis

☐ Tom **18.2.3** ($\mathscr{A}${M̃:1[ℝ][E]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.

Proof The same as Tom 18.2.1(p.122) due to Lemma 16.4.1(p.100). ∎

☐ Tom **18.2.4** ($\mathscr{A}${M̃:1[ℝ][E]}) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\triangle$.
(c) *Let $\beta\mu > a$.*
  1. *Let $\beta = 1$.*
    i. *Let $\mu + s \geq b$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
    ii. *Let $\mu + s < b$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
    i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.
    ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*†
      1. *Let $\beta\mu + s \geq b$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\parallel$.
      2. *Let $\beta\mu + s < b$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.
    iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
      1. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\triangle$.
      2. *Let $\beta\mu + s < b$ and $s < s_{\tilde{\mathcal{L}}}$. Then* S$_2$ [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 18.2.2. ∎

18.2.3.2 Market Restriction

18.2.3.2.1 Positive Restriction

☐ Pom **18.2.3** ($\mathscr{A}${M̃:1[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ → → ⓢ

Proof The same as Tom 18.2.3 due to Lemma 16.4.1(p.100). ∎

☐ Pom **18.2.4** ($\mathscr{A}${M̃:1[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$. Then* [● dOITd$_{\tau>1}\langle 1\rangle$]$_\triangle$ → → ⓓ

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
   ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\texttt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ⊛^{\parallel} \mid ⊛^{\vartriangle} \mid ⊛^{\blacktriangle}}$ *is true* $\rightarrow$ $\rightarrow$ ⓢ / ⊛
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$ $\rightarrow$ $\rightarrow$ ⓓ
   ii. *Let $\beta\mu + s < b$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\texttt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ⊛^{\parallel} \mid ⊛^{\vartriangle} \mid ⊛^{\blacktriangle}}$ *is true* (see `Numerical Example` 18.2.1(p.132)) $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Suppose $a > 0 \cdots$ **(1)**, hence $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 11.6.6(p.68)(a). Here note that $\mu\beta \leq a$ and $\mu\beta > a$ are both possible due to [5(p.101)].

(a,b) The same as `Tom` 18.2.4(a,b).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(3)** due to Lemma 11.6.5(c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have `Tom` 18.2.4(c1i,c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.101)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of `Tom` 18.2.4.

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of `Tom` 18.2.4. ∎

18.2.3.2.2 Mixed Restriction

□ `Mim` **18.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^{\pm}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$. $\rightarrow$ $\rightarrow$ ⓢ

Proof The same as `Tom` 18.2.3(p.126) due to Lemma 16.4.1(p.100). ∎

□ `Mim` **18.2.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu + s \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$. $\rightarrow$ ⓓ
   ii. *Let $\mu + s < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < -\beta\tilde{T}(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let $\beta\mu + s < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
   iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\vartriangle}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\texttt{S}_2$ $\boxed{ⓢ^{\blacktriangle} \mid ⊛^{\parallel} \mid ⊛^{\vartriangle} \mid ⊛^{\blacktriangle}}$ *is true* $\rightarrow$ $\rightarrow$ ⓢ / ⊛

Proof Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.

(a) The same as `Tom` 18.2.4(a).

(b,c) Always $\beta\mu > a$ due to [8(p.101)], hence $\beta\mu \leq a$ is impossible. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(1)** due to Lemma 11.6.5(p.68)(c).

(c1-c1ii) The same as `Tom` 18.2.4(c-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of `Tom` 18.2.4. Let $a = 0$. Now, in this case, since $\beta\mu + s = \beta\mu < b$ due to [8(p.101)], it suffices to consider only (c2ii2) of `Tom` 18.2.4. Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions become true from `Tom` 18.2.4(c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.17)). ∎

18.2.3.2.3 Negative Restriction

☐ `Nem` **18.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^-\}$) *Suppose $b<0$. Let $\beta=1$ and $s=0$.*

(a) *V_t is nonincreasing in $t>0$.*

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ → ⓢ

Proof The same as `Tom` 18.2.4 due to Lemma 16.4.1(p.100). ▮

☐ `Nem` **18.2.4** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^-\}$) *Suppose $b<0$. Let $\beta<1$ or $s>0$.*

(a) *V_t is nonincreasing in $t>0$ and converges to a finite $V\le x_{\tilde{K}}$ as $t\to\infty$.*

(b) *Let $\beta\mu\le a$* (impossible). ———

(c) *Let $\beta\mu>a$* (always holds).

1. *Let $\beta=1$.*
   - i. *Let $\mu+s\ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
   - ii. *Let $\mu+s<b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ → ⓢ
2. *Let $\beta<1$ and $s=0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ → ⓢ
3. *Let $\beta<1$ and $s>0$.*
   - i. *Let $\beta\mu<-s$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\rightarrow$ → ⓢ
   - ii. *Let $\beta\mu\ge -s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$ → ⓓ

Proof Suppose $b<0\cdots(1)$, hence $a<b<0\cdots(2)$. Then $\tilde{\kappa}=\beta\mu+s\cdots(3)$ due to Lemma 11.6.6(a).

(a) The same as `Tom` 18.2.4(p.126) (a).

(b,c) Always $a<\beta\mu$ due to [15(p.101)], hence $\beta\mu\le a$ is impossible.

(c1-c1ii) The same as the proof of `Tom` 18.2.4(c1-c1ii).

(c2) Let $\beta<1$ and $s=0$. Then, due to (2) it suffices to consider only (c2i) of `Tom` 18.2.4.

(c3) Let $\beta<1$ and $s>0$.

(c3i) Let $\beta\mu<-s$, hence $\beta\mu+s<0$. Hence, since $\tilde{\kappa}<0$ due to (3), it suffices to consider only (c2i) of `Tom` 18.2.4.

(c3ii) Let $\beta\mu\ge -s$, hence $\beta\mu+s\ge 0$. Let $\beta\mu+s=0$. Then, since $\tilde{\kappa}=0$ due to (3) and $\beta\mu+s>b$ due to (2), it suffices to consider only (c2ii1) of `Tom` 18.2.4. Let $\beta\mu+s>0$. Then, since $\tilde{\kappa}>0$ due to (3), it suffices to consider only (c2iii) of `Tom` 18.2.4. Then, since $\beta\mu+s>0>b$ due to (1), it suffices to consider only (c2iii1) of `Tom` 18.2.4. Accordingly, whether $\beta\mu+s=0$ or $\beta\mu+s>0$, we have the same result. ▮

18.2.4 M:1[ℙ][E]

18.2.4.1 Analysis

☐ `Tom` **18.2.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta=1$ and $s=0$.*

(a) *V_t is nondecreasing in $t>0$.*

(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

Proof The same as `Tom` 18.2.1. ▮

☐ `Tom` **18.2.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta<1$ or $s>0$.*

(a) *V_t is nondecreasing in $t>0$ and converges to a finite $V=x_K$ as $t\to\infty$.*

(b) *Let $\beta a\ge b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c) *Let $\beta a<b$.*

1. *Let $\beta=1$.*
   - i. *Let $a-s\le a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   - ii. *Let $a-s>a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta<1$ and $s=0$ ($s>0$).*
   - i. *Let $b>0$ ($\kappa>0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   - ii. *Let $b=0$ ($\kappa=0$).*
     1. *Let $\beta a-s\le a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
     2. *Let $\beta a-s>a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   - iii. *Let $b<0$ ($\kappa<0$).*
     1. *Let $\beta a-s\le a^\star$ or $s_{\mathscr{L}}\le s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.
     2. *Let $\beta a-s>a^\star$ and $s<s_{\mathscr{L}}$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^{\parallel} \mid ⊛^\vartriangle \mid ⊛^\blacktriangle}$ *is true.* Tend

Proof by analogy Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (15.3.3(p.98))) to `Tom` 18.2.2. ▮

Lemma 18.2.1 (optimal price to propose) *The optimal price to propose z_t is nondecreasing in $t>0$.* ☐

Proof Immediate from `Tom`'s 18.2.5(a) and 18.2.6(a) and from (6.2.28(p.23)) and Lemma 12.1.3(p.73). ▮

18.2.4.2 Market Restriction

18.2.4.2.1 Positive Restriction

☐ Pom **18.2.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ

Proof The same as Tom 18.2.5 due to Lemma 16.4.1(p.100). ∎

☐ Pom **18.2.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$* (impossible).
(c) *Let $\beta a < b$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ → → ⓓ
    ii. *Let $a - s > a^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < \beta T(0)$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
    ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ → → ⓓ
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
    iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$ → → ⓓ
      2. *Let $\beta a - s > a^\star$ and $s < s_{\mathcal{L}}$. Then* $\mathtt{S}_2$ $\boxed{Ⓢ^{\blacktriangle} \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^{\blacktriangle}}$ *is true* → → Ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots$ **(1)**.

(a) The same as Tom 18.2.6(a).

(b,c) Always $\beta a < b$ from [4(p.101)], hence $\beta a \geq b$ is impossible.

(c1-c1ii) The same as Tom 18.2.6(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 18.2.6.

(c3) Let $\beta < 1$ and $s > 0$.

(c3i-c3iii2) Immediate from Tom 18.2.6(c2i-c2iii2) due to (2) with $\kappa = \beta T(0) - s \cdots$ **(2)** from (5.1.23(p.18)). ∎

18.2.4.2.2 Mixed Restriction

☐ Mim **18.2.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ

Proof The same as Tom 18.2.5 due to Lemma 16.4.1(p.100). ∎

☐ Mim **18.2.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$* (impossible). ———
(c) *Let $\beta a < b$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ → → ⓓ
    ii. *Let $a - s > a^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
  2. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < \beta T(0)$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
    ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → Ⓢ
    iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$ → → ⓓ
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ $\boxed{Ⓢ^{\blacktriangle} \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^{\blacktriangle}}$ *is true* ☐

Proof Suppose $a \le 0 \le b$.

(a) The same as Tom 18.2.6(a).

(b,c) Always $\beta a < b$ due to [9(p.101)], hence $\beta a \ge b$ is impossible. .

(c1-c1ii) The same as Tom 18.2.6(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from Tom 18.2.6(c2i) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.101)], hence the assertion become true from Tom 18.2.6(c2ii2). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as Tom 18.2.6(c2i-c2iii2) with $\kappa = \beta T(0) - s$ from (5.1.23(p.18))) with $\lambda = 1$. ▮

18.2.4.2.3 Negative Restriction

☐ Nem **18.2.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ

Proof The same as Tom 18.2.5. ▮

☐ Nem **18.2.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \ge b$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$ → → ⓓ

(c) *Let $\beta a < b$.*

1. *Let $\beta = 1$.*
   i. *Let $a - s \le a^\star$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$ → → ⓓ
   ii. *Let $a - s > a^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^\parallel \mid ⊛^\vartriangle \mid ⊛^\blacktriangle}$ *is true* → → ⓢ / ⊛
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta a - s \le a^\star$ or $s_{\mathcal{L}} \le s$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$ → → ⓢ
   ii. *Let $\beta a - s > a^\star$ and $s < s_{\mathcal{L}}$. Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^\parallel \mid ⊛^\vartriangle \mid ⊛^\blacktriangle}$ *is true* → → ⓢ / ⊛

Proof Suppose $b < 0$. Then, $\kappa = -s \cdots$ **(1)** from Lemma 12.2.6(p.81) (a). In addition, $\beta a \ge b$ and $\beta a < b$ are both possible due to [18(p.101)].

(a,b) The same as Tom 18.2.6(a,b).

(c) Let $\beta a < b$.

(c1-c1ii) The same as Tom 18.2.6(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2iii-c2iii2) of Tom 18.2.6. In this case, since $\beta a - s = \beta a > a^\star$ due to [19(p.101)] and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 12.2.5(p.81) (c), it suffices to consider only (c2iii2) of Tom 18.2.6.

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (1). Hence, it suffices to consider only (c2iii1,c2iii2) of Tom 18.2.6. ▮

18.2.5 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$

18.2.5.1 Analysis

☐ Tom **18.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

Proof by symmetry Nothing changes even if applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Tom 18.2.5. ▮

☐ Tom **18.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\vartriangle$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   i. *Let $b + s \ge b^\star$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
   ii. *Let $b + s < b^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*
      1. *Let $\beta b + s \ge b^\star$.*[†] *Then* $\boxed{\bullet\ \text{dOITd}_{\tau>1}\langle 1\rangle}_\parallel$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{\text{ⓢ dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

iii. *Let* $a > 0\,(\!(\tilde{\kappa} > 0)\!)$.
  1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$.
  2. *Let* $\beta b + s < b^\star$ *and* $s < s_{\tilde{\mathcal{L}}}$. *Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Tom 18.2.6. ▮

Lemma 18.2.2 (optimal price to propose) *The optimal price to propose* z_t *is nonincreasing in* $t > 0$. □

Proof Immediate from Tom's 18.2.7(a) and 18.2.8(a) and
from (6.2.41(p.23)) and Lemma A 3.3(p.278)). ▮

18.2.5.2 Market Restriction

18.2.5.2.1 Positive Restriction

□ Pom **18.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ

Proof The same as Tom 18.2.7 due to Lemma 16.4.1(p.100). ▮

□ Pom **18.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta b \leq a$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$ → → ⓓ
(c) *Let* $\beta b > a$.
  1. *Let* $\beta = 1$.
    i. *Let* $b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ → → ⓓ
    ii. *Let* $b + s < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^\blacktriangle}$ *is true* → → ⓢ / ⊛
  3. *Let* $\beta < 1$ *and* $s > 0$.
    i. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$ → → ⓓ
    ii. *Let* $\beta b + s < b^\star$ *and* $s < s_{\tilde{\mathcal{L}}}$. *Then* $\mathtt{S}_2$ $\boxed{ⓢ^\blacktriangle \mid ⊛^{\parallel} \mid ⊛^{\triangle} \mid ⊛^\blacktriangle}$ *is true* → → ⓢ / ⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ ((15.3.2(p.98))) to Nem 18.2.6(p.130). ▮

Direct proof Suppose $a > 0 \cdots$ **(1)**. Then, $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 13.6.6(p.90) (a). In addition, $\beta b \leq a$ and $\beta b > a$ are both possible due to [6(p.101)].

(a,b) The same as Tom 18.2.8(a,b).

(c) Let $\beta b > a$.

(c1-c1ii) The same as Tom 18.2.8(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii-c2iii2) of Tom 18.2.8. In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.101)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from
Lemma 13.6.5(p.90) (c) with $\lambda = 1$, it suffices to consider only (c2iii2) of Tom 18.2.8.

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\tilde{\kappa} > 0$ due to (2). Hence, it suffices to consider only (c2iii1,c2iii2) of Tom 18.2.8. ▮

18.2.5.2.2 Mixed Restriction

□ Mim **18.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^\pm$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. → → ⓢ

Proof The same as Tom 18.2.7 due to Lemma 16.4.1(p.100). ▮

□ Mim **18.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^\pm\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta b \leq a$ (impossible). ______
(c) *Let* $\beta b > a$ (always holds).
  1. *Let* $\beta = 1$.
    i. *Let* $b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$. → ⓓ
    ii. *Let* $b + s < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
  3. *Let* $\beta < 1$ *and* $s > 0$.
    i. *Let* $s < -\beta\tilde{T}(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
    ii. *Let* $s = -\beta\tilde{T}(0)$.

1. *Let* $\beta b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
2. *Let* $\beta b + s < b^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

iii. *Let* $s > -\beta\tilde{T}(0)$.

1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$ $\rightarrow$ $\rightarrow$ ⓓ
2. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\mathtt{S}_2$ Ⓢ▲|⊛‖|⊛△|⊛▲ *is true.* $\rightarrow$ Ⓢ / ⊛

Proof Let $b \geq 0 \geq a \cdots$ **(1)**.

(a) The same as Tom 18.2.8(p.130) (a).

(b,c) Always $\beta b > a$ due to [10(p.101)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as Tom 18.2.8(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of Tom 18.2.8. Let $a < 0$. Then, the assertion is true from Tom 18.2.8(c2i). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.101)], it suffices to consider only (c2ii2) of Tom 18.2.8. Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 18.2.8(c2i-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.19)) with $\lambda = 1$. ▮

18.2.5.2.3 Negative Restriction

☐ **Nem 18.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^-\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

Proof The same as Tom 18.2.7 due to Lemma 16.4.1(p.100). ▮

☐ **Nem 18.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta b \leq a$ (impossible). ———
(c) *Let* $\beta b > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
   ii. *Let* $b + s < b^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < -\beta\tilde{T}(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
   ii. *Let* $s = -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
   iii. *Let* $-\beta\tilde{T}(0) < s$.
      1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_{\triangle}$ $\rightarrow$ $\rightarrow$ ⓓ
      2. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\mathtt{S}_2$ Ⓢ▲|⊛‖|⊛△|⊛▲ *is true.* $\rightarrow$ Ⓢ / ⊛

Proof Let $b < 0$, hence $a < b < 0 \cdots$ **(1)**.

(a) The same as Tom 18.2.8(p.130) (a).

(b,c) Always $\beta b > a$ due to [16(p.101)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as Tom 18.2.8(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 18.2.8.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from Tom 18.2.8(c2-c2iii2) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.19)) with $\lambda = 1$. ▮

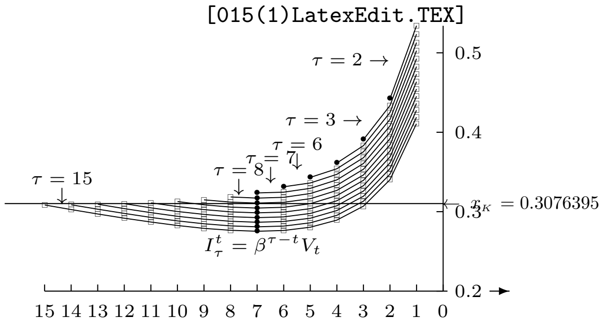
18.2.6 Numerical Calculation

■ **Numerical Example 18.2.1** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\}$ **[015(1)]**)

This is the example for Ⓢ▲ and ⊛▲ of $\mathtt{S}_2$(p.122) Ⓢ▲|⊛‖|⊛△|⊛▲ in Pom 18.2.4(p.126) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$ where $x_{\tilde{K}} = 0.3076395$ and $s_{\tilde{\mathcal{L}}} = 0.3232736$.[†] Note that the example is for the model of a buying problem with the cost minimization. The figure below is the graph of $I^t_\tau = \beta^{\tau-t}V_t$ where the symbol ▫ shows the optimal-initiating-time OIT for each $\tau = 2, 3, \cdots, 15$ (see t^*-column in the table below). In addition, note that each of polygonal curves for $\tau = 2, 3, \cdots, 7$ is strictly decreasing in $t = 1, 2, \cdots, 7$ and that each of polygonal curves for $\tau = 8, 9, \cdots, 15$ is *strictly* decreasing in $t = 1, 2, \cdots, 7$ and *strictly* increasing in $t = 7, 8, \cdots, 15$. The fact implies that the optimal-initiating-time t^*_τ degenerates to

[†]Note that $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. In addition, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu + s = 0.98 \times 0.505 + 0.05 = 0.5449 < 1.00 = b$. In addition, $s = 0.05 < 0.3232736 = s_{\tilde{\mathcal{L}}}$. Thus, the condition of the assertion is satisfied.

the starting $\tau = 2, 3, \cdots, 7$, i.e., $\boxed{\text{ⓢ dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ and that it is given by $t^*_\tau = 7$ (non-degenerate) for each of $\tau = 8, 9, \cdots, 15$, i.e., $\boxed{\text{⊛ ndOIT}_\tau\langle 7\rangle}_\blacktriangle$ (see t^*-column in the table below). Finally, note here that the leftmost point V_t in each curves converges to $x_K = 0.3076395$ as $\tau \to \infty$.



t	V_t	$\Delta_\beta V_t$	t^*_τ
0			
1	0.544900		
2	0.442388	−0.091614	2
3	0.391004	−0.042535	3
4	0.361335	−0.021849	4
5	0.343013	−0.011094	5
6	0.331264	−0.004889	6
7	0.323555	−0.001084	7
8	0.318422	+0.001338	7
9	0.314972	+0.002918	7
10	0.312638	+0.003965	7
11	0.311053	+0.004667	7
12	0.309973	+0.005141	7
13	0.309236	+0.005462	7
14	0.308732	+0.005681	7
15	0.308388	+0.005830	7

$\Delta_\beta V_t = V_t - \beta V_{t-1}$ [015(1)Data.DAT]

Figure 18.2.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$

18.2.7 Conclusion 2 (Search-Enforced-Model 1)

C1 Monotonicity

(a) The optimal reservation price V_t in M:1[ℝ][E] is nondecreasing in t.
(see Tom's 18.2.1(p.122) (a) and 18.2.2(p.122) (a).

(b) The optimal reservation price V_t in M̃:1[ℝ][E] is nonincreasing in t.
(see Tom's 18.2.3(p.126) (a) and 18.2.4(p.126) (a).

(c) The optimal price z_t to propose in M:1[ℙ][E] is nondecreasing in t.
(see Lemma 18.2.1(p.128)).

(d) The optimal price $\tilde{z}_t$ to propose in M̃:1[ℙ][E] is nonincreasing in t.
(see Lemma 18.2.2(p.131)).

C2 Inheritance and Collapse

a. On the positive market $\mathscr{F}^+$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.3(p.126) and 18.2.1(p.124)),} \tag{18.2.16}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.7(p.131) and 18.2.5(p.129)).} \tag{18.2.17}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.4(p.126) and 18.2.2(p.124)),} \tag{18.2.18}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.8(p.131) and 18.2.6(p.129)).} \tag{18.2.19}$$

2. Analogy

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.5(p.129) and 18.2.1(p.124)),} \tag{18.2.20}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.7(p.131) and 18.2.3(p.126)).} \tag{18.2.21}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\} \not\bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.6(p.129) and 18.2.2(p.124)),} \tag{18.2.22}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^+\} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^+\} \quad \text{(see Pom's 18.2.8(p.131) and 18.2.4(p.126)).} \tag{18.2.23}$$

b. On the mixed market $\mathscr{F}^\pm$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^\pm\} \quad \text{(see Mim's 18.2.3(p.127) and 18.2.1(p.125)),} \tag{18.2.24}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^\pm\} \quad \text{(see Mim's 18.2.7(p.131) and 18.2.5(p.129)),} \tag{18.2.25}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^\pm\} \quad \text{(see Mim's 18.2.4(p.127) and 18.2.2(p.125)),} \tag{18.2.26}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^\pm\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^\pm\} \quad \text{(see Mim's 18.2.8(p.131) and 18.2.6(p.129)).} \tag{18.2.27}$$

2. Analogy

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\} \quad (\text{see } \texttt{Mim}\text{'s } 18.2.5(\text{p.129}) \text{ and } 18.2.1(\text{p.125})), \tag{18.2.28}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\} \quad (\text{see } \texttt{Mim}\text{'s } 18.2.7(\text{p.131}) \text{ and } 18.2.3(\text{p.127})). \tag{18.2.29}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\} \quad (\text{see } \texttt{Mim}\text{'s } 18.2.6(\text{p.129}) \text{ and } 18.2.2(\text{p.125})), \tag{18.2.30}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\} \quad (\text{see } \texttt{Mim}\text{'s } 18.2.8(\text{p.131}) \text{ and } 18.2.4(\text{p.127})). \tag{18.2.31}$$

c. On the negative market $\mathscr{F}^-$:

1. Symmetry

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.3(\text{p.128}) \text{ and } 18.2.1(\text{p.125})), \tag{18.2.32}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.7(\text{p.132}) \text{ and } 18.2.5(\text{p.130})). \tag{18.2.33}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.3(\text{p.128}) \text{ and } 18.2.1(\text{p.125})), \tag{18.2.34}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.8(\text{p.132}) \text{ and } 18.2.6(\text{p.130})). \tag{18.2.35}$$

2. Analogy

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.5(\text{p.130}) \text{ and } 18.2.1(\text{p.125})), \tag{18.2.36}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.7(\text{p.132}) \text{ and } 18.2.3(\text{p.128})). \tag{18.2.37}$$

b. Let $\beta < 1$ or $s > 0$. Then we have:

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.6(\text{p.130}) \text{ and } 18.2.2(\text{p.126})), \tag{18.2.38}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\} \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{-}\} \quad (\text{see } \texttt{Nem}\text{'s } 18.2.8(\text{p.132}) \text{ and } 18.2.4(\text{p.128})). \tag{18.2.39}$$

C3 Occurrence of Ⓢ, ⊛, and ⓓ

a. Let $\beta = 1$ and $s = 0$. Then, from

`Pom` 18.2.1(p.124), `Mim` 18.2.1(p.125), `Nem` 18.2.1(p.125),
`Pom` 18.2.3(p.126), `Mim` 18.2.3(p.127), `Nem` 18.2.3(p.128),
`Pom` 18.2.5(p.129), `Mim` 18.2.5(p.129), `Nem` 18.2.5(p.130),
`Pom` 18.2.7(p.131), `Mim` 18.2.7(p.131), `Nem` 18.2.7(p.132)

we obtain the following table:

Table 18.2.1: Ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ ($\beta = 1$ and $s = 0$)

		$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\parallel}$	$Ⓢ_{\parallel}$			
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\triangle}$	$Ⓢ_{\triangle}$			
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\blacktriangle}$	$Ⓢ_{\blacktriangle}$	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\parallel}$	$⊛_{\parallel}$			
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\triangle}$	$⊛_{\triangle}$			
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	$⊛_{\blacktriangle}$			
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\parallel}$	$ⓓ_{\parallel}$			
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\triangle}$	$ⓓ_{\triangle}$			
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\blacktriangle}$	$ⓓ_{\blacktriangle}$			

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 18.2.2(p.124) , Mim 18.2.2(p.125) , Nem 18.2.2(p.126) ,
Pom 18.2.4(p.126) , Mim 18.2.4(p.127) , Nem 18.2.4(p.128) ,
Pom 18.2.6(p.129) , Mim 18.2.6(p.129) , Nem 18.2.6(p.130) ,
Pom 18.2.8(p.131) , Mim 18.2.8(p.131) , Nem 18.2.8(p.132)

we obtain the following table:

Table 18.2.2: ⓢ , ⊛ , and ⓓ on $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ ($\beta < 1$ or $s > 0$)

		$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
ⓢ $\text{dOITs}_\tau\langle\tau\rangle_{\parallel}$	ⓢ$_{\parallel}$			
ⓢ $\text{dOITs}_\tau\langle\tau\rangle_{\triangle}$	ⓢ$_{\triangle}$			
ⓢ $\text{dOITs}_\tau\langle\tau\rangle_{\blacktriangle}$	ⓢ$_{\blacktriangle}$	○	○	○
⊛ $\text{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\parallel}$	⊛$_{\parallel}$	○	○	○
⊛ $\text{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\triangle}$	⊛$_{\triangle}$	○	○	○
⊛ $\text{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	⊛$_{\blacktriangle}$	○	○	○
● $\text{dOITd}_\tau\langle 0\rangle_{\parallel}$	ⓓ$_{\parallel}$	○	○	○
● $\text{dOITd}_\tau\langle 0\rangle_{\triangle}$	ⓓ$_{\triangle}$	○	○	○
● $\text{dOITd}_\tau\langle 0\rangle_{\blacktriangle}$	ⓓ$_{\blacktriangle}$			

c. The table below is the list of the percents (frequencies) of ⓢ , ⊛ , and ⓓ appearing in Sections 18.2.2.2(p.124) and 18.2.3.2(p.126) .

Table 18.2.3: Percents (frequencies) of ⓢ , ⊛ , and ⓓ on $\mathscr{F}^+$

percent (total)	ⓢ	⊛	ⓓ
100% (31)	48% (15)	16% (5)	36% (11)

C4 Diagonal symmetry

Exercise 18.2.1 **(diagonal symmetry)** *Confirm by yourself that the relations below hold in fact:*

Pom 18.2.3(p.126) D-∾ Nem 18.2.1(p.125) ,
Pom 18.2.4(p.126) D-∾ Nem 18.2.2(p.126) ,
Pom 18.2.7(p.131) D-∾ Nem 18.2.5(p.129) ,
Pom 18.2.8(p.131) D-∾ Nem 18.2.6(p.129) ,
Pom 18.2.5(p.130) D-∾ Nem 18.2.7(p.131) ,
Pom 18.2.6(p.130) D-∾ Nem 18.2.8(p.131) . □

18.3 Conclusion 3 (The Whole Model 1)

Conclusions 18.1(p.119) and 18.2.7(p.133) are summed up as below.

C1 Monotonicity

From C1(p.119) and C1(p.133) we have, for whether s-A-model or s-E-model,

(a) The optimal reservation price V_t of M:1[ℝ][X] is nondecreasing in t.
(b) The optimal reservation price V_t of M̃:1[ℝ][X] is nonincreasing in t.
(c) The optimal price z_t to propose of M:1[ℙ][X] is nondecreasing in t.
(d) The optimal price $\tilde{z}_t$ to propose of M̃:1[ℙ][X] is nonincreasing in t.

C2 Inheritance and Collapse

a. (18.1.1(p.119))-(1) and (18.2.16(p.133))-(18.2.39) have the same structure of "inheritance and collapse" between s-A-model and s-E-model.

b. From (18.1.9(p.119))-(18.1.16) and (18.2.24(p.133))-(18.2.31) we see that each of symmetry and analogy is inherited on the mixed market $\mathscr{F}^\pm$ in whether s-A-model or s-E-model.

c. Let $\beta = 1$ and $s = 0$. Then:

1. For Model 1:
   a. Symmetry From C2a1a(p.119), C2b1a(p.119), and C2c1a(p.120) (C2a1a(p.133), C2b1a(p.133), and C2c1a(p.134)) we see that the symmetry is inherited ($\sim$) on all of $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ in s-A-model (s-E-model).
   b. Analogy From C2a2a(p.119), C2b2a(p.120), and C3(p.182) (C2a2a(p.133), C2b2a(p.134), and C2c2a(p.134)) we see that the analogy is inherited ($\sim$) on all of $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ in s-A-model (s-E-model).
2. For Model 2:
   a. Symmetry From C2a1(p.175) (C2a1(p.207)) we see that the symmetry is inherited ($\sim$) on $\mathscr{F}^+$ in s-A-model (s-E-model).
   b. Analogy From C2b(p.175) (C2b(p.207)) we see that the analogy collapses ($\bowtie$) on $\mathscr{F}^+$ in s-A-model (s-E-model).

C3 Occurrence of ⓢ, ⊛, and ⓓ

Joining Tables 18.1.3(p.121) and 18.2.3(p.135) produces Table 18.3.1 below.

Table 18.3.1: Ratios of ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$

ratio (total)	ⓢ	⊛	ⓓ
100% (122)	50% (68)	14% (17)	30% (37)

In other words, ⓢ, ⊛, and ⓓ occur at 50%, 14%, and 30% respectively.

C4 Diagonal symmetry

The diagonal symmetry holds in both s-A-model and s-E-model (see C4(p.121) and C4(p.135)).

Chapter 19

Model 2

19.1 Search-Allowed-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$

19.1.1 Theorems

As ones corresponding to Theorems 11.5.1(p.66) , 12.3.1(p.81) , 13.5.1(p.88) , and 14.2.1(p.94) , let us consider the following four theorems:

Theorem 19.1.1 (symmetry$[\mathbb{R} \to \tilde{\mathbb{R}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{19.1.1}$$

Theorem 19.1.2 (analogy$[\mathbb{R} \to \mathbb{P}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{19.1.2}$$

Theorem 19.1.3 (symmetry$[\mathbb{P} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{19.1.3}$$

Theorem 19.1.4 (analogy$[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square$$

In order for the above four theorems to hold, the following four relations *must* be satisfied for the same reason as in the search-Allowed-model 1 (see Chapter 10(p.47) -Chapter 14(p.93)):

$$\begin{aligned}
\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} &= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], && (19.1.4)\\
\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], && (19.1.5)\\
\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} &= \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}], && (19.1.6)\\
\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} &= \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}]. && (19.1.7)
\end{aligned}$$

19.1.2 Conditions

Lemma 19.1.1 ($\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$)

(a) Theorem 19.1.1 *holds.*
(b) Theorem 19.1.3 *holds.*
(c) *If* $\rho \le a^\star$ *or* $b \le \rho$, *then* Theorem 19.1.2 *holds.*
(d) *If* $a^\star < \rho < b$, *then* Theorem 19.1.2 *does not always hold.* $\square$

Proof (a) From Table 6.5.3(p.31) (I) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\} = \{V_0 = \rho,\, V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 0\} \tag{19.1.8}$$

First, applying the operation $\mathcal{R}$ (see Step 2 of Scenario$[\mathbb{R}]$(p.60)) to this leads to

$$\begin{aligned}
\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_0 = \rho,\, -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 0\}\\
&= \{-\hat{V}_0 = \rho,\, -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}\\
&= \{\hat{V}_0 = \hat{\rho},\, \hat{V}_0 = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}. \quad t > 0\}
\end{aligned} \tag{19.1.9}$$

Then, applying $\mathcal{C}_{\mathbb{R}}$ to this (see Step 11.5 (p.61)), we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\check{\tilde{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\} \quad t > 0\}. \tag{19.1.10}$$

Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this (see Step 11.5 (p.62)), we obtain

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\} \quad t > 0\}. \tag{19.1.11}$$

Since (19.1.11) holds for any $\rho \in (-\infty, \infty)$, it holds also for $\hat{\rho}$ due to $\hat{\rho} \in (-\infty, \infty)$. Accordingly, (19.1.11) holds for the $\hat{\hat{\rho}}$, so we have

$$\begin{aligned}\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] &= \{\hat{V}_0 = \hat{\hat{\rho}},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\} \quad t > 0\} \\ &= \{\hat{V}_0 = \rho,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\} \quad t > 0\}\end{aligned} \tag{19.1.12}$$

due to $\rho = \hat{\hat{\rho}}$. Now, we have $\hat{V}_0 = \rho = V_0$ from (6.5.17(p.31)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, the second term in the r.h.s. of (19.1.12(p.138)) can be rewritten as $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Thus, by induction $\hat{V}_t = V_t$ for $t \geq 0$. Accordingly (19.1.12) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] = \{V_0 = \rho,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad t > 0, \tag{19.1.13}$$

which is identical to $\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\texttt{A}]\}$ given by Table 6.5.3(p.31) (II), i.e.,

$$\begin{aligned}\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\texttt{A}]\} &= \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] \\ &= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] \quad \text{(see (11.5.32(p.63)))}.\end{aligned} \tag{19.1.14}$$

Hence, since (19.1.4) holds, it follows that Theorem 19.1.1 holds.

(b) From Table 6.5.3(p.31) (III) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\} = \left\{\begin{array}{l} V_0 = \rho, \\ V_1 = \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}, \\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 1 \end{array}\right\}$$

Applying the operation $\mathcal{R}$ to this leads to

$$\begin{aligned}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = \max\{\lambda\beta\max\{0, -\hat{a}-\rho\} + \beta\rho - s, \beta\rho\}, \\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = \max\{-\lambda\beta\min\{0, \hat{a}+\rho\} + \beta\rho - s, \beta\rho\}, \\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} -\hat{V}_0 = \rho, \\ -\hat{V}_1 = -\min\{\lambda\beta\min\{0, \hat{a}+\rho\} - \beta\rho + s, -\beta\rho\}, \\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} \hat{V}_0 = -\rho, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a}+\rho\} - \beta\rho + s, -\beta\rho\}, \\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\} \\ &= \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a}-\hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.\end{aligned}$$

Furthermore, applying $\mathcal{C}_{\mathbb{P}}$ to the above produces

$$\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \check{b}-\hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\check{\hat{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.$$

Finally, applying $\mathcal{I}_{\mathbb{P}}$ to the above produces

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, b-\hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\} \quad t > 1 \end{array}\right\}.$$

Moreover, for the same reason as in the proof of (a), we obtain.

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}] = \left\{\begin{array}{l} V_0 = \rho, \\ V_1 = \min\{\lambda\beta\min\{0, b-\rho\} + \beta\rho + s, \beta\rho\}, \\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad t > 1 \end{array}\right\}$$

The final result is the same as $\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ given by Table 6.5.3(p.31) (IV), hence we have

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} \;=\; \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}]. \tag{19.1.15}$$

Thus, it follows that Theorem 19.1.3 holds.

(c) Let $\rho \le a^\star$ or $b \le \rho$.

1. Let $\rho \le a^\star$. Then, since $\rho \le a^\star < a$ due to Lemma 12.2.1(p.77) (n), we have $\max\{0, a-\rho\} = a - \rho \cdots$ **(1)**. In addition, since $T_{\mathbb{R}}(\rho) = \mu - \rho$ from Lemma 9.1.1(p.41) (f) and $T_{\mathbb{P}}(\rho) = a - \rho$ from Lemma 12.2.1(f), we have $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mu-\rho] = a - \rho = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots$ **(2)** due to (1).
2. Let $b \le \rho$. Then, since $a < b < \rho$, we have $\max\{0, a-\rho\} = 0 \cdots$ **(3)**. In addition, since $T_{\mathbb{R}}(\rho) = 0$ from Lemma 9.1.1(g) and $T_{\mathbb{P}}(\rho) = 0$ from Lemma 12.2.1(g), we have $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = 0 = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots$ **(4)** due to (3).

From the above (2) and (4), whether $\rho \le a^\star$ or $b \le \rho$, we have

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\}, \tag{19.1.16}$$

hence from (5.1.4(p.17)) we have

$$\begin{aligned} \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\lambda\beta T_{\mathbb{R}}(\rho) - (1-\beta)\rho - s] \\ &= \lambda\beta\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] - (1-\beta)\rho - s \\ &= \lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s. \end{aligned} \tag{19.1.17}$$

Thus, we have

$$\begin{aligned} &\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.5.18(\text{p.31})) \text{ with } t = 1] \\ &\quad= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(V_0) + V_0, \beta V_0\}\}] \\ &\quad= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(\rho) + \rho, \beta\rho\}\}] \\ &\quad= \{V_1 = \max\{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] + \rho, \beta\rho\}\} \\ &\quad= \{V_1 = \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}\} \quad \text{(see (19.1.17))} \\ &\quad= \{(6.5.22)\}. \end{aligned}$$

The above result means that $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.5.18(\text{p.31})) \text{ with } t > 0]$ is separated into the two cases, (6.5.22) and (6.5.23). This fact implies that $\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ and $\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ is analogous, i.e.,

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \tag{19.1.18}$$

Accordingly, since (19.1.5) holds, it follows that Theorem 19.1.2 holds.

(d) Let $a^\star < \rho < b$. Then, the same reasoning as in the proof of (c) does not always hold, hence it follows that Theorem 19.1.2 does not always hold. ∎

Remark 19.1.1 (pseudo-reflective element ρ) Let us recall here that $\mathcal{R}$ is an operation applied *only* to attribute elements which depend on the distribution function F. Accordingly, the definition of the operation cannot be applied to the constant ρ which is not related to F; the $\hat{\rho}$ in the proofs of (a,b) is one resulting from *merely rearranging* the expression $-\hat{V}_1 = \rho$ as $\hat{V}_1 = -\rho \to \hat{V}_1 = \hat{\rho}$. However, superficially this transformation $\rho \to \hat{\rho}$ seems to be due to the application of the reflective operation $\mathcal{R}$ defined in Section 11.1.1(p.55). For this reason, regarding the ρ, which is originally a non-reflective element, as a *reflective element* of a sort (see Def. 11.3.3(p.59)), let us call it the *pseudo-reflective element*. □

19.1.3 Diagonal Symmetry

For the same reason as in Section 17.3(p.116), which provides the six equalities and one corollary for $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$, we see that the following equalities and corollary hold for $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{-} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{+}\}], \tag{19.1.19}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{\pm} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{\pm}\}], \tag{19.1.20}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{+} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{-}\}]. \tag{19.1.21}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^{+} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^{-}\}], \tag{19.1.22}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^{\pm} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^{\pm}\}], \tag{19.1.23}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^{-} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^{+}\}]. \tag{19.1.24}$$

Corollary 19.1.1 *We have:*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{+} \ \mathtt{D}\text{-}\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{-}\}, \tag{19.1.25}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{\pm} \ \mathtt{D}\text{-}\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{\pm}\}, \tag{19.1.26}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^{-} \ \mathtt{D}\text{-}\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^{+}\}. \ \square \tag{19.1.27}$$

19.1.4 $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$

19.1.4.1 Preliminary

From (6.5.18(p.31)) and (5.1.8(p.17)) we have

$$\begin{aligned} V_t &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{19.1.28}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 0. \tag{19.1.29}$$

Then, for $t > 0$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \quad (\text{see } (5.1.9(\text{p.17}))), \tag{19.1.30}$$

$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{19.1.31}$$

Finally, from (6.2.58(p.24)), (6.2.60(p.24)), and (6.2.58)we have

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \mathtt{Conduct}_{t\vartriangle}\ (\mathtt{Skip}_{t\vartriangle}), \quad t > 0, \tag{19.1.32}$$

$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle}), \quad t > 0. \tag{19.1.33}$$

19.1.4.2 Analysis

19.1.4.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 19.1.1** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho < b$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$, hence $x_L = x_K = b$ from Lemmas 9.2.1(p.43) (d) and 9.2.2(i). Then, since $K(x) = \lambda T(x) \cdots$ **(1)** for any x from (5.1.4(p.17)), due to Lemma 9.1.1(p.41) (g) we have $K(x) \geq 0 \cdots$ **(2)** for any x and $K(b) = 0 \cdots$ **(3)**.

(a) From (6.5.18(p.31)) we have $V_t \geq K(V_{t-1}) + V_{t-1}$ for $t > 0$, hence $V_t \geq V_{t-1}$ for $t > 0$ due to (2). Thus V_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq b$. Then, since $V_0 \geq b$ from (6.5.17(p.31)), we have $V_{t-1} \geq b$ for $t > 0$ from (a). Hence, since $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 9.2.1(d), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (19.1.29), thus $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$, i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$. Hence, since $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, we have $t^*_\tau = 0$ for $\tau > 0$ due to Preference Rule 7.2.1(p.35), i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) Let $\rho < b$. Then $V_0 < b$. Suppose $V_{t-1} < b$. Then, from Lemma 9.2.2(h) and (6.5.18(p.31)) with $\beta = 1$ we have $V_t < \max\{K(b) + b, b\} = \max\{b, b\}$ due to (3), hence $V_t < b$. Accordingly, by induction $V_{t-1} < b \cdots$ **(4)** for $t > 0$, hence $L(V_{t-1}) > 0 \cdots$ **(5)** for $t > 0$ from Lemma 9.2.1(d), so that $L(V_{t-1}) > 0 \cdots$ **(6)** for $\tau \geq t > 0$. Accordingly, from (19.1.29) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 0$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^\tau V_0$. Accordingly, we have $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. Then $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 0$ due to (6) and (19.1.33). ∎

19.1.4.2.2 Case of $\beta <$ or $s > 0$

For explanatory simplicity, let us define

$$\mathtt{S}_3\boxed{\circledS\blacktriangle}\boxed{\circledast\|} = \left\{\begin{array}{l}\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 0 \text{ such that} \\ (1)\ \boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 0}\langle\tau\rangle}_\blacktriangle \text{ where } \mathtt{Conduct}_{\tau \geq t > 0\blacktriangle}, \\ (2)\ \boxed{\circledast\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \mathtt{Conduct}_{t^\bullet_\tau \geq t > 0\blacktriangle}.\end{array}\right\}.$$

□ **Tom 19.1.2** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$.*

(c) *Let $\rho < x_L$.*

1. $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $a < \rho$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ $(s > 0)$.*
   - i. *Let $a < \rho$.*
     1. *Let $b \geq 0$ $(\kappa \geq 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     2. *Let $b < 0$ $(\kappa < 0)$. Then* $\mathtt{S}_3$(p.141) $\boxed{\circledS\blacktriangle}\boxed{\circledast\|}$ *is true.*
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $b > 0$ $(\kappa > 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
          2. *Let $b \leq 0$ $(\kappa \leq 0)$. Then* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $b \geq 0$ $(\kappa \geq 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
          2. *Let $b < 0$ $(\kappa < 0)$. Then* $\mathtt{S}_3$(p.141) $\boxed{\circledS\blacktriangle}\boxed{\circledast\|}$ *is true.*
     2. *Let $(\lambda\beta\mu - s)/\delta > a$.*
        - i. *Let $b \geq 0$ $(\kappa \geq 0)$. Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        - ii. *Let $b < 0$ $(\kappa < 0)$. Then* $\mathtt{S}_3$(p.141) $\boxed{\circledS\blacktriangle}\boxed{\circledast\|}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots$ **(1)**. Then $V_0 < x_K \cdots$ **(2)** from (6.5.17(p.31)) and $K(\rho) > 0 \cdots$ **(3)** due to Lemma 9.2.2(j1). Accordingly, from (6.5.18) with $t = 1$ we have $V_1 - V_0 = V_1 - \rho = \max\{K(\rho), \beta\rho - \rho\} \geq K(\rho) > 0$ due to (3), hence $V_1 > V_0 \cdots$ **(4)**.

(a) Note (4), hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.18(p.31)) and Lemma 9.2.2(p.43) (e) we have $V_t \leq \max\{K(V_t){+}V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$. Again note (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 9.2.2(f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, and if $a < \rho$, then $a < V_0$ from (6.5.17(p.31)), hence $a < V_t$ for $t \geq 0$ due to (a), thus from Lemma 9.2.2(g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \geq 0$. Consider a sufficiently large $M > 0$ with $\rho \leq M$ and $b \leq M$, hence $V_0 \leq M$ from (6.5.17(p.31)). Suppose $V_{t-1} \leq M$. Then, from Lemma 9.2.2(e) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (9.2.7 (2) (p.43)), hence $V_t \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Thus, by induction $V_t \leq M$ for $t \geq 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta V\}$ from (6.5.18), we have $0 = \max\{K(V), -(1-\beta)V\}$, hence $K(V) \leq 0$, so that $V \geq x_K$ due to Lemma 9.2.2(j1).

(b) Let $x_L \leq \rho$. Then, since $x_L \leq V_0$ from (6.5.17(p.31)), we have $x_L \leq V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 9.2.1(p.43) (a). Accordingly, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (19.1.29), we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, implying that $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$.

(c) Let $\rho < x_L \cdots$ **(5)**, hence $V_0 < x_L \cdots$ **(6)** from (6.5.17(p.31)).

(c1) Since $L(V_0) = L(\rho) > 0 \cdots$ **(7)**, from (5) and Corollary 9.2.1(a), we have $V_1 = K(\rho) + \rho \cdots$ **(8)** due to (19.1.30) with $t = 1$ and $V_1 - \beta V_0 > 0$ from (19.1.29) with $t = 1$, i.e., $V_1 > \beta V_0 \cdots$ **(9)**. Accordingly, we have $t^*_1 = 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_1\langle 1\rangle}_\blacktriangle \cdots$ **(10)** and $\mathtt{Conduct}_{1\blacktriangle} \cdots$ **(11)** due to (7) and (19.1.33) with $t = 1$. Below let $\tau > 1$.

(c2) Let $\beta = 1$, hence $s > 0 \cdots$ **(**12**)** due to the assumption of $\beta < 1$ or $s > 0$ in the lemma. Then $\delta = \lambda \cdots$ **(**13**)** from (9.2.1(p.42)) and $x_L = x_K \cdots$ **(**14**)** from Lemma 9.2.3(b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(**15**)**. Then, from (5) and (14) we have $\rho < x_K \cdots$ **(**16**)**.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.5.17(p.31)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.18) with $\beta = 1$ and Lemma 9.2.2(g) we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Hence, by induction $V_{t-1} < x_K \cdots$ **(**17**)** for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (14). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(e1), for almost the same reason as in the proof of `Tom` 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{CONDUCT}_{\tau\geq t>0\blacktriangle}$.

(c2ii) Let $\rho \leq a \cdots$ **(**18**)**. Then $V_0 \leq a \cdots$ **(**19**)** from (6.5.17(p.31)). In addition, from (8), (18), and (9.2.7 (1) (p.43)) with $\beta = 1$ we have $V_1 = \lambda\mu - s + (1-\lambda)\rho \cdots$ **(**20**)**

(c2ii1) Let $(\lambda\mu - s)/\lambda \leq a$. Then $x_K = (\lambda\mu - s)/\lambda \leq a \cdots$ **(**21**)** from Lemma 9.2.2(j2) and (13). Hence $K(a) \leq 0 \cdots$ **(**22**)** from Lemma 9.2.2(j1). Note (19). Suppose $V_{t-1} \leq a$. Then, from Lemma 9.2.2(e) and (6.5.18) with $\beta = 1$ we have $V_t \leq \max\{K(a) + a, a\} = a$ due to (22), hence by induction $V_{t-1} \leq a$ for $t > 0$. Accordingly, from (6.5.18) with $\beta = 1$ and (9.2.7 (1)) we have $V_t = \max\{\lambda\mu - s + (1-\lambda)V_{t-1}, V_{t-1}\} \cdots$ **(**23**)** for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, since $x_K = \mu - s$ from (21), we have $V_1 = \mu - s = x_K \cdots$ **(**24**)** from (20). In addition, from (23) we have $V_t = \max\{\mu - s, V_{t-1}\} = \max\{x_K, V_{t-1}\}$ for $t > 0$. Note (24). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} = x_L$ for $t > 1$ due to (14). Hence $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, so that $L(V_{t-1}) = 0 \cdots$ **(**25**)** for $\tau \geq t > 1$. Then, from (19.1.29) we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$, i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$. From this and (9) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$. Then, from (7) and (19.1.33) we have $\texttt{Conduct}_{1\blacktriangle}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, since $L(V_{t-1}) > 0$ due to Lemma 9.2.1(e1), from (19.1.30) and Lemma 9.2.2(f) we have $V_t = K(V_{t-1}) + V_{t-1} < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, so that $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(e1). Hence, for almost the same reason as in the proof of `Tom` 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a \cdots$ **(**26**)** from Lemma 9.2.2(j2). Note (6). Suppose $V_{t-1} < x_L$. Then $L(V_{t-1}) > 0$ from Lemma 9.2.1(e1), hence $V_t = K(V_{t-1}) + V_{t-1}$ from (19.1.30). Now, since $a < x_K = x_L$ due to (26) and (14), from Lemma 9.2.2(h) we have $V_t < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L \cdots$ **(**27**)** for $t > 0$, thus $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(e1). Hence, for almost the same reason as in the proof of `Tom` 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c3i) Let $a < \rho \cdots$ **(**28**)** from (6.5.17(p.31)). Then $a < V_0$, hence $a < V_{t-1} \cdots$ **(**29**)** for $t > 0$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 9.2.2(g) and (6.5.18) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_{t-1} is *strictly increasing* in $t > 0 \cdots$ **(**30**)**.

(c3i1) Let $b \geq 0$ ⟪$\kappa \geq 0$⟫. Then, $x_L \geq x_K \geq 0 \cdots$ **(**31**)** from Lemma 9.2.3(c ⟪d⟫). Note (2). Suppose $V_{t-1} < x_K$. Then, from (29) and Lemma 9.2.2(g) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (31), we have $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a). Consequently, for almost the same reason as in the proof of `Tom` 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ † and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3i2) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ **(**32**)** from Lemma 9.2.3(c ⟪d⟫). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$. Then, since $x_K \leq V \cdots$ **(**33**)** due to (a), we have the contradiction of $V \leq x_L < x_K \leq V$ from (32). Accordingly, it is impossible that $V_{t-1} \leq x_L$ for *all* $t > 0$. Therefore, from (6) and (30) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots .$$

Hence, for almost the same reason as in the proof of `Tom` 10.2.2(p.48) (c2iii2) we immediately see that $\texttt{S}_3$ is true.‡

(c3ii) Let $\rho \leq a \cdots$ **(**34**)**, hence $V_0 \leq a \cdots$ **(**35**)** from (6.5.17(p.31)). Then, from (8) and (9.2.7 (1) (p.43)) we have $V_1 = \lambda\beta\mu - s + (1-\lambda)\beta\rho \cdots$ **(**36**)**.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \leq a$. Then, since $x_K = (\lambda\beta\mu - s)/\delta \leq a \cdots$ **(**37**)** from Lemma 9.2.2(j2), we have $V_1 = \delta x_K + (1-\lambda)\beta\rho \cdots$ **(**38**)**.

(c3ii1i) Let $\lambda = 1$. Then, since $\delta = 1$ from (9.2.1(p.42)), we have $x_K = \beta\mu - s \leq a$ from (37) and $V_1 = x_K \leq a \cdots$ **(**39**)** from (38).

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†Note that we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ instead of $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ due to (c1).

‡Note the fine difference between $\texttt{S}_3$ and $\texttt{S}_1$(p.47).

(c3ii1i1) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots (40)$ due to Lemma 9.2.3(c (d)). Note (39). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Hence, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} < x_L$ for $t > 1$ due to (40). Accordingly $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 9.2.1(a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to (7). Therefore, for almost the same reason as in the proof of Tom 19.1.1(c) we have $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1i2) Let $b \leq 0$ ($\kappa \leq 0$). Then, since $x_L \leq x_K$ due to Lemma 9.2.3(c (d)), from (39) we have $V_1 \geq x_L$, hence $V_{t-1} \geq x_L$ for $t > 1$ from (a), so $V_{t-1} \geq x_L$ for $\tau \geq t > 1$. Accordingly, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from Corollary 9.2.1(a), we obtain $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (19.1.29) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$. From the result and (9) we obtain $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$. Then, we have $\mathtt{Conduct}_{1\blacktriangle}$ from (7) and (19.1.33) with $t = 1$.

(c3ii1ii) Let $\lambda < 1$. Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 9.2.2(f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_t is *strictly increasing* in $t \geq 0 \cdots (41)$.

(c3ii1iii1) Let $b \geq 0$ ($\kappa \geq 0$). Then $x_L \geq x_K \geq 0 \cdots (42)$ from Lemma 9.2.3(c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from Lemma 9.2.2(f) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (42). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for almost the same reason as in the proof of Tom 19.1.1(c) we have $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1ii2) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots (43)$ from Lemma 9.2.3(c (d)). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$. Then, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} \leq x_L$ for all $t > 0$. Therefore, from (6) and (41) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

hence for almost the same reason as in the proof of Tom 10.2.2(p.48) (c2iii2) we have $\mathtt{S}_3{}^\ddagger$ is true.

(c3ii2) Let $(\lambda\beta\mu - s)/\lambda > a \cdots (44)$. Then $x_K > (\lambda\beta\mu - s)/\delta > a \cdots (45)$ from Lemma 9.2.2(j2). Let us note here that:

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \geq 0$ for the same reason as in the proof of (c3ii1ii).
2. Let $\lambda = 1$. Then $\beta\mu - s > a \cdots (46)$ from (44). Now, since $K(\rho) + \rho = \beta\mu - s$ from (34) and (9.2.7 (1) (p.43)), we have $V_1 = \beta\mu - s$ from (8), hence $V_1 > a$ from (46), so that $V_{t-1} > a$ for $t > 1$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 9.2.2(g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t > 0$.

Consequently, whether $\lambda < 1$ or $\lambda = 1$, it follows that V_t is *strictly increasing* in $t > 0 \cdots (47)$.

(c3ii2i) Let $b \geq 0$ ($\kappa \geq 0$). Then $x_L \geq x_K \geq 0 \cdots (48)$ from Lemma 9.2.3(c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then from (45) and Lemma 9.2.2(h) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (48), so that $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a). Hence, for almost the same reason as in the proof of Tom 19.1.1(c) we have $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ and $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii2ii) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots (49)$ from Lemma 9.2.3(c (d)). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots (50)$. Now, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. Therefore, from (47) and (6) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

hence for almost the same reason as in the proof of Tom 10.2.2(p.48) (c2iii2) we have $\mathtt{S}_3$ is true. ∎

□ **Tom 19.1.3** ($\mathscr{A}\{$M:2[ℝ][A]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   1. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   2. *Let $b \leq 0$ ($\kappa \leq 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then $V_0 = x_K \cdots (1)$ from (6.5.17(p.31)), hence $K(V_0) = K(x_K) = 0 \cdots (2)$.

(a) We obtain $V_1 \geq K(V_0) + V_0 = V_0 \cdots (3)$ from (6.5.18(p.31)) with $t = 1$ and (2). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 9.2.2(e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$ in the lemma. Then $x_L = x_K$ from Lemma 9.2.3(b). Note (1). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} = x_L$ for $t > 0$, so that $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$. Accordingly, for the same reason as in the proof of Tom 19.1.1(b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) Let $\beta < 1$ and $s = 0$ ($s > 0$) .

(c1) Let $b > 0$ ($\kappa > 0$) . Then $x_L > x_K > 0 \cdots$ (4) from Lemma 9.2.3(c (d)). Note (1). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (4), so that $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 9.2.1(a). Therefore, for the same reason as in the proof of Tom 19.1.1(c) we have $\boxed{\text{ⓢ dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ and $\text{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c2) Let $b \leq 0$ ($\kappa \leq 0$) . Then, since $x_L \leq x_K$ from Lemma 9.2.3(c (d)), we have $x_L \leq V_0$ from (1), hence $x_L \leq V_{t-1}$ for $t > 0$ from (a), so that $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 9.2.1(a). Then, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from(19.1.29), for the same reason as in the proof of Tom 19.1.1(b) we obtain $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$. ∎

$$\text{S}_4\boxed{ⓢ\blacktriangle \mid \bullet\parallel \mid \text{pS}\triangle \mid \text{pS}\blacktriangle} = \left\{\begin{array}{l}\text{There exist } t^\bullet_\tau \text{ and } t^\circ_\tau\ (t^\bullet_\tau > t^\circ_\tau \geq 0) \text{ such that}\\ (1)\ \boxed{\bullet\ \text{dOITd}_{t^\bullet_\tau\geq\tau>0}\langle 0\rangle}_\parallel,\\ (2)\ \boxed{ⓢ\ \text{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_\blacktriangle \text{ where } \text{Conduct}_{\tau\geq t>t^\bullet_\tau\blacktriangle} \text{ and}\\ \qquad\qquad\qquad\quad \text{where } \text{pSkip}_{t^\bullet_\tau\geq t>t^\circ_\tau\triangle}, \text{ and } \text{pSkip}_{t^\circ_\tau\geq t>0\triangle}\ (\text{pSkip}_{t^\circ_\tau\geq t>0\blacktriangle}).\end{array}\right\}$$

□ **Tom 19.1.4** ($\mathscr{A}\{\text{M:2}[\mathbb{R}][\text{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*[†]

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.
3. *Let $x_L > \rho$. Then* $\boxed{ⓢ\ \text{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\text{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($s > 0$) .*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $b \leq 0$ ($\kappa \leq 0$) . Then* $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.
3. *Let $b > 0$ ($\kappa > 0$) .*
   i. *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \text{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\text{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho = x_L$. Then* $\boxed{ⓢ\ \text{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\text{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let $x_L < \rho$. Then* $\text{S}_4\boxed{ⓢ\blacktriangle \mid \bullet\parallel \mid \text{pS}\triangle \mid \text{pS}\blacktriangle}$ *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ($s > 0$) .*

1. *V_t is nondecreasing in $t \geq 0$.*
2. *Let $b \leq 0$ ($\kappa \leq 0$) . Then* $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.
3. *Let $b > 0$ ($\kappa > 0$) . Then* $\boxed{ⓢ\ \text{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\text{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho > x_K \cdots$ (1). Hence $V_0 > x_K \cdots$ (2) from (6.5.17(p.31)) and $K(\rho) < 0 \cdots$ (3) due to Lemma 9.2.2(j1). Suppose $V_{t-1} \geq x_K$. Then, from (6.5.18) and Lemma 9.2.2(e) we have $V_t \geq K(V_{t-1}) + V_{t-1} \geq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \geq x_K \cdots$ (4) for $t > 0$. From (6.5.18) with $t = 1$ we have $V_1 - V_0 = V_1 - \rho = \max\{K(V_0) + V_0, \beta V_0\} - \rho = \max\{K(\rho) + \rho, \beta\rho\} - \rho = \max\{K(\rho), -(1-\beta)\rho\} \cdots$ (5).

(a) Let $\beta = 1$ or $\rho = 0$.

(a1) Then, since $-(1-\beta)\rho = 0$, we have $V_1 - V_0 = 0$ from (5) and (3), i.e., $V_0 = V_1$. Suppose $V_{t-1} = V_t$. Then $V_t = \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Thus, by induction $V_{t-1} = V_t$ for $t > 0$, i.e., $V_0 = V_1 = V_2 = \cdots$, hence $V_t = V_0 = \rho$ for $t \geq 0$.

(a2) Let $x_L \leq \rho$. Then, since $x_L \leq V_{t-1}$ for $t > 0$ from (a1), we have $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 9.2.1(a), hence $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (19.1.29). Accordingly, for the same reason as in the proof of Tom 19.1.1(b) we obtain $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(a3) Let $x_L > \rho$. Then, since $x_L > V_{t-1}$ for $t > 0$ from (a1), we have $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 9.2.1(a), hence for the same reason as in the proof of Tom 19.1.1(c) we obtain $\boxed{ⓢ\ \text{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ and $\text{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) Let $\beta < 1 \cdots$ (6) and $\rho > 0 \cdots$ (7) and let $s = 0$ ($s > 0$) . Then, since $-(1-\beta)\rho < 0 \cdots$ (8), from (5) and (3) we have $V_1 - V_0 < 0$, hence $\rho = V_0 > V_1 \cdots$ (9) from (6.5.17(p.31)).

(b1) We have $V_0 \geq V_1$ from (9). Suppose $V_{t-1} \geq V_t$. Then, from (6.5.18) and Lemma 9.2.2(e) we have $V_t \geq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 0$, i.e., V_t is nonincreasing in $t \geq 0$. In addition, since V_t is lower bounded in t due to (4), it follows that V_t converges to a finite V as $t \to \infty$. Accordingly, from (6.5.18) we have $V = \max\{K(V)+V, \beta V\}$, from which $0 = \max\{K(V), -(1-\beta)V\}$, so that $K(V) \leq 0$, hence $V \leq x_K$ due to Lemma 9.2.2(j1).

(b2) Let $b \leq 0$ ($\kappa \leq 0$) . Then, since $x_L \leq x_K$ due to Lemma 9.2.3(c (d)), from (4) we have $V_{t-1} \geq x_L$ for $t > 0$. Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 9.2.1(a), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (19.1.29), hence for the same reason as in the proof of Tom 19.1.1(b) we obtain $\boxed{\bullet\ \text{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.

(b3) Let $b > 0$ ($\kappa > 0$) . Then $x_L > x_K > 0 \cdots$ (10) from Lemma 9.2.3(c (d)).

[†] The inverse of the condition "$\beta = 1$ or $\rho = 0$" is "$\beta < 1$ and $\rho \neq 0$", which is classified into the two cases of "$\beta < 1$ and $\rho > 0$" and "$\beta < 1$ and $\rho < 0$", leading to the conditions in (b) and (c) that follows.

(b3i) Let $\rho < x_L$. Then, since $V_0 < x_L$ from (6.5.17(p.31)), we have $V_{t-1} < x_L$ for $t > 0$ due to (b1). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of $\texttt{Tom}$ 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ and $\texttt{CONDUCT}_{\tau\geq t>0\blacktriangle}$.

(b3ii) Let $\rho = x_L \cdots \mathbf{(}11\mathbf{)}$. Then, since $V_0 = x_L$ from (6.5.17(p.31)), we have $L(V_0) = L(x_L) = 0 \cdots \mathbf{(}12\mathbf{)}$, hence from (19.1.31) with $t = 1$ we have $V_1 = \beta V_0 \cdots \mathbf{(}13\mathbf{)}$, so that $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$. Below let $\tau > 1$. From (9) and (11) we have $V_1 < V_0 = x_L$. Accordingly, since $V_{t-1} < x_L$ for $t > 1$ from (b1), we have $L(V_{t-1}) > 0 \cdots \mathbf{(}14\mathbf{)}$ for $t > 1$ from Corollary 9.2.1(a), hence $L(V_{t-1}) > 0 \cdots \mathbf{(}15\mathbf{)}$ for $\tau \geq t > 1$. Therefore, $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (19.1.29), hence $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, so that $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. From the result and (13) we obtain $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 = \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (15) and (19.1.33).

(b3iii) Let $x_L < \rho$, hence $x_L < V_0 \cdots \mathbf{(}16\mathbf{)}$ from (6.5.17(p.31)), so that $x_L \leq V_0$. Suppose $x_L \leq V_{t-1} \cdots \mathbf{(}17\mathbf{)}$ for *all* $t > 0$. Then, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 9.2.1(a), we have $V_t = \beta V_{t-1}$ for $t > 0$ from (19.1.31), hence $V_t = \beta^t V_0 = \beta^t \rho > 0$ for $t \geq 0$ due to (7). Then, since $\lim_{t\to\infty} V_t = 0$ due to (6), from (10) we have $x_L > x_K > V_t > 0$ for a sufficiently large t, which contradicts (17). Hence, it is impossible that $x_L \leq V_{t-1}$ for *all* $t > 0$. Accordingly, from (16) and (b1) we see that there exist t_τ° and $t_\tau^\bullet$ ($t_\tau^\circ < t_\tau^\bullet$) such that

$$V_0 \geq V_1 \geq \cdots \geq V_{t_\tau^\circ - 1} > V_{t_\tau^\circ} = V_{t_\tau^\circ+1} = \cdots = V_{t_\tau^\bullet - 1} = x_L > V_{t_\tau^\bullet} \geq V_{t_\tau^\bullet+1} \geq \cdots\cdots \mathbf{(}18\mathbf{)}$$

Hence, we have

$$x_L > V_{t_\tau^\bullet},\ \ x_L > V_{t_\tau^\bullet+1},\ \cdots,$$
$$V_{t_\tau^\circ} = x_L,\ \ V_{t_\tau^\circ+1} = x_L,\ \cdots,\ \ V_{t_\tau^\bullet-1} = x_L,$$
$$V_0 > x_L,\ \ V_1 > x_L,\ \cdots,\ \ V_{t_\tau^\circ-1} > x_L,$$

or equivalently

$$x_L > V_{t-1} \cdots \mathbf{(}19\mathbf{)}, \qquad t > t_\tau^\bullet,$$
$$V_{t-1} = x_L \cdots \mathbf{(}20\mathbf{)}, \qquad t_\tau^\bullet \geq t > t_\tau^\circ,$$
$$V_{t-1} > x_L \cdots \mathbf{(}21\mathbf{)}, \qquad t_\tau^\circ \geq t > 0.$$

Accordingly, we have:

1. Let $t_\tau^\bullet \geq \tau > 0$. Then, since $V_{t-1} \geq x_L$ for $\tau \geq t > 0$ from (20) and (21), we have $L(V_{t-1}) \leq 0 \cdots \mathbf{(}22\mathbf{)}$ for $\tau \geq t > 0$ from Corollary 9.2.1(a), hence $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$ from (19.1.29), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0 \cdots \mathbf{(}23\mathbf{)}$, hence $t_\tau^* = 0$ for $t_\tau^\bullet \geq \tau > 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t_\tau^\bullet\geq\tau>0}\langle 0\rangle}_{\parallel}$. Accordingly, $\texttt{S}_4(1)$ is true. Then, from (23) with $\tau = t_\tau^\bullet$ we have $V_{t_\tau^\bullet} = \beta V_{t_\tau^\bullet-1} = \cdots = \beta^{t_\tau^\bullet} V_0 \cdots \mathbf{(}24\mathbf{)}$,
2. Let $\tau > t_\tau^\bullet$. Then, since $x_L > V_{t-1}$ for $\tau \geq t > t_\tau^\bullet$ from (19), we have $L(V_{t-1}) > 0 \cdots \mathbf{(}25\mathbf{)}$ for $\tau \geq t > t_\tau^\bullet$ from Corollary 9.2.1(a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t_\tau^\bullet$ from (19.1.29), i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > t_\tau^\bullet$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} \cdots \mathbf{(}26\mathbf{)}$. From this and (24) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t_\tau^\bullet} V_{t_\tau^\bullet} = \beta^{\tau - t_\tau^\bullet + 1} V_{t_\tau^\bullet - 1} = \cdots = \beta^\tau V_0,$$

   hence $t_\tau^* = \tau$ for $\tau > t_\tau^\bullet$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>t^\circ}\langle\tau\rangle}_\blacktriangle$.

   (i) We have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t_\tau^\bullet \cdots \mathbf{(}27\mathbf{)}$ form (25) and (19.1.33). Hence, the former half of $\texttt{S}_4(2)$ is true.

   The latter half is shown as follows. First, note here (27). Then we have:

   (ii) If $t_\tau^\bullet \geq t > t_\tau^\circ$, then $L(V_{t-1}) = L(x_L) = 0$ from (20), hence we have $\texttt{Skip}_{t\vartriangle}$ from (19.1.32), implying $\texttt{pSkip}_{t\vartriangle}$ (see Figure 7.2.1(p.34) (II)) or equivalently $\texttt{pSkip}_{t_\tau^\bullet\geq t>t_\tau^\circ\vartriangle}$.

   (iii) If $t_\tau^\circ \geq t > 0$, then $L(V_{t-1}) = (\!(<)\!)\ 0$ ‡ from (21) and Lemma 9.2.1(p.43) (d (\!(e1)\!)), hence we have $\texttt{Skip}_{t\vartriangle}$ (\!($\texttt{Skip}_{t\blacktriangle}$)\!) from (19.1.32) (\!((19.1.33))\!), implying $\texttt{pSkip}_{t\vartriangle}$ (\!($\texttt{pSkip}_{t\blacktriangle}$)\!) or equivalently $\texttt{pSkip}_{t_\tau^\circ\geq t>0\vartriangle}$ (\!($\texttt{pSkip}_{t_\tau^\circ\geq t>0\blacktriangle}$)\!).

   Accordingly, the later half of $\texttt{S}_4(2)$ is true.

(c) Let $\beta < 1$ and $\rho < 0 \cdots \mathbf{(}28\mathbf{)}$ and let $s = 0$ (\!($s > 0$)\!).

(c1) Then, since $-(1-\beta)\rho > 0$, from (5) we have $V_1 - V_0 > 0$, i.e., $V_0 < V_1 \cdots \mathbf{(}29\mathbf{)}$, hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.18) and Lemma 9.2.2(e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(c2) Let $b \leq 0$ (\!($\kappa \leq 0$)\!). Then $x_L \leq x_K$ due to Lemma 9.2.3(c (\!(d)\!)), hence from (4) we have $V_{t-1} \geq x_L$ for $t > 0$. Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 9.2.1(a), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (19.1.29), hence for the same reason as in the proof of $\texttt{Tom}$ 19.1.1(b) we obtain $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c3) Let $b > 0$ (\!($\kappa > 0$)\!). Then $x_L > x_K > 0 \cdots \mathbf{(}30\mathbf{)}$ from Lemma 9.2.3(c (\!(d)\!)). Then, we have $\rho < 0 < x_K$ from (28) and (30), hence $V_0 < x_K$ from (6.5.17(p.31)), so that $V_0 \leq x_K$. Suppose $V_{t-1} \leq x_K$, hence $V_{t-1} < x_L$ form (30), thus $L(V_{t-1}) > 0$ from Corollary 9.2.1(a). Accordingly, from (19.1.30) and Lemma 9.2.2(e) we have $V_t = K(V_{t-1}) + V_{t-1} \leq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \leq x_K$ for $t > 0$, so that $V_{t-1} < x_L$ for $t > 0$ from (30). Therefore, since $L(V_{t-1}) > 0 \cdots \mathbf{(}31\mathbf{)}$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of $\texttt{Tom}$ 19.1.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. ∎

‡ If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

19.1.4.3 Market Restriction

19.1.4.3.1 Positive Restriction

□ Pom **19.1.1** ($\mathscr{A}${M:2[ℝ][A]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. → ⓓ
(c) *Let $\rho < b$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ

Proof The same as Tom 19.1.1(p.140) due to Lemma 16.4.1(p.100). ∎

□ Pom **19.1.2** ($\mathscr{A}${M:2[ℝ][A]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a \leq \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ
(c) *Let $\rho < x_L$.*
  1. $\boxed{Ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → Ⓢ
  2. *Let $\beta = 1$, hence $s > 0$.*
    i. *Let $a \leq \rho$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
    ii. *Let $\rho < a$.*
      1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
        i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
        ii. *Let $\lambda < 1$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
      2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
  3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
  4. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $a < \rho$.*
      1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
      2. *Let $\lambda\beta\mu < s$. Then* S3(p.141) $\boxed{Ⓢ^{\blacktriangle}\,\vert\,⊛^{\|}}$ *is true* → → Ⓢ / ⊛
    ii. *Let $\rho \leq a$.*
      1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        i. *Let $\lambda = 1$.*
          1. *Let $\beta\mu > s$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
          2. *Let $\beta\mu \leq s$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
        ii. *Let $\lambda < 1$.*
          1. *Let $\lambda\beta\mu \geq s$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
          2. *Let $\lambda\beta\mu < s$. Then* S3(p.141) $\boxed{Ⓢ^{\blacktriangle}\,\vert\,⊛^{\|}}$ *is true* → → Ⓢ / ⊛
      2. *Let $(\lambda\beta\mu - s)/\delta > a$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → Ⓢ

Proof Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$. Then, we have $\kappa = \lambda\beta\mu - s \cdots (3)$ from Lemma 9.3.1(p.45) (a).

(a-c2ii2) The same as Tom 19.1.2(p.141) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c3i1,c3ii1i1,c3ii1ii1,c3ii2i) of Tom 19.1.2.

(c4) Let $\beta < 1$ and $s > 0$.

(c4i-c4ii1ii2) Immediate from (3) and Tom 19.1.2(c3i-c3ii1ii2) with κ.

(c4ii2) Let $(\lambda\beta\mu - s)/\delta > a$. Then, since$(\lambda\beta\mu - s)/\delta > a > 0$ due to (1), we have $\lambda\beta\mu - s > 0$, so that $\kappa > 0$ due to (3). Hence, it suffices to consider only (c3ii2i) of Tom 19.1.2. ∎

□ Pom **19.1.3** ($\mathscr{A}${M:2[ℝ][A]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ
(c) *Let $\beta < 1$ and $s = 0$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
(d) *Let $\beta < 1$ and $s > 0$.*
  1. *Let $\lambda\beta\mu > s$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → Ⓢ
  2. *Let $\lambda\beta\mu \leq s$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then, we have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a,b) The same as Tom 19.1.3(p.143) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 19.1.3.

(d) Let $\beta < 1$ and $s > 0$.

(d1,d2) Immediate from (2) and Tom 19.1.3(c1,c2) with κ. ∎

☐ Pom **19.1.4** ($\mathscr{A}\{$M:2[$\mathbb{R}$][A]$^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_L \leq \rho$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ → ⓓ
3. *Let* $x_L > \rho$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$.*
1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\rho < x_L$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ
3. *Let* $\rho = x_L$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓓ/ⓢ
4. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ [ⓢ▲ | ●‖ | pS△ | pS▲] *is true* $\rightarrow$ → ⓢ/ⓓ/pS

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s > 0$.*
1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\lambda\beta\mu \leq s$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ → ⓓ
3. *Let* $\lambda\beta\mu > s$.
   i. *Let* $\rho < x_L$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ
   ii. *Let* $\rho = x_L$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle\ \vartriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ → ⓓ/ⓢ
   iii. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ [ⓢ▲ | ●‖ | pS△ | pS▲] *is true* (see `Numerical Example` 19.1.1(p.174)) $\rightarrow$ → ⓢ/ⓓ/pS

(d) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$.*
1. V_t *is nondecreasing in* $t \geq 0$.
2. $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ

(e) *Let $\beta < 1$ and $\rho < 0$ and let $s > 0$.*
1. V_t *is nondecreasing in* $t \geq 0$.
2. *Let* $\lambda\beta\mu \leq s$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ → ⓓ
3. *Let* $\lambda\beta\mu > s$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ

Proof Suppose $a > 0$, hence $b > \mu > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a-a3) The same as `Tom` 19.1.4(p.144) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. First, (b1) is the same as `Tom` 19.1.4(b1). Next, due to (1) it suffices to consider only (b3i-b3iii) of `Tom` 19.1.4.

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. First, (c1) is the same as `Tom` 19.1.4(b1). Next, due to (1) it suffices to consider only (b3i-b3iii) of `Tom` 19.1.4.

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. First, (d1) is the same as `Tom` 19.1.4(c1). Next, since $\kappa = \lambda\beta\mu > 0$ due to (2) and (1), it suffices to consider only (c3) of `Tom` 19.1.4.

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. First, (e1) is the same as `Tom` 19.1.4(c1). Next, (e2,e3) are the same as `Tom` 19.1.4(c2,c3) with κ. ∎

19.1.4.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.1.4.3.3 Negative Restriction

☐ Nem **19.1.1** ($\mathscr{A}\{$M:2[$\mathbb{R}$][A]$^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) V_t *is nondecreasing in* $t \geq 0$.
(b) *Let* $\rho \geq b$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ → ⓓ
(c) *Let* $\rho < b$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ

Proof The same as `Tom` 19.1.1(p.140) due to Lemma 16.4.1(p.100). ∎

☐ Nem **19.1.2** ($\mathscr{A}\{$M:2[$\mathbb{R}$][A]$^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t \geq 0$ *if* $\lambda < 1$ *or* $a \leq \rho$, *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
(b) *Let* $x_L \leq \rho$. *Then* $\boxed{\bullet\, \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ → ⓓ
(c) *Let* $\rho < x_L$.
1. $\boxed{ⓢ\, \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\rightarrow$ → ⓢ
2. *Let* $\beta = 1$.
   i. *Let* $a \leq \rho$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ
   ii. *Let* $\rho < a$.
      1. *Let* $(\lambda\mu - s)/\lambda \leq a$.
         i. *Let* $\lambda = 1$. *Then* $\boxed{⊛\, \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\rightarrow$ → ⊛
         ii. *Let* $\lambda < 1$. *Then* $\boxed{ⓢ\, \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → ⓢ

2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

3. *Let $\beta < 1$ and $s = 0$. Then we have* $\texttt{S}_3$(p.141) $\boxed{Ⓢ^\blacktriangle | ⊛^{\parallel}}$.

4. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $a < \rho$. Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ^\blacktriangle | ⊛^{\parallel}}$ *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ⊛
   - ii. *Let $\rho \leq a$.*
     1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
        - i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⊛
        - ii. *Let $\lambda < 1$. Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ^\blacktriangle | ⊛^{\parallel}}$ *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ⊛
     2. *Let $(\lambda\beta\mu - s)/\delta > a$. Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ^\blacktriangle | ⊛^{\parallel}}$ *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ⊛

Proof Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$ and $\kappa = -s \cdots (3)$ from Lemma 9.3.1(p.45) (a).

(a-c2ii2) The same as Tom 19.1.2(p.141) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu - s)/\delta \leq a$. Then, since $\lambda\beta\mu/\delta \leq a$, we have $\lambda\beta\mu \leq \delta a$, hence $\lambda\beta\mu \leq \delta a \leq \lambda a$ due to (2) and (9.2.2 (1) (p.42)), so that $\beta\mu \leq a$, which contradicts [15(p.101)]. Thus it must be that $(\lambda\beta\mu - s)/\delta > a$. From this and (1) it suffices to consider only (c3i2,c3ii2ii) of Tom 19.1.2(p.141).

(c4-c4ii2) Let $\beta < 1$ and $s > 0$. Then $\kappa < 0$ due to (3). Hence, it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of Tom 19.1.2(p.141) with κ. ▮

☐ **Nem 19.1.3** ($\mathscr{A}\{\text{M:2}[\mathbb{R}][\text{A}]^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *We have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. $\rightarrow$ ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a) The same as Tom 19.1.3(p.143) (a).

(b) Let $\beta = 1$. Then, it suffices to consider only (b) of Tom 19.1.3(p.143), we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.3 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.3. Thus, whether $s = 0$ or $s > 0$, we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. Accordingly, whether $\beta = 1$ or $\beta < 1$, it eventually follows that we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. ▮

☐ **Nem 19.1.4** ($\mathscr{A}\{\text{M:2}[\mathbb{R}][\text{A}]^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
1. *$V_t = \rho$ for $t \geq 0$.*
2. *$x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
3. *Let $x_L > \rho$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$.*
1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
2. *We have Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

(c) *Let $\beta < 1$ and $\rho < 0$.*
1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
2. *We have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a-a3) The same as Tom 19.1.4(p.144) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as Tom 19.1.4(b1).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 19.1.4 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (b2) of Tom 19.1.4. Thus, whether $s = 0$ or $s > 0$, it eventually follows that we have the same result.

(c) Let $\beta < 1$ and $\rho < 0$.

(c1) The same as Tom 19.1.4(c1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.4 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.4. Thus, whether $s = 0$ or $s > 0$, it eventually follows that we have the same result. ▮

19.1.5 $\tilde{\mathsf{M}}$:2[ℝ][A]

19.1.5.1 Preliminary

Due to Lemma 19.1.1(p.137) (a) we can use Theorem 19.1.1(p.137) in the proof of `Tom`'s 19.1.5 – 19.1.8 that follows.

19.1.5.2 Analysis

19.1.5.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 19.1.5** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][A]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *Let $\rho \le a$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Tom` 19.1.1(p.140). ∎

19.1.5.2.2 Case of $\beta < 1$ or $s > 0$

□ **Tom 19.1.6** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][A]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$ or $b > \rho$, and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \ge \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho > x_{\tilde{L}}$.*
  1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
    i. *Let $b > \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
    ii. *Let $\rho \ge b$.*
      1. *Let $(\lambda\mu + s)/\lambda \ge b$.*
        i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
        ii. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
      2. *Let $(\lambda\mu + s)/\lambda < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
  3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
    i. *Let $b > \rho$.*
      1. *Let $a \le 0$ ($\tilde{\kappa} \le 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
      2. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* S_3(p.141) $\boxed{ⓢ^{\blacktriangle}\,|\,⊛^{\parallel}}$ *is true.*
    ii. *Let $\rho \ge b$.*
      1. *Let $(\lambda\beta\mu + s)/\delta \ge b$.*
        i. *Let $\lambda = 1$.*
          1. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
          2. *Let $a \ge 0$ ($\tilde{\kappa} \ge 0$). Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
        ii. *Let $\lambda < 1$.*
          1. *Let $a \le 0$ ($\tilde{\kappa} \le 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
          2. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* S_3(p.141) $\boxed{ⓢ^{\blacktriangle}\,|\,⊛^{\parallel}}$ *is true.*
      2. *Let $(\lambda\beta\mu + s)/\delta < b$.*
        i. *Let $a \le 0$ ($\tilde{\kappa} \le 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
        ii. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* S_3(p.141) $\boxed{ⓢ^{\blacktriangle}\,|\,⊛^{\parallel}}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Tom` 19.1.2(p.141). ∎

□ **Tom 19.1.7** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][A]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
  1. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
  2. *Let $a \ge 0$ ($\tilde{\kappa} \ge 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Tom` 19.1.3(p.143). ∎

□ **Tom 19.1.8** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][A]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
  1. *$V_t = \rho$ for $t \ge 0$.*
  2. *Let $x_{\tilde{L}} \ge \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
(b) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ($s > 0$).*
  1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

2. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let* $a < 0$ $(\tilde{\kappa} < 0)$.
   i. *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\|}$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let* $x_{\tilde{L}} > \rho$. *Then* $\texttt{S}_4$ $\boxed{Ⓢ\blacktriangle \mid \bullet\| \mid \text{pS}\triangle \mid \text{pS}\blacktriangle}$ *is true.*

(c) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $(s > 0)$.
1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $a \geq 0$ $(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
3. *Let* $a < 0$ $(\tilde{\kappa} < 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Tom` 19.1.4(p.144). ∎

19.1.5.3 Market Restriction

19.1.5.3.1 Positive Restriction

☐ `Pom` **19.1.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *Let* $\rho \leq a$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ
(c) *Let* $\rho > a$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Nem` 19.1.1(p.147) (see (17.1.22(p.113))). ∎

Direct proof The same as `Tom` 19.1.5(p.149) due to Lemma 16.4.1(p.100). ∎

☐ `Pom` **19.1.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$ *or* $b \geq \rho$, *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ
(c) *Let* $\rho > x_{\tilde{L}}$.
1. $\boxed{Ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$ → → ⓢ
2. *Let* $\beta = 1$.
   i. *Let* $b > \rho$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ
   ii. *Let* $\rho \geq b$.
      1. *Let* $(\lambda\mu + s)/\lambda \geq b$.
         i. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → → ⊛
         ii. *Let* $\lambda < 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⊛
      2. *Let* $(\lambda\mu + s)/\lambda < b$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ
3. *Let* $\beta < 1$ *and* $s = 0$. *Then we have* $\texttt{S}_3$(p.141) $\boxed{Ⓢ\blacktriangle \mid ⊛\|}$.
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $b > \rho$. *Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ\blacktriangle \mid ⊛\|}$ *is true* → → ⓢ / ⊛
   ii. *Let* $\rho \geq b$.
      1. *Let* $(\lambda\beta\mu + s)/\delta \geq b$.
         i. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → → ⊛
         ii. *Let* $\lambda < 1$. *Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ\blacktriangle \mid ⊛\|}$ *is true* → → ⓢ / ⊛
      2. *Let* $(\lambda\beta\mu + s)/\delta < b$. *Then* $\texttt{S}_3$(p.141) $\boxed{Ⓢ\blacktriangle \mid ⊛\|}$ *is true.* ☐ → → ⓢ / ⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Nem` 19.1.2(p.147) (see (17.1.22(p.113))). ∎

Direct proof Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$ and $\tilde{\kappa} = s \cdots (3)$ from Lemma 11.6.6(p.68) (a).

(a-c2ii2) The same as `Tom` 19.1.6(p.149) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu + s)/\delta \geq b$. Then, since $\lambda\beta\mu/\delta \geq b$, we have $\lambda\beta\mu \geq \delta b$, hence $\lambda\beta\mu \geq \delta b \geq \lambda b$ due to (2) and (9.2.2 (1) (p.42)), so that $\beta\mu \geq b$, which contradicts [3(p.101)]. Thus, it must be that $(\lambda\beta\mu + s)/\delta < b$. From this and (1) it suffices to consider only (c3ii2ii) of `Tom` 19.1.6(p.149).

(c4-c4ii2) If $\beta < 1$ and $s > 0$, then $\kappa > 0$ due to (3), hence it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) with κ. ∎

☐ `Pom` **19.1.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *We have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ☐ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Nem` 19.1.3(p.148) (see (17.1.22(p.113))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 11.6.6(p.68) (a).

(a) The same as `Tom` 19.1.7(p.149) (a).

(b) Let $\beta = 1$. Then it suffices to consider only (b) of `Tom` 19.1.7. Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (c2) of `Tom` 19.1.7 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of `Tom` 19.1.7, thus, whether $s = 0$ or $s > 0$ we have the same result. Accordingly, whether $\beta = 1$ or $\beta < 1$, it follows that we have the same result. ∎

□ **Pom 19.1.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ
3. *Let* $x_{\tilde{L}} < \rho$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ

(b) *Let $\beta < 1$ and $\rho < 0$.*

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

(c) *Let $\beta < 1$ and $\rho > 0$.*

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to **Nem** 19.1.4(p.148) (see (a(p.89))). ▮

Direct proof Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 11.6.6(p.68) (a).

(a-a3) The same as **Tom** 19.1.8(p.149) (a-a3).

(b) Let $\beta < 1$ and $\rho < 0$.

(b1) The same as **Tom** 19.1.8(b1).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of **Tom** 19.1.8 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b2) of **Tom** 19.1.8. Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $\beta < 1$ and $\rho > 0$.

(c1) The same as **Tom** 19.1.8 (c1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of **Tom** 19.1.8 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of **Tom** 19.1.8. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ▮

19.1.5.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.1.5.3.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

19.1.6 M:2[ℙ][A]

19.1.6.1 Preliminary

From (6.5.23(p.31)) and from (5.1.21(p.18)) and (5.1.20) we have

$$\begin{aligned} V_t &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{19.1.34}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 1. \tag{19.1.35}$$

Then, for $t > 1$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \tag{19.1.36}$$

$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{19.1.37}$$

Now, from (6.2.86(p.26)) and (6.2.84) we have

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \mathtt{Conduct}_{t\vartriangle}(\mathtt{Skip}_{t\vartriangle}), \tag{19.1.38}$$

$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}(\mathtt{Skip}_{t\blacktriangle}). \tag{19.1.39}$$

From (6.5.22(p.31)) we have

$$V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - s, 0\} + \beta\rho, \tag{19.1.40}$$

hence

$$V_1 - \beta V_0 = V_1 - \beta\rho = \max\{\lambda\beta\max\{0, a-\rho\} - s, 0\} \geq 0. \tag{19.1.41}$$

From the comparison of the two terms within { } in the right side of (19.1.40) it can be seen that

$$\mathbb{S}_1 \overset{\text{def}}{=} \lambda\beta\max\{0, a-\rho\} \geq (\leq)\ s \Rightarrow \mathtt{Conduct}_{1\vartriangle}(\mathtt{Skip}_{1\vartriangle}), \tag{19.1.42}$$

$$\mathbb{S}_1 \overset{\text{def}}{=} \lambda\beta\max\{0, a-\rho\} > (<)\ s \Rightarrow \mathtt{Conduct}_{1\blacktriangle}(\mathtt{Skip}_{1\blacktriangle}). \tag{19.1.43}$$

19.1.6.2 Analysis

19.1.6.2.1 Case of $\beta = 1$ and $s = 0$

19.1.6.2.1.1 Preliminary

Let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.18)), (5.1.20), and Lemma 12.2.1(p.77)(g) we have

$$K(x) = L(x) = \lambda T(x) \geq 0 \quad \text{for any } x. \tag{19.1.44}$$

In addition, from (19.1.35) we have

$$V_t - \beta V_{t-1} = \max\{\lambda T(V_{t-1}), 0\} = \lambda T(V_{t-1}) \geq 0, \quad t > 1. \tag{19.1.45}$$

Finally, from (19.1.40) we have

$$V_1 = \max\{\lambda \max\{0, a-\rho\}, 0\} + \rho \tag{19.1.46}$$

$$= \lambda \max\{0, a-\rho\} + \rho \quad (\text{due to } \lambda \max\{0, a-\rho\} \geq 0) \tag{19.1.47}$$

$$= \max\{\rho, \lambda a + (1-\lambda)\rho\}. \tag{19.1.48}$$

19.1.6.2.1.2 Case of $\rho \leq a^\star$

In this case, due to Lemma 19.1.1(p.137)(c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (15.3.3(p.98))) in Theorem 19.1.2(p.137) to `Tom` 19.1.1(p.140).

Lemma 19.1.2 ($\mathscr{A}${M:2[ℙ][A]}) *Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.

(a) The same as `Tom` 19.1.1(a).

(b) Due to the assumption $\rho \leq a^\star$ we have $\rho \leq a^\star < a < b$ from Lemma 12.2.1(p.77)(n). Hence it suffices to consider only (c) of `Tom` 19.1.1. ∎

19.1.6.2.1.3 Case of $b \leq \rho$

In this case, due to Lemma 19.1.1(p.137)(c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.1.2(p.137) to `Tom` 19.1.1(p.140).

Lemma 19.1.3 ($\mathscr{A}${M:2[ℙ][A]}) *Assume $b \leq \rho$ and let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

Proof Assume $b \leq \rho \cdots (1)$ and let $\beta = 1$ and $s = 0$.

(a) The same as `Tom` 19.1.1(a).

(b) Due to (1) it suffices to consider only (b) of `Tom` 19.1.1. ∎

19.1.6.2.1.4 Case of $a^\star < \rho < b$

In this case, due to Lemma 19.1.1(p.137)(d) we cannot apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.1.2(p.137) to `Tom` 19.1.1.

Lemma 19.1.4 ($\mathscr{A}${M:2[ℙ][A]}) *Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $a \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ *and* $\texttt{pSKIP}_{1\vartriangle}$.
(c) *Let $\rho < a$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof Assume $a^\star < \rho < b \cdots (1)$ and let $\beta = 1$ and $s = 0$. Then $L(x) = K(x) = \lambda T(x) \geq 0 \cdots (2)$ for any x from (5.1.20(p.18)) and (5.1.21) and from Lemma 12.2.1(g). Then, since $\rho < b$ and $a < b$, from (19.1.48) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ due to (1), we have $V_t < \max\{K(b) + b, b\}$ from (6.5.23) with $\beta = 1$ and Lemma 12.2.3(p.80)(h), hence $V_t < \max\{\beta b - s, b\}$ from (12.2.13 (2) (p.79)), so that $V_{t-1} < \max\{b, b\} = b$ due to the assumption of $\beta = 1$ and $s = 0$. Accordingly, by induction we have $V_{t-1} < b \cdots (3)$ for $t > 1$, hence $T(V_{t-1}) > 0 \cdots (4)$ for $t > 1$ from Lemma 12.2.1(g). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (19.1.45), i.e., $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots (5)$ for $\tau > 1$. In addition, since $L(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots (6)$ for $\tau \geq t > 1$ due to (4), we have $\texttt{Conduct}_{\tau\geq t>1\blacktriangle} \cdots (7)$ from (19.1.39).

(a) From (19.2.24) and (6.5.21(p.31)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a-\rho\} \geq 0$, hence $V_1 \geq V_0 \cdots (8)$. Since $V_2 \geq K(V_1) + V_1$ from (6.5.23(p.31)) with $t = 2$, we have $V_2 - V_1 \geq K(V_1) \geq 0$ due to (2), hence $V_2 \geq V_1 \cdots (9)$. Suppose $V_t \geq V_{t-1}$. Then from (6.5.23) and Lemma 12.2.3(e) we have $V_{t+1} = \max\{K(V_t) + V_t, \beta V_t\} \geq \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 1$. From this and (8) we have $V_t \geq V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho \cdots (10)$, hence $V_1 = \rho$ from (19.1.47), so that $V_1 < b$ due to (1). Then $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$ from (6.5.21(p.31)), hence $V_1 = \beta V_0 \cdots (11)$, so that $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$. Below let $\tau > 1$. Then, from (5) and (11) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$. Here note $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ from (7). In addition, since $\lambda \max\{0, a - \rho\} = 0$ due to (10), we have $\lambda \max\{0, a - \rho\} = 0 \leq s$ for any $s \geq 0$, hence $\texttt{Skip}_{1\vartriangle}$ due to (19.1.42). Hence it follows that we have $\texttt{pSkip}_{1\vartriangle}$ (see Remark 7.2.1(p.34)).

(c) Let $\rho < a \cdots (12)$, hence $V_1 = \lambda(a - \rho) + \rho$ due to (19.1.47). Then, from (6.5.21(p.31)) we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a - \rho) > 0$, i.e., $V_1 > \beta V_0 \cdots (13)$, hence $t_1^* = 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle} \cdots (14)$. Below let $\tau > 1$. Then, from (5) and (13) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$. From the result and (14) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. Since $a - \rho > 0$ due to (12), we have $\lambda \max\{0, a - \rho\} > 0$, implying that we have $\texttt{Conduct}_{1\blacktriangle}$ due to (19.1.43(p.151)). From this and (7) it follows that $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. ∎

19.1.6.2.1.5 Summary of Lemmas 19.1.2 − 19.1.4

□ **Tom 19.1.9** ($\mathscr{A}\{$M:2[ℙ][A]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \leq a^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
(c) *Let $b \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ *and* $\texttt{pSKIP}_{1\vartriangle}$.
  2. *Let $\rho < a$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof (a) The same as Lemmas 19.1.2(a), 19.1.3(a), and 19.1.4(a).
(b) The same as Lemma 19.1.2(b).
(c) The same as Lemma 19.1.3(b).
(d-d2) The same as Lemma 19.1.4(b,c). ∎

Corollary 19.1.2 *Let $\beta = 1$ and $s = 0$. Then, the optimal price to propose z_t is nondecreasing in t.* □

Proof Immediate from Tom 19.1.9(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.1.6.2.2 Case of $\beta < 1$ or $s > 0$

19.1.6.2.2.1 Case of $\rho \leq a^\star$

In this case, due to Lemma 19.1.1(p.137) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.1.2(p.137) to Tom's 19.1.2(p.141) −19.1.4(p.144).

□ **Tom 19.1.10** ($\mathscr{A}\{$M:2[ℙ][A]$\}$) *Assume $\rho \leq a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho < x_L$.*
  1. $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
     i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
        1. *Let $\lambda = 1$. Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
        2. *Let $\lambda < 1$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *and* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
  3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
     i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
        1. *Let $\lambda = 1$.*
           i. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
           ii. *Let $b \leq 0$ ($\kappa \leq 0$). Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
        2. *Let $\lambda < 1$.*
           i. *Let $b \geq 0$ ($\kappa \geq 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
           ii. *Let $b < 0$ ($\kappa < 0$). Then* $\texttt{S}_3$(p.141) $\boxed{\circledS^{\blacktriangle} \mid \circledast^{\parallel}}$ *is true.*
     ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
        1. *Let $b \geq 0$ ($\kappa \geq 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        2. *Let $b < 0$ ($\kappa < 0$). Then* $\texttt{S}_3$(p.141) $\boxed{\circledS^{\blacktriangle} \mid \circledast^{\parallel}}$ *is true.* □

Proof When $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to Tom 19.1.2(p.141), "$a < \rho$" in (c2i,c3i) of Tom 19.1.2(p.141) changes into "$a^\star < \rho$", which contradicts the assumption $\rho \le a^\star$. Accordingly, removing all the assertions related to "$a < \rho$" from Tom 19.1.2 leads to this Tom. ∎

Corollary 19.1.3 *Assume $\rho \le a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$. Then, the optimal price to propose z_t is nondecreasing in $t \ge 0$.* ∎

Proof Immediate from Tom 19.1.10(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **Tom 19.1.11** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$\}$) *Assume $\rho \le a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\beta < 1$ and $s = 0$ ($\!(s > 0)\!$).*
  1. *Let $b > 0$ ($\!(\kappa > 0)\!$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
  2. *Let $b \le 0$ ($\!(\kappa \le 0)\!$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

Proof Since both a and μ are not included in Tom 19.1.3(p.143), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it, no change occurs. ∎

Corollary 19.1.4 *Assume $\rho \le a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$. Then, the optimal price to propose z_t is nondecreasing in $t \ge 0$.* ∎

Proof Immediate from Tom 19.1.11(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **Tom 19.1.12** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$\}$) *Assume $\rho \le a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
  1. *$V_t = \rho$ for $t \ge 0$.*
  2. *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $x_L > \rho$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($\!(s > 0)\!$).*
  1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
  2. *Let $b \le 0$ ($\!(\kappa \le 0)\!$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $b > 0$ ($\!(\kappa > 0)\!$).*
     i. *Let $\rho < x_L$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
     ii. *Let $\rho = x_L$. Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
     iii. *Let $\rho > x_L$. Then* $\texttt{S}_4$ $\boxed{\circledS\blacktriangle \mid \bullet\parallel \mid \text{pS}\triangle \mid \text{pS}\blacktriangle}$ *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ($\!(s > 0)\!$).*
  1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
  2. *Let $b \le 0$ ($\!(\kappa \le 0)\!$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $b > 0$ ($\!(\kappa > 0)\!$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$. □

Proof Since both a and μ are not included in Tom 19.1.4(p.144), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it, no change occurs. ∎

Corollary 19.1.5 *Assume $\rho \le a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then $z_t = z(\rho)$ for $t \ge 0$, i.e., constant in $t \ge 0$.*
(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($\!(s > 0)\!$). Then z_t is nonincreasing in $t \ge 0$.*
(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ($\!(s > 0)\!$). Then z_t is nondecreasing in $t \ge 0$.* □

Proof Immediate from Tom 19.1.12(a1,b1,c1) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.1.6.2.2.2 Case of $b \le \rho$

In this case, due to Lemma 19.1.1(p.137) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.1.2(p.137) to Tom's 19.1.2(p.141)-19.1.4(p.144).

□ **Tom 19.1.13** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$\}$) *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V \ge x_K$ as $t \to \infty$.*
(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho < x_L$.*
  1. $\boxed{\circledS\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\,\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
  3. *Let $\beta < 1$ and $s = 0$ ($\!(s > 0)\!$).*
     i. *Let $b \ge 0$ ($\!(\kappa \ge 0)\!$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\,\blacktriangle}$.
     ii. *Let $b < 0$ ($\!(\kappa < 0)\!$). Then* $\texttt{S}_3$(p.141) $\boxed{\circledS\blacktriangle \mid \circledast\parallel}$ *is true.* □

Proof When $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to Tom 19.1.2(p.141), $\rho \leq a$ in (c2ii,c3ii) of Tom 19.1.2(p.141) changes into $\rho \leq a^\star$. Then, since $\rho \leq a^\star < a$ due to Lemma 12.2.1(p.77) (n), we have $\rho < a < b$, which contradicts $b \leq \rho$. Thus, it follows that all the assertions related to $\rho \leq a$ must be removed from Tom 19.1.2. ∎

Corollary 19.1.6 *Assume $b \leq \rho$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$. Then z_t is nondecreasing in $t \geq 0$.* □

Proof Immediate from Tom 19.1.13(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **Tom 19.1.14** ($\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume $b \leq \rho$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*
  1. *Let $b > 0$ ⟪$\kappa > 0$⟫. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0}{}_{\blacktriangle}$.
  2. *Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

Proof Since both a and μ are not included in Tom 19.1.3(p.143), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it, no change occurs. ∎

Corollary 19.1.7 *Assume $b \leq \rho$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$. Then z_t is nondecreasing in $t \geq 0$.* □

Proof Immediate from Tom 19.1.14(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **Tom 19.1.15** ($\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume $b \leq \rho$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
  1. *$V_t = \rho$ for $t \geq 0$.*
  2. *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $x_L > \rho$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0}{}_{\blacktriangle}$.
(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟪$s > 0$⟫.*
  1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
  2. *Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $b > 0$ ⟪$\kappa > 0$⟫.*
     i. *Let $\rho < x_L$. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0}{}_{\blacktriangle}$.
     ii. *Let $\rho = x_L$. Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0}{}_{\blacktriangle}$.
     iii. *Let $x_L < \rho$. Then* $\texttt{S}_4$ | Ⓢ▲ | ●‖ | pS△ | pS▲ | *is true.*
(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟪$s > 0$⟫.*
  1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
  2. *Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
  3. *Let $b > 0$ ⟪$\kappa > 0$⟫. Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0}{}_{\blacktriangle}$. □

Proof Since both a and μ are not included in Tom 19.1.4(p.144), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it, no change occurs. ∎

Corollary 19.1.8 *Assume $b \leq \rho$, let $\beta < 1$ or $s > 0$, and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then $z_t = z(\rho)$ for $t \geq 0$.*
(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟪$s > 0$⟫. Then z_t is nonincreasing in $t \geq 0$.*
(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟪$s > 0$⟫. Then z_t is nondecreasing in $t \geq 0$.* □

Proof Immediate from Tom 19.1.15(a1,b1,c1) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.1.6.2.2.3 Case of $a^\star < \rho < b$

In this case, due to Lemma 19.1.1(p.137) (d) we cannot apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ of Theorem 19.1.2(p.137) to Tom's 19.1.1(p.140) – 19.1.4(p.144). Below, let us note

$$V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - s, 0\} + \beta\rho \quad \text{(the same as (19.1.40(p.151)))}. \tag{19.1.49}$$

Lemma 19.1.5

(a) *Let $V_1 \leq x_K$. Then V_t is nondecreasing in $t > 0$.*
(b) *Let $V_1 > x_K$.*
  1. *Let $\beta = 1$ or $V_1 = 0$. Then $V_t = V_1$ for $t > 0$.*
  2. *Let $\beta < 1$ and $V_1 > 0$. Then V_t is nonincreasing in $t > 0$.*
  3. *Let $\beta < 1$ and $V_1 < 0$. Then V_t is nondecreasing in $t > 0$.* □

Proof (a) Let $V_1 \leq x_K$. Then, $K(V_1) \geq 0$ due to Corollary 12.2.2(p.80)(b), hence from (6.5.23(p.31)) with $t = 2$ we have $V_2 \geq K(V_1) + V_1 \geq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.23(p.31)) and Lemma 12.2.3(e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Let $V_1 > x_K$. Then $K(V_1) \leq 0 \cdots$ **(1)** due to Corollary 12.2.2(a). Hence, from (6.5.23) with $t = 2$, hence $V_2 - V_1 = \max\{K(V_1) + V_1, \beta V_1\} - V_1 = \max\{K(V_1), -(1-\beta)V_1\} \cdots$ **(2)**.

(b1) Let $\beta = 1$ or $V_1 = 0$. Then, since $-(1-\beta)V_1 = 0$, we have $V_2 - V_1 = \max\{K(V_1), 0\} = 0$ due to (1), hence $V_2 = V_1$. Suppose $V_{t-1} = V_1$. Then from (6.5.23) we have $V_t = \max\{K(V_1) + V_1, \beta V_1\} = V_2 = V_1$. Hence, by induction we have $V_t = V_1$ for $t > 0$. Below note that $\overline{\beta = 1 \text{ or } V_1 = 0}$, the negation of $\beta = 1$ or $V_1 = 0$, is "$\beta < 1$ and $V_1 \neq 0$", which can be classified into the two cases "$\beta < 1$ and $V_1 > 0$" and "$\beta < 1$ and $V_1 > 0$".

(b2) Let $\beta < 1$ and $V_1 > 0$. Then, since $-(1-\beta)V_1 < 0$, from (2) and (1) we have $V_2 - V_1 \leq 0$, hence $V_2 \leq V_1$. Suppose $V_{t-1} \leq V_{t-2}$. Then, from (6.5.23) and Lemma 12.2.3(p.80)(e) we have $V_t \leq \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \leq V_{t-1}$ for $t > 1$, thus V_t nonincreasing in $t > 0$.

(b3) Let $\beta < 1$ and $V_1 < 0$. Then, since $-(1-\beta)V_1 > 0$, from (2) we have $V_2 - V_1 > 0$ or equivalently $V_2 > V_1$, so that $V_2 \geq V_1$. Suppose $V_{t-1} \geq V_{t-2}$. Then from (6.5.23)and Lemma 12.2.3(p.80)(e) we have $V_t \geq \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \geq V_{t-1}$ for $t > 1$, thus V_t nondecreasing in $t > 0$. ∎

Let us define:

$$
\mathtt{S}_5\boxed{\circledS\blacktriangle \mid \circledast\|} = \left\{ \begin{array}{l} \text{There exists } t^\bullet_\tau > 1 \text{ such that:} \\ (1)\ \ t^\bullet_\tau \geq \tau > 1 \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\geq t>1\blacktriangle} \\ (2)\ \ \tau > t^\bullet_\tau\ \Rightarrow \boxed{\circledast\ \mathtt{ndOIT}_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\|} \text{ where } \mathtt{Conduct}_{t^\bullet_\tau\geq t>1\blacktriangle}. \end{array} \right\}
$$

$$
\mathtt{S}_6\boxed{\circledS\blacktriangle \mid \circledast\| \mid \bullet\| \mid \mathrm{pS}\vartriangle \mid \mathrm{pS}\underline{\underline{\blacktriangle}}} \left\{ \begin{array}{l} \text{There exists } t^{\bullet\,\dagger}_\tau \text{ and } t^\circ_\tau\ (t^\bullet_\tau > t^\circ_\tau > 1) \text{ such that:} \\ (1)\ \ t^\bullet_\tau \geq \tau > 1 \Rightarrow \text{If } \lambda\beta\max\{0, a-\rho\} \leq s, \text{ then } \boxed{\bullet\, \mathtt{dOITd}_{t^\bullet_\tau\geq\tau>1}\langle 0\rangle}_{\|}. \\ \qquad\qquad\qquad\quad \text{If } \lambda\beta\max\{0, a-\rho\} > s, \text{ then } \boxed{\circledast\ \mathtt{ndOIT}_{t^\bullet_\tau\geq\tau>1}\langle 1\rangle}_{\|} \text{ where } \mathtt{Conduct}_{1\blacktriangle}. \\ (2)\ \ \tau > t^\bullet_\tau\ \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\geq t>t^\bullet_\tau\blacktriangle} \text{ and} \\ \qquad\qquad\qquad\qquad\qquad \text{where } \mathtt{pSKIP}_{t^\bullet_\tau\geq\tau>t^\circ_\tau\vartriangle} \text{ and } \mathtt{pSKIP}_{t^\circ_\tau\geq t>1\vartriangle}\ \llparenthesis\mathtt{pSKIP}_{t^\circ_\tau\geq t>1\blacktriangle}\rrparenthesis. \end{array} \right\}
$$

$$
\mathtt{S}_7\boxed{\circledS\blacktriangle \mid \circledast\| \mid \bullet\| \mid \mathrm{pS}\vartriangle} = \left\{ \begin{array}{l} \text{There exists } t^\bullet_\tau > 1 \text{ such that:} \\ (1)\ \ t^\bullet_\tau \geq \tau > 1 \Rightarrow \text{If } \lambda\beta\max\{0, a-\rho\} \leq s, \text{ then } \boxed{\bullet\, \mathtt{dOITd}_{t^\bullet_\tau\geq\tau>1}\langle 0\rangle}_{\|}. \\ \qquad\qquad\qquad\quad \text{If } \lambda\beta\max\{0, a-\rho\} > s, \text{ then } \boxed{\circledast\ \mathtt{ndOIT}_{t^\bullet_\tau\geq\tau>1}\langle 1\rangle}_{\|} \text{ where } \mathtt{Conduct}_{1\blacktriangle}. \\ (2)\ \ \tau > t^\bullet_\tau\ \Rightarrow \boxed{\circledS\ \mathtt{dOITs}_{\tau>t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\geq t>t^\bullet_\tau\blacktriangle} \text{ and where } \mathtt{pSKIP}_{t^\bullet_\tau\geq\tau>1\vartriangle}. \end{array} \right\}
$$

Remark 19.1.2 "$\beta = 1$ or $V_1 = 0$" can be rewritten as $\{\beta = 1 \cup V_1 = 0\}$. Then the negation of $\{\beta = 1 \cup V_1 = 0\}$ (i.e., $\overline{\{\beta = 1 \cup V_1 = 0\}}$) can be written as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap V_1 \neq 0\} = \{\beta < 1 \cap V_1 > 0\} \cup \{\beta < 1 \cap V_1 < 0\},$$

which can be expressed, without loss of generality, as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap s \geq 0 \cap V_1 > 0\} \cup \{\beta < 1 \cap s \geq 0 \cap V_1 < 0\}.$$

For explanatory convenience, let us denote $\{s \geq 0\}$ by $\{s = 0 \llparenthesis s > 0\rrparenthesis\}$. Then the above expression can be rewritten as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap \{s = 0 \llparenthesis s > 0\rrparenthesis\} \cap \{V_1 > 0\}\} \cup \{\beta < 1 \cap \{s = 0 \llparenthesis s > 0\rrparenthesis\} \cap \{V_1 < 0\}\}. \quad \square$$

□ **Tom 19.1.16** ($\mathscr{A}\{\mathsf{M{:}2}[\mathbb{P}][\mathtt{A}]\}$) *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \leq s$, then $\boxed{\bullet\,\mathtt{dOITd}_1\langle 0\rangle}_{\|}$, or else $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{1\blacktriangle}$. Below let $\tau > 1$.*

(b) *Let $V_1 \leq x_K$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
2. *Let $V_1 \geq x_L$. Then, if $\lambda\beta\max\{0, a-\rho\} \leq s$, we have $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.*
   - ii. *Let $\beta < 1$ and $s = 0 \llparenthesis s > 0\rrparenthesis$.*
     1. *Let $b > 0 \llparenthesis \kappa > 0\rrparenthesis$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.*
     2. *Let $b \leq 0 \llparenthesis \kappa \leq 0\rrparenthesis$. Then $\mathtt{S}_5\boxed{\circledS\blacktriangle \mid \circledast\|}$ is true.*

(c) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$.*
   - i. *$V_t = V_1$ for $t > 0$.*
   - ii. *If $\lambda\max\{0, a-\rho\} \leq s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
2. *Let $\beta < 1$ and $s = 0 \llparenthesis s > 0\rrparenthesis$* (see Remark 19.1.2 above)
   - i. *Let $V_1 > 0$.*
     1. *V_t is nonincreasing in $t \geq 0$ and converges to $V \geq x_K$ as $t \to \infty$.*
     2. *Let $b > 0 \llparenthesis \kappa > 0\rrparenthesis$. Then*
        - i. *Let $V_1 > x_L$. Then $\mathtt{S}_6\boxed{\circledS\blacktriangle \mid \circledast\| \mid \bullet\| \mid \mathrm{pS}\vartriangle \mid \mathrm{pS}\blacktriangle}$ is true.*
        - ii. *Let $V_1 = x_L$. Then $\mathtt{S}_7\boxed{\circledS\blacktriangle \mid \circledast\| \mid \bullet\| \mid \mathrm{pS}\vartriangle}$ is true.*
        - iii. *Let $V_1 < x_L$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.*
     3. *Let $b \leq 0 \llparenthesis \kappa \leq 0\rrparenthesis$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
   - ii. *Let $V_1 < 0$.*
     1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
     2. *Let $b > 0 \llparenthesis \kappa > 0\rrparenthesis$.*
        - i. *Let $V_1 \geq x_L$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.*
        - ii. *Let $V_1 < x_L$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.*
     3. *Let $b \leq 0 \llparenthesis \kappa \leq 0\rrparenthesis$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, or else $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ where $\mathtt{Conduct}_{1\blacktriangle}$.* □

Proof Assume $a^\star < \rho < b \cdots$ **(**1**)** and let $\beta < 1$ or $s > 0$.

(a) i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, since $\lambda\beta\max\{0, a-\rho\} - s \le 0$, we have $V_1 - \beta V_0 = 0$ from (19.1.41(p.151)), i.e., $V_1 = \beta V_0 \cdots$ **(**2**)**, hence $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, since $\lambda\beta\max\{0, a-\rho\} - s > 0$, we have $V_1 - \beta V_0 > 0$ from (19.1.41), i.e., $V_1 > \beta V_0 \cdots$ **(**3**)**, hence $t_1^* = 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. Then, since $\lambda\beta\max\{0, a-\rho\} - s > 0$, from the comparison of the two terms within { } in the r.h.s. of (19.1.40) we see that conducting the search is *strictly* optimal at time $t = 1$, i.e., $\texttt{Conduct}_{1\blacktriangle} \cdots$ **(**4**)**.

Below let $\tau > 1$.

(b) Let $V_1 \le x_K \cdots$ **(**5**)**.

(b1) V_t is nondecreasing in $t > 0$ due to Lemma 19.1.5(p.155) (a). Consider a sufficiently large $M > 0$ with $b \le M$ and $V_1 \le M$. Suppose $V_{t-1} \le M$. Then, from (6.5.23) and Lemma 12.2.3(e) we have $V_t \le \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (12.2.13 (2) (p.79)), hence $V_t \le \max\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Accordingly, by induction $V_t \le M$ for $t > 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta M\} \cdots$ **(**6**)** from (6.5.23(p.31)), we have $0 = \max\{K(V), -(1-\beta)V\} \cdots$ **(**7**)**, hence $K(V) \le 0$, so that $V \ge x_K$ due to Lemma 12.2.3(p.80) (j1).

(b2) Let $V_1 \ge x_L$. Then, since $V_{t-1} \ge x_L$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 12.2.1(a), hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (19.1.35(p.151)), i.e., $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots$ **(**8**)**.

i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, from (8) and (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t_\tau^* = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (8) and (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$. In addition, we have $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(b3) Let $V_1 < x_L \cdots$ **(**9**)**.

(b3i) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$, thus $x_L = x_K \cdots$ **(**10**)** from Lemma 12.2.4(p.80) (b). Now, since $V_1 \ge \beta\rho$ from (6.5.22(p.31)), we have $V_1 \ge \rho$ due to the assumption $\beta = 1$, hence $a^\star < V_1$ due to (1). Accordingly, it follows that $a^\star \le V_{t-1}$ for $t > 1$ due to (b1). Note $V_1 < x_K$ from (9) and (10). Suppose $V_{t-1} < x_K$. Then, from Lemma 12.2.3(p.80) (f) and (6.5.23) with $\beta = 1$ we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (10), so that $L(V_{t-1}) > 0$ for $t > 1$ from Lemma 12.2.2(e1). Then, since $L(V_{t-1}) > 0 \cdots$ **(**11**)** for $\tau \ge t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > 1$ from (19.1.35(p.151)), i.e., $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1$. In addition, since $V_1 \ge \beta V_0$ from (19.1.41(p.151)), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \ge \beta^\tau V_0$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. Then, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$ from (11) and (19.1.39).

(b3ii) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(b3ii1) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K > 0 \cdots$ **(**12**)** from Lemma 12.2.4(p.80) (c ⟪d⟫). Here note (9) and (b1). Then suppose there exists a t' such that $V_{t-1} \ge x_L$ for $t \ge t'$. Then $L(V_{t-1}) \le 0$ for $t \ge t'$ from Corollary 12.2.1(p.80) (a), hence $V_t = \beta V_{t-1}$ for $t \ge t'$ due to (19.1.37(p.151)). Hence, we have $V_t = \beta^{t-t'+1}V_{t'-1}$ for $t \ge t'$, leading to $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (12), which contradicts $V \ge x_K$ in (b1). Accordingly, it follows that $V_{t-1} < x_L$ for all $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 12.2.1(a). Thus, for the same reason as in the proof of (b3i) we have $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ and $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.

(b3ii2) Let $b \le 0$ ⟪$\kappa \le 0$⟫.

- Let $b = 0$ ⟪$\kappa = 0$⟫. Then $x_L = x_K = 0 \cdots$ **(**13**)** from Lemma 12.2.4(p.80) (c ⟪d⟫), hence $V \ge x_K = x_L = 0$ from (b1). Here assume $V > x_K = 0$. Then, since $-(1-\beta)V < 0$, we have $K(V) = 0$ from (7), leading to the contradiction $V = x_K$ due to Lemma 12.2.3(j1). Thus we have $V = x_K = 0$. Accordingly, due to (b1) and due to $V_1 < x_L = x_K = V$ from (9) and (13) it follows that there exists a $t_\tau^\bullet > 1$ such that

$$V_1 \le V_2 \le \cdots \le V_{t_\tau^\bullet - 1} < x_K = x_L = V_{t_\tau^\bullet} = V_{t_\tau^\bullet + 1} = \cdots,$$

  where $t_\tau^\bullet$ might be infinity (i.e., $t_\tau^\bullet = \infty$). Hence $V_{t-1} < x_L$ for $t_\tau^\bullet \ge t > 1$ and $V_{t-1} = x_L$ for $t > t_\tau^\bullet$. Thus, from Corollary 12.2.1(a) we have

$$L(V_{t-1}) > 0 \text{ for } t_\tau^\bullet \ge t > 1 \text{ and } L(V_{t-1}) = 0 \text{ for } t > t_\tau^\bullet \cdots \textbf{(}14\textbf{)}.$$

- Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K$ from Lemma 12.2.4(c ⟪d⟫). Since $V_1 < x_L$ from (9) and since $x_L < x_K \le V$ from (b1), there exists $t_\tau^\bullet$ such that

$$V_1 \le V_2 \le \cdots \le V_{t_\tau^\bullet - 1} < x_L \le V_{t_\tau^\bullet} \le V_{t_\tau^\bullet + 1} \le \cdots,$$

  hence $V_{t-1} < x_L$ for $t_\tau^\bullet \ge t > 1$ and $x_L \le V_{t-1}$ for $t > t_\tau^\bullet$. Accordingly, from Corollary 12.2.1(a) we have

$$L(V_{t-1}) > 0 \text{ for } t_\tau^\bullet \ge t > 1 \text{ and } L(V_{t-1}) \le 0 \text{ for } t > t_\tau^\bullet \cdots \textbf{(}15\textbf{)}.$$

From (14) and (15) we have, whether $b=0\,(\kappa=0)$ or $b<0\,(\kappa<0)$,

$$L(V_{t-1})>0\cdots(16)\ \text{for}\ t^{\bullet}_{\tau}\geq t>1,$$
$$L(V_{t-1})\leq 0\cdots(17)\ \text{for}\ t>t^{\bullet}_{\tau}.$$

Accordingly, from (19.1.35(p.151)) we have $V_t-\beta V_{t-1}>0$ for $t^{\bullet}_{\tau}\geq t>1$ due to (16) and $V_t-\beta V_{t-1}=0$ for $t>t^{\bullet}_{\tau}$ due to (17) or equivalently

$$V_t>\beta V_{t-1}\cdots(18),\quad t^{\bullet}_{\tau}\geq t>1,\qquad V_t=\beta V_{t-1}\cdots(19),\quad t>t^{\bullet}_{\tau}.$$

1. Let $t^{\bullet}_{\tau}\geq\tau>1$. Then, since $V_t>\beta V_{t-1}\cdots(20)$ for $\tau\geq t>1$ due to (18), for the same reason as in the proof of (b3i) we have $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$. Hence (1) of $\texttt{S}_5$ holds. From $V_t>\beta V_{t-1}$ for $t^{\bullet}_{\tau}\geq t>1$ due to (20) with $\tau=t^{\bullet}_{\tau}$ we have $V_{t^{\bullet}_{\tau}}>\beta V_{t^{\bullet}_{\tau}-1}>\cdots>\beta^{t^{\bullet}_{\tau}-1}V_1\cdots(21)$.
2. Let $\tau>t^{\bullet}_{\tau}$. Then $V_t=\beta V_{t-1}$ for $\tau\geq t>t^{\bullet}_{\tau}$ due to (19), hence $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-t^{\bullet}_{\tau}}V_{t^{\bullet}_{\tau}}\cdots(22)$. Hence, due to (21) and the fact that $V_1\geq\beta V_0$ from (2) and (3) we obtain

$$V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-t^{\bullet}_{\tau}}V_{t^{\bullet}_{\tau}}>\beta^{\tau-t^{\bullet}_{\tau}+1}V_{t^{\bullet}_{\tau}-1}>\cdots>\beta^{\tau-1}V_1\geq\beta^{\tau}V_0,$$

so that we have $t^*_\tau=t^{\bullet}_{\tau}$ for $\tau>t^{\bullet}_{\tau}$, i.e., $\boxed{⊛\ \texttt{ndOIT}_{\tau>t^{\bullet}_{\tau}}\langle t^{\bullet}_{\tau}\rangle}_{\parallel}$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $t^{\bullet}_{\tau}\geq t>1$ due to (16) and (19.1.39(p.151)). From the above we see that (2) of $\texttt{S}_5$ holds.

(c) Let $V_1>x_K\cdots(23)$..

(c1) Let $\beta=1$ or $V_1=0$.

(c1i) The same as Lemma 19.1.5(p.155) (b1).

(c1ii) Since $V_\tau=V_{\tau-1}=\cdots=V_1$ for $\tau>0$ from (c1i), we have $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1\cdots(24)$.

i. Let $\lambda\max\{0,a-\rho\}\leq s$. Then, from (24) and (2) we have $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1=\beta^{\tau}V_0$, hence $t^*_\tau=0$ for $\tau>1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\max\{0,a-\rho\}>s$. Then, from (24) and (3) we have $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1>\beta^{\tau}V_0$, hence $t^*_\tau=1$ for $\tau>1$, i.e., $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ where $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2) Let $\beta<1\cdots(25)$ and $s=0\,(s>0)$.

(c2i) Let $V_1>0$.

(c2i1) The former half is the same as Lemma 19.1.5(p.155) (b2). The latter half can be proven as follows. Note (23), hence $V_1\geq x_K$. Suppose $V_{t-1}\geq x_K$. Then from (6.5.23(p.31)) we have $V_t\geq K(V_{t-1})+V_{t-1}\geq K(x_K)+x_K$ due to Lemma 12.2.3(e), hence $V_t\geq x_K$ since $K(x_K)=0$. Accordingly, by induction $V_t\geq x_K$ for $t>0$, i.e., V_t is lower bounded in t. Hence V_t converges to a finite V as $t\to\infty$. Then, since $V=\max\{K(V)+V,\beta V\}$ from (6.5.23(p.31)), we have $0=\max\{K(V),-(1-\beta)V\}$, hence $K(V)\leq 0$, so that $V\geq x_K$ due to Lemma 12.2.3(p.80) (j1).

(c2i2) Let $b>0\,(\kappa>0)$. Then $x_L>x_K>0\cdots(26)$ from Lemma 12.2.4(c (d)).

(c2i2i) Let $V_1>x_L\cdots(27)$, hence $V_1\geq x_L$. Suppose $V_{t-1}\geq x_L$ for *all* $t>1$. Then, since $L(V_{t-1})\leq 0$ for $t>1$ from Corollary 12.2.1(p.80) (a), we have $V_t-\beta V_{t-1}=0$ for $t>1$ from (19.1.35), i.e., $V_t=\beta V_{t-1}$ for all $t>1$, hence $V_t=\beta^{t-1}V_1$, hence $V=\lim_{t\to\infty}V_t=0<x_K$ due to (25) and (26), which contradicts $V\geq x_K$ in (c2i1). Hence, it is impossible that $x_L\leq V_{t-1}$ for *all* $t>0$. Accordingly, due to (27) and (c2i1) it follows that there exist $t^{\bullet}_{\tau}$ and t°_{τ} ($t^{\bullet}_{\tau}>t^{\circ}_{\tau}>0$) such that

$$V_1\geq V_2\geq\cdots\geq V_{t^{\circ}_{\tau}-1}>x_L=V_{t^{\circ}_{\tau}}=V_{t^{\circ}_{\tau}+1}=\cdots=V_{t^{\bullet}_{\tau}-1}>V_{t^{\bullet}_{\tau}}\geq V_{t^{\bullet}_{\tau}+1}\geq\cdots.$$

Hence, we have

$$x_L>V_{t^{\bullet}_{\tau}},\ x_L>V_{t^{\bullet}_{\tau}+1},\cdots,$$
$$V_{t^{\circ}_{\tau}}=x_L,\ V_{t^{\circ}_{\tau}+1}=x_L,\cdots,V_{t^{\bullet}_{\tau}-1}=x_L,$$
$$V_1>x_L,\ V_2>x_L,\ \cdots,\ V_{t^{\circ}_{\tau}-1}>x_L,$$

or equivalently

$$x_L>V_{t-1}\cdots(28),\qquad t>t^{\bullet}_{\tau},$$
$$V_{t-1}=x_L\cdots(29),\qquad t^{\bullet}_{\tau}\geq t>t^{\circ}_{\tau},$$
$$V_{t-1}>x_L\cdots(30),\qquad t^{\circ}_{\tau}\geq t>1.$$

Accordingly, we have:

1. Let $t^{\bullet}_{\tau}\geq\tau>1$. Then, since $V_{t-1}\geq x_L$ for $\tau\geq t>1$ from (29) and (30), we have $L(V_{t-1})\leq 0\cdots(31)$ for $\tau\geq t>1$ from Corollary 12.2.1(a), hence $V_t-\beta V_{t-1}=0$ for $\tau\geq t>1$ from (19.1.35), i.e., $V_t=\beta V_{t-1}$ for $\tau\geq t>1$, leading to $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1\cdots(32)$.
   i. Let $\lambda\max\{0,a-\rho\}\leq s$. Then, from (32) and (2) we have $V_\tau=\beta V_{\tau-1}=\cdots=\beta^{\tau-1}V_1=\beta^{\tau}V_0$, hence $t^*_\tau=0$ for $t^{\bullet}_{\tau}\geq\tau>1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t^{\bullet}_{\tau}\geq\tau>1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda \max\{0, a-\rho\} > s$. Then, from (32) and (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_{\parallel}$ where $\texttt{Conduct}_{1\blacktriangle}$ from (4).

Accordingly $\texttt{S}_6(1)$ holds. From (32) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1}V_1 \cdots (33)$.

2. Let $\tau > t^\bullet_\tau$. Then, since $x_L > V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (28), due to Corollary 12.2.1(a) we have $L(V_{t-1}) > 0 \cdots (34)$ for $\tau \geq t > t^\bullet_\tau$. Accordingly, from (19.1.35) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau}$. From this and (33) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1. \cdots (35).$$

Since $V_1 \geq \beta V_0$ due to (2) and (3), from (35) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1 \geq \beta^\tau V_0.$$

Hence, we have $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\blacktriangle}$, thus the former half of $\texttt{S}_6(2)$ holds.

(i) If $\tau \geq t > t^\bullet_\tau$, then $\texttt{Conduct}_{t\blacktriangle}$ from (34) and (19.1.39).

The latter half is shown as follows. First, note here (27). Then we have:

(ii) If $t^\bullet_\tau \geq t > t^\circ_\tau$, then $V_{t-1} = x_L$ from (29), hence $L(V_{t-1}) = L(x_L) = 0$, hence $\texttt{Skip}_{t\vartriangle}$ from (19.1.38), implying that we have $\texttt{pSKIP}_{t^\bullet_\tau \geq t > t^\circ_\tau\vartriangle}$ (see Figure 7.2.1(p.34) (II).

(iii) If $t^\circ_\tau \geq t > 1$, then $V_{t-1} > x_L$ from (30), hence $L(V_{t-1}) = (\!(<)\!)\ 0$[‡] from Lemma 12.2.2(p.80) (d (\!(e1)\!)); i.e., $\texttt{Skip}_{t\vartriangle}$ (\!($\texttt{Skip}_{t\blacktriangle}$)\!) due to (19.1.38) (\!((19.1.39))\!), implying that we have $\texttt{pSKIP}_{t^\circ_\tau \geq t > 1\vartriangle}$ (\!($\texttt{pSKIP}_{t^\circ_\tau \geq t > 1\blacktriangle}$)\!).

From the above results we see that the latter half of $\texttt{S}_6(2)$ holds.

(c2i2ii) Let $V_1 = x_L$. Suppose $V_{t-1} = x_L$ for *all* $t > 1$. Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t - \beta V_{t-1} = 0$ for *all* $t > 1$ from (19.1.35(p.151)), i.e., $V_t = \beta V_{t-1}$ for all $t > 1$, hence $V_t = \beta^{t-1}V_1$. Then $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (26), which contradicts $V \geq x_K$ in (c2i1). Hence, since V_{t-1} is not equal to x_L for *all* $t > 1$, due to (c2i1) it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau - 1} = x_L > V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

or equivalently $V_{t-1} = x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Thus, due to Corollary 12.2.1(p.80) (a) we have

$$L(V_{t-1}) = L(x_L) = 0 \cdots (36), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) > 0 \cdots (37), \quad t > t^\bullet_\tau.$$

Accordingly, we have:

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, from (36) and (19.1.35) we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, from which we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$.

   i. Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_{\parallel}$.

   ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_{\parallel}$. In addition, we have $\texttt{Conduct}_{1\blacktriangle}$ from (4).

   Accordingly, it follows that $\texttt{S}_7(1)$ holds.

2. Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) > 0 \cdots (38)$ for $\tau \geq t > t^\bullet_\tau$ from (37), hence due to (19.1.35) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\circ} \cdots (39)$. In addition, since $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \geq t > 1$ from (36) and (19.1.35), we have $V_t = \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, leading to $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1}V_1 \cdots (40)$. From (39) and (40) we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1$. In addition, since $V_1 \geq \beta^\tau V_0$ from (2) and (3), we eventually obtain $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1 \geq \beta^\tau V_0 \cdots (41)$. Thus $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\blacktriangle}$, hence the former half of $\texttt{S}_7(2)$ holds. Then, we have that $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t^\bullet_\tau$ due to (38) and (19.1.39(p.151)). Moreover, we have $\texttt{Skip}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (36) and (19.1.38), so that it follows that we have $\texttt{pSKIP}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ (see Figure 7.2.1(p.34))(II) or equivalently $\texttt{pSKIP}_{t^\bullet_\tau \geq t > 1\vartriangle}$. Hence the latter half of $\texttt{S}_7(2)$ holds.

(c2i2iii) Let $V_1 < x_L$. Then $V_{t-1} < x_L$ for $t > 1$ due to (c2i1), hence $L(V_{t-1}) > 0 \cdots (42)$ for $t > 1$ from Corollary 12.2.1(p.80) (a). Accordingly, since $L(V_{t-1}) > 0 \cdots (43)$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (19.1.35) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1$. Since $V_1 \geq \beta V_0$ from (2) and (3), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \geq \beta^\tau V_0$, hence we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > 1}\langle \tau\rangle}_{\blacktriangle}$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (43) and (19.1.39).

(c2i3) Let $b \leq 0$ (\!($\kappa \leq 0$)\!), hence $x_L \leq x_K \cdots (44)$ from Lemma 12.2.4(p.80) (c (\!(d)\!)). Then, from (23) and (c2i1) we have $V_{t-1} \geq x_K$ for all $t > 1$, hence $V_{t-1} \geq x_L$ for all $t > 1$ due to (44), thus $L(V_{t-1}) \leq 0$ for all $t > 1$ from Corollary 12.2.1(p.80) (a). Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (19.1.35) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$.

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‡If $s = 0$, then "$= 0$", or else "< 0".

i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$. Then $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii) Let $V_1 < 0$.

(c2ii1) The same as the proof of (c2i1).

(c2ii2) Let $b > 0$ ($\kappa > 0$), hence $x_L > x_K > 0 \cdots$ (45) from Lemma 12.2.4(p.80) (c (d)).

(c2ii2i) Let $V_1 \ge x_L$. Then, since $V_{t-1} \ge x_L$ for $t > 1$ due to (c2ii1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 12.2.1(p.80) (a), hence $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$. Thus $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 1$ from (19.1.35), i.e., $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, so $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$.

i. Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\texttt{dOIT}_{\tau>1}\langle 0\rangle_{\parallel}$.

ii. Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$. Then $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii2ii) Let $V_1 < x_L$. Suppose that there exists $t' > 1$ such that $x_L \le V_{t-1}$ for $t > t'$. Then, since $L(V_{t-1}) \le 0$ for $t > t'$ from Corollary 12.2.1(p.80) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > t'$ due to (19.1.35), hence $V_t = \beta V_{t-1}$ for $t > t'$, so $V_t = \beta V_{t-1} = \beta^2 V_{t-2} = \cdots = \beta^{t-t'}V_{t'}$. Accordingly $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (45), which contradicts $V \ge x_K$ in (c2ii1), hence it must be that $V_{t-1} < x_L$ for $t > 1$. Then, since $V_{t-1} < x_L$ for $\tau \ge t > 1$, we have $L(V_{t-1}) > 0 \cdots$ (46) for $\tau \ge t > 1$ from Corollary 12.2.1(p.80) (a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > 1$ from (19.1.35) or equivalently $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, thus $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1$. Since $V_1 \ge \beta V_0$ from (2) and (3), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1 \ge \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$. From (46) and (19.1.39) we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii3) Let $b \le 0$ ($\kappa \le 0$), hence $x_L \le x_K \cdots$ (47) from Lemma 12.2.4(p.80) (c (d)). Then, due to (23) and (c2ii1) we have $V_{t-1} > x_K$ for $t > 1$, hence $V_{t-1} > x_L$ for $t > 1$ from (47), thus $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 12.2.1(p.80) (a). Accordingly, the assertion is true for the same reason as in the proof of (c2ii2i). ∎

Corollary 19.1.9 *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $V_1 \le x_K$. Then z_t is nondecreasing in $t > 0$.*
(b) *Let $V_1 > x_K$.*
  1. *Let $\beta = 1$ or $V_1 = 0$. Then $z_t = z(V_1)$ for $t > 0$.*
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
     i. *Let $V_1 > 0$. Then z_t is nonincreasing in $t > 0$.*
     ii. *Let $V_1 < 0$. Then z_t is nondecreasing in $t > 0$.* □

Proof Immediate from Tom 19.1.16(b1,c1i,c2i1,c2ii1) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.1.6.3 Market Restriction

19.1.6.3.1 Positive Restriction

19.1.6.3.1.1 Case of $\beta = 1$ and $s = 0$

□ Pom **19.1.9** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \ge 0$.*
(b) *Let $\rho \le a^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ → → ⓢ
(c) *Let $b \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → → ⓓ
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ *and* $\texttt{pSKIP}_1$ → → ⓓ/ⓢ
  2. *Let $\rho < a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ → → ⓢ

Proof The same as Tom 19.1.9(p.153) due to Lemma 16.4.1(p.100). ∎

19.1.6.3.1.2 Case of $\beta < 1$ or $s > 0$

19.1.6.3.1.2.1 Case of $\rho \le a^\star$

□ Pom **19.1.10** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*
(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → → ⓓ
(c) *Let $\rho < x_L$.*
  1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$* → → ⓢ
  2. *Let $\beta = 1$.*
     i. *Let $(\lambda a - s)/\lambda \le a^\star$.*

1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⊛
2. *Let* $\lambda < 1$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

ii. *Let* $(\lambda a - s)/\lambda > a^\star$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *and* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $(\lambda\beta a - s)/\delta \leq a^\star$.
   1. *Let* $\lambda = 1$.
      i. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
      ii. *Let* $s \geq \lambda\beta T(0)$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⊛
   2. *Let* $\lambda < 1$.
      i. *Let* $s \leq \lambda\beta T(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
      ii. *Let* $s > \lambda\beta T(0)$. *Then* $\mathtt{S}_3$(p.141) [Ⓢ▲ | ⊛‖] *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ⊛
   ii. *Let* $(\lambda\beta a - s)/\delta > a^\star$.
   1. *Let* $s \geq \lambda\beta T(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
   2. *Let* $s < \lambda\beta T(0)$. *Then* $\mathtt{S}_3$(p.141) [Ⓢ▲ | ⊛‖] *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a-c2ii) The same as `Tom` 19.1.10(p.153) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of `Tom` 19.1.10.

(c4-c4ii2) The same as `Tom` 19.1.10(c3-c3ii2) with κ. ▮

□ `Pom` **19.1.11** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose* $a > 0$. *Assume* $\rho \leq a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ $\rightarrow$ ❶
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
(d) *Let* $\beta < 1$ *and* $s > 0$.
1. *Let* $s < \beta\mu T(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
2. *Let* $s \geq \beta\mu T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau}\langle 0\rangle}_{\|}$ $\rightarrow$ $\rightarrow$ ❶

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a,b) The same as `Tom` 19.1.11(p.154) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of `Tom` 19.1.11.

(d-d2) The same as `Tom` 19.1.11(c1,c2) with κ. ▮

□ `Pom` **19.1.12** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose* $a > 0$. *Assume* $\rho \leq a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.
1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_L \leq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ $\rightarrow$ ❶
3. *Let* $x_L > \rho$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$.
1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \rightarrow \infty$.
2. *Let* $\rho < x_L$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
3. *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\|}$ *and* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\vartriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ ❶ / Ⓢ
4. *Let* $x_L < \rho$. *Then* $\mathtt{S}_4$ [Ⓢ▲ | ●‖ | pS△ | pS▲] *is true* $\rightarrow$ $\rightarrow$ Ⓢ / ❶ / pS

(c) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s > 0$.
1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \rightarrow \infty$.
2. *Let* $s \geq \beta\mu T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ $\rightarrow$ ❶
3. *Let* $s < \beta\mu T(0)$.
   i. *Let* $\rho < x_L$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ
   ii. *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\|}$ *and* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\vartriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ ❶ / Ⓢ
   iii. *Let* $x_L < \rho$. *Then* $\mathtt{S}_4$ [Ⓢ▲ | ●‖ | pS△ | pS▲] *is true* $\rightarrow$ Ⓢ / ❶ / pS

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.
1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \rightarrow \infty$.
2. $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.
1. V_t *is nondecreasing in* t $(\tau \geq t \geq 0)$ *and converges to a finite* V *as* $t \rightarrow \infty$.
2. *Let* $s \geq \beta\mu T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ $\rightarrow$ $\rightarrow$ ❶
3. *Let* $s < \beta\mu T(0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a-a3) The same as Tom 19.1.12(p.154) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of Tom 19.1.12.

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as Tom 19.1.12(b1-b3iii) with κ.

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of Tom 19.1.12.

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as Tom 19.1.12(c1-c3) with κ. ∎

19.1.6.3.1.2.2 Case of $b \le \rho$

☐ **Pom 19.1.13** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ

(c) *Let $\rho < x_L$.*

1. [ⓢ dOITs$_1\langle 1\rangle$]$_\blacktriangle$ *where* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$* → → ⓢ
2. *Let $\beta = 1$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
3. *Let $\beta < 1$ and $s = 0$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s \le \lambda\beta T(0)$. Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
   ii. *Let $s > \lambda\beta T(0)$. Then* S$_3$(p.141) [ⓢ▲ | ⊛‖] *is true* → → ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a-c2) The same as Tom 19.1.13(p.154) (a-c2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i) of Tom 19.1.13.

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then, we have the same as Tom 19.1.13(c3i,c3ii) with κ. ∎

☐ **Pom 19.1.14** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ

(c) *Let $\beta < 1$ and $s = 0$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $s < \lambda\beta T(0)$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
2. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a,b) The same as Tom 19.1.14(p.155) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 19.1.14.

(d-d2) Let $\beta < 1$ and $s > 0$. Then, we have the same as Tom 19.1.14(c1,c2) with κ. ∎

☐ **Pom 19.1.15** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_L \le \rho$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ
3. *Let $x_L > \rho$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $\rho < x_L$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
3. *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_{\|}$ *and* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓓ / ⓢ
4. *Let $x_L < \rho$. Then* S$_4$ [ⓢ▲ | ●‖ | pS△ | pS▲] *is true* → → ⓢ / ⓓ / pS

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s > 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ
3. *Let $s < \lambda\beta T(0)$.*
   i. *Let $\rho < x_L$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ
   ii. *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_{\|}$ *and* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → ⓓ / ⓢ
   iii. *Let $x_L < \rho$. Then* S$_4$ [ⓢ▲ | ●‖ | pS△ | pS▲] *is true* → ⓢ / ⓓ / pS

(d) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ

(e) *Let $\beta < 1$ and $\rho < 0$ and let $s > 0$.*

1. *V_t is nondecreasing in t $(\tau \geq t \geq 0)$.*
2. *Let $s \geq \lambda\beta T(0)$. Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\qquad\rightarrow$ ⓓ
3. *Let $s < \lambda\beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a-a3) The same as `Tom` 19.1.15(p.155) (a-a3).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of `Tom` 19.1.15.

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as `Tom` 19.1.15(b1-b3iii) with κ.

(d,d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of `Tom` 19.1.15.

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as `Tom` 19.1.15(c1-c3) with κ. ∎

19.1.6.3.1.2.3 Case of $a^\star < \rho < b$

□ `Pom` **19.1.16** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $a^\star \leq \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$*. Below let $\tau > 1$.* $\qquad\rightarrow$ ⓓ / ⓢ

(b) *Let $V_1 \leq x_K$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*
2. *Let $V_1 \geq x_L$. Then, if $\lambda\beta\max\{0, a-\rho\} \leq s$, we have* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
   - ii. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
   - iii. *Let $\beta < 1$ and $s > 0$.*
     1. *Let $s < \lambda\beta T(0)$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
     2. *Let $s \geq \lambda\beta T(0)$. Then* $\texttt{S}_5$ [ⓢ▲ | ⊛‖] *is true* $\rightarrow$ $\qquad\rightarrow$ ⓢ / ⊛

(c) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$.*
   - i. *$V_t = V_1$ for $t > 0$.*
   - ii. *If $\lambda\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓓ / ⊛
2. *Let $\beta < 1$ and $s = 0$.*
   - i. *Let $V_1 > 0$.*
     1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*
     2. *Let $V_1 > x_L$. Then* $\texttt{S}_6$ [ⓢ▲ | ⊛‖ | ●‖ | pSΔ | pS▲] *is true* $\rightarrow$ $\qquad\rightarrow$ ⓢ / ⓓ / ⊛ / pS
     3. *Let $V_1 = x_L$. Then* $\texttt{S}_7$ [ⓢ▲ | ⊛‖ | ●‖ | pSΔ] *is true* $\rightarrow$ $\qquad\rightarrow$ ⓢ / ⓓ / ⊛ / pS
     4. *Let $V_1 < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
   - ii. *Let $V_1 < 0$.*
     1. *Then V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*
     2. *Let $V_1 \geq x_L$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. $\qquad\rightarrow$ ⓓ / ⊛
     3. *Let $V_1 < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $V_1 > 0$.*
     1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*
     2. *Let $s < \lambda\beta T(0)$.*
        - i. *Let $V_1 > x_L$. Then* $\texttt{S}_6$ [ⓢ▲ | ⊛‖ | ●‖ | pSΔ | pS▲] *is true* $\rightarrow$ $\qquad\rightarrow$ ⓢ / ⓓ / ⊛ / pS
        - ii. *Let $V_1 = x_L$. Then* $\texttt{S}_7$ [ⓢ▲ | ⊛‖ | ●‖ | pSΔ] *is true* $\rightarrow$ $\qquad\rightarrow$ ⓢ / ⓓ / ⊛ / pS
        - iii. *Let $V_1 < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
     3. *Let $s \geq \lambda\beta T(0)$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\qquad\rightarrow$ ⓓ / ⊛
   - ii. *Let $V_1 < 0$.*
     1. *Then V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*
     2. *Let $s < \lambda\beta T(0)$.*
        - i. *Let $V_1 \geq x_L$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. $\qquad\rightarrow$ ⓓ / ⊛
        - ii. *Let $V_1 < x_L$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ $\rightarrow$ $\qquad\rightarrow$ ⓢ
     3. *Let $s \geq \lambda\beta T(0)$. If $\lambda\beta\max\{0, a-\rho\} \leq s$, then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\qquad\rightarrow$ ⓓ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.18)).

(a-b3i) The same as Tom 19.1.16(p.157) (a-b3i).

(b3ii) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (b3ii1) of Tom 19.1.16.

(b3iii-b3iii2) Let $\beta < 1$ and $s > 0$. Then, the two assertions are immediate from Tom 19.1.16(b3ii1,b3ii2) with κ.

(c-c1ii) The same as Tom 19.1.16(c-c1ii).

(c2-c2i4) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i-c2i1,c2i2i-c2i2iii) of Tom 19.1.16.

(c2ii-c2ii3) Due to (1) it suffices to consider only (c2ii,c2ii1,c2ii2i,c2ii2ii) of Tom 19.1.16.

(c3-c3i3) Let $\beta < 1$ and $s > 0$. Then, we have the same as Tom 19.1.16(c2-c2i1,c2i2i-c2i2iii) with κ.

(c3ii-c3ii3) We have the same as Tom 19.1.16(c2ii-c2ii2ii) with κ. ∎

19.1.6.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.1.6.3.3 Negative Restriction

19.1.6.3.3.1 Case of $\beta = 1$ and $s = 0$

□ **Nem 19.1.5** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \leq a^\star$. Then* [ⓢ $\mathtt{dOITs}_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
(c) *Let $b \leq \rho$. Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\parallel}$ → → ⓓ
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \leq \rho$. Then* [● $\mathtt{dOITd}_1\langle 0\rangle$]$_{\parallel}$ *and* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ *and* $\mathtt{pSKIP}_1$ → → ⓓ/ⓢ
  2. *Let $\rho < a$. Then* [ⓢ $\mathtt{dOITs}_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ

Proof The same as Tom 19.1.9(p.153) due to Lemma 16.4.1(p.100). ∎

19.1.6.3.3.2 Case of $\beta < 1$ or $s > 0$

19.1.6.3.3.2.1 Case of $\rho \leq a^\star$

□ **Nem 19.1.6** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose $b < 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $x_L \leq \rho$. Then* [● $\mathtt{dOITd}_{\tau>0}\langle 0\rangle$]$_{\parallel}$ → → ⓓ
(c) *Let $\rho < x_L$.*
  1. [ⓢ $\mathtt{dOITs}_1\langle 1\rangle$]$_\blacktriangle$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$* → → ⓢ
  2. *Let $\beta = 1$.*
     i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
        1. *Let $\lambda = 1$. Then* [⊛ $\mathtt{ndOIT}_{\tau>1}\langle 1\rangle$]$_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
        2. *Let $\lambda < 1$. Then* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ
     ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* [ⓢ $\mathtt{dOITs}_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$ *and* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ
  3. *Let $\beta < 1$ and $s = 0$. Then we have* $\mathtt{S}_3$(p.141) [ⓢ▲ | ⊛‖].
  4. *Let $\beta < 1$ and $s > 0$.*
     i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
        1. *Let $\lambda = 1$. Then* [⊛ $\mathtt{ndOIT}_{\tau>1}\langle 1\rangle$]$_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
        2. *Let $\lambda < 1$. Then* $\mathtt{S}_3$(p.141) [ⓢ▲ | ⊛‖] *is true* → → ⓢ/⊛
     ii. *Let $(\lambda\beta a - s)/\delta > a^\star$. Then* $\mathtt{S}_3$(p.141) [ⓢ▲ | ⊛‖] *is true* → → ⓢ/⊛

Proof Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$ and $\kappa = -s \cdots (3)$ from Lemma 12.2.6(p.81) (a). Then $a^\star < 0 \cdots (4)$ due to Lemma 12.2.1(p.77) (n) and (2).

(a,c2ii) The same as Tom 19.1.10(p.153) (a,c2ii) due to Lemma 16.4.1(p.100).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta a - s)/\delta \leq a^\star$. Then, since $\lambda\beta a/\delta \leq a^\star$, we have $\lambda\beta a \leq \delta a^\star$ due to (9.2.2 (1) (p.42)), hence $\lambda\beta a \leq \delta a^\star \leq \lambda a^\star$ due to (9.2.2 (1) (p.42)) and (4), so that $\beta a \leq a^\star$, which contradicts [19(p.101)]. Thus, it must be that $(\lambda\beta a - s)/\delta > a^\star$. From this it suffices to consider only (c3ii2) of Tom 19.1.10(p.153).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\kappa < 0$ due to (3), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 19.1.10 with κ. ∎

□ Nem **19.1.7** ($\mathscr{A}$\{M:2[ℙ][A]$^-$\}) *Suppose $b < 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *We have* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a) The same as Tom 19.1.11(p.154) (a).

(b) Let $\beta = 1$. Then, the assertion is the same as Tom 19.1.11(b). Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.11 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.11; accordingly, whether $s = 0$ or $s > 0$, we have the same result. Thus, whether $\beta = 1$ or $\beta < 1$, it eventually follows that we have the same result. ∎

□ Nem **19.1.8** ($\mathscr{A}$\{M:2[ℙ][A]$^-$\}) *Suppose $b < 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_L \le \rho$. Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
3. *Let $x_L > \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ → ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *We have* $\boxed{\bullet\,\texttt{dOITd}_1\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ

(c) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *We have* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-a3) The same as Tom 19.1.12(p.154) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as Tom 19.1.12(b1).

(b2) If $s = 0$, it suffices to consider only (b2) of Tom 19.1.12 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (b2) of Tom 19.1.12. Accordingly, whether $s = 0$ or $s > 0$, it eventually follows that we have the same results.

(c) Let $\beta < 1$ and $\rho < 0$.

(c1) The same as Tom 19.1.12(c1).

(c2) If $s = 0$, it suffices to consider only (c2) of Tom 19.1.12 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.12. Accordingly, whether $s = 0$ or $s > 0$, it eventually follows that we have the same results. ∎

19.1.6.3.3.2.2 Case of $b \le \rho$

□ Nem **19.1.9** ($\mathscr{A}$\{M:2[ℙ][A]$^-$\}) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ

(c) *Let $\rho < x_L$.*

1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.* → ⓢ
2. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ → ⓢ
3. *Let $\beta < 1$. Then* $\texttt{S}_3$(p.141) $\boxed{ⓢ\blacktriangle}\boxed{⊛\parallel}$ *is true* $\rightarrow$ → ⓢ / ⊛

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-c2) The same as Tom 19.1.13(p.154) (a-c2).

(c3) Let $\beta < 1$. If $s = 0$, it suffices to consider only (c3ii) of Tom 19.1.13 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 19.1.13. Accordingly, whether $s = 0$ or $s > 0$, it eventually follows that we have the same results. ∎

□ Nem **19.1.10** ($\mathscr{A}$\{M:2[ℙ][A]$^-$\}) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *We have* $\boxed{\bullet\,\texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a) The same as Tom 19.1.14(p.155) (a).

(b) First, let $\beta = 1$. Then, the assertion is the same as Tom 19.1.14(b). Next, let $\beta < 1$. If $s = 0$, then it suffices to consider only (c2) of Tom 19.1.14 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.14. Thus, whether $s = 0$ or $s > 0$, we have the same results. Accordingly, whether $\beta = 1$ or $\beta < 1$, it eventually follows that we have the same result. ∎

☐ Nem **19.1.11** ($\mathscr{A}${M:2[ℙ][A]$^-$}) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
1. $V_t = \rho$ *for* $t \ge 0$.
2. *Let* $x_L \le \rho$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ
3. *Let* $x_L > \rho$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$ → → ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$.*
1. V_t *is nonincreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *We have* [● dOITd$_1\langle 0\rangle$]$_{\|}$ → → ⓓ

(c) *Let $\beta < 1$ and $\rho < 0$.*
1. V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *We have Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-a3) The same as Tom 19.1.15(p.155) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as Tom 19.1.15(b1).

(b2) If $s = 0$, then it suffices to consider only (b2) of Tom 19.1.15 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (b2) of Tom 19.1.15. Thus, whether $s = 0$ or $s > 0$, it eventually follows that we have the same result.

(c) Let $\beta < 1$ and $\rho < 0$.

(c1) The same as Tom 19.1.15(c1).

(c2) If $s = 0$, then it suffices to consider only (c2) of Tom 19.1.15 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.15. Thus, whether $s = 0$ or $s > 0$, it eventually follows that we have the same result. ∎

19.1.6.3.3.2.3 Case of $a^\star < \rho < b$

☐ Nem **19.1.12** ($\mathscr{A}${M:2[ℙ][A]$^-$}) *Suppose $b < 0$. Assume $a^\star \le \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* [● dOITd$_1\langle 0\rangle$]$_{\|}$, *or else* [ⓢ dOITs$_1\langle 1\rangle$]$_{\blacktriangle}$ *where* Conduct$_{1\blacktriangle}$. *Below let* $\tau > 1$. → ⓓ/ⓢ

(b) *Let* $V_1 \le x_K$.
1. V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $V_1 \ge x_L$. *Then, if* $\lambda\beta\max\{0, a-\rho\} \le s$, *we have* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\|}$, *or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|}$ *where* Conduct$_{1\blacktriangle}$ → ⓓ/⊛
3. *Let* $V_1 < x_L$.
   i. *Let* $\beta = 1$. *Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$ *where* Conduct$_{\tau\ge t>1\blacktriangle}$ → → ⓢ
   ii. *Let* $\beta < 1$. *Then* S$_5$ [ⓢ$^\blacktriangle$|⊛$^{\|}$] *is true* → → ⓢ/⊛

(c) *Let* $V_1 > x_K$.
1. *Let* $\beta = 1$ *or* $V_1 = 0$. *Then:*
   i. $V_t = V_1$ *for* $t > 0$.
   ii. *If* $\lambda\max\{0, a-\rho\} \le s$, *then* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\|}$, *or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|}$ *where* Conduct$_{1\blacktriangle}$ → ⓓ/⊛
2. *Let* $\beta < 1$.
   i. *Let* $V_1 > 0$.
      1. V_t *is nonincreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
      2. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\|}$, *or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|}$ *where* Conduct$_{1\blacktriangle}$ → ⓓ/⊛
   ii. *Let* $V_1 < 0$.
      1. *Then* V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
      2. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\|}$, *or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|}$ *where* Conduct$_{1\blacktriangle}$ → ⓓ/⊛

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-b3i) The same as Tom 19.1.16(p.157) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (b3ii2) of Tom 19.1.16(p.157) and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (b3ii2) of Tom 19.1.16(p.157) with κ. Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $V_1 > x_K$.

(c1-c1ii) The same as Tom 19.1.16(p.157) (c1-c1ii).

(c2) Let $\beta < 1$.

(c2i) Let $V_1 > 0$.

(c2i1) The same as Tom 19.1.16(p.157) (c2i1).

(c2i2) If $s = 0$, then it suffices to consider only (c2i3) of Tom 19.1.16 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2i3) of Tom 19.1.16. Consequently, whether $s = 0$ or $s > 0$, we have the same result.

(c2ii) Let $V_1 < 0$.

(c2ii1) The same as Tom 19.1.16(p.157) (c2ii1).

(c2ii2) If $s = 0$, then it suffices to consider only (c2ii3) of Tom 19.1.16 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2ii3) of Tom 19.1.16. Consequently, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.1.7 $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]

19.1.7.1 Preliminary

Due to (19.1.15(p.139)) we see that Theorem 19.1.3(p.137) holds, hence $\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{A}]\}$ can be obtained by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to $\mathscr{A}\{\mathsf{M}:2[\mathbb{P}][\mathtt{A}]\}$.

19.1.7.2 Analysis

19.1.7.2.1 Case of $\beta = 1$ and $s = 0$

☐ **Tom 19.1.17** ($\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*
(b) *Let $\rho \geq b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
(c) *Let $a \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(d) *Let $b^\star > \rho > a$.*
  1. *Let $b \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\|}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$ *and* $\mathtt{pSKIP}_{1\vartriangle}$.
  2. *Let $\rho > b$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. ☐

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.9(p.153). ∎

Corollary 19.1.10 *Let $\beta = 1$ and $s = 0$. Then z_t is nonincreasing in $t \geq 0$.* ☐

Proof Immediate from Tom 19.1.17(a), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

19.1.7.2.2 Case of $\beta < 1$ or $s > 0$

19.1.7.2.2.1 Case of $\rho \geq b^\star$ [†]

☐ **Tom 19.1.18** ($\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let $\rho > x_{\tilde{L}}$.*
  1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
    i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
    ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
  3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
    i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
      1. *Let $\lambda = 1$.*
        i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        ii. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
      2. *Let $\lambda < 1$.*
        i. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
        ii. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* S_3(p.141) $\boxed{ⓢ\blacktriangle \mid ⊛\|}$ *is true.*
    ii. *Let $(\lambda\beta b + s)/\delta < b^\star$.*
      1. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      2. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* S_3(p.141) $\boxed{ⓢ\blacktriangle \mid ⊛\|}$ *is true.* ☐

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.10(p.153). ∎

Corollary 19.1.11 *Assume $\rho \geq b^\star$, let $\beta < 1$ or $s > 0$, and let $\rho > x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* ☐

Proof Immediate from Tom 19.1.18(a), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

☐ **Tom 19.1.19** ($\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then, for a given starting time $\tau > 0$:*

(a) *V_t is nonincreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
  1. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *and* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
  2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ☐

Proof by symmetry Clear from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.11(p.154). ∎

[†] The condition of $\rho \geq b^\star$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $\rho \leq a^\star$ in Section 19.1.6.2.2.1(p.153).

Corollary 19.1.12 *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

Proof Immediate from Tom 19.1.19(a), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

□ **Tom 19.1.20** ($\mathscr{A}\{\tilde{\text{M}}{:}2[\mathbb{P}][\text{A}]\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ($s > 0$).*
1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $a < 0$ ($\tilde{\kappa} < 0$).*
   i. *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *where* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   iii. *Let $\rho < x_{\tilde{L}}$. Then* $\texttt{S}_4$ $\boxed{ⓢ\blacktriangle \mid \bullet\parallel \mid \text{pS}\Delta \mid \text{pS}\blacktriangle}$ *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ($s > 0$).*
1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.12(p.154). ∎

Corollary 19.1.13 *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then z_t is constant in t ($z_t = \tilde{z}(\rho)$ for $t \geq 0$).*
(b) *Let $\beta < 1$ and $\rho > 0$. Then z_t is nondecreasing in $t \geq 0$ for any $s \geq 0$.*
(c) *Let $\beta < 1$ and $\rho < 0$. Then z_t is nonincreasing in $t \geq 0$ for any $s \geq 0$.* □

Proof by symmetry Evident from Tom 19.1.20(a1,b1,c1), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

19.1.7.2.2.2 Case of $a \geq \rho$[†]

□ **Tom 19.1.21** ($\mathscr{A}\{\tilde{\text{M}}{:}2[\mathbb{P}][\text{A}]\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho > x_{\tilde{L}}$.*
1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $a \leq 0$ ($\tilde{\kappa} \leq 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $a > 0$ ($\tilde{\kappa} > 0$). Then* $\texttt{S}_3$(p.141) $\boxed{ⓢ\blacktriangle \mid ⊛\parallel}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.13(p.154). ∎

Corollary 19.1.14 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

Proof Evident from Tom 19.1.21(a), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

□ **Tom 19.1.22** ($\mathscr{A}\{\tilde{\text{M}}{:}2[\mathbb{P}][\text{A}]\}$) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
1. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *and* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
2. *Let $a \geq 0$ ($\tilde{\kappa} \geq 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.14(p.155). ∎

Corollary 19.1.15 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* □

Proof Evident from Tom 19.1.22(a), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

[†] The condition of $a \geq \rho$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $b \leq \rho$ in Section 19.1.6.2.2.2(p.154).

□ **Tom 19.1.23** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.

1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $x_{\tilde{L}} < \rho$. *Then* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $⦅s > 0⦆$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $a \geq 0$ $⦅\tilde{\kappa} \geq 0⦆$. *Then* $\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $a < 0$ $⦅\tilde{\kappa} < 0⦆$.
   - i. *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\,\mathtt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *where* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - iii. *Let* $x_{\tilde{L}} > \rho$. *Then* $\mathtt{S}_4$ $\boxed{\text{ⓢ}\blacktriangle \mid \bullet\parallel \mid \text{pS}\Delta \mid \text{pS}\blacktriangle}$ *is true.*

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $⦅s > 0⦆$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $a \geq 0$ $⦅\tilde{\kappa} \geq 0⦆$. *Then* $\boxed{\bullet\,\mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
3. *Let* $a < 0$ $⦅\tilde{\kappa} < 0⦆$. *Then* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.15(p.155). ∎

Corollary 19.1.16 *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$. *Then* $z_t = \tilde{z}(\rho)$ *for* $t \geq 0$.
(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $⦅s > 0⦆$. *Then* z_t *is nondecreasing in* $t \geq 0$.
(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $⦅s > 0⦆$. *Then* z_t *is nonincreasing in* $t \geq 0$. □

Proof Evident from Tom 19.1.23(a1,b1,c1), (6.2.90(p.26)), and Lemma A 3.3(p.278). ∎

19.1.7.2.2.3 Case of $b^\star > \rho > a$[†]

Let us here note that (19.1.49(p.155)) changes as follows.

$$V_1 = \min\{\lambda\beta\min\{0, b-\rho\} + s, 0\} + \beta\rho.^{\dagger} \tag{19.1.50}$$

□ **Tom 19.1.24** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume* $b^\star > \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\min\{0, \rho - b\} \geq -s$, *then* $\boxed{\bullet\,\mathtt{dOITd}_{1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$.

(b) *Let* $V_1 \geq x_{\tilde{K}}$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $V_1 \leq x_{\tilde{L}}$. *Then, if* $\lambda\beta\min\{0, \rho - b\} \geq -s$, *we have* $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledast\,\mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
3. *Let* $V_1 > x_{\tilde{L}}$.
   - i. *Let* $\beta = 1$. *Then* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let* $\beta < 1$ *and* $s = 0$ $⦅s > 0⦆$.
     1. *Let* $a < 0$ $⦅\tilde{\kappa} < 0⦆$. *Then* $\boxed{\text{ⓢ}\,\mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
     2. *Let* $a \geq 0$ $⦅\tilde{\kappa} \geq 0⦆$. *Then* $\mathtt{S}_5$ $\boxed{\text{ⓢ}\blacktriangle \mid \circledast\parallel}$ *is true.*

(c) *Let* $V_1 < x_{\tilde{K}}$.

1. *Let* $\beta = 1$ *or* $V_1 = 0$.
   - i. $V_t = V_1$ *for* $t > 0$.
   - ii. *If* $\lambda\min\{0, \rho - b\} \geq -s$, *then* $\boxed{\bullet\,\mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\parallel}$, *or else* $\boxed{\circledast\,\mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ $⦅s > 0⦆$.[†]
   - i. *Let* $V_1 < 0$.
     1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $\tau \to \infty$.

[†]The condition of $b^\star > \rho > a$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $a^\star < \rho < b$ in Section 19.1.6.2.2.3(p.155).

[†]
$$\begin{aligned}
-\hat{V}_1 &= \max\{\lambda\beta\max\{0, -\hat{a} + \hat{\rho}\} - s, 0\} - \beta\hat{\rho} && \text{(apply the reflection to (19.1.49(p.155)))}\\
\hat{V}_1 &= -\max\{\lambda\beta\max\{0, -\hat{a} + \hat{\rho}\} - s, 0\} + \beta\hat{\rho} && \text{(multiply the above by } -1)\\
&= \min\{-\lambda\beta\max\{0, -\hat{a} + \hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(arrangement the above)}\\
&= \min\{\lambda\beta\min\{0, \hat{a} - \hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(arrangement the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, \check{b} - \hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(apply } \mathcal{I}_{\mathbb{R}} \text{ to the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, b - \hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(apply } \mathcal{C}_{\mathbb{R}} \text{ to the above)}\\
V_1 &= \min\{\lambda\beta\min\{0, b - \rho\} + s, 0\} + \beta\rho && \text{(remove the hat symbol } \hat{\ })
\end{aligned}$$

2. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then*

i. *Let* $V_1 < x_{\tilde{L}}$. *Then* $\mathtt{S}_6$ $\boxed{ⓢ\blacktriangle \mid \bullet\| \mid ⊛\| \mid \text{pS}\vartriangle \mid \text{pS}\blacktriangle}$ *is true.*

ii. *Let* $V_1 = x_{\tilde{L}}$. *Then* $\mathtt{S}_7$ $\boxed{ⓢ\blacktriangle \mid \bullet\| \mid ⊛\| \mid \text{pS}\vartriangle}$ *is true.*

iii. *Let* $V_1 > x_{\tilde{L}}$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

3. *Let* $a \geq 0\,(\!(\tilde{\kappa} \geq 0)\!)$. *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\|$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.

ii. *Let* $V_1 > 0$.

1. *Then* V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $\tau \to \infty$.
2. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then*

i. *Let* $V_1 \leq x_{\tilde{L}}$. *If* $\lambda\beta\min\{0, \rho - b\} \geq -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\|$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.

ii. *Let* $V_1 > x_{\tilde{L}}$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.

3. *Let* $a \geq 0\,(\!(\tilde{\kappa} \geq 0)\!)$. *If* $\lambda\beta\min\{0, \rho-b\} \geq -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\|$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.1.16(p.157) . ∎

Corollary 19.1.17 *Assume* $b^\star > \rho > a$. *Let* $\beta < 1$ *or* $s > 0$:

(a) *Let* $V_1 \geq x_{\tilde{K}}$. *Then* z_t *is nonincreasing in* $t > 0$.
(b) *Let* $V_1 < x_{\tilde{K}}$. *Then*

1. *Let* $\beta = 1$ *or* $V_1 = 0$. *Then* z_t *is constant in* $t > 0$ ($z_t = \tilde{z}(V_1)$ *for* $t > 0$).
2. *Let* $\beta < 1$.

i. *Let* $V_1 < 0$. *Then* z_t *is nondecreasing in* $t > 0$ *for any* $s \geq 0$.

ii. *Let* $V_1 > 0$. *Then* z_t *is nonincreasing in* $t > 0$ *for any* $s \geq 0$. □

Proof Immediate from Tom 19.1.24(b1,c1i,c2i1,c2ii1), (6.2.90(p.26)), and Lemma A 3.3(p.278) . ∎

19.1.7.3 Market Restriction

19.1.7.3.1 Positive Restriction

19.1.7.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 19.1.17** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *Let* $\rho \geq b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ
(c) *Let* $a \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$ → → ⓓ
(d) *Let* $b^\star > \rho > a$.

1. *Let* $b \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\|$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ *and* $\mathtt{pSKIP}_{1\vartriangle}$ → → ⓓ/ⓢ/ⓟⓢ
2. *Let* $\rho > b$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$. □ → → ⓢ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.5(p.165) (see (17.3.7(p.116))). ∎

Direct proof The same as Tom 19.1.17(p.168) due to Lemma 16.4.1(p.100) . ∎

19.1.7.3.1.2 Case of $\beta < 1$ or $s > 0$

19.1.7.3.1.2.1 Case of $\rho \geq b^\star$

□ **Pom 19.1.18** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\|$ → → ⓓ
(c) *Let* $\rho > x_{\tilde{L}}$.

1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$ *and* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$ → → ⓢ
2. *Let* $\beta = 1$.

i. *Let* $(\lambda b + s)/\lambda \geq b^\star$.

1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
2. *Let* $\lambda < 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ

ii. *Let* $(\lambda b + s)/\lambda < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ

3. *Let* $\beta < 1$ *and* $s > 0$. *Then we have* $\mathtt{S}_3$(p.141) $\boxed{ⓢ\blacktriangle \mid ⊛\|}$.
4. *Let* $\beta < 1$ *and* $s > 0$.

i. *Let* $(\lambda\beta b + s)/\delta \geq b^\star$.

1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ → → ⊛
2. *Let* $\lambda < 1$. *Then* $\mathtt{S}_3$(p.141) $\boxed{ⓢ\blacktriangle \mid ⊛\|}$ *is true* → → ⓢ/⊛

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†See Remark 19.1.2(p.157) .

ii. *Let* $(\lambda\beta b + s)/\delta < b^\star$. *Then* $\mathtt{S}_3$(p.141) [ⓢ▲ | ⊛‖] *is true.* ▯ → → ⓢ / ⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.6(p.165) (see (17.3.7(p.116))). ▮

Direct proof Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$ and $b^\star > 0 \cdots (3)$ from Lemma 13.6.1(p.89) (n) and (2). Then we have $\tilde{\kappa} = s \cdots (4)$ from Lemma 13.6.6(p.90) (a).

(a-c2ii) The same as Tom 19.1.18(p.168) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta b + s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (9.2.2 (1) (p.42)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (3), so that $\beta b \geq b^\star$, which contradicts [7(p.101)]. Thus it must be that $(\lambda\beta b + s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of Tom 19.1.18(p.168).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\kappa > 0$ due (2), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 19.1.18(p.168); accordingly, whether $s = 0$ or $s > 0$, we have the same result. ▮

□ Pom **19.1.19** ($\mathscr{A}\{\tilde{\mathrm{M}}{:}2[\mathbb{P}][\mathrm{A}]^+\}$) *Suppose* $a > 0$. *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *We have* [● dOITd$_{\tau>0}\langle 0\rangle$]‖. ▯ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.7(p.166) (see (17.3.7(p.116))). ▮

Direct proof Suppose $a > 0$. Then $\tilde{\kappa} = s \cdots (1)$ from Lemma 13.6.6(p.90) (a).

(a) The same as Tom 19.1.19(p.168) (a).

(b) Let $\beta = 1$. Then, we have [● dOITd$_{\tau>0}\langle 0\rangle$]‖ from Tom 19.1.19(p.168) (b). Let $\beta < 1$. Then, if $s = 0$, it suffices to consider only (c2) of Tom 19.1.19 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (1), hence it suffices to consider only (c2) of Tom 19.1.19; accordingly, whether $s = 0$ or $s > 0$, we have the same results. Therefore, whether $\beta = 1$ or $\beta < 1$, we have the same result. ▮

□ Pom **19.1.20** ($\mathscr{A}\{\tilde{\mathrm{M}}{:}2[\mathbb{P}][\mathrm{A}]^+\}$) *Suppose* $a > 0$. *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.
 1. $V_t = \rho$ *for* $t \geq 0$.
 2. *Let* $x_{\tilde{L}} \geq \rho$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]‖ → → ⓓ
 3. *Let* $x_{\tilde{L}} < \rho$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]▲ *where* Conduct$_{\tau\geq t>0\blacktriangle}$ → → ⓢ

(b) *Let* $\beta < 1$ *and* $\rho > 0$.
 1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
 2. [● dOITd$_{\tau>0}\langle 0\rangle$]‖ → → ⓓ

(c) *Let* $\beta < 1$ *and* $\rho < 0$.
 1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
 2. [● dOITd$_{\tau>0}\langle 0\rangle$]‖. ▯ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.8(p.166) (see (17.3.7(p.116))). ▮

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a-a3) The same as Tom 19.1.20(p.169) (a-a3).

(b-b2) Let $\beta < 1$ and $\rho > 0$. First, we have the same as Pom 19.1.20(b1). Next, if $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 19.1.20 and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (b2) of Tom 19.1.20. Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c-c2) Let $\beta < 1$ and $\rho < 0$. First, we have the same as Pom 19.1.20(c1). Next, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.20 and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (c2) of Tom 19.1.20. Thus, whether $s = 0$ or $s > 0$, we have the same result. ▮

19.1.7.3.1.2.2 Case of $a \geq \rho$

□ Pom **19.1.21** ($\mathscr{A}\{\tilde{\mathrm{M}}{:}2[\mathbb{P}][\mathrm{A}]^+\}$) *Suppose* $a > 0$. *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]‖ → → ⓓ
(c) *Let* $\rho > x_{\tilde{L}}$.
 1. [ⓢ dOITs$_1\langle 1\rangle$]▲ *where* Conduct$_{1\blacktriangle}$. *Below let* $\tau > 1$ → → ⓢ
 2. *Let* $\beta = 1$. *Then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]▲ *where* Conduct$_{\tau\geq t>0\blacktriangle}$ → → ⓢ
 3. *Let* $\beta < 1$. *Then* $\mathtt{S}_3$(p.141) [ⓢ▲ | ⊛‖] *is true.* ▯ → → ⓢ / ⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.9(p.166) (see (17.3.7(p.116))). ▮

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a-c2) The same as Tom 19.1.21(p.169) (a-c2).

(c3) Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c3ii) of Tom 19.1.21 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 19.1.21. Thus, whether $s = 0$ or $s > 0$, we have the same result. ▮

□ Pom **19.1.22** ($\mathscr{A}\{\tilde{\text{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$, and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*
(b) *We have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ◻ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.10(p.166) (see (19.1.21(p.140))). ▮

Direct proof Suppose $a > 0 \cdots$ **(**1**)**. Then $\tilde{\kappa} = s \cdots$ **(**2**)** from Lemma 13.6.6(p.90) (a).

(a) The same as Tom 19.1.22(p.169) (a).

(b) Let $\beta = 1$. Then $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ from Tom 19.1.22(p.169) (b). Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.22, and if $s > 0$, then $\tilde{\kappa} \geq 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.1.22 with $\tilde{\kappa}$; accordingly, whether $s = 0$ or $s > 0$, we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. Thus, whether $\beta = 1$ or $\beta < 1$, we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ▮

□ Pom **19.1.23** ($\mathscr{A}\{\tilde{\text{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*
  1. *$V_t = \rho$ for $t \geq 0$.*
  2. *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ
  3. *Let $x_{\tilde{L}} < \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$ → → ⓢ

(b) *Let $\beta < 1$ and $\rho > 0$.*
  1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
  2. $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$ → → ⓓ

(c) *Let $\beta < 1$ and $\rho < 0$.*
  1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
  2. $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$. ◻ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.11(p.167) (see (17.3.7(p.116))). ▮

Direct proof Suppose $a > 0 \cdots$ **(**1**)**, hence $b > a > 0 \cdots$ **(**2**)**. Then $\tilde{\kappa} = s \cdots$ **(**3**)** from Lemma 13.6.6(p.90) (a).

(a-a3) The same as Tom 19.1.23(p.170) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as Pom 19.1.23(b1).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 19.1.23 and if $s > 0$, then $\tilde{\kappa} > 0$ from (3), hence it suffices to consider only (b2) of Tom 19.1.23. Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as Pom c1(b1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.1.23 and if $s > 0$, then $\tilde{\kappa} > 0$ from (3), hence it suffices to consider only (c2) of Tom 19.1.23. Thus, whether $s = 0$ or $s > 0$, we have the same result. ▮

19.1.7.3.1.2.3 Case of $b^\star > \rho > b$

□ Pom **19.1.24** ($\mathscr{A}\{\tilde{\text{M}}$:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $b^\star > \rho > b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\|}$, *or else* $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$* → → ⓓ/ⓢ

(b) *Let $V_1 \geq x_{\tilde{K}}$.*
  1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
  2. *Let $V_1 \leq x_{\tilde{L}}$. Then, if $\lambda\beta\max\{0, \rho - b\} \leq s$, we have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. → ⓓ/⊛
  3. *Let $V_1 > x_{\tilde{L}}$.*
     i. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ → → ⓢ
     ii. *Let $\beta < 1$. Then* $\texttt{S}_5$ $\boxed{ⓢ\blacktriangle \mid ⊛\|}$ *is true* → → ⓢ/⊛

(c) *Let $V_1 < x_{\tilde{K}}$.*
  1. *Let $\beta = 1$ or $V_1 = 0$. Then:*
     i. *$V_t = V_1$ for $t > 0$.*
     ii. *If $\lambda\min\{0, \rho - b\} \geq -s$, then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. → ⓓ/⊛
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*†
     i. *Let $V_1 < 0$.*
        1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
        2. *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$. → ⓓ/⊛
     ii. *Let $V_1 > 0$.*
        1. *Then V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$ where $V = x_{\tilde{K}}$ if the immediate initiation is strictly optimal for any $\tau \gg 0$.*
        2. *If $\lambda\beta\min\{0, \rho - b\} \geq -s$, then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\|}$, *or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → ⓓ/⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.1.12(p.167) (see (17.3.7(p.116))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a-b3i) The same as Tom 19.1.24(p.170) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (b3ii2) of Tom 19.1.24 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii2) of Tom 19.1.24. Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $V_1 < x_{\tilde{K}}$.

(c1-c1ii) The same as Tom 19.1.24(c1-c1ii).

(c2) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c2i,c2i1) The same as Tom 19.1.24(c2i,c2i1).

(c2i2) The same as Tom 19.1.24(p.170) (20.2.3).

(c2ii,c2ii1) The same as Tom 19.1.24(p.170) (c2ii,c2ii1).

(c2ii2) If $s = 0$, then due to (1) it suffices to consider only (c2ii3) of Tom 19.1.24 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence t suffices to consider only (c2ii3) of Tom 19.1.24. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.1.7.3.1.2.4 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.1.7.3.1.2.5 Negative Restriction

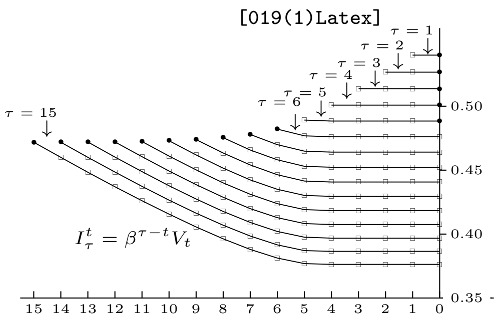
Omitted (see Section 17.2.3(p.116)).

19.1.8 Numerical Calculation

■ Numerical Example 19.1.1 ($\mathscr{A}${M:2[ℝ][A]$^+$} [019(1)])

This is the example for [pS▲] (pSKIP▲) of S_4 [Ⓢ▲ | ●‖ | pS△ | pS▲] in Pom 19.1.4(p.147) (c3iii) in which $a > 0$, $\rho > x_K$, $\beta < 1$, $\rho > 0$, $s > 0$, and $x_L < \rho$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$[†] where $x_L = 0.462767$ and $x_K = 0.439640$. The graph below is for $I^t_\tau = \beta^{\tau-t}V_t$, $\tau = 1, 2, \cdots, 15$ and $t = 0, 1, \cdots, \tau$, where • represents the optimal-initiating-time OIT for each $\tau = 1, 2, \cdots, 15$ (see t^*_τ-column in the table below).

1. Since $\Delta_\beta V_1 = \Delta_\beta V_2 = \Delta_\beta V_3 = \Delta_\beta V_4 = 0$ (see $\Delta_\beta V_t$-column in the table below), we have $V_4 = \beta V_3$, $V_3 = \beta V_2$, $V_2 = \beta V_1$, and $V_1 = \beta V_0$, implying that it becomes *indifferent* to skip the search up to the deadline $t_d = 0$ on $t = 4, 3, 2, 1$ (see Preference Rule 7.2.1(p.35)), i.e., [● dOITd$_{\tau=4,3,2,1}\langle 0\rangle$]▲. On the other hand, since $L(V_{t-1}) < 0$ for $1 \le t \le 4$ (see $L(V_{t-1})$-column in the table below), it follows that it is *strictly optimal* to skip the search for $1 \le t \le \tau = 4$, i.e., [● dOITd$_{\tau=4,3,2,1}\langle 0\rangle$]▲. Although the above two results "*indifferent*" and "*strictly optimal*" seem to contradict each other at a glance, it is what is caused by the jumble of intuition and theory (see Alice 3(p.36)).
2. Each of the graphs for $\tau = 5, 6, \cdots, 15$ shows that the optimal-initiating-time is *strictly*, i.e., [Ⓢ dOITs$_{5\le\tau\le15}\langle\tau\rangle$]▲, meaning that the immediate initiation is strictly optimal and that conducting the search is *strictly optimal* at time $t = 5, 6, \cdots, 15$ (Conduct▲) and skipping the search becomes *strictly optimal* at time $t = 4, 3, 2, 1$ after that (see $L(V_{t-1})$-column in the table below), implying that the strictly-posterior-skip-of search (pSKIP▲ (see Remark 7.2.1(p.34))) occurs.



$\Delta_\beta V_t = V_t - \beta V_{t-1}$ [019(1)Data.DAT]

t	V_t	$\Delta_\beta V_t$	t^*_τ	$L(V_{\tau-1})$
0	0.5000000			
1	0.4900000	0.0000000	0	−0.0133838 (pSKIP▲)
2	0.4802000	0.0000000	0	−0.0098846 (pSKIP▲)
3	0.4705960	0.0000000	0	−0.0063880 (pSKIP▲)
4	0.4611841	0.0000000	0	−0.0028969 (pSKIP▲)
5	0.4525469	+0.0005865	5	+0.0005865 (Conduct▲)
6	0.4473331	+0.0038371	6	+0.0038371 (Conduct▲)
7	0.4442109	+0.0058244	7	+0.0058244 (Conduct▲)
8	0.4423501	+0.0070235	8	+0.0070235 (Conduct▲)
9	0.4412444	+0.0077413	9	+0.0077413 (Conduct▲)
10	0.4405885	+0.0081690	10	+0.0081690 (Conduct▲)
11	0.4401998	+0.0084231	11	+0.0084231 (Conduct▲)
12	0.4399696	+0.0085738	12	+0.0085738 (Conduct▲)
13	0.4398333	+0.0086631	13	+0.0086631 (Conduct▲)
14	0.4397527	+0.0087160	14	+0.0087160 (Conduct▲)
15	0.4397049	+0.0087473	15	+0.0087473 (Conduct▲)

Figure 19.1.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \ge \tau > 1, \tau \ge t > 0)$

[†] See Remark 19.1.2(p.157).

[†] Note that $a = 0.01 > 0$, $\rho = 0.5 > 0$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. In addition, since $\mu = (1.00 + 0.01)/2 = 0.505$, we have $\lambda\beta\mu = 0.34643 > 0.1 = s$. Furthermore, we have $x_L = 0.4627674 < 0.5 = \rho$. Thus the condition of the assertion is satisfied.

19.1.9 Conclusion 4 (Search-Allowed-Model 2)

C1 Monotonicity

On the total market $\mathscr{F}$ we have:

a. The optimal reservation price V_t in M:2[ℝ][A] is nondecreasing $\blacktriangle^{a}$, constant $\|^{a}$, or nonincreasing $\blacktriangledown^{a}$.
b. The optimal reservation price V_t in $\tilde{\mathsf{M}}$:2[ℝ][A] is nondecreasing $\blacktriangle^{b}$, constant $\|^{b}$, or nonincreasing $\blacktriangledown^{b}$.
c. The optimal price z_t in M:2[ℙ][A] is nondecreasing $\blacktriangle^{c}$, constant $\|^{c}$, or nonincreasing $\blacktriangledown^{c}$.
d. The optimal price z_t in $\tilde{\mathsf{M}}$:2[ℙ][A] is nondecreasing $\blacktriangle^{d}$, constant $\|^{d}$, or nonincreasing $\blacktriangledown^{d}$.

- $\blacktriangle^{a}$ ← Tom's 19.1.1(p.140) (a), 19.1.2(p.141) (a), 19.1.3(p.143) (a), 19.1.4(p.144) (c1).
  $\|^{a}$ ← Tom 19.1.4(p.144) (a1)).
  $\blacktriangledown^{a}$ ← Tom's 19.1.4(p.144) (b1).
- $\blacktriangle^{b}$ ← Tom 19.1.8(p.149) (b1).
  $\|^{b}$ ← Tom 19.1.8(p.149) (a1).
  $\blacktriangledown^{b}$ ← Tom's 19.1.5(p.149) (a), 19.1.6(p.149) (a), 19.1.7(p.149) (a), 19.1.8(p.149) (c1).
- $\blacktriangle^{c}$ ← Corollaries 19.1.2(p.153) , 19.1.3(p.154) , 19.1.4(p.154) ,19.1.5(p.154) (c), 19.1.6(p.155) , 19.1.7(p.155) , 19.1.8(p.155) (c), 19.1.9(p.161) (a,b2ii).
  $\|^{c}$ ← Corollary 19.1.5(p.154) (a), 19.1.8(p.155) (a), 19.1.9(p.161) (b1).
  $\blacktriangledown^{c}$ ← Corollaries 19.1.5(p.154) (b), 19.1.8(p.155) (b), 19.1.9(p.161) (b2i).
- $\blacktriangle^{d}$ ← Corollaries 19.1.13(p.169) (b), 19.1.16(p.170) (b),19.1.17(p.171) (b2i).
  $\|^{d}$ ← Corollaries 19.1.16(p.170) (a), 19.1.17(p.171) (b1).
  $\blacktriangledown^{d}$ ← Corollaries 19.1.10(p.168) , 19.1.11(p.168) , 19.1.12(p.169) , 19.1.13(p.169) (c), 19.1.14(p.169) , 19.1.15(p.169) , 19.1.16(p.170) (c), 19.1.17(p.171) (a),b2ii).

C2 Inheritance and Collapse

On the positive market $\mathscr{F}^+$ we have:

a. Symmetry

1. Let $\beta = 1$ and $s = 0$. Then, the symmetry is inherited (∼) in whether ℝ-model or ℙ-model where

Pom 19.1.1(p.146) ∼ Pom 19.1.5(p.150) (ℝ-model),
Pom 19.1.9(p.161) ∼ Pom 19.1.17(p.171) (ℙ-model).

2. Let $\beta < 1$ or $s > 0$. Then, the symmetry collapses (≁) in whether ℝ-model or ℙ-model where

Pom 19.1.2(p.146) ≁ Pom 19.1.6(p.150) (ℝ-model),
Pom 19.1.3(p.146) ≁ Pom 19.1.7(p.150) (ℝ-model),
Pom 19.1.4(p.147) ≁ Pom 19.1.8(p.151) (ℝ-model),
Pom 19.1.10(p.161) ≁ Pom 19.1.18(p.171) (ℙ-model),
Pom 19.1.11(p.162) ≁ Pom 19.1.19(p.172) (ℙ-model),
Pom 19.1.12(p.162) ≁ Pom 19.1.20(p.172) (ℙ-model),
Pom 19.1.13(p.163) ≁ Pom 19.1.21(p.172) (ℙ-model),
Pom 19.1.14(p.163) ≁ Pom 19.1.22(p.173) (ℙ-model),
Pom 19.1.15(p.163) ≁ Pom 19.1.23(p.173) (ℙ-model),
Pom 19.1.16(p.164) ≁ Pom 19.1.24(p.173) (ℙ-model).

b. Analogy

For whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$", the analogy collapses (⋈) in whether S-model or B-model where

Pom 19.1.1(p.146) ⋈ Pom 19.1.9(p.161) (S-model),
Pom 19.1.5(p.150) ⋈ Pom 19.1.17(p.171) (B-model),
Pom 19.1.2(p.146) ⋈ Pom 19.1.10(p.161) (S-model),
Pom 19.1.6(p.150) ⋈ Pom 19.1.18(p.171) (B-model), .

C3 Occurrence of ⓢ, ⊛, and ⓓ

On the positive market $\mathscr{F}^+$ we have:

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 19.1.1(p.146) , Pom 19.1.5(p.150) , Pom 19.1.9(p.161) , Pom 19.1.17(p.171) ,

we have the following table:

Table 19.1.1: ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$ ($\beta = 1$ and $s = 0$)

		$\mathscr{A}$\{M:2[ℝ][A]$^+$\}	$\mathscr{A}$\{$\tilde{\mathsf{M}}$:2[ℝ][A]$^+$\}	$\mathscr{A}$\{M:2[ℙ][A]$^+$\}	$\mathscr{A}$\{$\tilde{\mathsf{M}}$:2[ℙ][A]$^+$\}
ⓢ dOITs$_\tau\langle\tau\rangle$ ‖	ⓢ‖				
ⓢ dOITs$_\tau\langle\tau\rangle$ △	ⓢ△				
ⓢ dOITs$_\tau\langle\tau\rangle$ ▲	ⓢ▲	○	○	○	○
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ ‖	⊛‖				
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ △	⊛△				
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ ▲	⊛▲				
● dOITd$_\tau\langle 0\rangle$ ‖	ⓓ‖	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ △	ⓓ△				
● dOITd$_\tau\langle 0\rangle$ ▲	ⓓ▲				

1. What is amazing is here that, even in the most simple case "$\beta = 1$ and $s = 0$", the deadline-falling ⓓ occurs in all of quadruple-asset-trading models.

b. Let $\beta < 1$ or $s > 0$. Then, from

`Pom` 19.1.2(p.146), `Pom` 19.1.3(p.146), `Pom` 19.1.4(p.147), `Pom` 19.1.5(p.150), `Pom` 19.1.6(p.150), `Pom` 19.1.7(p.150), `Pom` 19.1.8(p.151), `Pom` 19.1.10(p.161), `Pom` 19.1.11(p.162), `Pom` 19.1.12(p.162), `Pom` 19.1.13(p.163), `Pom` 19.1.14(p.163), `Pom` 19.1.15(p.163), `Pom` 19.1.16(p.164), `Pom` 19.1.19(p.172), `Pom` 19.1.20(p.172), `Pom` 19.1.21(p.172), `Pom` 19.1.22(p.173), `Pom` 19.1.23(p.173), `Pom` 19.1.24(p.173),

we have the following table:

Table 19.1.2: ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$ ($\beta <$ or $s > 0$)

		$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\parallel}$	ⓢ$_{\parallel}$				
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\vartriangle}$	ⓢ$_{\vartriangle}$				
ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\blacktriangle}$	ⓢ$_{\blacktriangle}$	○	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t_\tau^\bullet\rangle_{\parallel}$	⊛$_{\parallel}$	○	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$	⊛$_{\vartriangle}$				
⊛ $\mathtt{ndOIT}_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$	⊛$_{\blacktriangle}$				
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\parallel}$	ⓓ$_{\parallel}$	○	○	○	○
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\vartriangle}$	ⓓ$_{\vartriangle}$				
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\blacktriangle}$	ⓓ$_{\blacktriangle}$				

1. In addition to ⓢ and ⓓ, the non-degenerate `OIT` ⊛ occurs in all of quadruple-asset-trading models.

c. The table below is the list of percents (frequencies) of ⓢ, ⊛, and ⓓ that appear in Sections 19.1.6.3(p.161) and 19.1.7.3(p.171).

Table 19.1.3: Percents (frequencies) of ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$

ratio (total)	ⓢ	⊛	ⓓ
100% (249)	47% (117)	19% (47)	34% (85)

C4 Posterior-skip-of-search

On the positive market $\mathscr{F}^+$ we have:

From `Pom`'s 19.1.4(p.147), 19.1.12(p.162), 19.1.15(p.163), and 19.1.16(p.164), if $\beta < 1$ or $s > 0$, we have the following table:

Table 19.1.4: Posterior-skip-of-search ($\beta < 1$ or $s > 0$)

	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
$\mathtt{pSKIP}_{t_\vartriangle}$ (pS)	○		○	
$\mathtt{pSKIP}_{t_\blacktriangle}$ (pS)	○		○	

a. The posterior-skip-of-search $\mathtt{pSKIP}_t$ occurs only in $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+$ and $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+$ which are both selling models. What is amazing is here is that $\mathtt{pSKIP}_{t_\blacktriangle}$ can occurs.

C5 Diagonal symmetry

*Exercise **19.1.1*** *Confirm by yourself that the following relations hold in fact.*

`Pom` 19.1.5(p.150) ∾ `Nem` 19.1.1(p.147) ($\mathbb{R}$-model),
`Pom` 19.1.6(p.150) ∾ `Nem` 19.1.2(p.147) ($\mathbb{R}$-model),
`Pom` 19.1.7(p.150) ∾ `Nem` 19.1.3(p.148) ($\mathbb{R}$-model),
`Pom` 19.1.8(p.151) ∾ `Nem` 19.1.4(p.148) ($\mathbb{R}$-model).

`Pom` 19.1.17(p.171) ∾ `Nem` 19.1.5(p.165) ($\mathbb{P}$-model),
`Pom` 19.1.18(p.171) ∾ `Nem` 19.1.6(p.165) ($\mathbb{P}$-model),
`Pom` 19.1.19(p.172) ∾ `Nem` 19.1.7(p.166) ($\mathbb{P}$-model),
`Pom` 19.1.20(p.172) ∾ `Nem` 19.1.8(p.166) ($\mathbb{P}$-model),
`Pom` 19.1.21(p.172) ∾ `Nem` 19.1.9(p.166) ($\mathbb{P}$-model),
`Pom` 19.1.22(p.173) ∾ `Nem` 19.1.10(p.166) ($\mathbb{P}$-model),
`Pom` 19.1.23(p.173) ∾ `Nem` 19.1.11(p.167) ($\mathbb{P}$-model),
`Pom` 19.1.24(p.173) ∾ `Nem` 19.1.12(p.167) ($\mathbb{P}$-model). □

a. The diagonal symmetry holds in whether $\mathbb{R}$-model or $\mathbb{P}$-model.

19.2 Search-Enforced-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathsf{E}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}], \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$

19.2.1 Theorems

As ones corresponding to Theorems 18.2.1(p.122), 18.2.2, 18.2.3, and 18.2.4, let us herein consider the following four theorems:

Theorem 19.2.1 (symmetry$[\mathbb{R} \to \mathbb{R}]$)) *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{19.2.1}$$

Theorem 19.2.2 (analogy$[\mathbb{R} \to \mathbb{P}]$) *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{19.2.2}$$

Theorem 19.2.3 (symmetry$[\mathbb{P} \to \mathbb{P}]$) *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}]. \quad \square \tag{19.2.3}$$

Theorem 19.2.4 (analogy$[\mathbb{R} \to \mathbb{P}]$) *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square$$

In order for the four theorems above to hold, the following four relations *must* hold for the same reason as in the search-**A**llowed-model 1 (see Part 2 (see p.38):

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}], \tag{19.2.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}], \tag{19.2.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}], \tag{19.2.6}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}], \tag{19.2.7}$$

From the comparisons between (I) and (II) in Table 6.5.4(p.31) and between (III) and (IV) we see that respectively (19.2.4) and (19.2.6) hold, hence Theorems 19.2.1 and 19.2.3 hold. But, from the comparison of (I) and (III) we see that (19.2.5) does not always hold, hence it follows that Theorem 19.2.2 does not always hold.

19.2.2 Conditions

Lemma 19.2.1

(a) Theorem 19.2.1 *always hold.*
(b) Theorem 19.2.3 *always hold.*
(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$. *Then* Theorem 19.2.2 *holds.*
(d) *Let* $a^\star < \rho < b$. *Then* Theorem 19.2.2 *does not always hold.* $\square$

Proof Almost the same as the proof of Lemma 19.1.1(p.137). ∎

19.2.3 Diagonal Symmetry

For the same reason as in Section 19.1.3(p.140), which provides the six equalities and one corollary for $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$, we see that the following equalities and corollary hold for $\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]$ and $\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]$:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^- = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}], \tag{19.2.8}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^\pm = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^\pm\}], \tag{19.2.9}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^+ = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^-\}]. \tag{19.2.10}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}^+ = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^-\}], \tag{19.2.11}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}^\pm = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^\pm\}], \tag{19.2.12}$$

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}^- = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}]. \tag{19.2.13}$$

Corollary 19.2.1 *We have*:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^+ \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^-\}, \tag{19.2.14}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^\pm \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^\pm\}, \tag{19.2.15}$$

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}^- \ \mathtt{D}\text{-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]^+\}. \quad \square \tag{19.2.16}$$

19.2.4 M:2[ℝ][E]

19.2.4.1 Preliminary

From (6.5.28(p.31)) and (5.1.8) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 0. \tag{19.2.17}$$

19.2.4.2 Analysis

19.2.4.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 19.2.1** ($\mathscr{A}${M:2[ℝ][E]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let $\rho < b$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x) \cdots$ **(1)** from (5.1.4), we have $K(x) \geq 0 \cdots$ **(2)** for any x due to Lemma 9.1.1(p.41) (g).

(a) From (6.5.28) and (2) we obtain $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq b$. Then, since $b \leq V_0$ from (6.5.27), we have $b \leq V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 9.2.1(d), thus $V_t = \beta V_{t-1}$ for $t > 0$ from (19.2.17). Then, since $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ (see Preference Rule 7.2.1(p.35)).

(c) Let $\rho < b$. Then $V_0 < b \cdots$ **(3)**. Let $V_{t-1} < b$. Then, since $V_t < K(b) + b$ from (6.5.28) and Lemma 9.2.2(h), we have $V_t < \beta b - s = b$ from (9.2.7 (2) (p.43)) and the assumptions of $\beta = 1$ and $s = 0$. Hence, by induction $V_{t-1} < b$ for $t > 0$, so $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(d). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 0$ from (19.2.17) or equivalently $V_t > \beta V_{t-1}$ for $t > 0$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. ∎

19.2.4.2.2 Case of $\beta < 1$ or $s > 0$

For explanatory simplicity, let us define

$$\mathtt{S}_8\boxed{ⓢ\blacktriangle|⊛\parallel|⊛\triangle|⊛\blacktriangle} = \left\{\begin{array}{l}\text{For any } \tau > 0 \text{ there exists } t^\bullet_\tau > 0 \text{ such that}\\ (1)\ \boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 0}\langle \tau\rangle}_{\blacktriangle},\\ (2)\ \boxed{⊛\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\triangle},\\ (3)\ \boxed{⊛\ \texttt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\parallel}\ (\!(\boxed{⊛\ \texttt{ndOIT}_{\tau>t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\blacktriangle})\!).\end{array}\right\}.$$

Remark 19.2.1 $\mathtt{S}_8$ is the same as $\mathtt{S}_2$(p.122) except that the inequalities of $\tau > 1$, $t^\bullet_\tau > 1$, and $t^\bullet_\tau \geq \tau > 1$ in $\mathtt{S}_2$ changes into respectively $\tau > 0$, $t > 0$, and $t^\bullet_\tau \geq \tau > 0$ in $\mathtt{S}_8$. □

□ **Tom 19.2.2** ($\mathscr{A}${M:2[ℝ][E]}) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.
(c) *Let $\rho < x_L$.*
  1. $\boxed{ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
     i. *Let $a < \rho$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
     ii. *Let $\rho \leq a$.*
        1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
           i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$.
           ii. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
        2. *Let $(\lambda\mu - s)/\lambda > a$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
  3. *Let $\beta < 1$ and $s = 0$ (\!($s > 0$)\!).*
     i. *Let $a < \rho$.*
        1. *Let $b \geq 0$ (\!($\kappa \geq 0$)\!). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
        2. *Let $b < 0$ (\!($\kappa < 0$)\!). Then* $\mathtt{S}_8\boxed{ⓢ\blacktriangle|⊛\parallel|⊛\triangle|⊛\blacktriangle}$ *is true.*
     ii. *Let $\rho \leq a$.*
        1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
           i. *Let $\lambda = 1$.*
              1. *Let $b > 0$ (\!($\kappa > 0$)\!). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
              2. *Let $b \leq 0$ (\!($\kappa \leq 0$)\!). Then* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\triangle}$.
           ii. *Let $\lambda < 1$.*
              1. *Let $b \geq 0$ (\!($\kappa \geq 0$)\!). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
              2. *Let $b < 0$ (\!($\kappa < 0$)\!). Then* $\mathtt{S}_8\boxed{ⓢ\blacktriangle|⊛\parallel|⊛\triangle|⊛\blacktriangle}$ *is true.*

2. *Let* $(\lambda\beta\mu - s)/\delta > a$.
   i. *Let* $b \geq 0\,(\kappa \geq 0)$. *Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$
   ii. *Let* $b < 0\,(\kappa < 0)$. *Then* $\texttt{S}_8$ $\boxed{\circledS^{\blacktriangle}\,|\,\circledast^{\|}\,|\,\circledast^{\vartriangle}\,|\,\circledast^{\blacktriangle}}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots$ **(1)**. Then $V_0 < x_K \cdots$ **(2)** from (6.5.27(p.31)) and $K(\rho) > 0$ due to Lemma 9.2.2(p.43) (j1). Since $V_1 = K(\rho) + \rho \cdots$ **(3)** from (6.5.28) with $t = 1$, we have $V_1 - V_0 = V_1 - \rho = K(\rho) > 0$, hence $V_1 > V_0 \cdots$ **(4)**.

(a) Note (4), hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from Lemma 9.2.2(p.43) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$. Note again (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 9.2.2(f) we have $V_t < K(V_t) + V_t = V_{t+1}$. If $a < \rho$, then $a < V_0$, hence $a < V_{t-1}$ for $t > 0$ due to the nondecreasingness of V_t, so that from Lemma 9.2.2(g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \geq 0$. Consider a sufficiently large $M > 0$ with $\rho \leq M$ and $b \leq M$, hence $V_0 \leq M$. Suppose $V_{t-1} \leq M$. Then, from Lemma 9.2.2(e) we have $V_t \leq K(M) + M = \beta M - s$ due to (9.2.7 (2) (p.43)), hence $V_t \leq M$ due to the assumptions of $\beta \leq 1$ and $s \geq 0$. Accordingly, by induction $V_t \leq M$ for $t \geq 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Thus $V = K(V) + V$ from (6.5.28), hence $K(V) = 0$, so that $V = x_K$ due to Lemma 9.2.2(j1).

(b) Let $x_L \leq \rho$. Then, since $x_L \leq V_0$, we have $x_L \leq V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 9.2.1(a(p.43)), thus $V_t - \beta V_{t-1} \leq 0$ for $t > 0$ from (19.2.17) or equivalently $V_t \leq \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.

(c) Let $\rho < x_L \cdots$ **(5)**. Then $V_0 < x_L \cdots$ **(6)** from (6.5.27(p.31)), hence $L(V_0) > 0 \cdots$ **(7)** due to Corollary 9.2.1(a).

(c1) Since $V_1 - \beta V_0 = L(V_0) > 0$ from (19.2.17) with $t = 1$ and (7), we have $V_1 > \beta V_0 \cdots$ **(8)**, hence $t^*_1 = 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle} \cdots$ **(9)**. Below let $\tau > 1 \cdots$ **(10)**.

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$. Then $\delta = \lambda$ from (9.2.1(p.42)) and $x_L = x_K \cdots$ **(11)** from Lemma 9.2.3(p.44) (b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(12)**.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.5.27(p.31)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from Lemma 9.2.2(g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K \cdots$ **(13)** for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (11), we have $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(e1), hence for the same reason as in the proof of `Tom` 19.2.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2ii) Let $\rho \leq a \cdots$ **(14)**, hence $V_0 \leq a \cdots$ **(15)** from (6.5.27(p.31)). Then from (3) and (9.2.7 (1) (p.43)) we have $V_1 = \lambda\mu - s + (1-\lambda)\rho \cdots$ **(16)**

(c2ii1) Let $(\lambda\mu - s)/\lambda \leq a$. Then $x_K = (\lambda\mu - s)/\lambda \leq a \cdots$ **(17)** from Lemma 9.2.2(j2). Hence $K(a) \leq 0$ from Lemma 9.2.2(j1). Note (15). Suppose $V_{t-1} \leq a$. Then, from (6.5.28(p.31)) and Lemma 9.2.2(e) we have $V_t \leq K(a) + a \leq a$, hence by induction $V_{t-1} \leq a$ for $t > 0$. Accordingly, from (6.5.28) and (9.2.7 (1)) we have $V_t = \lambda\mu - s + (1-\lambda)V_{t-1} \cdots$ **(18)** for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, we have $x_K = \mu - s$ from (17) and $V_t = \mu - s$ for $t > 0$ from (18), hence $V_t = x_K$ for $t > 0$, so that $V_{t-1} = x_K$ for $t > 1$. Accordingly, $V_{t-1} = x_L$ for $t > 1$ due to (11). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (19.2.17) or equivalently $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau = \beta V_{\tau-1} \cdots = \beta^{\tau-1} V_1$ for $\tau > 1$. From the result and (4) we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\|}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from Lemma 9.2.2(f) and (12). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 9.2.1(e1). Thus, for the same reason as in the proof of `Tom` 19.2.1(c) we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a \cdots$ **(19)** from Lemma 9.2.2(j2), hence $x_L > a$ from (11). Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from Lemma 9.2.2(h) and (12). Accordingly, by induction $V_{t-1} < x_L \cdots$ **(20)** for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ due to Lemma 9.2.1(e1). Consequently, for the same reason as in the proof of `Tom` 19.2.1(c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3) Let $\beta < 1$ and $s = 0\,(s > 0)$.

(c3i) Let $a < \rho \cdots$ **(21)**. Then, since $a < V_0$ from (6.5.27(p.31)), we have $a < V_{t-1}$ for $t > 0$ due to (a).

(c3i1) Let $b \geq 0\,(\kappa \geq 0)$. Then $x_L \geq x_K \cdots$ **(22)** from Lemma 9.2.3(c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.28) and Lemma 9.2.2(g) we have $V_t < K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (22). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of `Tom` 19.2.1(c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3i2) Let $b < 0\,(\kappa < 0)$. Then $x_L < x_K \cdots$ **(23)** from Lemma 9.2.3(c (d)). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots$ **(24)**. Now, since $V = x_K$ due to (a), we have $x_L < V$ due to (23), which is a contradiction. Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. In addition, from (6) and the *strict increasingness* of V_t due to (a), it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau -1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau+1} < V_{t^\bullet_\tau+2} < \cdots.$$

from which we have

$$V_{t-1} < x_L,\ t^\bullet_\tau \geq t > 0, \quad x_L \leq V_{t^\bullet_\tau}, \quad x_L < V_{t-1},\ t > t^\bullet_\tau + 1. \tag{19.2.18}$$

Hence, we have

$$L(V_{t-1}) > 0, \qquad \cdots (25)\ t^\bullet_\tau \geq t > 0 \quad \text{(due to Corollary 9.2.1(a))}$$
$$L(V_{t^\bullet_\tau}) \leq 0, \qquad \cdots (26) \qquad \text{(due to Corollary 9.2.1(a))}$$
$$L(V_{t-1}) = (\!(< 0)\!),^\dagger \cdots (27)\ t > t^\bullet_\tau + 1 \quad \text{(due to Lemma 9.2.1(d(\!(e1)\!)))}$$

- Let $t^\bullet_\tau \geq \tau > 0$. Then $L(V_{t-1}) > 0 \cdots (28)$ for $\tau \geq t > 0$ from (25). Hence, for the same reason as in `Tom` 19.2.1(c) we obtain $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $t^\bullet_\tau \geq \tau > 0$. Accordingly, $\texttt{S}_8(1)$ is true. Now, since $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 0$ from (19.2.17) and (28), we have $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, hence $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0$. Accordingly, when $\tau = t^\bullet_\tau$, we have $V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau -1} > \cdots > \beta^{t^\bullet_\tau} V_0 \cdots (29)$.
- Let $\tau = t^\bullet_\tau + 1$. From (19.2.17) with $t = t^\bullet_\tau + 1$ and (26) we have $V_{t^\bullet_\tau+1} - \beta V_{t^\bullet_\tau} = L(t^\bullet_\tau) \leq 0$, hence $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$. Accordingly, from (29) we have

  $$V_{t^\bullet_\tau+1} \leq \boxed{\beta V_{t^\bullet_\tau}} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_0 \cdots (30),$$

  thus $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_\vartriangle$, so that $\texttt{S}_8(2)$ is true.
- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau+1}) = (\!(<)\!)\ 0$ from (27) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1}$ from (19.2.17), hence from (30) we have

  $$V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+1} \leq \boxed{\beta^2 V_{t^\bullet_\tau}} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+2} V_0$$

  Similarly we have

  $$V_{t^\bullet_\tau+3} = (\!(<)\!)\ \beta V_{t^\bullet_\tau+2} = (\!(<)\!)\ \beta^2 V_{t^\bullet_\tau+1} \leq \boxed{\beta^3 V_{t^\bullet_\tau}} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+3} V_0.$$

  By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain

  $$V_\tau = (\!(<)\!)\ \beta V_{\tau-1} = (\!(<)\!)\ \cdots = (\!(<)\!)\ \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = (\!(<)\!)$$
  $$\beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \boxed{\beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^\tau V_0. \cdots (31)$$

  - Let $s = 0$. Then (31) can be written as

    $$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \boxed{\beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^\tau V_0,$$

    hence $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_{\parallel}$ (see Preference Rule 7.2.1(p.35)), hence $\texttt{S}_8(3)$ is true.
  - Let $s > 0$. Then (31) can be written as

    $$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} < \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \boxed{\beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^\tau V_0, \tag{19.2.19}$$

    hence $t^*_\tau = t^\bullet_\tau$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau+1}\langle t^\circ\rangle}_\blacktriangle$, hence $\texttt{S}_8(3)$ is true.

(c3ii) Let $\rho \leq a \cdots (32)$, hence $V_0 \leq a \cdots (33)$ from (6.5.27(p.31)). Then, from (3) and (9.2.7 (1) (p.43)) we have $V_1 = \lambda\beta\mu - s + (1-\lambda)\beta\rho \cdots (34)$.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \leq a$. Then $x_K = (\lambda\beta\mu - s)/\delta \leq a \cdots (35)$ from Lemma 9.2.2(j2(p.43)). Hence $V_1 = \delta x_K + (1 - \lambda)\beta\rho \cdots (36)$.

(c3ii1i) Let $\lambda = 1$, hence $\delta = 1$ from (9.2.1(p.42)). Thus, from (35) and (36) we have $x_K = \beta\mu - s \leq a$ and $V_1 = x_K \leq a \cdots (37)$.

(c3ii1i1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K \cdots (38)$ due to Lemma 9.2.3(c (\!(d)\!)). Note (37). Suppose $V_{t-1} = x_K$. Then, from (6.5.28(p.31)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (38), thus $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 9.2.1(a). Hence, from (7) we obtain $L(V_{t-1}) > 0$ for $t > 0$. Accordingly, for almost the same reason as in the proof of `Tom` 19.2.1(c) we obtain $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.

(c3ii1i2) Let $b \leq 0\,(\!(\kappa \leq 0)\!)$. Then, since $x_L \leq x_K$ from Lemma 9.2.3(c (\!(d)\!)), we have $V_1 \geq x_L$ from (37), hence $V_{t-1} \geq x_L$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 9.2.1(a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, thus $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (19.2.17), i.e., $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$. Hence $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1$. Now, from (6.5.27(p.31)), (4), (37), and (38) we have $\rho = V_0 < V_1 = x_K < x_L$, hence $L(\rho) > 0$ from Corollary 9.2.1(p.43) (a). In addition, from (3) and (6.5.27(p.31)) we have $V_1 - \beta V_0 = V_1 - \beta\rho = K(\rho) + \rho - \beta\rho = K(\rho) + (1-\beta)\rho = L(\rho) > 0$ from (5.1.8(p.17)), hence $V_1 > \beta V_0$. Accordingly, we have $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \boxed{\beta^{\tau-1} V_1} > \beta^\tau V_0$ for $\tau > 1$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$.

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† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

(c3ii1ii) Let $\lambda < 1$.

(c3ii1ii1) Let $b \geq 0\,(\kappa \geq 0)$. Then $x_L \geq x_K \cdots (39)$ from Lemma 9.2.3(c(d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from Lemma 9.2.2(f) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (39). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of Tom 19.2.1(c) we obtain $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3ii1ii2) Let $b < 0\,(\kappa < 0)$. Then $x_L < x_K \cdots (40)$ from Lemma 9.2.3(c(d)). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots (41)$. Now, since $V = x_K$ from (a), we have the contradiction $x_L < V$ from (40). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From the result and the strict increasingness of V_t due to (a), it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < V_{t^\bullet_\tau + 2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we have $\mathsf{S}_8 \boxed{Ⓢ\blacktriangle | ⊛\parallel | ⊛\triangle | ⊛\blacktriangle}$.

(c3ii2) Let $(\lambda\beta\mu - s)/\lambda > a \cdots (42)$. Then $x_K > (\lambda\beta\mu - s)/\delta > a$ from Lemma 9.2.2(j2).

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \geq 0$ due to (a).
2. Let $\lambda = 1$, hence $\delta = 1$ from (9.2.1(p.42)), so $\beta\mu - s > a$ from (42). Now, since $K(x) \geq \beta\mu - s - x$ for any x from (9.2.4(p.42)) or equivalently $K(x) + x \geq \beta\mu - s$ for any x, we have $V_1 \geq \beta\mu - s > a$ from (3). Accordingly $V_{t-1} > a$ for $t > 1$ due to (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 9.2.2(g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \geq 0$.

From the above, whether $\lambda < 1$ or $\lambda = 1$, we see that V_t is *strictly increasing* in $t > 0$.

(c3ii2i) Let $b \geq 0\,(\kappa \geq 0)$. Then $x_L \geq x_K \cdots (43)$ from Lemma 9.2.2(c(d)). From (2) and the above strict increasingness of V_t in $t \geq 0$ we have $V_{t-1} < V = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (43). Thus, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of Tom 19.2.1(c) we obtain $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c3ii2ii) Let $b < 0\,(\kappa < 0)$. Then $x_L < x_K \cdots (44)$ from Lemma 9.2.3(c(d)). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots (45)$. Now, since $V = x_K$ from (a), we have $x_L < V$ from (44), which is a contradiction. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From the result, (6), and the above strict increasingness of V_t in $t \geq 0$ it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < V_{t^\bullet_\tau + 2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we can immediately see that the assertion holds true. ▮

□ **Tom 19.2.3** ($\mathscr{A}\{$M:2[ℝ][E]$\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) $V_t = x_K = \rho$ *for* $t \geq 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.
  1. *Let* $b > 0\,(\kappa > 0)$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
  2. *Let* $b \leq 0\,(\kappa \leq 0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$. □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Hence $V_0 = \rho = x_K \cdots (1)$ from (6.5.27(p.31)).

(a) Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.5.28) we have $V_t = K(x_K) + x_K = x_K$. Hence, by induction $V_t = x_K = \rho$ for $t \geq 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$ in the lemma. Then $x_L = x_K$ from Lemma 9.2.3(p.44)(b). Accordingly, since $V_{t-1} = x_L$ for $t > 0$ from (a), we have $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$, hence for the same reason as in the proof of Tom 19.2.1(p.178)(b) we obtain $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) Let $\beta < 1$ and $s = 0\,(s > 0)$.

(c1) Let $b > 0\,(\kappa > 0)$. Then, since $x_L > x_K$ from Lemma 9.2.3(c(d)), we have $x_L > x_K = V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 9.2.1(a), thus for the same reason as in the proof of Tom 19.2.1(c) we obtain $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.

(c2) Let $b \leq 0\,(\kappa \leq 0)$. Then $x_L \leq x_K$ from Lemma 9.2.3(c(d)). Hence, since $x_L \leq x_K = V_{t-1}$ for $t > 0$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 9.2.1(a), hence $V_t - \beta V_{t-1} \leq 0$ for $t > 0$ from (19.2.17(p.178)) or equivalently $V_t \leq \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 0$, we have $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^\tau V_0$, thus $t^*_\tau = 0$ for $\tau > 0$, i.e., $\mathtt{dOIT}_{\tau>0}\langle 0\rangle_{\triangle}$. ▮

$$\mathsf{S}_9 \boxed{Ⓢ\triangle | \bullet\triangle | \bullet\blacktriangle} = \left\{ \begin{array}{l} \text{For any } \tau > 0 \text{ there exists } t^\bullet > 0 \text{ such that} \\ (1)\ \boxed{\bullet\ \mathtt{dOITd}_{t^\bullet \geq \tau > 0}\langle 0\rangle}_{\triangle}\ (\boxed{\bullet\ \mathtt{dOITd}_{t^\bullet \geq \tau > 0}\langle 0\rangle}_{\blacktriangle}), \\ (2)\ \boxed{Ⓢ\ \mathtt{dOITs}_{\tau > t^\bullet}\langle\tau\rangle}_{\triangle} \text{ or } \boxed{\bullet\ \mathtt{dOITd}_{\tau > t^\bullet}\langle 0\rangle}_{\triangle}. \end{array} \right\}$$

□ **Tom 19.2.4** ($\mathscr{A}${M:2[ℝ][E]}) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then, for a given starting time $\tau > 0$:*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as to $\to \infty$.*

(b) *Let $\rho < x_L$. Then* [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.

(c) *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_\|$ *and* [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.

(d) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\vartriangle$.
2. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
   i. *Let $b \le 0\,(\!(\kappa \le 0)\!)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_\vartriangle$ (\!([● dOITd$_{\tau>0}\langle 0\rangle$]$_\blacktriangle$)\!) .
   ii. *Let $b > 0\,(\!(\kappa > 0)\!)$. Then* S_9 [Ⓢ△ | ●△ | ●▲] *is true.* □

Proof Let $\beta < 1$ or $s > 0$ and let $\rho > x_K \cdots$ **(1)**. Then $V_0 > x_K \cdots$ **(2)** from (6.5.27(p.31)) and $K(\rho) < 0 \cdots$ **(3)** from Lemma 9.2.2(j1). From (6.5.28(p.31)) with $t = 1$ we have $V_1 - V_0 = K(V_0) = K(\rho) < 0 \cdots$ **(4)**, hence $V_1 < V_0 \cdots$ **(5)**. In addition, from (19.2.17) with $t = 1$ we have $V_1 - \beta V_0 = L(V_0) = L(\rho) \cdots$ **(6)**.

(a) Note (5), hence $V_0 \ge V_1$. Suppose $V_{t-1} \ge V_t$. Then, from Lemma 9.2.2(e) we have $V_t \ge K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 0$, i.e., V_t is nonincreasing in $t \ge 0$. Let $\lambda < 1$. Note again (5). Suppose $V_{t-1} > V_t$. Then, from Lemma 9.2.2(f) we have $V_t > K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} > V_t$ for $t > 0$, i.e., V_t is *strictly decreasing* in $t \ge 0$. Note (2), hence $V_0 \ge x_K$. Suppose $V_{t-1} \ge x_K$. Then, from Lemma 9.2.2(e) we have $V_t \ge K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \ge x_K \cdots$ **(7)** for $t > 0$, i.e., V_t is lower bounded in t. Thus, it follows that V_t converges to a finite V as $t \to \infty$. Hence, since $V = K(V) + V$, we have $K(V) = 0$, thus $V = x_K$ due to Lemma 9.2.2(j1).

(b) Let $\rho < x_L$. Then, since $V_0 < x_L$ from (6.5.27(p.31)), we have $V_{t-1} < x_L$ for $t > 0$ due to (a). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 9.2.1(a), for the same reason as in the proof of Tom 19.2.1(c) we obtain [Ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$.

(c) Let $\rho = x_L \cdots$ **(8)**. Then, since $L(\rho) = L(x_L) = 0$, we have $V_1 = \beta V_0 \cdots$ **(9)** from (6), hence [● dOITd$_1\langle 0\rangle$]$_\|$. Below, let $\tau > 1$. Now, since $V_1 = K(\rho) + \rho < \rho$ from (6.5.28(p.31)) with $t = 1$ and (3), we have $V_{t-1} < \rho$ for $t > 1$ from (a), hence $V_{t-1} < x_L$ for $t > 1$ due to (8), so that $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 9.2.1(p.43)(a). Accordingly, since $L(V_{t-1}) > 0$ for $\tau \ge t > 1$, we have $V_t - \beta V_t > 0$ for $\tau \ge t > 1$ due to (19.2.17) or equivalently $V_t > \beta V_t$ for $\tau \ge t > 1$, from which we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. Hence, from (9) we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$. Accordingly, we obtain $t^*_\tau = \tau$ for $\tau > 1$, i.e., [Ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_\blacktriangle$.

(d) Let $x_L < \rho \cdots$ **(10)**, hence $x_L < V_0 \cdots$ **(11)** from (6.5.27(p.31)). Thus, if $s = (\!(>)\!)\ 0$, then $L(V_0) = (\!(<)\!)\ 0 \cdots$ **(12)** from Lemma 9.2.1(d(\!(e1)\!)), hence $V_1 = (\!(<)\!)\ \beta V_0 \cdots$ **(13)** from (6).

(d1) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$. Then $L(V_0) < 0$ from (12), hence $V_1 < \beta V_0 \cdots$ **(14)** from (19.2.17(p.178)). Now, since $x_L = x_K$ due to Lemma 9.2.3(b), from (7) we have $V_{t-1} \ge x_L$ for $t > 0$, hence $L(V_{t-1}) \le 0$ for $t > 0$ due to Lemma 9.2.1(e1), thus $V_t - \beta V_{t-1} \le 0$ for $t > 0$ from (19.2.17). Then, since $V_t - \beta V_{t-1} \le 0$ for $\tau \ge t > 0$, we have $V_t \le \beta V_{t-1}$ for $\tau \ge t > 0$, leading to $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \le \beta^\tau V_0$. Hence we have $t^*_\tau = 0$ for $\tau > 0$, i.e., [● dOITd$_{\tau>0}\langle 0\rangle$]$_\vartriangle$.

(d2) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(d2i) Let $b \le 0\,(\!(\kappa \le 0)\!)$. Then $x_L \le x_K \cdots$ **(15)** due to Lemma 9.2.3(c(\!(d)\!)). Hence, from (7) we have $V_{t-1} \ge x_L$ for $t > 0$, hence $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 9.2.1(a), so that $V_t - \beta V_{t-1} \le 0$ for $t > 0$ from (19.2.17). Then, since $V_t - \beta V_{t-1} \le 0$ for $\tau \ge t > 0$, we have $V_t \le \beta V_{t-1}$ for $\tau \ge t > 0$, leading to $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \le \beta^\tau V_0$. Due to (13) the inequality can be rewritten as $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 = (\!(<)\!)\ \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., [● dOITd$_{\tau>0}\langle 0\rangle$]$_\vartriangle$ (\!([● dOITd$_{\tau>0}\langle 0\rangle$]$_\blacktriangle$)\!) .

(d2ii) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots$ **(16)** from Lemma 9.2.3(c(\!(d)\!)). Hence, from (5) and (11) and from the nonincreasingness of V_t, and the convergency of V_t to $V = x_K$ due to (a) we see that there exists $t^\bullet > 0$ such that

$$V_0 > V_1 \ge V_2 \ge \cdots \ge V_{t^\bullet-1} \ge x_L > V_{t^\bullet} \ge V_{t^\bullet+1} \ge \cdots \to x_K \cdots \textbf{(17)}$$

or equivalently $V_0 > x_L$, $V_{t-1} \ge x_L$ for $t^\bullet \ge t > 1$, and $x_L > V_{t-1}$ for $t > t^\bullet$. Hence, we have

$$\begin{aligned} &L(V_{t-1}) > 0, \quad t > t^\bullet, && \text{due to Corollary 9.2.1(a)},\\ &L(V_{t-1}) \le 0, \quad t^\bullet \ge t > 1, && \text{due to Corollary 9.2.1(a)},\\ &L(V_0) = (\!(<)\!)\ 0 && \text{due to Lemma 9.2.1(p.43) (d(\!(e1)\!))}. \end{aligned}$$

Hence, from (19.2.17) we have

$$V_t > \beta V_{t-1} \cdots \textbf{(18)}, \quad t > t^\bullet, \qquad V_t \le \beta V_{t-1} \cdots \textbf{(19)}, \quad t^\bullet \ge t > 1, \qquad V_1 = (\!(<)\!)\ \beta V_0 \cdots \textbf{(20)}.$$

⟨A⟩ Let $t^\bullet \ge \tau > 0$. Then, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$ from (19), we have $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1$, hence $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 = (\!(<)\!)\ \beta^\tau V_0 \cdots$ **(21)** from (20) or equivalently

$$I^\tau_\tau \le I^{\tau-1}_\tau \le \cdots \le I^1_\tau = (\!(<)\!)\ I^0_\tau \cdots \textbf{(22)}, \quad t^\bullet \ge \tau > 0.$$

Thus $t^*_\tau = 0$ for $t^\bullet \ge \tau > 0$, i.e., [● dOITd$_{t^\bullet_\tau \ge \tau > 0}\langle 0\rangle$]$_\vartriangle$ (\!([● dOITd$_{t^\bullet_\tau \ge \tau > 0}\langle 0\rangle$]$_\blacktriangle$)\!), hence (1) of S_9 holds. Now, from (21) with $\tau = t^\bullet$ we have

$$V_{t^\bullet} \le \beta V_{t^\bullet-1} \le \cdots \le \beta^{t^\bullet-1} V_1 = (\!(<)\!)\ \beta^{t^\bullet} V_0 \cdots \textbf{(23)}.$$

⟨B⟩ Let $\tau > t^\bullet$ (> 0), hence $\tau > 1$. From (18) with $\tau \geq t > t^\bullet$ we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet-1} V_{t^\bullet+1} > \beta^{\tau-t^\bullet} V_{t^\bullet_\tau} \cdots (24), \quad \tau > t^\bullet.$$

Combining (24) and (23) leads to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet-1} V_{t^\bullet+1} > \beta^{\tau-t^\bullet} V_{t^\bullet} \leq \beta^{\tau-t^\bullet+1} V_{t^\bullet-1} \leq \cdots \leq \beta^{\tau-1} V_1 = (\!(<)\!)\ \beta^\tau V_0, \quad \tau > t^\bullet,$$

or equivalently

$$I_\tau^\tau > I_\tau^{\tau-1} > I_\tau^{\tau-2} > \cdots > I_\tau^{t^\bullet+1} > I_\tau^{t^\bullet} \leq I_\tau^{t^\bullet-1} \leq \cdots \leq I_\tau^1 = (\!(<)\!)\ I_\tau^0 \cdots (25), \quad \tau > t^\bullet.$$

Hence we have ⓢ dOITs$_\tau\langle\tau\rangle$ or ● dOITd$_\tau\langle 0\rangle$, so that hence (2) of S$_9$ holds. ∎

19.2.4.3 Market Restriction

19.2.4.3.1 Positive Restriction

19.2.4.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 19.2.1** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \geq b$. Then* ● dOITd$_{\tau>0}\langle 0\rangle_{\parallel}$ → → ⓓ
(c) *Let $\rho < b$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ

Proof The same as Tom 19.2.1(p.178) due to Lemma 16.4.1(p.100). ∎

19.2.4.3.1.2 Case of $\beta < 1$ or $s > 0$

□ **Pom 19.2.2** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a \leq \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* ● dOITd$_{\tau>0}\langle 0\rangle_{\triangle}$ → → ⓓ
(c) *Let $\rho < x_L$.*
  1. ⓢ dOITs$_1\langle 1\rangle_{\blacktriangle}$. *Below let $\tau > 1$* → → ⓢ
  2. *Let $\beta = 1$.*
     i. *Let $a \leq \rho$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
     ii. *Let $\rho < a$.*
        1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
           i. *Let $\lambda = 1$. Then* ⊛ ndOIT$_{\tau>1}\langle 1\rangle_{\parallel}$ → → ⊛
           ii. *Let $\lambda < 1$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
        2. *Let $(\lambda\mu - s)/\lambda > a$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
  3. *Let $\beta < 1$ and $s = 0$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
  4. *Let $\beta < 1$ and $s > 0$.*
     i. *Let $a \leq \rho$.*
        1. *Let $\lambda\beta\mu \geq s$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
        2. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.178) ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲ *is true* → → ⓢ / ⊛
     ii. *Let $\rho < a$.*
        1. *Let $(\lambda\beta\mu - s)/\delta \leq a$.*
           i. *Let $\lambda = 1$.*
              1. *Let $\beta\mu > s$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
              2. *Let $\beta\mu \leq s$. Then* ⊛ ndOIT$_{\tau>1}\langle 1\rangle_{\triangle}$ → → ⊛
           ii. *Let $\lambda < 1$.*
              1. *Let $\lambda\beta\mu \geq s$. Then* ⓢ dOITs$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
              2. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.178) ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲ *is true* → → ⓢ / ⊛
        2. *Let $(\lambda\beta\mu - s)/\delta > a$.*
           i. *Let $\lambda\beta\mu \geq s$. Then* ⓢ dOITs$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ → → ⓢ
           ii. *Let $\lambda\beta\mu < s$. Then* S$_8$(p.178) ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲ *is true* → → ⓢ/⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a-c2ii2) The same as Tom 19.2.2(p.178) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1,c3ii1i1,c3ii1ii1,c3ii2i) of Tom 19.2.2.

(c4-c4ii2ii) Let $\beta < 1$ and $s < 0$. Then, due to (2) it suffices to consider only (c3-c3ii2ii) of Tom 19.2.2 with κ. ∎

☐ Pom **19.2.3** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) $V_t = x_K = \rho$ *for* $t > 0$.
(b) *Let* $\beta = 1$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
(d) *Let* $\beta < 1$ *and* $s > 0$.
  1. *Let* $\lambda\beta\mu > s$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
  2. *Let* $\lambda\beta\mu \le s$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ → → ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a,b) The same as Tom 19.2.3(p.181) (a,b).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 19.2.3.

(d,d2) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (c1,c2) of Tom 19.2.3. ∎

☐ Pom **19.2.4** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.
(b) *Let* $\rho < x_L$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
(c) *Let* $\rho = x_L$. *Then* [● dOITd$_1\langle 0\rangle$]$_{\|}$ *and* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$ *for* $\tau > 1$ → → ⓢ/ⓓ
(d) *Let* $\rho > x_L$.
  1. *Let* $\beta = 1$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ → → ⓓ
  2. *Let* $\beta < 1$ *and* $s = 0$. *Then* S$_9$(p.181) [ⓢ△ | ●△ | ●▲] *is true* → → ⓢ/ⓓ
  3. *Let* $\beta < 1$ *and* $s > 0$.
    i. *Let* $\lambda\beta\mu \le s$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\blacktriangle}$ → → ⓓ
    ii. *Let* $\lambda\beta\mu > s$. *Then* S$_9$(p.181) [ⓢ△ | ●△ | ●▲] *is true* (see Numerical Example 19.2.1(p.206)) → → ⓢ/ⓓ

Proof Suppose $a > 0$. Then $b > a > 0 \cdots (1)$. We have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a-d1) The same as Tom 19.2.4(a-d1).

(d2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (d2ii) of Tom 19.2.4.

(d3,d3ii) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (d2i,d2ii) of Tom 19.2.4 with κ. ∎

19.2.4.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.2.4.3.3 Negative Restriction

19.2.4.3.3.1 Case of $\beta = 1$ and $s = 0$

☐ Nem **19.2.1** ($\mathscr{A}${M:2[ℝ][E]$^-$}) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) V_t *is nondecreasing in* $t \ge 0$.
(b) *Let* $\rho \ge b$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\|}$ → → ⓓ
(c) *Let* $\rho < b$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ

Proof The same as Tom 19.2.1(p.178) due to Lemma 16.4.1(p.100). ∎

19.2.4.3.3.2 Case of $\beta < 1$ or $s > 0$

☐ Nem **19.2.2** ($\mathscr{A}${M:2[ℝ][E]$^-$}) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $a \le \rho$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
(b) *Let* $x_L \le \rho$. *Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ → → ⓓ
(c) *Let* $\rho < x_L$.
  1. [ⓢ dOITs$_1\langle 1\rangle$]$_{\blacktriangle}$. *Below let* $\tau > 1$ → → ⓢ
  2. *Let* $\beta = 1$.
    i. *Let* $a \le \rho$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
    ii. *Let* $\rho < a$.
      1. *Let* $(\lambda\mu - s)/\lambda \le a$.
        i. *Let* $\lambda = 1$. *Then* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\|}$ → → ⊛
        ii. *Let* $\lambda < 1$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
      2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
  3. *Let* $\beta < 1$ *and* $s = 0$. S$_8$(p.178) [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true* → → ⓢ/⊛
  4. *Let* $\beta < 1$.
    i. *Let* $a \le \rho$. S$_8$(p.178) [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true* → → ⓢ/⊛
    ii. *Let* $\rho < a$.

1. *Let* $(\lambda\beta\mu - s)/\delta \le a$.
   i. *Let* $\lambda = 1$. *Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$ $\rightarrow$ $\rightarrow \circledast$
   ii. *Let* $\lambda < 1$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ\blacktriangle \mid \circledast \parallel \mid \circledast \vartriangle \mid \circledast \blacktriangle}$ *is true* $\rightarrow$
2. *Let* $(\lambda\beta\mu - s)/\delta > a$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ\blacktriangle \mid \circledast \parallel \mid \circledast \vartriangle \mid \circledast \blacktriangle}$ *is true* $\rightarrow$ $\rightarrow$ ⓢ / $\circledast$

Proof Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\kappa = -s \cdots (3)$ from Lemma 9.3.1(p.45) (a).

(a-c2ii2) The same as `Tom` 19.2.2(p.178) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu - s)/\delta \le a$. Then, since $\lambda\beta\mu/\delta \le a$, we have $\lambda\beta\mu \le \delta a$ due to (9.2.2 (1) (p.42)), hence $\lambda\beta\mu \le \delta a \le \lambda a$ due to (9.2.2 (1) (p.42)) and (2), so $\beta\mu \le a$, which contradicts [15(p.101)]. Thus, it must be that $(\lambda\beta\mu - s)/\delta > a$. From this it suffices to consider only (c3i2,c3ii2ii) of `Tom` 19.2.2(p.178).

(c4-c4ii2) Let $\beta < 1$ and $s > 0$. Then $\kappa < 0$ due to (3), hence it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of `Tom` 19.2.2. ∎

☐ **`Nem` 19.2.3** ($\mathscr{A}\{$M:2[$\mathbb{R}$][E]$^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t > 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ ❶
(c) *Let* $\beta < 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\rightarrow$ $\rightarrow$ ❶

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a,b) The same as `Tom` 19.2.3(p.181) (a,b).

(c) If $s = 0$, then due to (1) it suffices to consider only (c2) of `Tom` 19.2.3 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of `Tom` 19.2.3. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ **`Nem` 19.2.4** ($\mathscr{A}\{$M:2[$\mathbb{R}$][E]$^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.
(b) *Let* $\rho < x_L$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ ⓢ
(c) *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$ *for* $\tau > 1$ $\rightarrow$ $\rightarrow$ ❶/ⓢ
(d) *Let* $\rho > x_L$.
   1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\rightarrow$ $\rightarrow$ ❶
   2. *Let* $\beta < 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$) $\rightarrow$ $\rightarrow$ ❶

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a-d1) The same as `Tom` 19.2.4(p.182) (a-d1).

(d2) If $s = 0$, then due to (1) it suffices to consider only (d2i) of `Tom` 19.2.4 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (d2i) of `Tom` 19.2.4. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.2.5 $\tilde{\text{M}}$:2[$\mathbb{R}$][E]

Due to Lemma 19.2.1(p.177) (a) we can apply $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ in Theorem 19.2.1(p.177) to `Tom`'s 19.2.1(p.178) − 19.2.4.

19.2.5.1 Analysis

19.2.5.1.1 Case of $\beta = 1$ and $s = 0$

☐ **`Tom` 19.2.5** ($\mathscr{A}\{\tilde{\text{M}}$:2[$\mathbb{R}$][E]$\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \ge 0$.
(b) *Let* $\rho \le a$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.
(c) *Let* $\rho > a$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. ☐

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 19.2.1(p.178). ∎

19.2.5.1.2 Case of $\beta < 1$ or $s > 0$

☐ **`Tom` 19.2.6** ($\mathscr{A}\{\tilde{\text{M}}$:2[$\mathbb{R}$][E]$\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $b > \rho$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \ge \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
(c) *Let* $\rho > x_{\tilde{L}}$.
   1. $\boxed{ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.
   2. *Let* $\beta = 1$.
      i. *Let* $b > \rho$. *Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
      ii. *Let* $\rho \ge b$.

1. *Let* $(\lambda\mu + s)/\lambda \geq b$.

i. *Let* $\lambda = 1$. *Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\rightarrow$ $\rightarrow$ Ⓢ

ii. *Let* $\lambda < 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$ $\rightarrow$ $\rightarrow$ Ⓢ

2. *Let* $(\lambda\mu + s)/\lambda < b$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.

i. *Let* $b > \rho$.

1. *Let* $a \leq 0\,(\tilde{\kappa} \leq 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

2. *Let* $a > 0\,(\tilde{\kappa} > 0)$. *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid \circledast\parallel \mid \circledast\triangle \mid \circledast\blacktriangle}$ *is true.*

ii. *Let* $\rho \geq b$.

1. *Let* $(\lambda\beta\mu + s)/\delta \geq b$.

i. *Let* $\lambda = 1$.

1. *Let* $a < 0\,(\tilde{\kappa} < 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

2. *Let* $a \geq 0\,(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\triangle}$.

ii. *Let* $\lambda < 1$.

1. *Let* $a \leq 0\,(\tilde{\kappa} \leq 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

2. *Let* $a > 0\,(\tilde{\kappa} > 0)$. *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid \circledast\parallel \mid \circledast\triangle \mid \circledast\blacktriangle}$ *is true.*

2. *Let* $(\lambda\beta\mu + s)/\delta < b$.

i. *Let* $a \leq 0\,(\tilde{\kappa} \leq 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

ii. *Let* $a > 0\,(\tilde{\kappa} > 0)$. *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid \circledast\parallel \mid \circledast\triangle \mid \circledast\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 19.2.2(p.178). ∎

□ `Tom` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.

1. *Let* $a < 0\,(\tilde{\kappa} < 0)$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

2. *Let* $a \geq 0\,(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 19.2.3(p.181). ∎

□ `Tom` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\parallel}$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.

1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.

2. *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.

i. *Let* $a \geq 0\,(\tilde{\kappa} \geq 0)$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$ $\left(\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}\right)$.

ii. *Let* $a < 0\,(\tilde{\kappa} < 0)$. *Then* $\texttt{S}_9$ $\boxed{Ⓢ\triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 19.2.4(p.182). ∎

19.2.5.2 Market Restriction

19.2.5.2.1 Positive Restriction

19.2.5.2.1.1 Case of $\beta = 1$ and $s = 0$

□ `Pom` **19.2.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.

(b) *Let* $\rho \leq a$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$. $\rightarrow$ $\rightarrow$ ⓓ

(c) *Let* $\rho > a$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. $\rightarrow$ $\rightarrow$ Ⓢ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to `Nem` 19.2.1(p.184) (see (17.1.22(p.113))). ∎

Direct proof The same as `Tom` 19.2.5(p.185) due to Lemma 16.4.1(p.100). ∎

19.2.5.2.1.2 Case of $\beta < 1$ or $s > 0$

☐ Pom **19.2.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$ or $b \ge \rho$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \ge \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$. → → ⓓ
(c) *Let $\rho > x_{\tilde{L}}$.*
  1. $\boxed{Ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let $\tau > 1$.* → → Ⓢ
  2. *Let $\beta = 1$.*
    i. *Let $b \ge \rho$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle \tau\rangle}_\blacktriangle$. → → Ⓢ
    ii. *Let $\rho > b$.*
      1. *Let $(\lambda\mu + s)/\lambda \ge b$.*
        i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\parallel$. → → ⊛
        ii. *Let $\lambda < 1$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$. → → Ⓢ
      2. *Let $(\lambda\mu + s)/\lambda < b$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$. → Ⓢ
  3. *Let $\beta < 1$ and $s = 0$. Then we have* $\mathtt{S}_8$(p.178) [Ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲] → → Ⓢ / ⊛
  4. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $b \ge \rho$. Then* $\mathtt{S}_8$(p.178) [Ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲] *is true.* → → Ⓢ / ⊛
    ii. *Let $\rho > b$.*
      1. *Let $(\lambda\beta\mu + s)/\delta \ge b$.*
        i. *Let $\lambda = 1$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$. → → ⊛
        ii. *Let $\lambda < 1$. Then* $\mathtt{S}_8$(p.178) [Ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲] *is true.* → → Ⓢ / ⊛
      2. *Let $(\lambda\beta\mu + s)/\delta < b$. Then* $\mathtt{S}_8$(p.178) [Ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲] *is true.* ☐ → → Ⓢ / ⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to Nem 19.2.2(p.184) (see (17.1.22(p.113))). ∎

Direct proof Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$. Then $\tilde{\kappa} = s \cdots (3)$ from Lemma 11.6.6(p.68) (a).

(a-c2ii2) The same as Tom 19.2.6(p.185) (a-c2ii2).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu + s)/\delta \ge b$. Then, since $\lambda\beta\mu/\delta \ge b$, we have $\lambda\beta\mu \ge \delta b$ from (9.2.2 (1) (p.42)), hence $\lambda\beta\mu \ge \delta b \ge \lambda b$ due to (2), so $\beta\mu \ge b$, which contradicts [3(p.101)]. Thus, it must be that $(\lambda\beta\mu + s)/\delta < b$. From this it suffices to consider only (c3i2,c3ii2ii) of Tom 19.2.6(p.185).

(c4-c4ii2) Let $\beta < 1$ and $s > 0$. Then $\kappa < 0$ due to (3), hence it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of Tom 19.2.2 with κ. ∎

☐ Pom **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$ → → ⓓ
(c) *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to Nem 19.2.3(p.185) (see (17.1.22(p.113))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 11.6.6(p.68) (a).

(a,b) The same as Tom 19.2.7(p.186) (a,b).

(c) If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.2.7 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.2.7 with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ Pom **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*
(b) *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>0}\langle \tau\rangle}_\blacktriangle$ → → Ⓢ
(c) *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\parallel$ *and* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau>1}\langle \tau\rangle}_\blacktriangle$ → → ⓓ/Ⓢ
(d) *Let $\rho < x_{\tilde{L}}$.*
  1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ → → ⓓ
  2. *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$) → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to Nem 19.2.4(p.185) (see (17.1.22(p.113))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 11.6.6(p.68) (a).

(a-d1) The same as Tom 19.2.8(p.186) (a-d1).

(d2) If $s = 0$, due to (1) it suffices to consider only (d2i) of Tom 19.2.8 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of Tom 19.2.8(d2i) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

Remark 19.2.2 (diagonal symmetry) The diagonal symmetry holds between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^{+}\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^{-}\}$, i.e.,

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^{+}\}\ \texttt{D-}\!\sim\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^{-}\}. \tag{19.2.20}$$

In fact it can be confirmed that the following relations hold:

$$\begin{aligned}
&\mathscr{A}\{\texttt{Pom}\ 19.2.5(\text{p.186})\} &&= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\texttt{Nem}\ 19.2.1(\text{p.184})\}] \quad \cdots(1),\\
&\mathscr{A}\{\texttt{Pom}\ 19.2.6(\text{p.187})\} &&= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\texttt{Nem}\ 19.2.2(\text{p.184})\}] \quad \cdots(2),\\
&\mathscr{A}\{\texttt{Pom}\ 19.2.7(\text{p.187})\} &&= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\texttt{Nem}\ 19.2.3(\text{p.185})\}] \quad \cdots(3),\\
&\mathscr{A}\{\texttt{Pom}\ 19.2.8(\text{p.187})\} &&= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\texttt{Nem}\ 19.2.4(\text{p.185})\}] \quad \cdots(4).\ \Box
\end{aligned}$$

19.2.5.2.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.2.5.2.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

19.2.6 M:2[ℙ][E]

19.2.6.1 Preliminary

From (6.5.33(p.31)) and from (5.1.21(p.18)) and (5.1.20) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 1. \tag{19.2.21}$$

From (6.5.32) we have

$$V_1 - \beta V_0 = V_1 - \beta\rho = \lambda\beta \max\{0, a-\rho\} - s. \tag{19.2.22}$$

19.2.6.2 Analysis

19.2.6.2.1 Case of $\beta = 1$ and $s = 0$

Let $\beta = 1$ and $s = 0$. Then, from (19.2.21) and (5.1.20(p.18)) we have

$$V_t - \beta V_{t-1} = \lambda T(V_{t-1}) \geq 0, \quad t > 1, \tag{19.2.23}$$

due to Lemma 12.2.1(p.77) (g). From (6.5.32) we have

$$\begin{aligned}
V_1 &= \lambda \max\{0, a-\rho\} + \rho && (19.2.24)\\
&= \max\{\rho, \lambda a + (1-\lambda)\rho\}. && (19.2.25)
\end{aligned}$$

19.2.6.2.1.1 Case of $\rho \leq a^\star$

In this case, due to Lemma 19.2.1(p.177) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.2.2(p.177) to `Tom` 19.2.1(p.178) with $\rho \leq a^\star$.

□ **`Tom` 19.2.9** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$) *Assume $\rho \leq a^\star$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) [Ⓢ `dOITs`$_{\tau>0}\langle\tau\rangle$]$_\blacktriangle$. □

Proof Assume $\rho \leq a^\star$. Let $\beta = 1$ and $s = 0$.

(a) The same as `Tom` 19.2.1(p.178) (a).

(b) Since (b,c) of `Tom` 19.2.1 have none of a and μ, even if $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied the two assertions, no change occurs. In addition, from the assumption $\rho \leq a^\star$ we have $\rho \leq a^\star < a < b$ due to Lemma 12.2.1(p.77) (n), hence only (c) of `Tom` 19.2.1 holds. ∎

19.2.6.2.1.2 Case of $b \leq \rho$

In this case, due to Lemma 19.2.1(p.177) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.2.2(p.177) to `Tom` 19.2.1(p.178) with $b \leq \rho$.

□ **`Tom` 19.2.10** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$) *Assume $b \leq \rho$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) [● `dOITd`$_{\tau>0}\langle 0\rangle$]$_{\parallel}$. □

Proof Assume $b \leq \rho$. Let $\beta = 1$ and $s = 0$.

(a) The same as `Tom` 19.2.1(a).

(b) Due to the assumption $b \leq \rho$ it follows that only (b) of `Tom` 19.2.1 holds. ∎

19.2.6.2.1.3 Case of $a^\star < \rho < b$

In this case, due to Lemma 19.2.1(p.177) (d) we cannot use Theorem 19.2.2.

□ **Tom 19.2.11** ($\mathscr{A}\{$M:2[ℙ][E]$\}$) *Assume $a^\star < \rho < b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $a \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\shortparallel}$ *and* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
(c) *Let $\rho < a$. Then* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. □

Proof Assume $a^\star < \rho < b \cdots$ **(1)**. Let $\beta = 1$ and $s = 0$. Then $L(x) = K(x) = \lambda T(x) \geq 0 \cdots$ **(2)** for any x from (5.1.20(p.18)) and (5.1.21) and from Lemma 12.2.1(g). Since $V_0 < b$ from (1) and (6.5.31(p.31)), we have $L(V_0) = \lambda T(V_0) = \lambda T(\rho) > 0 \cdots$ **(3)** from (1) and Lemma 12.2.1(p.77) (g). Then, since $\rho < b$ and $a < b$, from (19.2.25) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ from (1), we have $V_t < K(b) + b$ from (6.5.33) and Lemma 12.2.3(h), hence $V_t < \beta b - s$ from (12.2.13 (2) (p.79)), so that $V_{t-1} < b$ due to the assumption of $\beta = 1$ and $s = 0$. Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $T(V_{t-1}) > 0 \cdots$ **(4)** for $t > 1$ from Lemma 12.2.1(g). Thus $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (19.2.23) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots$ **(5)** for $\tau > 1$. In addition, from (2) we have $L(V_{t-1}) = K(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots$ **(6)** for $t > 1$ due to (4), so $L(V_{t-1}) > 0$ for $t > 0$ due to (3).

(a) From (19.2.24) and (6.5.31(p.31)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a - \rho\} \geq 0$, hence $V_1 \geq V_0 \cdots$ **(7)**. From (6.5.33(p.31)) with $t = 2$ we have $V_2 - V_1 = K(V_1) \geq 0$ due to (6) with $t = 2$ and (2), hence $V_2 \geq V_1 \cdots$ **(8)**. Suppose $V_t \geq V_{t-1}$. Then from (6.5.33) and Lemma 12.2.3(p.80) (e) we have $V_{t+1} = K(V_t) + V_t \geq K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 1$. From this and (7) we have $V_t \geq V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho$. Then $V_1 = \rho$ from (19.2.24), hence $V_1 < b$ due to (1). Then, since $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$, we have $V_1 = \beta V_0 \cdots$ **(9)**, hence $t_1^* = 0$, i.e., $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\shortparallel}$. Let $\tau > 1$. Then, from (5) and (9) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

(c) Let $\rho < a$. Then, since $V_1 = \lambda(a - \rho) + \rho$ due to (19.2.24), we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a - \rho) > 0$, i.e., $V_1 > \beta V_0$, hence $t_1^* = 1 \cdots$ **(10)**. Let $\tau > 1$. Then, from (5) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$. From this and (10) we have $t_\tau^* = \tau$ for $\tau > 0$, i.e., $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. ∎

19.2.6.2.1.4 Summary of Tom's 19.2.9-19.2.11

□ **Tom 19.2.12** ($\mathscr{A}\{$M:2[ℙ][E]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \leq a^\star$. Then* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$.
(c) *Let $b \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\shortparallel}$.
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_{\shortparallel}$ *and* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.
  2. *Let $\rho < a$. Then* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>0}\langle \tau\rangle}_{\blacktriangle}$. □

Proof (a) The same as Tom's 19.2.9(a), 19.2.10(a), and 19.2.11(a).

(b) The same as Tom 19.2.9(b).

(c) The same as Tom 19.2.10(b).

(d-d2) The same as Tom 19.2.11(b,c). ∎

Corollary 19.2.2 (M:2[ℙ][E]) *Let $\beta = 1$ and $s = 0$. Then, z_t is nondecreasing in $t \geq 0$.* □

Proof Immediate from Tom 19.2.12(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.2.6.2.2 Case of $\beta < 1$ or $s > 0$

19.2.6.2.2.1 Case of $\rho \leq a^\star$

In this case, due to Lemma 19.2.1(p.177) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.2.2(p.177) to Tom's 19.2.2(p.178) -19.2.4(p.182).

□ **Tom 19.2.13** ($\mathscr{A}\{$M:2[ℙ][E]$\}$) *Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
(c) *Let $\rho < x_L$.*
  1. $\boxed{\text{ⓢ}\ \texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
     i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
        1. *Let $\lambda = 1$. Then* $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\shortparallel}$.
        2. *Let $\lambda < 1$. Then* $\boxed{\text{ⓢ}\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_{\blacktriangle}$.

ii. *Let* $(\lambda a - s)/\lambda > a^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
   i. *Let* $(\lambda\beta a - s)/\delta \le a^\star$.
      1. *Let* $\lambda = 1$.
         i. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
         ii. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
      2. *Let* $\lambda < 1$.
         i. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
         ii. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{ⓢ\blacktriangle}\boxed{⊛\|}\boxed{⊛\vartriangle}\boxed{⊛\blacktriangle}$ *is true.*
   ii. *Let* $(\lambda\beta a - s)/\delta > a^\star$.
      1. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
      2. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{ⓢ\blacktriangle}\boxed{⊛\|}\boxed{⊛\vartriangle}\boxed{⊛\blacktriangle}$ *is true.* □

Proof When $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to `Tom` 19.2.2(p.178), the condition of $a < \rho$ in `Tom` 19.2.2(c2i,c3i) changes into $a^\star < \rho$, which contradicts the assumption $\rho \le a^\star$. Hence, it follows that the `Tom` can be obtained by removing assertions related to $a^\star < \rho$ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the inequality $a < \rho$ in `Tom` 19.2.2. ∎

Corollary 19.2.3 (M:2[ℙ][E]) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$. *Then,* z_t *is nondecreasing in* $t \ge 0$. □

Proof Immediate from `Tom` 19.2.13(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **`Tom` 19.2.14** ($\mathscr{A}${M:2[ℙ][E]}) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
   1. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
   2. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. □

Proof Since a and μ are not included in `Tom` 19.2.3(p.181), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it `Tom`, any change does not occurs. ∎

Corollary 19.2.4 (M:2[ℙ][E]) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$. *Then,* $z_t = z(\rho)$ *for* $t \ge 0$. □

Proof Immediate from `Tom` 19.2.14(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ **`Tom` 19.2.15** ($\mathscr{A}${M:2[ℙ][E]}) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.
(b) *Let* $\rho < x_L$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
(c) *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
(d) *Let* $\rho > x_L$.
   1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
   2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
      i. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $(\!(\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle})\!)$.
      ii. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\mathtt{S}_9$ $\boxed{ⓢ\vartriangle}\boxed{\bullet\vartriangle}\boxed{\bullet\blacktriangle}$ *is true.* □

Proof Since a and μ are not included in `Tom` 19.2.4(p.182), even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to it, no change occurs. ∎

Corollary 19.2.5 (M:2[ℙ][E]) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$. *Then,* z_t *is nonincreasing in* $t \ge 0$. □

Proof Immediate from `Tom` 19.2.15(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.2.6.2.2.2 Case of $b \le \rho$

In this case, due to Lemma 19.2.1(p.177) (c) we can apply $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 19.2.2(p.177) to `Tom`'s 19.2.2(p.178) -19.2.4(p.182) with $b \le \rho$.

□ **`Tom` 19.2.16** ($\mathscr{A}${M:2[ℙ][E]}) *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
(b) *Let* $x_L \le \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
(c) *Let* $\rho < x_L$.
   1. $\boxed{ⓢ\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.
   2. *Let* $\beta = 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   3. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
      i. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
      ii. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{ⓢ\blacktriangle}\boxed{⊛\|}\boxed{⊛\vartriangle}\boxed{⊛\blacktriangle}$ *is true.* □

Proof When $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to `Tom` 19.2.2(p.178), the condition $\rho \le a$ in `Tom` 19.2.2(p.178) (c2ii,c3ii) changes into $\rho < a^\star$, hence $\rho < a^\star < a < b$ due to Lemma 12.2.1(p.77) (n), which contradicts the assumption $b \le \rho$. Hence, only to the case with the condition $a < \rho$ can be applied $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, so that it follows that the `Tom` can be obtained by removing assertions related to $\rho < a$ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the inequality $\rho \le a$ in `Tom` 19.2.2. ∎

Corollary 19.2.6 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$. Then, z_t is nondecreasing in $t \ge 0$.* □

Proof Immediate from `Tom` 19.2.16(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ `Tom` **19.2.17** ($\mathscr{A}${M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \ge 0$.*

(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$ ($s > 0$).*

1. *Let $b \ge 0$ ($\kappa \ge 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let $b < 0$ ($\kappa < 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$. □

Proof The same as `Tom` 19.2.3(p.181) since it has not both a and μ, even if applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to this, no change occurs. ∎

Corollary 19.2.7 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then, $z_t = z(\rho)$ for $t \ge 0$.* □

Proof Immediate from `Tom` 19.2.17(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

□ `Tom` **19.2.18** ($\mathscr{A}${M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho = x_L$. Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\triangle}$ *and* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.

(c) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   i. *Let $b \le 0$ ($\kappa \le 0$). Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$ ($\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$).
   ii. *Let $b > 0$ ($\kappa > 0$). Then* $\texttt{S}_9$ $\boxed{\circledS\triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* □

Proof Assume $\rho < x_L$. Then, since $b \le \rho < x_L \cdots$ (1) due to the assumption $b \le \rho$, we have that $0 = L(x_L) = \lambda\beta T(x_L) - s$ and that $T(x_L) = 0$ from Lemma 12.2.1(p.77) (g). Hence, since $0 = -s$ or equivalently $s = 0$, we have $x_L = b$ due to Lemma 12.2.2(d), which is a contradicts (1). Thus, the inequality $\rho < x_L$ becomes impossible, i.e., it must be that $\rho \ge x_L$, so that the assertion (b) of `Tom` 19.2.4(p.182) must be omitted; accordingly, it follows that we have the lemma. ∎

Corollary 19.2.8 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then, z_t is nonincreasing in $t \ge 0$.* □

Proof Immediate from `Tom` 19.2.18(a) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.2.6.2.2.3 Case of $a^\star < \rho < b$

In this case we cannot use Theorem 19.2.2 due to Lemma 19.2.1(p.177) (d). For explanatory convenience let us define:

$\texttt{S}_{10}$ $\boxed{\circledS\triangle \mid \bullet\triangle}$ = {
We have:
(1) Let $\lambda \max\{0, a-\rho\} < s$. Then $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\triangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.
(2) Let $\lambda \max\{0, a-\rho\} \ge$. Then $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\triangle}$.
}

$\texttt{S}_{11}$ $\boxed{\circledS\triangle \mid \circledS\blacktriangle \mid \circledast\triangle \mid \bullet\triangle}$ = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta \max\{0, a-\rho\} < s$, then
  i. $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_{\triangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle}_{\triangle}$,
  ii. $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\triangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_{\triangle}$.
(2) If $\lambda\beta \max\{0, a-\rho\} \ge s$, then
  i. $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_{\blacktriangle}$,
  ii. $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\triangle}$.
}

$\texttt{S}_{12}$ $\boxed{\circledS\triangle \mid \circledS\blacktriangle \mid \circledast\triangle \mid \bullet\triangle \mid \bullet\underline{\underline{\blacktriangle}}}$ {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta \max\{0, a-\rho\} < s$, then
  i. $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \ge \tau > 0}\langle 0\rangle}_{\blacktriangle}$,
  ii. $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle\tau\rangle}_{\triangle}$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_{\triangle}$.
(2) If $\lambda\beta \max\{0, a-\rho\} \ge s$, then
  i. $\boxed{\circledast\ \texttt{ndOIT}_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle}_{\|}$,
  ii. $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle\tau\rangle}_{\blacktriangle}$.
}

$$
\texttt{S}_{13}\,\boxed{\circledS\vartriangle\,|\,\circledast\vartriangle\,|\,\bullet\vartriangle\,|\,\bullet\blacktriangle} = \left\{\begin{array}{l}
\text{There exists } t^\bullet_\tau > 1 \text{ and } t^\star_\tau > 1 \text{ such that:}\\
\text{(1) If } \lambda\beta\max\{0, a-\rho\} < s, \text{ then}\\
\qquad \text{i. } \boxed{\bullet\,\texttt{dOITd}_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle}_{\blacktriangle},\\
\qquad \text{ii. } \boxed{\circledS\,\texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\vartriangle} \text{ or } \boxed{\bullet\,\texttt{dOITd}_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle}_{\vartriangle}.\\
\text{(2) If } \lambda\beta\max\{0, a-\rho\} \ge s, \text{ then}\\
\qquad \text{i. } \boxed{\circledast\,\texttt{ndOIT}_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle}_{\vartriangle},\\
\qquad \text{ii. } \boxed{\circledS\,\texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_{\vartriangle} \text{ or } \boxed{\bullet\,\texttt{dOITd}_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle}_{\vartriangle}.
\end{array}\right\}
$$

For convenience of reference, below let us copy (6.5.32(p.31))

$$V_1 = \lambda\beta\max\{0, a-\rho\} + \beta\rho - s. \tag{19.2.26}$$

□ **Tom 19.2.19** ($\mathscr{A}${M:2[$\mathbb{P}$][E]}) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \le s$, then $\boxed{\bullet\,\texttt{dOITd}_1\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledS\,\texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. Below let $\tau > 1$.*

(b) *Let $V_1 \le x_K$.*

1. *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $V_1 \ge x_L$. If $\lambda\beta\max\{0, a-\rho\} \le s$, then $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledast\,\texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.*
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then $\texttt{S}_{10}\,\boxed{\circledS\vartriangle\,|\,\bullet\vartriangle}$ is true.*
   - ii. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
     1. *Let $b > 0$ ($\kappa > 0$). Then $\texttt{S}_{10}\,\boxed{\circledS\vartriangle\,|\,\bullet\vartriangle}$ is true.*
     2. *Let $b = 0$ ($\kappa = 0$). If $\lambda\beta\max\{0, a-\rho\} < s$, then $\boxed{\circledS\,\texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledS\,\texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.*
     3. *Let $b < 0$ ($\kappa < 0$). Then $\texttt{S}_{11}\,\boxed{\circledS\vartriangle\,|\,\circledS\blacktriangle\,|\,\circledast\vartriangle\,|\,\bullet\vartriangle}$ is true.*

(c) *Let $V_1 > x_K$.*

1. *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $\beta = 1$. If $\lambda\max\{0, a-\rho\} < s$, then $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\blacktriangle}$, or else $\boxed{\circledast\,\texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.*
3. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $b > 0$ ($\kappa > 0$).*
     1. *Let $V_1 < x_L$. Then $\texttt{S}_{10}\,\boxed{\circledS\vartriangle\,|\,\bullet\vartriangle}$ is true.*
     2. *Let $V_1 = x_L$. Then $\texttt{S}_{12}\,\boxed{\circledS\vartriangle\,|\,\circledS\blacktriangle\,|\,\circledast\vartriangle\,|\,\bullet\vartriangle\,|\,\bullet\blacktriangle}$ is true.*
     3. *Let $V_1 > x_L$. Then $\texttt{S}_{13}\,\boxed{\circledS\vartriangle\,|\,\circledast\vartriangle\,|\,\bullet\vartriangle\,|\,\bullet\blacktriangle}$ is true.*
   - ii. *Let $b \le 0$ ($\kappa \le 0$). If $\lambda\beta\max\{0, a-\rho\} \le s$, then $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, or else $\boxed{\circledast\,\texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.* ∎

Proof Assume $a^\star < \rho < b \cdots$ **(1)** and let $\beta < 1$ or $s > 0$.

(a) If $\lambda\beta\max\{0, a-\rho\} \le s$, then $V_1 \le \beta V_0 \cdots$ **(2)** from (19.2.22(p.188)), hence $t^*_1 = 0$, i.e., $\boxed{\bullet\,\texttt{dOITd}_1\langle 0\rangle}_{\vartriangle} \cdots$ **(3)**, or else $V_1 > \beta V_0 \cdots$ **(4)**, hence $t^*_1 = 1$, i.e., $\boxed{\circledS\,\texttt{dOITs}_1\langle 1\rangle}_{\blacktriangle} \cdots$ **(5)**. Below let $\tau > 1$.

(b) Let $V_1 \le x_K \cdots$ **(6)**, hence $K(V_1) \ge 0 \cdots$ **(7)** from Lemma 12.2.3(p.80) (j1).

(b1) From (6.5.33) with $t = 2$ we have $V_2 = K(V_1) + V_1 \ge V_1$ due to (7). Suppose $V_t \ge V_{t-1}$. Then $V_{t+1} \ge K(V_{t-1}) + V_{t-1} = V_t$ from Lemma 12.2.3(e), hence by induction $V_t \ge V_{t-1}$ for $t > 1$, so V_t is nondecreasing in $t > 0$. Note (6). Suppose $V_{t-1} \le x_K$. Then, from (6.5.33) and Lemma 12.2.3(e) we have $V_t \le K(x_K) + x_K = x_K$. Hence, by induction $V_t \le x_K \cdots$ **(8)** for $t > 0$, i.e., V_t is upper bounded in t, hence V_t converges to a finite V as $t \to \infty$. Then, since $V = K(V) + V$ as $\tau \to \infty$ from (6.5.33), we have $V = K(V) + V$, hence $K(V) = 0$ thus $V = x_K$ from Lemma 12.2.3(p.80) (j1).

(b2) Let $V_1 \ge x_L$. Then, since $x_L \le V_{t-1}$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 12.2.1(a), thus $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$ from (19.2.21(p.188)), we have $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \cdots$ **(9)** for $\tau > 1$.

(1) Let $\lambda\beta\max\{0, a-\rho\} \le s$. Then, from (9) and (2) we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1} V_1 \le \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\,\texttt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$.

(2) Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, from (9) and (4) we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\,\texttt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.

(b3) Let $V_1 < x_L \cdots$ **(10)**.

(b3i) Let $\beta = 1 \cdots$ **(11)**, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$ in the Tom. Then $x_L = x_K \cdots$ **(12)** from Lemma 12.2.4(b), hence $V_{t-1} \le x_L$ for $t > 1$ due to (8). Accordingly, since $V_{t-1} \le x_L$ for $\tau \ge t > 1$, we have $L(V_{t-1}) \ge 0$ for $\tau \ge t > 1$ from Lemma 12.2.2(e1), hence $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 1$ from (19.2.21(p.188)), so

$$V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{\tau-2} \ge \cdots \ge \beta^{\tau-1} V_1 \cdots \textbf{(13)}, \quad \tau > 1.$$

(1) Let $\lambda \max\{0, a-\rho\} < s$, hence $\lambda\beta \max\{0, a-\rho\} < s$ due to (11). Then $V_1 - \beta V_0 < 0 \cdots (14)$ from (19.2.22(p.188)) or equivalently $V_1 < \beta V_0 \cdots (15)$. Then, from (13) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 < \beta^\tau V_0 \cdots (16), \quad \tau > 1.$$

Thus, we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$, hence (1) of $\texttt{S}_{10}$ is true.

(2) Let $\lambda \max\{0, a-\rho\} \geq s$, hence $\lambda\beta \max\{0, a-\rho\} \geq s$ due to (11). Then $V_1 - \beta V_0 \geq 0$ from (19.2.22(p.188)) or equivalently $V_1 \geq \beta V_0$ from (19.2.22). Then, from (13) we have $V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \geq \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$, thus (2) of $\texttt{S}_{10}$ holds.

(b3ii) Let $\beta < 1 \cdots (17)$ and $s = 0\ (s > 0)$.

(b3ii1) Let $b > 0\ (\kappa > 0)$. Then $x_L > x_K > 0 \cdots (18)$ from Lemma 12.2.4(p.80) (c (d)). Accordingly, from (8) we have $V_{t-1} \leq x_K < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 12.2.1(a), thus $L(V_{t-1}) > 0$ for $\tau \geq t > 1$. Accordingly, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$ from (19.2.21(p.188)), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (19)$ for $\tau > 1$.

(1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then $V_1 - \beta V_0 < 0 \cdots (20)$ from (19.2.22(p.188)) or equivalently $V_1 < \beta V_0 \cdots (21)$. Hence, from (19) and (21) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0 \cdots (22), \quad \tau > 1.$$

Hence, we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$, so that (1) of $\texttt{S}_{10}$ holds.

(2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then $V_1 - \beta V_0 \geq 0$ from (19.2.22) or equivalently $V_1 \geq \beta V_0$. Hence, from (19) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0$, hence $t^* = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$. Hence (2) of $\texttt{S}_{10}$ holds.

(b3ii2) Let $b = 0\ (\kappa = 0)$. Then $x_L = x_K = 0$ from Lemma 12.2.4(p.80) (c (d)). Accordingly, due to (10) we have $V_1 < x_K$, so $V_1 \leq x_K$. From this and (b1) we have $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} \leq x_K = x_L$ for $\tau \geq t > 1$. Therefore, from Corollary 12.2.1(b) we have $L(V_{t-1}) \geq 0 \cdots (23)$ for $\tau \geq t > 1$, hence $V_t - \beta V_{t-1} \geq 0$ for $\tau \geq t > 1$ from (19.2.21(p.188)) or equivalently $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_t \geq \beta V_{t-1} \geq \cdots \geq \beta^{t-1} V_1$.

(1) Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, since $V_1 \leq \beta V_0$ from (19.2.22), we have $V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \leq \beta^\tau V_0$, hence $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$.

(2) Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (19.2.22), we have $V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$.

(b3ii3) Let $b < 0\ (\kappa < 0)$, hence $x_L < x_K \leq 0 \cdots (24)$ from Lemma 12.2.4(p.80) (c (d)). Then, from (10) we have $V_1 < x_L < x_K = V$ due to (b1). Accordingly, due to the nondecreasingness of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} \leq V_{t^\bullet_\tau + 1} \leq \cdots.$$

Hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \leq V_{t-1}$ for $t > t^\bullet_\tau$. Therefore, from Corollary 12.2.1(a) we have

$$L(V_{t-1}) > 0 \cdots (25), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \leq 0 \cdots (26), \quad t > t^\bullet_\tau.$$

- Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0$ for $\tau \geq t > 1$ from (25), we have $V_t - \beta V_{t-1} >$ for $\tau \geq t > 1$ from (19.2.21) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, so that $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (27)$.
  (1) Let $\lambda\beta \max\{0, a-\rho\}\rho < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0$ from (27), hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau>1}\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau>1}\langle 0\rangle}_\vartriangle$. Accordingly (1i) of $\texttt{S}_{11}$ holds.
  (2) Let $\lambda\beta \max\{0, a-\rho\}\rho \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0$ from (27), hence $t^*_\tau = \tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau>1}\langle\tau\rangle}_\blacktriangle$. Accordingly (2i) of $\texttt{S}_{11}$ holds.
- Let $\tau > t^\bullet_\tau$. Since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (26), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (19.2.21), hence $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (28)$ for $\tau > t^\bullet_\tau$. From (25) and (19.2.21) we have $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence $V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (29)$. From (28) and (29) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1. \cdots (30)$$

  (1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

  Hence, we have $t^*_\tau = t^\bullet_\tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_\vartriangle$. Accordingly (1ii) of $\texttt{S}_{11}$ holds.

  (2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22), from (30) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$

  hence $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle$. Accordingly (2ii) of $\texttt{S}_{11}$ holds.

(c) Let $V_1 > x_K \cdots (31)$, hence $K(V_1) < 0 \cdots (32)$ due to Lemma 12.2.3(p.80) (j1).

(c1) From (6.5.33(p.31)) with $t = 2$ we have $V_2 = K(V_1) + V_1 < V_1 \cdots (33)$ due to (32), hence $V_2 \leq V_1$. Suppose $V_t \leq V_{t-1}$. Then, from Lemma 12.2.3(e) we have $V_{t+1} = K(V_t) + V_t \leq K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \leq V_{t-1}$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Note (31), hence $V_1 \geq x_K$. Suppose $V_{t-1} \geq x_K$. Then, since $V_t \geq K(x_K) + x_K = x_K$ from Lemma 12.2.3(e), by induction we have $V_t \geq x_K \cdots (34)$ for $t > 0$, i.e., V_t is lower bounded in t, hence V_t converges to a finite V. Then, we have $V = x_K$ for the same reason as in the proof of (b1).

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption of $\beta < 1$ or $s > 0$ in the Tom. Then, since $x_L = x_K \cdots (35)$ from Lemma 12.2.4(b), we have $V_{t-1} \geq x_L$ for $t > 1$ from (34). Accordingly $L(V_{t-1}) \leq 0$ for $t > 1$ from Lemma 12.2.2(e1), hence $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, so $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$ from (19.2.21), leading to $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1$.

(1) Let $\lambda\beta \max\{0, a - \rho\} < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 < \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_\blacktriangle$.

(2) Let $\lambda\beta \max\{0, a - \rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22) we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \geq \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\triangle$.

(c3) Let $\beta < 1 \cdots (36)$ and $s = 0$ ($s > 0$).

(c3i) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots (37)$ from Lemma 12.2.4(c (d)).

(c3i1) Let $V_1 < x_L$, hence $x_L > V_{t-1}$ for $t > 1$ from (c1). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 12.2.1(a), we have $V_t - \beta V_{t-1} > 0$ for $t > 1$ due to (19.2.21) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$, hence $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (38)$.

(1) Let $\lambda\beta \max\{0, a - \rho\} < s$. Then for the same reason as in (1(p.193)) we see that (1) of $\texttt{S}_{10}$ is true.

(2) Let $\lambda\beta \max\{0, a - \rho\} \geq s$. Then for the same reason as in (2(p.193)) we see that (2) of $\texttt{S}_{10}$ is true.

(c3i2) Let $V_1 = x_L$. Then, since $V_1 = x_L > x_K = V$ from (37) and (c1), there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau - 1} = x_L > V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

from which $V_{t-1} = x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 12.2.1(a) we have

$$L(V_{t-1}) = 0 \cdots (39), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) > 0 \cdots (40), \quad t > t^\bullet_\tau.$$

Accordingly, from (19.2.21(p.188)) we have $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \geq t > 1$ and $V_t - \beta V_{t-1} > 0$ for $t > t^\bullet_\tau$ or equivalently $V_t = \beta V_{t-1} \cdots (41)$ for $t^\bullet_\tau \geq t > 1$ and $V_t > \beta V_{t-1} \cdots (42)$ for $t > t^\bullet_\tau$.

- Let $t^\bullet_\tau \geq \tau > 1$. Then, we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$ from (41), leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (43)$.

(1) Let $\lambda\beta \max\{0, a - \rho\} < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 < \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_\blacktriangle$, hence (1i) of $\texttt{S}_{12}$ holds.

(2) Let $\lambda\beta \max\{0, a - \rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22), we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $t^\bullet_\tau \geq \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\circledast\ \texttt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_\parallel$, hence (2i) of $\texttt{S}_{12}$ holds.

From (43) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1} V_1 \cdots (44)$.

- Let $\tau > t^\bullet_\tau$. Then, we have $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (42), leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (45)$. From this and (44) we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1$.

(1) Let $\lambda\beta \max\{0, a - \rho\} < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 < \beta^\tau V_0$, hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\triangle$ or $\boxed{\bullet\ \texttt{dOITd}_{\tau > t^\bullet_\tau}\langle 0\rangle}_\triangle$, thus (1ii) of $\texttt{S}_{12}$ holds.

(2) Let $\lambda\beta \max\{0, a - \rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 \geq \beta^\tau V_0$ for $\tau > t^\bullet_\tau$, hence $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\boxed{\circledS\ \texttt{dOITs}_{\tau > t^\bullet_\tau}\langle \tau\rangle}_\blacktriangle$, hence (2ii) of $\texttt{S}_{12}$ holds.

(c3i3) Let $V_1 > x_L \cdots (46)$. Then, since $V_1 > x_L > x_K = V$ from (37) and (c1), due to the nonincreasingness of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \geq V_2 \geq \cdots \geq V_{t^\bullet_\tau - 1} > x_L \geq V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

from which $V_{t-1} > x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \geq V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 12.2.1(a) we have

$$L(V_{t-1}) \leq 0 \cdots (47), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \geq 0 \cdots (48), \quad t > t^\bullet_\tau.$$

- Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from (47), hence $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (19.2.21), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$. Hence $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1} V_1 \cdots (49)$.

(1) Let $\lambda\beta \max\{0, a - \rho\} < s$. Then, since $V_1 < \beta V_0$ from (19.2.22), we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 < \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle}_\blacktriangle$, so (1i) of $\texttt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (19.2.22), we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \geq \beta^\tau V_0$ for $t^\bullet_\tau \geq \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle}_\vartriangle$, hence (2i) of $\mathtt{S}_{13}$ holds.

From (49) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} \leq \beta V_{t^\bullet_\tau - 1} \leq \cdots \leq \beta^{t^\bullet_\tau - 1}V_1 \cdots$ (50).

- Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (48), hence $V_t - \beta V_{t-1} \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (19.2.21) or equivalently $V_t \geq \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}$. Hence, from (50) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \cdots (51).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Since $V_1 - \beta V_0 < 0 \cdots$ (52) from (19.2.22) or equivalently $V_1 < \beta V_0 \cdots$ (53). Then, from (51) and (53) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 < \beta^\tau V_0.$$

Thus, we obtain $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$, hence (1ii) of $\mathtt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then $V_1 - \beta V_0 \geq 0$ from (19.2.22), hence $V_1 \geq \beta V_0$. Then, from (51) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} \leq \cdots \leq \beta^{\tau-2}V_2 \leq \beta^{\tau-1}V_1 \geq \beta^\tau V_0.$$

Thus, we have $\boxed{ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ or $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 0\rangle}_\vartriangle$, hence (2ii) of $\mathtt{S}_{13}$ holds.

(c3ii) Let $b \leq 0$ ($\kappa \leq 0$). Then, since $x_L \leq x_K$ from Lemma 12.2.4(c (d)), we have $V_1 > x_K \geq x_L$ from (31), hence $V_{t-1} \geq x_K \geq x_L$ for $t > 1$ due to (c1). Accordingly $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 12.2.1(p.80) (a), hence $V_t - \beta V_{t-1} \leq 0$ for $t > 1$ from (19.2.21) or equivalently $V_t \leq \beta V_{t-1}$ for $t > 1$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \cdots$ (54).

(1) Let $\lambda\beta\max\{0, a-\rho\} \leq s$. Then, since $V_1 \leq \beta V_0$ from (19.2.22), we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \leq \beta^\tau V_0$ from (54), hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (19.2.22), we have $V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 > \beta^\tau V_0$ from (54), hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$. ∎

Corollary 19.2.9 (M:2[ℙ][E]) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.* :

(a) *Let $x_K \geq V_1$. Then z_t is nondecreasing in $t > 0$.*
(b) *Let $x_K < V_1$. Then z_t is nonincreasing in $t > 0$.* ∎

Proof Immediate from `Tom` 19.2.19(b1,c1) and from (6.2.76(p.25)) and Lemma 12.1.3(p.73). ∎

19.2.6.3 Market Restriction

19.2.6.3.1 Positive Restriction

19.2.6.3.1.1 Case of $\beta = 1$ and $s = 0$

□ `Pom` **19.2.9** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \leq a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
(c) *Let $b \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$ → → ⓓ
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\parallel$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ → → ⓢ/ⓓ
  2. *Let $\rho < a$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ → → ⓢ

Proof The same as `Tom` 19.2.12(p.189) due to Lemma 16.4.1(p.100). ∎

19.2.6.3.1.2 Case of $\beta < 1$ or $s > 0$

19.2.6.3.1.2.1 Case of $\rho \leq a^\star$

□ `Pom` **19.2.10** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a^\star < \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ → → ⓓ
(c) *Let $\rho < x_L$.*
  1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let $\tau > 1$* → → ⓢ
  2. *Let $\beta = 1$.*
     i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
        1. *Let $\lambda = 1$. Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\parallel$ → ⊛
        2. *Let $\lambda < 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
     ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ → → ⓢ
  3. *Let $\beta < 1$ and $s = 0$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ → → ⓢ

4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $(\lambda\beta a - s)/\delta \le a^\star$.
      1. *Let* $\lambda = 1$.
         i. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
         ii. *Let* $s \ge \lambda\beta T(0)$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$ $\to$ $\to$ ⊛
      2. *Let* $\lambda < 1$.
         i. *Let* $s \le \lambda\beta T(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
         ii. *Let* $s > \lambda\beta T(0)$. *Then* $\mathtt{S}_8$(p.178) ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲ *is true* $\to$ $\to$ ⓢ / ⊛
   ii. *Let* $(\lambda\beta a - s)/\delta > a^\star$.
      1. *Let* $s \le \lambda\beta T(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ ⓢ
      2. *Let* $s > \lambda\beta T(0)$. *Then* $\mathtt{S}_8$(p.178) ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲ *is true* $\to$ $\to$ ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a-c2ii) The same as `Tom` 19.2.13(p.189) (a-c2ii).

(c3) Due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of `Tom` 19.2.13.

(c4-c4ii2) Immediate from (2) and `Tom` 19.2.13(c3-c3ii2) with κ due to (2). ∎

☐ **Pom 19.2.11** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$ $\to$ $\to$ ⓓ
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
(d) *Let* $\beta < 1$ *and* $s > 0$.
   1. *Let* $s < \lambda\beta T(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
   2. *Let* $s \ge \lambda\beta T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ $\to$ $\to$ ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a,b) The same as `Tom` 19.2.14(p.190) (a,b).

(c) Due to (1) it suffices to consider only (c1) of `Tom` 19.2.14.

(d-d2) Immediate from (2) and `Tom` 19.2.14(c1,c2) with κ. ∎

☐ **Pom 19.2.12** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.
(b) *Let* $\rho < x_L$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
(c) *Let* $\rho = x_L$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\vartriangle$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓓ/ⓢ
(d) *Let* $\rho > x_L$.
   1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ $\to$ $\to$ ⓓ
   2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_9$(p.181) ⓢ▵ | ●▵ | ●▲ *is true* $\to$ $\to$ ⓢ / ⓓ
   3. *Let* $\beta < 1$ *and* $s > 0$.
      i. *Let* $s \ge \lambda\beta T(0)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$) $\to$ $\to$ ⓓ
      ii. *Let* $s < \lambda\beta T(0)$. *Then* $\mathtt{S}_9$(p.181) ⓢ▵ | ●▵ | ●▲ *is true* $\to$ $\to$ ⓢ / ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a-d1) The same as `Tom` 19.2.15(a-d1).

(d2) Due to (1) it suffices to consider only (d2ii) of `Tom` 19.2.15.

(d3,d3ii) Immediate from (2) and `Tom` 19.2.15(d2i,d2ii) with κ. ∎

19.2.6.3.1.2.2 Case of $b \le \rho$

☐ **Pom 19.2.13** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
(b) *Let* $x_L \le \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ $\to$ $\to$ ⓓ
(c) *Let* $\rho < x_L$.
   1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let* $\tau > 1$ $\to$ $\to$ ⓢ
   2. *Let* $\beta = 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
   3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
   4. *Let* $\beta < 1$ *and* $s > 0$.
      i. *Let* $s \le \lambda\beta T(0)$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$ $\to$ $\to$ ⓢ
      ii. *Let* $s > \lambda\beta T(0)$. *Then* $\mathtt{S}_8$(p.178) ⓢ▲ | ⊛‖ | ⊛▵ | ⊛▲ *is true* $\to$ $\to$ ⓢ / ⊛

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a-c2) The same as Tom 19.2.16(p.190) (a-c2).

(c3) Due to (1) it suffices to consider only (c3i) of Tom 19.2.16.

(c4-c4ii) Immediate from (2) and Tom 19.2.16(c3i,c3ii) with κ. ∎

☐ **Pom 19.2.14** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \ge 0$.*
(b) *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\parallel}$ → → ⓓ
(c) *Let $\beta < 1$ and $s = 0$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
(d) *Let $\beta < 1$ and $s > 0$.*
 1. *Let $s < \lambda\beta T(0)$. Then* [ⓢ dOITs$_{\tau>0}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ
 2. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ → → ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a,b) The same as Tom 19.2.17(p.191) (a,b).

(c) Due to (1) it suffices to consider only (c1) of Tom 19.2.17.

(d-d2) Immediate from (2) and Tom 19.2.17(c1,c2) with κ. ∎

☐ **Pom 19.2.15** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*
(b) *Let $\rho = x_L$. Then* [● dOITd$_1\langle 0\rangle$]$_{\triangle}$ *and* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_{\blacktriangle}$ → → ⓢ / ⓓ
(c) *Let $\rho > x_L$.*
 1. *Let $\beta = 1$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ → → ⓓ
 2. *Let $\beta < 1$ and $s = 0$. Then* S$_9$(p.181) [ⓢ△ | ●△ | ●▲] *is true* → → ⓢ / ⓓ
 3. *Let $\beta < 1$ and $s > 0$.*
  i. *Let $s \ge \lambda\beta T(0)$. Then* [● dOITd$_{\tau>0}\langle 0\rangle$]$_{\triangle}$ ⟪[● dOITd$_{\tau>0}\langle 0\rangle$]$_{\blacktriangle}$⟫ → → ⓓ
  ii. *Let $s < \lambda\beta T(0)$. Then* S$_9$(p.181) [ⓢ△ | ●△ | ●▲] *is true* → → ⓢ / ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.18)).

(a-c1) The same as Tom 19.2.18(p.191) (a-c1).

(c2) Due to (1) it suffices to consider only (c2ii) of Tom 19.2.18.

(c3-c3ii) Immediate from (2) and Tom 19.2.18(c2i,c2ii) with κ. ∎

19.2.6.3.1.2.3 Case of $a^\star < \rho < b$

☐ **Pom 19.2.16** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $a^\star \le \rho < a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} < s$, then* [● dOITd$_1\langle 0\rangle$]$_{\triangle}$*, or else* [ⓢ dOITs$_1\langle 1\rangle$]$_{\blacktriangle}$*. Below let $\tau > 1$* → → ⓢ / ⓓ
(b) *Let $x_K \ge V_1$.*
 1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_K$ as $t \to \infty$*
 2. *Let $x_L \le V_1$. If $\lambda\beta\max\{0, a-\rho\} \le s$, then* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\triangle}$*, or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\triangle}$ → → ⓓ / ⊛
 3. *Let $x_L > V_1$.*
  i. *Let $\beta = 1$. Then* S$_{10}$(p.191) [ⓢ△ | ●△] *is true* → → ⓢ / ⓓ
  ii. *Let $\beta < 1$ and $s = 0$. Then* S$_{10}$(p.191) [ⓢ△ | ●△] *is true* → → ⓢ / ⓓ
  iii. *Let $\beta < 1$ and $s > 0$.*
   1. *Let $s < \lambda\beta T(0)$. Then* S$_{10}$(p.191) [ⓢ△ | ●△] *is true* → → ⓢ / ⓓ
   2. *Let $s = \lambda\beta T(0)$. If $\lambda\beta\max\{0, a-\rho\} < s$, then* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_{\triangle}$ *or* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\triangle}$*, or else* [ⓢ dOITs$_{\tau>1}\langle\tau\rangle$]$_{\triangle}$ → ⓢ / ⓓ
   3. *Let $s > \lambda\beta T(0)$. Then* S$_{11}$(p.191) [ⓢ△ | ⓢ▲ | ⊛△ | ●△] *is true* → → ⓢ / ⊛ / ⓓ

(c) *Let $x_K < V_1$.*
 1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
 2. *Let $\beta = 1$. If $\lambda\beta\max\{0, a-\rho\} < s$, then* [● dOITd$_{\tau>1}\langle 0\rangle$]$_{\blacktriangle}$*, or else* [⊛ ndOIT$_{\tau>1}\langle 1\rangle$]$_{\triangle}$ → → ⊛ / ⓓ
 3. *Let $\beta < 1$ and $s = 0$.*
  i. *Let $x_L > V_1$. Then* S$_{10}$(p.191) [ⓢ△ | ●△] *is true* → → ⓢ / ⓓ
  ii. *Let $x_L = V_1$. Then* S$_{12}$(p.191) [ⓢ△ | ⓢ▲ | ⊛△ | ●△ | ●▲] *is true* → → ⓢ / ⊛ / ⓓ
  iii. *Let $x_L < V_1$. Then* S$_8$(p.178) [ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲] *is true* → → ⓢ / ⊛
 4. *Let $\beta < 1$ and $s > 0$.*
  i. *Let $s < \lambda\beta T(0)$.*
   1. *Let $x_L > V_1$. Then* S$_{10}$(p.191) [ⓢ△ | ●△] *is true* → → ⓢ / ⓓ
   2. *Let $x_L = V_1$. Then* S$_{12}$(p.191) [ⓢ△ | ⓢ▲ | ⊛△ | ●△ | ●▲] *is true* → → ⓢ / ⊛ / ⓓ

3. *Let* $x_L < V_1$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ▲ | ⊛∥ | ⊛△ | ⊛▲}$ *is true* → → ⓢ / ⊛

ii. *Let* $s \ge \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\triangle}$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\triangle}$ → → ⊛ / ⓓ

Proof Suppose $a > 0$, hence $b > a > 0 \cdots$ **(1)**. Then, we have $\kappa = \lambda\beta T(0) - s \cdots$ **(2)** from (5.1.23(p.18)).

(a-b3i) The same as `Tom` 19.2.19(p.192) (a-b3i).

(b3ii) Due to (1) it suffices to consider only (b3ii1) of `Tom` 19.2.19.

(b3iii-b3iii3) The same as `Tom` 19.2.19(b3ii1-b3ii3).

(c-c2) Immediate from (2) and `Tom` 19.2.19(c-c2).

(c3-c3iii) Due to (1) it suffices to consider only (c3i1-c3i3) of `Tom` 19.2.19.

(c4-c4ii) Immediate from (2) and `Tom` 19.2.19(c3i-c3ii). ∎

19.2.6.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

19.2.6.3.3 Negative Restriction

19.2.6.3.3.1 Case of $\beta = 1$ and $s = 0$

☐ `Nem` **19.2.5** ($\mathscr{A}$\{M:2[ℙ][E]$^-$\}) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t \ge 0$.

(b) *Let* $\rho \le a^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ

(c) *Let* $b \le \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → → ⓓ

(d) *Let* $a^\star < \rho < b$.

1. *Let* $a \le \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ / ⓓ
2. *Let* $\rho < a$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ

Proof The same as `Tom` 19.2.12(p.189) due to Lemma 16.4.1(p.100). ∎

19.2.6.3.3.2 Case of $\beta < 1$ or $s > 0$

19.2.6.3.3.2.1 Case of $\rho \le a^\star$

☐ `Nem` **19.2.6** ($\mathscr{A}$\{M:2[ℙ][E]$^-$\}) *Suppose* $b < 0$. *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $a^\star < \rho$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \le \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$ → ⓓ

(c) *Let* $\rho < x_L$.

1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$ → → ⓢ
2. *Let* $\beta = 1$.
   i. *Let* $(\lambda a - s)/\lambda \le a^\star$.
      1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ → ⊛
      2. *Let* $\lambda < 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ
   ii. *Let* $(\lambda a - s)/\lambda > a^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ
3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ▲ | ⊛∥ | ⊛△ | ⊛▲}$ → → ⓢ / ⊛
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $(\lambda\beta a - s)/\delta \le a^\star$.
      1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\triangle}$. ⊛
      2. *Let* $\lambda < 1$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ▲ | ⊛∥ | ⊛△ | ⊛▲}$ *is true* → → ⓢ / ⊛
   ii. *Let* $(\lambda\beta a - s)/\delta > a^\star$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ▲ | ⊛∥ | ⊛△ | ⊛▲}$ *is true* → → ⓢ / ⊛

Proof Suppose $b < 0 \cdots$ **(1)**, hence $a^\star < a < b < 0 \cdots$ **(2)** from Lemma 12.2.1(p.77) (n). Then $\kappa = -s \cdots$ **(3)** from Lemma 12.2.6(p.81) (a).

(a-c2ii) The same as `Tom` 19.2.13(p.189) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta a - s)/\delta \le a^\star$. Then, since $\lambda\beta a/\delta \le a^\star$, we have $\lambda\beta a \le \delta a^\star$ from (9.2.2 (1) (p.42)), hence $\lambda\beta a \le \delta a^\star \le \lambda a^\star$ due to (2), so $\beta a \le a^\star$, which contradicts [19(p.101)]. Thus it must be that $(\lambda\beta\mu - s)/\delta > a^\star$. From this it suffices to consider only (c3ii2) of `Tom` 19.2.13(p.189).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\kappa < 0$ due to (3), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of `Tom` 19.2.13 with κ. ∎

☐ `Nem` **19.2.7** ($\mathscr{A}$\{M:2[ℙ][E]$^-$\}) *Suppose* $b < 0$. *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *We have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a) The same as Tom 19.2.14(p.190) (a,b).

(b) Let $\beta = 1$. Then we have $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ from Tom 19.2.14(b). Let $\beta < 1$. Then, if $s = 0$, due to (1) it suffices to consider only (c2) of Tom 19.2.14 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.2.14 with κ. Thus, whether $s = 0$ or $s > 0$, we have the same result. Accordingly, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

□ **Nem 19.2.8** ($\mathscr{A}\{$M:2[$\mathbb{P}$][E]$^-\}$) *Suppose $b < 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*
(b) *Let $\rho < x_L$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\rightarrow$ → ⓢ
(c) *Let $\rho = x_L$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ $\rightarrow$ → ⓓ/ⓢ
(d) *Let $\rho > x_L$.*
  1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\rightarrow$ → ⓓ
  2. *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$) $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-d1) The same as Tom 19.2.15(p.190) (a-d1).

(d2) If $s = 0$, then due to (1) it suffices to consider only (d2i) of Tom 19.2.15 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (d2i) of Tom 19.2.15. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.2.6.3.3.2.2 Case of $b \le \rho$

□ **Nem 19.2.9** ($\mathscr{A}\{$M:2[$\mathbb{P}$][E]$^-\}$) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \le \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\rightarrow$ → ⓓ
(c) *Let $\rho < x_L$.*
  1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$* $\rightarrow$ → ⓢ
  2. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\rightarrow$ → ⓢ
  3. *Let $\beta < 1$. Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ\blacktriangle \mid ⊛\parallel \mid ⊛\vartriangle \mid ⊛\blacktriangle \mid *\blacktriangle}$ *is true* $\rightarrow$ → ⓢ/⊛

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = \kappa_{\mathbb{P}} = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a,c2) The same as Tom 19.2.16(p.190) (a,c2).

(c3) If $s = 0$, then due to (1) it suffices to consider only (c3ii) of Tom 19.2.16 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 19.2.16. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ **Nem 19.2.10** ($\mathscr{A}\{$M:2[$\mathbb{P}$][E]$^-\}$) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \ge 0$.*
(b) *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\rightarrow$ → ⓓ
(c) *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a) The same as Tom 19.2.17(p.191) (a).

(b) Let $\beta = 1$. Then we have $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ from Tom 23.1.17(p.236) (b).

(c) Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 19.2.17and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.2.17. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ **Nem 19.2.11** ($\mathscr{A}\{$M:2[$\mathbb{P}$][E]$^-\}$) *Suppose $b < 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*
(b) *Let $\rho = x_L$. Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ $\rightarrow$ → ⓢ/ⓓ
(c) *Let $\rho > x_L$.*
  1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.. → ⓓ
  2. *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$) $\rightarrow$ → ⓓ

Proof Suppose $b < 0 \cdots (1)$. Then $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-c1) The same as Tom 19.2.18(p.191) (a-c1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2i) of Tom 19.2.18 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (c2i) of Tom 19.2.18. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.2.6.3.3.2.3 Case of $a^\star < \rho < b$

□ **Nem 19.2.12** ($\mathscr{A}\{$M:2[$\mathbb{P}$][E]$^-\}$) *Suppose $b < 0$. Assume $a^\star \le \rho < a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a - \rho\} < s$, then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$* $\rightarrow$ → ⓢ/ⓓ

(b) *Let* $V_1 \leq x_K$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $V_1 \geq x_L V_1$. *If* $\lambda\beta \max\{0, a-\rho\} \leq s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$, *or else* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$ $\to$ $\to$ ⊛ / ⓓ
3. *Let* $V_1 < x_L$.
   i. *Let* $\beta = 1$. *Then* $\mathtt{S}_{10}$(p.191) ⓢ△ | ●△ *is true* $\to$ $\to$ ⓢ / ⓓ
   ii. *Let* $\beta < 1$. *Then* $\mathtt{S}_{11}$(p.191) ⓢ△ | ⓢ▲ | ⊛△ | ●△ *is true* $\to$ $\to$ ⓢ / ⊛ / ⓓ

(c) *Let* $V_1 > x_K$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$
2. *If* $\lambda\beta \max\{0, a-\rho\} < s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_\blacktriangle$, *or else* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$ $\to$ $\to$ ⊛ / ⓓ

Proof Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 12.2.6(p.81) (a).

(a-b3i) The same as Tom 19.2.19(p.192) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (b3ii3) of Tom 19.2.19 and if $s > 0$, then $\kappa < 0$ due to (2), hence it suffices to consider only (b3ii3) of Tom 19.2.19. Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $V_1 > x_K$.

(c1) The same as Tom 19.2.19(c1)

(c2) Let $\beta = 1$. Then, we have the same as Tom 19.2.19(c2). Let $\beta < 1$. Then, if $s = 0$, then due to (1) it suffices to consider only (c3ii) of Tom 19.2.19 and if $s > 0$, then $\kappa < 0$ from (2), hence it suffices to consider only (c3ii) of Tom 19.2.19. Thus, whether $s = 0$ or $s > 0$, we have the same result. Accordingly, whether $\beta = 1$ or $\beta < 1$, it eventually follows that we have the same result. ∎

19.2.7 $\tilde{\mathsf{M}}$:2[ℙ][E]

19.2.7.1 Preliminary

Since (19.2.6(p.177)) can be known to hold through the comparison of (III) and (IV) of Table 6.5.4(p.31), we see that Theorem 19.2.3(p.177) holds, hence $\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℙ][E]$\}$ can be derived by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to $\mathscr{A}\{\mathsf{M}$:2[ℙ][E]$\}$.

19.2.7.2 Analysis

19.2.7.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 19.2.20** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℙ][E]$\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *Let* $\rho \geq b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$.
(c) *Let* $a \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\parallel$.
(d) *Let* $b^\star > \rho > a$.

1. *Let* $b \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_\parallel$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
2. *Let* $\rho > b$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_\blacktriangle$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.12(p.189). ∎

Corollary 19.2.10 ($\tilde{\mathsf{M}}$:2[ℙ][E]) *Let* $\beta = 1$ *and* $s = 0$. *Then,* z_t *is nonincreasing in* $t \geq 0$. □

Proof Immediate from Tom 19.2.20(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

19.2.7.2.2 Case of $\beta < 1$ or $s > 0$

19.2.7.2.2.1 Case of $\rho \geq b^\star$

□ **Tom 19.2.21** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℙ][E]$\}$) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$.
(c) *Let* $\rho > x_{\tilde{L}}$.

1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_\blacktriangle$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$.
   i. *Let* $(\lambda b + s)/\lambda \geq b^\star$.
      1. *Let* $\lambda = 1$. *Then* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\parallel$.
      2. *Let* $\lambda < 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
   ii. *Let* $(\lambda b + s)/\lambda < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
3. *Let* $\beta < 1$ *and* $s = 0$ ($s > 0$).
   i. *Let* $(\lambda\beta b + s)/\delta \geq b^\star$.
      1. *Let* $\lambda = 1$.
         i. *Let* $a < 0$ ($\tilde{\kappa} < 0$). *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$.
         ii. *Let* $a \geq 0$ ($\tilde{\kappa} \geq 0$). *Then* $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$.

2. *Let* $\lambda < 1$.
   i. *Let* $a \leq 0$ ⟅$\tilde{\kappa} \leq 0$⟆ . *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   ii. *Let* $a > 0$ ⟅$\tilde{\kappa} > 0$⟆ . *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid ⊛\| \mid ⊛\triangle \mid ⊛\blacktriangle}$ *is true.*

ii. *Let* $(\lambda\beta b + s)/\delta < b^\star$.
   1. *Let* $a \leq 0$ ⟅$\tilde{\kappa} \leq 0$⟆ . *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   2. *Let* $a > 0$ ⟅$\tilde{\kappa} > 0$⟆ . *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid ⊛\| \mid ⊛\triangle \mid ⊛\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.13(p.189) . ∎

Corollary 19.2.11 ($\tilde{\text{M}}$:2[ℙ][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$. *Then,* z_t *is nonincreasing in* $t \geq 0$. □

Proof Immediate from Tom 19.2.21(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

□ Tom **19.2.22** ($\mathscr{A}${$\tilde{\text{M}}$:2[ℙ][E]}) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆ .
   1. *Let* $a < 0$ ⟅$\tilde{\kappa} < 0$⟆ . *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
   2. *Let* $a \geq 0$ ⟅$\tilde{\kappa} \geq 0$⟆ . *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$. □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.14(p.190) . ∎

Corollary 19.2.12 ($\tilde{\text{M}}$:2[ℙ][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$. *Then,* $z_t = \tilde{z}(\rho)$ *for* $t \geq 0$. □

Proof Immediate from Tom 19.2.22(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

□ Tom **19.2.23** ($\mathscr{A}${$\tilde{\text{M}}$:2[ℙ][E]}) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$*, is strictly increasing in* $t > 0$ *if* $\lambda < 1$*, and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.
(b) *Let* $\rho > x_{\tilde{L}}$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
(c) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{1}\langle 0\rangle}_{\triangle}$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
(d) *Let* $\rho < x_{\tilde{L}}$.
   1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.
   2. *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆ .
      i. *Let* $a \geq 0$ ⟅$\tilde{\kappa} \geq 0$⟆. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$ ⟅$\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$⟆ .
      ii. *Let* $a < 0$ ⟅$\tilde{\kappa} < 0$⟆. *Then* $\texttt{S}_9$ $\boxed{Ⓢ\triangle \mid \bullet\triangle \mid \bullet\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.15(p.190) . ∎

Corollary 19.2.13 ($\tilde{\text{M}}$:2[ℙ][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$. *Then,* z_t *is nondecreasing in* $t \geq 0$. □

Proof Immediate from Tom 19.2.23(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

19.2.7.2.2.2 Case of $a \geq \rho$

□ Tom **19.2.24** ($\mathscr{A}${$\tilde{\text{M}}$:2[ℙ][E]}) *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$*, is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$*, and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\triangle}$.
(c) *Let* $\rho > x_{\tilde{L}}$.
   1. $\boxed{Ⓢ\ \texttt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.
   2. *Let* $\beta = 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
   3. *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆ .
      i. *Let* $a \leq 0$ ⟅$\tilde{\kappa} \leq 0$⟆ . *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
      ii. *Let* $a > 0$ ⟅$\tilde{\kappa} > 0$⟆ . *Then* $\texttt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid ⊛\| \mid ⊛\triangle \mid ⊛\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.16(p.190) .† ∎

Corollary 19.2.14 ($\tilde{\text{M}}$:2[ℙ][E]) *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$. *Then,* z_t *is nonincreasing in* $t \geq 0$.

Proof Immediate from Tom 19.2.24(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

□ Tom **19.2.25** ($\mathscr{A}${$\tilde{\text{M}}$:2[ℙ][E]}) *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.
(b) *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_{\|}$.
(c) *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆ .

† $\texttt{S}_8$ does not change by the application of the operation.

1. *Let* $a \le 0\,(\!(\tilde{\kappa} \le 0)\!)$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$.
2. *Let* $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. □

Proof by symmetry Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.17(p.191) . ▮

Corollary 19.2.15 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $a \ge \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$. *Then,* $z_t = \tilde{z}(\rho)$ *for* $t \ge 0$. □

Proof Immediate from Tom 19.2.25(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278) . ▮

□ Tom **19.2.26** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$\}$) *Assume* $a \ge \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.
(b) *Let* $\rho = x_{\tilde{L}}$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$.
(c) *Let* $\rho < x_{\tilde{L}}$.
  1. *Let* $\beta = 1$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
  2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
     i. *Let* $a \ge 0\,(\!(\tilde{\kappa} \ge 0)\!)$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $(\!(\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle})\!)$.
     ii. *Let* $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. *Then* $\mathtt{S}_9$ $\boxed{\circledS\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.* □

Proof by symmetry Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 19.2.18(p.191) .‡ ▮

Corollary 19.2.16 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $a \ge \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$. *Then,* z_t *is nondecreasing in* $t \ge 0$. □

Proof Immediate from Tom 19.2.26(a) and from (6.2.90(p.26)) and Lemma A 3.3(p.278) . ▮

19.2.7.2.2.3 Case of $b^\star > \rho > a$

By applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 19.2.3, we see that $\mathtt{S}_{10}$(p.191) $-\mathtt{S}_{13}$ change as follows respectively:

$\mathtt{S}_{14}$ $\boxed{\circledS\vartriangle \mid \bullet\vartriangle}$ =
- We have:
- (1) Let $\lambda\min\{0,\rho-b\} > -s$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$.
- (2) Let $\lambda\min\{0,\rho-b\} \le -s$. Then $\boxed{\circledS\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.

$\mathtt{S}_{15}$ $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledast\vartriangle \mid \bullet\vartriangle}$ =
- There exists $t_\tau^\bullet > 1$ such that:
- (1) If $\lambda\beta\min\{0,\rho-b\} > -s$, then
  - i. $\boxed{\circledS\ \mathtt{dOITs}_{t_\tau^\bullet\ge\tau>1}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet\ge\tau>1}\langle 0\rangle}_{\vartriangle}$,
  - ii. $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>t_\tau^\bullet}\langle t_\tau^\bullet\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>t_\tau^\bullet}\langle 0\rangle}_{\vartriangle}$.
- (2) If $\lambda\beta\min\{0,\rho-b\} \le -s$, then
  - i. $\boxed{\circledS\ \mathtt{dOITs}_{t_\tau^\bullet\ge\tau>1}\langle\tau\rangle}_{\blacktriangle}$,
  - ii. $\boxed{\circledast\ \mathtt{ndOIT}_{\tau>t_\tau^\bullet}\langle t_\tau^\bullet\rangle}_{\vartriangle}$.

$\mathtt{S}_{16}$ $\boxed{\circledS\vartriangle \mid \circledS\blacktriangle \mid \circledast\vartriangle \mid \bullet\vartriangle \mid \bullet\underline{\blacktriangle}}$ =
- There exists $t_\tau^\bullet > 1$ such that:
- (1) If $\lambda\beta\min\{0,\rho-b\} > -s$, then
  - i. $\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet\ge\tau>0}\langle 0\rangle}_{\blacktriangle}$,
  - ii. $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t_\tau^\bullet}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>t_\tau^\bullet}\langle 0\rangle}_{\vartriangle}$.
- (2) If $\lambda\beta\min\{0,\rho-b\} \le -s$, then
  - i. $\boxed{\circledast\ \mathtt{ndOIT}_{t_\tau^\bullet\ge\tau>1}\langle 1\rangle}_{\parallel}$,
  - ii. $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t_\tau^\bullet}\langle\tau\rangle}_{\blacktriangle}$.

$\mathtt{S}_{17}$ $\boxed{\circledS\vartriangle \mid \circledast\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ =
- There exists $t_\tau^\bullet > 1$ and $t_\tau^\star > 1$ such that:
- (1) If $\lambda\beta\min\{0,\rho-b\} > -s$, then
  - i. $\boxed{\bullet\ \mathtt{dOITd}_{t_\tau^\bullet\ge\tau>1}\langle 0\rangle}_{\blacktriangle}$,
  - ii. $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t_\tau^\bullet}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>t_\tau^\bullet}\langle t_\tau^\star\rangle}_{\vartriangle}$.
- (2) If $\lambda\beta\min\{0,\rho-b\} \le -s$, then
  - i. $\boxed{\circledast\ \mathtt{ndOIT}_{t_\tau^\bullet\ge\tau>1}\langle 1\rangle}_{\vartriangle}$,
  - ii. $\boxed{\circledS\ \mathtt{dOITs}_{\tau>t_\tau^\bullet}\langle\tau\rangle}_{\vartriangle}$ or $\boxed{\bullet\ \mathtt{dOITd}_{\tau>t_\tau^\bullet}\langle t_\tau^\star\rangle}_{\vartriangle}$.

Moreover, note that (19.2.26(p.192)) can be changed into

$$V_1 = \lambda\beta\min\{0,\rho-b\} + \beta\rho + s. \tag{19.2.27}$$

□ Tom **19.2.27** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$\}$) *Assume* $b^\star \ge \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\min\{0,\rho-b\} \ge -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{\circledS\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$. *Below let* $\tau > 1$.
(b) *Let* $V_1 \ge x_{\tilde{K}}$.†

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‡$\mathtt{S}_9$ does not change by the application of the operation.
†$V_1 = \lambda\beta\min\{0,b-\rho\} + \beta\rho + s$ (see (6.5.25(p.31))).

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $V_1 \leq x_{\tilde{L}}$. *If* $\lambda\beta\min\{0, \rho - b\} \geq -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$.
3. *Let* $V_1 > x_{\tilde{L}}$.
   i. *Let* $\beta = 1$. *Then* $\mathtt{S}_{14}$ $\boxed{ⓢ\vartriangle \mid \bullet\vartriangle}$ *is true.*
   ii. *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.
      1. *Let* $a < 0\,(\breve{\kappa} < 0)$. *Then* $\mathtt{S}_{14}$ $\boxed{ⓢ\vartriangle \mid \bullet\vartriangle}$ *is true.*
      2. *Let* $a = 0\,(\breve{\kappa} = 0)$. *If* $\lambda\beta\min\{0, \rho{-}b\} > -s$, *then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$ *or* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.
      3. *Let* $a > 0\,(\breve{\kappa} > 0)$. *Then* $\mathtt{S}_{15}$ $\boxed{ⓢ\vartriangle \mid ⓢ\blacktriangle ⊛\vartriangle \mid \bullet\vartriangle}$ *is true.*

(c) *Let* $V_1 < x_{\tilde{K}}$.
1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
2. *Let* $\beta = 1$. *If* $\lambda\beta\min\{0, \rho - b\} > -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0\,(s > 0)$.
   i. *Let* $a < 0\,(\breve{\kappa} < 0)$.
      1. *Let* $V_1 \geq x_{\tilde{L}}$. *Then* $\mathtt{S}_{14}$ $\boxed{ⓢ\vartriangle \mid \bullet\vartriangle}$ *is true.*
      2. *Let* $V_1 = x_{\tilde{L}}$. *Then* $\mathtt{S}_{16}$ $\boxed{ⓢ\vartriangle \mid ⓢ\blacktriangle ⊛\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.*
      3. *Let* $V_1 < x_{\tilde{L}}$. *Then* $\mathtt{S}_{17}$ $\boxed{ⓢ\vartriangle \mid ⊛\vartriangle \mid \bullet\vartriangle \mid \bullet\blacktriangle}$ *is true.*
   ii. *Let* $a \geq 0\,(\breve{\kappa} \geq 0)$. *If* $\lambda\beta\min\{0, \rho - b\} > -s$, *then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 0\rangle}_{\vartriangle}$, *or else* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\vartriangle}$. □

Proof by symmetry Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 19.2.19(p.192). ∎

Corollary 19.2.17 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $b^\star \geq \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.
(a) *Let* $V_1 \geq x_{\tilde{K}}$. *Then* z_t *is nonincreasing in* $t > 0$.
(b) *Let* $V_1 < x_{\tilde{K}}$. *Then* z_t *is nondecreasing in* $t > 0$. □

Proof Immediate from `Tom` 19.2.27(b1,c1) and from (6.2.90(p.26)) and Lemma A 3.3(p.278). ∎

19.2.7.3 Market Restriction

19.2.7.3.1 Positive Restriction

19.2.7.3.1.1 $\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$^+\}$

19.2.7.3.1.1.1 Case of $\beta = 1$ and $s = 0$

□ `Pom` **19.2.17** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.
(a) V_t *is nonincreasing in* $t \geq 0$.
(b) *Let* $\rho \geq b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\to$ $\to$ ⓢ
(c) *Let* $a \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ $\to$ $\to$ ⓓ
(d) *Let* $b^\star > \rho > a$.
1. *Let* $b \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_1\langle 0\rangle}_{\parallel}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ $\to$ $\to$ ⓓ/ⓢ
2. *Let* $\rho > b$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\to$ $\to$ ⓢ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to `Nem` 19.2.5(p.198). ∎
Direct proof The same as `Tom` 19.2.20(p.200) due to Lemma 16.4.1(p.100). ∎

19.2.7.3.1.1.2 Case of $\beta < 1$ or $s > 0$

19.2.7.3.1.1.2.1 Case of $\rho \geq b^\star$

□ `Pom` **19.2.18** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]$^+\}$) *Suppose* $a > 0$. *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.
(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ $\to$ $\to$ ⓓ
(c) *Let* $\rho > x_{\tilde{L}}$.
1. $\boxed{ⓢ\ \mathtt{dOITs}_1\langle 1\rangle}_{\blacktriangle}$ *and* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$ $\to$ $\to$ ⓢ
2. *Let* $\beta = 1$.
   i. *Let* $(\lambda b + s)/\lambda \geq b^\star$.
      1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau>1}\langle 1\rangle}_{\parallel}$ $\to$ ⊛
      2. *Let* $\lambda < 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\to$ $\to$ ⓢ
   ii. *Let* $(\lambda b + s)/\lambda < b^\star$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ $\to$ $\to$ ⓢ
3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_8$ $\boxed{ⓢ\blacktriangle \mid ⊛\parallel \mid ⊛\vartriangle \mid ⊛\blacktriangle}$ *is true* $\to$ $\to$ ⓢ/⊛
4. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $(\lambda\beta b + s)/\delta \geq b^\star$.
      1. *Let* $\lambda = 1$. *Then* $\boxed{⊛\ \mathtt{ndOIT}_{\tau}\langle 1\rangle}_{\vartriangle}$ $\to$ ⊛
      2. *Let* $\lambda < 1$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ\blacktriangle \mid ⊛\parallel \mid ⊛\vartriangle \mid ⊛\blacktriangle}$ *is true* $\to$ $\to$ ⓢ/⊛
   ii. *Let* $(\lambda\beta b + s)/\delta < b^\star$. *Then* $\mathtt{S}_8$(p.178) $\boxed{ⓢ\blacktriangle \mid ⊛\parallel \mid ⊛\vartriangle \mid ⊛\blacktriangle}$ *is true* $\to$ $\to$ ⓢ/⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.6(p.198) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$, hence $b^\star > b > a > 0 \cdots (2)$ from Lemma 13.6.1(p.89) (n). Then we have $\tilde{\kappa} = s \cdots (3)$ from Lemma 13.6.6(p.90) (a).

(a-c2ii) The same as Tom 19.2.21(p.200) (a-c2ii).

(c3) Let $\beta < 1$ and $s = 0$, hence $\tilde{\kappa} = 0$ due to (3). Assume $(\lambda\beta b+s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (9.2.2 (1) (p.42)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (2), so $\beta b \geq b^\star$, which contradicts [7(p.101)]. Thus it must be that $(\lambda\beta b+s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of Tom 19.2.21(p.200).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} > 0$ from (3), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 19.2.21(p.200) with κ. ∎

☐ Pom **19.2.19** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*
(b) *We have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.7(p.198) (see (19.2.10(p.177))). ∎

Direct proof Let $a > 0 \cdots (1)$, then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a) The same as Tom 19.2.22(p.201) (a).

(b) Let $\beta = 1$. Then we have Tom 19.2.22(a). Let $\beta < 1$. Then, if $s = 0$, due to (1) it suffices to consider only (c2) of Tom 19.2.22 and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (c2 of Tom 19.2.22. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ Pom **19.2.20** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*
(b) *Let $\rho > x_{\tilde{L}}$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>0}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ
(c) *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{1}\langle 0\rangle}_{\vartriangle}$ *and* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ/ⓓ
(d) *Let $\rho < x_{\tilde{L}}$.*
  1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$. → ⓓ
  2. *Let $\beta < 1$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ ($\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\blacktriangle}$) → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.8(p.199) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 13.6.6(p.90) (a).

(a-d1) The same as Tom 19.2.23(p.201) (a-d1).

(d2) If $s = 0$, due to (1) it suffices to consider only (d2i) of Tom 19.2.23 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of Tom 19.2.23. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.2.7.3.1.1.2.2 Case of $a \geq \rho$

☐ Pom **19.2.21** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ → → ⓓ
(c) *Let $\rho > x_{\tilde{L}}$.*
  1. $\boxed{ⓢ\ \mathtt{dOITs}_{1}\langle 1\rangle}_{\blacktriangle}$. *Below let $\tau > 1$* → → ⓢ
  2. *Let $\beta = 1$. Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ → → ⓢ
  3. *Let $\beta < 1$. Then* S_8(p.178) | ⓢ▲ | ⊛‖ | ⊛△ | ⊛▲ | *is true* → → ⓢ/⊛

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.9(p.199) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a-c2) The same as Tom 19.2.24(p.201) (a-c2).

(c3) If $s = 0$, due to (1) it suffices to consider only (c3ii) of Tom 19.2.24 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 19.2.24. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ Pom **19.2.22** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*
(b) *Let $\beta = 1$. Then we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\parallel}$ → → ⓓ
(c) *Let $\beta < 1$. Then we have* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>0}\langle 0\rangle}_{\vartriangle}$ → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.10(p.199) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 13.6.6(p.90) (a).

(a) The same as Tom 19.2.25(a).

(b) The same as Tom 19.2.25(p.201) (b).

(c) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (c2) of Tom 19.2.25. If $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 19.2.25. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ **Pom 19.2.23** ($\mathscr{A}\{\tilde{\text{M}}{:}2[\mathbb{P}][\text{E}]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho = x_{\tilde{L}}$. Then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_\vartriangle$ *and* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle \tau\rangle}_\blacktriangle$ → → Ⓢ / ⓓ

(c) *Let $\rho < x_{\tilde{L}}$.*

1. *Let $\beta = 1$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ → → ⓓ
2. *Let $\beta < 1$ and let $s = 0 (s > 0)$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ ($\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\blacktriangle$) → → ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.11(p.199) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 13.6.6(p.90) (a).

(a,b) The same as Tom 19.2.26(a,b).

(c) Let $\rho < x_{\tilde{L}}$.

(c1) Let $\beta = 1$. Then we have $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 0\rangle}_\vartriangle$ from Tom 19.2.26(c1).

(c2) Let $\beta < 1$. If $s = 0$, then due to (2) it suffices to consider only (c2i) of Tom 19.2.26 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2i) of Tom 19.2.26. Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

19.2.7.3.1.1.2.3 Case of $b^\star > \rho > a$

□ **Pom 19.2.24** ($\mathscr{A}\{\tilde{\text{M}}{:}2[\mathbb{P}][\text{E}]^+\}$) *Suppose $a > 0$. Assume $b^\star \geq \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta \max\{0, \rho - b\} \leq s$, then* $\boxed{\bullet\ \texttt{dOITd}_1\langle 0\rangle}_\vartriangle$*, or else* $\boxed{Ⓢ\ \texttt{dOITs}_1\langle 1\rangle}_\blacktriangle$*. Below let $\tau > 1$* → → Ⓢ / ⓓ

(b) *Let $V_1 \geq x_{\tilde{K}}$.*[†]

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \geq x_{\tilde{L}}$. If $\lambda\beta \max\{0, \rho - b\} \leq s$, then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_\vartriangle$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$ → → ⊛ / ⓓ
3. *Let $V_1 > x_{\tilde{L}}$.*
   i. *Let $\beta = 1$. Then $\texttt{S}_{14}$(p.202)* [Ⓢ△ | ●△] *is true* → → Ⓢ / ⓓ
   ii. *Let $\beta < 1$. Then $\texttt{S}_{15}$(p.202)* [Ⓢ△ | Ⓢ▲ ⊛△ | ●△] *is true* → → Ⓢ / ⊛ / ⓓ

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *If $\lambda\beta \max\{0, \rho - b\} < s$, then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 0\rangle}_\blacktriangle$*, or else* $\boxed{⊛\ \texttt{ndOIT}_{\tau>1}\langle 1\rangle}_\vartriangle$ → → ⊛ / ⓓ

Proof by diagonal-symmetry Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (15.3.2(p.98))) to Nem 19.2.12(p.199) (see (19.2.10(p.177))). ∎

Direct proof Suppose $a > 0 \cdots (1)$, hence $b > a > 0$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 13.6.6(p.90) (a).

(a-b3i) The same as Tom 19.2.27(p.202) (a-b3i).

(b3ii) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (b3ii3) of Tom 19.2.27 and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii3) of Tom 19.2.27. Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as Tom 19.2.27(p.202) (c1).

(c2,1) If $\beta = 1$, then it suffices to consider only (c2) of Tom 19.2.27 and if $\beta < 1$, whether $s = 0$ or $s > 0$, it suffices to consider only (c3ii) of Tom 19.2.27(p.202). Accordingly, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

19.2.7.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

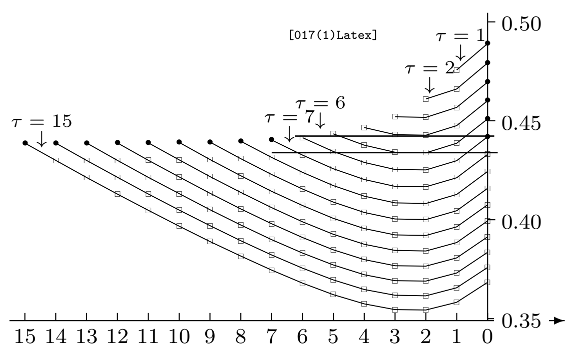
19.2.7.3.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

[†] $V_1 = \lambda\beta \min\{0, b - \rho\} + \beta\rho + s$ (see (6.5.25(p.31))).

19.2.7.4 Numerical Example

■ **Numerical Example 19.2.1** ($\mathscr{A}${M:2[ℝ][E]$^+$} **[019(1)]**) This example is for the assertion in `Pom` 19.2.4(p.184) (d3ii) in which $a > 0$, $\rho > x_K$, $\rho > x_L$, $\beta < 1$, $s > 0$, and $\lambda\beta\mu > s$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$.[†] where $x_L = 0.462767$ and $x_K = 0.439640$. The symbols • in the figure below shows the optimal-initiating-times $t^*_{15 \geq \tau \geq 1}$ (see the t^*_τ-column in the table of Figure 19.2.2 below).



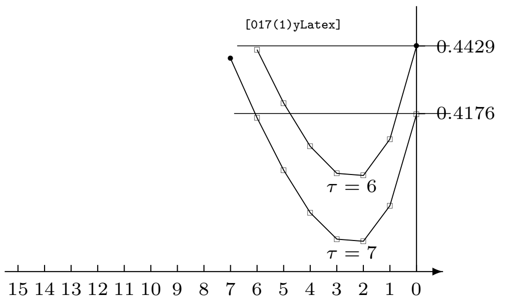
Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $15 \geq \tau > 0$ and $\tau \geq t \geq 0$ [FIG7498x]

t	V_t	$\Delta_\beta V_t$	t^*_τ
0	0.5000000		
1	0.4766162	−0.0133838	1
2	0.4619911	−0.0050927	1
3	0.4530367	+0.0002854	1
4	0.4476274	+0.0036514	1
5	0.4443866	+0.0057117	1
6	0.4424547	+0.0069558	1
7	0.4413065	+0.0077009	7
8	0.4406253	+0.0081449	8
9	0.4402216	+0.0084088	9
10	0.4399825	+0.0085653	10
11	0.4398410	+0.0086581	11
12	0.4397572	+0.0087130	12
13	0.4397076	+0.0087456	13
14	0.4396783	+0.0087648	14
15	0.4396609	+0.0087762	15

$\Delta_\beta V_t = V_t - \beta V_{t-1}$ [017(1)Data.DAT]

Figure 19.2.2: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $15 \geq \tau \geq 2$ and $\tau \geq t \geq 1$

Scaling up the graphs for $\tau = 6$ and $\tau = 7$ in the above figure, we have the figure below. This figure shows that the optimal initiating time *shifts* from 0 to 7 when the starting time changes from $\tau = 6$ to $\tau = 7$.



Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 6, 7$ [FIG7498y]

$\tau = 6$		$\tau = 7$	
t	$\beta^{6-t}V_t$	t	$\beta^{7-t}V_t$
0	0.4429212	0	0.4340628
1	0.4308233	1	0.4222069
2	0.4261259	2	0.4176034
3	0.4263946	3	0.4178667
4	0.4299014	4	0.4213034
5	0.4354989	5	0.4267889
6	0.4424547	6	0.4336056
		7	0.4413065

Figure 19.2.3: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 6$ and $\tau = 7$

19.2.7.5 Conclusion 5 (Search-Enforced-Model 2)

C1 Monotonicity

On the total market $\mathscr{F}$ we have:

a. The optimal reservation price in M:2[ℝ][E] is nondecreasing in t ▲[a], constant ‖[a], or nonincreasing in t ▼[a].
b. The optimal reservation price in M̃:2[ℝ][E] is nondecreasing in t ▲[b], constant ‖[b], or nonincreasing in t ▼[b].
c. The optimal price in M:2[ℙ][E] is nondecreasing in t ▲[c], constant ‖[c], or nonincreasing in t ▼[c].
d. The optimal price in M̃:2[ℙ][E] is nondecreasing in t ▲[d], constant ‖[d], or nonincreasing in t ▼[d].

- ▲[a] ← `Tom` 19.2.1(p.178) (a), 19.2.2(p.178) (a).
  ‖[a] ← `Tom` 19.2.3(p.181) (a)).
  ▼[a] ← `Tom` 19.2.4(p.182) (a).
- ▲[b] ← `Tom` 19.2.8(p.186) (a).
  ‖[b] ← `Tom` 19.2.7(p.186) (a).
  ▼[b] ← `Tom` 19.2.5(p.185) (a), 19.2.6(p.185) (a).
- ▲[c] ← Corollary 19.2.2(p.189), 19.2.3(p.190), 19.2.6(p.191), 19.2.9(p.195) (a).
  ‖[c] ← Corollary 19.2.4(p.190), 19.2.7(p.191).
  ▼[c] ← Corollary 19.2.5(p.190), 19.2.8(p.191), 19.2.9(p.195) (b).
- ▲[d] ← Corollary 19.2.13(p.201), 19.2.16(p.202), 19.2.17(p.203) (b).
  ‖[c] ← Corollary 19.2.12(p.201), 19.2.15(p.202).
  ▼[d] ← Corollary 19.2.10(p.200), 19.2.11(p.201), 19.2.14(p.201), 19.2.17(p.203) (a).

[†] We have $\rho = 0.5 > 0.462767 = x_L$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. Since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\lambda\beta\mu = 0.7 \times 0.98 \times 0.505 = 0.34634 > 0.1 = s$. Thus the condition of this assertion is confirmed.

C2 Inheritance and Collapse

On the positive market $\mathscr{F}^+$ we have:

a. Symmetry

1. Let $\beta = 1$ and $s = 0$. Then, the symmetry is inherited ($\sim$) in whether $\mathbb{R}$-model or $\mathbb{P}$-model where

   Pom 19.2.1(p.183) $\sim$ Pom 19.2.5(p.186) ($\mathbb{R}$-model),
   Pom 19.2.9(p.195) $\sim$ Pom 19.2.17(p.203) ($\mathbb{P}$-model),

2. Let $\beta < 1$ or $s > 0$. Then, the symmetry collapses ($\not\sim$) in whether $\mathbb{R}$-model or $\mathbb{P}$-model where

   Pom 19.2.2(p.183) $\not\sim$ Pom 19.2.6(p.187) ($\mathbb{R}$-model),
   Pom 19.2.3(p.184) $\not\sim$ Pom 19.2.7(p.187) ($\mathbb{R}$-model),
   Pom 19.2.4(p.184) $\not\sim$ Pom 19.2.8(p.187) ($\mathbb{R}$-model),
   Pom 19.2.10(p.195) $\not\sim$ Pom 19.2.18(p.203) ($\mathbb{R}$-model),
   Pom 19.2.11(p.196) $\not\sim$ Pom 19.2.19(p.204) ($\mathbb{R}$-model),
   Pom 19.2.12(p.196) $\not\sim$ Pom 19.2.20(p.204) ($\mathbb{P}$-model),
   Pom 19.2.13(p.196) $\not\sim$ Pom 19.2.21(p.204) ($\mathbb{P}$-model),
   Pom 19.2.14(p.197) $\not\sim$ Pom 19.2.22(p.204) ($\mathbb{P}$-model),
   Pom 19.2.15(p.197) $\not\sim$ Pom 19.2.23(p.205) ($\mathbb{P}$-model),
   Pom 19.2.16(p.197) $\not\sim$ Pom 19.2.24(p.205) ($\mathbb{P}$-model),

b. Analogy

Whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$", the analogy collapses ($\not\bowtie$) in whether S-model or B-model where

   Pom 19.2.1(p.183) $\not\bowtie$ Pom 19.2.9(p.195) (S-model),
   Pom 19.2.5(p.186) $\not\bowtie$ Pom 19.2.17(p.203) (B-model),
   Pom 19.2.2(p.183) $\not\bowtie$ Pom 19.2.10(p.195) (S-model),
   Pom 19.2.3(p.184) $\not\bowtie$ Pom 19.2.11(p.196) (S-model),
   Pom 19.2.4(p.184) $\not\bowtie$ Pom 19.2.12(p.196) (S-model),
   "all Pom's in Section 19.2.5.2.1.2(p.187)" $\not\bowtie$ "all Pom's in Section 19.2.7.3.1.1(p.203)" (B-model),

C3 Occurrence of Ⓢ, ⊛, and ❶d

On the positive market $\mathscr{F}^+$ we have:

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 19.2.1(p.183), Pom 19.2.5(p.186), Pom 19.2.9(p.195), Pom 19.2.17(p.203),

Table 19.2.3: OIT ($\beta = 1$ and $s = 0$)

		$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\parallel}$	Ⓢ$_{\parallel}$				
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\vartriangle}$	Ⓢ$_{\vartriangle}$				
Ⓢ $\mathtt{dOITs}_\tau\langle\tau\rangle_{\blacktriangle}$	Ⓢ$_{\blacktriangle}$	○	○	○	○
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\parallel}$	⊛$_{\parallel}$				
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\vartriangle}$	⊛$_{\vartriangle}$				
⊛ $\mathtt{ndOIT}_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	⊛$_{\blacktriangle}$				
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\parallel}$	❶d$_{\parallel}$	○	○	○	○
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\vartriangle}$	❶d$_{\vartriangle}$				
● $\mathtt{dOITd}_\tau\langle 0\rangle_{\blacktriangle}$	❶d$_{\blacktriangle}$				

1. What is amazing is here that, even in the most simple case "$\beta = 1$ and $s = 0$", the deadline-falling ❶d occurs in all of quadruple-asset-trading models.

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 19.2.4(p.184), Pom 19.2.12(p.196), Pom 19.2.15(p.197), Pom 19.2.16(p.197), Pom 19.2.24(p.205), Pom 19.2.2(p.183), Pom 19.2.3(p.184), Pom 19.2.4(p.184), Pom 19.2.6(p.187), Pom 19.2.8(p.187), Pom 19.2.10(p.195), Pom 19.2.11(p.196), Pom 19.2.13(p.196), Pom 19.2.14(p.197), Pom 19.2.16(p.197), Pom 19.2.18(p.203), Pom 19.2.20(p.204), Pom 19.2.23(p.205), Pom 19.2.16(p.197), Pom 19.2.21(p.204), Pom 19.2.7(p.187), Pom 19.2.19(p.204), Pom 19.2.22(p.204), Pom 19.2.22(p.204),

we obtain the following table:

Table 19.2.4: OIT ($\beta < 1$ or $s > 0$)

		$\mathscr{A}${M:2[ℝ][E]$^+$}	$\mathscr{A}${M̃:2[ℝ][E]$^+$}	$\mathscr{A}${M:1[ℙ][E]$^+$}	$\mathscr{A}${M̃:2[ℙ][E]$^+$}
ⓢ dOITs$_\tau\langle\tau\rangle$ ‖	Ⓢ‖				
ⓢ dOITs$_\tau\langle\tau\rangle$ △	Ⓢ△	○	○	○	○
ⓢ dOITs$_\tau\langle\tau\rangle$ ▲	Ⓢ▲	○	○	○	○
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ ‖	⊛‖	○	○	○	○
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ △	⊛△	○	○	○	○
⊛ ndOIT$_\tau\langle t^\bullet_\tau\rangle$ ▲	⊛▲	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ ‖	ⓓ‖	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ △	ⓓ△	○	○	○	○
● dOITd$_\tau\langle 0\rangle$ ▲	ⓓ▲	○	○	○	○

1. In addition to ⓢ and ⓓ, all kinds of OIT excluding the two occurs in all of quadruple-asset-trading models.

c. Table 19.2.5 below is the list of the percents (frequencies) of ⓢ, ⊛, and ⓓ that appear in Sections 19.2.4.3(p.183), 19.2.5.2(p.186), 19.2.6.3(p.195), and 19.2.7.3(p.203).

Table 19.2.5: Percents (frequencies) of ⓢ, ⊛, and ⓓ on $\mathscr{F}^+$

ratio (total)	ⓢ	⊛	ⓓ
100% (218)	43% (93)	17% (38)	4%0 (87)

C4 Diagonal symmetry

Exercise ***19.2.1*** *Confirm by yourself that the following relations hold in fact.*

Pom 19.2.5(p.186) ∾ Nem 19.2.1(p.184),
Pom 19.2.6(p.187) ∾ Nem 19.2.2(p.184),
Pom 19.2.7(p.187) ∾ Nem 19.2.3(p.185),
Pom 19.2.8(p.187) ∾ Nem 19.2.4(p.185),

Pom 19.2.17(p.203) ∾ Nem 19.2.5(p.198),
Pom 19.2.18(p.203) ∾ Nem 19.2.6(p.198),
Pom 19.2.19(p.204) ∾ Nem 19.2.7(p.198),
Pom 19.2.20(p.204) ∾ Nem 19.2.8(p.199),
Pom 19.2.21(p.204) ∾ Nem 19.2.9(p.199),
Pom 19.2.22(p.204) ∾ Nem 19.2.10(p.199),
Pom 19.2.23(p.205) ∾ Nem 19.2.11(p.199),
Pom 19.2.24(p.205) ∾ Nem 19.2.12(p.199). □

a. The diagonal symmetry always holds in whether ℝ-model or ℙ-model.

19.3 Conclusion 6 (The whole Model 2)

Conclusions 19.1.9(p.175) and 19.2.7.5(p.206) can be summed up as below.

C1 Monotonicity

On the total market $\mathscr{F}$, from C1(p.175) and C1(p.206) we have:

a. The optimal reservation price V_t in M:2[ℝ][X] is nondecreasing in $t^\blacktriangle$, constant$^{\|}$, or nonincreasing in $t^\blacktriangledown$.
b. The optimal reservation price V_t in M̃:2[ℝ][X] is nondecreasing in $t^\blacktriangle$, constant$^{\|}$, or nonincreasing in $t^\blacktriangledown$.
c. The optimal price to propose z_t in M:2[ℙ][X] is nondecreasing in $t^\blacktriangle$, constant$^{\|}$, or nonincreasing in $t^\blacktriangledown$.
d. The optimal price to propose $\tilde{z}_t$ in M̃:2[ℙ][X] is nondecreasing in $t^\blacktriangle$, constant$^{\|}$, or nonincreasing in $t^\blacktriangledown$.

C2 Inheritance and Collapse

On the positive market $\mathscr{F}^+$, in whether s-A-model or s-E-model we have:

a. Symmetry

If $\beta = 1$ and $s = 0$, the symmetry is inherited (∾) (see C2a1(p.175) and C2a1(p.207)), or else ($\beta < 1$ or $s > 0$) collapses (≁) (see C2a2(p.175) and C2a2(p.207)).

b. Analogy

For whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$", the analogy collapses (⋈) (see C2b(p.175) and C2b(p.207)).

C3 Occurrence of ⓢ, ⊛, and 🅓

On the positive market $\mathscr{F}^+$, in both s-A-model and s-E-model we have:

a. Let $\beta = 1$ and $s = 0$. Then only ⓢ$_\blacktriangle$ and 🅓$_{\|}$ are possible for both s-A-model and s-E-model (see Tables 19.1.1(p.175) and 19.2.3(p.207)). It is especially noteworthy that 🅓$_{\|}$ is possible even in the *simplest case* of $\beta = 1$ and $s = 0$; it should be noted that such event was impossible in Model 1.

b. Let $\beta < 1$ or $s > 0$. Then ⊛ and 🅓 are possible only for s-E-model (see Table 19.2.4(p.207)). Here it should be noted that the two are both *strictly optimal*, i.e., ⊛$_\blacktriangle$ and 🅓$_\blacktriangle$ (see Section 7.3(p.37)).

C4 On $\mathscr{F}^+$, joining Tables 19.1.3(p.176) and 19.2.5 produces the table below.

Table 19.3.1: Percents (frequencies) of ⓢ, ⊛, and 🅓 on $\mathscr{F}^+$

ratio (total)	ⓢ	⊛	🅓
100% (467)	45% (210)	18% (85)	37% (172)

In other words, ⓢ, ⊛, and 🅓 occur at 45%, 18%, and 37% respectively.

C5 Diagonal symmetry

See C5(p.176) and C4(p.208).

Chapter 20

Model 3

20.1 Search-Allowed-Model 3: $\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\} = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}3[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

20.1.1 Preliminary I

As ones corresponding to Theorems 11.5.1(p.66), 12.3.1(p.81), and 13.5.1(p.88), and 14.2.1(p.94) let us consider the following four theorems:

Theorem 20.1.1 (symmetry$[\mathbb{R} \to \tilde{\mathbb{R}}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.1}$$

Theorem 20.1.2 (analogy$[\mathbb{R} \to \mathbb{P}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.2}$$

Theorem 20.1.3 (symmetry$(\mathbb{P} \to \tilde{\mathbb{P}}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{20.1.3}$$

Theorem 20.1.4 (analogy$[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}]. \quad \square$$

In addition, as ones corresponding to (11.5.38(p.63)), (12.2.4(p.77)), (13.5.4(p.87)), and (14.2.5 (2) (p.94)), let us consider the following four relations:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.4}$$
$$\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.5}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \tag{20.1.6}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}]. \tag{20.1.7}$$

If (20.1.4) - (20.1.7) are all satisfied, then Theorems 20.1.1 - 20.1.4 can be derived for the same reason as in Parts 1 and 2. Now, from the comparison of (I) and (II) of Table 6.5.5(p.31) and from the comparison of (III) and (IV) it can be easily seen that (20.1.4) and (20.1.6) hold; accordingly, it follows that Theorems 20.1.1 and 20.1.3 hold. However, from the comparison of (I) and (III) we see that (20.1.5) does *not always* hold, hence it follows that Theorem 20.1.2 cannot be used. Similarly, from the comparison of (II) and (IV) we see that (20.1.7) does *not always* hold, hence it follows that Theorem 20.1.4 cannot be used.

The following lemma provides conditions on whether or not each of the four theorems holds.

Lemma 20.1.1

(a) Theorem 20.1.1 *always hold.*
(b) Theorem 20.1.3 *always hold.*
(c) *Let $\rho \le a^\star$ or $b \le \rho$. Then* Theorem 20.1.2 *holds.*
(d) *Let $a^\star < \rho < b$. Then* Theorem 20.1.4 *does not always hold.* $\square$

Proof Almost the same as the proof of Lemma 19.1.1(p.137). ∎

20.1.2 M:3[ℝ][A]

Definition 20.1.1 (reduction)

(a) `model reduction`
Model 1 can be regarded as Model 2 if it is possible that the terminal quitting penalty ρ in Model 2 can be rejected; moreover Model 2 can be regarded as Model 3 if it is possible that the intervening quitting penalty ρ in Model 3 can be rejected. The above two interpretations imply the inclusion relation Model 1 $\subseteq$ Model 2 $\subseteq$ Model 3; in other words, it follows that Model 3 can be reduced to Model 2 and Model 2 can be reduced to Model 1, schematized as

$$\text{Model 3} \twoheadrightarrow \text{Model 2} \quad \text{and} \quad \text{Model 2} \twoheadrightarrow \text{Model 1}. \tag{20.1.8}$$

Let us refer to the above `model reduction` as the `model-running-back`, implying that a model in "upstream" runs back to a model in "downstream".

(b) `optdr reduction`
`Tom` 20.1.1(b) implies that the optimal decision is reduced to

$$\texttt{Accept}_t(\rho)/\texttt{Stop} \overset{\text{def}}{=} \{\text{Accept the intervening quitting penalty } \rho \text{ at time } t \text{ and stop the process}\}. \tag{20.1.9}$$

Let us represent the reduction as the `optdr-Accept/Stop`.

Let us schematize the above two reductions as below.

$$\text{Reduction} \begin{cases} \texttt{model reduction}: \rightarrow \texttt{model-running-back} & (\,\text{M:3}[\mathbb{R}][\texttt{A}] \twoheadrightarrow \text{M:2}[\mathbb{R}][\texttt{A}]\,) \\ \texttt{optdr reduction}: \rightarrow \texttt{optdr-Accept/Stop} & (\,\texttt{odr} \mapsto \texttt{Accept}_t(\rho)/\texttt{Stop}\,) \end{cases} \square \tag{20.1.10}$$

□ `Tom` **20.1.1** ($\mathscr{A}\{$M:3[ℝ][A]$\}$)

(a) *Let $\rho \le x_K$ or $\rho \le 0$. Then* M:3[ℝ][A] ↠ M:2[ℝ][A].†

(b) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then* [Ⓢ $\texttt{dOITs}_{\tau\ge 0}\langle\tau\rangle$] *and* `odr` ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ ‡ → Ⓢ

Proof From (6.5.39(p.31)) with $t = 1$ we have $U_1 = \max\{K(\rho)+\rho, \beta\rho\} \cdots$ **(1)**, hence $U_1 - \rho = \max\{K(\rho), -(1-\beta)\rho\} \cdots$ **(2)**. From (6.5.38) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.5.39) with $t = 2$ and Lemma 9.2.2(e) we have $U_2 = \max\{K(V_1)+V_1, \beta V_1\} \ge \max\{K(V_0)+V_0, \beta V_0\} = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence from (6.5.38(p.31)) we have $V_{t-1} = \max\{\rho, U_{t-1}\} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$. Then, from (6.5.39(p.31)) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$ due to Lemma 9.2.2(e). Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., we have that U_t is nondecreasing in $t > 0 \cdots$ **(3)**.

(a) Let $\rho \le x_K$, hence $K(\rho) \ge 0 \cdots$ **(4)** from Corollary 9.2.2(b). Then, from (1) we have $U_1 \ge K(\rho)+\rho \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (3). Let $\rho \le 0$, hence $-(1-\beta)\rho \ge 0$. Then, noting (4), from (2) we have $U_1 - \rho \ge 0$, i.e., $U_1 \ge \rho$, so that $U_t \ge \rho$ for $t > 0$ due to (3). Accordingly, whether $\rho \le x_K$ or $\rho \le 0$, we have $U_t \ge \rho$ for $t > 0$, meaning that it is optimal to reject the intervening quitting penalty ρ for any $t > 0$. This fact is the same as the event "The intervening quitting penalty ρ does not exist on any time $t > 0$"; in other words, it follows that M:3[ℝ][A] is substantially reduced to M:2[ℝ][A], which has not an intervening quitting penalty ρ (see Section 6.2.2.1(p.24)).

(b) Let $\rho \ge x_K$ and $\rho \ge 0 \cdots$ **(5)**, hence $K(\rho) \le 0 \cdots$ **(6)** from Corollary 9.2.2(a) and $-(1-\beta)\rho \le 0 \cdots$ **(7)**. Then, since $U_1 - \rho = \max\{K(\rho), -(1-\beta)\rho\} \le 0$ from (1), we have $U_1 - \rho \le 0$ i.e., $U_1 \le \rho \cdots$ **(8)**. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.5.38), hence from (6.5.39(p.31)) we have $U_t = \max\{K(\rho)+\rho, \beta\rho\} = U_1 \le \rho$ due to (1) and (8). Accordingly, by induction $U_t \le \rho$ for $t > 0$, hence $V_t = \rho \cdots$ **(9)** for $t > 0$ from (6.5.44(p.31)), so $I_\tau^t = \beta^{\tau-t}\rho$ from (7.2.9(p.34)). Therefore, due to (5) we see that the largest of I_τ^t on $\tau \ge t \ge 0$ is given by $t = \tau$, i.e., $t_\tau^* = \tau$ or equivalently [Ⓢ $\texttt{dOITs}_{\tau\ge 0}\langle\tau\rangle$]. Thus, we have `odr` ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$. ▮

Lemma 20.1.2 *Suppose we have $\texttt{Accept}_t(\rho)/\texttt{Stop}$ for any t $(\tau \ge t \ge 0)$. Then*

(a) *Let $\rho \ge 0$. Then we have* [Ⓢ $\texttt{dOITs}_{\tau\ge 0}\langle\tau\rangle$], *i.e.,* Ⓢ*-falling.*

(b) *Let $\rho \le 0$. Then we have* [● $\texttt{dOITd}_{\tau\ge 0}\langle 0\rangle$], *i.e.,* ⓓ*-falling.* □

Proof Since $V_t = \rho$ for $\tau \ge t \ge 0$ under the condition of the lemma (see (9)) and since $\beta^0 \ge \beta^1 \ge \cdots \ge \beta^{\tau-t} \ge \cdots \ge \beta^\tau$ for any $\tau \ge t \ge 0$. Then, we have $\beta^0\rho \ge \beta^1\rho \ge \cdots \ge \beta^{\tau-t}\rho \ge \cdots \ge \beta^\tau\rho$ if $\rho \ge 0$ and $\beta^0\rho \le \beta^1\rho \le \cdots \le \beta^{\tau-t}\rho \le \cdots \le \beta^\tau\rho$ if $\rho \le 0$. Therefore, $\beta^0 V_\tau \ge \beta^1 V_{\tau-1} \ge \cdots \ge \beta^{\tau-t}V_t \ge \cdots \ge \beta^\tau V_0$ if $\rho \ge 0$ and $\beta^0 V_\tau \le \beta^1 V_{\tau-1} \le \cdots \le \beta^{\tau-t}V_t \le \cdots \le \beta^\tau V_0$ if $\rho \le 0$, which can be rewritten as respectively $I_\tau^\tau \ge I_\tau^{\tau-1} \ge \cdots \ge I_\tau^{\tau-t} \ge \cdots \ge I_\tau^0$ if $\rho \ge 0$ and $I_\tau^\tau \le I_\tau^{\tau-1} \le \cdots \le I_\tau^{\tau-t} \le \cdots \le I_\tau^0$ if $\rho \le 0$. Accordingly, it follows that $t_\tau^* = \tau$ if $\rho \ge 0$, i.e., [Ⓢ $\texttt{dOITs}_\tau\langle\tau\rangle$] (Ⓢ) and $t_\tau^* = 0$ if $\rho \le 0$, i.e., [● $\texttt{dOITd}_\tau\langle 0\rangle$] (ⓓ).

20.1.3 M̃:3[ℝ][A]

□ `Tom` **20.1.2** ($\mathscr{A}\{$M̃:3[ℝ][A]$\}$)

(a) *Let $\rho \ge x_{\tilde{K}}$ or $\rho \ge 0$. Then* M̃:3[ℝ][A] ↠ M̃:2[ℝ][A].

(b) *Let $\rho \le x_{\tilde{K}}$ and $\rho \le 0$. Then* [Ⓢ $\texttt{dOITs}_{\tau\ge 0}\langle\tau\rangle$] *and* `odr` ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ → → Ⓢ

Proof Due to Lemma 20.1.1(a) and Lemma 16.4.1(p.100), immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ in Theorem 20.1.1 to `Tom` 20.1.1. ▮

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†See (a) of Def. 20.1.1 just below.
‡See (b) of Def. 20.1.1 just below.

20.1.4 M:3[ℙ][A]

20.1.4.1 Case of $\rho \le a^\star$ or $b \le \rho$

□ Tom **20.1.3** ($\mathscr{A}${M:3[ℙ][A]}) *Assume $\rho \le a^\star$ or $b \le \rho$. Then:*

(a) *Let $\rho \le x_K$ or $\rho \le 0$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A].

(b) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then* [ⓢ dOITs$_{\tau\ge 0}\langle\tau\rangle$] *and* odr ↦ Accept$_\tau(\rho)$/Stop → → ⓢ

Proof Due to Lemma 20.1.1(c) and Lemma 16.4.1(p.100), immediately obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 20.1.2 to Tom 20.1.1. ∎

20.1.4.2 Case of $a^\star < \rho < b$

□ Tom **20.1.4** ($\mathscr{A}${M:3[ℙ][A]}) *Assume $a^\star < \rho < b$. Let $\beta = 1$ and $s = 0$. Then* M:3[ℙ][A] ↦ M:2[ℙ][A]. □

Proof Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.18)) we have $K(x) = \lambda T(x) \ge 0 \cdots$ **(1)** for any x due to Lemma 12.2.1(p.77) (g). From (6.5.45(p.31)) we have $U_1 \ge \beta\rho = \rho$. Suppose $U_{t-1} \ge \rho$. Then, from (6.5.44) we have $V_{t-1} = U_{t-1} \ge \rho$, hence from (6.5.46) we obtain $U_t \ge \beta V_{t-1} = V_{t-1} \ge \rho$. Thus, by induction $U_t \ge \rho$ for $t > 0$. Accordingly, for the same reason as in the proof of Lemma 20.1.1(a) it follows that M:3[ℙ][A] is reduced to M:2[ℙ][A] ∎

□ Tom **20.1.5** ($\mathscr{A}${M:3[ℙ][A]}) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ or $-(1-\beta)\rho \ge 0$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A].

(b) *Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ and $-(1-\beta)\rho \le 0$.*

1. *Let $\tau = 1$. Then* [ⓢ dOITs$_1\langle\tau\rangle$] *and* odr ↦ Accept$_1(\rho)$/Stop *if $\rho \ge 0$ and* [● dOITd$_0\langle 0\rangle$] *if $\rho \le 0$* → → ⓢ/ⓓ
2. *Let $\tau > 1$. Then:*
   i. *Let $\rho \le x_K$. Then* M:3[ℙ][A] ↠ M:2[ℙ][A]
   ii. *Let $\rho \ge x_K$. Then* [ⓢ dOITs$_{\tau\ge 0}\langle\tau\rangle$] *and* odr ↦ *5921*Accept$_\tau(\rho)$/Stop *if $\rho \ge 0$ and* [● dOITd$_{\tau\ge 0}\langle 0\rangle$] *if $\rho \le 0$* → → ⓢ /ⓓ

Proof Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.5.45(p.31)) we have

$$U_1 - \rho = \max\{\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s, -(1-\beta)\rho\} \cdots \mathbf{(1)}.$$

(a) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ or $-(1-\beta)\rho \ge 0$, hence $U_1 - \rho \ge 0$ from (1) or equivalently $U_1 \ge \rho \cdots$ **(2)**. Then, since $V_1 = U_1 \cdots$ **(3)** from (6.5.44) with $t = 1$, from (6.5.46(p.31)) with $t = 2$ we have $U_2 = \max\{K(U_1) + U_1, \beta U_1\} \cdots$ **(4)**. Hence, from Lemma 12.2.3(p.80) (e) and (5.1.21(p.18)) we have $U_2 \ge \max\{K(\rho) + \rho, \beta\rho\} = \max\{\lambda\beta T(\rho) - (1-\beta)\rho - s + \rho, \beta\rho\} = \max\{\lambda\beta T(\rho) + \beta\rho - s, \beta\rho\}$. Then, from Lemma 12.2.1(p.77) (h) we have $U_2 \ge \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\} = U_1$ due to (6.5.45). Suppose $U_{t-1} \ge U_{t-2}$, hence $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.5.44). Hence, from (6.5.46(p.31)) and Lemma 12.2.3(p.80) (e) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$. Accordingly, by induction $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t \ge \rho$ for $t > 0$. Therefore, for almost the same reason as in the proof of Lemma 20.1.1(a) it follows that M:3[ℙ][A] is reduced to M:2[ℙ][A].

(b) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ and $-(1-\beta)\rho \le 0 \cdots$ **(5)**. Then $U_1 - \rho \le 0$ from (1), i.e., $U_1 \le \rho \cdots$ **(6)**.

(b1) Let $\tau = 1$. Then (6) implies that "Accept the intervening quitting penalty ρ at $t = 1$ and stop the process" is optimal, i.e., [Accept$_1(\rho)$∖Stop].

(b2) Let $\tau > 1$. Due to (6) we have $V_1 = \rho$ from (6.5.44) with $t = 1$, hence $U_2 = \max\{K(\rho) + \rho, \beta\rho\} \cdots$ **(7)** from (6.5.46) with $t = 2$.

(b2i) Let $\rho \le x_K$. Then $K(\rho) > 0$ from Lemma 12.2.3(j1), hence from (7) we have $U_2 \ge K(\rho) + \rho \ge \rho$. Suppose $U_{t-1} \ge \rho$, hence $V_{t-1} = U_{t-1}$ from (6.5.44). Then, from Lemma 12.2.3(e) we have $U_t \ge \max\{K(\rho) + \rho, \beta\rho\} \ge K(\rho) + \rho \ge \rho$. Accordingly, by induction we have $U_t \ge \rho$ for $t > 1$. Thus the assertion holds for the same reason as in the proof of Lemma 20.1.1(a).

(b2ii) Let $\rho \ge x_K$, hence $K(\rho) \le 0$ from Lemma 12.2.3(p.80) (j1). Then, from (7) we have $U_2 \le \max\{\rho, \beta\rho\} \cdots$ **(8)**. If $\beta < 1$, then $\rho \ge 0$ from (5), hence $U_2 \le \max\{\rho, \rho\} = \rho$ and if $\beta = 1$, then $U_2 \le \max\{\rho, \rho\} = \rho$. Accordingly, whether $\beta < 1$ or $\beta = 1$, we have $U_2 \le \rho$ for $t > 0$. Suppose $U_{t-1} \le \rho$, hence $V_{t-1} = \rho$ from (6.5.44). Then, from (6.5.46) we have $U_t = \max\{K(\rho) + \rho, \beta\rho\} = U_2 \le \rho$. Accordingly, by induction we have $U_t \le \rho$ for $t > 1$. Hence, from (6) we have $U_t \le \rho$ for $t > 0$. Thus, for the same reason as in the proof of Tom 20.1.1(b) it follows that the assertion holds. ∎

20.1.5 M̃:3[ℙ][A]

20.1.5.1 Case of $\rho \ge b^\star$ or $a \ge \rho$

□ Tom **20.1.6** ($\mathscr{A}${M̃:3[ℙ][A]}) *Assume $\rho \ge b^\star$ or $a \ge \rho$. Let $\rho \ge b^\star$ or $a \ge \rho$.*

(a) *Let $\rho \ge x_{\tilde{K}}$ or $\rho \ge 0$. Then* M̃:3[ℙ][A] ↠ M̃:2[ℙ][A].

(b) *Let $\rho \le x_{\tilde{K}}$ and $\rho \le 0$. Then* [ⓢ dOITs$_{\tau\ge 0}\langle\tau\rangle$] *and* odr ↦ Accept$_\tau(\rho)$/Stop → → ⓢ

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.1.3 to Tom 20.1.3. ∎

20.1.5.2 Case of $b^\star > \rho > a$

□ `Tom` **20.1.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume* $b^\star > \rho > b$. *Let* $\beta = 1$ *and* $s = 0$. *Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \mapsto \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$. □

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.1.3 to `Tom` 20.1.4. ∎

□ `Tom` **20.1.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume* $b^\star > \rho > a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *Let* $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \ge 0$ *or* $(1-\beta)\rho \ge 0$. *Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.

(b) *Let* $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \le s$ *and* $(1-\beta)\rho \le 0$.

1. *Let* $\tau = 1$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_1\langle\tau\rangle}$ *and* `odr` $\mapsto$ $\mathtt{Accept}_1(\rho)/\mathtt{Stop}$ *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \mathtt{dOITd}_0\langle\tau\rangle}$ *if* $\rho \le 0$ → Ⓢ / ⓓ
2. *Let* $\tau > 1$.
   i. *Let* $\rho > x_{\tilde{K}}$. *Then* $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.
   ii. *Let* $\rho \le x_{\tilde{K}}$. *Then* $\boxed{Ⓢ\ \mathtt{dOITs}_{\tau\ge 0}\langle\tau\rangle}$ *and* `odr` $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Stop}$ *if* $\rho \le 0$ *and* $\boxed{\bullet\ \mathtt{dOITd}_{\tau\ge 0}\langle\tau\rangle}$ *if* $\rho \ge 0$ → Ⓢ / ⓓ

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.1.3 to `Tom` 20.1.5. ∎

20.2 Search-Enforced-Model 3: $\mathcal{Q}\{\mathsf{M}{:}3[\mathrm{E}]\} = \{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}], \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}], \mathsf{M}{:}3[\mathbb{P}][\mathrm{E}], \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\}$

20.2.1 Preliminary

As the ones corresponding to Theorems 18.2.1(p.122), 18.2.2, 18.2.3, and 18.2.4, the following four theorems can be considered:

Theorem 20.2.1 (symmetry[$\mathbb{R} \to \mathbb{R}$]) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}]. \quad □ \tag{20.2.1}$$

Theorem 20.2.2 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}]. \quad □ \tag{20.2.2}$$

Theorem 20.2.3 (symmetry[$\mathbb{P} \to \mathbb{P}$])) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\}]. \quad □ \tag{20.2.3}$$

Theorem 20.2.4 (analogy($\mathbb{R} \to \mathbb{P}$])) *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\}]. \quad □$$

In addition, as ones corresponding to (20.1.4)-(20.1.7), let us consider the following four relations:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}], \tag{20.2.4}$$
$$\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathrm{E}]\}], \tag{20.2.5}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathrm{E}]\}]. \tag{20.2.6}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\}]. \tag{20.2.7}$$

If (20.2.4) - (20.2.7) are satisfied, then Theorems 20.2.1 - 20.2.4 can be easily derived for the same reason as in Parts 1 and 2. Now, from the comparison of (I) and (II) of Table 6.5.6(p.31) and from the comparison of (III) and (IV) it can be easily seen that (20.2.4) and (20.2.6) hold; accordingly, it follows that Theorem 20.1.1 and Theorem 20.1.3 hold. However, from the comparison of (I) and (III) we see that (20.2.5) does *not always* hold, hence it follows that Theorem 20.2.2 cannot be used. The following lemma provides conditions on whether or not each of the three theorems holds.

Lemma 20.2.1

(a) Theorem 20.2.1 *always hold.*
(b) Theorem 20.2.3 *always hold.*
(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$. *Then* Theorem 20.2.2 *holds.*
(d) *Let* $a^\star < \rho < b$. *Then* Theorem 20.2.2 *does not always hold.* □

Proof Almost the same as the proof of Lemma 19.1.1(p.137). ∎

20.2.2 M:3[ℝ][E]

□ Tom **20.2.1** ($\mathscr{A}${M:3[ℝ][E]})

(a) *Let* $\rho \le x_K$. *Then* M:3[ℝ][E] ↠ M:2[ℝ][E].

(b) *Let* $\rho \ge x_K$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau\ge0}\langle\tau\rangle}$ *and* odr ↦ $\texttt{Accept}_\tau(\rho)$/Stop *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \text{dOITd}_{\tau\ge0}\langle\tau\rangle}$ *and* $\texttt{Accept}_0(\rho)$/Terminate *if* $\rho \le 0$ → → ⓢ /ⓓ

Proof From (6.5.53(p.31)) with $t = 1$ and (6.5.57) we have $U_1 = K(\rho) + \rho \cdots$ **(1)** and from (6.5.52) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.5.53) with $t = 2$ and Lemma 9.2.2(e) we have $U_2 = K(V_1) + V_1 \ge K(\rho) + \rho = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$. Then $U_t = K(V_{t-1}) + V_{t-1} \ge K(V_{t-2}) + V_{t-2} = U_{t-1}$. Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0 \cdots$ **(2)**.

(a) Let $\rho \le x_K$, hence $K(\rho) \ge 0$ from Corollary 9.2.2(b). Then, from (1) we have $U_1 \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (2). Accordingly, for almost the same reason as in the proof of Tom 20.1.1(a) it follows that M̃:3[ℝ][E] ↦ M̃:2[ℝ][E].

(b) Let $\rho \ge x_K$, hence $K(\rho) \le 0 \cdots$ **(3)** from Corollary 9.2.2(a). Then, from (1) we have $U_1 \le \rho$. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.5.52), hence $U_t = K(\rho) + \rho \le \rho$ due to (3). Accordingly, by induction $U_t \le \rho$ for $t > 0$. Hence, since $V_t = \rho$ for $t > 0$ from (6.5.52(p.31)), we have $I_\tau^t = \beta^{\tau-t}\rho$ for $t \ge 0$. Therefore, if $\rho \ge 0$, the largest of I_τ^t on $\tau \ge t \ge 0$ is given by $t = \tau$ (i.e., $t_\tau^* = \tau$) or equivalently $\boxed{\text{ⓢ dOITs}_{\tau\ge0}\langle\tau\rangle}$, hence we have $\texttt{Accept}_\tau(\rho)$/Stop and if $\rho \le 0$, then the largest of I_τ^t on $\tau \ge t \ge 0$ is given by $t = 0$ (i.e., $t_\tau^* = 0$) or equivalently $\boxed{\bullet\ \text{dOITd}_{\tau\ge0}\langle0\rangle}$, hence we have $\texttt{Accept}_0(\rho)$/Terminate. ∎

20.2.3 M̃:3[ℝ][E]

□ Tom **20.2.2** ($\mathscr{A}${M̃:3[ℝ][E]}) *For any* $\beta \le 1$ *and* $s \ge 0$ *we have:*

(a) *Let* $\rho \le x_{\tilde K}$. *Then* M̃:3[ℝ][E] ↠ M̃:2[ℝ][E].

(b) *Let* $\rho \le x_{\tilde K}$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau\ge0}\langle\tau\rangle}$ *and* odr ↦ $\texttt{Accept}_\tau(\rho)$/Stop *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \text{dOITd}_{\tau\ge0}\langle\tau\rangle}$ *if* $\rho \le 0$ → → ⓢ /ⓓ

Proof Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ in Theorem 20.2.1 to Tom 20.2.1. ∎

20.2.4 M:3[ℙ][E]

20.2.4.1 Case of $\rho \le a^\star$ or $b \le \rho$

□ Tom **20.2.3** ($\mathscr{A}${M:3[ℙ][E]}) *Assume* $\rho \le a^\star$ *or* $b \le \rho$.

(a) *Let* $\rho \le x_K$. *Then* M:3[ℙ][E] ↠ M:2[ℙ][E].

(b) *Let* $\rho \ge x_K$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau\ge0}\langle\tau\rangle}$ *and* odr ↦ $\texttt{Accept}_\tau(\rho)$/Stop *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \text{dOITd}_{\tau\ge0}\langle0\rangle}$ *if* $\rho \le 0$ → ⓢ /ⓓ

Proof Due to Lemma 20.1.1(a) and Lemma c(p.211), immediately obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ in Theorem 20.2.2 to Tom 20.2.1. ∎

20.2.4.2 Case of $a^\star < \rho < b$

□ Tom **20.2.4** ($\mathscr{A}${M:3[ℙ][E]}) *Assume* $a^\star < \rho < b$ *and let* $\beta = 1$ *and* $s = 0$. *Then* M:3[ℙ][E] ↦ M:2[ℙ][E]. □

Proof Suppose $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. Then we can not use Theorem 20.2.2 due to Lemma 20.2.1(d). From (5.1.21(p.18)) we have $K(x) = \lambda T(x) \ge 0 \cdots$ **(1)** for any x due to Lemma 12.2.1(p.77)(g). Now, from (6.5.59(p.31)) we have $U_1 = \lambda\max\{0, a-\rho\} + \rho \ge \rho$ due to $\max\{0, a-\rho\} \ge 0$. Suppose $U_{t-1} \ge \rho$. Then, since $V_{t-1} = U_{t-1}$ from (6.5.58(p.31)), we have $U_t = K(U_{t-1}) + U_{t-1} \ge U_{t-1}$ due to (1), hence $U_t \ge \rho$. Accordingly, by induction $U_t \ge \rho$ for $t > 0$. Thus, for almost the same as in the proof of Tom 20.1.1(a) it follows that M:3[ℙ][E] is reduced to M:2[ℙ][E]. ∎

□ Tom **20.2.5** ($\mathscr{A}${M:3[ℙ][E]}) *Assume* $a^\star < \rho < b$ *and let* $\beta < 1$ *or* $s > 0$.

(a) *Let* $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$. *Then* M:3[ℙ][E] ↠ M:2[ℙ][E].

(b) *Let* $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$.

1. *Let* $\tau = 1$. *Then* $\boxed{\text{ⓢ dOITs}_1\langle\tau\rangle}$ *and* odr ↦ $\texttt{Accept}_1(\rho)$/Stop *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \text{dOITd}_1\langle\tau\rangle}$ *if* $\rho \le 0$ → ⓢ /ⓓ
2. *Let* $\tau > 1$. *Then*
   i. *Let* $\rho \le x_K$. *Then* M:3[ℙ][E] ↠ M:2[ℙ][E].
   ii. *Let* $\rho \ge x_K$. *Then* $\boxed{\text{ⓢ dOITs}_{\tau\ge0}\langle\tau\rangle}$ *and* odr ↦ $\texttt{Accept}_\tau(\rho)$/Stop *if* $\rho \ge 0$ *and* $\boxed{\bullet\ \text{dOITd}_{\tau\ge0}\langle\tau\rangle}$ *if* $\rho \le 0$ → ⓢ /ⓓ

Proof Suppose $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.5.59(p.31)) we have

$$U_1 - \rho = \lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s \cdots \textbf{(1)}.$$

(a) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$, hence $U_1 \ge \rho \cdots$ **(2)** from (1). Then, since $V_1 = U_1 \cdots$ **(3)** from (6.5.58) with $t = 1$, we have $U_2 = K(U_1) + U_1 \cdots$ **(4)** from (6.5.60(p.31)) with $t = 2$. Hence, from (2), Lemma 12.2.3(p.80)(e), and (5.1.21(p.18)) we have $U_2 \ge K(\rho) + \rho = \lambda\beta T(\rho) - (1-\beta)\rho - s + \rho = \lambda\beta T(\rho) + \beta\rho - s$. Then, from Lemma 12.2.1(p.77)(h) we have $U_2 \ge \lambda\beta\max\{0, a-\rho\} + \beta\rho - s = U_1$ due to (6.5.59). Suppose $U_{t-1} \ge U_{t-2}$, hence $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.5.58(p.31)). Then, from Lemma 12.2.3(p.80)(e) we have $U_t \ge K(V_{t-2}) + V_{t-2} = U_{t-1}$. Accordingly, by induction $U_t \ge U_{t-1}$ for

$t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t > \rho$ for $t > 0$. Therefore, for the same as in the proof of Tom 20.1.1(a) it follows that M:3[ℙ][E] is reduced to M:2[ℙ][E].

(b) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \leq s \cdots (5)$. Then $U_1 - \rho \leq 0$ from (1), i.e., $U_1 \leq \rho \cdots (6)$.

(b1) Let $\tau = 1$. (6) implies that "Accept the intervening quitting penalty ρ at the starting time $t = 1$ and the process stops" is optimal, i.e., $\texttt{Accept}_1(\rho)/\texttt{Stop}$.

(b2) Let $\tau > 1$. Now, due to (6) we have $V_1 = \rho$ from (6.5.58) with $t = 1$, thus $U_2 = K(\rho) + \rho \cdots (7)$ from (6.5.60) with $t = 2$.

(b2i) Let $\rho \leq x_K$, hence $K(\rho) \geq 0$ from Lemma 12.2.3(p.80) (j1). Then, from (7) we have $U_2 \geq \rho$. Suppose $U_{t-1} \geq \rho$, hence $V_{t-1} = U_{t-1}$ from (6.5.58). Then, from Lemma 12.2.3(e) we have $U_t = K(U_{t-1}) + U_{t-1} \geq K(\rho) + \rho \geq \rho$. Hence, by induction $U_t \geq \rho$ for $t > 1$. Therefore, we have that "Reject the intervening quitting penalty ρ for any $t > 1$" is optimal. Thus, for almost the same as in the proof of Lemma 20.1.1(a) we have M:3[ℙ][E] is reduced to M:2[ℙ][E].

(b2ii) Let $\rho \geq x_K$. Then $K(\rho) \leq 0 \cdots (8)$ from Lemma 12.2.3(p.80) (j1). Hence $U_2 \leq \rho$ from (7). Suppose $U_{t-1} \leq \rho$, hence $V_{t-1} = \rho$ from (6.5.58). Then, from (6.5.60) we have $U_t = K(\rho) + \rho \leq \rho \cdots (9)$ due to (8). Thus, by induction $U_t \leq \rho$ for $t > 1$. From this and (6) we have $U_t \leq \rho$ for $t > 0$. Accordingly, for the same reason as in the proof of Tom 20.1.1(b) we have that the assertion holds. ∎

20.2.5 M̃:3[ℙ][E]

20.2.5.1 Case of $\rho \geq b^\star$ or $a \geq \rho$

□ Tom **20.2.6** ($\mathscr{A}${M̃:3[ℙ][E]}) *Assume $\rho \geq b^\star$ or $a \geq \rho$ and let $\beta \leq 1$ and $s \geq 0$.*

(a) *Let $\rho \geq x_{\tilde{K}}$. Then* M̃:3[ℙ][E] ↠ M̃:2[ℙ][E].
(b) *Let $\rho \leq x_{\tilde{K}}$. Then* [ⓢ $\texttt{dOITs}_{\tau\geq 0}\langle\tau\rangle$] *and* odr ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ *if $\rho \geq 0$ and* [● $\texttt{dOITd}_{\tau\geq 0}\langle 0\rangle$] *if $\rho \leq 0$* → ⓢ / ⓓ

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.2.3 to Tom 20.2.3. ∎

20.2.5.2 Case of $b^\star > \rho > a$

□ Tom **20.2.7** ($\mathscr{A}${M̃:3[ℙ][E]}) *Assume $b^\star > \rho \geq b$ and let $\beta = 1$ and $s = 0$. Then* M̃:3[ℙ][E] ↦ M̃:2[ℙ][E]. □

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.2.3 to Tom 20.2.4. ∎

□ Tom **20.2.8** ($\mathscr{A}${M̃:3[ℙ][E]}) *Assume $b^\star > \rho > a$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \geq s$. Then* M̃:3[ℙ][E] ↠ M̃:2[ℙ][E].
(b) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \leq s$.*
  1. *Let $\tau = 1$. Then* [ⓢ $\texttt{dOITs}_1\langle\tau\rangle$] *and* odr ↦ $\texttt{Accept}_1(\rho)/\texttt{Stop}$ *if $\rho \geq 0$ and* [● $\texttt{dOITd}_1\langle 0\rangle$] *if $\rho \leq 0$* → ⓢ / ⓓ
  2. *Let $\tau > 1$. Then*
     i. *Let $\rho > x_{\tilde{K}}$. Then* M̃:3[ℙ][E] ↠ M̃:2[ℙ][E]
     ii. *Let $\rho \leq x_{\tilde{K}}$. Then* [ⓢ $\texttt{dOITs}_{\tau\geq 0}\langle\tau\rangle$] *and* odr ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ *if $\rho \geq 0$ and* [● $\texttt{dOITd}_{\tau\geq 0}\langle 0\rangle$] *if $\rho \leq 0$* → ⓢ / ⓓ

Proof Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.2.3 to Tom 20.2.5. ∎

20.3 Conclusion 7 (The whole Model 3)

For S-model and B-model, for ℝ-model and ℙ-model, and for s-A-model and s-E-model:

C1 we have Model 3 ↠ Model 2 where

a. $\mathscr{A}${M:3[ℝ][A]} ↠ $\mathscr{A}${rM:2[ℝ][A]} (see Tom 20.1.1(p.212) (a)),
b. $\mathscr{A}${M̃:3[ℝ][A]} ↠ $\mathscr{A}${rM̃:2[ℝ][A]} (see Tom 20.1.2(p.212) (a)),
c. $\mathscr{A}${M:3[ℙ][A]} ↠ $\mathscr{A}${rM:2[ℝ][A]} (see Tom 20.1.3(p.213) (a)), 20.1.4, 20.1.5(a,b2i),
d. $\mathscr{A}${M̃:3[ℙ][A]} ↠ $\mathscr{A}${rM̃:2[ℝ][A]} (see Tom 20.1.6(p.213) (a)), 20.1.7, 20.1.8(a,b2i),
e. $\mathscr{A}${M:3[ℝ][E]} ↠ $\mathscr{A}${rM:2[ℝ][E]} (see Tom 20.2.1(p.215) (a)),
f. $\mathscr{A}${M̃:3[ℝ][E]} ↠ $\mathscr{A}${rM̃:2[ℝ][E]} (see Tom 20.2.2(p.215) (a)),
g. $\mathscr{A}${M:3[ℙ][E]} ↠ $\mathscr{A}${rM:2[ℝ][E]} (see Tom 20.2.3(p.215) (a)), 20.2.4, 20.2.5(a,b2i),
h. $\mathscr{A}${M̃:3[ℙ][E]} ↠ $\mathscr{A}${rM̃:2[ℝ][E]} (see Tom 20.2.6(p.216) (a)), 20.2.7, 20.2.8(a,b2i).
i. The above results implies that all discussions of Model 3 is reduced to those of Model 2 in Chapter 19(p.137), in other words, becomes unnecessary as well redundant.

C2 We have optdr ↦ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ where

a. $\mathscr{A}${M:3[ℝ][A]} (see Tom 20.1.1(p.212) (b)),
b. $\mathscr{A}${M̃:3[ℝ][A]} (see Tom 20.1.2(p.212) (b)),
c. $\mathscr{A}${M:3[ℙ][A]} (see Tom 20.1.3(p.213) (b),20.1.5(b1,b2ii),
d. $\mathscr{A}${M̃:3[ℙ][A]} (see Tom 20.1.6(p.213) (b),20.1.8(b1.b2ii),
e. $\mathscr{A}${M:3[ℝ][E]} (see Tom 20.2.1(p.215) (b)),
f. $\mathscr{A}${M̃:3[ℝ][E]} (see Tom 20.2.2(p.215) (b)),
g. $\mathscr{A}${M:3[ℙ][E]} (see Tom 20.2.3(p.215) (b),20.2.5(b1,b2ii),
h. $\mathscr{A}${M̃:3[ℙ][E]} (see Tom 20.2.6(p.216) (b),20.2.8(b1,b2ii).
i. The above results implies that it is optimal to stop the trading process in Model 3 by accepting the intervening quitting penalty ρ at the starting time τ.

Chapter 21

The Whole Conclusion of No-Recall-Model

This chapter summarizes all conclusions for the whole no-recall-model (see Conclusions 18.1(p.119) – 20.3(p.216)).

21.1 Conclusion 8

C1 **Reduction**

a. We have Model 3 $\twoheadrightarrow$ Model 2 (see C1 (p.216)).
b. We have optdr $\mapsto$ $\texttt{Accept}_\tau(\rho)/\texttt{Stop}$ (see C2 (p.216)).

C1a implies that discussions for Model 3 become unnecessary as well as redundant; in other words, it does not become necessary to discuss any more for Model 3; accordingly, below we make discussions only for Model 1 and Model 2.

C2 **Integration theory**

Here let us recall Motive 2(p.4) of this study "Can the theory integrating quadruple-asset-trading-problems exist ?", and the motivation was put an end with successfully constructing it. The whole flow of its construction is summarized as below (see Figure 15.1.1(p.97)). First, the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling-model) selected as a *seed* is directly proven (see Chapter 10(p.47)), next $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying-model) is derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Chapter 11(p.55)), then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ ($\mathbb{P}$-mech-model) is derived so as to become *analogous* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ ($\mathbb{R}$-mech-model) (see Chapter 12(p.73)), and finally $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (see Chapter 13(p.83)). Herein note the following two epilegomenas concerning symmetry and analogy.

a. **Symmetry**
The introduction of the concept of symmetry between a selling model and a buying model was first touched off by a vague inspiration from *yin-yang principle*, the ancient Chinese philosophy.† Before long, this rather superstitious concept was mathematically reified by the introduction of the *reflection operation* $\mathcal{R}$ (see Step 11.5 (p.61)). Through trial-and-errors, this operation led us to the *correspondence replacement operation* $\mathcal{C}_\mathbb{R}$ (see Step 11.5 (p.61)) and the *identity replacement operation* $\mathcal{I}_\mathbb{R}$ (see Step 11.5 (p.62)). Finally, the above three operations were compiled into a single operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_\mathbb{R}\mathcal{C}_\mathbb{R}\mathcal{R}$ (see (11.5.32(p.63))), called the *symmetry transformation operation*, leading to Theorem 11.5.1(p.66) (*symmetry theorem*) which combines the selling problem and the buying problem. The above is only for $\mathbb{P}$-mech-model. The same as the above discussion holds also for $\mathbb{P}$-mech-model (see Chapter 13(p.83)), yielding the *symmetry transformation operation* $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (13.5.3(p.87))).

b. **Analogy**
In the beginning of this study, we had not any anticipation at all for the existence of the analogous relation between $\mathbb{R}$-mech-model and $\mathbb{P}$-mech-model. However, in the process of proceeding with analyses of both models, we noticed the existence of some similarities between the two procedures of treating both models, then were led, as if in the *jigsaw puzzle*, to an analogous relationship between Lemmas 9.1.1(p.41) and 12.2.1(p.77), and lastly obtained Theorem 12.3.1(p.81) (*analogy theorem*) which combines the above two models.

C3 **Inheritance and Collapse**

Let us note here the fact that the integration theory can be constructed under the basic premise that the price, whether reservation price or posted price, is defined on the total market $\mathscr{F} = (-\infty, \infty)$ (see Section 16.3(p.99)). Now, we showed that under this premise the assertion system $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model) is derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model), and then the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ ($\mathbb{P}$-mechanism) is derived so as to become *analogous* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ ($\mathbb{R}$-mechanism). Accordingly, without this basic premise, it follows that the integration theory might not be successfully constructed. Now, since the premise allows the negative price, the theory seems to be *unrealistic* and *imaginary* since trading on the normal market in the real world is usually made on the positive market $\mathscr{F}^+$ (see (16.3.1(p.99))).

†The yin-yang principle is a philosophical mindset not a procedural technique; it provides conceptual guidance for harmonizing opposites as appropriate and useful, but is not a rule that all opposites must be harmonized. Design and evaluation are not intrinsic opposites, but are typically treated as separate and sequential (bluemarbleeval.org/principles/operating-principles/yin-yang-principle)

To resolve the unnaturalness, in this paper we employ the methodology of restricting results obtained on the total market $\mathscr{F}$ to the positive market $\mathscr{F}^+$. Then, since the market restriction compels a change in the inner structure of the above basic premise, it can be naturally foreseen that the symmetrical relation and the analogous relation, which are both obtained on the total market $\mathscr{F}$, might collapse by the application of the market restriction.

a. Symmetry

1. In whether Model 1 or Model 2 and in whether s-A-model or s-E-model, on $\mathscr{F}^+$, if $\beta = 1$ and $s = 0$, the symmetry is inherited ($\sim$), or else ($\beta < 1$ or $s > 0$) collapses ($\not\sim$) (see C2c1a(p.136) and C2a(p.208)).
2. It is proven that the assertion systems of M (S-model) and $\tilde{\mathsf{M}}$ (B-model) are symmetrical for both $\mathbb{R}$-model and $\mathbb{P}$-model, i.e., SOE{M} $\sim$ SOE{$\tilde{\mathsf{M}}$}

b. Analogy

1. In whether Model 1 or Model 2 and in whether s-A-model or s-E-model, on $\mathscr{F}^+$, if $\beta = 1$ and $s = 0$, the analogy collapses ($\not\bowtie$) (see C2c2(p.136) and C3a(p.244)).
2. It is proven that the assertion systems of M[$\mathbb{R}$] ($\mathbb{R}$-model) and M[$\mathbb{P}$] ($\mathbb{P}$-model) are not always analogous SOE{M[$\mathbb{R}$]} $\not\bowtie$ SOE{M[$\mathbb{P}$]} and that the SOE's of $\tilde{\mathsf{M}}$[$\mathbb{R}$] ($\mathbb{R}$-model) and $\tilde{\mathsf{M}}$[$\mathbb{P}$] ($\mathbb{P}$-model) are not always analogous, i.e., SOE{$\tilde{\mathsf{M}}$[$\mathbb{R}$]} $\not\bowtie$ SOE{$\tilde{\mathsf{M}}$[$\mathbb{P}$]}.

C4 **Diagonal Symmetry**

In both $\mathbb{R}$-mech-model and $\mathbb{P}$-mech-model the symmetry is inherited between the selling-problem on the negative market $\mathscr{F}^+$ ($\mathscr{F}^-$) and the buying problem on the positive market $\mathscr{F}^-$ ($\mathscr{F}^+$) (see Figure 17.1.5(p.114) and Figure 17.3.1(p.117)).

C5 **Null-time-zone and deadline-falling**

Suppose that the existence of a decision-making problem has been recognized (see A1(p.9)) and then that all preparations for the exertion of the decision-making have been completed. Then, normally a decision-maker will immediately try to initiate an effort toward the solution of the problem. In this case, it is usual that one has no void space into which the feeling of postponing its initiation might become better penetrates and that and he *unconsciously* understand without hesitating that the immediate initiation is quite a natural behaviour (see A2bi(p.9)). However, in this rather cursory way of thinking, the possibility is not taken into account at all that it may become better to postpone the initiation of process. Now, in Section 7.2.4.6(p.36) we already stated that the introduction of the concept of the OIT inevitably leads us to the existence of the null-time-zone on which any decision-making activity is made *quite meaningless*. Now, in the usual theory of a decision-making in which the concept of OIT has not been being taken into account at all. This fact implies that this meaningless decision-making activity is *unconsciously* and *understandably* taken into consideration. This conventional theory of decision-making includes a ***fatal defect***. In Section A 5(p.291) we will discuss this serious problem from a viewpoint of Markovina decision process which can be regarded as the most basic and general model of decision processes. Now, we pointed out that there exist three possibilities of the optimal initiating time t^*_τ, symbolized as ⓢ, ⊛, and ⓓ (see Section 7.2.4.4(p.35)). Here it should be noted that the existence of ⊛ and ⓓ *inevitably* leads us to the existence of the null-time-zone (Section 7.2.4.6(p.36)) and that it leads us, as its inevitable consequence, to the existence of the deadline-falling (see Figures 7.2.3(p.36) and 7.2.4). This should be said to be one of the most *striking* findings in this paper, and this fact prompts us to the overall re-examination of the whole theory of decision processes that have been investigated so far without knowing the existence of the deadline-falling (see Section A 5(p.291)).

C6 **Occurrence of ⓢ, ⊛, and ⓓ**

a. Let $\beta = 1$ and $s = 0$. Then, from Tables 18.1.1(p.120) and 18.2.1(p.134) we see that only ⓢ$_\blacktriangle$ and ⓓ$_\parallel$. What is amazing is here that ⓓ$_\parallel$ appears even in the simplest case of "$\beta = 1$ and $s = 0$".

b. Let $\beta < 1$ or $s > 0$. Then, from Tables 19.1.2(p.176) and 19.2.4(p.207) we see that ⊛ and ⓓ are possible for both s-A-model and s-E-model; however, ⊛$_\blacktriangle$ and ⓓ$_\blacktriangle$ (both are strong assertion) are possible only for s-E-model.

c. Summing up Tables 19.1.3(p.176) and 19.2.5(p.208) yields Tables 21.1.1(p.218) below.

Table 21.1.1: Percents (frequencies) of Respective OIT's

ratio (total)	ⓢ	⊛	ⓓ
100% (596)	46% (274)	17% (102)	35% (209)

In other words, we have ⓢ, ⊛, and ⓓ at 46%, 17%, and 35% respectively.

C7 **Posterior-skip-of-search (pSkip)**

The posterior-skip-of-search (see Remark 7.2.1(p.34)) is possible *only* for $\mathscr{A}${M:2[$\mathbb{R}$][A]$^+$} and $\mathscr{A}${M:2[$\mathbb{P}$][A]$^+$} (selling model) with $\beta < 1$ or $s > 0$ (see C4(p.176)). It is usual to consider that once conducting the search is optimal, it will become also optimal to continue conducting the search after that. However, we demonstrated that there exists a case that this expectation does not always hold; in other words, it is possible, although being very rare, that it can become optimal to skip the search after having conducted the search for a while.

Part 4

Recall-Model

Chapter 22

Definitions of Models

22.1 Introduction

In the recall-model with $\mathbb{R}$-mechanism it suffices to memorize only the best of prices which have been rejected so far. Against this, in the recall-model with $\mathbb{P}$-mechanism it is hard to define the best price itself. For this reason, in this part we exclude the application of the integration theory to the latter model, which is left as a subject to be tackled in the future. For convenience of reference, below let us copy Table 3.3.2(p.12) where ——— represents the model excluded.

Table 22.1.1: Twelve recall-models

	$\mathtt{ASP}[\mathbb{R}]$	$\mathtt{ABP}[\mathbb{R}]$	~~$\mathtt{ASP}[\mathbb{P}]$~~	~~$\mathtt{ABP}[\mathbb{P}]$~~
$\mathcal{Q}\{\mathsf{rM}{:}1[\mathtt{A}]\} = \{$	$\mathsf{rM}{:}1[\mathbb{R}][\mathtt{A}]$,	$\mathsf{r\tilde{M}}{:}1[\mathbb{R}][\mathtt{A}]$,	~~$\mathsf{rM}{:}1[\mathbb{P}][\mathtt{A}]$,~~	~~$\mathsf{r\tilde{M}}{:}1[\mathbb{P}][\mathtt{A}]$~~ $\}$
$\mathcal{Q}\{\mathsf{rM}{:}1[\mathtt{E}]\} = \{$	$\mathsf{rM}{:}1[\mathbb{R}][\mathtt{E}]$,	$\mathsf{r\tilde{M}}{:}1[\mathbb{R}][\mathtt{E}]$,	~~$\mathsf{rM}{:}1[\mathbb{P}][\mathtt{E}]$,~~	~~$\mathsf{r\tilde{M}}{:}1[\mathbb{P}][\mathtt{E}]$~~ $\}$
$\mathcal{Q}\{\mathsf{rM}{:}2[\mathtt{A}]\} = \{$	$\mathsf{rM}{:}2[\mathbb{R}][\mathtt{A}]$,	$\mathsf{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]$,	~~$\mathsf{rM}{:}2[\mathbb{P}][\mathtt{A}]$,~~	~~$\mathsf{r\tilde{M}}{:}2[\mathbb{P}][\mathtt{A}]$~~ $\}$
$\mathcal{Q}\{\mathsf{rM}{:}2[\mathtt{E}]\} = \{$	$\mathsf{rM}{:}2[\mathbb{R}][\mathtt{E}]$,	$\mathsf{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{E}]$,	~~$\mathsf{rM}{:}2[\mathbb{P}][\mathtt{E}]$,~~	~~$\mathsf{r\tilde{M}}{:}2[\mathbb{P}][\mathtt{E}]$~~ $\}$
$\mathcal{Q}\{\mathsf{rM}{:}3[\mathtt{A}]\} = \{$	$\mathsf{rM}{:}3[\mathbb{R}][\mathtt{A}]$,	$\mathsf{r\tilde{M}}{:}3[\mathbb{R}][\mathtt{A}]$,	~~$\mathsf{rM}{:}3[\mathbb{P}][\mathtt{A}]$,~~	~~$\mathsf{r\tilde{M}}{:}3[\mathbb{P}][\mathtt{A}]$~~ $\}$
$\mathcal{Q}\{\mathsf{rM}{:}3[\mathtt{E}]\} = \{$	$\mathsf{rM}{:}3[\mathbb{R}][\mathtt{E}]$,	$\mathsf{r\tilde{M}}{:}3[\mathbb{R}][\mathtt{E}]$,	~~$\mathsf{rM}{:}3[\mathbb{P}][\mathtt{E}]$,~~	~~$\mathsf{r\tilde{M}}{:}3[\mathbb{P}][\mathtt{E}]$~~ $\}$

22.2 Three Models

Below, we provide the strict definitions of recall-models treated in this part.

22.2.1 Model 1

22.2.1.1 Search-Enforced-Model 1: $\mathcal{Q}\{\mathsf{rM}{:}1[\mathtt{E}]\} = \{\mathbf{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}],\ \mathbf{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}],$ ~~$\mathbf{r}\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}],\ \mathbf{r}\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\}$~~

22.2.1.1.1 $\mathbf{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]$

This is the most basic model of the selling model with recall, which is the same as $\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]$ (see Section 4.1.1.1(p.13)) except that the price to be accepted is the best of prices rejected so far by buyers.

22.2.1.1.2 $\mathbf{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$

This is the most basic model of the buying model with recall, which is the same as $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$ (see Section 4.1.1.2(p.14)) except that the price to be accepted is the best of prices rejected so far by sellers.

22.2.1.2 Search-Allowed-Model 1: $\mathcal{Q}\{\mathsf{rM}{:}1[\mathtt{A}]\} = \{\mathbf{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}],\ \mathbf{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}],$ ~~$\mathbf{r}\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}],\ \mathbf{r}\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$~~

This is the same as the model in Section 22.2.1.1 except that the search is allowed.

22.2.2 Model 2

This is the model defined by adding the terminal quitting penalty ρ to Model 1 in Section 22.2.1.

22.2.3 Model 3

This is the model defined by adding the intervening quitting penalty ρ to Model 2 in Section 22.2.2.

22.3 Systems of Optimality Equations

22.3.1 Model 1

22.3.1.1 Search-Enforced-Model 1

22.3.1.1.1 rM:1[ℝ][E]

This is the most basic model with recall [43, Sak1961][0005], which is defined as below. By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y (best price) and with no highest buying price y respectively, expressed as

$$v_0(y) = y, \tag{22.3.1}$$

$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{22.3.2}$$

$$V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s, \quad t > 0, \tag{22.3.3}$$

where $V_t(y)$ is the maximum total expected present discounted *profit* from rejecting the highest buying price y, expressed as

$$V_t(y) \;=\; \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s, \quad t > 0. \tag{22.3.4}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\text{R}\} = \{(22.3.1) - (22.3.4)\}. \tag{22.3.5}$$

For convenience, let us define

$$V_0(y) = y. \tag{22.3.6}$$

Then (22.3.2) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.7}$$

From (22.3.3) and (22.3.4) with $t = 1$ we have respectively

$$V_1 \;=\; \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s, \tag{22.3.8}$$

$$V_1(y) \;=\; \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s \tag{22.3.9}$$

$$= K(y) + y \quad \text{(from (5.1.10(p.17)) with } \lambda = 1) \tag{22.3.10}$$

$$= L(y) + \beta y \quad \text{(from (5.1.9(p.17)))}. \tag{22.3.11}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.2) we see that the decision "whether or not to accept the highest buying price y" can be prescribed as follows:

$$\begin{cases} y \geq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \leq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted.} \end{cases} \tag{22.3.12}$$

22.3.1.1.2 r$\tilde{\text{M}}$:1[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y (best price) and with no lowest selling price y respectively, expressed as

$$v_0(y) = y, \tag{22.3.13}$$

$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{22.3.14}$$

$$V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s, \quad t > 0, \tag{22.3.15}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the lowest selling price y, expressed as

$$V_t(y) \;=\; \beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s, \quad t > 0. \tag{22.3.16}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \{(22.3.13) - (22.3.16)\}. \tag{22.3.17}$$

For convenience, let us define

$$V_0(y) = y. \tag{22.3.18}$$

Then (22.3.14) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.19}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.14) we see that the decision "whether or not to accept the lowest selling price y" can be prescribed as follows:

$$\begin{cases} y \leq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \geq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted.} \end{cases} \tag{22.3.20}$$

22.3.1.2 Search-Allowed-Model 1

22.3.1.2.1 rM:1[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y and with no highest buying price y respectively, expressed as

$$v_0(y) = y, \tag{22.3.21}$$
$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{22.3.22}$$
$$V_t = \max\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s, \beta V_{t-1}\} \quad t > 0, \tag{22.3.23}$$

where $V_t(y)$ is the maximum total expected present discounted *profit* from rejecting the highest buying price y, expressed as

$$V_t(y) \;=\; \max\{\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.24}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\texttt{A}]\text{R}\} = \{(22.3.21) - (22.3.24)\}. \tag{22.3.25}$$

For convenience, let us define

$$V_0(y) = y. \tag{22.3.26}$$

Then (22.3.22) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.27}$$

From (22.3.23) and (22.3.24) with $t = 1$ we have respectively

$$V_1 \;=\; \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s, \tag{22.3.28}$$
$$V_1(y) \;=\; \max\{\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s, \beta y\} \tag{22.3.29}$$
$$=\; \max\{K(y) + y, \beta y\} \qquad (\text{from } (5.1.10(\text{p.17})) \text{ with } \lambda = 1) \tag{22.3.30}$$
$$=\; \max\{L(y) + \beta y, \beta y\} \qquad (\text{from } (5.1.9(\text{p.17}))). \tag{22.3.31}$$
$$=\; \max\{L(y), 0\} + \beta y. \tag{22.3.32}$$

Let us here define

$$\mathbb{S}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \tag{22.3.33}$$

Then, (22.3.23) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{22.3.34}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{22.3.35}$$

Furthermore, let us define

$$\mathbb{S}_t(y) \;=\; \beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) - s, \quad t > 0. \tag{22.3.36}$$

Then (22.3.24) can be rewritten as

$$V_t(y) = \max\{\mathbb{S}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{22.3.37}$$

implying that

$$\mathbb{S}_t(y) \geq (\leq)\ 0 \Rightarrow \texttt{CONDUCT}_t\ (\texttt{SKIP}_t). \tag{22.3.38}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.22) we see that the decision "whether or not to accept the highest buying price y" can be prescribed as follows:

$$\begin{cases} y \geq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \leq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{CONDUCT}_t/\texttt{SKIP}_t.^{\dagger} \end{cases} \tag{22.3.39}$$

22.3.1.2.2 r$\tilde{\text{M}}$:1[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y and with no lowest selling price y respectively, expressed as

$$v_0(y) = y, \tag{22.3.40}$$
$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{22.3.41}$$
$$V_t = \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s, \beta V_{t-1}\} \quad t > 0, \tag{22.3.42}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the lowest selling price y, expressed as

$$V_t(y) \;=\; \min\{\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.43}$$

†The symbol "/" means "or", i.e., "$\texttt{CONDUCT}_t$ or $\texttt{SKIP}_t$".

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{r}\tilde{\mathsf{M}}\text{:1}[\mathbb{R}][\texttt{A}]\text{R}\} = \{(22.3.40) - (22.3.43)\}. \tag{22.3.44}$$

For convenience, let us define

$$V_0(y) = y. \tag{22.3.45}$$

Then (22.3.41) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.46}$$

Let us define

$$\tilde{\mathbb{S}}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{22.3.47}$$

Then, (22.3.42) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{22.3.48}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{22.3.49}$$

Let us define

$$\tilde{\mathbb{S}}_t(y) \;=\; \beta(\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) + s, \quad t > 0. \tag{22.3.50}$$

Then, (22.3.43) can be rewritten as, for any y,

$$V_t(y) = \min\{\tilde{\mathbb{S}}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{22.3.51}$$

implying that

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\ 0 \Rightarrow \texttt{CONDUCT}_t\ (\texttt{SKIP}_t). \tag{22.3.52}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.41) we see that the decision "whether or not to accept the lowest selling price y" can be prescribed as follows:

$$\begin{cases} y \leq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \geq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{CONDUCT}_t/\texttt{SKIP}_t. \end{cases} \tag{22.3.53}$$

22.3.2 Mode 2

22.3.2.1 Search-Enforced-Model 2

22.3.2.1.1 rM:2[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y and with no highest buying price y respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{22.3.54}$$

$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{22.3.55}$$

$$V_0 = \rho, \tag{22.3.56}$$

$$V_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \quad t > 0, \tag{22.3.57}$$

where $V_t(y)$ $(t > 0)$ is the maximum total expected present discounted *profit* from rejecting the highest buying price y, expressed as

$$V_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \quad t > 0. \tag{22.3.58}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\text{rM:2}[\mathbb{R}][\text{E}]\text{R}\} = \{(22.3.54) - (22.3.58)\}. \tag{22.3.59}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{22.3.60}$$

Then (22.3.55) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.61}$$

From (22.3.57) and (22.3.58) with $t = 1$ we have respectively

$$\begin{aligned} V_1 &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s \\ &= K(\rho) + \rho \quad \text{(from (5.1.10(p.17)))} \qquad (22.3.62) \\ &= L(\rho) + \beta\rho \quad \text{(from (5.1.9(p.17))),} \qquad (22.3.63) \end{aligned}$$

$$\begin{aligned} V_1(y) &= \lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s \\ &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}\}] + (1-\lambda)\beta\max\{y, \rho\} - s \\ &= K(\max\{y, \rho\}) + \max\{y, \rho\} \quad \text{(from (5.1.10(p.17)))} \qquad (22.3.64) \\ &= L(\max\{y, \rho\}) + \beta\max\{y, \rho\} \quad \text{(from (5.1.9(p.17))).} \qquad (22.3.65) \end{aligned}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.55) we see that the decision "whether or not to accept the highest buying price y" can be prescribed as follows:

$$\begin{cases} y \geq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \leq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted.} \end{cases} \tag{22.3.66}$$

22.3.2.2 r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y and with no lowest selling price y respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{22.3.67}$$
$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{22.3.68}$$
$$V_0 = \rho, \tag{22.3.69}$$
$$V_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \quad t > 0, \tag{22.3.70}$$

where $V_t(y)$ is the minimum total expected present discounted *cost* from rejecting the lowest selling price y, expressed as

$$V_t(y) \;=\; \lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0. \tag{22.3.71}$$

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\mathrm{R}\} = \{(22.3.67) - (22.3.71)\}. \tag{22.3.72}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{22.3.73}$$

Then (22.3.68) holds also for $t \geq$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.74}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.68) we see that the decision "whether or not to accept the lowest selling price y" can be prescribed as follows:

$$\begin{cases} y \leq V_t(y) & \Rightarrow \ \mathtt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \geq V_t(y) & \Rightarrow \ \mathtt{Reject}_t\langle y\rangle \text{ and the search is conducted.} \end{cases} \tag{22.3.75}$$

22.3.2.3 Search-Allowed-Model 2

22.3.2.3.1 rM:2[$\mathbb{R}$][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y and with no highest buying price y respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{22.3.76}$$
$$v_t(y) = \max\{y, V_t(y)\}, \quad t > 0, \tag{22.3.77}$$
$$V_0 = \rho, \tag{22.3.78}$$
$$V_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0, \tag{22.3.79}$$

where $V_t(y)$ $(t > 0)$ is the maximum total expected present discounted *profit* from rejecting the highest buying price y, expressed as

$$V_t(y) \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.80}$$

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{r}\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\} = \{(22.3.76) - (22.3.80)\}. \tag{22.3.81}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{22.3.82}$$

Then (22.3.77) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, V_t(y)\}, \quad t \geq 0, \tag{22.3.83}$$

From (22.3.79) and (22.3.80) with $t = 1$ we have respectively

$$\begin{aligned} V_1 &= \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} && (22.3.84)\\ &= \max\{K(\rho) + \rho, \beta\rho\} \quad \text{(see (5.1.10))} && (22.3.85)\\ &= \max\{L(\rho) + \beta\rho, \beta\rho\} \quad \text{(see (5.1.9))} && (22.3.86)\\ &= \max\{L(\rho), 0\} + \beta\rho, && (22.3.87) \end{aligned}$$

$$\begin{aligned} V_1(y) &= \max\{\lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y,\rho\} - s, \beta\max\{y,\rho\}\} && (22.3.88)\\ &= \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y,\rho\}\}] + (1-\lambda)\beta\max\{y,\rho\} - s, \beta\max\{y,\rho\}\} && \\ &= \max\{K(\max\{y,\rho\}) + \max\{y,\rho\}, \beta\max\{y,\rho\}\} \quad \text{(see (5.1.10))} && (22.3.89)\\ &= \max\{L(\max\{y,\rho\}) + \beta\max\{y,\rho\}, \beta\max\{y,\rho\}\} \quad \text{(see (5.1.9))} && (22.3.90)\\ &= \max\{L(\max\{y,\rho\}), 0\} + \beta\max\{y,\rho\}. && (22.3.91) \end{aligned}$$

Now, let us define

$$\mathbb{S}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \tag{22.3.92}$$

Then, (22.3.79) can be rewritten as

$$V_t = \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{22.3.93}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{22.3.94}$$

In addition, let us define

$$\mathbb{S}_t(y) \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) - s, \quad t > 0. \tag{22.3.95}$$

Then, (22.3.80) can be rewritten as, for any y,

$$V_t(y) = \max\{\mathbb{S}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{22.3.96}$$

implying that

$$\mathbb{S}_t(y) \geq (\leq)\ 0 \Rightarrow \texttt{CONDUCT}_t\ (\texttt{SKIP}_t). \tag{22.3.97}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.77) we see that the decision "whether or not to accept the highest buying price y" can be prescribed as follows:

$$\begin{cases} y \geq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \leq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{CONDUCT}_t/\texttt{SKIP}_t. \end{cases} \tag{22.3.98}$$

22.3.2.4 r$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y and with no lowest selling price y respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{22.3.99}$$

$$v_t(y) = \min\{y, V_t(y)\}, \quad t > 0, \tag{22.3.100}$$

$$V_0 = \rho, \tag{22.3.101}$$

$$V_t = \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\}, \quad t > 0, \tag{22.3.102}$$

where $V_t(y)$ $(t > 0)$ is the minimum total expected present discounted *cost* from rejecting the lowest selling price y, expressed as

$$V_t(y) \;=\; \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.103}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\mathrm{R}\} = \{(22.3.99) - (22.3.103)\}. \tag{22.3.104}$$

For convenience, let us define

$$V_0(y) = \rho. \tag{22.3.105}$$

Then (22.3.100) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, V_t(y)\}, \quad t \geq 0. \tag{22.3.106}$$

Let us define

$$\tilde{\mathbb{S}}_t \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{22.3.107}$$

Then, (22.3.102) can be rewritten as

$$V_t = \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \tag{22.3.108}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{22.3.109}$$

In addition, let us define

$$\tilde{\mathbb{S}}_t(y) \;=\; \lambda\beta(\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] - v_{t-1}(y)) + s, \quad t > 0. \tag{22.3.110}$$

Then, (22.3.103) can be rewritten as, for any y,

$$V_t(y) = \min\{\tilde{\mathbb{S}}_t(y), 0\} + \beta v_{t-1}(y), \quad t > 0, \tag{22.3.111}$$

implying that

$$\tilde{\mathbb{S}}_t(y) \leq (\geq)\ 0 \Rightarrow \texttt{CONDUCT}_t\ (\texttt{SKIP}_t). \tag{22.3.112}$$

From the comparison of the two terms within { } in the right-hand side of (22.3.100) we see that the decision "whether or not to accept the lowest selling price y" can be prescribed as follows:

$$\begin{cases} y \leq V_t(y) & \Rightarrow \quad \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \blacksquare. \\ y \geq V_t(y) & \Rightarrow \quad \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{CONDUCT}_t/\texttt{SKIP}_t. \end{cases} \tag{22.3.113}$$

22.3.3 Mode:3

22.3.3.1 Search-Enforced-Model 3

22.3.3.1.1 rM:3[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y and with no highest buying price y respectively, expressed as

$$v_0(y) = \max\{y, \rho\}, \tag{22.3.114}$$
$$v_t(y) = \max\{y, \rho, U_t(y)\}, \quad t > 0, \tag{22.3.115}$$
$$V_0 = \rho, \tag{22.3.116}$$
$$V_t = \max\{\rho, U_t\}, \quad t > 0. \tag{22.3.117}$$

where $U_t(y)$ in (22.3.115) is the maximum total expected present discounted *profit* from rejecting both y and ρ, expressed as

$$U_t(y) \ = \ \lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \quad t > 0. \tag{22.3.118}$$

and where U_t in (22.3.117) is the maximum total expected present discounted *profit* from rejecting ρ, expressed as

$$U_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \quad t > 0. \tag{22.3.119}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathrm{rM{:}3[\mathbb{R}][E]R}\} = \{(22.3.114) - (22.3.119)\}. \tag{22.3.120}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{22.3.121}$$

Then (22.3.115) and (22.3.117) hold also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \max\{\rho, U_t\} \cdots (2), \quad t \geq 0. \tag{22.3.122}$$

22.3.3.2 r$\tilde{\text{M}}$:3[ℝ][E]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y and with no lowest selling price y respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{22.3.123}$$
$$v_t(y) = \min\{y, \rho, U_t(y)\}, \quad t > 0, \tag{22.3.124}$$
$$V_0 = \rho, \tag{22.3.125}$$
$$V_t = \min\{\rho, U_t\}. \tag{22.3.126}$$

where $U_t(y)$ in (22.3.124) is the minimum total expected present discounted *cost* from rejecting both y and ρ, expressed as

$$U_t(y) \ = \ \lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0. \tag{22.3.127}$$

and where U_t in (22.3.126) is the minimum total expected present discounted *cost* from rejecting ρ, expressed as

$$U_t = \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \quad t > 0, \tag{22.3.128}$$

The system of optimality equations of this model is given by

$$\texttt{SOE}\{\mathrm{r\tilde{M}{:}3[\mathbb{R}][E]R}\} = \{(22.3.123) - (22.3.128)\}. \tag{22.3.129}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{22.3.130}$$

Then (22.3.124) and (22.3.126) hold also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \min\{y, U_t\} \cdots (2), \quad t \geq 0. \tag{22.3.131}$$

22.3.3.3 Search-Allowed-Model 3

22.3.3.3.1 rM:3[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximum total expected present discounted *profit* from initiating the process at time t with the highest buying price y and with no highest buying price $y$respectively, expressed as

$$v_0(y) = \max\{y, \rho\} \tag{22.3.132}$$
$$v_t(y) = \max\{y, \rho, U_t(y)\}, \quad t > 0, \tag{22.3.133}$$
$$V_0 = \rho, \tag{22.3.134}$$
$$V_t = \max\{\rho, U_t\}, \quad t > 0. \tag{22.3.135}$$

(22.3.136)

where $U_t(y)$ in (22.3.133) is the maximum total expected present discounted *profit* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) - s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.137}$$

where U_t in (22.3.135) is the maximum total expected present discounted *profit* from rejecting ρ, expressed as

$$U_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0, \tag{22.3.138}$$

(22.3.139)

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{rM{:}3[\mathbb{R}][A]R}\} = \{(22.3.132) - (22.3.138)\}. \tag{22.3.140}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{22.3.141}$$

Then (22.3.133) holds also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \max\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \max\{\rho, U_t\} \cdots (2), \quad t \geq 0. \tag{22.3.142}$$

22.3.3.4 r$\tilde{\mathsf{M}}$:3[ℝ][A]

By $v_t(y)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimum total expected present discounted *cost* from initiating the process at time t with the lowest selling price y and with no lowest selling price y respectively, expressed as

$$v_0(y) = \min\{y, \rho\} \tag{22.3.143}$$
$$v_t(y) = \min\{y, \rho, U_t(y)\}, \quad t > 0, \tag{22.3.144}$$
$$V_0 = \rho, \tag{22.3.145}$$
$$V_t = \min\{\rho, U_t\}, \quad t > 0, \tag{22.3.146}$$

where $U_t(y)$ in (22.3.144) is the minimum total expected present discounted *cost* from rejecting both y and ρ, expressed as

$$U_t(y) \;=\; \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}(y) + s, \beta v_{t-1}(y)\}, \quad t > 0. \tag{22.3.147}$$

and where U_t in (22.3.146) is the minimum total expected present discounted *cost* from rejecting ρ, expressed as

$$U_t = \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\}, \quad t > 0, \tag{22.3.148}$$

The system of optimality equations of this model is given by

$$\mathtt{SOE}\{\mathrm{r\tilde{M}{:}3[\mathbb{R}][A]R}\} = \{(22.3.143) - (22.3.148)\}. \tag{22.3.149}$$

For convenience, let us define

$$U_0(y) = \rho \cdots (1), \qquad U_0 = \rho \cdots (2). \tag{22.3.150}$$

Then (22.3.144) and (22.3.146) hold also for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(y) = \min\{y, \rho, U_t(y)\} \cdots (1), \qquad V_t = \min\{y, U_t\} \cdots (2), \quad t \geq 0. \tag{22.3.151}$$

22.3.4 Reservation Value

⟨a⟩ `t-reservation-value` (no-recall-model).
Consider the selling model with no recall. Then note (7.2.5(p.34)), i.e.,

$$\mathit{\Delta}_t(w) \geq (\leq)\ 0 \Leftrightarrow w \geq (\leq)\ V_t \Rightarrow \mathtt{Accept}_t\langle w\rangle\ \ (\mathtt{Reject}_t\langle w\rangle) \quad (\mathsf{S}\text{-model}), \tag{22.3.152}$$

implying that the reservation value of the model is given by V_t which depends on t, so let us refer to this as the t-dependent `reservation-value` or `t-reservation-value` for short.

⟨b⟩ `t-reservation-value` (recall-model).
Consider the selling model with recall. Here, by $\mathtt{A}_t(y)$ let us represent the profit from accepting the current highest buying price y (best price) at a given time t and by $\mathtt{R}_t(y)$ the profit from rejecting the current highest buying price y (best price) at a given time t. Then, we have $\mathtt{A}_t(y) = y$ and $\mathtt{R}_t(y) = V_t(y)$ from (22.3.2(p.222)), and then let us define

$$\mathtt{AR}_t(y) \overset{\text{def}}{=} \mathtt{A}_t(y) - \mathtt{B}_t(y) = y - V_t(y). \tag{22.3.153}$$

Here, suppose that there exists a y_t^* such that

$$\mathtt{AR}_t(y) \geq (\leq)\ 0 \Leftrightarrow y \geq (\leq)\ V_t(y) \Leftrightarrow y \geq (\leq)\ y_t^* \Leftrightarrow \mathtt{Accept}_t\langle y\rangle\ (\mathtt{Reject}_t\langle y\rangle) \quad \text{(see (22.3.12(p.222)))}, \tag{22.3.154}$$

implying that the reservation value of the model is given by y_t^* which depends on t, so let us also refer to this as `t-reservation-value`.

⟨c⟩ `c-reservation-value`.
If V_t = constant in t and y_t^* = constant in t, then let us refer to each of the two as the constant `reservation-value` or the `c-reservation-value` for short(see C1a(p.206) with ‖[a] for the former and Pom 23.1.1(p.234) (a) for the latter).

Chapter 23

Model 1

23.1 Search-Enforced-Model 1

23.1.1 rM:1[ℝ][E]

Below let us define

$$\mathbb{V}_t \overset{\text{def}}{=} V_t - \beta V_{t-1}, \quad t > 1. \tag{23.1.1}$$

23.1.1.1 Some Lemmas

Lemma 23.1.1 (rM:1[ℝ][E])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \geq 0$.*
(b) *$v_t(y)$ and $V_t(y)$ are nondecreasing in $t \geq 0$ and $t > 0$ respectively for any y.*[†]
(c) *$\mathbb{V}_t$ is nondecreasing in $t > 0$.* □

Proof (a) $v_0(y)$ is nondecreasing in y from (22.3.1(p.222)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (22.3.4(p.222)), hence $v_t(y)$ is nondecreasing in y from (22.3.2(p.222)). Thus, by induction $v_t(y)$ is nondecreasing in y and $t \geq 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is also nondecreasing in y for $t > 0$ from (22.3.4). In addition, $V_0(y)$ is nondecreasing in y from (22.3.6(p.222)), hence it follows that $V_t(y)$ is nondecreasing in y for $t \geq 0$.

(b) Clearly $v_1(y) \geq y = v_0(y)$ for any y from (22.3.2) with $t = 1$ and (22.3.1). Suppose $v_{t-1}(y) \geq v_{t-2}(y)$ for any y. Then, from (22.3.4) we have $V_t(y) \geq \beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s = V_{t-1}(y)$ for any y. Hence, from (22.3.2) we have $v_t(y) \geq \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y) \geq v_{t-1}(y)$ for $t > 0$ and any y, i.e., $v_t(y)$ is nondecreasing in $t \geq 0$ for any y. Accordingly, since $v_{t-1}(y) \geq v_{t-2}(y)$ for $t > 1$ and any y, from (22.3.4) we have $V_t(y) \geq \beta\,\mathbf{E}[v_{t-2}(y)] - s = V_{t-1}(y)$ for $t > 1$ and any y, hence $V_t(y)$ is nondecreasing in $t > 0$ for any y.

(c) We have $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y due to (b), hence V_t is nondecreasing in $t > 0$ from (22.3.3). ∎

Lemma 23.1.2 (rM:1[ℝ][E])

(a) *Let $x_K \leq y$. Then $V_t(y) \leq y$ for $t \geq 0$.*
(b) *Let $y \leq x_K$. Then $y \leq V_t(y) \leq x_K$ for $t \geq 0$.* □

Proof [‡] (a) Let $x_K \leq y$. Then $K(y) \leq 0 \cdots \mathbf{(1)}$ from Corollary 9.2.2(p.44)(a). From (22.3.6(p.222)) we clearly have $V_0(y) \leq y$. Suppose $V_{t-1}(y) \leq y$, hence $v_{t-1}(y) = y$ from (22.3.2(p.222)). Then, since $x_K \leq y \leq \max\{\boldsymbol{\xi}, y\}$ for any $\boldsymbol{\xi}$, we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) = \max\{\boldsymbol{\xi}, y\}$. Accordingly, from (22.3.4(p.222)) we have $V_t(y) = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y \cdots \mathbf{(2)}$ due to (5.1.10(p.17)) with $\lambda = 1$, hence $V_t(y) \leq y$ due to (1). This completes the induction.

(b) Let $y \leq x_K \cdots \mathbf{(3)}$. Then $K(y) \geq 0 \cdots \mathbf{(4)}$ from Corollary 9.2.2(p.44)(b). Now, from (22.3.7) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \geq \max\{\boldsymbol{\xi}, y\}$ for any $t > 0$, $\boldsymbol{\xi}$, and y, hence from (22.3.4) and (5.1.10) with $\lambda = 1$ we have $V_t(y) \geq \beta[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y$ for $t > 0$, so that $V_t(y) \geq y$ for $t > 0$ due to (4). In addition, since $V_0(y) \geq y$ from (22.3.6), it follows that $V_t(y) \geq y$ for $t \geq 0$. Now, since $\max\{\boldsymbol{\xi}, y\} \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (3), from Lemma 23.1.1(a) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \cdots \mathbf{(5)}$ for any $\boldsymbol{\xi}$ and $t > 0$. Since $x_K \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, due to (a) we have $V_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$, hence $v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$ from (22.3.7(p.222)), so that from (5) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ and $t > 0$. Thus, from (22.3.4) and (5.1.10(p.17)) with $\lambda = 1$ we have $V_t(y) \leq \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s = K(x_K) + x_K = x_K$ for $t > 0$. ∎

Since $V_t(y)$ is nondecreasing in $t > 0$ from Lemma 23.1.1(b) and is upper bounded in t from Lemma 23.1.2(a,b), it converges to a finite $V(y)$ as $t \to \infty$, hence so also do $v_t(y)$, V_t, and $\mathbb{V}_t$ (see (23.1.1(p.229))). Then, defining these limits by $v(y)$, V, and $\mathbb{V}$, from (22.3.3(p.222)), (22.3.2), (22.3.4), and (23.1.1) we have:

[†]It cannot be always guaranteed that $V_1(y) \geq V_0(y)$. For example, let $\beta < 1$ or $s > 0$ and let $y > x_K$. Then, from (22.3.10(p.222)) and (22.3.6(p.222)) we have $V_1(y) - V_0(y) = K(y) < 0$ due to Lemma 9.2.2(p.43)(j1), i.e., $V_1(y) < V_0(y)$.

[‡]Although (a) and (b) are already proven in [43,Sakaguchi,1961][0005], we anew prove herein the two by using properties of the underlying function $K(x)$ (see (5.1.4(p.17))).

$$
\begin{aligned}
V(y) &= \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})] - s, && (23.1.2)\\
v(y) &= \max\{y, V(y)\}, && (23.1.3)\\
V &= \beta\,\mathbf{E}[v(\boldsymbol{\xi})] - s, && (23.1.4)\\
\mathbb{V} &= (1-\beta)V. && (23.1.5)
\end{aligned}
$$

Lemma 23.1.3 (rM:1[ℝ][E])

(a) *Let $x_K \le y$. Then $V(y) \le y$.*
(b) *Let $y \le x_K$. Then $y \le V(y) \le x_K$.* □

Proof Immediate from Lemma 23.1.2. ∎

Lemma 23.1.4 (rM:1[ℝ][E]) *Let $\beta < 1$.*

(a) *Let $y \le x_K$. Then $V(y) = x_K$.*
(b) *$v(y) = \max\{y, x_K\}$ for any y.*
(c) *$V = x_K$.*
(d) *Let $\kappa > (= (<))\ 0$. Then $\mathbb{V} > (= (<))\ 0$.* □

Proof Let $\beta < 1 \cdots$ **(1)**.

(a) Let $y \le x_K \cdots$ **(2)**. Now, (23.1.2) can be rewritten as

$$V(y) = \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \le x_K)] - s \cdots \textbf{(3)}.$$

If $x_K < \boldsymbol{\xi}$, then $y < \boldsymbol{\xi}$ from (2), hence $x_K < \boldsymbol{\xi} = \max\{\boldsymbol{\xi}, y\}$. Thus, from Lemma 23.1.3(a) we have $V(\max\{\boldsymbol{\xi}, y\}) \le \max\{y, \boldsymbol{\xi}\} = \boldsymbol{\xi}$, so that $v(\max\{\boldsymbol{\xi}, y\}) = \max\{y, \boldsymbol{\xi}\} = \boldsymbol{\xi}$ due to (23.1.3). Therefore, (3) can rewritten as

$$V(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[v(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \le x_K)] - s \cdots \textbf{(4)}.$$

In addition, since $v(\max\{\boldsymbol{\xi}, y\}) = \max\{\max\{\boldsymbol{\xi}, y\}, V(\max\{\boldsymbol{\xi}, y\})\}$ from (23.1.3) for ξ and y, we can rewrite (4) as

$$V(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, V(\max\{\boldsymbol{\xi}, y\})\}I(\boldsymbol{\xi} \le x_K)] - s. \cdots \textbf{(5)}$$

To prove (a) it suffices to show the following two:

1. *The function $V(y) = x_K \cdots$ **(6)** with $y \le x_K$ is a solution of the functional equation (5)* To prove this it suffices to show that, substituting the relation $V(y) = x_K$ with $y \le x_K$ for the r.h.s. of (5) yields x_K, implying that the r.h.s. rustlingly becomes equal to the $V(y)$ in its l.h.s., i.e., $V(y) = x_K$. Below, let us show this. Let $\boldsymbol{\xi} \le x_K$. Then $\max\{y, \boldsymbol{\xi}\} \le \max\{x_K, x_K\} = x_K \cdots$ **(7)** due to (2), hence $V(\max\{y, \boldsymbol{\xi}\}) = x_K \cdots$ **(8)** due to (6), so $\max\{\max\{y, \boldsymbol{\xi}\}, V(\max\{y, \boldsymbol{\xi}\})\} = \max\{\max\{y, \boldsymbol{\xi}\}, x_K\} = x_K$ due to (8) and (7). Consequently, we eventually get

$$
\begin{aligned}
\text{r.h.s of (5)} &= \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[x_K I(\boldsymbol{\xi} \le x_K)] - s\\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}I(\boldsymbol{\xi} \le x_K)] - s\\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s\\
&= K(x_K) + x_K \quad \text{(See (5.1.10(p.17))) with } \lambda = 1\\
&= x_K.
\end{aligned}
$$

Accordingly, it follows that $V(y) = x_K$ with $y \le x_K$ is a solution of the functional equation (5).

2. *The solution is unique* Suppose there exists *another* solution $Z(y)$ with $y \le x_K$ where $V(y) \ne Z(y)$ for at least one $y \le x_K$. Then, let $z(y) \overset{\text{def}}{=} \max\{y, Z(y)\} \cdots$ **(9)** with $y \le x_K$ (see (23.1.3)). Accordingly, we have (see (4))

$$Z(y) = \beta\,\mathbf{E}[\boldsymbol{\xi} I(x_K < \boldsymbol{\xi})] + \beta\,\mathbf{E}[z(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \le x_K)] - s. \cdots \textbf{(10)}$$

Hence, from (4) and (10) we have

$$
\begin{aligned}
|V(y) - Z(y)| &= \big|\beta\,\mathbf{E}[(v(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\}))I(\boldsymbol{\xi} \le x_K)]\big|\\
&\le \beta\,\mathbf{E}[\,\big|v(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\})\big|I(\boldsymbol{\xi} \le x_K)]. \cdots \textbf{(11)}.
\end{aligned}
$$

Now, in general

$$|v(y) - z(y)| = |\max\{y, V(y)\} - \max\{y, Z(y)\}| \le \max\{0, |V(y) - Z(y)|\} = |V(y) - Z(y)|$$

for any y, hence we have

$$\big|v(\max\{\boldsymbol{\xi}, y\}) - z(\max\{\boldsymbol{\xi}, y\})\big| \le \big|V(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})\big| \cdots \textbf{(12)}.$$

for any y and $\boldsymbol{\xi}$. Thus, from (11) we have

$$|V(y) - Z(y)| \le \beta\,\mathbf{E}[\,\big|V(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})\big|I(\boldsymbol{\xi} \le x_K)] \cdots \textbf{(13)}.$$

Let $\nu = \max_{y \le x_K} |V(y) - Z(y)| \cdots$ **(**14**)** where $0 < \nu \cdots$ **(**15**)**, hence $|V(y) - Z(y)| \le \nu \cdots$ **(**16**)** for $y \le x_K$. If $\boldsymbol{\xi} \le x_K$, then $\max\{\boldsymbol{\xi}, y\} \le \max\{x_K, x_K\} = x_K \cdots$ **(**17**)**, hence $|V(\max\{\boldsymbol{\xi}, y\}) - Z(\max\{\boldsymbol{\xi}, y\})| \le \nu$ due to (16). Accordingly, from (13) we have

$$|V(y) - Z(y)| \le \beta\,\mathbf{E}[\nu I(\boldsymbol{\xi} \le x_K)] = \beta\nu\,\mathbf{E}[I(\boldsymbol{\xi} \le x_K)] = \beta\nu \Pr\{\boldsymbol{\xi} \le x_K\} = \beta\nu F(x_K).$$

Thus, we have $\nu \le \beta\nu F(x_K) \cdots$ **(**18**)** due to the definition (14). In addition, since $\beta\nu F(x_K) \le \beta\nu$ due to $F(x_K) \le 1$, we have $\nu \le \beta\nu$ from (18), leading to the contradiction of $1 \le \beta$ due to (15). Accordingly, $V(y) = x_K$ with $y \le x_K$ must be the unique solution of (5).

(b) If $x_K \le y$, from Lemma 23.1.3(a) and (23.1.3) we have $v(y) = y = \max\{y, x_K\}$. If $y \le x_K$, from Lemma 23.1.3(b) and (23.1.3) we have $v(y) = V(y) \le x_K$ and from (a) we have $V(y) = x_K$, hence it follows that $v(y) = V(y) = x_K = \max\{y, x_K\}$. Thus, whether $x_K \le y$ or $y \le x_K$, we have $v(y) = \max\{y, x_K\}$.

(c) Since $v(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (b), from (23.1.4) we have $V = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s = K(x_K) + x_K = x_K$ (see (5.1.10(p.17))).

(d) Let $\kappa > (= (<))\ 0$. Then, since $x_K > (= (<))\ 0$ due to Lemma 9.3.1(p.45) (b), from (c) we have $V > (= (<))\ 0$, hence the assertion becomes true from (23.1.5). ∎

Here, let us define

$$\ell_t(y) \stackrel{\text{def}}{=} v_t(y) - \beta v_{t-1}(y), \quad t > 0. \tag{23.1.6}$$

Then, from (23.1.1) and (22.3.3(p.222)) we have

$$\begin{aligned}
\mathbb{V}_t &= \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s - \beta(\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] - s) && (23.1.7)\\
&= \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi}) - \beta v_{t-2}(\boldsymbol{\xi})] - (1-\beta)s && (23.1.8)\\
&= \beta\,\mathbf{E}[\ell_{t-1}(\boldsymbol{\xi})] - (1-\beta)s, \quad t > 1. && (23.1.9)
\end{aligned}$$

Furthermore, for any y let us define

$$A(y) \stackrel{\text{def}}{=} \ell_2(y) - \ell_1(y). \tag{23.1.10}$$

Lemma 23.1.5 (rM:1[ℝ][E])

(a) *Let $x_K \le y$. Then $A(y) = 0$.*
(b) *Let $y \le x_K$. Then $A(y)$ is nondecreasing in y.*
(c) *$A(y) \le 0$ for any y.* □

Proof (a) Let $x_K \le y$. Then $V_2(y) \le y$ and $V_1(y) \le y$ from Lemma 23.1.2(a), hence from (22.3.7(p.222)) we have $v_2(y) = v_1(y) = y$. In addition, $v_0(y) = y$ from (22.3.1). Thus, since $\ell_2(y) = v_2(y) - \beta v_1(y) = (1-\beta)y$ and $\ell_1(y) = v_1(y) - \beta v_0(y) = (1-\beta)y$, we have $A(y) = 0 \cdots$ **(**1**)**.

(b) Let $y \le x_K \cdots$ **(**2**)**. Now, from Lemma 23.1.2(b) and (22.3.2(p.222)) we have

$$\begin{aligned}
v_1(y) &= V_1(y) = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s && (\leftarrow (22.3.9(\text{p.222}))) && (23.1.11)\\
&= K(y) + y && (\leftarrow (5.1.10)), && (23.1.12)\\
v_2(y) &= V_2(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s && (\leftarrow (22.3.4(\text{p.222})) \text{ with } t = 2). && (23.1.13)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\ell_1(y) &= v_1(y) - \beta v_0(y) = v_1(y) - \beta y,\\
\ell_2(y) &= v_2(y) - \beta v_1(y) = \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s - \beta v_1(y),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
A(y) &= \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})] - s - (1+\beta)v_1(y) + \beta y\\
&= \beta\,\mathbf{E}[v_1(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + v_1(\max\{\boldsymbol{\xi}, y\})I(x_K \le \boldsymbol{\xi})] - s - (1+\beta)v_1(y) + \beta y.
\end{aligned}$$

If $\boldsymbol{\xi} < x_K$, due to (2) we have $\max\{\boldsymbol{\xi}, y\} \le \max\{x_K, x_K\} = x_K$, hence from (23.1.12) we have

$$v_1(\max\{\boldsymbol{\xi}, y\}) = K(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}.$$

If $x_K \le \boldsymbol{\xi}$, then $x_K \le \boldsymbol{\xi} \le \max\{\boldsymbol{\xi}, y\}$ for any y, hence from Lemma 23.1.2(a) we have $V_1(\max\{\boldsymbol{\xi}, y\}) \le \max\{\boldsymbol{\xi}, y\}$, so that $v_1(\max\{\boldsymbol{\xi}, y\}) = \max\{\boldsymbol{\xi}, y\}$ from (22.3.2(p.222)) with $t = 1$. Accordingly, we have

$$\begin{aligned}
A(y) &= \beta\,\mathbf{E}[(K(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}I(x_K \le \boldsymbol{\xi})] - s - (1+\beta)v_1(y) + \beta y\\
&= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}(I(\boldsymbol{\xi} < x_K) + I(x_K \le \boldsymbol{\xi}))] - s - (1+\beta)v_1(y) + \beta y\\
&= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) + \max\{\boldsymbol{\xi}, y\}] - s - (1+\beta)v_1(y) + \beta y\\
&= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s - (1+\beta)v_1(y) + \beta y.
\end{aligned}$$

Using (23.1.11), we can rewrite the above as

$$\begin{aligned} A(y) &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] + v_1(y) - (1+\beta)v_1(y) + \beta y \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] - \beta(v_1(y) - y). \end{aligned}$$

Furthermore, since $v_1(y) - y = K(y)$ due to (23.1.12), we can rewrite the above as

$$\begin{aligned} A(y) &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K)] - \beta K(y) \\ &= \beta\,\mathbf{E}[K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) - K(y)] \\ &= \beta\,\mathbf{E}[B(\boldsymbol{\xi}, y)] \end{aligned} \tag{23.1.14}$$

where

$$B(\boldsymbol{\xi}, y) \stackrel{\text{def}}{=} K(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} < x_K) - K(y).$$

Now we have:

1 Let $x_K \le \boldsymbol{\xi}$. Then $I(\boldsymbol{\xi} < x_K) = 0$, hence $B(\boldsymbol{\xi}, y) = -K(y)$, which is nondecreasing in $y \le x_K$ from Lemma 9.2.2(p.43) (b).

2 Let $\boldsymbol{\xi} < x_K$. Then $I(\boldsymbol{\xi} < x_K) = 1$, hence $B(\boldsymbol{\xi}, y) = K(\max\{\boldsymbol{\xi}, y\}) - K(y)$ for $y \le x_K$. Thus, if $y \le \boldsymbol{\xi}$, then $B(\boldsymbol{\xi}, y) = K(\boldsymbol{\xi}) - K(y)$, which is nondecreasing in $y \le \boldsymbol{\xi}$ due to Lemma 9.2.2(p.43) (b) and if $\boldsymbol{\xi} < y\,(\le x_K)$, then $B(\boldsymbol{\xi}, y) = K(y) - K(y) = 0$ for $y \le x_K$, which can be regarded as nondecreasing in $y > \boldsymbol{\xi}$. Therefore, it follows that $B(\boldsymbol{\xi}, y)$ is nondecreasing in $y \le x_K$ whether $y \le \boldsymbol{\xi}$ or $y > \boldsymbol{\xi}$.

From the above two results we have that, whether $x_K \le \boldsymbol{\xi}$ or $\boldsymbol{\xi} < x_K$, $B(\boldsymbol{\xi}, y)$ is nondecreasing in $y \le x_K$. Hence, from (23.1.14) it eventually follows that $A(y)$ is nondecreasing in $y \le x_K$.

(c) Immediate from (a,b) and the fact that $A(y)$ is continuous on $(-\infty, \infty)$. ∎

Lemma 23.1.6 (rM:1[ℝ][E])

(a) *$\ell_t(y)$ is nonincreasing in $t > 0$ for any y.*

(b) *$\mathbb{V}_t$ is nonincreasing in $t > 1$.* □

Proof (a) From Lemma 23.1.5(c) and (23.1.10) we have $\ell_2(y) \le \ell_1(y)$ for any y. Suppose that $\ell_{t-1}(y) \le \ell_{t-2}(y)$ for any y.

1. Let $x_K \le y$. Then, since $V_t(y) \le y$ for $t \ge 0$ due to Lemma 23.1.2(a), we have $V_{t-1}(y) \le y$ for $t \ge 1$, hence $v_t(y) = y$ for $t \ge 0$ and $v_{t-1}(y) = y$ for $t \ge 1$ from (22.3.7(p.222)). Thus, from (23.1.6) we have $\ell_t(y) = (1-\beta)y$ for $t \ge 1$ and $\ell_{t-1}(y) = (1-\beta)y$ for $t \ge 2$, so that $\ell_t(y) = \ell_{t-1}(y)$ for $t \ge 2$, hence $\ell_t(y) \le \ell_{t-1}(y)$ for $t \ge 2$. Accordingly, it follows that $\ell_t(y)$ is nonincreasing in $t \ge 1$ or equivalently in $t > 0$ on $x_K \le y$.
2. Let $y \le x_K$. Then, since $y \le V_t(y)$ for $t \ge 0$ and $y \le V_{t-1}(y)$ for $t > 0$ from Lemma 23.1.2(b), we have $v_t(y) = V_t(y)$ for $t \ge 0$ and $v_{t-1}(y) = V_{t-1}(y)$ for $t > 0$ from (22.3.7), hence from (23.1.6) and (22.3.4(p.222)) we have

$$\begin{aligned} \ell_t(y) &= V_t(y) - \beta V_{t-1}(y) \\ &= \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s - \beta(v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s) \\ &= \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - \beta v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s \\ &= \beta\,\mathbf{E}[\ell_{t-1}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s, \quad t > 0. \end{aligned}$$

Thus, we have

$$\ell_{t-1}(y) = \beta\,\mathbf{E}[\ell_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s, \quad t > 1.$$

Here, since $\ell_{t-1}(\max\{\boldsymbol{\xi}, y\}) \le \ell_{t-2}(\max\{\boldsymbol{\xi}, y\})$ due to the induction hypothesis, we have $\ell_t(y) \le \beta\,\mathbf{E}[\ell_{t-2}(\max\{\boldsymbol{\xi}, y\})] - (1-\beta)s = \ell_{t-1}(y)$ for $t > 1$. Accordingly, by induction we have $\ell_t(y) \le \ell_{t-1}(y)$ for $t > 1$ on $y \le x_K$, i.e., $\ell_t(y)$ is nonincreasing in $t > 0$ on $y \le x_K$.

From the above two results it eventually follows, whether $x_K \le y$ or $y \le x_K$, $\ell_t(y)$ is nonincreasing in $t > 0$.

(b) Immediate from (a) and (23.1.9). ∎

23.1.1.2 Analysis

From (22.3.3(p.222)) with $t = 2$ we have

$$\begin{aligned} V_2 &= \beta\,\mathbf{E}[v_1(\boldsymbol{\xi})] - s \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, V_1(\boldsymbol{\xi})\}] - s \qquad \text{(see (22.3.2) with } t = 1) \\ &= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, K(\boldsymbol{\xi}) + \boldsymbol{\xi}\}] - s \quad \text{(see (22.3.10) with } y = \boldsymbol{\xi}) \\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\} + \boldsymbol{\xi}] - s \\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\,\mathbf{E}[\boldsymbol{\xi}] - s \\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\mu - s. \end{aligned}$$

Then (23.1.1(p.229)) with $t = 2$ can be rewritten as

$$\begin{aligned}\mathbb{V}_2 &= V_2 - \beta V_1\\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + \beta\mu - s - \beta(\beta\mu - s) \quad \text{(see (22.3.8))}\\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}] + (1-\beta)(\beta\mu - s)\\ &= \beta\,\mathbf{E}[\max\{0, K(\boldsymbol{\xi})\}I(\boldsymbol{\xi} < x_K)] + \max\{0, K(\boldsymbol{\xi})\}I(x_K \le \boldsymbol{\xi})] + (1-\beta)(\beta\mu - s).\end{aligned}$$

Due to Corollary 9.2.2(p.44) (a) we have $K(\boldsymbol{\xi}) > 0$ for $\boldsymbol{\xi} < x_K$ and $K(\boldsymbol{\xi}) \le 0$ for $x_K \le \boldsymbol{\xi}$, hence we have

$$\mathbb{V}_2 = \beta\,\mathbf{E}[K(\boldsymbol{\xi})I(\boldsymbol{\xi} < x_K)] + (1-\beta)(\beta\mu - s). \tag{23.1.15}$$

Let us define

$$\mathtt{S}_{18}\boxed{\circledS\blacktriangle|\circledast\vartriangle|\circledast\blacktriangle} = \left\{\begin{array}{l}\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau\ (t^\circ_\tau \ge t^\bullet_\tau > 1) \text{ such that}\\ \boxed{\circledS\ \mathtt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_\blacktriangle,\ \boxed{\circledast\ \mathtt{ndOIT}_{t^\circ_\tau \ge \tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle, \text{ and } \boxed{\circledast\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\blacktriangle.\end{array}\right\}$$

□ **Tom 23.1.1** ($\mathscr{A}${rM:1[ℝ][E]}) *For any* $\tau > 1$:

(a) *We have*:

1. ♣ *Let* $y \ge x_K$. *Then* $y \ge V_t(y)$ *for* $t \ge 0$.
2. ♣ *Let* $y \le x_K$. *Then* $y \le V_t(y)$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu - s \ge 0$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.
2. *Let* $\beta\mu - s < 0$ *and* $\beta\mu - s < a$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_\blacktriangle$.
3. *Let* $\beta\mu - s < 0$ *and* $\beta\mu - s \ge a$ (*hence* $a < 0$).
   i. *Let* $\mathbb{V}_2 \le 0$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_\vartriangle$.
   ii. *Let* $\mathbb{V}_2 > 0$.
      1. *Let* $\kappa \ge 0$. *Then* $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.
      2. *Let* $\kappa < 0$. *Then we have* $\mathtt{S}_{18}$(p.233)$\boxed{\circledS\blacktriangle|\circledast\vartriangle|\circledast\blacktriangle}$. ∎

Proof Since $\lambda = 1$ is assumed in the model, we have $\delta = 1$ (See (9.2.1(p.42))). Hence $(\lambda\beta\mu - s)/\delta = \beta\mu - s \cdots$ **(1)** and $K(a) = \beta\mu - s - a \cdots$ **(2)** from (9.2.4 (1) (p.42)).

(a1,a2) The same as Lemma 23.1.2(a,b).

(b) Let $\beta = 1$. Then, from (23.1.1) we have $\mathbb{V}_t = V_t - \beta V_{t-1} = V_t - V_{t-1}$ for $t > 1$, hence $\mathbb{V}_t \ge 0$ for $t > 1$ due to Lemma 23.1.1(c) or equivalently $V_t \ge \beta V_{t-1}$ for $t > 1$. Thus, since $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 1$, we have $V_\tau \ge \beta V_{\tau-1}$, $V_{\tau-1} \ge \beta V_{\tau-2}, \cdots, V_2 \ge \beta V_1$, hence $V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{\tau-2} \ge \cdots \ge \beta^{\tau-1}V_1$, so that $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.

(c) Let $\beta < 1$.

(c1) Let $\beta\mu - s \ge 0$, hence $V_1 \ge 0$ from (22.3.8(p.222)). Then $V_t \ge V_{t-1} \ge V_1 \ge 0$ for $t > 1$ from Lemma 23.1.1(c). Hence, from (23.1.1) we have $\mathbb{V}_t = V_t - \beta V_{t-1} \ge V_{t-1} - \beta V_{t-1} = (1-\beta)V_{t-1} \ge 0$ for $t > 1$. Then, since $V_t \ge \beta V_{t-1}$ for $t > 1$, for the same reason as in the proof of (b) we have $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$.

(c2) Let $\beta\mu - s < 0 \cdots$ **(3)** and $\beta\mu - s < a$ from (22.3.8(p.222)). Then, from (2) we have $K(a) < 0$, hence $x_K < a$ from Lemma 9.2.2(j1). Below consider only y such that $x_K < a \le y \in [a, b]$. Then, since $V_t(y) \le y$ for $t \ge 0$ from Lemma 23.1.2(a), we have $v_t(y) = y$ for $t \ge 0$ from (22.3.7(p.222)), hence $v_{t-1}(y) = y$ for $t > 0$, so that from (23.1.6) we have $\ell_t(y) = v_t(y) - \beta v_{t-1}(y) = y - \beta y = (1-\beta)y$ for $t > 0$. Accordingly, since $\ell_{t-1}(\boldsymbol{\xi}) = (1-\beta)\boldsymbol{\xi}$ for $t > 1$ and $\boldsymbol{\xi} \in [a, b]$, from (23.1.9) we obtain $\mathbb{V}_t = V_t - \beta V_{t-1} = \beta\,\mathbf{E}[(1-\beta)\boldsymbol{\xi}] - (1-\beta)s = \beta(1-\beta)\,\mathbf{E}[\boldsymbol{\xi}] - (1-\beta)s = \beta(1-\beta)\mu - (1-\beta)s = (1-\beta)(\beta\mu - s) < 0$ for $t > 1$ due to (3). Then, since $V_t < \beta V_{t-1}$ for $t > 1$, we have $V_t < \beta V_{t-1}$ for $\tau \ge t > 1$. Accordingly, since $V_\tau < \beta V_{\tau-1}$, $V_{\tau-1} < \beta V_{\tau-2}, \cdots, V_2 < \beta V_1$, we have $V_\tau < \beta V_{\tau-1} < \beta^2 V_{\tau-2} < \cdots < \beta^{\tau-1}V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_\blacktriangle$.

(c3) Let $\beta\mu - s < 0 \cdots$ **(4)** and $\beta\mu - s \ge a$, hence $a < 0$. Then $K(a) \ge 0$ from (2), so $a \le x_K \cdots$ **(5)** from Lemma 9.2.2(j1).

(c3i) Let $\mathbb{V}_2 \le 0$. Then, since $\mathbb{V}_t \le 0$ for $t > 1$ from Lemma 23.1.6(b), we have $\mathbb{V}_t \le 0$ for $\tau \ge t > 1$. Hence, since $V_\tau - \beta V_{\tau-1} \le 0$ for $\tau \ge t > 1$ from (23.1.1), we have $V_\tau \le \beta V_{\tau-1}$ for $\tau \ge t > 1$. Accordingly, since $V_\tau \le \beta V_{\tau-1}$, $V_{\tau-1} \le \beta V_{\tau-2}$, $\cdots, V_2 \le \beta V_1$, we have $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1}V_1$, so that $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \mathtt{dOITd}_\tau\langle 1\rangle}_\vartriangle$.

(c3ii) Let $\mathbb{V}_2 > 0 \cdots$ **(6)**.

(c3ii1) Let $\kappa \ge 0$. Then $\mathbb{V} \ge 0$ due to Lemma 23.1.4(d). Hence, from (6) and Lemma 23.1.6(b) we have $\mathbb{V}_t \ge 0$ for $t > 1$, so we obtain $\boxed{\circledS\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\vartriangle$ for the same reason as in the proof of (c1).

(c3ii2) Let $\kappa < 0$. Then $\mathbb{V} < 0$ due to Lemma 23.1.4(d). Hence, from (6), and Lemma 23.1.6(b) it follows that there exist t°_τ and $t^\bullet_\tau$ ($t^\circ_\tau \ge t^\bullet_\tau > 1$) such that

$$\mathbb{V}_2 \ge \cdots \ge \mathbb{V}_{t^\bullet_\tau - 1} \ge \mathbb{V}_{t^\bullet_\tau} > 0 \ge \mathbb{V}_{t^\bullet_\tau + 1} \ge \mathbb{V}_{t^\bullet_\tau + 1} \ge \cdots \ge \mathbb{V}_{t^\circ_\tau} > \mathbb{V}_{t^\circ_\tau + 1} \ \ge\ \cdots$$

or equivalently

$$\mathbb{V}_t > 0 \cdots (1^*), \quad t^\bullet_\tau \ge t > 1, \qquad 0 \ge \mathbb{V}_t \ \cdots (2^*), \quad t^\circ_\tau \ge t > t^\bullet_\tau, \qquad 0 > \mathbb{V}_t \ \cdots (3^*), \quad t > t^\circ_\tau.$$

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $\mathbb{V}_t > 0$ for $\tau \geq t > 1$ due to (1^*), for almost the same reason as in the proof of (b) we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (7)$, hence $t^*_\tau = \tau$ for $t^\bullet_\tau \geq \tau > 1$, i.e., $\boxed{ⓢ\ \mathtt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \cdots (8)$. From (7) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \beta^2 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau - 1} V_1.$$

2. Since $\mathbb{V}_{t^\bullet_\tau+1} \leq 0$ due to (2^*), we have $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$ from (23.1.1). Hence

$$V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau-1} > \beta^3 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau} V_1 \cdots (9),$$

so $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$ or equivalently $\boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (10)$. Since $\mathbb{V}_{t^\bullet_\tau+2} \leq 0$ due to (2^*), we have $V_{t^\bullet_\tau+2} \leq \beta V_{t^\bullet_\tau+1}$. Hence, from (9) we have

$$V_{t^\bullet_\tau+2} \leq \beta V_{t^\bullet_\tau+1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+1} V_1 \cdots (11),$$

so $t^*_{t^\bullet_\tau+2} = t^\bullet_\tau$ or equivalently we have $\boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau+2}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (12)$. Similarly we obtain $\boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau+3}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (13)$, $\boxed{⊛\ \mathtt{ndOIT}_{t^\bullet_\tau+4}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (14)$, $\cdots$. Since $\mathbb{V}_{t^\circ_\tau} \leq 0$ due to (2^*), we have $V_{t^\circ_\tau} \leq \beta V_{t^\circ_\tau-1}$. Hence

$$V_{t^\circ_\tau} \leq \beta V_{t^\circ_\tau-1} \leq \cdots \leq \beta^{t^\circ_\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{t^\circ_\tau - t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\circ_\tau-1} V_1 \cdots (15),$$

so $t^*_{t^\circ_\tau} = t^\bullet_\tau$ or equivalently $\boxed{⊛\ \mathtt{ndOIT}_{t^\circ_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (16)$. Hence, we have $\boxed{⊛\ \mathtt{ndOIT}_{t^\circ_\tau \geq \tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\vartriangle \cdots (17)$ from (10)-(16).

3. Since $\mathbb{V}_{t^\circ_\tau+1} < 0$ due to (3), we have $V_{t^\circ_\tau+1} < \beta V_{t^\circ_\tau}$, hence from (15) we get

$$V_{t^\circ_\tau+1} < \beta V_{t^\circ_\tau} \leq \beta^2 V_{t^\circ_\tau-1} \leq \cdots \leq \beta^{t^\circ_\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{t^\circ_\tau - t^\bullet_\tau+1} V_{t^\bullet_\tau} > \beta^{t^\circ_\tau - t^\bullet_\tau+2} V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\circ_\tau} V_1,$$

so $t^*_{t^\bullet+1} = t^\circ_\tau$ or equivalently $\boxed{⊛\ \mathtt{ndOIT}_{t^\circ_\tau+1}\langle t^\bullet_\tau\rangle}_\blacktriangle$. Similarly, since $\mathbb{V}_{t^\circ_\tau+2} < 0$, we have $\boxed{⊛\ \mathtt{ndOIT}_{t^\circ_\tau+3}\langle t^\bullet_\tau\rangle}_\blacktriangle$. In general, we have $\boxed{⊛\ \mathtt{ndOIT}_{\tau > t^\circ_\tau}\langle t^\bullet_\tau\rangle}_\blacktriangle \cdots (18)$.

(8), (17), and (18) can be summarized as $\mathtt{S}_{18}$(p.233) ⓢ▲ | ⊛△ | ⊛▲. ∎

23.1.1.3 Flow of Optimal Decision Rules

♣ `Flow-ODR` **23.1.1** (**r**M:1[ℝ][E]) (`c-reservation-price`) *From* `Tom` 23.1.1(♣a1,♣a2) *and* (22.3.12(p.222)) *we have the following decision rule for* $\tau \geq t > 0$:

$$\begin{cases} y \geq x_K & \Rightarrow \mathtt{Accept}_t\langle y\rangle \text{ and the process stops } \mathsf{I} \\ y \leq x_K & \Rightarrow \mathtt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{cases}$$

This yields the following scenario. First the process is initiated at the optimal initiating time t^*_τ *and the search is conducted at that time, and then a buyer appearing at time* $t^*_\tau - 1$ *proposes a price* ξ, *hence the best price at that time is* $y = \xi$. *After that, the following condition branching follows.*

- *Let* $y \geq x_K$. *Then* $\mathtt{Accept}_{t^*_\tau-1}\langle y\rangle$ *and the process stops* I
- *Let* $y \leq x_K$. *Then* $\mathtt{Reject}_{t^*_\tau-1}\langle y\rangle$ *and the search is conduct, and a buyer appearing at time* $t^*_\tau - 2$ *proposes the price* ξ, *hence the best price* y *is* <u>*enlarged*</u> *to* $y \overset{\text{def}}{=} \max\{\xi, y\}$. *After that, the following condition branching follows.*
  - *Let* $y \geq x_K$. *Then* $\mathtt{Accept}_{t^*_\tau-2}\langle y\rangle$ *and the process stops* I
  - *Let* $y \leq x_K$. *Then* $\mathtt{Reject}_{t^*_\tau-2}\langle y\rangle$ *and the search is conducted.*

    ⋮

    - $\mathtt{Accept}_0\langle y\rangle$ *and the process terminates* II □

23.1.1.4 Market Restriction

23.1.1.4.1 Positive Restriction

□ `Pom` **23.1.1** ($\mathscr{A}${**r**M:1[ℝ][E]$^+$}) *Suppose* $a > 0$.

(a) *We have* `c-reservation-price` (♣ `Flow-ODR` 23.1.1).

(b) *Let* $\beta = 1$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ → → ⓢ

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu - s \geq 0$. *Then* $\boxed{ⓢ\ \mathtt{dOITs}_{\tau>1}\langle\tau\rangle}_\vartriangle$ → → ⓢ
2. *Let* $\beta\mu - s < 0$. *Then* $\boxed{\bullet\ \mathtt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$ → → ⓓ

Proof Suppose $a > 0 \cdots (1)$. Then $\kappa = \kappa_{\mathbb{R}} = \beta\mu - s \cdots (2)$ from Lemma 9.3.1(p.45) (a).

(a) See ♣`Flow-ODR` 23.1.1.

(b) The same as `Tom` 23.1.1(b).

(c) Let $\beta < 1$.

(c1) The same as `Tom` 23.1.1(c1).

(c2) Let $\beta\mu - s < 0$. Then $\beta\mu - s < a$ due to (1), hence we have `Tom` 23.1.1(c2). ∎

23.1.1.4.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

23.1.1.4.3 Negative Restriction

□ `Nem` **23.1.1** ($\mathscr{A}\{$rM:1[ℝ][E]$^-\}$) *Suppose* $b < 0$.

(a) *We have* `c-reservation-price` (♣ `Flow-ODR` 23.1.1).
(b) *Let* $\beta = 1$. *Then* ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle_\vartriangle$ → → ⓢ
(c) *Let* $\beta < 1$.
 1. *Let* $\beta\mu - s < a$. *Then* ● `dOITd`$_{\tau>1}\langle 1\rangle_\blacktriangle$ → → ⓓ
 2. *Let* $\beta\mu - s \geq a$.
  i. *Let* $\mathbb{V}_2 \leq 0$. *Then* ● `dOITd`$_{\tau>1}\langle 1\rangle_\vartriangle$ → → ⓓ
  ii. *Let* $\mathbb{V}_2 > 0$.
   1. *Let* $s = 0$. *Then* ⓢ `dOITs`$_{\tau>1}\langle\tau\rangle_\vartriangle$ → → ⓢ
   2. *Let* $s > 0$. *Then we have* $\texttt{S}_{18}$(p.233) ⓢ▲ ⊛△ ⊛▲ → → ⓢ/⊛

Proof Suppose $b < 0$. Then $\mu < b < 0$, hence $\beta\mu < 0$, so that $\beta\mu - s < 0 \cdots$ **(1)** for any $s \geq 0$. Then $\kappa = \kappa_{\mathbb{R}} = -s \cdots$ **(2)** from Lemma 9.3.1(p.45) (a).

(a) See ♣ `Flow-ODR` 23.1.1.
(b) The same as `Tom` 23.1.1(b).
(c) Let $\beta < 1$.
(c1) Let $\beta\mu - s < a$. Then, due to (1) we have `Tom` 23.1.1(c2).
(c2) Let $\beta\mu - s \geq a$. Then, due to (1) `Tom` 23.1.1(c3i-c3ii2) hold.
(c2i) Let $\mathbb{V}_2 \leq 0$. Then we have `Tom` 23.1.1(c3i).
(c2ii) Let $\mathbb{V}_2 > 0$.
(c2ii1) Let $s = 0$. Then $\kappa = 0$ due to (2), hence we have `Tom` 23.1.1(c3ii1).
(c2ii2) Let $s > 0$. Then $\kappa < 0$ due to (2), hence we have `Tom` 23.1.1(c3ii2). ∎

23.1.2 r$\tilde{\text{M}}$:1[ℝ][E]

23.1.2.1 Preliminary I

Here let us show that `SOE`{r$\tilde{\text{M}}$:1[ℝ][E]} (see (22.3.17(p.222))) is symmetrical to `SOE`{rM:1[ℝ][E]} (see (22.3.5(p.222))), which is a necessary condition under which $\mathscr{A}\{$r$\tilde{\text{M}}$:1[ℝ][E]$\}$ can be derived by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (15.3.1(p.98))) to $\mathscr{A}\{$rM:1[ℝ][E]$\}$ (see `Tom` 23.1.1(p.233)).

1. For convenience of reference, below let us copy (22.3.1(p.222))-(22.3.4):

   (1^*): $v_0(y) = y$, (2^*): $v_t(y) = \max\{y, V_t(y)\}$, (3^*): $V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s$,
   (4^*): $V_t(y) = \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$;

   $$\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\} = \{(1^*), (2^*), (3^*), (4^*)\}.$$

2. Applying the reflection operation $\mathcal{R}$ to the above four yields:

   $(1^*)' : -\hat{v}_0(-\hat{y}) = -\hat{y}$, $(2^*)' : -\hat{v}_t(-\hat{y}) = \max\{-\hat{y}, -\hat{V}_t(-\hat{y})\} = -\min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)' : -\hat{V}_t = \beta\,\mathbf{E}[-\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] - s$,
   $(4^*)' : -\hat{V}_t(-\hat{y}) = \beta\,\mathbf{E}[-\hat{v}_{t-1}(\max\{-\hat{\boldsymbol{\xi}}, -\hat{y}\})] - s = \beta\,\mathbf{E}[-\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] - s$,

   which can be rearranged as:

   $(1^*)'$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)'$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)'$: $\hat{V}_t = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] + s$,
   $(4^*)'$: $\hat{V}_t(-\hat{y}) = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] + s$;

   $$\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)', (2^*)', (3^*)', (4^*)'\}.$$

3. We have $\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] = \mathbf{E}[\hat{v}_{t-1}(\boldsymbol{\xi})] = \int_{-\infty}^{\infty} \hat{v}_{t-1}(\xi) f(\xi) d\xi = \int_{-\infty}^{\infty} \hat{v}_{t-1}(\xi) \check{f}(\hat{\xi}) d\xi$ from (11.1.10(p.55)), i.e., the application of the correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$ (see Lemma 11.3.1(p.57)). Let $\eta = \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then $\mathbf{E}[\hat{v}_{t-1}(-\hat{\boldsymbol{\xi}})] = -\int_{\infty}^{-\infty} \hat{v}_{t-1}(-\eta)\check{f}(\eta)d\eta = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\eta)\check{f}(\eta)d\eta\,\check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\xi})] \cdots (\blacklozenge)$. Similarly we have $\mathbf{E}[\hat{v}_{t-1}(-\min\{\hat{\boldsymbol{\xi}}, \hat{y}\})] = \check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta}, \hat{y}\})]$. Hence $(1^*)'$ - $(4^*)'$ can be rewritten as:

   $(1^*)''$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)''$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)''$: $\hat{V}_t = \beta\,\check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\eta})] + s$,
   $(4^*)''$: $\hat{V}_t(-\hat{y}) = \beta\,\check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta}, \hat{y}\})] + s$;

   $$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)'', (2^*)'', (3^*)'', (4^*)''\}.$$

4. Let us replace $\check{f}(\eta)$ by $f(\eta)$ in (♦) (see (11.1.12(p.56))), i.e., the application of the identity replacement operation $\mathcal{I}_{\mathbb{R}}$ (see Lemma 11.3.3(p.59)). Then, from (♦) we have $\check{\mathbf{E}}[\hat{v}_{t-1}(-\boldsymbol{\eta})] = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\eta)f(\eta)d\eta = \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\xi)f(\xi)d\xi = \mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})]$ without loss of generality. Similarly $\check{\mathbf{E}}[\hat{v}_{t-1}(-\min\{\boldsymbol{\eta},\hat{y}\})] + s = \mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi},\hat{y}\})] + s$. Accordingly $(1^*)''$ - $(4^*)''$ can be rewritten as;

   $(1^*)'''$: $\hat{v}_0(-\hat{y}) = \hat{y}$, $(2^*)'''$: $\hat{v}_t(-\hat{y}) = \min\{\hat{y}, \hat{V}_t(-\hat{y})\}$, $(3^*)'''$: $\hat{V}_t = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})] + s$,
   $(4^*)'''$: $\hat{V}_t(-\hat{y}) = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi},\hat{y}\})] + s$;

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)''', (2^*)''', (3^*)''', (4^*)'''\}.$$

5. Since $(1^*)'''$ - $(4^*)'''$ hold for any given $y \in (-\infty,\infty)$, they holds also for $\hat{y} \in (-\infty,\infty)$, hence $(1^*)'''$ - $(4^*)'''$ hold for $\hat{\hat{y}} \in (-\infty,\infty)$. Accordingly, since $\hat{\hat{y}} = y$, it follows that they hold also for any given y. Thus, we obtain the following:

   $(1^*)''''$: $\hat{v}_0(-y) = y$, $(2^*)''''$: $\hat{v}_t(-y) = \min\{y, \hat{V}_t(-y)\}$, $(3^*)''''$: $\hat{V}_t = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\boldsymbol{\xi})] + s$,
   $(4^*)''''$: $\hat{V}_t(-y) = \beta\,\mathbf{E}[\hat{v}_{t-1}(-\min\{\boldsymbol{\xi},y\})] + s$;

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\text{E}]\}] = \{(1^*)'''', (2^*)'''', (3^*)'''', (4^*)''''\}. \tag{23.1.16}$$

6. Note here that $\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$ can be given by (22.3.13(p.222))-(22.3.16), i.e.,

   $(1^*)'''''$: $v_0(y) = y$, $(2^*)'''''$: $v_t(y) = \min\{y, V_t(y)\}$, $(3^*)'''''$: $V_t = \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s$,
   $(4^*)'''''$: $V_t(y) = \beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi},y\})] + s$;

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \{(1^*)''''', (2^*)''''', (3^*)''''', (4^*)'''''\}. \tag{23.1.17}$$

7. From $(1^*)''''$ and $(1^*)'''''$ we have $\hat{v}_0(-y) = v_0(y) = y$ for any y, i.e., $(1^*)'''' = (1^*)'''''$. Suppose $\hat{v}_{t-1}(-y) = v_{t-1}(y)$ for any y. Then, from $(3^*)''''$ we have $\hat{V}_t(-y) = \beta\,\mathbf{E}[v_{t-1}(\min\{\boldsymbol{\xi},y\})] + s = V_t(y)$, so $(3^*)''''$ becomes identical to $(3^*)'''''$for any y. Hence, from $(2^*)''''$, we have $\hat{v}_t(-y) = \min\{y, V_t(y)\} = v_t(y)$, so $(2^*)''''$ becomes identical to $(2^*)'''''$. Accordingly, by induction $\hat{v}_{t-1}(-y) = v_{t-1}(y)$ for any $t > 0$. Therefore, we have $(1^*)'''' = (1^*)'''''$. Thus we see that (23.1.16) is identical to (23.1.17), i.e.,

$$\texttt{SOE}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\} = \mathcal{ICR}[\texttt{SOE}\{\text{rM:1}[\mathbb{R}][\texttt{A}]\}].$$

23.1.2.2 Preliminary II

First, applying the reflection operation $\mathcal{R}$ to (23.1.15(p.233)) yields

$$\begin{aligned}\mathcal{R}[\mathbb{V}_2] = \hat{\mathbb{V}}_2 = -\mathbb{V}_2 &= \beta\int_{-\infty}^{\infty} -K(\xi)I(-\xi > -x_K)f(\xi)d\xi + (1-\beta)(-\beta\mu + s)\\ &= \beta\int_{-\infty}^{\infty} \hat{K}(\xi)I(\hat{\xi} > \hat{x}_K)f(\xi)d\xi + (1-\beta)(\beta\hat{\mu} + s).\end{aligned}$$

Then, applying the replacement $\eta = \hat{\xi} = -\xi$ (hence $d\eta = -d\xi$), $\hat{\mu} = \check{\mu}$, $\hat{K}(\xi) = \check{\tilde{K}}(\hat{\xi})$, and $\hat{x}_K = x_{\check{\tilde{K}}}$ (see Lemma 11.3.1(p.57) (e,h) to this leads to

$$\begin{aligned}\mathcal{R}[\mathbb{V}_2] &= -\beta\int_{\infty}^{-\infty} \check{\tilde{K}}(\hat{\xi})I(\eta > x_{\check{\tilde{K}}})\check{f}(\eta)d\eta + (1-\beta)(\beta\check{\mu} + s)\\ &= \beta\int_{-\infty}^{\infty} \check{\tilde{K}}(\eta)I(\eta > x_{\check{\tilde{K}}})\check{f}(\eta)d\eta + (1-\beta)(\beta\check{\mu} + s)\\ &= \beta\int_{-\infty}^{\infty} \check{\tilde{K}}(\xi)I(\xi > x_{\check{\tilde{K}}})\check{f}(\xi)dx + (1-\beta)(\beta\check{\mu} + s) \quad \text{(without loss of generality)}\end{aligned}$$

Since the above replacement is the same as the application of $\mathcal{C}_{\mathbb{R}}$ to $\mathcal{R}[\mathbb{V}_2]$, i.e., $\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] = \mathcal{R}[\mathbb{V}_2]$. Thus, we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] = \beta\int_{-\infty}^{\infty} \check{\tilde{K}}(\xi)I(\xi > x_{\check{\tilde{K}}})\check{f}(\xi)d\xi + (1-\beta)(\beta\check{\mu} + s).$$

Furthermore, applying the identity replacement operation $\mathcal{I}_{\mathbb{R}}$ to this yields

$$\begin{aligned}\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] &= \beta\int_{-\infty}^{\infty} \tilde{K}(\xi)I(\xi > x_{\check{\tilde{K}}})f(\xi)d\xi + (1-\beta)(\beta\mu + s))\\ &= \beta\,\mathbf{E}[\tilde{K}(\boldsymbol{\xi})I(\boldsymbol{\xi} > x_{\tilde{K}})] + (1-\beta)(\beta\mu + s).\end{aligned}$$

Noting (11.5.32(p.63)), we can rewrite the above as

$$\tilde{\mathbb{V}}_2 \stackrel{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathbb{V}_2] = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathbb{V}_2] = \beta\,\mathbf{E}[\tilde{K}(\boldsymbol{\xi})I(\boldsymbol{\xi} > x_{\tilde{K}})] + (1-\beta)(\beta\mu + s).$$

23.1.2.3 Derivation of $\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$

Taking into consideration the results in Preliminaries I and II, we immediately see that Scenario[$\mathbb{R}$](p.60) can be applied also to $\mathscr{A}\{\text{rM:1}[\mathbb{R}][\text{E}]\}$. Accordingly, we can obtain the following `Tom`.

□ `Tom` **23.1.2** ($\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}$)

(a) *We have:*
   1. ♣ *Let* $y \le x_{\tilde{K}}$. *Then* $y \le V_t(y)$ *for* $t \ge 0$.
   2. ♣ *Let* $y \ge x_{\tilde{K}}$. *Then* $y \ge V_t(y)$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $\boxed{\text{ⓢ } \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\vartriangle}$.

(c) *Let* $\beta < 1$.

1. *Let* $\beta\mu + s \le 0$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$.
2. *Let* $\beta\mu + s > 0$ *and* $\beta\mu + s > b$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$.
3. *Let* $\beta\mu + s > 0$ *and* $\beta\mu + s \le b$ *(hence* $b > 0$*).*
   i. *Let* $\tilde{\mathbb{V}}_2 \ge 0$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$.
   ii. *Let* $\tilde{\mathbb{V}}_2 < 0$.
      1. *Let* $\tilde{\kappa} \le 0$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$.
      2. *Let* $\tilde{\kappa} > 0$. *Then we have* $\texttt{S}_{18}$(p.233) $\boxed{Ⓢ\blacktriangle}\boxed{⊛\triangle}\boxed{⊛\blacktriangle}$. □

Proof Immediately obtained from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 23.1.1. ∎

23.1.2.4 Flow of Optimal Decision Rules

♣ Flow-ODR **23.1.2** ($\mathbf{r}\tilde{\mathsf{M}}$:1[ℝ][E]) (c-reservation-price) *From* Tom 23.1.2(♣a1,♣a2) *and* (22.3.20(p.222)) *we have the following decision rule for* $\tau \ge t > 0$.

$$\begin{cases} y \le x_{\tilde{K}} \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \mathbf{I} \\ y \ge x_{\tilde{K}} \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted.} \end{cases}$$

The rest is the same as Flow-ODR 23.1.1(p.234) *except that "⋯ is enlarged to ⋯" is replaced by "⋯ is reduced to ⋯".* □

23.1.2.5 Market Restriction

23.1.2.5.1 Positive Restriction

□ Pom **23.1.2** ($\mathscr{A}\{\mathbf{r}\tilde{\mathsf{M}}$:1[ℝ][E]$^+\}$) *Suppose* $a > 0$.

(a) *We have* c-reservation-price (♣ Flow-ODR 23.1.2).
(b) *Let* $\beta = 1$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$ → → ⓢ
(c) *Let* $\beta < 1$.
   1. *Let* $\beta\mu + s > b$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\blacktriangle$ → → ⓓ
   2. *Let* $\beta\mu + s \le b$.
      i. *Let* $\tilde{\mathbb{V}}_2 \ge 0$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\triangle$ → → ⓓ
      ii. *Let* $\tilde{\mathbb{V}}_2 < 0$.
         1. *Let* $s = 0$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\triangle$ → → ⓢ
         2. *Let* $s > 0$. *Then we have* $\texttt{S}_{18}$(p.233) $\boxed{Ⓢ\blacktriangle}\boxed{⊛\triangle}\boxed{⊛\blacktriangle}$ → → ⓢ/⊛

Proof Suppose $a > 0$. Then $\mu > a > 0$, hence $\beta\mu > 0$, so that $\beta\mu + s > 0 \cdots$ **(1)** for any $s \ge 0$. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 11.6.6(p.68) (a).

(a) See Flow-ODR 23.1.2.
(b) The same as Tom 23.1.2(b).
(c) Let $\beta < 1$.
(c1) Let $\beta\mu + s > b$. Hence, due to (1) we have Tom 23.1.2(c2).
(c2) Let $\beta\mu + s \le b$. Hence due to (1) we have Tom 23.1.2(c3i-c3ii2).
(c2i) The same as Tom 23.1.2(c3i).
(c2ii) Let $\tilde{\mathbb{V}}_2 < 0$.
(c2ii1) Let $s = 0$. Then $\kappa = 0$ due to (2), hence we have Tom 23.1.2(c3ii1).
(c2ii2) Let $s > 0$. Then $\kappa > 0$ due to (2), hence we have Tom 23.1.2(c3ii2). ∎

Remark 23.1.1 (diagonal symmetry) Since Pom 23.1.2 can be derived by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Nem 23.1.1(p.235) (see (17.1.22 (1) (p.113))), we see that the diagonal symmetry holds between both, i.e.,

$$\mathscr{A}\{\text{Pom } 23.1.2(\text{p.237})\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\text{Nem } 23.1.1(\text{p.235})\}] \quad □$$

23.1.2.5.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

23.1.2.5.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

23.1.3 Conclusion 9 (Search-Enforced-Model 1)

C1. We have $\mathscr{A}\{\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]\}^+ \;\not\sim\; \mathscr{A}\{\text{rM:1}[\mathbb{R}][\text{E}]\}^+$.
C2.
 a. We have ⓢ for $\text{rM:1}[\mathbb{R}][\text{E}]^+$ and $\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]^+$.
 b. We have ⊛ for $\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]^+$.
 c. We have ⓓ for $\text{rM:1}[\mathbb{R}][\text{E}]^+$ and $\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]^+$.
C3. We have `c-reservation-price` for $\text{rM:1}[\mathbb{R}][\text{E}]^+$ and $\text{r}\tilde{\text{M}}\text{:1}[\mathbb{R}][\text{E}]^+$. □

C1	Compare Pom's 23.1.2(p.237) with 23.1.1(p.234).
C2a	See Pom's 23.1.1(p.234) (b,c1) and 23.1.2(p.237) (b,c2ii1,c2ii2).
C2b	See Pom's 23.1.2(p.237) (c2ii2).
C2c	See Pom's 23.1.1(p.234) (c2) and 23.1.2(p.237) (c1,c2i).
C3	See Pom's 23.1.1(p.234) (a) and 23.1.2(p.237) (a). ∎

23.2 Search-Allowed-Model 1

23.2.1 rM:1[ℝ][A]

23.2.1.1 Some Lemmas

23.2.1.1.1 Preliminary

Lemma 23.2.1 (rM:1[ℝ][A]) *We have* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\triangle$. □

Proof Since $V_t \ge \beta V_{t-1}$ for $t>0$ from (22.3.23(p.223)), we have $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 0$, hence $V_\tau \ge \beta V_{\tau-1}$, $V_{\tau-1} \ge \beta V_{\tau-2}$, $\cdots$, $V_1 \ge \beta V_0$, leading to $V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{\tau-1} \ge \cdots \ge \beta^{\tau-1} V_1 \ge \beta^\tau V_0$. Thus, we have $t^*_\tau = \tau$ for $\tau > 0$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\triangle$. ∎

Lemma 23.2.2 (rM:1[ℝ][A])
(a) $v_t(y)$ *and* $V_t(y)$ *are nondecreasing in* y *for* $t \ge 0$.
(b) $v_t(y)$ *and* $V_t(y)$ *are nondecreasing in* $t \ge 0$ *and* $t > 0$ *respectively.*[†]
(c) V_t *is nondecreasing in* $t > 0$. □

Proof (a) $v_0(y)$ is nondecreasing in y from (22.3.21(p.223)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (22.3.24), hence $v_t(y)$ is nondecreasing in y from (22.3.27). Accordingly, by induction $v_t(y)$ is nondecreasing in y for $t \ge 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is nondecreasing in y for $t > 0$ from (22.3.24). In addition, $V_0(y)$ is nondecreasing in y from (22.3.26), hence it follows that $V_t(y)$ is nondecreasing in y for $t \ge 0$

(b) Clearly $v_1(y) \ge y = v_0(y)$ for any y from (22.3.22) with $t=1$ and (22.3.21). Suppose $v_{t-1}(y) \ge v_{t-2}(y)$ for any y. Then, from (22.3.24) we have $V_t(y) \ge \max\{\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] - s, \beta v_{t-2}(y)\} = V_{t-1}(y)$ for any y. Hence, from (22.3.27) we have $v_t(y) \ge \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y)$ is nondecreasing in $t \ge 0$ for any y. Since $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y, it follows that $V_t(y)$ is nondecreasing in $t > 0$ for any y from (22.3.24).

(c) From (22.3.23) with $t=2$ we have $V_2 \ge \beta\,\mathbf{E}[v_1(\boldsymbol{\xi})] - s$. In addition, since $v_1(\boldsymbol{\xi}) \ge \boldsymbol{\xi}$ for any $\boldsymbol{\xi}$ from (22.3.22) with $t=1$, we have $V_2 \ge \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s = V_1$ due to (22.3.28). Suppose $V_{t-1} \ge V_{t-2}$. Then, since $v_{t-1}(\boldsymbol{\xi}) \ge \max\{\boldsymbol{\xi}, V_{t-2}\} = v_{t-2}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}$ due to (22.3.27), from (22.3.23) we have $V_t \ge \max\{\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] - s, \beta V_{t-2}\} = V_{t-1}$. Thus, by induction $V_t \ge V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t > 0$. ∎

Since $1 = \mathbf{E}[1] = \mathbf{E}[I(\boldsymbol{\xi} > y) + I(\boldsymbol{\xi} \le y)]$, we can rewrite (22.3.36(p.223)) as follows.

$$\begin{aligned}
\mathbb{S}_t(y) &= \beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} > y) + v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \le y)] - v_{t-1}(y)\,\mathbf{E}[I(\boldsymbol{\xi} > y) + I(\boldsymbol{\xi} \le y)]) - s \\
&= \beta(\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} > y) + v_{t-1}(\max\{\boldsymbol{\xi}, y\})I(\boldsymbol{\xi} \le y)] - \,\mathbf{E}[v_{t-1}(y)I(\boldsymbol{\xi} > y) + v_{t-1}(y)I(\boldsymbol{\xi} \le y)]) - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(\boldsymbol{\xi} > y) + (v_{t-1}(\max\{\boldsymbol{\xi}, y\}) - v_{t-1}(y))I(\boldsymbol{\xi} \le y)] - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y) + (v_{t-1}(y) - v_{t-1}(y))I(\boldsymbol{\xi} \le y)] - s \\
&= \beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y)] - s, \quad t > 0. \qquad (23.2.1)
\end{aligned}$$

Note here that $\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} = \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} > y) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} \le y)$. Now, due to Lemma 23.2.2(a), if $\boldsymbol{\xi} > y$, then $v_{t-1}(\boldsymbol{\xi}) \ge v_{t-1}(y)$ or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \ge 0$ and if $\boldsymbol{\xi} \le y$, then $v_{t-1}(\boldsymbol{\xi}) \le v_{t-1}(y)$ or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le 0$. Hence $\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} = (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(\boldsymbol{\xi} > y)$. Thus (23.2.1) can be rewritten as

$$\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] - s, \quad t > 0. \qquad (23.2.2)$$

Then, we have

$$\begin{aligned}
\mathbb{S}_1(y) &= \beta\,\mathbf{E}[\max\{v_0(\boldsymbol{\xi}) - v_0(y), 0\}] - s \\
&= \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}] - s \qquad (\leftarrow\ (22.3.21(\text{p.223}))) \\
&= \beta T(y) - s \qquad (\leftarrow\ (5.1.1(\text{p.17}))) \\
&= L(y) \qquad (\leftarrow\ (5.1.3) \text{ with } \lambda = 1). \qquad (23.2.3)
\end{aligned}$$

[†] From (22.3.30) and (22.3.26) we have $V_1(y) - V_0(y) = \max\{K(y), -(1-\beta)y\}$. Let $x_K < y$ and $\beta < 1$. Then $K(y) < 0$ due to Lemma 9.2.2(p.43) (j1) and $-(1-\beta)y < 0$ for a $y > 0$, hence $V_1(y) - V_0(y) < 0$, i.e., $V_1(y) < V_0(y)$. Thus $V_t(y)$ does not become nondecreasing in $t \ge 0$ for any y.

Lemma 23.2.3 (rM:1[$\mathbb{R}$][$\mathtt{A}$])

(a) $\mathbb{S}_t(y)$ *is nonincresing in* y *for* $t > 0$.
(b) $\mathbb{S}_t(y) \le L(y)$ *for any* $t > 0$ *and* y.
(c) *Let* $x_L \le y$. *Then* $\mathbb{S}_t(y) \le 0$ *for* $t > 0$. □

Proof (a) Immediate from (23.2.2) and Lemma 23.2.2(a).

(b) First, (23.2.2) can be rewritten as

$$\begin{aligned}\mathbb{S}_t(y) &= \beta\,\mathbf{E}\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi}) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] - s \\ &= \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] - s \cdots \textbf{(1)}.\end{aligned}$$

Next, we have:

- Let $y \le \boldsymbol{\xi} \cdots \textbf{(2)}$.[†] Now $v_0(\boldsymbol{\xi}) - v_0(y) = \boldsymbol{\xi} - y \le \boldsymbol{\xi} - y$ from (22.3.21). Suppose $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le \boldsymbol{\xi} - y \cdots \textbf{(3)}$. From (22.3.27) we have $v_t(\boldsymbol{\xi}) - v_t(y) \le \max\{\boldsymbol{\xi} - y, V_t(\boldsymbol{\xi}) - V_t(y)\} \cdots \textbf{(4)}$. Then, from (22.3.24) we have

$$\begin{aligned}V_t(\boldsymbol{\xi}) - V_t(y) &= \max\left\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\})] - s, \beta v_{t-1}(\boldsymbol{\xi})\right\} - \max\left\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', y\})] - s, \beta v_{t-1}(y)\right\}^{\ddagger} \\ &\le \max\{\beta\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\})], \beta(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))\} \\ &= \beta\max\{\,\mathbf{E}_{\xi'}[v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\})], v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y)\}.\end{aligned}$$

  Since $\max\{\boldsymbol{\xi}', y\} \le \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}$ for any $\boldsymbol{\xi}'$ due to (2), from (3) we have $v_{t-1}(\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}) - v_{t-1}(\max\{\boldsymbol{\xi}', y\}) \le \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\} - \max\{\boldsymbol{\xi}', y\}$. Hence we obtain

$$\begin{aligned}V_t(\boldsymbol{\xi}) - V_t(y) &\le \beta\max\{\,\mathbf{E}_{\xi'}[\max\{\boldsymbol{\xi}', \boldsymbol{\xi}\} - \max\{\boldsymbol{\xi}', y\}], \boldsymbol{\xi} - y\} \\ &\le \beta\max\{\,\mathbf{E}_{\xi'}[\max\{0, \boldsymbol{\xi} - y\}], \boldsymbol{\xi} - y\} \\ &= \beta\max\{\max\{0, \boldsymbol{\xi} - y\}, \boldsymbol{\xi} - y\} = \beta\max\{\boldsymbol{\xi} - y, 0\}.\end{aligned}$$

  Then, since $\boldsymbol{\xi} - y \ge 0$ due to (2), we have $V_t(\boldsymbol{\xi}) - V_t(y) \le \beta(\boldsymbol{\xi} - y) \le \boldsymbol{\xi} - y$. Thus, from (4) we have $v_t(\boldsymbol{\xi}) - v_t(y) \le \boldsymbol{\xi} - y$. Accordingly, by induction it follows that $v_t(\boldsymbol{\xi}) - v_t(y) \le \boldsymbol{\xi} - y$ for $t \ge 0$, so that $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le \boldsymbol{\xi} - y \cdots \textbf{(5)}$ for $t > 1$. Thus $\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(y \le \boldsymbol{\xi})] \le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(y \le \boldsymbol{\xi})] \cdots \textbf{(6)}$.

- Let $\boldsymbol{\xi} < y \cdots \textbf{(7)}$. Then $v_{t-1}(\boldsymbol{\xi}) \le v_{t-1}(y)$ from Lemma 23.2.2(a or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \le 0$, hence $\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] = \beta\,\mathbf{E}[0 \times I(\boldsymbol{\xi} < y)] = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] \le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] \cdots \textbf{(8)}$ since $\max\{\boldsymbol{\xi} - y\} = 0$ due to $\boldsymbol{\xi} - y < 0$ from (7).
- From (6) and (8), whether $y \le \boldsymbol{\xi}$ or $\boldsymbol{\xi} < y$, we have $\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}I(\boldsymbol{\xi} < y)] \le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] \cdots \textbf{(9)}$

From (1) and (9) we have $\mathbb{S}_t(y) \le \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(y \le \boldsymbol{\xi})] + \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}I(\boldsymbol{\xi} < y)] - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}(I(y \le \boldsymbol{\xi}) + I(\boldsymbol{\xi} < y))] - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - y, 0\}] - s = \beta T(y) - s \cdots \textbf{(10)}$ from (5.1.1(p.17)), hence $\mathbb{S}_t(y) \le L(y)$ from (5.1.3) with $\lambda = 1$.

(c) If $x_L \le y$, then $L(y) \le 0$ from Corollary 9.2.1(p.43) (a), hence $\mathbb{S}_t(y) \le 0$ from (b). ∎

23.2.1.1.2 Case of $s = 0$

Lemma 23.2.4 (rM:1[$\mathbb{R}$][$\mathtt{A}$]) *Let* $s = 0$. *Then* $\mathbb{S}_t(y) \ge 0$ *for* $t > 0$. □

Proof If $s = 0$, from (23.2.2) we have $\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\}] \ge 0$ for $t > 0$. ∎

23.2.1.1.3 Case of $\beta = 1$ and $s > 0$

Lemma 23.2.5 (rM:1[$\mathbb{R}$][$\mathtt{A}$]) *Let* $\beta = 1$ *and* $s > 0$.

(a) *Let* $y \ge x_K$. *Then* $y = V_t(y)$ *for* $t \ge 0$.
(b) *Let* $y \le x_K$. *Then* $y \le V_t(y)$ *for* $t \ge 0$.
(c) $y \le V_t(y)$ *for any* y *and* $t > 0$. □

Proof Let $\beta = 1$ and $s > 0$.

(a,b) Evident for $t = 0$ from (22.3.26). Suppose that $y \ge (\le)\ x_K \Rightarrow y = (\le)\ V_{t-1}(y)$ (induction hypothesis).

- Let $y \ge x_K$, hence $K(y) \le 0 \cdots \textbf{(1)}$ from Lemma 9.2.2(p.43) (j1). Due to the induction hypothesis we have $v_{t-1}(y) = y \cdots \textbf{(2)}$ from (22.3.22). Then, from Lemma 23.2.3(b) we have $\mathbb{S}_t(y) \le L(y) = T(y) - s = K(y)$ from (5.1.3)) and (5.1.4) due to the assumptions $\beta = 1$ and $\lambda = 1$, so that $\mathbb{S}_t(y) \le 0$ due to (1). Hence, from (22.3.37) and the assumption $\beta = 1$ we have $V_t(y) = \beta v_{t-1}(y) = v_{t-1}(y)$, thus $V_t(y) = y$ from (2). This completes the induction.
- Let $y \le x_K$, hence $K(y) \ge 0 \cdots \textbf{(3)}$ from Lemma 9.2.2(j1). From (22.3.24) we have $V_t(y) \ge \mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$. Since $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \ge \max\{\boldsymbol{\xi}, y\}$ for any $\boldsymbol{\xi}$ and y from (22.3.27), we get $V_t(y) \ge \mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y$ from (5.1.10(p.17)) with $\beta = 1$ and $\lambda = 1$. Thus, we obtain $V_t(y) \ge y$ due to (3). This completes the induction.

(c) Immediate from (a,b). ∎

[†]Consider a group of all pairs $(\boldsymbol{\xi}, y)$ satisfying the inequality $y \le \boldsymbol{\xi}$". Then, if $\max\{\boldsymbol{\xi}', y\} \le \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\}$, the pair $(\max\{\boldsymbol{\xi}', y\}, \max\{\boldsymbol{\xi}', \boldsymbol{\xi}\})$ is also an element of the group.

[‡] $\mathbf{E}_{\xi'}$ represent the expectation as to $\boldsymbol{\xi}'$.

23.2.1.1.4 Case of $\beta < 1$ and $s > 0$

23.2.1.1.4.1 Case of $\kappa > 0$

Lemma 23.2.6 ($\mathscr{A}\{$rM:1[$\mathbb{R}$][A]$\}$) *Let $\beta < 1$ and $s > 0$ and let $\kappa > 0$.*

(a) *Let $y \geq x_K$. Then $y \geq V_t(y)$ for $t \geq 0$.*
(b) *Let $y \leq x_K$. Then $x_K \geq V_t(y) \geq y$ for $t \geq 0$.* □

Proof Let $\beta < 1$ and $s > 0$ and let $\kappa > 0$. Then, from Lemma 9.2.3(p.44) (d) we have $x_L > x_K > 0 \cdots$ **(1)**.

(a,b) The two assertions are evident for $t = 0$ from (22.3.26). Suppose that

$$y \geq (\leq)\ x_K \Rightarrow y \geq V_{t-1}(y) \cdots \textbf{(2)}\ (y \leq V_{t-1}(y) \leq x_K \cdots \textbf{(3)})\ \text{(induction hypothesis)},$$

hence $v_{t-1}(y) = y \cdots$ **(4)** ($v_{t-1}(y) = V_{t-1}(y) \cdots$ **(5)**) from (22.3.27).

- Let $y \geq x_K \cdots$ **(6)**, hence $0 < y \cdots$ **(7)** from (1). Then $v_{t-1}(y) = y \cdots$ **(8)** due to (2) and (22.3.22(p.223)).
  1. Let $x_L \geq y \geq x_K \cdots$ **(9)**. Then $L(y) \geq 0 \cdots$ **(10)** due to Lemma 9.2.1(p.43) (e1) and $K(y) \leq 0 \cdots$ **(11)** due to Lemma 9.2.2(j1). Now $\mathbb{S}_t(y) \leq L(y) \cdots$ **(12)** for any y from Lemma 23.2.3(p.239) (b), hence, from (22.3.37), (12), and (10) we have $V_t(y) \leq \max\{L(y), 0\} + \beta y = L(y) + \beta y = K(y) + y \leq y$ due to (5.1.9) and (11).
  2. Let $y \geq x_L\ (> x_K) \cdots$ **(13)**, hence $L(y) \leq 0 \cdots$ **(14)** due to Lemma 9.2.1(e1). Then $\mathbb{S}_t(y) \leq L(y) \leq 0 \cdots$ **(15)** from Lemma 23.2.3(p.239) (b), hence from (22.3.37) we have $V_t(y) = \beta v_{t-1}(y) = \beta y \leq y$ due to (4) and (7).

  From the above, whether for $x_L \geq y \geq x_K$ or for $y \geq x_L\ (\geq x_K)$, it follows that $y \geq V_t(y)$ for $t \geq 0$. This completes the induction, i.e., it follows that (a) holds.
- Let $y \leq x_K \cdots$ **(16)**, hence $K(y) \geq 0 \cdots$ **(17)** from Lemma 9.2.2(p.43) (j1). Since $V_t(y) \geq \beta\,\mathbf{E}[v_{t-1}(\max\{\boldsymbol{\xi}, y\})] - s$ from (22.3.24) and since $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \geq \max\{\boldsymbol{\xi}, y\}$ from (22.3.27), we have $V_t(y) \geq \beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] - s = K(y) + y$ from (5.1.10(p.17))) with $\lambda = 1$, hence $V_t(y) \geq y$ due to (17). Since $\max\{\boldsymbol{\xi}, y\} \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$ due to (16), from Lemma 23.2.2(a) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \cdots$ **(18)**. Furthermore, since $\max\{\boldsymbol{\xi}, x_K\} \geq x_K$ for any $\boldsymbol{\xi}$, due to (2) we have $V_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, hence from (22.3.27) we have $v_{t-1}(\max\{\boldsymbol{\xi}, x_K\}) = \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$, so that from (18) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) \leq \max\{\boldsymbol{\xi}, x_K\}$ for any $\boldsymbol{\xi}$. In addition, since $v_{t-1}(y) = V_{t-1}(y) \leq x_K$ due to (5), from (22.3.24) we have $V_t(y) \leq \max\{\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x_K\}] - s, \beta x_K\}$, hence from (5.1.10(p.17)) with $\lambda = 1$ we have $V_t(y) \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ since $x_K > 0$ due to (1). This completes the induction. ∎

23.2.1.1.4.2 Case of $\kappa \leq 0$

Lemma 23.2.7 ($\mathscr{A}\{$rM:1[$\mathbb{R}$][A]$\}$) *Let $\beta < 1$ and $s > 0$ and let $\kappa \leq 0$.*

(a) *Let $y \geq 0$. Then $y \geq V_t(y)$ for $t \geq 0$.*
(b) *Let $y \leq 0$. Then $y \leq V_t(y)$ for $t \geq 0$.* □

Proof Let $\beta < 1$ and $s > 0$ and let $\kappa \leq 0$. Then, from Lemma 9.2.3(p.44) (d) we have $x_L \leq x_K \leq 0 \cdots$ **(1)**. Due to (22.3.26) the two assertions clearly hold for $t = 0$. Suppose that $y \geq (\leq)\ 0 \Rightarrow V_{t-1}(y) \leq (\geq)\ y$ (induction hypothesis), hence $v_{t-1}(y) = y$ ($v_{t-1}(y) = V_{t-1}(y)$).

(a) Let $y \geq 0$. Then, since $x_L \leq y$ from (1), we have $L(y) \leq 0$ from Lemma 9.2.1(p.43) (e1), hence $\mathbb{S}_t(y) \leq 0$ for $t > 0$ due to Lemma 23.2.3(c). Therefore, from (22.3.34) we obtain $V_t(y) = \beta v_{t-1}(y) = \beta y$ due to the induction hypothesis, hence $V_t(y) \leq y$ due to $\beta < 1$ and $y \geq 0$. This completes the induction.

(b) Let $y \leq 0$. Now, since $V_t(y) \geq \beta v_{t-1}(y)$ from (22.3.24) and since $v_{t-1}(y) \geq y$ from (22.3.27), we have $V_t(y) \geq \beta y \geq y$ due to $\beta < 1$ and $y \leq 0$. This completes the induction. ∎

23.2.1.2 Analysis

□ `Tom` **23.2.1** ($\mathscr{A}\{$rM:1[$\mathbb{R}$][A]$\}$)

(a) *Let $s = 0$. Then $\mathscr{A}\{r$M:1[$\mathbb{R}$][A]$\} \looparrowright \mathscr{A}\{r$M:1[$\mathbb{R}$][E]$\}$.*
(b) *Let $s > 0$.*
  1. *We have* $\boxed{\text{ⓢ } \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_{\vartriangle}$.
  2. ♣ *Let $\beta = 1$. Then $y \leq V_t(y)$ for $t \geq 0$ and any y.*
  3. *Let $\beta < 1$.*
     i. *Let $\kappa > 0$.*
        1. ♠ *Let $y \geq x_K$. Then $y \geq V_t(y)$ for $t \geq 0$.*
        2. ♠ *Let $y \leq x_K$. Then $y \leq V_t(y)$ for $t \geq 0$.*
     ii. *Let $\kappa \leq 0$.*
        1. *Let $y \geq 0$. Then $y \geq V_t(y)$ for $t \geq 0$.*
        2. ♣ *Let $y \leq 0$. Then $y \leq V_t(y)$ for $t \geq 0$.* □

Proof (a) Let $s = 0$. Then, from Lemma 23.2.4 we have $\mathbb{S}_t(y) \geq 0$ for all y and $t > 0$, hence it is optimal to $\texttt{CONDUCT}_t$ the search for all y and $t > 0$ due to (22.3.38(p.223)). This fact implies that rM:1[ℝ][A] which is originally a search-**A**llowed-model migrates (↬) over to rM:1[ℝ][E] which is a search-**E**nforced-model (see Def. 23.2.1 below).

(b) Let $s > 0$.

(b1) The same as Lemma 23.2.1.†

(b2) The same as Lemma 23.2.5(c).

(b3) Let $\beta < 1$.

(b3i-b3i2) The same as Lemma 23.2.6(p.240).

(b3ii-b3ii2) The same as Lemma 23.2.7(p.240). ∎

23.2.1.3 Flow of Optimal Decision Rules

♠ `Flow-ODR` **23.2.1 (r$\tilde{\text{M}}$:1[ℝ][A])** (`c-reservation-price`) *From* `Tom` 23.2.1(♠b3i1,♠b3i2) *and* (22.3.39(p.223)) *we have the following relations for $\tau \geq t \geq 0$:*

$$\begin{cases} y \geq x_K \ \Rightarrow \ \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \text{I} \\ y \leq x_K \ \Rightarrow \ \texttt{Reject}_t\langle y\rangle \text{ and } \texttt{CONDUCT}_t/\texttt{SKIP}_t \end{cases}$$

which yields the following scenario. First the process is initiated at the optimal initiating time t^*_τ, and then $\texttt{Conduct}_{t^*_\tau}/\texttt{Skip}_{t^*_\tau}$ follows (see (22.3.35(p.223))).

⋆ *Let $\texttt{Skip}_{t^*_\tau}$. Then the process goes to time $t^*_\tau - 1$ and $\texttt{Conduct}_{t^*_\tau-1}/\texttt{Skip}_{t^*_\tau-1}$ follows.*

⋆ *Let $\texttt{Conduct}_{t^*_\tau}$, and a seller appearing at time $t^*_\tau - 1$ proposes the price ξ; hence the best price at that time is $y = \xi$. After that, the following condition branching follows.*

i. *Let $y \geq x_K$. Then $\texttt{Accept}_{t^*_\tau-1}\langle y\rangle$ and stop the process* I

ii. *Let $y \leq x_K$. Then $\texttt{Reject}_{t^*_\tau-1}\langle y\rangle$, and then $\texttt{CONDUCT}_{t^*_\tau-1}/\texttt{SKIP}_{t^*_\tau-1}$ follows* (see (22.3.38(p.223)))

⋆⋆ *Let $\texttt{SKIP}_{t^*_\tau-1}$. Then the process goes to time $t^*_\tau - 2$. After that, $\texttt{CONDUCT}_{t^*_\tau-2}/\texttt{SKIP}_{t^*_\tau-2}$ follows.*

⋆⋆ *Let $\texttt{CONDUCT}_{t^*_\tau-1}$, and a seller appearing at time $t^*_\tau - 2$ proposes the price ξ, hence the best price y at that time is enlarged to $y \overset{\text{def}}{=} \max\{\xi, y\}$. After that, the following condition branching follows.*

i. *Let $y \geq x_K$. Then $\texttt{Accept}_{t^*_\tau-2}\langle y\rangle$ and stop the process* I

ii. *Let $y \leq x_K$. Then $\texttt{Reject}_{t^*_\tau-2}\langle y\rangle$ and $\texttt{CONDUCT}_t/\texttt{SKIP}_t$*

⋮

$\texttt{Accept}_0\langle y\rangle$ *and the process terminates* II □

♣ `Flow-ODR` **23.2.2 (rM:1[ℝ][A])** ($\texttt{Accept}_0(y)/\texttt{Terminate}$) *The inequality $y \leq V_t(y)$ in* `Tom` 23.2.1(♣b2,♣b3ii2) *yields the following flow of the optimal decision rule. First the process is initiated at the optimal initiating time t^*_τ, and then $\texttt{Conduct}_{t^*_\tau}/\texttt{Skip}_{t^*_\tau}$ follows* (see (22.3.35(p.223))).

⋆ *Let $\texttt{Skip}_{t^*_\tau}$. Then the process goes to time $t^*_\tau - 1$ and $\texttt{Conduct}_{t^*_\tau-1}/\texttt{Skip}_{t^*_\tau-1}$ follows.*

⋆ *Let $\texttt{Conduct}_{t^*_\tau}$, and a buyer appearing at time $t^*_\tau - 1$ proposes the price ξ; hence the best price at that time is $y = \xi$.*

- $\texttt{Reject}_{t^*_\tau-1}\langle y\rangle$ ($y \leq V_t(y)$). *After that, $\texttt{CONDUCT}_{t^*_\tau-1}/\texttt{SKIP}_{t^*_\tau-1}$ follows* (see (22.3.38(p.223))).

⋆⋆ *Let $\texttt{SKIP}_{t^*_\tau-1}$. Then the process goes to time $t^*_\tau - 2$ and $\texttt{CONDUCT}_{t^*_\tau-2}/\texttt{SKIP}_{t^*_\tau-2}$ follows.*

⋆⋆ *Let $\texttt{CONDUCT}_{t^*_\tau-1}$, and a buyer appearing at time $t^*_\tau - 2$ propose the price ξ, hence the best price y at that time is enlarged to $y \overset{\text{def}}{=} \max\{\xi, y\}$.*

- $\texttt{Reject}_{t^*_\tau-2}\langle y\rangle$ ($y \leq V_t(y)$). *After that, $\texttt{CONDUCT}_{t^*_\tau-1}/\texttt{SKIP}_{t^*_\tau-1}$ follows.*

⋆⋆ *Let $\texttt{SKIP}_{t^*_\tau-2}$. Then* ⋯

⋆⋆ *Let $\texttt{CONDUCT}_{t^*_\tau-2}$, and* ⋯

⋮

$\texttt{Accept}_0\langle y\rangle$ *and the process terminates* II. □

Remark 23.2.1 In `Flow-ODR` 23.2.2, first let us consider following two extreme cases:

Case 1 Suppose that $\texttt{Skip}_{t^*_\tau}$, $\texttt{Skip}_{t^*_\tau-1}$, $\texttt{Skip}_{t^*_\tau-2}$, ⋯ continue, and the process finally arrives in $\texttt{Accept}_0\langle y\rangle$ and stop.

Case 2 Suppose that $\texttt{Conduct}_{t^*_\tau}$ is made and then, whether $\texttt{SKIP}_{t^*_\tau-2}$, $\texttt{SKIP}_{t^*_\tau-3}$, ⋯ continue (Case 2.1) or $\texttt{CONDUCT}_{t^*_\tau-2}$, $\texttt{CONDUCT}_{t^*_\tau-3}$, ⋯ continue (Case 2.2), the process finally arrives in $\texttt{Accept}_0\langle y\rangle$ and terminates. In Case 2.1 the best price $y = \rho$ (terminal quitting penalty) at $t = 0$ and in Case 2.2 the best price y continues to be cumulatively enlarged every time `CONDUCT` is made and is finally accepted at the deadline $t = 0$, i.e., $\texttt{Accept}_0\langle y\rangle$ and terminate.

†Note that we have $\boxed{\circledS\ \texttt{dOITs}_{\tau>0}\langle\tau\rangle}_\triangle$ for any $s \geq 0$.

Different intermediate cases with the mixture of `Skip`, `Conduct`, `SKIP`, and `CONDUCT` can be considered between Case 1 and Case 2; however, they are all led also to the decision "$\texttt{Accept}_0\langle y\rangle$ and terminate", denoted by $\texttt{Accept}_0(y)/\texttt{Terminate}$ (*reduction of optimal decision rule*). □

Definition 23.2.1 (`model-migration`) The reduction of model in `Tom` 23.2.1(a) implies that search-Allowed-model (Model[A]) migrates over to Emodel (Model[E]), represented as

$$\text{rM:1}[\mathbb{R}][\texttt{A}] \looparrowright \text{rM:1}[\mathbb{R}][\texttt{E}]. \tag{23.2.4}$$

Replacing "`model-running-back`" and "`optdr-Accept/Stop`" in (20.1.10(p.212)) by "`model-migration`" and "`Accept/Terminate`" respectively leads to

$$\text{Reduction}\begin{cases}\texttt{model reduction}: \to \texttt{model-migration} & (\text{rM:1}[\mathbb{R}][\texttt{A}] \looparrowright \text{rM:1}[\mathbb{R}][\texttt{E}])\\ \texttt{optdr reduction}: \to \texttt{Accept/Terminate} & (\texttt{optdr} \mapsto \texttt{Accept}_0(y)/\texttt{Terminate})\end{cases} \tag{23.2.5}$$

Moreover, combining (20.1.10(p.212)) and (23.2.5), we have the following classification map:

$$\text{Reduction}\begin{cases}\texttt{model reduction}\begin{cases}\texttt{model-running-back}\\ \texttt{model-migration}\end{cases}\\ \\ \texttt{optdr reduction}\begin{cases}\texttt{Accept/Stp}\\ \texttt{Accept/Terminate} \quad \square\end{cases}\end{cases} \tag{23.2.6}$$

23.2.1.4 Market Restriction

23.2.1.4.1 Positive Restriction

□ `Pom` **23.2.1** ($\mathscr{A}\{\text{rM:1}[\mathbb{R}][\texttt{A}]^+\}$) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}\{r\text{M:1}[\mathbb{R}][\texttt{A}]^+\} \looparrowright \mathscr{A}\{r\text{M:1}[\mathbb{R}][\texttt{E}]^+\}$.*

(b) *Let $s > 0$.*

1. *We have* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\Delta$ $\to$ → Ⓢ
2. *Let $\beta = 1$. Then we have* $\texttt{optdr} \mapsto \texttt{Accept}_0(y)/\texttt{Terminate}$.
3. *Let $\beta < 1$.*
   i. *Let $\beta\mu > s$. Then we have* `c-reservation-price`.
   ii. *Let $\beta\mu \le s$. Then we have* $\boxed{\bullet\ \texttt{dOITd}_{\tau>0}\langle 1\rangle}$ $\to$ → ⓓ

Proof Suppose $a > 0$, hence $\kappa = \kappa_{\mathbb{R}} = \beta\mu - s \cdots$ **(1)** from Lemma 9.3.1(p.45) (a) with $\lambda = 1$.

(a) The same as `Tom` 23.2.1(a).

(b) Let $s > 0$.

(b1) The same as `Tom` 23.2.1(b1).

(b2) Evident from the fact that `Tom` 23.2.1(b2) can be rewritten as `Flow-ODR` 23.2.2.

(b3) Let $\beta < 1$.

(b3i) Let $\beta\mu > s$, hence $\kappa > 0$ due to (1). Thus, it suffices to consider only (♣b3i1,♣b3i2) of `Tom` 23.2.1, hence we have ♣`Flow-ODR` 23.2.2.

(b3ii) Let $\beta\mu \le s$, hence $\kappa \le 0 \cdots$ **(2)** due to (1). Thus, it suffices to consider only `Tom` 23.2.1(b3ii-b3ii2). Below consider only $y \in [a, b]$, i.e., $0 < a < y < b$. Let ξ be such that $0 < a \le \xi \le b \cdots$ **(3)**. Then $\xi \ge V_{t-1}(\xi)$ from `Tom` 23.2.1(p.240) (b3ii1), hence we have $v_{t-1}(\xi) = \xi$ from (22.3.27(p.223)). Thus, from (22.3.23(p.223)) we have $V_t = \max\{\beta\,\mathbf{E}[\boldsymbol{\xi}] - s, \beta V_{t-1}\} = \max\{\beta\mu - s, \beta V_{t-1}\} = \max\{\kappa, \beta V_{t-1}\}$ for $t > 1$. First $V_1 = \beta\mu - s = \kappa \le 0$ from (22.3.28(p.223)), (1), and (2) or equivalently $V_1 = \beta^0\kappa \le 0$. Suppose $V_{t-1} = \beta^{t-2}\kappa \le 0$. Then $V_t = \max\{\kappa, \beta\beta^{t-2}\kappa\} = \max\{\kappa, \beta^{t-1}\kappa\} = \beta^{t-1}\kappa \le 0$ due to (2). Thus by induction we have $V_t = \beta^{t-1}\kappa \le 0$ for $t > 1$. Accordingly, we have $V_t - \beta V_{t-1} = \beta^{t-1}\kappa - \beta\beta^{t-2}\kappa = \beta^{t-1}\kappa - \beta^{t-1}\kappa = 0$, hence $V_t = \beta V_{t-1}$ for $t > 1$. Accordingly, we get $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1}V_1$, i.e., $t^*_\tau = 1$ for $\tau > 1$ or equivalently $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}$. ∎

23.2.1.4.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

23.2.1.4.3 Negative Restriction

□ `Nem` **23.2.1** ($\mathscr{A}\{\text{rM:1}[\mathbb{R}][\texttt{A}]^-\}$) *Suppose $b < 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}\{r\text{M:1}[\mathbb{R}][\texttt{A}]^-\} \looparrowright \mathscr{A}\{r\text{M:1}[\mathbb{R}][\texttt{E}]^-\}$.*

(b) *Let $s > 0$.*

1. *We have* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\Delta$ $\to$ → Ⓢ
2. *We have* $\texttt{optdr} \mapsto \texttt{Accept}_0(y)/\texttt{Terminate}$. □

Proof Suppose $b < 0$.

(a) The same as Tom 23.2.1(a).

(b) Let $s > 0$.

(b1) The same as Tom 23.2.1(b1).

(b2) Here consider only $0 < a < y < b$. Then, since $y \le b < 0 \cdots (1)$, it suffices to consider only (♣b3ii2) of Tom 23.2.1. Moreover, since $\kappa = -s$ from Lemma 9.3.1(p.45) (a), we have $\kappa \le 0 \cdots (2)$ for any $s > 0$. If $\beta = 1$, then $y \le V_t(y)$ for $t \ge 0$ from Tom 23.2.1(b2) and if $\beta < 1$, then from Tom 23.2.1(♣b3ii2) we have $y \le V_t(y)$ for $t \ge 0$. Hence, whether $\beta = 1$ or $\beta < 1$, we have $y \le V_t(y)$ for $t \ge 0$. Accordingly, it follows that we have optdr $\mapsto$ $\texttt{Accept}_0(y)/\texttt{Terminate}$ (♣Flow-ODR 23.2.2). ∎

23.2.2 r$\tilde{\mathsf{M}}$:1[ℝ][A]

23.2.2.1 Preliminary

For almost the same reason as in Section 23.1.2.1(p.235) it can be confirmed that SOE{r$\tilde{\mathsf{M}}$:1[ℝ][A]} (see (22.3.44(p.224))) is symmetrical to SOE{rM:1[ℝ][A]} (see (22.3.25(p.223))). Taking into consideration the results, we immediately see that Scenario[ℝ](p.60) can be applied also to $\mathscr{A}${rM:1[ℝ][A]}. Accordingly, we can obtain the following Tom.

23.2.2.2 Derivation of $\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][A]}

□ **Tom 23.2.2** ($\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][A]})

(a) *Let $s = 0$. Then $\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][A]} ↬ $\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][E]}.*

(b) *Let $s > 0$.*

1. *We have* ⓢ dOITs$_{\tau>1}\langle\tau\rangle$ $_\triangle$ → → ⓢ
2. ♣ *Let $\beta = 1$. Then $y \ge V_t(y)$ for $t \ge 0$ and any t.*
3. *Let $\beta < 1$.*
   - i. *Let $\tilde{\kappa} < 0$.*
     1. ♠ *Let $y \le x_{\tilde{K}}$. Then $y \le V_t(y)$ for $t \ge 0$.*
     2. ♠ *Let $y \ge x_{\tilde{K}}$. Then $y \ge V_t(y)$ for $t \ge 0$.*
   - ii. *Let $\tilde{\kappa} \ge 0$.*
     1. *Let $y \le 0$. Then $y \le V_t(y)$ for $t \ge 0$.*
     2. ♣ *Let $y \ge 0$. Then $y \ge V_t(y)$ for $t \ge 0$.*

Proof Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see in (15.3.1(p.98))) to Tom 23.2.1(p.240). ∎

23.2.2.3 Flow of Optimal Decision Rules

♠ Flow-ODR **23.2.3** (r$\tilde{\mathsf{M}}$:1[ℝ][A]) (c-reservation-price) *From* Tom 23.2.2(♠b3i1,♠b3i2) *and* (22.3.53(p.224)) *we have the following optimal decision rule for $\tau \ge t \ge 0$:*

$$\begin{cases} y \le x_{\tilde{K}} \;\Rightarrow\; \texttt{Accept}_t\langle y\rangle \text{ and the process stops.} \\ y \ge x_{\tilde{K}} \;\Rightarrow\; \texttt{Reject}_t\langle y\rangle \text{ and then } \texttt{CONDUCT}_t/\texttt{SKIP}_t. \end{cases}$$

The rest is the same as Flow-ODR 23.2.1(p.241) *except that "⋯ is enlarged to ⋯" is replaced by "⋯ is reduced to ⋯".* □

♠ Flow-ODR **23.2.4** (r$\tilde{\mathsf{M}}$:1[ℝ][A]) ($\texttt{Accept}_0(y)/\texttt{Terminate}$) *The inequality $y \ge V_t(y)$ in* Tom 23.2.2(♣b2,♣b3ii2) *yields the same decision rule for $\tau \ge t \ge 0$ as in* Flow-ODR 23.2.2(p.241). *The rest is the same as* Flow-ODR 23.2.2(p.241) *except that "⋯ is enlarged to ⋯" is replaced by "⋯ is reduced to ⋯".* □

23.2.2.4 Market Restriction

23.2.2.4.1 Positive Restriction

□ **Pom 23.2.2** ($\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][A]$^+$}) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][A]$^+$} ↬ $\mathscr{A}${r$\tilde{\mathsf{M}}$:1[ℝ][E]$^+$}.*

(b) *Let $s > 0$.*

1. *We have* ⓢ dOITs$_{\tau>1}\langle\tau\rangle$ $_\triangle$ → → ⓢ
2. *We have* $\texttt{Accept}_0(y)/\texttt{Terminate}$. □

Proof Suppose $a > 0$. Below consider only $y \in [a, b]$, hence $0 < a \le y \le b$, so $y \ge 0 \cdots (1)$. Moreover, since $\tilde{\kappa} = s$ from Lemma 11.6.6(p.68) (a), we have $\tilde{\kappa} \ge 0 \cdots (2)$ for any $s \ge 0$.

(a) The same as Tom 23.2.2(a).

(b) Let $s > 0$.

(b1) The same as Tom 23.2.2(b1).

(b2) If $\beta = 1$, then $y \ge V_t(y)$ for $t \ge 0$ from Tom 23.2.2(b2). If $\beta < 1$, then due to (2) and (1) it suffices to consider only (♣b3ii2) of Tom 23.2.2, hence we have $y \ge V_t(y)$ for $t \ge 0$. Accordingly, whether $\beta = 1$ or $\beta < 1$, we have $y \ge V_t(y)$ for $t \ge 0$. Thus, it follows that we have $\texttt{Accept}_0(y)/\texttt{Terminate}$ (♣Flow-ODR 23.2.3). ∎

Remark 23.2.2 (diagonal symmetry) Pom 23.2.2 can be also derived by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Nem 23.2.1 (see (17.1.22(p.113))). □

23.2.2.4.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

23.2.2.4.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

23.2.3 Conclusion 10 (Search-Allowed-Model 1)

C1 We have $\mathscr{A}\{\mathrm{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}\} \curvearrowright \mathscr{A}\{\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}\}$.

C2

a. Let $s = 0$. Then we have s-A-model 1 $\looparrowright$ s-E-model 1 for $\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$ and $\mathrm{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}$.

b. Let $s > 0$. Then we have optdr $\mapsto$ $\mathtt{Accept}_0(y)$/Terminate for $\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$ with $\beta = 1$ and $\mathrm{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}$ for any β.

C3 Let $s > 0$.

a. We have ⓢ for $\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$ and $\mathrm{r}\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^{+}$.

b. We have ⓓ for $\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$.

c. We have c-reservation-price is possible for $\mathrm{r}\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]^{+}$. □

C1 Compare Pom's 23.2.2(p.243) with 23.2.1(p.242) .

C2a See Pom's 23.2.1(p.242) (a) and 23.2.2(p.243) (a).

C2b See Pom's 23.2.1(p.242) (b2) and 23.2.2(p.243) (b2).

C3a See Pom's 23.2.1(p.242) (b1) and 23.2.2(p.243) (b1).

C3b See Pom's 23.2.1(p.242) (b3ii).

C3c See Pom's 23.2.1(p.242) (b3i). ▮

Chapter 24

Model 2

24.1 Search-Enforced-Model 2

24.1.1 rM:2[ℝ][E]

24.1.1.1 Preliminary

Let us define

$$v_t^\diamond(y) \;=\; v_t(y) - y, \quad t \geq 0, \tag{24.1.1}$$

$$V_t^\diamond(y) \;=\; V_t(y) - y, \quad t \geq 0. \tag{24.1.2}$$

Then, from (22.3.61(p.224)) we have

$$v_t^\diamond(y) \;=\; \max\{0, V_t^\diamond(y)\} \geq 0, \quad t \geq 0, \tag{24.1.3}$$

where

$$v_0^\diamond(y) \;=\; v_0(y) - y = \max\{0, \rho - y\} \quad \text{(see (22.3.54))}, \tag{24.1.4}$$

$$V_0^\diamond(y) \;=\; V_0(y) - y = \rho - y \quad \text{(see (22.3.60))} \tag{24.1.5}$$

Furthermore, from (22.3.58) we have

$$\begin{aligned} V_t^\diamond(y) \;&=\; \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta(v_{t-1}^\diamond(y) + y) - s - y \\ &=\; \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta y - s - y \\ &=\; \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + K(y) + y - y \quad t > 0 \quad (\leftarrow (5.1.10\text{(p.17)})) \\ &=\; \lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta\, v_{t-1}^\diamond(y) + K(y), \quad t \geq 0. \end{aligned} \tag{24.1.6}$$

By $y_t^\diamond$ let us denote the solution of the equation $V_t^\diamond(y) = 0$ if it exists, i.e.,

$$V_t^\diamond(y_t^\diamond) = 0, \quad t \geq 0. \tag{24.1.7}$$

If multiple solutions exist, it is defined to be the *smallest* of them.

24.1.1.2 Some Lemmas

Lemma 24.1.1 (rM:2[ℝ][E])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t \geq 0$.*
(b) *$V_t^\diamond(y)$ is nonincreasing in y for $t \geq 0$.* □

Proof (a) Clearly $v_0(y)$ is nondecreasing in y from (22.3.54). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (22.3.58), hence $v_t(y)$ is also nondecreasing in y from (22.3.61). Thus, by induction $v_t(y)$ is nondecreasing in y for $t \geq 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t > 0$, hence $V_t(y)$ is also nondecreasing in y for $t > 0$ from (22.3.58). In addition, since $V_0(y)$ can be regarded as nondecreasing in y from (22.3.60), it follows that $V_t(y)$ is nondecreasing in y for $t \geq 0$.

(b) $V_0^\diamond(y)$ is nonincreasing in y from (24.1.4). Suppose $V_{t-1}^\diamond(y)$ is nonincreasing in y, hence $v_{t-1}^\diamond(y)$ is also nonincreasing in y from (24.1.3). Accordingly, from (24.1.6) and Lemma 9.2.2(p.43) (b)) we see that $V_t^\diamond(y)$ is also nonincreasing in y. This completes the induction. ∎

Lemma 24.1.2 (rM:2[ℝ][E]) *Let $\beta = 1$ and $s = 0$. Then $V_t(y) \geq y$ for any y and $t > 0$.* □

Proof Let $\beta = 1$ and $s = 0$, hence $K(y) = \lambda T(y)$ from (5.1.4). Then, from (24.1.6) we have $V_t^\diamond(y) = \lambda\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)v_{t-1}^\diamond(y) + \lambda T(y)$ for $t \geq 0$. Now, for any $\boldsymbol{\xi}$ and y we have that $v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) \geq 0$ and $v_{t-1}^\diamond(y) \geq 0$ for $t > 0$ from (24.1.3) and that $T(y) \geq 0$ due to Lemma 9.1.1(p.41) (g), hence it follows that $V_t^\diamond(y) \geq 0$ for any y and $t > 0$ or equivalently $V_t(y) \geq y$ for any y and $t > 0$ from (24.1.2). ∎

Lemma 24.1.3 (rM:2[ℝ][E]) *Let $\beta < 1$ or $s > 0$.*

(a) $\lim_{y\to-\infty} V_t^\diamond(y) = \infty$ *for* $t \ge 0$.
(b) $\lim_{y\to\infty} V_t^\diamond(y) < 0$ *for* $t > 0$.
(c) *The sequence* $y_1^\diamond, y_2^\diamond, \cdots$ *exists where*

$$y \le (\ge)\ y_t^\diamond \ \Rightarrow\ V_t^\diamond(y) \ge (\le)\ 0. \ \square \tag{24.1.8}$$

Proof Let $\beta < 1$ or $s > 0$.

(a) Obviously $V_0^\diamond(y) \to \infty$ as $y \to -\infty$ from (24.1.5). Suppose $V_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$. Then $v_{t-1}^\diamond(y) \to \infty$ as $y \to -\infty$ from (24.1.3). In addition, since $K(y) \to \infty$ as $y = -\infty$ due to (9.2.4 (1) (p.42)), from (24.1.6) we see that $V_t^\diamond(y) \to \infty$ as $y \to -\infty$. This completes the induction.

(b) Evidently $v_0^\diamond(y) \to 0$ as $y \to \infty$ from (24.1.4). Suppose $v_{t-1}^\diamond(y) \to 0$ as $y \to \infty$. Then, the first and second terms of the right-hand side of (24.1.6) converge to 0 as $y \to \infty$. In addition, due to (9.2.5 (2) (p.42)), if $\beta = 1$, then $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$", hence $K(y) = -s < 0$ for any y and if $\beta < 1$, then $K(y) \to -\infty < 0$ as $y \to \infty$, so that $\lim_{y\to\infty} K(y) < 0$ whether $\beta = 1$ or $\beta < 1$. Hence, it follows that $\lim_{y\to\infty} V_t^\diamond(y) < 0$. Thus, from (24.1.3) we have $v_t^\diamond(y) \to 0$ as $y \to \infty$. Hence, by induction we have $v_t^\diamond(y) \to 0$ as $y \to \infty$ for $t \ge 0$. Accordingly, since $v_{t-1}^\diamond(y) \to 0$ as $y \to \infty$ for $t > 0$, for quite the same reason as the above we have $\lim_{y\to\infty} V_{t-1}^\diamond(y) < 0$ for $t > 0$.

(c) Immediate from (a,b) and Lemma 24.1.1(b). ∎

Lemma 24.1.4 (rM:2[ℝ][E]) *Let $\rho \le x_K$. Then for any $y \in [a, b]$ we have:*

(a) $v_t(y)$ *and* $V_t(y)$ *are nondecreasing in* $t \ge 0$.
(b) $v_t(y)$ *and* $V_t(y)$ *converges to finite* $v(y)$ *and* $V(y)$ *respectively as* $t \to \infty$.
(c) $V_t^\diamond(y)$ *is nondecreasing in* $t \ge 0$.
(d) $y_t^\diamond$ *is nondecreasing in* $t > 0$.
(e) V_t *is nondecreasing in* $t \ge 0$.

Proof Let $\rho \le x_K$ and consider only $y \in [a, b] \cdots$ **(1)**. Then $K(\rho) \ge 0 \cdots$ **(2)** from Corollary 9.2.2(b).

(a) Since $\max\{y, \rho\} \ge \rho$ for any y, from (22.3.64(p.224)) and Lemma 9.2.2(p.43) (e) we have $V_1(y) \ge K(\rho) + \rho \ge \rho \cdots$ **(3)** due to (2) . Hence, from (22.3.55(p.224)) with $t = 1$ we have $v_1(y) = \max\{y, V_1(y)\} \ge \max\{y, \rho\} = v_0(y)$ for any y from (22.3.54). Suppose $v_{t-1}(y) \ge v_{t-2}(y)$ for any y. Then, from (22.3.58) we have $V_t(y) \ge \lambda\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-2}(y) - s = V_{t-1}(y)$ for any y. Hence, from (22.3.61) we have $v_t(y) \ge \max\{y, V_{t-1}(y)\} = v_{t-1}(y)$ for any y. Thus, by induction $v_t(y)$ is nondecreasing in $t \ge 0$ for any y. Then $v_{t-1}(y)$ is nondecreasing in $t > 0$ for any y, hence $V_t(y)$ is nondecreasing in $t > 0$ for any y from (22.3.58). From (3) and (22.3.60) we have $V_1(y) \ge V_0(y)$. Accordingly, it follows that $V_t(y)$ is nondecreasing in $t \ge 0$ for any y.

(b) Below let us consider only y and $\boldsymbol{\xi}$ such that $y \in [a, b]$ and $\boldsymbol{\xi} \in [a, b]$† and, in addition, consider a sufficiently large $M > 0$ such that $b \le M$ and $\rho \le M$. Then we have $V_0(y) \le M$ from (22.3.60(p.224)). Suppose $V_{t-1}(y) \le M$ for any $y \in [a, b]$, hence from (22.3.55(p.224)) we have $v_{t-1}(y) \le \max\{M, M\} = M$. Then, since $\max\{\boldsymbol{\xi}, y\} \le \max\{M, M\} = M$ and $\max\{\boldsymbol{\xi}, y\} \in [a, b]$, we have $V_{t-1}(\max\{\boldsymbol{\xi}, y\}) \le M$. Thus, from (22.3.55) we have $v_{t-1}(\max\{\boldsymbol{\xi}, y\}) = \max\{\max\{\boldsymbol{\xi}, y\}, V_{t-1}(\max\{\boldsymbol{\xi}, y\})\} \le \max\{M, M\} = M$. Hence, from (22.3.58) we have $V_t(y) \le \lambda\beta\,\mathbf{E}[M] + (1-\lambda)\beta M - s = \lambda\beta M + (1-\lambda)\beta M - s = \beta M - s \le M$, i.e., $V_t(y)$ is upper bounded in t. Accordingly, due to (a) it follows that $V_t(y)$ converge to a finite $V(y)$ as $t \to \infty$.

(c) Immediate from (24.1.2) and (a).

(d) Evident from Lemma 24.1.1(b), (c), and Lemma 24.1.3(c) (see Figure A 7.2(p.295) (I)).

(e) From (22.3.62) and (2) we have $V_1 \ge \rho = V_0$ from (22.3.56(p.224)). Suppose $V_{t-1} \ge V_{t-2}$. Since $v_{t-1}(\boldsymbol{\xi}) \ge v_{t-2}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}$ due to (a), from (22.3.57) we have $V_t \ge \lambda\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-2} - s = V_{t-1}$. This completes the induction. ∎

Lemma 24.1.5 (rM:2[ℝ][E]) *Let $\beta < 1$ or $s > 0$.*

(a) *Let* $y \ge y_t^\diamond$. *Then* $y \ge V_t(y)$ *for* $t > 0$.
(b) *Let* $y \le y_t^\diamond$. *Then* $y \le V_t(y)$ *for* $t > 0$. □

Proof Let $\beta < 1$ or $s > 0$.

(a,b) From Lemmas 24.1.1(b) and 24.1.3(c) we have that if $y \le (\ge)\ y_t^\diamond$, then $V_t^\diamond(y) \ge (\le)\ 0$ for $t > 0$, hence from (24.1.2) we have $V_t(y) \ge (\le)\ y$ for $t > 0$. ∎

24.1.1.3 Analysis

□ Tom **24.1.1** ($\mathscr{A}${rM:2[ℝ][E]})

(a) ♣ *Let* $\beta = 1$ *and* $s = 0$. *Then* $y \le V_t(y)$ *for any* y *and* $t \ge 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$.
  1. ♠ *Let* $y \ge y_t^\diamond$. *Then* $y \ge V_t(y)$ *for* $t \ge 0$.
  2. ♠ *Let* $y \le y_t^\diamond$. *Then* $y \le V_t(y)$ *for* $t \ge 0$. □

Proof (a) The same as Lemma 24.1.2.

(b-c2ii1) The same as Lemma 24.1.5(a,b). ∎

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† $a \le y \le b \le M$ and $a \le \boldsymbol{\xi} \le b \le M$.

24.1.1.4 Flow of Optimal Decision Rules

♠ `Flow-ODR` **24.1.1** (rM:2[ℝ][E]) (t`-reservation-price`) *From* `Tom` 24.1.1(♠ b1,♠ b2) *and* (22.3.66(p.224)) *we have the following decision rule for* $\tau \ge t \ge 0$,

$$\begin{cases} y \ge y_t^{\diamond} \;\Rightarrow y \ge V_t \;\Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops } \mathsf{I} \\ y \le y_t^{\diamond} \;\Rightarrow y \le V_t \;\Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{cases}$$

which yields the following scenario. First the process is initiated at the optimal initiating time t_τ^**, and then the search is conducted at that time.*

∗ *Assume that a buyer appears at time* $t_\tau^* - 1$ *with* λ *and that he proposes the price* ξ*, hence the best price at that time is* $y = \xi$*. After that, the following condition branching follows.*

- ∘ *Let* $y \ge y_{t_\tau^*-1}^{\diamond}$*. Then* $\texttt{Accept}_{t_\tau^*-1}\langle y\rangle$ *and the process stops* I
- ∘ *Let* $y \le y_{t_\tau^*-1}^{\diamond}$*. Then* $\texttt{Reject}_{t_\tau^*-1}\langle y\rangle$ *and the search is conducted.*

  ∗ *Assume that a buyer appears at time* $t_\tau^* - 2$ *with* λ *and that he proposes the price* ξ*, hence the best price* y *at that time is* <u>*enlarged*</u> *to* $y \overset{\text{def}}{=} \max\{\xi, y\}$*. After that, the following condition branching follows.*

  - ∘ *Let* $y \ge y_{t_\tau^*-2}^{\diamond}$*. Then* $\texttt{Accept}_{t_\tau^*-2}\langle y\rangle$ *and the process stops* I
  - ∘ *Let* $y \le y_{t_\tau^*-2}^{\diamond}$*. Then* $\texttt{Reject}_{t_\tau^*-2}\langle y\rangle$ *and the search is conducted.*

  ∗ *Assume that no buyer appears at time* $t_\tau^* - 2$ *with* $1-\lambda$*. Then the process goes to time* $t_\tau^* - 3$*, and the search is conducted.*

  ⋮

∗ *Assume that no buyer appears at time* $t_\tau^* - 1$ *with* $1-\lambda$*. Then the process goes to time* $t_\tau^* - 2$*, and the search is conducted*

⋮

∘ $\texttt{Accept}_0\langle y\rangle$ *and the process terminates* II □

♣ `Flow-ODR` **24.1.2** (rM:2[ℝ][E]) ($\texttt{Accept}_0(y)$`/Terminate`) *The inequality* $y \le V_t(y)$ *in* `Tom` 24.1.1(♣ a) *yields the following flow of the optimal decision rule. First the process is initiated at the optimal initiating time* t_τ^**.*

∗ *Assume that no buyer appears at time* $t_\tau^* - 1$ *with* $1-\lambda$*. Then the process goes to time* $t_\tau^* - 2 \cdots$

- • $\texttt{Reject}_{t_\tau^*-2}\langle y\rangle$ $(y \le V_t(y))$*.*

  ∗ *Assume that a buyer appearing at time* $t_\tau^* - 3$ *with* λ *proposes the price* ξ*, hence the best price* y *at that time is* <u>*enlarged*</u> *to* $y \overset{\text{def}}{=} \max\{\xi, y\}$*.*

  - • $\texttt{Reject}_{t_\tau^*-3}\langle y\rangle$ $(y \le V_t(y))$*.*

    ⋮

  ∗ *Assume that no buyer appears at time* $t_\tau^* - 3$ *with* $1-\lambda$*. Then the process goes to time* $t_\tau^* - 4$*.*

    ⋮

  - • $\texttt{Accept}_0\langle y\rangle$ *and the process terminates* II.

∗ *Assume that a buyer appearing at time* $t_\tau^* - 1$ *with* λ *proposes the price* ξ*, hence the best price at that time is* $y = \xi$*.*

- • $\texttt{Reject}_{t_\tau^*-1}\langle y\rangle$ $(y \le V_t(y))$*.*

  ⋮

  ∗ *Assume that a buyer appearing at time* $t_\tau^* - 2$ *with* λ *proposes the price* ξ*, hence the best price* y *at that time is* <u>*enlarged*</u> *to* $y \overset{\text{def}}{=} \max\{\xi, y\}$*.*

  - • $\texttt{Reject}_{t_\tau^*-2}\langle y\rangle$ $(y \le V_t(y))$*.*

    ⋮

  ∗ *Assume that no buyer appears at time* $t_\tau^* - 2$ *with* $1-\lambda$*. Then the process goes to time* $t_\tau^* - 3$*.*

    ⋮

- • $\texttt{Accept}_0\langle y\rangle$ *and the process terminates* II. □

Remark 24.1.1 ($\texttt{Accept}_0(y)$`/Terminate`) In `Flow-ODR` 24.1.2, first let us consider following two extreme cases:

Case 1 Suppose that no buyer appears at times $t_\tau^* - 1$, $t_\tau^* - 2$, $\cdots$, 1. Then the process finally arrives in $\texttt{Accept}_0\langle y\rangle$ and terminate II.

Case 2 Suppose that, even if buyers appear at all times $t_\tau^* - 1$, $t_\tau^* - 2$, $\cdots$, 1, they are all rejected (`Reject`), hence the process eventually arrive in $\texttt{Accept}_0\langle y\rangle$ and terminate II. Here note that the best price y is cumulatively <u>enlarged</u> every time `Reject` is made and that the best price which continues to be enlarged is lastly accepted at the deadline $t = 0$, i.e., $\texttt{Accept}_0\langle y\rangle$ and the process terminates II.

Different intermediate cases can be considered between the two extreme cases. In these cases, however, evidently they are all eventually led also to $\texttt{Accept}_0\langle y\rangle$ and the process terminates II. Let us denote each of these decisions by $\texttt{Accept}_0(y)$`/Terminate` (reduction of optimal decision rule). □

24.1.1.5 Market Restriction

24.1.1.5.1 Positive Restriction

□ Pom **24.1.1** ($\mathscr{A}$\{rM:2[ℝ][E]\}$^+$) *Suppose $a > 0$.*

(a) *Let $\beta = 1$ and $s = 0$. Then we have* `Accept`$_0(y)$/`Terminate`.
(b) *Let $\beta < 1$ or $s > 0$. Then we have t*-`reservation-price`.

Proof Suppose $a > 0$.

(a) Obvious from the fact that Tom 24.1.1(♣ a) can be rewritten as Flow-ODR 24.1.2.

(b) Evident from the fact that Tom 24.1.1(♠ b1,♠ b2) can be rewritten as Flow-ODR 24.1.1. ▮

24.1.1.5.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

24.1.1.5.3 Negative Restriction

□ Nem **24.1.1** ($\mathscr{A}$\{rM:2[ℝ][E]\}$^-$) *Suppose $b < 0$.*

(a) *Let $\beta = 1$ and $s = 0$. Then we have* `Accept`$_0(y)$/`Terminate`.
(b) *Let $\beta < 1$ or $s > 0$. Then we have t*-`reservation-price`.

Proof The same as the proof of Pom 24.1.1. ▮

24.1.2 r$\tilde{\mathsf{M}}$:2[ℝ][E]

24.1.2.1 Preliminary I

Let us define

$$\tilde{v}_t^\diamond(y) = v_t(y) - y, \quad t \geq 0, \tag{24.1.9}$$
$$\tilde{V}_t^\diamond(y) = V_t(y) - y, \quad t \geq 0. \tag{24.1.10}$$

Then, from (22.3.74(p.225)) we have

$$\tilde{v}_t^\diamond(y) = \min\{0, \tilde{V}_t^\diamond(y)\}, \quad t \geq 0. \tag{24.1.11}$$

By $\tilde{y}_t^\diamond$ let us denote the solution of the equation $\tilde{V}_t^\diamond(y) = 0$, $t > 0$, it exists, i.e.,

$$\tilde{V}_t^\diamond(\tilde{y}_t^\diamond) = 0. \tag{24.1.12}$$

If multiple solutions exist, it is defined to be the *largest* of them. Now, we have

$$\tilde{v}_0^\diamond(y) = \min\{0, \rho - y\} \quad (\leftarrow (22.3.67)), \tag{24.1.13}$$
$$\tilde{V}_0^\diamond(y) = \rho - y \quad (\leftarrow (22.3.73)). \tag{24.1.14}$$

Lemma 24.1.6 (r$\tilde{\mathsf{M}}$:2[ℝ][E]) *We have $\tilde{y}_t^\diamond = \hat{y}_t^\diamond$ for $t > 0$.* □

Proof First, note that (22.3.71(p.225)) can be rewritten as follows.

$$V_t(y) = \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{\xi, y\}) f(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s, \quad t > 0.$$

Here replacing $f(\xi)$ by $\check{f}(\hat{\xi})$ (see (11.1.10(p.55))) leads to

$$\begin{aligned} V_t(y) &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{\xi, y\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(\min\{-\hat{\xi}, -\hat{y}\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\hat{\xi}, \hat{y}\}) \check{f}(\hat{\xi}) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \cdots \mathbf{(1)}, \quad t > 0. \end{aligned}$$

Next, let $\eta \overset{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then, the above expression can be rearranged as

$$\begin{aligned} V_t(y) &= -\lambda\beta \int_{\infty}^{-\infty} v_{t-1}(-\max\{\eta, \hat{y}\}) \check{f}(\eta) d\eta + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\eta, \hat{y}\}) \check{f}(\eta) d\eta + (1-\lambda)\beta v_{t-1}(y) + s \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\xi, \hat{y}\}) \check{f}(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \quad \text{(without loss of generality).} \\ &= \lambda\beta \int_{-\infty}^{\infty} v_{t-1}(-\max\{\xi, \hat{y}\}) f(\xi) d\xi + (1-\lambda)\beta v_{t-1}(y) + s \quad \text{(see (11.1.12(p.56))).} \end{aligned}$$

Applying the reflection operation $\mathcal{R}$ to the above expression yields

$$\begin{aligned} -\hat{V}_t(-\hat{y}) &= -\lambda\beta \int_{-\infty}^{\infty} \hat{v}_{t-1}(-\max\{\xi, \hat{y}\}) f(\xi) d\xi - (1-\lambda)\beta \hat{v}_{t-1}(-\hat{y}) + s \\ &= -\lambda\beta\, \mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi}, \hat{y}\})] - (1-\lambda)\beta \hat{v}_{t-1}(-\hat{y}) + s, \quad t > 0. \end{aligned}$$

Multiplying the above by -1 yields

$$\hat{V}_t(-\hat{y}) \;=\; \lambda\beta\,\mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi},\hat{y}\})] + (1-\lambda)\beta\hat{v}_{t-1}(-\hat{y}) - s, \quad t>0.\cdots\textbf{(2)}.$$

Now, since (2) holds for any y with $-\infty < y < \infty$, it holds also for $\hat{y}$ since $-\infty < \hat{y} < \infty$, hence we have

$$\hat{V}_t(-\hat{\hat{y}}) \;=\; \lambda\beta\,\mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi},\hat{\hat{y}}\})] + (1-\lambda)\beta\hat{v}_{t-1}(-\hat{\hat{y}}) - s, \quad t>0.\cdots\textbf{(3)}.$$

Since $\hat{\hat{y}} = y$, we can rewrite (3) as

$$\hat{V}_t(-y) \;=\; \lambda\beta\,\mathbf{E}[\hat{v}_{t-1}(-\max\{\boldsymbol{\xi},y\})] + (1-\lambda)\beta\hat{v}_{t-1}(-y) - s\cdots\textbf{(4)}.$$

1. Below let us temporarily represent the symbols "v" and "V" used in rM:2[ℝ][E] in Section 22.3.2.1.1(p.224) by "z" and "Z" respectively. Then (22.3.54) and (22.3.55) can be rewritten as respectively

$$z_0(y) \;=\; \max\{y,\rho\}\cdots\textbf{(5)},$$
$$z_t(y) \;=\; \max\{y, Z_t(y)\}\cdots\textbf{(6)}, \quad t>0,$$

where (22.3.60) and (22.3.58) can be rewritten as respectively

$$Z_0(y) \;=\; \rho\cdots\textbf{(7)},$$
$$Z_t(y) \;=\; \lambda\beta\,\mathbf{E}[z_{t-1}(\max\{\boldsymbol{\xi},y\})] + (1-\lambda)\beta z_{t-1}(y) - s\cdots\textbf{(8)}, \quad t>0.$$

In addition, let $Z^\diamond_t(y) \overset{\text{def}}{=} Z_t(y)-y\cdots\textbf{(9)}$ and $z^\diamond_t(y) \overset{\text{def}}{=} z_t(y)-y = \max\{0, Z^\diamond_t(y)\}$. Then we have $Z^\diamond_t(y^\diamond_t) = 0$ and $z_t(y^\diamond_t)-y^\diamond_t = 0$ (see (24.1.7(p.245))).

2. Since $V_0(y) = \rho\cdots\textbf{(10)}$ from (22.3.73(p.225)), we have $-\hat{V}_0(-\hat{y}) = -\hat{\rho}$, hence $\hat{V}_0(-\hat{y}) = \hat{\rho}$. Since the equality holds for any $y \in (-\infty,\infty)$ and any $\rho \in (-\infty,\infty)$, so also does for $\hat{y} \in (-\infty,\infty)$ and $\hat{\rho} \in (-\infty,\infty)$, hence $\hat{V}_0(-\hat{\hat{y}}) = \hat{\hat{\rho}}$, thus $\hat{V}_0(-y) = \rho\cdots\textbf{(11)}$.

3. From (11) and (7) we have $\hat{V}_0(-y) = Z_0(y)\,(=\rho)$. Suppose $\hat{V}_{t-1}(-y) = Z_{t-1}(y)$. Then, from (22.3.74(p.225)) we have $v_{t-1}(y) = \min\{y, V_{t-1}(y)\} = \min\{-\hat{y}, -\hat{V}_{t-1}(-\hat{y})\} = -\max\{\hat{y}, \hat{V}_{t-1}(-\hat{y})\} = -\max\{\hat{y}, Z_{t-1}(\hat{y})\} = -z_{t-1}(\hat{y})$. Hence, since $\hat{v}_{t-1}(y) = z_{t-1}(\hat{y})$, we have $\hat{v}_{t-1}(-y) = \hat{v}_{t-1}(\hat{y}) = z_{t-1}(\hat{\hat{y}}) = z_{t-1}(y)$; accordingly, (4) can be rewritten as

$$\hat{V}_t(-y) \;=\; \lambda\beta\,\mathbf{E}[z_{t-1}(\max\{\boldsymbol{\xi},y\})] + (1-\lambda)\beta z_{t-1}(y) - s = Z_t(y).$$

Hence, since $-V_t(-y) = Z_t(y)$, we have $V_t(-y) = -Z_t(y)$. Since the equality holds for any $y \in (-\infty,\infty)$, so also does for $\hat{y} \in (-\infty,\infty)$, so that $V_t(-\hat{y}) = -Z_t(\hat{y})$, hence $V_t(y) = -Z_t(\hat{y})$. Now, from (24.1.12) we have $0 = \tilde{V}_t(\tilde{y}^\diamond_t) = V_t(\tilde{y}^\diamond_t) - \tilde{y}^\diamond_t = -Z_t(\hat{\tilde{y}}^\diamond_t) - \tilde{y}^\diamond_t = -Z_t(\hat{\tilde{y}}^\diamond_t) + \hat{\tilde{y}}^\diamond_t = -(Z_t(\hat{\tilde{y}}^\diamond_t) - \hat{\tilde{y}}^\diamond_t) = -Z^\diamond_t(\hat{\tilde{y}}^\diamond_t)$ due to (9) or equivalently $0 = Z^\diamond_t(\hat{\tilde{y}}^\diamond_t)$. Hence, we have $y^\diamond_t = \hat{\tilde{y}}^\diamond_t$ by definition or equivalently $\hat{\tilde{y}}^\diamond_t = y^\diamond_t$, so that $-\tilde{y}^\diamond_t = y^\diamond_t$, hence $\tilde{y}^\diamond_t = -y^\diamond_t = \hat{y}^\diamond_t$. ∎

24.1.2.2 Derivation of $\mathscr{A}${r$\tilde{\text{M}}$:2[ℝ][E]}

For almost the same reason as in Section 23.1.2.1(p.235) it can be confirmed that SOE{r$\tilde{\text{M}}$:1[ℝ][E]} (see (22.3.17(p.222))) is symmetrical to SOE{rM:1[ℝ][E]} (see (22.3.5(p.222))). Taking into consideration the result, we immediately see that Scenario[ℝ](p.60) can be applied also to $\mathscr{A}${rM:1[ℝ][E]}. Accordingly, we can obtain the following Tom.

□ **Tom 24.1.2** ($\mathscr{A}${r$\tilde{\text{M}}$:2[ℝ][E]})

(a) ♣ *Let $\beta = 1$ and $s = 0$. Then $y \geq V_t(y)$ for $t \geq 0$ and any y.*
(b) *Let $\beta < 1$ or $s > 0$.*
   1. ♠ *Let $y \leq \tilde{y}^\diamond_t$. Then $V_t(y) \geq y$ for $t \geq 0$.*
   2. ♠ *Let $y \geq \tilde{y}^\diamond_t$. Then $y \geq V_t(y)$ for $t \geq 0$.* □

Proof Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see p.98) to Tom 24.1.1(p.246). ∎

24.1.2.3 Flow of Optimal Decision Rules

♠ Flow-ODR **24.1.3** (r$\tilde{\text{M}}$:2[ℝ][E]) *From* Tom 24.1.2(♠b1,♠b2) *and* (22.3.75(p.225)) *we have the following decision rule for $\tau \geq t \geq 0$.*

$$\begin{cases} y \leq y^\diamond_t \;\Rightarrow y \geq V_t \;\Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \\ y \geq y^\diamond_t \;\Rightarrow y \leq V_t \;\Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{cases}$$

The rest is the same as Flow-ODR 24.1.1(p.247) *except that "$\cdots$ is enlarged to $\cdots$" is replaced by "$\cdots$ is reduced to $\cdots$".* □

♣ Flow-ODR **24.1.4** (r$\tilde{\text{M}}$:2[ℝ][E]) (Accept$_0(y)$/Terminate *We have the inequality $y \geq V_t(y)$ in* Tom 24.1.2(♣a). *The rest is the same as* Flow-ODR 24.1.2(p.247) *except that "$\cdots$ is enlarged to $\cdots$" is replaced by "$\cdots$ is reduced to $\cdots$".* □

24.1.2.4 Market Restriction

24.1.2.4.1 Positive Restriction

□ Pom **24.1.2** ($\mathscr{A}\{$r$\tilde{\text{M}}$:2[$\mathbb{R}$][E]$\}^+$) *Assume $a > 0$.*

(a) *Let $\beta = 1$ and $s = 0$. Then we have* $\texttt{Accept}_0(y)/\texttt{Terminate}$.
(b) *Let $\beta < 1$ or $s > 0$. Then we have* t-`reservation-price`.

Proof (a) The same as Tom 24.1.2(♣ a).
(b) See Tom 24.1.2(♠ b-b2). ∎

Remark 24.1.2 (diagonal symmetry) Pom 24.1.2 can be also obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Nem 24.1.1 (see (17.1.22 (1) (p.113))). □

24.1.2.4.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

24.1.2.4.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

24.1.3 Conclusion 11 (Search-Enforced-Model 2)

C1 We have $\mathscr{A}\{$r$\tilde{\text{M}}$:2[$\mathbb{R}$][E]$\}^+ \sim \mathscr{A}\{$rM:2[$\mathbb{R}$][E]$\}^+$.
C2 Let $\beta = 1$ and $s = 0$. Then we have `optdr` ↦ $\texttt{Accept}_0(y)/\texttt{Terminate}$ for rM:2[$\mathbb{R}$][E]$^+$ and r$\tilde{\text{M}}$:2[$\mathbb{R}$][E]$^+$.
C3 Let $\beta < 1$ or $s > 0$. Then we have t-`reservation-price` for rM:2[$\mathbb{R}$][E]$^+$ and r$\tilde{\text{M}}$:2[$\mathbb{R}$][E]$^+$. □

C1 Compare Pom's 24.1.2(p.250) with 24.1.1(p.248) .
C2 See Pom's 24.1.1(p.248) (a) and 24.1.2(p.250) (a).
C3 See Pom's 24.1.1(p.248) (b) and 24.1.2(p.250) (b). ∎

24.2 Search-Allowed-Model 2

24.2.1 rM:2[$\mathbb{R}$][A]

24.2.1.1 Preliminary

Let us define

$$V_t^\diamond(y) \overset{\text{def}}{=} V_t(y) - y, \quad t \geq 0, \tag{24.2.1}$$

$$v_t^\diamond(y) \overset{\text{def}}{=} v_t(y) - y = \max\{0, V_t^\diamond(y)\}, \quad t \geq 0, \quad \text{(see (22.3.83))} \tag{24.2.2}$$

where

$$V_0^\diamond(y) = V_0(y) - y = \rho - y \qquad \text{(see (22.3.82(p.225))),} \tag{24.2.3}$$

$$v_0^\diamond(y) = v_0(y) - y = \max\{0, \rho - y\} \quad \text{(see (22.3.76)).} \tag{24.2.4}$$

Then, from (22.3.80) we have

$$\begin{aligned} V_t^\diamond(y) &= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\}) + \max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta(v_{t-1}^\diamond(y) + y) - s, \beta(v_{t-1}^\diamond(y) + y)\} - y \\ &= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, y\}] + (1-\lambda)\beta y - s, \beta v_{t-1}^\diamond(y) + \beta y\} - y \\ &= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + K(y) + y, \beta v_{t-1}^\diamond(y) + \beta y\} - y \quad \text{(see (5.1.10(p.17)))} \\ &= \max\{\lambda\beta\,\mathbf{E}[v_{t-1}^\diamond(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-1}^\diamond(y) + K(y), \beta v_{t-1}^\diamond(y) - (1-\beta)y\}, \quad t > 0. \end{aligned} \tag{24.2.5}$$

By $y_t^\diamond$ let us denote the solution of the equation $V_t^\diamond(y) = 0$ for $t \geq 0$ if it exists, i.e.,

$$V_t^\diamond(y_t^\diamond) = 0, \quad t > 0. \tag{24.2.6}$$

If multiple solutions exist, it is defined to be the *smallest* of them. Let us define

$$\mathbb{V}_t \overset{\text{def}}{=} V_t - \beta V_{t-1}, \quad t > 0. \tag{24.2.7}$$

Then, from (22.3.87) and (22.3.78) we have

$$\mathbb{V}_1 = V_1 - \beta V_0 = \max\{L(\rho), 0\}. \tag{24.2.8}$$

From (22.3.76) and (22.3.78) we have $v_0(\boldsymbol{\xi}) - V_0 = \max\{\boldsymbol{\xi}, \rho\} - \rho = \max\{\boldsymbol{\xi} - \rho, 0\}$, hence from (22.3.92(p.226)) with $t = 1$ we get

$$\begin{aligned} \mathbb{S}_1 &= \lambda\beta\,\mathbf{E}[v_0(\boldsymbol{\xi}) - V_0] - s \\ &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - \rho, 0\}] - s \\ &= \lambda\beta T(\rho) - s = L(\rho) \quad \text{(see (5.1.1(p.17)) and (5.1.3)).} \end{aligned} \tag{24.2.9}$$

Now (22.3.95) can be rewritten as

$$
\begin{aligned}
\mathbb{S}_t(y) &= \lambda\beta\,\mathbf{E}[(v_{t-1}(\max\{\boldsymbol{\xi},y\}) - v_{t-1}(y))I(y<\boldsymbol{\xi}) + (v_{t-1}(\max\{\boldsymbol{\xi},y\}) - v_{t-1}(y))I(\boldsymbol{\xi}\le y)] - s \\
&= \lambda\beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y<\boldsymbol{\xi}) + (v_{t-1}(y) - v_{t-1}(y))I(\boldsymbol{\xi}\le y)] - s \\
&= \lambda\beta\,\mathbf{E}[(v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y<\boldsymbol{\xi})] - s. \qquad (24.2.10)
\end{aligned}
$$

From (22.3.76) we have $v_0(\boldsymbol{\xi}) - v_0(y) = \max\{\boldsymbol{\xi},\rho\} - \max\{y,\rho\} \le \max\{\boldsymbol{\xi}-y,0\}$, hence from (24.2.10) with $t=1$ we have $\mathbb{S}_1(y) = \lambda\beta\,\mathbf{E}[(v_0(\boldsymbol{\xi}) - v_0(y))I(y<\boldsymbol{\xi})] - s \le \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}-y,0\}I(y<\boldsymbol{\xi})] - s$. Then, since $\max\{\boldsymbol{\xi}-y,0\}\ge 0$ and $I(y<\boldsymbol{\xi})\le 1$, we get $\max\{\boldsymbol{\xi}-y,0\}I(y<\boldsymbol{\xi}) \le \max\{\boldsymbol{\xi}-y,0\}$, hence

$$
\begin{aligned}
\mathbb{S}_1(y) &\le \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}-y,0\}] - s \qquad (24.2.12)\\
&= \lambda\beta T(y) - s = L(y) \quad \text{(see (5.1.1(p.17)) and (5.1.3))}. \qquad (24.2.13)
\end{aligned}
$$

24.2.1.2 Some Lemmas

Lemma 24.2.1 (rM:2[ℝ][A])

(a) *$v_t(y)$ and $V_t(y)$ are nondecreasing in y for $t\ge 0$.*
(b) *$V_t^\diamond(y)$ is nonincreasing in y for $t\ge 0$.* □

Proof (a) Clearly $v_0(y)$ is nondecreasing in y from (22.3.76(p.225)). Suppose $v_{t-1}(y)$ is nondecreasing in y. Then $V_t(y)$ is nondecreasing in y from (22.3.80), hence $v_t(y)$ is nondecreasing in y from (22.3.83). Thus by induction $v_t(y)$ is nondecreasing in y for $t\ge 0$. Then $v_{t-1}(y)$ is nondecreasing in y for $t>0$, hence $V_t(y)$ is also nondecreasing in y for $t>0$ from (22.3.80). In addition, since $V_0(y)$ can be regarded as nondecreasing in y from (22.3.82), it follows that $V_t(y)$ is nondecreasing in y for $t\ge 0$.

(b) $V_0^\diamond(y)$ is nonincreasing in y from (24.2.3). Suppose $V_{t-1}^\diamond(y)$ is nonincreasing in y, hence $v_{t-1}^\diamond(y)$ is also nonincreasing in y from (24.2.2). Accordingly, since $K(y)$ and $-(1-\beta)y$ are both nonincreasing in y (see Lemma 9.2.2(b), it follows from (24.2.5) that $V_t^\diamond(y)$ is also nonincreasing in y. Thus, by induction $V_t^\diamond(y)$ is also nonincreasing in y for $t\ge 0$. ∎

If $y<(\ge)\ \boldsymbol{\xi}$, then $v_{t-1}(\boldsymbol{\xi}) \ge (\le)\ v_{t-1}(y)$ due to Lemma 24.2.1(a) or equivalently $v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y) \ge (\le)\ 0$ for $t>0$. Then, since

$$
\begin{aligned}
&\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y), 0\} \\
&\quad= \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y),0\}I(y<\boldsymbol{\xi}) + \max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y),0\}I(y\ge\boldsymbol{\xi}) \\
&\quad= (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y<\boldsymbol{\xi}) + 0\times I(y\ge\boldsymbol{\xi}) \\
&\quad= (v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y))I(y<\boldsymbol{\xi}),
\end{aligned}
$$

we can rewrite (24.2.10) as

$$
\mathbb{S}_t(y) = \lambda\beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y),0\}] - s, \quad t>0. \qquad (24.2.15)
$$

Lemma 24.2.2 (rM:2[ℝ][A]) *Let $\beta=1$ or $s=0$.*

(a) *Let $s=0$. Then $\mathbb{S}_t(y)\ge 0$ for any y and $t>0$.*
(b) *Let $\beta=1$. Then $y\le V_t(y)$ for any y and $t>0$.* □

Proof (a) If $s=0$, from (24.2.15) we have $\mathbb{S}_t(y) = \beta\,\mathbf{E}[\max\{v_{t-1}(\boldsymbol{\xi}) - v_{t-1}(y),0\}] \ge 0$ for any y and $t>0$.

(b) If $\beta=1$, from (22.3.80) and (22.3.77) we have $V_t(y)\ge\beta v_{t-1}(y) = v_{t-1}(y)\ge y$ for any y and any $t>0$. ∎

Lemma 24.2.3 (rM:2[ℝ][A]) *Let $\beta<1$ and $s>0$.*

(a) $\lim_{y\to-\infty} V_t^\diamond(y) = \infty$ *for $t\ge 0$.*
(b) $\lim_{y\to\infty} V_t^\diamond(y) = -\infty$ *for $t>0$.*
(c) *The solution $y_t^\diamond$ exists for $t>0$ such that*
  1. *Let $y\ge y_t^\diamond$. Then $V_t(y)\le y$ for $t>0$.*
  2. *Let $y\le y_t^\diamond$. Then $V_t(y)\ge y$ for $t>0$.* □

Proof Let $\beta<1$ and $s>0$.

(a) Obviously $V_0^\diamond(y)\to\infty$ as $y\to-\infty$ from (24.2.3). Suppose $V_{t-1}^\diamond(y)\to\infty$ as $y\to-\infty$. Then $v_{t-1}^\diamond(y)\to\infty$ as $y\to-\infty$ from (24.2.2). Hence, from (24.2.5) we have $V_t^\diamond(y)\to\infty$ as $y\to-\infty$ due to the facts that $K(y)\to\infty$ as $y=-\infty$ due to (9.2.4 (1) (p.42)) and that $-(1-\beta)y\to\infty$ as $y\to-\infty$. Thus, by induction $V_{t-1}^\diamond(y)\to\infty$ as $y\to-\infty$ for $t\ge 0$, so we have $\lim_{y\to-\infty} V_t^\diamond(y)=\infty$ for $t\ge 0$.

(b) Evidently $v_0^\diamond(y)\to 0$ as $y\to\infty$ from (24.2.4). Suppose $v_{t-1}^\diamond(y)\to 0$ as $y\to\infty$. Noting that $K(y)\to-\infty$ as $y\to\infty$ from (9.2.5 (2) (p.42)) and that $-(1-\beta)y\to-\infty$ as $y\to\infty$, from (24.2.5) we have $V_t^\diamond(y)\to-\infty$ for $t\ge 0$ as $y\to\infty$. Hence, from (24.2.2) we have $v_t^\diamond(y)\to 0$ as $y\to\infty$. Thus, by induction $v_t^\diamond(y)\to 0$ for any $t\ge 0$ as $y\to\infty$, hence $v_{t-1}^\diamond(y)\to 0$ for any $t>0$ as $y\to\infty$. Then, for the same reason as just above we have $V_t^\diamond(y)\to-\infty$ for $t>0$ as $y\to\infty$, so we have $\lim_{y\to\infty} V_t^\diamond(y) = -\infty$ for $t>0$.

(c) From (a,b) and Lemma 24.2.1(b) we see that there exist $y_t^\diamond$ such that $y\ge(\le)\ y_t^\diamond \Rightarrow V_t^\diamond(y)\le(\ge)\ 0 \Leftrightarrow V_t(y)\le(\ge)\ y$ for $t>0$ from (24.2.1). ∎

Lemma 24.2.4 (rM:2[ℝ][A]) *Let $\beta < 1$ and $s > 0$.*

(a) *Let $y \le 0$. Then $V_t(y) \ge y$ for $t > 0$.*

(b) *Let $y > 0$.*

1. *Let $y \ge y_t^\diamond$. Then $V_t(y) \le y$ for $t > 0$,*
2. *Let $y \le y_t^\diamond$. Then $V_t(y) \ge y$ for $t > 0$*

where $y_t^\diamond \ge 0$ for $t > 0$. □

Proof Let $\beta < 1$ and $s > 0$. Since $V_1(y) \ge K(\max\{y, \rho\}) + \max\{y, \rho\}$ for any y from (22.3.89(p.225)) and since $\max\{y, \rho\} \ge y$ for any y, we obtain $V_1(y) \ge K(y) + y \cdots$ **(1)** for any y due to Lemma 9.2.2(e).

(a) Let $y \le 0 \cdots$ **(2)**. Since $V_t(y) \ge \beta v_{t-1}(y)$ for $t > 0$ from (22.3.80(p.225)) and since $v_{t-1}(y) \ge y$ for $t > 0$ from (22.3.77(p.225)), we have $V_t(y) \ge \beta v_{t-1}(y) \ge \beta y$ for $t > 0$. Then, since $\beta y \ge y$ due to (2), we have $V_t(y) \ge y$ for $t > 0$.

(b) Let $y > 0 \cdots$ **(3)**. Here note that the result in (a) implies $y_t^\diamond \ge 0$ for all $t > 0$ because if $y_{t'}^\diamond < 0$ for a $t' > 0$, then for $y_{t'}^\diamond < y < 0$ we have $V_{t'}^\diamond(y) < 0$ or equivalently $V_{t'}(y) - y < 0$, leading to the contradiction $V_{t'}(y) < y$.

(b1,b2) See Lemma 24.2.3(c1,b1). ∎

24.2.1.3 Analysis

□ `Tom` **24.2.1** ($\mathscr{A}${rM:2[ℝ][A]})

(a) *Let $s = 0$. Then $\mathscr{A}${rM:2[ℝ][A]} ↬ $\mathscr{A}${rM:2[ℝ][E]}.*

(b) ♣ *Let $\beta = 1$. Then $y \le V_t(y)$ for any y and $t \ge 0$.*

(c) *Let $\beta < 1$ and $s > 0$.*

1. *We have* ⓢ `dOITs`$_{\tau\ge 0}\langle\tau\rangle$$_\Delta$.
2. ♣ *Let $y \le 0$. Then $y \le V_t(y)$ for $t \ge 0$.*
3. *Let $y \ge 0$.*
   i. ♠ *Let $y \ge y_t^\diamond$. Then $V_t(y) \le y$ for $t \ge 0$.*
   ii. ♠ *Let $y \le y_t^\diamond$. Then $y \le V_t(y)$ for $t \ge 0$.* □

Proof (a) Let$s = 0$. Then, from Lemma 24.2.2(a) we see that it is always optimal to `CONDUCT`$_t$ the search due to (22.3.97(p.226)), implying that rM:2[ℝ][A], which is originally a search-Allowed-model, is substantially reduced to rM:2[ℝ][E], which is a search-Enforced-model.

(b) The same as Lemma 24.2.2(p.251) (b).

(c) Let $\beta < 1$ and $s > 0$.

(c1) From (22.3.79(p.225)) we have $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 0$, hence $V_\tau \ge \beta V_{\tau-1}$, $V_{\tau-1} \ge \beta V_{t-2}$, $\cdots$, $V_1 \ge \beta V_0$, so $V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{t-2} \ge \cdots \ge \beta^\tau V_0$. Accordingly, we have $t_\tau^* = \tau$ for $\tau \ge 0$, i.e., ⓢ `dOITs`$_{\tau\ge 0}\langle\tau\rangle$$_\Delta$.

(c2) The same as Lemma 24.2.4(a).

(c3-c3ii) The same as Lemma 24.2.4(b-b2). ∎

24.2.1.4 Flow of Optimal Decision Rules

♠ `Flow-ODR` **24.2.1** (rM:2[ℝ][A]) (*t*-`reservation-price`) *From* `Tom` 24.2.1(♠c3i,♠c3ii) *and* (22.3.98(p.226)) *we have the following decision rule for $\tau \ge t \ge 0$.*

$$\begin{cases} y \ge y_t^\diamond \ \Rightarrow y \ge V_t \ \Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \,| \\ y \le y_t^\diamond \ \Rightarrow y \le V_t \ \Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and the search is conducted} \end{cases}$$

which yields the following scenario. First the process is initiated at the optimal initiating time t_τ^*, and then the condition branching below follows.

⋆ *Let* `Skip`$_{t_\tau^*}$. *Then the process goes to time $t_\tau^* - 1$, and then* `Conduct`$_{t_\tau^*-1}$/`Skip`$_{t_\tau^*-1}$ *follows* (see (22.3.94(p.226))):.

⋆ *Let* `Conduct`$_{t_\tau^*}$.

- *Assume that a buyer appearing at time $t_\tau^* - 1$ with λ proposes the price ξ; hence the best price at that time is $y = \xi$. After that, the following condition branching follows.*
  1. *Let $y \ge y_{t_\tau^*-1}^\diamond$. Then* `Accept`$_{t_\tau^*-1}\langle y\rangle$ *and the process stops* |
  2. *Let $y \le y_{t_\tau^*-1}^\diamond$. Then* `Reject`$_{t_\tau^*-1}\langle y\rangle$ *and then* `CONDUCT`$_{t_\tau^*-1}$/`SKIP`$_{t_\tau^*-1}$ *follows* (see (22.3.97(p.226))).
     - ⋆⋆ *Let* `SKIP`$_{t_\tau^*-1}$. *Then the process goes to time $t_\tau^* - 2$, and then* `CONDUCT`$_{t_\tau^*-2}$/`SKIP`$_{t_\tau^*-2}$ *follows* (see (22.3.97(p.226))).
     - ⋆⋆ *Let* `CONDUCT`$_{t_\tau^*-1}$.
       - *Assume that a buyer appearing at time $t_\tau^* - 2$ with λ proposes the price ξ, hence the best price y at that time is enlarged to $y \overset{\text{def}}{=} \max\{\xi, y\}$. After that, the following condition branching follows.*
         1. *Let $y \ge y_{t_\tau^*-2}^\diamond$. Then* `Accept`$_{t_\tau^*-2}\langle y\rangle$ *and the process stop* |
         2. *Let $y < y_{t_\tau^*-2}^\diamond$. Then* `Reject`$_{t_\tau^*-2}\langle y\rangle$ *and* `CONDUCT`$_{t_\tau^*-1}$/`SKIP`$_{t_\tau^*-1}$ *follows* (see (22.3.97(p.226))).
       - *Assume that no buyer appears at time $t_\tau^* - 2$ with $1 - \lambda$, then the process goes to time $t_\tau^* - 3$*

.

∗ *Assume that no buyer appears at time $t^*_\tau - 1$ with $1-\lambda$, then the process goes to time $t^*_\tau - 2$.*

⋮

• $\texttt{Accept}_0\langle y\rangle$ *and the process terminates* ‖ □

♣ `Flow-ODR` **24.2.2 (rM:2[ℝ][A])** ($\texttt{Accept}_0(y)$/`Terminate`) *The inequality $y \le V_t(y)$ in* `Tom` 24.2.1(♣ b,♣ c2) *yields the following flow of the optimal decision rule. First the process is initiated at the optimal initiating time t^*_τ, and then* $\texttt{Conduct}_{t^*_\tau}/\texttt{Skip}_{t^*_\tau}$ *follows* (see (22.3.94(p.226))):

⋆ *Let* $\texttt{Skip}_{t^*_\tau}$. *Then the process goes to time $t^*_\tau - 1$, and then* $\texttt{Conduct}_{t^*_\tau-1}/\texttt{Skip}_{t^*_\tau-1}$ *follows* (see (22.3.94(p.226))):

⋆ *Let* $\texttt{Conduct}_{t^*_\tau}$.

∗ *Assume that a buyer appearing at time $t^*_\tau - 1$ with λ proposes the price ξ; hence the best price at that time is $y = \xi$.*

• $\texttt{Reject}_{t^*_\tau-1}\langle y\rangle$ *and then* $\texttt{CONDUCT}_{t^*_\tau-1}/\texttt{SKIP}_{t^*_\tau-1}$ *follows*(see (22.3.97(p.226))).

⋆⋆ *Let* $\texttt{SKIP}_{t^*_\tau-1}$. *Then the process goes to time $t^*_\tau - 2$ and* $\texttt{CONDUCT}_{t^*_\tau-2}/\texttt{SKIP}_{t^*_\tau-2}$ *follows* (see (22.3.97(p.226))):

⋆⋆ *Let* $\texttt{CONDUCT}_{t^*_\tau-1}$.

∗ *Assume that a buyer appearing at time $t^*_\tau - 2$ with λ proposes the price ξ, hence the best price y at that time is enlarged to $y \overset{\text{def}}{=} \max\{\xi, y\}$.*

• $\texttt{Reject}_{t^*_\tau-2}\langle y\rangle$ *and then* $\texttt{CONDUCT}_{t^*_\tau-1}/\texttt{SKIP}_{t^*_\tau-1}$ *follows* (see (22.3.97(p.226))).

⋆⋆ *Let* $\texttt{SKIP}_{t^*_\tau-2}$. ⋯

⋆⋆ *Let* $\texttt{CONDUCT}_{t^*_\tau-2}$.

∗ *Assume that no buyer appears at time $t^*_\tau - 2$ with $1-\lambda$ and the process goes to time $t^*_\tau - 3$*

⋮

∗ *Assume that no buyer appears at time $t^*_\tau - 1$ with $1-\lambda$ and the process goes to time $t^*_\tau - 2$.*

⋮

• $\texttt{Accept}_0\langle y\rangle$ *and the process terminates* ‖. □

Remark 24.2.1 ($\texttt{Accept}_0(y)$/`Terminate`) In `Flow-ODR` 24.2.2, first let us consider following two extreme cases:

Case 1 Suppose that the process starts with $\texttt{Skip}_{t^*_\tau}$ and then $\texttt{Skip}_{t^*_\tau-1}$, $\texttt{Skip}_{t^*_\tau-2}$, ⋯ continue, and the process arrives finally in $\texttt{Accept}_0\langle y\rangle$ and terminates.

Case 2 Suppose that $\texttt{Conduct}_{t^*_\tau}$ is made and then that $\texttt{CONDUCT}_{t^*_\tau-1}$y, $\texttt{CONDUCT}_{t^*_\tau-2}$, ⋯ continue. Then the process arrives finally in $\texttt{Accept}_0\langle y\rangle$ and terminates. Here note that the best price y is cumulatively enlarged every time `CONDUCT` is made and that the best price which continues to be preserved and enlarged is lastly accepted at the deadline $t = 0$, i.e., $\texttt{Accept}_0\langle y\rangle$.

Different intermediate cases can be considered between the two cases. Then it is evident that they are all led also to $\texttt{Accept}_0\langle y\rangle$ and the process terminates. □

24.2.1.5 Market Restriction

24.2.1.5.1 Positive Restriction

□ `Pom` **24.2.1** ($\mathscr{A}\{$**rM:2[ℝ][A]**$\}^+$) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}\{r$M:2[ℝ][A]$^+\} \hookleftarrow \mathscr{A}\{$M:2[ℝ][E]$^+\}$.*

(b) *Let $\beta = 1$. Then we have* $\texttt{Accept}_0(y)$/`Terminate`.

(c) *Let $\beta < 1$ and $s > 0$.*

1. *We have* [ⓢ $\texttt{dOITs}_{\tau\ge 0}\langle\tau\rangle$] → → ⓢ
2. *We have* t-`reservation-price`.

Proof Suppose $a > 0$. Then $y > a > 0$ for any $y \in [a, b]$, hence the case "$y \le 0$" should be removed, so that it suffices to consider only `Tom` 24.2.1(c3i,c3ii).

(a) The same as `Tom` 24.2.1(a).

(b) See `Flow-ODR` 24.2.2.

(c) Let $\beta < 1$ and $s > 0$.

(c1) The same as `Tom` 24.2.1(c1).

(c2) See `Tom` 24.2.1(c3i,c3ii). ∎

24.2.1.5.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

24.2.1.5.3 Negative Restriction

□ Nem **24.2.1** ($\mathscr{A}_{\text{Tom}}\{$rM:2[ℝ][A]$^-\}$) *Suppose $b < 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}\{r$M:2[ℝ][A]$^-\} \hookleftarrow \mathscr{A}\{r$M:2[ℝ][E]$^-\}$.*
(b) *Let $\beta = 1$. Then we have* Accept$_0(y)$/Terminate.
(c) *Let $\beta < 1$ and $s > 0$.*
  1. *We have* [Ⓢ dOITs$_{\tau\geq 0}\langle\tau\rangle$] → → Ⓢ
  2. *We have* Accept$_0(y)$/Terminate.

Proof Suppose $b < 0$. Then $y < b < 0$ for any $y \in [a, b]$, hence the case "$y \geq 0$ should be removed, so that it suffices to consider only Tom 24.2.1(c2).

(a) The same as Tom 24.2.1(a).

(b) See Flow-ODR 24.2.2.

(c) Let $\beta < 1$ and $s > 0$.

(c1) The same as Tom 24.2.1(c1).

(c2) See Flow-ODR 24.2.2. ∎

24.2.2 r$\tilde{\text{M}}$:2[ℝ][A]

24.2.2.1 Derivation of $\mathscr{A}\{$r$\tilde{\text{M}}$:2[ℝ][A]$\}$

For almost the same reason as in Section 23.1.2.1(p.235) it can be confirmed that SOE{r$\tilde{\text{M}}$:2[ℝ][A]} (see (22.3.104(p.226))) is symmetrical to SOE{rM:2[ℝ][A]} (see (22.3.81(p.225))). Taking into consideration the result, we immediately see that Scenario[ℝ](p.60) can be applied also to $\mathscr{A}\{$rM:2[ℝ][A]$\}$. Accordingly, we can obtain the following Tom.

□ Tom **24.2.2** ($\mathscr{A}\{$r$\tilde{\text{M}}$:2[ℝ][A]$\}$)

(a) *Let $s = 0$. Then $\mathscr{A}\{r\tilde{\text{M}}$:2[ℝ][A]$\} \hookleftarrow \mathscr{A}\{r\tilde{\text{M}}$:2[ℝ][E]$\}$.*
(b) ♣ *Let $\beta = 1$. Then $y \geq V_t(y)$ for $t \geq 0$.*
(c) *Let $\beta < 1$ and $s > 0$.*
  1. *We have* [Ⓢ dOITs$_{\tau\geq 0}\langle\tau\rangle$]$_\triangle$.
  2. ♣ *Let $y \geq 0$. Then $y \geq V_t(y)$ for $t \geq 0$.*
  3. *Let $y \leq 0$.*
     i. ♠ *Let $y \leq \tilde{y}_t^\diamond$. Then $y \leq V_t(y)$ for $t \geq 0$.*
     ii. ♠ *Let $y \geq \tilde{y}_t^\diamond$. Then $y \geq V_t(y)$ for $t \geq 0$.* □

Proof Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 24.2.1. ∎

24.2.2.2 Flow of Optimal Decision Rules

♠ Flow-ODR **24.2.3** (r$\tilde{\text{M}}$:2[ℝ][A]) (t-reservation-price) *From* Tom 24.2.2(♠c3i,♠c3ii) *and* (22.3.113(p.226)) *we have the following decision rule for $\tau \geq t \geq 0$.*

$$\begin{cases} y \leq \tilde{y}_t^\diamond \;\Rightarrow y \leq V_t(y) \;\Rightarrow \texttt{Accept}_t\langle y\rangle \text{ and the process stops} \,\mathbf{|} \\ y \geq \tilde{y}_t^\diamond \;\Rightarrow y \geq V_t(y) \;\Rightarrow \texttt{Reject}_t\langle y\rangle \text{ and then } \texttt{CONDUCT}_t/\texttt{SKIP}_t \end{cases}$$

The rest is the same as Flow-ODR 24.2.1(p.252) *except that "⋯ is enlarged to ⋯" is replaced by "⋯ is reduced to ⋯".* □

♣ Flow-ODR **24.2.4** (r$\tilde{\text{M}}$:2[ℝ][A]) (Accept$_0(y)$/Terminate) *We have the inequality $y \geq V_t(y)$ in* Tom 24.2.2(♣b,♣c2). *The rest is the same as* Flow-ODR 24.2.2(p.253) *except that "⋯ is enlarged to ⋯" is replaced by "⋯ is reduced to ⋯".* □

24.2.2.3 Market Restriction

24.2.2.3.1 Positive Restriction

□ Pom **24.2.2** ($\mathscr{A}\{$r$\tilde{\text{M}}$:2[ℝ][A]$^+\}$) *Suppose $a > 0$.*

(a) *Let $s = 0$. Then $\mathscr{A}\{r\tilde{\text{M}}$:2[ℝ][A]$^+\} \hookleftarrow \mathscr{A}\{r\tilde{\text{M}}$:2[ℝ][E]$^+\}$.*
(b) *Let $\beta = 1$. Then we have* Accept$_0(y)$/Terminate.
(c) *Let $\beta < 1$ and $s > 0$.*
  1. *We have* [Ⓢ dOITs$_{\tau\geq 0}\langle\tau\rangle$] → → Ⓢ
  2. *We have* Accept$_0(y)$/Terminate.

Proof Suppose $a > 0$. Then $y > a > 0$ for any $y \in [a, b]$, hence the case "$y \leq 0$" should be removed, so it suffices to consider only Tom 24.2.2(c2).

(a) The same as Tom 24.2.2(a).

(b) Immediate from Tom 24.2.2(b) and Flow-ODR 24.2.4.

(c) Let $\beta < 1$ and $s > 0$.

(c1) The same as Tom 24.2.2(c1).

(c2) Immediate Tom 24.2.2(c2) and Flow-ODR 24.2.4. ∎

Remark 24.2.2 (diagonal symmetry) Pom 24.2.2 can be also obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Nem 24.2.1. □

24.2.2.3.2 Mixed Restriction

Omitted (see Section 17.2.3(p.116)).

24.2.2.3.3 Negative Restriction

Omitted (see Section 17.2.3(p.116)).

24.2.3 Conclusion 12 (Search-Allowed-Model 2)

C1 We have $\mathscr{A}\{\text{r}\tilde{\mathsf{M}}\text{:2}[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]\}^+$.

C2

- a. Let $s = 0$. Then we have s-A-model 2 ↬ s-E-model 2 for $\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ and $\text{r}\tilde{\mathsf{M}}\text{:2}[\mathbb{R}][\mathtt{A}]^+$.
- b. Let $\beta = 1$. Then we have optdr $\mapsto$ $\mathtt{Accept}_0(y)/\mathtt{Terminate}$ for both $\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ and $\text{r}\tilde{\mathsf{M}}\text{:2}[\mathbb{R}][\mathtt{A}]^+$.

C3 Let $\beta < 1$ or $s > 0$. Then we have ⓢ for $\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ and $\text{r}\tilde{\mathsf{M}}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ → → ⓢ

C4 Let $\beta < 1$ or $s > 0$.

- a. We have t-reservation-price for $\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]^+$.
- b. We have optdr $\mapsto$ $\mathtt{Accept}_0(y)/\mathtt{Terminate}$ for $\text{r}\mathsf{M}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ and $\text{r}\tilde{\mathsf{M}}\text{:2}[\mathbb{R}][\mathtt{A}]^+$ with $\beta = 1$. □

C1	Compare Pom's 24.2.2(p.254) with 24.2.1(p.253) .
C2a	See Pom's 24.2.1(p.253) (a) and 24.2.2(p.254) (a).
C2b	See Pom's 24.2.1(p.253) (b) and 24.2.2(p.254) (b).
C3	See Pom's 24.2.1(p.253) (c1) and 24.2.2(p.254) (c1).
C4a	See Pom's 24.2.1(p.253) (c2).
C4b	See Pom's 24.2.1(p.253) (b) and 24.2.2(p.254) (b). ∎

Chapter 25

Model 3

25.1 Search-Enforced-Model 3

25.1.1 rM:3[ℝ][E]

Lemma 25.1.1 *Let $\rho \geq x_K$. Then $U_t \leq \rho$ and $v_t(y) \leq \max\{y, \rho\}$ for $t \geq 0$.* ∎

Proof Let $\rho \geq x_K$, hence $\max\{y, \rho\} \geq \rho \geq x_K$ for any y. Accordingly, from Corollary 9.2.2(p.44) (a) we have $K(\rho) \leq 0 \cdots (1)$ and $K(\max\{y, \rho\}) \leq 0 \cdots (2)$. Now $U_0 \leq \rho$ from (22.3.121 (2) (p.227)) and $v_0(y) \leq \max\{y, \rho\}$ for any y from (22.3.114(p.227)). Suppose $U_{t-1} \leq \rho$ and $v_{t-1}(y) \leq \max\{y, \rho\}$ for any y, hence $V_{t-1} = \rho$ from (22.3.117(p.227)) and $v_{t-1}(\max\{\xi, y\}) \leq \max\{\max\{\xi, y\}, \rho\}$ for any ξ and y. Then, from (22.3.119(p.227)) we have $U_t \leq \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s = K(\rho) + \rho$ from (5.1.10(p.17)), hence $U_t \leq \rho$ due to (1). In addition, from (22.3.118(p.227)) we have $U_t(y) \leq \lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s = \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}\}] + (1-\lambda)\beta\max\{y, \rho\} - s = K(\max\{y, \rho\}) + \max\{y, \rho\}$ from (5.1.10(p.17)), hence $U_t(y) \leq \max\{y, \rho\}$ from (2). Accordingly, from (22.3.115(p.227)) we have $v_t(y) \leq \max\{y, \rho, \max\{y, \rho\}\} = \max\{y, \rho\}$. This complete the inductions. ∎

□ **Tom 25.1.1** ($\mathscr{A}\{$**rM**:3[ℝ][E]$\}$)

(a) *Let $\rho \leq x_K$. Then we have $\mathscr{A}\{r$M:3[ℝ][E]$\} \twoheadrightarrow \mathscr{A}\{r$M:2[ℝ][E]$\}$.*
(b) *Let $\rho \geq x_K$.*
1. *We have* `optdr` ↦ `Accept`$_\tau(\rho)$`/Stop`.
2. *Let $\rho \geq 0$. Then we have* ⓢ.
3. *Let $\rho \leq 0$. Then we have* ⓓ. □

Proof (a) Let $\rho \leq x_K$, hence $K(\rho) \geq 0 \cdots (1)$ from Corollary 9.2.2(p.44) (b). Since $V_{t-1} \geq \rho$ for $t > 0$ from (22.3.117(p.227))) and since $v_{t-1}(y) \geq \max\{y, \rho\}$ for any y, ρ, and $t > 0$ from (22.3.115(p.227))), from (22.3.119(p.227))) we have $U_t \geq \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s = K(\rho) + \rho$ for $t > 0$ from (5.1.10(p.17)), hence $U_t \geq \rho$ for $t > 0$ from (1). This fact means that "Reject the intervening quitting penalty ρ for all $t > 0$", implying "Behave as if there does not exist the intervening quitting penalty ρ"; in other words, it follows that rM:3[ℝ][E] is reduced to rM:2[ℝ][E].

(b) Let $\rho \geq x_K$.

(b1) Then, we have $U_t \leq \rho$ for $\tau \geq t \geq 0$ from Lemma 25.1.1, meaning that "Accept the intervening quitting penalty ρ and the process stops" is optimal for $\tau \geq t > 0$; in other words, we have `Accept`$_\tau(\rho)$`/Stop` for $\tau \geq \tau > 0$ (see (20.1.9(p.212))).

(b2,b3) The same as Lemma 20.1.2(p.212). ∎

□ **Pom 25.1.1** ($\mathscr{A}\{$**rM**:3[ℝ][E]$\}^+$)

(a) *Let $\rho \leq x_K$. Then we have $\mathscr{A}\{r$M:3[ℝ][E]$\}^+ \twoheadrightarrow \mathscr{A}\{r$M:2[ℝ][E]$\}^+$.*
(b) *Let $\rho \geq x_K$.*
1. *We have* `optdr` ↦ `Accept`$_\tau(\rho)$`/Stop`.
2. *Let $\rho \geq 0$. Then we have* ⓢ.
3. *Let $\rho \leq 0$. Then we have* ⓓ. □

Proof Immediate from `Tom` 25.1.1(p.257). ∎

25.1.2 r$\tilde{\text{M}}$:3[ℝ][E]

In the same way as in Section 23.1.2.1(p.235) we can easily verify that `SOE`{r$\tilde{\text{M}}$:3[ℝ][E]} $= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$[`SOE`{rM:3[ℝ][E]}] (see (22.3.129(p.227)) and (22.3.120(p.227)))), hence, applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 25.1.1 yields the following `Tom`.

□ **Tom 25.1.2** ($\mathscr{A}\{$**r**$\tilde{\text{M}}$:3[ℝ][E]$\}$)

(a) *Let $\rho \geq x_{\tilde{K}}$. Then we have $\mathscr{A}\{r\tilde{\text{M}}$:3[ℝ][E]$\} \twoheadrightarrow \mathscr{A}\{r\tilde{\text{M}}$:2[ℝ][E]$\}$.*
(b) *Let $\rho \leq x_{\tilde{K}}$.*
1. *We have* `optdr` ↦ `Accept`$_\tau(\rho)$`/Stop`.

2. *Let $\rho \le 0$. Then we have* ⓢ.
3. *Let $\rho \ge 0$. Then we have* ⓓ $\square$

$\square$ Pom **25.1.2** ($\mathscr{A}\{\mathsf{r}\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]\}^+$)

(a) *Let $\rho \ge x_{\tilde{K}}$. Then we have* $\mathscr{A}\{r\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathrm{E}]^+\} \twoheadrightarrow \mathscr{A}\{r\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]^+\}$.
(b) *Let $\rho \le x_{\tilde{K}}$.*
1. *We have* optdr $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Stop}$.
2. *Let $\rho \le 0$. Then we have* ⓢ.
3. *let $\rho \ge 0$. Then we have* ⓓ. $\square$

Proof Immediate from Tom 25.1.2(p.257). ∎

25.1.3 Conclusion 13 (Search-Enforced-Model 3)

In a selling model (buying model) we have:

C1. Let $\rho \le x_K$ ($\rho \ge x_{\tilde{K}}$). Then we have Model 3 $\twoheadrightarrow$ Model 2.
C2. Let $\rho \ge x_K$ ($\rho \le x_{\tilde{K}}$). Then we have optdr $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Stop}$.
C3. Let $\rho \ge 0$ ($\rho \le 0$). Then we have ⓢ.
C4. Let $\rho \le 0$ ($\rho \ge 0$). Then we have ⓓ. $\square$

C1 See Pom 25.1.1(p.257) (a) (Pom 25.1.2(p.258) (a)).
C2 See Pom 25.1.1(p.257) (b1) (Pom 25.1.2(p.258) (b1)).
C3 See Pom 25.1.1(p.257) (b2) and (Pom 25.1.2(p.258) (b2)).
C4 See Pom 25.1.1(p.257) (b3) and Pom 25.1.2(p.258) (b3)). ∎

25.2 Search-Allowed-Model 3

Lemma 25.2.1 *We have*

(a) *$v_t(y)$ is nondecreasing in $t \ge 0$ for any y.*
(b) *Let $\rho \le 0$. Then U_t is nondecreasing in $t \ge 0$.*
(c) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then $U_t \le \rho$ for $t \ge 0$ and $v_t(y) \le \max\{y, \rho\}$ for $t \ge 0$.* ∎

Proof (a) From (22.3.133(p.227)) with $t = 1$ and (22.3.132(p.227)) we have $v_1(y) \ge \max\{y, \rho\} = v_0(y)$ for any y. Suppose $v_{t-1}(y) \ge v_{t-2}(y)$ for any y. Then, from (22.3.137(p.228)) we have
$U_t(y) \ge \max\{\lambda\beta\,\mathbf{E}[v_{t-2}(\max\{\boldsymbol{\xi}, y\})] + (1-\lambda)\beta v_{t-2}(y) - s, \beta v_{t-2}(y)\} = U_{t-1}(y)$, so that from (22.3.133(p.227)) we have $v_t(y) \ge \max\{y, \rho, U_{t-1}(y)\} = v_{t-1}(y)$. Thus, by induction we have $v_t(y) \ge v_{t-1}(y)$ for $t > 0$. Accordingly, it follows that $v_t(y)$ is nondecreasing in $t \ge 0$.

(b) Let $\rho \le 0$. from (22.3.138(p.228)) with $t = 1$ and (22.3.134(p.227)) we have $U_1 \ge \beta V_0 = \beta\rho \ge \rho = U_0$ from (22.3.141 (2) (p.228)). Suppose $U_t \ge U_{t-1}$. Then, since $v_{t-1}(\xi) \ge v_{t-2}(\xi)$ for any ξ from (a) and since $V_t \ge \max\{\rho, U_{t-1}\} = V_{t-1}$ from (22.3.135(p.227)), we have $U_t \ge \max\{\lambda\beta\,\mathbf{E}[v_{t-2}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-2} - s, \beta V_{t-2}\} = U_{t-1}$ from (22.3.138(p.228)). This completes the induction.

(c) Let $\rho \ge x_K$ and $\rho \ge 0 \cdots$ **(1)**. Then, we have $K(\rho) \le 0 \cdots$ **(2)** from Corollary 9.2.2(p.44) (a) and we have $K(\max\{y, \rho\}) \le 0 \cdots$ **(3)** for any y due to $\max\{y, \rho\} \ge \rho \ge x_K$. Clearly, we have $U_0 \le \rho$ from (22.3.141 (2) (p.228)) and $v_0(y) \le \max\{y, \rho\}$ for any y from (22.3.132(p.227)). Suppose $U_{t-1} \le \rho$ and $v_{t-1}(y) \le \max\{y, \rho\}$ for any y, hence $V_{t-1} = \rho$ from (22.3.135(p.227)). Then, from (22.3.138(p.228)) we have $U_t \le \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} = \max\{K(\rho) + \rho, \beta\rho\}$ from (5.1.10(p.17)), hence $U_t \le \max\{\rho, \beta\rho\} = \rho$ due to (2) and (1). Moreover, from (22.3.137(p.228)) we have $U_t(y) \le \max\{\lambda\beta\,\mathbf{E}[\max\{\max\{\boldsymbol{\xi}, y\}, \rho\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} = \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \max\{y, \rho\}\}] + (1-\lambda)\beta\max\{y, \rho\} - s, \beta\max\{y, \rho\}\} = \max\{K(\max\{y, \rho\}) + \max\{y, \rho\}, \beta\max\{y, \rho\}\}$ from (5.1.10(p.17)). Hence $U_t(y) \le \max\{\max\{y, \rho\}, \beta\{\max\{y, \rho\}\} = \max\{y, \rho\}$ due to (3) and $\max\{y, \rho\} \ge \rho \ge 0$ for any y. Accordingly, from (22.3.133(p.227)) we have $v_t(y) \le \max\{y, \rho, \max\{y, \rho\}\} = \max\{y, \rho\}$. This complete the inductions. ∎

$\square$ Tom **25.2.1** ($\mathscr{A}\{\mathsf{rM}{:}3[\mathbb{R}][\mathtt{A}]\}$)

(a) *Let $\rho \le x_K$ or $\rho \le 0$. Then we have* $\mathscr{A}\{r\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\} \twoheadrightarrow \mathscr{A}\{r\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$.
(b) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then we have* odr $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Stop}$. $\square$

Proof From (22.3.138) with $t = 1$, (22.3.132) with $t = 1$, and (22.3.134(p.227)) we have
$U_1 = \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} = \max\{K(\rho) + \rho, \beta\rho\} \cdots$ **(1)** due to (5.1.10(p.17)).

(a) Let $\rho \le x_K$, hence $K(\rho) \ge 0 \cdots$ **(2)** from Corollary 9.2.2(p.44) (b). Since $v_t(y) \ge \max\{y, \rho\}$ for any y and for $t > 0$ from (22.3.133(p.227))) and $V_t \ge \rho$ for $t > 0$ from (22.3.135(p.227))), from
(22.3.138(p.228))) and (5.1.10(p.17)) we have $U_t \ge \max\{\lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, \rho\}] + (1-\lambda)\beta\rho - s, \beta\rho\} = \max\{K(\rho) + \rho, \beta\rho\} \ge K(\rho) + \rho \ge \rho$ for any $t > 0$ due to (2). Let $\rho \le 0$, hence $-(1-\beta)\rho \ge 0$. From (1) we have $U_1 - \rho = \max\{K(\rho), -(1-\beta)\rho\} \ge 0$, so $U_1 \ge \rho$; accordingly, we have $U_t \ge \rho$ for $t > 0$ from Lemma 25.2.1(b). Consequently, whether $\rho \le x_K$ or $\rho \le 0$, it follows that $U_t \ge \rho$ for $t > 0$. This fact means that "Reject the intervening quitting penalty ρ for all $t > 0$", implying "Behave as if there does not exist the intervening quitting penalty ρ"; in other words, it follows that rM:3[ℝ][A] is reduced to rM:2[ℝ][A].

(b) Let $\rho \ge x_K$ and $\rho \ge 0$. Then, we have $U_t \le \rho$ for $\tau \ge t \ge 0$ from Lemma 25.2.1(c), meaning "Accept the intervening quitting penalty ρ and the process stops" for $\tau \ge t > 0$; in other words, we have $\mathtt{Accept}_t(\rho)/\mathtt{Stop}$ for $\tau \ge \tau > 0$ (see (20.1.9(p.212))). Accordingly, it follows that the assertion holds due to Lemma 20.1.2(p.212). ∎

□ Pom **25.2.1** ($\mathscr{A}\{$**r**M:3[$\mathbb{R}$][A]$\}^+$)

(a) *Let* $\rho \le x_K$ *or* $\rho \le 0$. *Then we have* $\mathscr{A}\{r$M:3[$\mathbb{R}$][A]$^+\} \twoheadrightarrow \mathscr{A}\{r$M:2[$\mathbb{R}$][A]$^+\}$.
(b) *Let* $\rho \ge x_K$ *and* $\rho \ge 0$. *Then we have* odr $\mapsto$ Accept$_\tau(\rho)$/Stop. □

Proof See Lemma 16.4.1(p.100). ▮

25.2.1 r$\tilde{\text{M}}$:3[$\mathbb{R}$][A]

In the same way as in Section 23.1.2.1(p.235) we can easily verify that SOE{r$\tilde{\text{M}}$:3[$\mathbb{R}$][A]} $= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$[SOE{rM:3[$\mathbb{R}$][A]}] (see (22.3.149(p.228)) and (22.3.140)), hence, applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to Tom 25.2.1 yields the following Tom.

□ Tom **25.2.2** ($\mathscr{A}\{$**r**$\tilde{\text{M}}$:3[$\mathbb{R}$][A]$\}$)

(a) *Let* $\rho \ge x_{\tilde{K}}$ *or* $\rho \le 0$. *Then we have* $\mathscr{A}\{r\tilde{\text{M}}$:3[$\mathbb{R}$][A]$\} \twoheadrightarrow \mathscr{A}\{r\tilde{\text{M}}$:2[$\mathbb{R}$][A]$\}$.
(b) *Let* $\rho \le x_{\tilde{K}}$ *and* $\rho \ge 0$. *Then we have* odr $\mapsto$ Accept$_\tau(\rho)$/Stop. □

□ Pom **25.2.2** ($\mathscr{A}\{$**r**$\tilde{\text{M}}$:3[$\mathbb{R}$][A]$\}^+$)

(a) *Let* $\rho \ge x_{\tilde{K}}$ *or* $\rho \le 0$. *Then we have* $\mathscr{A}\{r\tilde{\text{M}}$:3[$\mathbb{R}$][A]$^+\} \twoheadrightarrow \mathscr{A}\{r\tilde{\text{M}}$:2[$\mathbb{R}$][A]$^+\}$.
(b) *Let* $\rho \le x_{\tilde{K}}$ *and* $\rho \ge 0$. *Then we have* odr $\mapsto$ Accept$_\tau(\rho)$/Stop. □

Proof See Lemma 16.4.1(p.100). ▮

25.2.2 Conclusion 14 (Search-Allowed-Model 3)

In a selling model (buying model) we have:

C1. Let $\rho \le x_K$ or $\rho \le 0$ ($\rho \ge x_{\tilde{K}}$ or $\rho \le 0$). Then we have Model 3 $\twoheadrightarrow$ Model 2.
C2. Let $\rho \ge x_K$ and $\rho \ge 0$ ($\rho \le x_{\tilde{K}}$ and $\rho \ge 0$). Then we have optdr $\mapsto$ Accept$_\tau(\rho)$/Stop. □

C1 See Pom 25.2.1(p.258) (a) (Pom 25.2.2(p.259) (a)).

C2 See Pom 25.2.1(p.258) (b) (Pom 25.2.2(p.259) (b)). ▮

Chapter 26

The Whole Conclusion of Recall-Model

26.1 Conclusion 15

■ Conclusions 23.1.3(p.238) -25.2.2(p.259) are summarized as below.

C1 **Reduction**

a. (model reduction) We have Model 3 $\twoheadrightarrow$ Model 2 (see C1 (p.258) ,C1 (p.259)).
b. (optdr reduction) We have `optdr` $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Stop}$ (see C2 (p.259)).
c. optdr reduction) We have `optdr` $\mapsto$ $\mathtt{Accept}_\tau(\rho)/\mathtt{Terminate}$ (see C2b (p.244) ,C4b (p.255)).
d. (chain of reduction) We have

$\mathscr{A}\{\mathrm{rM}{:}3[\mathbb{R}][\mathtt{A}]^+\} \twoheadrightarrow \mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+\}$ (see Pom 25.2.1(p.259) (a)) $\cdots$ (1°).
$\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+\} \looparrowright \mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ (see Pom 24.2.1(p.253) (a)) $\cdots$ (2°).
$\mathscr{A}\{\mathrm{r\tilde{M}}{:}3[\mathbb{R}][\mathtt{A}]^+\} \twoheadrightarrow \mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$ (see Pom 25.2.2(p.259) (a)) $\cdots$ (1•).
$\mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\} \looparrowright \mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ (see Pom 24.2.2(p.254) (a)) $\cdots$ (2•).

From (1°) and (2°) we have

$$\mathscr{A}\{\mathrm{rM}{:}3[\mathbb{R}][\mathtt{A}]^+\} \twoheadrightarrow \mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+\} \looparrowright \mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{E}]^+\},$$

i.e, $\mathscr{A}\{\mathrm{rM}{:}3[\mathbb{R}][\mathtt{A}]^+\}$ is reduced to $\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ via $\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+\}$. Likewise, from (1•) and (2•) we have

$$\mathscr{A}\{\mathrm{r\tilde{M}}{:}3[\mathbb{R}][\mathtt{A}]^+\} \twoheadrightarrow \mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\} \looparrowright \mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{E}]^+\},$$

i.e., $\mathscr{A}\{\mathrm{r\tilde{M}}{:}3[\mathbb{R}][\mathtt{A}]^+\}$ is reduced to $\mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ via $\mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$. Let us refer to this reduction flows as the chain of reductions.

■ C1i(p.216) implies that it is not necessary to discuss any more for Model 3, hence, below we make discussions only for Model 1 and Model 2.

C2 **Monotonicity**

a. We have t-`reservation-price` (see C3 (p.250) ,C4a (p.255)).
b. We have `c-reservation-price` (see C3 (p.238) ,C3c (p.244)).

Remark 26.1.1 (myopic property) Thus far, it has been considered as a common sense that the reservation price is t-dependent (see ⟨a(p.228)⟩). However, from the fact that it can become constant (see ⟨c(p.228)⟩) for the recall-model (see C3(p.238)) we noticed that the common sense does not always hold. Now, the constant reservation price implies that the optimal decision of any point in time $t > 0$ is identical to that of time 1 at which the process terminates a period hence, i.e., the deadline, implying that the optimal decision is the same as "behave as if the process terminates a period hence", called the *myopic property*. Herein it should be noted that although the property has been thought of as quite an isolated one which appears only for $\mathrm{rM}{:}1[\mathbb{R}][\mathtt{E}]$ (recall-model; see Section 22.3.1.1.1(p.222)), it appears also for a <u>no-recall-model</u> (see C1a(p.208)).

C3 **Inheritance and Collapse**

a. The symmetry collapses ($\not\sim$) between $\mathscr{A}\{\mathrm{rM}{:}1[\mathbb{R}][\mathtt{E}]^+\}$ and $\mathscr{A}\{\mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathtt{E}]^+\}$ (see C1(p.238)).
b. The symmetry collapses ($\not\sim$) between $\mathscr{A}\{\mathrm{rM}{:}1[\mathbb{R}][\mathtt{A}]^+\}$ and $\mathscr{A}\{\mathrm{r\tilde{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\}$ (see C1(p.244)).
c. The symmetry is inherited ($\sim$) between $\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ and $\mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{E}]^+\}$ (see C1(p.250)).
d. The symmetry collapses ($\not\sim$) between $\mathscr{A}\{\mathrm{rM}{:}2[\mathbb{R}][\mathtt{A}]^+\}$ and $\mathscr{A}\{\mathrm{r\tilde{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$ (see C1(p.255)).

C4 **Diagonal symmetry**

a. Confirm by yourself that the diagonal symmetry holds by comparing `Pom` 23.1.2(p.237) and `Nem` 23.1.1(p.235).

b. Confirm by yourself that the diagonal symmetry holds by comparing `Pom` 23.2.2(p.243) and `Nem` 23.2.1(p.242).

c. Confirm by yourself that the diagonal symmetry holds by comparing `Pom` 24.1.2(p.250) and `Nem` 24.1.1(p.248).

d. Confirm by yourself that the diagonal symmetry holds by comparing `Pom` 24.2.2(p.254) and `Nem` 24.2.1(p.254). □

C5 **Occurrence of ⓢ, ⊛, and 🅓**

a. We have ⓢ (see C2a(p.238) C3a(p.244), C3(p.255), C3(p.258)).

b. We have ⊛ (see C2b(p.238)),

c. We have 🅓 (see C2c(p.238), C3b(p.244), C4a(p.255)).

Part 5

Epilogue

In the epilogue we state the whole conclusion of the present paper and the subjects of future studies

Chapter 27

Kernals in The Whole Conclusion of This Paper

This chapter summarizes kernels in the whole conclusion of this paper (see Conclusions 18.1(p.119) to 26.1(p.261)).

27.1 Introduction of Some Novel Concepts

In order to attain the two study goals stated in Section 1.3(p.4), we introduced some novel concepts as stated below, which are all what have not been taken into consideration so far at all by any researchers.

[1] Quitting penalty (see A6(p.7))

[2] Enforced-case and allowed-case (see Concept 2(p.9))

[3] Quadruple-asset-trading-models and structured-unit-of-models (see Section 1.2(p.3), Table 3.3.1(p.11), and Section 3.4(p.12)).

[4] Four kinds of points in time (see Concept 1(p.9) and Section 7.1(p.33))

[5] Market restriction (see Section 16.4(p.100))

[6] Strong assertion and weak assertion (see Section 7.3(p.37))

These concepts serve as the impetus for opening the way to quite a new horizon for discussions of decision processes.

27.2 Main Findings

Below let us list main findings that have been obtained in this paper.

[7] Underlying functions

One of the most noteworthy results in this paper is the finding of the underlying functions T, L, K, and $\mathcal{L}$ (see Chapter 5(p.17)), the properties of which (see Lemmas 9.2.1(p.43)-9.3, Lemmas 11.6.1(p.66)-11.6.6, Lemmas 12.2.1(p.77)-12.2.6, and Lemmas 13.6.1(p.89)-13.6.6) play a central role in the analysis of all models.

[8] Symmetry and Analogy

Another one of the most noteworthy results in this paper is the finding of the following eight successive theorems:

· Theorems 11.5.1(p.66) and 11.8.1(p.72) (symmetry).
· Theorems 12.3.1(p.81) and 12.3.2(p.82) (analogy).
· Theorems 13.5.1(p.88) and 13.5.2(p.88) (symmetry).
· Theorems 14.2.1(p.94) and 14.2.2(p.94) (analogy).

The first and third theorems (symmetry) above provide the answer to the question " Is a buying problem always symmetrical to a selling problem ?" in Motive 1(p.4) of this study. Note here that these theorems are what were derived on the premise that prices are defined on the *total market* $\mathscr{F} = \{-\infty, \infty\}$.

[9] Integration Theory

The whole of the *integration theory* depicted by Figure 15.1.1(p.97) consists of the flow of the eight successive theorems in [8]. This is the answer to the question "Can the theory integrating quadruple-asset-trading-problems exist ?" in Motive 2(p.4).

[10] Inheritance and Collapse

Recall here that the symmetry and the analogy can be derived on the *total market* $\mathscr{F}$, implying that if the market restriction (see Section 16.4(p.100)) is applied, it is questioned whether each of the symmetry and the analogy *is inherited* or *collapses*. In fact , "inheritance" and "collapses" are both possible (see C2(p.119), C2(p.133), C2(p.175), C2(p.207), and C3(p.261)). From this result it follows that the answer to the question in the Motive 1(p.4) is "No !" on the *restricted markets* $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$. However, the following should be noted.

On the positive market $\mathscr{F}^+$ we have:

a. Symmetry In the most simple case "$\beta = 1$ and $s = 0$" the symmetry is inherited (∾) for both Model 1 (see C2c1a(p.136)) and Model 2 (see C2a(p.208)).

b. Analogy In the most simple case "$\beta = 1$ and $s = 0$", although the analogy is inherited (⋈) for Model 1 (see C2c2(p.136)), it can collapse (⋈̸) for Model 2 (see C2b(p.209)).

[11] Diagonal symmetry

As seen in [10], the symmetry *cannot be always inherited* between the selling model and the buying model on $\mathscr{F}^+$ (see [10]). However, it is proven that the symmetry is *always inherited* between the selling-problem on $\mathscr{F}^-$ and the buying problem on $\mathscr{F}^+$ (see Chapter 17(p.111)), called the *diagonal symmetry.*

[12] Null-time-zone and deadline-falling

We showed that there exist three possibilities of the optimal initiating time, ⓢ , ⊛ , and ⓓ (see Section 7.2.4.4(p.35)). Here it should be noted that the existence of ⊛ and ⓓ *inevitably* leads us to the existence of the null-time-zone (Section 7.2.4.6(p.36)) and that it also leads us, as its inevitable consequence, to the existence of the deadline-falling (see Figures 7.2.3(p.36) and 7.2.4). This should be said to be one of the most *striking* findings in this paper, and this fact prompts us to the overall re-examination of the whole theory of decision processes that have been investigated so far without knowing the existence of the deadline-falling (see Section A 5(p.291)). Furthermore it should be noted that the deadline falling ⓓ$_{\|}$can occur even in the most simple case "$\beta = 1$ and $s = 0$" (see C3a(p.209)).

[13] Posterior-skip-of-search

It is usual to consider that once conducting the search becomes optimal, it will become also optimal to continue conducting the search after that. However, there exists a case that this expectation does not always hold (see Remark 7.2.1(p.34)); in other words, it is possible, although being very rare, that it can become optimal to skip the search after having conducted the search for a while (see C7(p.218)).

[14] Reduction

The reduction for models and optimal decision rules is another noteworthy finding in the paper (see Defs 20.1.1(p.212) , 23.2.1(p.242) , C1(p.217) , and C1(p.261)).

27.3 Alice's Adventures in Wonderland

Herein recall the confusions and wonders upon which Alice fell and the suggestions which Dr. Rabbit told to her:

[15] Alice 1(p.9) (**discount factor for cost**)
See [39, Ross][0535] for the description concerning a managerial and economic implication of introducing the discount factor β for *profit.* Strangely enough, there exists no reference, as far as we know, in which the persuasive explanation has been stated for introducing the discount factor β for *cost.*

[16] Alice 2(p.36) (**first-search-conducting-time**)
Maybe some readers might consider that the optimal initiating time can be replaced by the optimal first-search-conducting-time; however, by Dr. Rabbit's instruction many of them will immediately notice that this way of thinking is not always proper.

[17] Alice 3(p.36) (**jumble of intuition and theory**)
The two questions that Alice raised is what was caused by the jumble of intuition and theory. Fortunately, almost researchers soon notice the mistake; however, unfortunately there might exist ones who lapse into the confusion and do not obstinately admit it; as a result, a submitted paper might be rejected if such a researcher is selected as a referee.

[18] Alice 4(p.36) (**deadline as the black hole**)
Perhaps this may be what should be said to be one of the biggest findings in the present paper, which leads us not only the re-examination of conventional theory of decision processes but also to quite a new horizon of the theory of decision processes .

[19] Alice 5(p.37) (**strong assertion and weak assertion**)
It might seem to be a molehill at a glance to dare to define the two kinds of assertions; however, readers will know that the two are at any price necessary in order to make discussions clearer by avoiding the half-and-half standpoint.

[20] Alice 6(p.53) (**unknown box**)
What is pointed out by Dr. Rabbit is not the vulnerability of the theory but what should be said to be the proof of its expansivity.

Above confusions and wonders of Alice and suggestions of Dr. Rabbit are also the ventilation of philosophical background which we have for the whole of this paper.

27.4 Decision Theory as Natural Science

Almost all decision theories that have been made thus far by many researchers are discussed as *mathematical* subjects of study; however, in this paper we take a hold on "decision" as a subject of study in *natural science*. Here, it should not be forgot that the truth of mathematics lies in mathematics itself and the truth of physics lies in physics itself. Originally, no relation exists between the two truths; if that helps, a part of physicists sometimes refer to the term "mathematics" as "arithmetic". For this reason we should cast our mind back to the apothegm of Einstein that was quoted in the title page of the paper, "*As far as the lows of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.*" Throughout the whole of the present paper, this philosophy is reflected on the following point:

[23] Finite planning horizon (see Remark 7.1.2(p.33))
A decision process with the *infinite* planning horizon is a *product of fantasy* created by mathematicians, which does not exist in the real world at all; for this reason, all of decision processes treated in the present paper are, *right down the line*, of finite planning horizon.

Chapter 28

Future Studies

28.1 Recall-Model with $\mathbb{P}$-mechanism

In Part 4(p.219) we tried the application of the integration theory to the recall-model with $\mathbb{R}$-mechanism in which it suffices to memorize only the best of prices once rejected. However, in the recall-model with $\mathbb{P}$-mechanism we will confront the difficult problem "Should which of prices once rejected be memorized ?". It remains as a subject of future study how to tackle this difficulty.

28.2 Overall Re-examination

We demonstrated that the introduction of the optimal initiating time inevitably leads us to "deadline-falling ⓓ" (see Section 7.2.4.7(p.36)), which is not a rare case but a phenomenon which is very often possible (see Table 21.1.1(p.218)). Moreover, we also pointed out that this phenomenon might occur not only in the decision processes in this paper but also in more generalized decision processes such as Markovian decision processes [23,Howard,1960][0528] (see Section A 5(p.291)). This tells to us the necessity of the overall re-examination for the whole of conventional discussions in which the concept of the optimal initiating time has not been taken into account at all.

28.3 Different Variations of Basic Models

In Section 4.5(p.16) we showed the 10 variations of the basic models of asset trading problems. Since each variation has the 2 kinds of models (s-E-model and s-A-model), it follows that the $20 = 10 \times 2$ variations can be considered. Moreover, adding the two models (model with terminal quitting penalty and model with intervening quitting penalty) to each of these variations, it follows that we have in all the $40 = 20 \times 2$ variations. Furthermore, since each variation has $\mathbb{R}$-model and $\mathbb{P}$-model, it eventually follows that $80 = 40 \times 2$ variations can be defined. Additionally, since each conditions for each of these variations can be independently specified, we can propose different mixed variations; for example

- Model with several search areas and limited search budget
- Model with uncertain deadline and mechanism switching
- Model with limited search budget, uncertain deadline, and mechanism switching
- Model with several search areas, limited search budget, uncertain deadline, and mechanism switching
- Model with recall, several search areas, limited search budget, uncertain deadline, and mechanism switching
- Model with uncertain recall, uncertain deadline, and mechanism switching

⋮

Accordingly, variations that can be tackled with amounts to an astronomical number. In addition to what were stated above, we can add discussions involved with the optimal initiating time OIT. Taking into consideration all variations stated above, we believe that our integration theory will become a strong tools to tackle the vast amount of these variations—analyzing these variations without this theory will be *almost beyond the realm of possibility.*

Appendix

[1]

A 1 Direct Proof of Underlying Functions of Type $\mathbb{R}$

A 1.1 $\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$

For convenience of reference, below let us copy Lemma 11.6.1(p.66).

Lemma A 1.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *strictly increasing on* $(-\infty, b]$.
(f) $\tilde{T}(x) = \mu - x$ *on* $[b, \infty)$ *and* $\tilde{T}(x) < \mu - x$ *on* $(-\infty, b)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $\tilde{T}(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \leq \min\{0, \mu - x\}$ *on* $x \in (-\infty, \infty)$.
(i) $\tilde{T}(0) = 0$ *if* $a > 0$ *and* $\tilde{T}(0) = \mu$ *if* $b < 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b > y$, *then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b > \mu$. □

Proof First, for any x and y let us prove the following two inequalities:

$$-(x-y)F(y) \geq \tilde{T}(x) - \tilde{T}(y) \geq -(x-y)F(x) \cdots \textbf{(1)},$$

$$(x-y)(1-F(y)) \geq \tilde{T}(x) + x - \tilde{T}(y) - y \geq (x-y)(1-F(x)) \cdots \textbf{(2)}.$$

Note here that $\tilde{T}(x)$ defined by (5.1.11(p.17)) can be rewritten as $\tilde{T}(x) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < x)]$ for any x.[†] Then, for any x and y let $\tilde{T}(x, y) \stackrel{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} \leq y)]$. Since $0 \leq I(\boldsymbol{\xi} \leq y) \leq 1$, $\min\{\boldsymbol{\xi} - x, 0\} \leq 0$, and $\min\{\boldsymbol{\xi} - x, 0\} \leq \boldsymbol{\xi} - x$, multiplying the both-sides of $I(\boldsymbol{\xi} \leq y) \leq 1$ by $\min\{\boldsymbol{\xi} - x, 0\}$ leads to $\min\{\boldsymbol{\xi} - x, 0\} = \min\{\boldsymbol{\xi} - x, 0\} \times 1 \leq \min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > y)$ and then multiplying the both-sides of $\min\{\boldsymbol{\xi} - x, 0\} \leq \boldsymbol{\xi} - y$ by $I(\boldsymbol{\xi} > y)$ leads to $\min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > y) \leq (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)$. Hence, since $\min\{\boldsymbol{\xi} - x, 0\} \leq (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)$, from (5.1.11) we have $\tilde{T}(x) \leq \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] = \tilde{T}(x, y)$. Accordingly, for any x and y we have

$$\tilde{T}(x) - \tilde{T}(y) \leq \tilde{T}(x, y) - \tilde{T}(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} \leq y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} \leq y)] = -(x-y)\,\mathbf{E}[I(\boldsymbol{\xi} \leq y)].$$

In addition, since

$$\mathbf{E}[I(\boldsymbol{\xi} \leq y)] = \textstyle\int_{-\infty}^{\infty} I(\xi \leq y)f(\xi)d\xi = \int_{-\infty}^{y+} 1 \times f(\xi)d\xi + \int_{y-}^{\infty} 0 \times f(\xi)d\xi = \int_{-\infty}^{y+} f(\xi)d\xi = F(y)$$

for any y, we have $\tilde{T}(x) - \tilde{T}(y) \leq -(x-y)F(y)$ for any x and y, hence the former half of (1) is true. Multiplying both sides of the inequality by -1 leads to $\tilde{T}(y) - \tilde{T}(x) \geq (x-y)F(y) = -(y-x)F(y)$, and then interchanging the notations x and y yields $\tilde{T}(x) - \tilde{T}(y) \geq -(x-y)F(x)$, hence the latter half of (1) is true. (2) is immediate from adding $x - y$ to the both-sides of (1). Let us note here that $\tilde{T}(x)$ defined by (5.1.11) can be rewritten as

$$\begin{aligned}\tilde{T}(x) &= \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(b \geq \boldsymbol{\xi})] + \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > b)]. \cdots \textbf{(3)} \\ &= \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} \geq a)] + \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(a > \boldsymbol{\xi})]. \cdots \textbf{(4)}.\end{aligned}$$

(a,b) Immediate from the fact that $\min\{\boldsymbol{\xi} - x, 0\}$ is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any given $\boldsymbol{\xi}$.

(c) Let $x > y > a$. Then, since $-(x-y) < 0$ and $F(y) > 0$ due to (2.1.2 (2,3) (p.8)), we have $-(x-y)F(y) < 0$, hence $0 > \tilde{T}(x) - \tilde{T}(y)$ from (1), i.e., $\tilde{T}(y) > \tilde{T}(x)$, so $\tilde{T}(x)$ is *strictly* decreasing on $(a, \infty) \cdots$ **(5)**. Suppose $\tilde{T}(a) = \tilde{T}(x)$ for any $x > a$, hence $x - a > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > 2\varepsilon > 0$ we have $a < a + \varepsilon < x - \varepsilon < x$, hence $\tilde{T}(a) = \tilde{T}(x) < \tilde{T}(a+\varepsilon) \leq \tilde{T}(a)$ from (5) and (b), which is a contradiction. Thus it must be that $\tilde{T}(a) \neq \tilde{T}(x)$ for any $x > a$, i.e., $\tilde{T}(a) > \tilde{T}(x)$ for any $x > a$ or $\tilde{T}(a) < \tilde{T}(x)$ for any $x > a$. Since the latter is impossible due to (b), it follows that $\tilde{T}(a) > \tilde{T}(x)$ for any $x > a$, hence together with (5) it eventually follows that $\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$ instead of (a, ∞).

(d) Evident from the fact that $\tilde{T}(x) + x = \mathbf{E}[\min\{\boldsymbol{\xi}, x\}]$ from (5.1.11) and that $\min\{\boldsymbol{\xi}, x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $b > x > y$, hence $F(x) < 1$ due to (2.1.2 (1,2) (p.8)). Then, since $(x-y)(1-F(x)) > 0$, we have $\tilde{T}(x) + x > \tilde{T}(y) + y$ from (2), i.e., $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b) \cdots$ **(6)**. Suppose $\tilde{T}(b) + b = \tilde{T}(x) + x$ for any $x < b$. Then, for any sufficiently small $\varepsilon > 0$ such that $b - x > \varepsilon$ we have $x < x + \varepsilon < b$, hence $\tilde{T}(b) + b = \tilde{T}(x) + x < \tilde{T}(x+\varepsilon) + x + \varepsilon \leq \tilde{T}(b) + b$ due to (6) and (d), which is a contradiction. Thus, $\tilde{T}(x) + x \neq \tilde{T}(b) + b$ for $x < b$, i.e., $\tilde{T}(x) + x > \tilde{T}(b) + b$ for $x < b$ or $\tilde{T}(x) + x < \tilde{T}(b) + b$ for $x < b$. Since the former is impossible due to (d), it must be that $\tilde{T}(x) + x < \tilde{T}(b) + b$ for $x < b$, hence, together with (6) it follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b]$.

(f) Let $x \geq b$. If $b \geq \boldsymbol{\xi}$, then $x \geq \boldsymbol{\xi}$, hence $\min\{\boldsymbol{\xi} - x, 0\} = \boldsymbol{\xi} - x$, and if $\boldsymbol{\xi} > b$, then $f(\xi) = 0$ due to (2.1.4 (3) (p.8)). Thus, from (3) we have $\tilde{T}(x) = \mathbf{E}[(\boldsymbol{\xi} - x)I(b \geq \boldsymbol{\xi})] + 0 = \mathbf{E}[(\boldsymbol{\xi} - x)I(b \geq \boldsymbol{\xi})] + \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > b)] = \mathbf{E}[(\boldsymbol{\xi} - x)(I(b \geq \boldsymbol{\xi}) + I(\boldsymbol{\xi} >$

[†] In general, if a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

$b))] = \mathbf{E}[\boldsymbol{\xi} - x] = \mu - x$,[†] hence the former half is true. Then, since $\tilde{T}(b) = \mu - b$ or equivalently $\tilde{T}(b) + b = \mu$, if $b > x$, from (e) we have $\tilde{T}(x) + x < \tilde{T}(b) + b = \mu$, hence $\tilde{T}(x) < \mu - x$, so the latter half is true.

(g) Let $a \geq x$. If $\boldsymbol{\xi} \geq a$, then since $\boldsymbol{\xi} \geq x$, we have $\min\{\boldsymbol{\xi} - x, 0\} = 0$ and if $a > \boldsymbol{\xi}$, then since $f(\boldsymbol{\xi}) = 0$ due to (2.1.4 (1) (p.8)), we have $\mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\} I(a > \boldsymbol{\xi})] = 0$. Accordingly, we have $\tilde{T}(x) = 0$ from (4), hence the latter half is true. Let $x > a$. Then, since $\tilde{T}(x) < \tilde{T}(a)$ from (c) and since $\tilde{T}(a) = 0$ from the fact stated just above, we have $\tilde{T}(x) < 0$ for $x > a$, hence the former half is true.

(h) From (f) we have $\tilde{T}(x) \leq \mu - x$ for any x and from (g) we have $\tilde{T}(x) \leq 0$ for any x, thus it follows that $\tilde{T}(x) \leq \min\{0, \mu - x\}$ for any x.

(i) From (5.1.11(p.17)) we have $\tilde{T}(0) = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\} I(a \leq \boldsymbol{\xi} \leq b)]$. If $a > 0$, then $\min\{\boldsymbol{\xi}, 0\} I(a \leq \boldsymbol{\xi} \leq b) = \min\{\boldsymbol{\xi}, 0\} I(0 < a \leq \boldsymbol{\xi} \leq b) = 0 \times I(0 < a \leq \boldsymbol{\xi} \leq b) = 0$, hence $\tilde{T}(0) = \mathbf{E}[0] = 0$, and if $b < 0$, then $\min\{\boldsymbol{\xi}, 0\} I(a \leq \boldsymbol{\xi} \leq b) = \min\{\boldsymbol{\xi}, 0\} \times I(a \leq \boldsymbol{\xi} \leq b < 0) = \boldsymbol{\xi} \times I(a \leq \boldsymbol{\xi} \leq b < 0) = \boldsymbol{\xi}$, hence $\tilde{T}(0) = \mathbf{E}[\boldsymbol{\xi}] = \mu$.

(j) If $\beta = 1$, then $\beta\tilde{T}(x) + x = \tilde{T}(x) + x$, hence the assertion is true from (d).

(k) Since $\beta\tilde{T}(x) + x = \beta(\tilde{T}(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x > y$ and $b > y$. If $x \geq b$, then $\tilde{T}(x) + x \geq \tilde{T}(b) + b > \tilde{T}(y) + y$ due to (d,e), and if $b > x$, then $b > x > y$, hence $\tilde{T}(x) + x > \tilde{T}(y) + y$ due to (e).

(m) From (5.1.11(p.17)) we have

$$\begin{aligned} \lambda\beta\tilde{T}(\lambda\beta\mu + s) + s &= \lambda\beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - \lambda\beta\mu - s, 0\}] + s \\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu - \lambda\beta s, 0\}] + s \\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu + (1 - \lambda\beta)s, s\}], \end{aligned}$$

which is nondecreasing in s and strictly increasing in s if $\lambda\beta < 1$.

(n) Evident from (2.1.3(p.8)). ∎

A 1.2 $\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$, and $\tilde{\kappa}_{\mathbb{R}}$

From (5.1.13(p.17)) and (5.1.14) and from Lemma A 1.1(f) we obtain, noting (9.2.1(p.42)),

$$\tilde{L}(x) \begin{cases} = \lambda\beta\mu + s - \lambda\beta x & \text{on } [b, -\infty) \quad \cdots (1), \\ < \lambda\beta\mu + s - \lambda\beta x & \text{on } (-\infty, b) \quad \cdots (2), \end{cases} \tag{A 1.1}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta\mu + s - \delta x & \text{on } [b, \infty) \quad \cdots (1), \\ < \lambda\beta\mu + s - \delta x & \text{on } (-\infty, b) \quad \cdots (2). \end{cases} \tag{A 1.2}$$

In addition, from Lemma A 1.1(g) we have

$$\tilde{K}(x) \begin{cases} < -(1 - \beta)x + s & \text{on } (a, \infty) \quad \cdots (1), \\ = -(1 - \beta)x + s & \text{on } (-\infty, a] \quad \cdots (2), \end{cases} \tag{A 1.3}$$

hence we obtain

$$\tilde{K}(x) + x \leq \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{A 1.4}$$

Then, from (A 1.2 (1)) and (A 1.3 (2)) we get

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta\mu + s + (1 - \lambda)\beta x & \text{on } [b, \infty) \quad \cdots (1), \\ \beta x + s & \text{on } (-\infty, a] \quad \cdots (2). \end{cases} \tag{A 1.5}$$

Since $\tilde{K}(x) = \tilde{L}(x) - (1 - \beta)x$ from (5.1.14) and (5.1.13), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}(x_{\tilde{L}}) = -(1 - \beta)\, x_{\tilde{L}} \cdots (1), \qquad \tilde{L}(x_{\tilde{K}}) = (1 - \beta)\, x_{\tilde{K}} \cdots (2). \tag{A 1.6}$$

Lemma A 1.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *Let $s = 0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.*
(e) *Let $s > 0$.*

1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.*
2. *$(\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b$.* □

[†] $I(b \geq \boldsymbol{\xi}) + I(\boldsymbol{\xi} > b) = 1$.

Proof (a-c) Immediate from (5.1.13(p.17)) and Lemma A 1.1(a-c).

(d) Let $s = 0$. Then, since $\tilde{L}(x) = \lambda\beta\tilde{T}(x)$, from Lemma A 1.1(g) we have $\tilde{L}(x) = 0$ for $a \geq x$ and $\tilde{L}(x) < 0$ for $x > a$, hence $x_{\tilde{L}} = a$ by the definition of $x_{\tilde{L}}$ (see Section 5.2(p.19)(b)), so $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (=)\ 0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x) = 0 \Rightarrow \tilde{L}(x) \geq 0$, we have $\tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.

(e) Let $s > 0$.

(e1) From (A 1.1 (1)) and due to $\lambda > 0$ and $\beta > 0$ we have $\tilde{L}(x) < 0$ for a sufficiently large $x > 0$ such that $x \geq b$. In addition, we have $\tilde{L}(a) = \lambda\beta\tilde{T}(a) + s = s > 0$ from Lemma A 1.1(g). Hence, from (a,c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}} > a$ is immediate from $\tilde{L}(a) > 0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b$, from (A 1.1) we have $\tilde{L}((\lambda\beta\mu + s)/\lambda\beta) = (<)\ \lambda\beta\mu + s - \lambda\beta(\lambda\beta\mu + s)/\lambda\beta = 0$, hence $x_{\tilde{L}} = (<)\ (\lambda\beta\mu + s)/\lambda\beta \geq (<)\ b$ from (e1). ∎

Corollary A 1.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.
(b) $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. □

Proof (a) Clearly $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0$ from Lemma A 1.2(d,e1). The inverse holds by contraposition.

(b) Since $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0$ due to (a) and since clearly $\tilde{L}(x) < (\geq)\ 0 \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$, we have $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{L}} = x$, then $\tilde{L}(x) = \tilde{L}(x_{\tilde{L}}) = 0 \leq 0$ or equivalently $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) \leq 0$, hence it eventually follows that $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. ∎

Lemma A 1.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{K}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{K}(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$\tilde{K}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$\tilde{K}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(g) *$\tilde{K}(x) + x$ is strictly increasing on $(-\infty, b]$.*
(h) *If $x > y$ and $b > y$, then $\tilde{K}(x) + x > \tilde{K}(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{K}} = a$ where $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*
   1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (= (>))\ 0$.*
   2. *$(\lambda\beta\mu + s)/\delta \geq (<)\ b \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta\mu + s)/\delta$.*
   3. *Let $\tilde{\kappa} < (= (>))\ 0$. Then $x_{\tilde{K}} < (= (>))\ 0$.* □

Proof (a-c) Immediate from (5.1.14(p.17)) and Lemma A 1.1(a-c).

(d) Immediate from (5.1.14) and Lemma A 1.1(b).

(e) From (5.1.14) we have

$$\tilde{K}(x) + x \ = \ \lambda\beta\tilde{T}(x) + \beta x + s = \lambda\beta(\tilde{T}(x) + x) + (1 - \lambda)\beta x + s \cdots \mathbf{(1)},$$

hence the assertion holds from Lemma A 1.1(d).

(f) Obvious from (1) and Lemma A 1.1(d).

(g) Clearly from (1) and Lemma A 1.1(e).

(h) Let $x > y$ and $b > y$. If $x \geq b$, then $\tilde{K}(x) + x \geq \tilde{K}(b) + b > \tilde{K}(y) + y$ due to (e,g), and if $b > x$, then $b > x > y$, hence $\tilde{K}(x) + x > \tilde{K}(y) + y$ due to (g). Thus, whether $x \geq b$ or $b > x$, we have $\tilde{K}(x) + x > \tilde{K}(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $\tilde{K}(x) = \lambda\tilde{T}(x)$ due to (5.1.14(p.17)), from Lemma A 1.1(g) we have $\tilde{K}(x) = 0$ for $a \geq x$ and $\tilde{K}(x) < 0$ for $x > a$, so $x_{\tilde{K}} = a$ by the definition of $x_{\tilde{K}}$ (see Section 5.2(p.19)(b)). Hence $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (=)\ 0$. The inverse holds by contraposition. In addition, since $\tilde{K}(x) = 0 \Rightarrow \tilde{K}(x) \geq 0$, we have $\tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) First see (A 1.3 (2)). Then, if $\beta = 1$, then $s > 0$, hence $\tilde{K}(x) = s > 0$ for any $x \leq a$ and if $\beta < 1$, then $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ such that $x \leq a$. Hence, whether $\beta = 1$ or $\beta < 1$, we have $\tilde{K}(x) > 0$ for any sufficiently small x. Next, for any sufficiently large $x > 0$ such that $x \geq b$, from (A 1.2 (1)) we have $\tilde{K}(x) < 0$, whether $\beta =$ or $\beta < 1$, since $\delta > 0$ due to (9.2.2 (1) (p.42)). Hence, it follows that there exists the solution $x_{\tilde{K}}$ whether $\beta =$ or $\beta < 1$. Let $\beta < 1$. Then, the solution is unique due to (d). Let $\beta = 1$, hence $s > 0$. Then, since $\tilde{K}(a) = s > 0$ from (A 1.3 (2)), we have $x_{\tilde{K}} > a$, hence $\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x = x_{\tilde{K}}$ due to (c), implying that the solution $x_{\tilde{K}}$ is unique. Therefore, whether $\beta =$ or $\beta < 1$, the solution is unique. Thus the latter half is immediate.

(j2) Let $(\lambda\beta\mu + s)/\delta \geq (<)\ b$. Then, from (A 1.2 (1(2))) we have $\tilde{K}((\lambda\beta\mu + s)/\delta) = (<)\ \lambda\beta\mu + s - \delta(\lambda\beta\mu + s)/\delta = 0$, hence $x_{\tilde{K}} = (<)\ (\lambda\beta\mu + s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.17)), hence $x_{\tilde{K}} < (= (>))\ 0$ from (j1). ∎

Corollary A 1.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.
(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

Proof (a) Clearly $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to Lemma A 1.3(i,j1). The inverse holds by contraposition.

(b) Since $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to (a) and since $\tilde{K}(x) < (\geq)\ 0 \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$, we have $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{K}} = x$, then $\tilde{K}(x) = \tilde{K}(x_{\tilde{K}}) = 0 \leq 0$, hence it eventually follows that $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. ∎

Lemma A 1.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}} = a$.
(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}}$.
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $a < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (=))\ 0$.
(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (>))\ 0$. □

Proof (a) If $\beta = 1$ and $s = 0$, then $x_{\tilde{L}} = a$ from Lemma A 1.2(d) and $x_{\tilde{K}} = a$ from Lemma A 1.3(i), hence $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) Let $\beta = 1$ and $s > 0$. Then $\tilde{K}(x_{\tilde{L}}) = 0$ from (A 1.6 (1)), hence $x_{\tilde{K}} = x_{\tilde{L}}$ from Lemma A 1.3(j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_{\tilde{L}} = a \cdots (1)$ from Lemma A 1.2(d). Suppose $a < 0$. Then, since $x_{\tilde{L}} < 0$, we have $\tilde{K}(x_{\tilde{L}}) > 0$ from (A 1.6 (1)), hence $x_{\tilde{L}} < x_{\tilde{K}}$ from Lemma A 1.3(j1). Furthermore, from (5.1.16(p.17)) and (5.1.17(p.17)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) + s = \lambda\beta\tilde{T}(0) < 0$ due to Lemma A 1.1(g), hence $x_{\tilde{K}} < 0$ from Lemma A 1.3(j1). Suppose $a = (>)\ 0$. Then, since $x_{\tilde{L}} = (>)\ 0$ from (1), we have $\tilde{K}(x_{\tilde{L}}) = (<)\ 0$ from (A 1.6 (1)), thus $x_{\tilde{L}} = (>)\ x_{\tilde{K}}$ from Lemma A 1.3(j1). Furthermore, from (5.1.16(p.17)) and (5.1.17(p.17)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) = (=)\ 0$ due to Lemma A 1.1(g), hence $x_{\tilde{K}} = (=)\ 0$ from Lemma A 1.3(j1).

(d) Let $\beta < 1$ and $s > 0$. Then, from (5.1.16(p.17)) and (5.1.17(p.17)), if $\tilde{\kappa} < (= (>))\ 0$, we have $\tilde{K}(0) < (= (>))\ 0$, thus $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 1.3(j1). Accordingly $\tilde{L}(x_{\tilde{K}}) < (= (>))\ 0$ from (A 1.6 (2)), hence $x_{\tilde{L}} < (= (>))\ x_{\tilde{K}}$ from Lemma A 1.2(e1).

Lemma A 1.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s.
(b) *If* $\lambda\beta < 1$, *then* $\tilde{\mathcal{L}}(s)$ *is strictly increasing in* s.
(c) *Let* $\lambda\beta\mu \leq a$.
  1. $x_{\tilde{L}} \geq \lambda\beta\mu + s$.
  2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_{\tilde{L}} > \lambda\beta\mu + s$.
(d) *Let* $\lambda\beta\mu > a$. *Then, there exists a* $s_{\tilde{\mathcal{L}}} > 0$ *such that if* $s_{\tilde{\mathcal{L}}} > (\leq)\ s$, *then* $x_{\tilde{L}} < (\geq)\ \lambda\beta\mu + s$. □

Proof (a,b) From (5.1.15(p.17)) and (5.1.13(p.17)) we have $\tilde{\mathcal{L}}(s) = \lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$, hence the assertion holds from Lemma A 1.1(m).

(c) Let $\lambda\beta\mu \leq a$. Then, from (5.1.15(p.17)) and (5.1.13(p.17)) we have $\tilde{\mathcal{L}}(0) = \tilde{L}(\lambda\beta\mu) = \lambda\beta\tilde{T}(\lambda\beta\mu) = 0 \cdots (1)$ due to Lemma A 1.1(g).

(c1) Since $s \geq 0$, from (a) we have $\tilde{\mathcal{L}}(s) \geq \tilde{\mathcal{L}}(0) = 0$ due to (1) or equivalently $\tilde{L}(\lambda\beta\mu + s) \geq 0$ from (5.1.15(p.17)), hence $x_{\tilde{L}} \geq \lambda\beta\mu + s$ from Corollary A 1.1(a).

(c2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (b) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0$ due to (1) or equivalently $\tilde{L}(\lambda\beta\mu + s) > 0$ from (5.1.15), hence $x_{\tilde{L}} > \lambda\beta\mu + s$ from Lemma A 1.2(e1).

(d) Let $\lambda\beta\mu > a$. From (5.1.15(p.17)) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta\mu) < 0$ due to Lemma A 1.1(g). Noting (A 1.1 (1)), for any sufficiently large $s > 0$ such that $\lambda\beta\mu + s \geq b$ and $\lambda\beta\mu + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s) = \lambda\beta\mu + s - \lambda\beta(\lambda\beta\mu + s) = (1 - \lambda\beta)(\lambda\beta\mu + s) \geq 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\tilde{\mathcal{L}}} > 0$ of $\tilde{\mathcal{L}}(s) = 0$. Then $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \geq 0$ for $s \geq s_{\tilde{\mathcal{L}}}$ or equivalently $\tilde{L}(\lambda\beta\mu + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta\mu + s) \geq 0$ for $s \geq s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 1.1(a) we get $x_{\tilde{L}} < \lambda\beta\mu + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \geq \lambda\beta\mu + s$ for $s \geq s_{\tilde{\mathcal{L}}}$. ∎

Lemma A 1.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have*:

(a) $\tilde{\kappa} = s$ *if* $a > 0$ *and* $\tilde{\kappa} = \lambda\beta\mu + s$ *if* $b < 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$. □

Proof (a) Immediate from (5.1.16(p.17)) and Lemma A 1.1(i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} < (= (>))\ 0$, we have $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.17)), hence $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 1.3(j3). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 2 Direct Proof of Underlying Functions of Type $\mathbb{P}$

A 2.1 $\mathscr{A}\{T_{\mathbb{P}}\}$

For convenience of reference, below let us copy Lemma 12.2.1(p.77).

Lemma A 2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b]$.
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $T(x) + x$ *is strictly increasing on* $[a^\star, \infty)$.
(f) $T(x) = a - x$ *on* $(-\infty, a^\star]$ *and* $T(x) > a - x$ *on* $(a^\star, \infty)$.
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty)$.
(h) $T(x) \ge \max\{0, a - x\}$ *on* $(-\infty, \infty)$.
(i) $T(0) = a$ *if* $a^\star > 0$ *and* $T(0) = 0$ *if* $b < 0$.
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x < y$ *and* $a^\star < y$, *then* $T(x) + x < T(y) + y$.
(m) $\lambda\beta T(\lambda\beta a - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(n) $a^\star < a$. □

A 2.2 $\mathscr{A}\{L_{\mathbb{P}}\}$, $\mathscr{A}\{K_{\mathbb{P}}\}$, $\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$, and $\kappa_{\mathbb{P}}$

Noting Lemma A 2.1(f), from (5.1.20(p.18)) and (5.1.21) we obtain

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x & \text{on } (-\infty, a^\star] \quad \cdots (1), \\ > \lambda\beta a - s - \lambda\beta x & \text{on } (a^\star, \infty) \quad \cdots (2), \end{cases} \tag{A 2.1}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x & \text{on } (-\infty, a^\star] \quad \cdots (1), \\ > \lambda\beta a - s - \delta x & \text{on } (a^\star, \infty) \quad \cdots (2). \end{cases} \tag{A 2.2}$$

In addition, from (5.1.21(p.18)) and Lemma A 2.1(g) we have

$$K(x)\begin{cases} > -(1-\beta)x - s & \text{on } (-\infty, b) \quad \cdots (1), \\ = -(1-\beta)x - s & \text{on } [b, \infty) \quad \cdots (2), \end{cases} \tag{A 2.3}$$

from which we obtain

$$K(x) + x \ge \beta x - s \quad \text{on} \quad (-\infty, \infty). \tag{A 2.4}$$

Then, from (A 2.2 (1)) and (A 2.3 (2)) we get

$$K(x) + x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x & \text{on } (-\infty, a^\star] \quad \cdots (1), \\ \beta x - s & \text{on } [b, \infty) \quad \cdots (2). \end{cases} \tag{A 2.5}$$

Since $K(x) = L(x) - (1-\beta)x$ from (5.1.21) and (5.1.20), if x_L and x_K exist, then

$$K(x_L) = -(1-\beta)\,x_L \cdots (1), \quad L(x_K) = (1-\beta)\,x_K \cdots (2). \tag{A 2.6}$$

Lemma A 2.2 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) $L(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $L(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $L(x)$ *is strictly decreasing on* $(-\infty, b]$.
(d) *Let* $s = 0$. *Then* $x_L = b$ *where* $x_L > (\le)\ x \Leftrightarrow L(x) > (=)\ 0 \Rightarrow L(x) > (\le)\ 0$.
(e) *Let* $s > 0$.
   1. x_L *uniquely exists with* $x_L < b$ *where* $x_L > (= (<))\ x \Leftrightarrow L(x) > (= (<))\ 0$.
   2. $(\lambda\beta a - s)/\lambda\beta \le (>)\ a^\star \Leftrightarrow x_L = (>)\ (\lambda\beta a - s)/\lambda\beta > (\le)\ a^\star$. □

Proof (a-c) Immediate from (5.1.20(p.18)) and Lemma A 2.1(a-c).

(d) Let $s = 0$. Then, since $L(x) = \lambda\beta T(x)$, we have $L(x) = 0$ for $b \le x$ and $L(x) > 0$ for $x < b$ from Lemma A 2.1(g), hence $x_L = b$ by the definition of x_L (see Section 5.2(p.19) (a)), thus $x_L > (\le)\ x \Rightarrow L(x) > (=)\ 0$. The inverse is true by contraposition. In addition, since $L(x) = 0 \Rightarrow L(x) \le 0$, we have $L(x) > (=)\ 0 \Rightarrow L(x) > (\le)\ 0$.

(e) Let $s > 0$.

(e1) From (A 2.1 (1)) and the assumptions of $\lambda > 0$ and $\beta > 0$ we have $L(x) > 0$ for a sufficiently small $x < 0$ such that $x \le a^\star$. In addition, we have $L(b) = \lambda\beta T(b) - s = -s < 0$ from Lemma A 2.1(g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L < b$ is immediate from $L(b) < 0$. The latter half is evident.

(e2) If $(\lambda\beta a - s)/\lambda\beta \le (>)\ a^\star$, from (A 2.1 (1(2))) we have $L((\lambda\beta a - s)/\lambda\beta) = (>)\ \lambda\beta a - s - \lambda\beta(\lambda\beta a - s)/\lambda\beta = 0$, hence $x_L = (>)\ (\lambda\beta a - s)/\lambda\beta$ from (e1). ∎

Corollary A 2.1 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) $x_L > (\leq)\ x \Leftrightarrow L(x) > (\leq)\ 0$.
(b) $x_L \geq (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$. □

Proof (a) Clearly $x_L > (\leq)\ x \Rightarrow L(x) > (\leq)\ 0$ from Lemma A 2.2(d,e2). The inverse holds by contraposition.

(b) Since $x_L > (\leq)\ x \Rightarrow L(x) > (\leq)\ 0$ due to (a) and since $L(x) > (\leq)\ 0 \Rightarrow L(x) \geq (\leq)\ 0$, we have $x_L > (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$. In addition, if $x_L = x$, then $L(x) = L(x_L) = 0 \geq 0$ or equivalently $x_L = x \Rightarrow L(x) \geq 0$, hence it eventually follows that $x_L \geq (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$. ∎

Lemma A 2.3 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$K(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(g) *$K(x) + x$ is strictly increasing on $[a^\star, \infty)$.*
(h) *If $x < y$ and $a^\star < y$, then $K(x) + x < K(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*
1. *There uniquely exists x_K where $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.*
2. *$(\lambda\beta a - s)/\delta \leq (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.*
3. *Let $\kappa > (= (<))\ 0$. Then $x_K > (= (<))\ 0$.* □

Proof (a-c) Immediate from (5.1.21(p.18)) and Lemma A 2.1(a-c).

(d) Immediate from (5.1.21(p.18)) and Lemma A 2.1(b).

(e) From (5.1.21(p.18)) we have

$$K(x) + x = \lambda\beta T(x) + \beta x - s = \lambda\beta(T(x) + x) + (1-\lambda)\beta x - s \cdots \mathbf{(1)},$$

hence the assertion holds from Lemma A 2.1(d).

(f) Obvious from (1) and Lemma A 2.1(d).

(g) Clearly from (1) and Lemma A 2.1(e).

(h) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $K(x) + x \leq K(a^\star) + a^\star < K(y) + y$ due to (e,g), and if $a^\star < x$, then $a^\star < x < y$, hence $K(x) + x < K(y) + y$ due to (g). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $K(x) + x < K(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x)$ due to (5.1.21(p.18)), from Lemma A 2.1(g) we have $K(x) = 0$ for $b \leq x$ and $K(x) > 0$ for $x < b$, so that $x_K = b$ due to the definition in Section 5.2(p.19)(a). Hence $x_K > (\leq)\ x \Rightarrow K(x) > (=)\ 0$. The inverse holds by contraposition. In addition, since $K(x) = 0 \Rightarrow K(x) \leq 0$, we have $K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) First see (A 2.3 (2)). If $\beta = 1$, then $s > 0$, hence $K(x) = -s < 0$ for any $x \geq b$ and if $\beta < 1$, then $K(x) < 0$ for any sufficiently large $x > 0$ such that $x \geq b$, hence, whether $\beta = 1$ or $\beta < 1$, we have $K(x) < 0$ for any sufficiently large x. Next, for any sufficiently small $x < 0$ such that $x \leq a^\star$, from (A 2.1 (1)) we have $K(x) > 0$, whether $\beta = 1$ or $\beta < 1$, since $\delta > 0$ from (9.2.2 (1) (p.42)). Hence, it follows that there exists the solution x_K whether $\beta = 1$ or $\beta < 1$. Let $\beta < 1$. Then, the solution is unique due to (d). Let $\beta = 1$, hence $s > 0$. Then, since $K(b) = -s < 0$ from (A 2.3 (2)), we have $x_K < b$, hence $K(x)$ is strictly decreasing on the neighbourhood of $x = x_K$ due to (c), implying that the solution x_K is unique. Therefore, whether $\beta < 1$ or $\beta = 1$, the solution is unique. Thus the latter half is immediate.

(j2) Let $(\lambda\beta a - s)/\delta \leq (>)\ a^\star$. Then, from (A 2.2 (1(2))) we have $K((\lambda\beta a - s)/\delta) = (>)\ \lambda\beta a - s - \delta(\lambda\beta a - s)/\delta = 0$, hence $x_K = (>)\ (\lambda\beta a - s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$ from (5.1.24(p.18)), hence $x_K > (= (<))\ 0$ from (j1). ∎

Corollary A 2.2 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) $x_K > (\leq)\ x \Leftrightarrow K(x) > (\leq)\ 0$.
(b) $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. □

Proof (a) Clearly $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to Lemma A 2.3(i,j1). The inverse holds by contraposition.

(b) Since $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to (a) and since $K(x) > (\leq)\ 0 \Rightarrow K(x) \geq (\leq)\ 0$, we have $x_K > (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. In addition, if $x_K = x$, then $K(x) = K(x_K) = 0 \geq 0$ or equivalently $x_K = x \Rightarrow K(x) \geq 0$, hence it eventually follows that $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. ∎

Lemma A 2.4 ($\mathscr{A}\{L_{\mathbb{P}}/K_{\mathbb{P}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_L = x_K = b$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_L = x_K$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $b > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\kappa > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (<))\ 0$.* □

Proof (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma A 2.2(d) and $x_K = b$ from Lemma A 2.3(i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (A 2.6 (1)), hence $x_K = x_L$ from Lemma A 2.3(j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots (1)$ from Lemma A 2.2(d). Suppose $b > 0$. Then, since $x_L > 0$, we have $K(x_L) < 0$ from (A 2.6 (1)), hence $x_L > x_K$ from Lemma A 2.3(j1). Furthermore, from (5.1.24(p.18)) and (5.1.23(p.18)) we have $K(0) = \lambda\beta T(0) - s = \lambda\beta T(0) > 0$ due to Lemma A 2.1(g), hence $x_K > 0$ from Lemma A 2.3(j1). Suppose $b = (<)\ 0$. Then, since $x_L = (<)\ 0$ from (1), we have $K(x_L) = (>)\ 0$ from (A 2.6 (1)), thus $x_L = (<)\ x_K$ from Lemma A 2.3(j1). Furthermore, from (5.1.24(p.18)) and (5.1.23(p.18)) we have $K(0) = \lambda\beta T(0) = (=)\ 0$ due to Lemma A 2.1(g), hence $x_K = (=)\ 0$ from Lemma A 2.3(j1).

(d) Let $\beta < 1$ and $s > 0$. Now, from (5.1.24(p.18)) and (5.1.23(p.18)), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0$ from Lemma A 2.3(j1). Accordingly $L(x_K) > (= (<))\ 0$ from (A 2.6 (2)), hence $x_L > (= (<))\ x_K$ from Lemma A 2.2(e1

Lemma A 2.5 ($\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$)

(a) *$\mathcal{L}(s)$ is nonincreasing in s.*
(b) *If $\lambda\beta < 1$, then $\mathcal{L}(s)$ is strictly decreasing in s.*
(c) *Let $\lambda\beta a \geq b$.*
  1. *$x_L \leq \lambda\beta a - s$.*
  2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta a - s$.*

(d) *Let $\lambda\beta a < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\leq)\ s$, then $x_L > (\leq)\ \lambda\beta a - s$.* □

Proof (a,b) From (5.1.22(p.18)) and (5.1.20(p.18)) we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta T(\lambda\beta a - s) - s$, hence the assertion holds from Lemma A 2.1(m).

(c) Let $\lambda\beta a \geq b$. Then, from (5.1.22(p.18)) and (5.1.20(p.18)) we have $\mathcal{L}(0) = L(\lambda\beta a) = \lambda\beta T(\lambda\beta a) = 0 \cdots (1)$ due to Lemma A 2.1(g).

(c1) Since $s \geq 0$, from (a) and (5.1.20(p.18)) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) \leq 0$, hence $x_L \leq \lambda\beta a - s$ from Corollary A 2.1(a).

(c2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (b) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) < 0$, thus $x_L < \lambda\beta a - s$ from Lemma A 2.2(e1).

(d) Let $\lambda\beta a < b$. From (5.1.22(p.18)) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta a) > 0$ due to Lemma A 2.1(g). Noting (A 2.1 (1)), for any sufficiently large $s > 0$ such that $\lambda\beta a - s \leq a^\star$ and $\lambda\beta a - s < 0$ we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta a - s - \lambda\beta(\lambda\beta a - s) = (1 - \lambda\beta)(\lambda\beta a - s) \leq 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}} > 0$ of $\mathcal{L}(s) = 0$. Then $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta a - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta a - s) \leq 0$ for $s \geq s_{\mathcal{L}}$. Hence, from Corollary A 2.1(a) we get $x_L > \lambda\beta a - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta a - s$ for $s \geq s_{\mathcal{L}}$. ∎

Lemma A 2.6 ($\mathscr{A}\{\kappa_{\mathbb{P}}\}$) *We have:*

(a) *$\kappa = \lambda\beta a - s$ if $a^\star > 0$ and $\kappa = -s$ if $b < 0$.*
(b) *Let $\beta < 1$ or $s > 0$, Then $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$.* □

Proof (a) Immediate from (5.1.23(p.18)) and Lemma A 2.1(i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.24(p.18)) and (5.1.23(p.18)), hence $x_K > (= (<))\ 0$ from Lemma A 2.3(j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 3 Direct Proof of Underlying Functions of $\tilde{\text{T}}$ype $\mathbb{P}$

A 3.1 $\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$

Lemma A 3.1

(a) *Let $x \le a$. Then $\tilde{z}(x) = a$*
(b) *Let $a < x$. Then $a < \tilde{z}(x) < x$.*
(c) *$\tilde{z}(x) \le b$ for any x.* □

Proof (a) Let $x \le a$. If $a < z \cdots$ (II), then $x < z$, hence $\tilde{p}(z)(z-x) > 0$ due to (5.1.41 (2) (p.19)), and if $z \le a \cdots$ (I), then $\tilde{p}(z)(z-x) = 0$ due to (5.1.41 (1) (p.19)) (see Figure A 3.1 below). Hence $\tilde{z}(x) = a$ due to Def. 5.1.2(p.19).

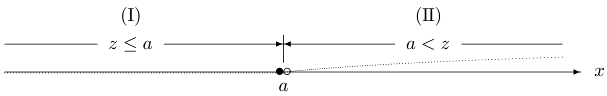

Figure A 3.1: Case $x \le a$

(b) Let $a < x$. If $x \le z \cdots$ (III), then $\tilde{p}(z)(z-x) \ge 0$, if $a < z < x \cdots$ (II), then $\tilde{p}(z)(z-x) < 0$ due to (5.1.41 (2) (p.19)), and if $z \le a \cdots$ (I), then $\tilde{p}(z)(z-x) = 0$ due to (5.1.41 (1) (p.19)) (see Figure A 3.2 just below). Hence, $\tilde{z}(x)$ is given by a z on $a < z < x$, i.e., $a < \tilde{z}(x) < x$.

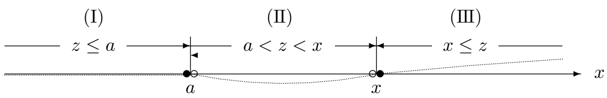

Figure A 3.2: Case $a < x$

(c) Assume that $\tilde{z}(x) > b$ for a certain x. Then, since $\tilde{p}(\tilde{z}(x)) = 1 = \tilde{p}(b)$ due to (5.1.42 (2) (p.19)), from (5.1.38(p.19)) we have $\tilde{T}(x) = \tilde{z}(x) - x > b - x = \tilde{p}(b)(b-x) \ge \tilde{T}(x)$, which is a contradiction. Hence, it must be that $\tilde{z}(x) \le b$ for any x. ∎

Corollary A 3.1 *$a \le \tilde{z}(x) \le b$ for any x.* □

Proof Evident from Lemma A 3.1. ∎

Lemma A 3.2 *$\tilde{p}(z)$ is nondecreasing on $(-\infty, \infty)$ and strictly increasing in $z \in [a, b]$.* □

Proof The former half is immediate from (5.1.31(p.18)). For $a \le z' < z \le b$ we have $\tilde{p}(z) - \tilde{p}(z') = \Pr\{\boldsymbol{\xi} \le z\} - \Pr\{\boldsymbol{\xi} \le z'\} = \Pr\{z' < \boldsymbol{\xi} \le z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.1.4 (2) (p.8))), hence $p(z) > p(z')$, i.e., $p(z)$ is strictly increasing on $[a, b]$. ∎

Lemma A 3.3 *$\tilde{z}(x)$ is nondecreasing on $(-\infty, \infty)$.* □

Proof From (5.1.38(p.19)), for any x and y we have

$$\begin{aligned}
\tilde{T}(x) &= \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) \\
&= \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - y) - (x-y)\tilde{p}(\tilde{z}(x)) \\
&\ge \tilde{T}(y) - (x-y)\tilde{p}(\tilde{z}(x)) \\
&= \tilde{p}(\tilde{z}(y))(\tilde{z}(y) - y) - (x-y)\tilde{p}(\tilde{z}(x)) \\
&= \tilde{p}(\tilde{z}(y))\big(\tilde{z}(y) - x + (x-y)\big) - (x-y)\tilde{p}(\tilde{z}(x)) \\
&= \tilde{p}(\tilde{z}(y))(\tilde{z}(y) - x) + (x-y)(\tilde{p}(\tilde{z}(y)) - \tilde{p}(\tilde{z}(x))) \\
&\ge \tilde{T}(x) + (x-y)(\tilde{p}(\tilde{z}(y)) - \tilde{p}(\tilde{z}(x))).
\end{aligned}$$

Hence $0 \ge (x-y)(\tilde{p}(\tilde{z}(y)) - \tilde{p}(\tilde{z}(x)))$. Let $x > y$. Then $0 \ge \tilde{p}(\tilde{z}(y)) - \tilde{p}(\tilde{z}(x))$ or equivalently $\tilde{p}(\tilde{z}(x)) \ge \tilde{p}(\tilde{z}(y)) \cdots$ **(**1**)**. Since $a \le \tilde{z}(x) \le b$ and $a \le \tilde{z}(y) \le b$ from Corollary A 3.1, if $\tilde{z}(x) < \tilde{z}(y)$, then $\tilde{p}(\tilde{z}(x)) < \tilde{p}(\tilde{z}(y))$ from Lemma A 3.2, which contradicts (1). Hence, it must be that $\tilde{z}(x) \ge \tilde{z}(y)$, i.e., $\tilde{z}(x)$ is nondecreasing in $x \in (-\infty, \infty)$. ∎

Lemma A 3.4

(a) *$\tilde{T}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{T}(x) < 0$ on (a, ∞) and $\tilde{T}(x) = 0$ on $(-\infty, a]$.*
(e) *$\tilde{T}(x) \le b - x$ on $(-\infty, \infty)$.*
(f) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(g) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(h) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(i) *$\tilde{T}(x) \le \min\{0, b - x\}$ for any $x \in (-\infty, \infty)$.*
(j) *$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing in s and is strictly increasing in s if $\lambda\beta < 1$.* □

Proof (a,b) Immediate from the fact that $\tilde{p}(z)(z-x)$ in (5.1.32(p.18)) is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any z.

(c) Let $x' > x > a$. Then $\tilde{z}(x) > a$ from Lemma A 3.1(b). Accordingly, since $\tilde{p}(\tilde{z}(x)) > 0$ due to (5.1.41 (2)) and since $\tilde{z}(x) - x > \tilde{z}(x) - x'$, from (5.1.38(p.19)) we have $\tilde{T}(x) = \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) > \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x') \geq \tilde{T}(x')$, i.e., $\tilde{T}(x)$ is strictly decreasing on $(a, \infty) \cdots$ **(1)**. Assume $\tilde{T}(a) = \tilde{T}(x)$ for a given $x > a$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > 2\varepsilon > 0$ we have $a < a + \varepsilon < x - \varepsilon < x$, hence $\tilde{T}(a) = \tilde{T}(x) < \tilde{T}(a + \varepsilon) \leq \tilde{T}(a)$ due to the *strict* decreasingness shown just above and the nonincreasingness in (b), which is a contradiction. Thus, since $\tilde{T}(x) \neq \tilde{T}(a)$ for any $x > a$, we have $\tilde{T}(x) < \tilde{T}(a)$ for any $x > a$ or $\tilde{T}(x) > \tilde{T}(a)$ for any $x > a$. However, the latter is impossible due to (b), hence only the former holds. Accordingly, it follows that $\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$ instead of on (a, ∞).

(d) Let $x \leq a$. Then, since $\tilde{z}(x) = a$ from Lemma A 3.1(a), we have $\tilde{p}(\tilde{z}(x)) = \tilde{p}(a) = 0$ due to (5.1.41 (1)), hence $\tilde{T}(x) = \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) = 0$ on $(-\infty, a]$. Let $x > a$. Then, from (c) we have $\tilde{T}(x) < \tilde{T}(a) = 0$, i.e., $\tilde{T}(x) < 0$ on (a, ∞).

(e) From (5.1.32(p.18)) and (5.1.42 (2) (p.19)) we see that $\tilde{T}(x) \leq \tilde{p}(b)(b - x) = b - x$ for any x on $(-\infty, \infty)$.

(f) For $x' < x$ we have, from (5.1.38(p.19)),

$$\begin{aligned}
\tilde{T}(x) + x &= \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) + x \\
&= \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1 - \tilde{p}(\tilde{z}(x)))x \\
&\geq \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1 - \tilde{p}(\tilde{z}(x)))x' \\
&= \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x') + x' \geq \tilde{T}(x') + x',
\end{aligned}$$

hence it follows that $\tilde{T}(x) + x$ is nondecreasing in x,

(g) If $\beta = 1$, then $\beta\tilde{T}(x) + x = T(x) + x$, hence the assertion is true from (f).

(h) Since $\beta\tilde{T}(x) + x = \beta(\tilde{T}(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Immediate from the fact that $\tilde{T}(x) \leq b - x$ for any x from (e) and $\tilde{T}(x) \leq 0$ for any x from (d).

(j) From (5.1.32(p.18)) we have

$$\lambda\beta\tilde{T}(\lambda\beta b + s) + s = \lambda\beta \min_z \tilde{p}(z)(z - \lambda\beta b - s) + s = \min_z \tilde{p}(z)(\lambda\beta z - (\lambda\beta)^2 b - \lambda\beta s) + s.$$

Then, for $s > s'$ we have

$$\begin{aligned}
&\lambda\beta\tilde{T}(\lambda\beta b + s) + s - \lambda\beta\tilde{T}(\lambda\beta b + s') - s' \\
&= \min_z p(z)(\lambda\beta z - (\lambda\beta)^2 b - \lambda\beta s) - \min_z p(z)(\lambda\beta z - (\lambda\beta)^2 b - \lambda\beta s') + (s - s') \\
&\geq \min_z -p(z)\lambda\beta(s - s') + (s - s')^{\dagger} \\
&\geq \min_z -(s - s')\lambda\beta + (s - s') \quad \text{(due to } p(z) \leq 1 \text{ and } s - s' > 0) \\
&= -(s - s')\lambda\beta + (s - s') \\
&= (s - s')(1 - \lambda\beta) \geq (>)\ 0 \text{ if } \lambda\beta \leq (<)\ 1.
\end{aligned}$$

Hence, since $\lambda\beta\tilde{T}(\lambda\beta b + s) + s \geq (>)\ \lambda\beta\tilde{T}(\lambda\beta b + s') + s'$ if $\lambda\beta \leq (<)\ 1$, it follows that $\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing (strictly increasing) in s if $\lambda\beta \leq (<)\ 1$. ∎

Let us define

$$\begin{aligned}
&\tilde{h}(z) = \tilde{p}(z)(z - b)/(1 - \tilde{p}(z)), \quad z < b, \\
&\tilde{h}^\star = \inf_{z<b} \tilde{h}(z), \\
&\underline{f} = \min_{a \leq w \leq b} f(w) > 0 \quad \text{due to (2.1.4 (2) (p.8))}.
\end{aligned}$$

Below, for a given x let us define the following successive four assertions:

$$\begin{aligned}
A_1(x) &= \langle\!\langle\, \tilde{z}(x) < b \,\rangle\!\rangle, \\
A_2(x) &= \langle\!\langle\, \tilde{T}(b, x) > \tilde{T}(z', x,) \text{ for at least one } z' < b \,\rangle\!\rangle, \\
A_3(x) &= \langle\!\langle\, b - \tilde{h}(z') > x \text{ for at least one } z' < b \,\rangle\!\rangle, \\
A_4(x) &= \langle\!\langle\, \sup_{z<b}\{b - \tilde{h}(z)\} > x \,\rangle\!\rangle.
\end{aligned}$$

Proposition A 3.1 For any given x we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. □

Proof Let $\tilde{T}(z, x) \overset{\text{def}}{=} \tilde{p}(z)(z - x)$. Then (5.1.38(p.19)) can be rewritten as $\tilde{T}(x) = \min_z \tilde{T}(z, x) = \tilde{T}(\tilde{z}(x), x)$.

1. Let $A_1(x)$ be true for any given x. Suppose $\tilde{T}(b, x) \leq \tilde{T}(z', x)$ for all $z' < b$. Then the minimum of $\tilde{T}(z, x)$ is attained at $z = b$, i.e., $\tilde{z}(x) = b$ (see Def. 5.1.2(p.19)), which contradicts $A_1(x)$. Hence it must be that $\tilde{T}(b, x) > \tilde{T}(z', x)$ for at least one $z' < b$, thus $A_2(x)$ becomes true; accordingly, we have $A_1(x) \Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any given x. Then, if $\tilde{z}(x) = b$, we have $\tilde{T}(b, x) > \tilde{T}(z', x) \geq \tilde{T}(x) = \tilde{T}(\tilde{z}(x), x) = \tilde{T}(b, x)$, which is a contradiction, so $\tilde{z}(x) \neq b$, hence it must be that $\tilde{z}(x) < b$ due to Lemma A 3.1(c); accordingly, we have $A_2(x) \Rightarrow A_1(x)$. Thus, it follows that we have $A_1(x) \Leftrightarrow A_2(x)$ for any given x.

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† In general we have $\min a(x) - \min b(x) \geq \min\{a(x) - b(x)\}$.

2. Since $\tilde{p}(b) = 1$ from (5.1.42 (2) (p.19)), for $z' < b$ (hence $1 > \tilde{p}(z')$ from (5.1.42 (1))) we have

$$\begin{aligned}
&\tilde{T}(b, x) - \tilde{T}(z', x)\\
&= \tilde{p}(b)(b - x) - \tilde{p}(z')(z' - x)\\
&= b - x - \tilde{p}(z')(z' - x)\\
&= b - x - \tilde{p}(z')(b - x + z' - b)\\
&= b - x - \tilde{p}(z')(b - x) - \tilde{p}(z')(z' - b)\\
&= (1 - \tilde{p}(z'))(b - x) - \tilde{p}(z')(z' - b)\\
&= (1 - \tilde{p}(z'))\big(b - x - \tilde{p}(z')(z' - b)/(1 - \tilde{p}(z'))\big)\\
&= (1 - \tilde{p}(z'))(b - x - \tilde{h}(z'))\\
&= (1 - \tilde{p}(z'))(b - \tilde{h}(z') - x).
\end{aligned}$$

Accordingly, it is immediate that $A_2(x) \Leftrightarrow A_3(x)$ for any given x.

3. Let $A_3(x)$ be true for any given x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x) \Rightarrow A_4(x)$. Let $A_4(x)$ be true for any given x. Then evidently $b - \tilde{h}(z') > x$ for at least one $z' < b$, hence $A_3(x)$ is true, so we have $A_4(x) \Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x) \Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. ▮

Lemma A 3.5

(a) $-\infty < \tilde{h}^\star < 0$.
(b) $\tilde{x}^\star = b - \tilde{h}^\star > b$.
(c) $\tilde{x}^\star > (\leq)\ x \Leftrightarrow \tilde{z}(x) < (=)\ b$.
(d) $b^\star > b$. □

Proof (a) For any infinitesimal $\varepsilon > 0$ such that $a < a + \varepsilon < b$ we have $0 < \tilde{p}(a + \varepsilon) < 1$ from (5.1.41 (2) (p.19)) and (5.1.42 (1) (p.19)). Hence, $\tilde{h}(a + \varepsilon) = \tilde{p}(a + \varepsilon)(a + \varepsilon - b)/(1 - \tilde{p}(a + \varepsilon)) < 0 \cdots$ **(1)**, so we see that $\tilde{h}^\star < 0 \cdots$ **(2)**. If $z \leq a \cdots$ (I), then $\tilde{p}(z) = 0$ due to (5.1.41 (1)), hence $\tilde{h}(z) = 0$ for $z \leq a$, implying that $\tilde{h}^\star$ can be rewritten as $\tilde{h}^\star = \inf_{a<z<b} \tilde{h}(z)$. Here let us define $\underline{f} = \inf_{a<z<b} f(\xi) > 0$ (see (2.1.4 (2) (p.8))). Assume that $\tilde{h}^\star = -\infty$. Then, there exists at least one z' on $a < z' < b$ such that $\tilde{h}(z') \leq -N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$ with any $M > 1$, i.e., $N = M/\underline{f}$, we have $\tilde{h}(z') \leq -M/\underline{f}$ or equivalently $\tilde{p}(z')(z' - b)/(1 - \tilde{p}(z')) \leq -M/\underline{f}$. Hence, noting (5.1.31(p.18)), we have

$$\tilde{p}(z')(z' - b) \ \leq\ -(1 - \tilde{p}(z'))M/\underline{f} = -(1 - \Pr\{\boldsymbol{\xi} \leq z'\})M/\underline{f} = -\Pr\{z' < \boldsymbol{\xi}\}M/\underline{f} \cdots (*)$$

where $\Pr\{z' < \boldsymbol{\xi}\} = \int_{z'}^{b} f(w)dw \geq \int_{z'}^{b} dw \times \underline{f} = (b - z')\underline{f}$. Accordingly, since $\tilde{p}(z')(z' - b) \leq -(b - z')\underline{f}M/\underline{f} = (z' - b)M$, we have $\tilde{p}(z') \geq M > 1$ due to $z' - b < 0$, which is a contradiction. Hence, it must follow that $\tilde{h}^\star > -\infty$.

(b) Since $A_1(x) \Rightarrow A_4(x)$ (see Proposition A 3.1), we can rewritten (5.1.40(p.19)) as

$$\begin{aligned}
\tilde{x}^\star &= \sup\{x \mid \sup_{z<b}\{b - \tilde{h}(z)\} > x\}\\
&= \sup_{z<b}\{b - \tilde{h}(z)\} \cdots \mathbf{(3)}\\
&= b - \inf_{z<b} \tilde{h}(z) = b - \tilde{h}^\star > b
\end{aligned}$$

due to (2) , hence (b) holds.

(c) Let $\tilde{x}^\star > x$, hence $\sup_{z<b}\{b - \tilde{h}(z)\} > x$ from (3) , so $\tilde{z}(x) < b$ due to $A_4(x) \Rightarrow A_1(x)$. Let $\tilde{x}^\star \leq x$, hence $\sup_{z<b}\{b - \tilde{h}(z)\} \leq x$ from (3) , so we have $\sup_{z<b}\{b - \tilde{h}(z)\} \leq x \Rightarrow \tilde{z}(x) \geq b$ (consider the contraposition of $A_1(x) \Leftrightarrow A_4(x)$), hence we obtain $\tilde{z}(x) = b$ due to Lemma A 3.1(c).

(d) First note $\tilde{T}(x) \leq \tilde{p}(z')(z' - x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a + \varepsilon < b$ we have $\tilde{T}(b) \leq \tilde{p}(a + \varepsilon)(a + \varepsilon - b) < 0$, hence, adding b to the inequality yields $\tilde{T}(b) + b < b$, so $\tilde{T}(x) + x \leq \tilde{T}(b) + b < b$ for $x \leq b$ due to Lemma A 3.4(f). Then, if $b^\star \leq b$, we have $\tilde{T}(b^\star) + b^\star \leq \tilde{T}(b) + b < b$, hence from Lemma A 3.4(a) we have $\tilde{T}(b^\star + \varepsilon) + b^\star + \varepsilon < b$ for any sufficiently small $\varepsilon > 0$, so $\tilde{T}(b^\star + \varepsilon) < b - (b^\star + \varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.19))). Therefore, it must follow that $b^\star > b$. ▮

Lemma A 3.6

(a) *$\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.*
(b) *$\tilde{T}(x) = b - x$ on $[b^\star, \infty)$ and $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$.*
(c) *$\tilde{T}(0) = b$ if $b^\star < 0$ and $\tilde{T}(0) = 0$ if $a > 0$.*
(d) *If $x > y$ and $b^\star > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$.* □

Proof (a) From (5.1.38(p.19)) we have

$$\tilde{T}(x) + x \ =\ \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) + x = \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1 - \tilde{p}(\tilde{z}(x)))x. \cdots \mathbf{(1)}$$

- Let $\tilde{x}^\star > x$. Then $\tilde{z}(x) < b$ from Lemma A 3.5(c), hence $\tilde{p}(\tilde{z}(x)) < 1$ due to (5.1.42 (1)) or equivalently $1 - \tilde{p}(\tilde{z}(x)) > 0$. If $x > x'$, from (1) we have

$$\tilde{T}(x) + x \;=\; \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1-\tilde{p}(\tilde{z}(x)))x > \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1-\tilde{p}(\tilde{z}(x)))x' = \tilde{p}(\tilde{z}(x))(\tilde{z}(x)-x') + x' \ge \tilde{T}(x') + x',$$

i.e., $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$, hence understandably so also on $(-\infty, b^\star]$.

- Let $\tilde{x}^\star \le x$. Then $\tilde{z}(x) = b$ from Lemma A 3.5(c), hence $\tilde{p}(\tilde{z}(x)) = 1$ from (5.1.42 (2)), so $\tilde{T}(x) = \tilde{p}(\tilde{z}(x))(\tilde{z}(x) - x) = b - x \cdots$ **(2)**. Suppose $b^\star > \tilde{x}^\star$. Then, since $b^\star > b^\star - 2\varepsilon > \tilde{x}^\star$ for an infinitesimal $\varepsilon > 0$, we have $b^\star > b^\star - \varepsilon > \tilde{x}^\star + \varepsilon > \tilde{x}^\star$ or equivalently $\tilde{x}^\star < b^\star - \varepsilon$; accordingly, due to (2) we obtain $\tilde{T}(b^\star - \varepsilon) = b - (b^\star - \varepsilon) \cdots$ **(3)**. Now, due to (5.1.39(p.19)) we have $\tilde{T}(b^\star - \varepsilon) < b - (b^\star - \varepsilon)$, which contradicts (3). Accordingly, it must be that $\tilde{x}^\star \ge b^\star$. Let $x' < x < b^\star$. Then, since $\tilde{x}^\star > x$, we have $\tilde{z}(x) < b$ Lemma A 3.5(c), hence $\tilde{p}(\tilde{z}(x)) < 1$ due to (5.1.42 (1)) or equivalently $1 - \tilde{p}(\tilde{z}(x)) > 0$. Thus, from (1) we have

$$\tilde{T}(x) + x \;=\; \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1-\tilde{p}(\tilde{z}(x)))x > \tilde{p}(\tilde{z}(x))\tilde{z}(x) + (1-\tilde{p}(\tilde{z}(x)))x' = \tilde{p}(\tilde{z}(x))(\tilde{z}(x)-x') + x' \ge \tilde{T}(x') + x',$$

i.e., $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star)$, hence so also on $(-\infty, b^\star]$ for almost the same reason as in the proof of Lemma 9.1.1(p.41) (c).

Accordingly, whether $\tilde{x}^\star > x$ or $\tilde{x}^\star \le x$, it follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.

(b) By the definition $b^\star$ (see (5.1.39(p.19))) we have $\tilde{T}(x) < b - x$ for $x < b^\star$, i.e., $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$. Here note that $\tilde{T}(x) \le b - x$ on $(-\infty, \infty)$ due to Lemma A 3.4(e), i.e., $\tilde{T}(x) + x \le b \cdots$ **(4)** on $(-\infty, \infty)$. Suppose $\tilde{T}(b^\star) + b^\star < b$. Then, for an infinitesimal $\varepsilon > 0$ we have $\tilde{T}(b^\star + \varepsilon) + b^\star + \varepsilon < b$ due to Lemma A 3.4(a), i.e., $\tilde{T}(b^\star + \varepsilon) < b - (b^\star + \varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.19))). Consequently, we have $\tilde{T}(b^\star) + b^\star = b \cdots$ **(5)** or equivalently $\tilde{T}(b^\star) = b - b^\star$. Let $x > b^\star$. Then, from Lemma A 3.4(f) we have $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star = b$. From the result and (4) we have $\tilde{T}(x) + x = b$, hence $\tilde{T}(x) = b - x$ on $(b^\star, \infty)$. Thus, from (5) it follows that $\tilde{T}(x) = b - x$ also on $[b^\star, \infty)$.

(c) Let $b^\star < 0$. Then, since $0 \in [b^\star, \infty)$, we have $\tilde{T}(0) = b$ from the former half of (b). Now, we have $\tilde{T}(0) = \min_z \tilde{p}(z)z$ from (5.1.32(p.18)). Let $a > 0$. Then, if $z \le a$, we have $\tilde{p}(z)z = 0$ from (5.1.41 (1) (p.19)) and if $z > a\,(> 0)$, then $\tilde{p}(z)z > 0$ from (5.1.41 (2)). Hence, from Def. 5.1.2(p.19) it follows that $\tilde{T}(0) = \min_x \tilde{p}(z)z = 0$.

(d) Let $x > y$ and $b^\star > y$. If $x \ge b^\star$, then $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star > \tilde{T}(y) + y$ due to Lemma A 3.4(f) and (a), and if $b^\star > x$, then $b^\star \ge x > y$, hence $\tilde{T}(x) + x > \tilde{T}(y) + y$ due to (a). Thus, whether $x \ge b^\star$ or $b^\star > x$, we have $\tilde{T}(x) + x > \tilde{T}(y) + y$. ∎

All the results obtained above (see Lemmas A 3.1(p.278)-A 3.6) can be complied into Lemma A 3.7 below.

Lemma A 3.7 ($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) *$\tilde{T}(x)$ is continuous on $(-\infty, \infty)$* ← ← Lemma A 3.4(a)
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$* ← ← Lemma A 3.4(b)
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$* ← ← Lemma A 3.4(c)
(d) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$* ← ← Lemma A 3.4(f)
(e) *$\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$* ← ← Lemma A 3.6(a)
(f) *$\tilde{T}(x) = b - x$ on $[b^\star, \infty)$ and $T(x) < b - x$ on $(-\infty, b^\star)$* ← ← Lemma A 3.6(b)
(g) *$\tilde{T}(x) < 0$ on (a, ∞) and $T(x) = 0$ on $(-\infty, a]$* ← ← Lemma A 3.4(d)
(h) *$\tilde{T}(x) \le \min\{0, b - x\}$ on $(-\infty, \infty)$* ← ← Lemma A 3.4(i)
(i) *$\tilde{T}(0) = b$ if $b^\star < 0$ and $T(0) = 0$ if $a > 0$* ← ← Lemma A 3.6(c)
(j) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$* ← ← Lemma A 3.4(g)
(k) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$* ← ← Lemma A 3.4(h)
(l) *If $x > y$ and $b^\star > y$, then $T(x) + x > T(y) + y$* ← ← Lemma A 3.6(d)
(m) *$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing in s and strictly increasing in s if $\lambda\beta < 1$* ← ← Lemma A 3.4(j)
(n) *$b^\star > b$* ← ← Lemma A 3.5(d)

A 3.2 $\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$, and $\tilde{\kappa}_{\mathbb{P}}$

From (5.1.33(p.19)) and (5.1.34(p.19)) and from Lemma A 3.7(f) we obtain, noting (9.2.1(p.42)),

$$\tilde{L}(x) \begin{cases} = \lambda\beta b + s - \lambda\beta x \text{ on } [b^\star, -\infty) & \cdots (1), \\ < \lambda\beta b + s - \lambda\beta x \text{ on } (-\infty, b^\star) & \cdots (2), \end{cases} \tag{A 3.1}$$

$$\tilde{K}(x) \begin{cases} = \lambda\beta b + s - \delta x \quad \text{on} \quad [b^\star, \infty) & \cdots (1), \\ < \lambda\beta b + s - \delta x \quad \text{on} \quad (-\infty, b^\star) & \cdots (2). \end{cases} \tag{A 3.2}$$

In addition, from (5.1.34(p.19)) and Lemma A 3.7(g) we have

$$\tilde{K}(x) \begin{cases} < -(1-\beta)x + s \text{ on } \quad (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s \text{ on } \quad (-\infty, a] & \cdots (2), \end{cases} \tag{A 3.3}$$

hence we obtain

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{A 3.4}$$

Then, from (A 3.2 (1)) and (A 3.3 (2)) we get

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta b + s + (1-\lambda)\beta x \text{ on } & [b^\star, \infty) \quad \cdots (1), \\ \beta x + s \qquad\qquad\qquad \text{ on } & (-\infty, a] \quad \cdots (2). \end{cases} \tag{A 3.5}$$

Since $\tilde{K}(x) = \tilde{L}(x) - (1-\beta)x$ from (5.1.34(p.19)) and (5.1.33(p.19)), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\, x_{\tilde{L}} \cdots (1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\, x_{\tilde{K}} \cdots (2). \tag{A 3.6}$$

Lemma A 3.8 ($\tilde{L}_{\mathbb{P}}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *Let $s = 0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.*
(e) *Let $s > 0$.*
1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.*
2. *$(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta < (\geq)\ b^\star$.* □

Proof (a-c) Immediate from (5.1.33(p.19)) and Lemma A 3.7(a-c).

(d) Let $s = 0$. Then, since $\tilde{L}(x) = \lambda\beta\tilde{T}(x)$, from Lemma A 3.7(g) we have $\tilde{L}(x) = 0$ for $a \geq x$ and $\tilde{L}(x) < 0$ for $x > a$, hence $x_{\tilde{L}} = a$ by definition (see Section 5.2(p.19) (b)), so $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (=)\ 0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x) = 0 \Rightarrow \tilde{L}(x) \geq 0$, we have $\tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.

(e) Let $s > 0$.

(e1) From (A 3.1 (1)) and the assumption of $\lambda > 0$ and $\beta > 0$ we have $\tilde{L}(x) < 0$ for a sufficiently large $x > 0$ such that $x > b^\star$. In addition, we have $\tilde{L}(a) = \lambda\beta\tilde{T}(a) + s = s > 0$ from Lemma A 3.7(g). Hence, from (a,c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}} > a$ is immediate from $\tilde{L}(a) > 0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^\star$, from (A 3.1) we have $\tilde{L}((\lambda\beta b + s)/\lambda\beta) = (<)\ \lambda\beta b + s - \lambda\beta(\lambda\beta b + s)/\lambda\beta = 0$, hence $x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta$ from (e1). ▮

Corollary A 3.2 ($\tilde{L}_{\mathbb{P}}$)

(a) *$x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.*
(b) *$x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$.* □

Proof (a) Clearly $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0 \cdots (*)$ from Lemma A 3.8(d,e1). The inverse is true by contraposition.

(b) Since $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0$ due to (a) and since $\tilde{L}(x) < (\geq)\ 0 \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$, we have $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{L}} = x$, then $\tilde{L}(x) = \tilde{L}(x_{\tilde{L}}) = 0 \leq 0$ or equivalently $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) \leq 0$, hence it eventually follows that $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. ▮

Lemma A 3.9 ($\tilde{K}_{\mathbb{P}}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{K}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{K}(x)$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$\tilde{K}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$\tilde{K}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(g) *$K(x) + x$ is strictly increasing on $(-\infty, b^\star]$.*
(h) *If $x > y$ and $b^\star > y$, then $\tilde{K}(x) + x > \tilde{K}(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{K}} = a$ where $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*
1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (= (>))\ 0$.*
2. *$(\lambda\beta b + s)/\delta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$.*
3. *Let $\tilde{\kappa} < (= (>))\ 0$. Then $x_{\tilde{K}} < (= (>))\ 0$.* □

Proof (a-c) Evident from (5.1.34(p.19)) and Lemma A 3.7(a-c).

(d) Evident from Lemma A 3.7(b) and (5.1.34(p.19)).

(e) From (5.1.34(p.19)) we have

$$\tilde{K}(x) + x \ = \ \lambda\beta\tilde{T}(x) + \beta x + s = \lambda\beta(\tilde{T}(x) + x) + (1-\lambda)\beta x + s \cdots \mathbf{(1)},$$

hence the assertion is immediate from Lemma A 3.7(d).

(f) Evident from (1) and Lemma A 3.7(d).

(g) Evident from (1) and Lemma A 3.7(e).

(h) Let $x > y$ and $b^\star > y$. If $x \geq b^\star$, then $\tilde{K}(x) + x \geq \tilde{K}(b^\star) + b^\star > \tilde{K}(y) + y$ due to (e,g), and if $b^\star > x$, then $b^\star > x > y$, hence $\tilde{K}(x) + x > \tilde{K}(y) + y$ due to (g).

(i) Let $\beta = 1$ and $s = 0$. Then, since $\tilde{K}(x) = \lambda\tilde{T}(x)$ due to (5.1.34(p.19)), from Lemma A 3.7(g) we have $\tilde{K}(x) = 0$ for $a \geq x$ and $\tilde{K}(x) < 0$ for $x > a$, so $x_{\tilde{K}} = a$ by the definition of $x_{\tilde{K}}$ (See Section 5.2(p.19) (b)). Hence $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (=)\ 0$. The inverse is immediate by contraposition. In addition, since $\tilde{K}(x) = 0 \Rightarrow \tilde{K}(x) \geq 0$, we have $\tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) First see (A 3.3 (2)). Then, if $\beta = 1$, then $s > 0$, hence $\tilde{K}(x) = s > 0$ for any $x \leq a$ and if $\beta < 1$, then $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ such that $x < a$. Hence, whether $\beta = 1$ or $\beta < 1$, we have $\tilde{K}(x) > 0$ for any sufficiently small x. Next, for any sufficiently large $x > 0$ such that $x \geq b^\star$, from (A 3.2 (1)) we have $\tilde{K}(x) < 0$ since to $\delta > 0$ due to (9.2.2 (1) (p.42)). Hence, it follows that there exists the solution $x_{\tilde{K}}$ whether $\beta = 1$ or $\beta < 1$. Let $\beta < 1$. Then, the solution is unique due to (d). Let $\beta = 1$, hence $s > 0$. Then, since $\tilde{K}(a) = s > 0$ from (A 3.3 (2)), we have $x_{\tilde{K}} > a$, hence $\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x = x_{\tilde{K}}$ due to (c), implying that the solution $x_{\tilde{K}}$ is unique. Therefore, whether $\beta = 1$ or $\beta < 1$, the solution is unique. Thus, the latter half is immediate.

(j2) Let $(\lambda\beta b + s)/\delta \geq (<)\ b^\star$. Then, from (A 3.2 (1(2))) we have $\tilde{K}((\lambda\beta b + s)/\delta) = (<)\ \lambda\beta b + s - \delta(\lambda\beta b + s)/\delta = 0$, hence $x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$ due to (j1). Its inverse is also true by contraposition.

(j3) If $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) = \tilde{\kappa} < (= (>))\ 0$ from (5.1.36(p.19)) and (5.1.37(p.19)), hence $x_{\tilde{K}} < (= (>))\ 0$ from (j1). ∎

The corollary below is used when it is not specified whether $s > 0$ or $s = 0$.

Corollary A 3.3 ($\tilde{K}_{\mathbb{P}}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.
(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

Proof (a) Clearly $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0 \cdots (*)$ due to Lemma A 3.9(i,j1). The inverse is immediate by contraposition.

(b) Since $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to (a) and since $\tilde{K}(x) < (\geq)\ 0 \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$, we have $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{K}} = x$, then $\tilde{K}(x) = \tilde{K}(x_{\tilde{K}}) = 0 \leq 0$, hence it eventually follows that $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. ∎

Lemma A 3.10 ($\tilde{L}_{\mathbb{P}} / \tilde{K}_{\mathbb{P}}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{L}} = x_{\tilde{K}} = a$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_{\tilde{L}} = x_{\tilde{K}}$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $a < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (>))\ 0$.* □

Proof (a) If $\beta = 1$ and $s = 0$, then $x_{\tilde{L}} = a$ from Lemma A 3.8(d) and $x_{\tilde{K}} = a$ from Lemma A 3.9(i), hence $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) Let $\beta = 1$ and $s > 0$. Then $\tilde{K}(x_{\tilde{L}}) = 0$ from (A 3.6 (1)), hence $x_{\tilde{K}} = x_{\tilde{L}}$ from Lemma A 3.9(j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_{\tilde{L}} = a \cdots (1)$ from Lemma A 3.8(d). Suppose $a < 0$. Then, since $x_{\tilde{L}} < 0$, we have $\tilde{K}(x_{\tilde{L}}) > 0$ from (A 3.6 (1)), hence $x_{\tilde{K}} > x_{\tilde{L}}$ from Lemma A 3.9(j1). Furthermore, from (5.1.37(p.19)) and (5.1.36(p.19)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) + s = \lambda\beta\tilde{T}(0) < 0$ due to Lemma A 3.7(g), hence $x_{\tilde{K}} < 0$ from Lemma A 3.9(j1). Suppose $a = (>)\ 0$. Then, since $x_{\tilde{L}} = (>)\ 0$ from (1), we have $\tilde{K}(x_{\tilde{L}}) = (<)\ 0$ due to (A 3.6 (1)), hence $x_{\tilde{L}} = (>)\ x_{\tilde{K}}$ from Lemma A 3.9(j1). Furthermore, from (5.1.37(p.19)) and (5.1.36(p.19)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) = 0$ due to Lemma A 3.7(g), hence $x_{\tilde{K}} = (=)\ 0$ from Lemma A 3.9(j1).

(d) Let $\beta < 1$ and $s > 0$. Since $\tilde{\kappa} = \tilde{K}(0)$ from (5.1.36(p.19)) and (5.1.37(p.19)), if $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$, hence $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 3.9(j1). Accordingly $\tilde{L}(x_{\tilde{K}}) < (= (>))\ 0$ from (A 3.6 (2)), so $x_{\tilde{L}} < (= (>))\ x_{\tilde{K}}$ from Lemma A 3.8(e1). ∎

Lemma A 3.11 ($\tilde{\mathcal{L}}_{\mathbb{P}}$)

(a) *$\tilde{\mathcal{L}}(s)$ is nondecreasing in s.*
(b) *If $\lambda\beta < 1$, then $\tilde{\mathcal{L}}(s)$ is strictly increasing in s.*
(c) *Let $\lambda\beta b \leq a$.*
   1. *$x_{\tilde{L}} \geq \lambda\beta b + s$.*
   2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_{\tilde{L}} > \lambda\beta b + s$.*

(d) *Let $\lambda\beta b > a$. Then, there exists a $s_{\tilde{\mathcal{L}}} > 0$ such that if $s_{\tilde{\mathcal{L}}} > (\leq)\ s$, then $x_{\tilde{L}} < (\geq)\ \lambda\beta b + s$.* □

Proof (a,b) From (5.1.35(p.19)) and (5.1.33(p.19)) we have $\tilde{\mathcal{L}}(s) = \lambda\beta\tilde{T}(\lambda\beta b + s) + s \cdots (1)$, hence the assertions are true from Lemma A 3.7(m).

(c) Let $\lambda\beta\mu \leq a$. Then, from (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) = 0 \cdots (2)$ due to Lemma A 3.7(g).

(c1) Since $s \geq 0$, from (a) we have $\tilde{\mathcal{L}}(s) \geq \tilde{\mathcal{L}}(0) = 0$ due to (2) or equivalently $\tilde{L}(\lambda\beta b + s) \geq 0$, hence $x_{\tilde{L}} \geq \beta b + s$ from Corollary A 3.2(a).

(c2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (b) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0$ due (2), hence $\tilde{L}(\lambda\beta b + s) > 0$, so $x_{\tilde{L}} > \lambda\beta b + s$ from Lemma A 3.8(e1).

(d) Let $\lambda\beta b > a$. From (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) < 0$ due to Lemma A 3.7(g). Noting (A 3.1 (1)), for any sufficiently large $s > 0$ such that $\lambda\beta b + s \ge b^\star$ and $\lambda\beta b + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta b + s) = \lambda\beta b + s - \lambda\beta(\lambda\beta b + s) = (1-\lambda\beta)(\beta b + s) \ge 0$. Accordingly, due to (a) it follows that there exists a $s_{\tilde{\mathcal{L}}} > 0$ where $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$, or equivalently, $\tilde{L}(\lambda\beta b + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta b + s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 3.2(a) we have $x_{\tilde{L}} < \beta b + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \ge \beta b + s$ for $s \ge s_{\tilde{\mathcal{L}}}$. ∎

Lemma A 3.12 ($\mathscr{A}\{\tilde{\kappa}_{\mathbb{P}}\}$) *We have*:

(a) $\tilde{\kappa} = \lambda\beta b + s$ *if* $b^\star < 0$ *and* $\tilde{\kappa} = s$ *if* $a > 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$. □

Proof (a) Immediate from (5.1.36(p.19)) and Lemma A 3.7(i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} > (= (<))\ 0$, we have $\tilde{K}(0) > (= (<))\ 0$ from (5.1.37(p.19)) and (5.1.36(p.19)), hence $x_{\tilde{K}} > (= (<))\ 0$ from Lemma A 3.9(j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 4 Direct Proof of Assertion Systems

A 4.1 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Since $\tilde{K}(x) + (1-\beta)x = \tilde{L}(x)$ for any x due to (5.1.14(p.17)) and (5.1.13(p.17)), from (6.5.4(p.31)) we have

$$V_t - \beta V_{t-1} = \min\{\tilde{L}(V_{t-1}), 0\} \le 0, \quad t > 1. \tag{A 4.1}$$

Accordingly:

1. If $\tilde{L}(V_{t-1}) \le 0$, then $V_t - \beta V_{t-1} = \tilde{L}(V_{t-1})$, hence

$$V_t = \tilde{L}(V_{t-1}) + \beta V_{t-1} = \tilde{K}(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.2}$$

2. If $\tilde{L}(V_{t-1}) \ge 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.3}$$

Now, from (6.5.4(p.31)) with $t = 2$ we have

$$V_2 - V_1 = \min\{\tilde{K}(V_1), -(1-\beta)V_1\}. \tag{A 4.4}$$

Finally, from (A 4.1) we see that

$$\tilde{L}(V_{t-1}) < (>)\, 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle})^\dagger. \tag{A 4.5}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.14) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.14})), \qquad \delta = 1 \cdots (2) \quad (\text{see } (9.2.1(\text{p.42}))). \tag{A 4.6}$$

□ `Tom` **A 4.1** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, from (5.1.14(p.17)) we have $\tilde{K}(x) = \tilde{T}(x) \le 0 \cdots$ **(1)** for any x due to Lemma A 1.1(p.271) (g), hence from (6.5.4(p.31)) and (1) we have $V_t = \min\{\tilde{T}(V_{t-1}) + V_{t-1}, V_{t-1}\} = \min\{\tilde{T}(V_{t-1}), 0\} + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1} \cdots$ **(2)** for $t > 1$.

(a) Since $V_2 = \tilde{T}(V_1) + V_1$, we have $V_2 \le V_1$ due to (1). Suppose $V_{t-1} \ge V_t$. Then, from Lemma A 1.1(p.271) (d) we have $V_t \ge \tilde{T}(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$.

(b) Since $V_1 = \mu$ from (6.5.3(p.31)), we have $V_1 > a$. Suppose $V_{t-1} > a$. Then, noting $b > a$, from (2) we have $V_t > \tilde{T}(a) + a = a$ due to Lemma A 1.1(p.271) (l,g). Accordingly, by induction $V_{t-1} > a$ for $t > 1$, hence $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Lemma A 1.2(d), so $\tilde{L}(V_{t-1}) < 0 \cdots$ **(3)** for $\tau \ge t > 1$. Hence, from (A 4.1) we obtain $V_t - \beta V_{t-1} < 0$ for $\tau \ge t > 1$, i.e., $V_t < \beta V_{t-1}$ for $\tau \ge t > 1$. Accordingly $V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{Ⓢ\ \mathtt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$. Then $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$ due to (3) and (A 4.5). ∎

Let us define

$$\mathsf{S}_{19}\boxed{Ⓢ\blacktriangle \mid ⊛\,\|} = \left\{ \begin{array}{ll} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} & \\ (1) \quad \boxed{Ⓢ\ \mathtt{dOITs}_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \mathtt{Conduct}_{\tau\ge t>1\blacktriangle}, & \\ (2) \quad \boxed{⊛\ \mathtt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_{\|} \text{ where } \mathtt{Conduct}_{\tau\ge t>1\blacktriangle}. & \end{array} \right\}$$

† See Section 6.1(p.21).

□ Tom **A 4.2** ($\mathscr{A}$\{M̃:1[ℝ][A]\}) *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \le x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta\mu \le a$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
(c) *Let* $\beta\mu > a$.

1. *Let* $\beta = 1$.
   i. *Let* $\mu + s \ge b$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
   ii. *Let* $\mu + s < b$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆.
   i. *Let* $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
   ii. *Let* $a = 0$ ⦅$\tilde{\kappa} = 0$⦆.
      1. *Let* $\beta\mu + s \ge b$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let* $\beta\mu + s < b$. *Then* $\boxed{Ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.
   iii. *Let* $a > 0$ ⦅$\tilde{\kappa} > 0$⦆.
      1. *Let* $\beta\mu + s \ge b$ *or* $s_{\tilde{\mathcal{L}}} \le s$. *Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_{\parallel}$.
      2. *Let* $\beta\mu + s < b$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\texttt{S}_{19}$(p.284) $\boxed{\circ\blacktriangle \mid *\parallel}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$. Note here (A 4.6 (1,2) (p.284)).

(a) Since $x_{\tilde{K}} \le (\beta\mu + s)/\delta = \beta\mu + s = V_1$ due to Lemma A 1.3(p.273) (j2) and (6.5.3(p.31)), we have $\tilde{K}(V_1) \le 0$ due to Lemma A 1.3(p.273) (j1), hence $V_2 - V_1 \le 0$ from (A 4.4), i.e., $V_1 \ge V_2$. Suppose $V_{t-1} \ge V_t$. Then, from (6.5.4(p.31)) and Lemma A 1.3(p.273) (e) we have $V_t \ge \min\{\tilde{K}(V_t)+V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta\mu + s \ge M$ and $a \ge M$, hence $V_1 \ge M$. Suppose $V_{t-1} \ge M$. Then, from Lemma A 1.3(p.273) (e) and (A 1.5 (2) (p.272)) we have $V_t \ge \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \ge \min\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Hence, by induction $V_t \ge M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.4(p.31)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \ge 0$, we have $V \le x_{\tilde{K}}$ from Lemma A 1.3(p.273) (j1).

(b) Let $\beta\mu \le a \cdots$ **(1)**. Then $x_{\tilde{L}} \ge \beta\mu + s = V_1$ from Lemma A 1.5(p.274) (c1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 1.1(p.273) (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > 1$. Hence, from (A 4.3) we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^1$. Hence $t_\tau^* = 1$ for $\tau > 1$ (see Preference Rule 7.2.1(p.35)), i.e., $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c) Let $\beta\mu > a$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" in the lemma. Then $(\lambda\beta\mu+s)/\delta = \mu+s \cdots$ **(3)** due to (2) and (A 4.6 (2)). In addition, we have $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(4)** from Lemma A 1.4(p.274) (b), we have $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots$ **(5)**.

(c1i) Let $\mu + s \ge b$. Then $x_{\tilde{L}} = x_{\tilde{K}} = \mu + s = V_1$ from (4), Lemma A 1.3(p.273) (j2), (3), and (6.5.3(p.31)). Accordingly, since $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Lemma A 1.2(p.272) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c1ii) Let $\mu + s < b$. Then $x_{\tilde{L}} = x_{\tilde{K}} < \mu + s = V_1 < b$ from (4), Lemma A 1.3(p.273) (j2), and (6.5.3(p.31)), hence $b > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Lemma A 1.2(p.272) (e1). Then, from (A 4.2), Lemma A 1.3(p.273) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Lemma A 1.2(p.272) (e1). Thus, for the same reason as in the proof of Tom A 4.1(b) we have $\boxed{Ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.

(c2i) Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots$ **(6)** from Lemma A 1.4(p.274) (c ⦅d⦆). Now, since $x_{\tilde{K}} \le \beta\mu + s$ due to Lemma A 1.3(p.273) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \le V_1$ from (6.5.3(p.31)). Suppose $x_{\tilde{K}} \le V_{t-1}$. Then, from Lemma A 1.3(p.273) (e) we have $V_t \ge \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \ge x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.273) (a). Hence, for the same reason as in the proof of Tom A 4.1(b) we have $\boxed{Ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{CONDUCT}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii) Let $a = 0$ ⦅$\tilde{\kappa} = 0$⦆. Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(7)** from Lemma A 1.4(p.274) (c ⦅d⦆).

(c2ii1) Let $\beta\mu + s \ge b$. Then, $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.273) (j2) and (6.5.3(p.31)). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.2) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.3), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\parallel}$ for $\tau > 1$.

(c2ii2) Let $\beta\mu + s < b$. Then, since $V_1 < b$ from (6.5.3(p.31)), we have $V_{t-1} < b$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta\mu + s = V_1$ from Lemma A 1.3(p.273) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Corollary A 1.2(p.272) (e1), from (A 4.2) and
Lemma A 1.3(p.273) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.273) (a), for the same reason as in the proof of Tom A 4.1(b) we have $\boxed{Ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{\tau\blacktriangle}$ for $\tau \ge t > 1$.

(c2iii) Let $a > 0$ ⦅$\tilde{\kappa} > 0$⦆. Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots$ **(8)** from Lemma A 1.4(p.274) (c ⦅d⦆).

(c2iii1) Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. First, let $\beta\mu + s \ge b$. Then, since $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.273) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \ge V_1$. Next, let $s_{\tilde{\mathcal{L}}} \le s$. Then, since $x_{\tilde{L}} \ge \beta\mu + s$ due to Lemma A 1.5(p.274) (d), we have $x_{\tilde{L}} \ge V_1$

from (6.5.3(p.31)). Accordingly, whether $\beta\mu + s \geq b$ or $s_{\tilde{L}} \leq s$, we have $x_{\tilde{L}} \geq V_1$, so $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ from Corollary A 1.1(p.273) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta\mu+s < b \cdots (9)$ and $s < s_{\tilde{L}}$. Then, from (8) and Lemma A 1.5(p.274) (d) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta\mu+s = V_1 \cdots (10)$, hence $\tilde{K}(V_1) < 0 \cdots (11)$ from Lemma A 1.3(p.273) (j1). In addition, since $V_1 < b$ due to (9) and (6.5.3(p.31)), we have $V_{t-1} < b$ for $t > 0$ from (a). Now, from (A 4.4) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from (6.5.4(p.31)) and Lemma A 1.3(p.273) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10), so $V_1 \geq x_{\tilde{L}}$. Assume that $V_{t-1} \geq x_{\tilde{L}}$ for *all* $t > 1$, hence $V \geq x_{\tilde{L}}$. Now, from (8) and $V \leq x_{\tilde{K}}$ in (a) we have the contradiction of $V \leq x_{\tilde{K}} < x_{\tilde{L}} \leq V$. Hence, it is impossible that $V_{t-1} \geq x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t^\bullet > 1$ such that

$$V_1 > V_2 > \cdots > V_{t^\bullet-1} > x_{\tilde{L}} \geq V_{t^\bullet} > V_{t^\bullet+1} > V_{t^\bullet+2} > \cdots, \tag{A 4.7}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t^\bullet \geq t > 1, \qquad x_{\tilde{L}} \geq V_{t-1}, \quad t > t^\bullet. \tag{A 4.8}$$

Therefore, from Corollary A 1.1(p.273) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots (12)$ for $t^\bullet \geq t > 1$ and $\tilde{L}(V_{t-1}) \geq 0 \cdots (13)$ for $t > t^\bullet$.

1. Let $t^\bullet \geq \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots (14)$ for $\tau \geq t > 1$ from (12), for the same reason as in the proof of `Tom` A 4.1(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $t^\bullet \geq \tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\texttt{S}_{19}$(p.284) (1) is true.
2. Let $\tau > t^\bullet$. First, let $\tau \geq t > t^\bullet$. Then, since $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet$ from (A 4.3), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet} V_{t^\bullet} \cdots (15).$$

Next, let $t^\bullet \geq t > 1$. Then, from (12) and (A 4.1) we have $V_t - \beta V_{t-1} < 0$ for $t^\bullet \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t^\bullet \geq t > 1$, hence

$$V_{t^\bullet} < \beta V_{t^\bullet-1} < \beta^2 V_{t^\bullet-2} < \cdots < \beta^{t^\bullet-1} V_1 \cdots (16).$$

From (15) and (16) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet} V_{t^\bullet} < \beta^{\tau-t^\bullet+1} V_{t^\bullet-1} < \beta^{\tau-t^\bullet+2} V_{t^\bullet-2} < \cdots < \beta^{\tau-1} V_1,$$

hence we obtain $t^*_\tau = t^\bullet$ for $\tau > t^\bullet$ due to Preference Rule 7.2.1(p.35), i.e., $\boxed{\circledast\ \texttt{ndOIT}_\tau\langle t^\bullet\rangle}_{\|}$ for $\tau > 1$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet \geq t > 1$ due to (12) and (A 4.5). Hence $\texttt{S}_{19}$(p.284) (2) is true. ∎

A 4.2 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

Since $K(x) + (1-\beta)x = L(x)$ for any x due to (5.1.21(p.18)) and (5.1.20(p.18)), from (6.5.6(p.31)) we have

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\} \geq 0, \quad t > 1. \tag{A 4.9}$$

Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.10}$$

2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.11}$$

Now, from (6.5.6(p.31)) with $t = 2$ we have

$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{A 4.12}$$

Finally, from (A 4.9) we see that

$$L(V_{t-1}) > (<)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{A 4.13}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.14) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.14})), \qquad \delta = 1 \cdots (2) \quad (\text{see } (9.2.1(\text{p.42}))). \tag{A 4.14}$$

□ `Tom` **A 4.3** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.18)) we have $K(x) = T(x) \geq 0 \cdots (1)$ for any x due to Lemma A 2.1(g), hence from (6.5.6(p.31)) and (1) we have
$V_t = \max\{T(V_{t-1}) + V_{t-1}, V_{t-1}\} = \max\{T(V_{t-1}), 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1} \cdots (2)$ for $t > 1$.

(a) Since $V_2 = T(V_1) + V_1$, we have $V_2 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma A 2.1(d) we have $V_t \leq T(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Since $V_1 = a$ from (6.5.5), we have $V_1 < b$. Suppose $V_{t-1} < b$. Then, noting $a^\star < a < b$ due to Lemma A 2.1(p.275) (n), from (2) we have $V_t < T(b) + b = b$ due to Lemma A 2.1(p.275) (3,g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma A 2.2(d), so $L(V_{t-1}) > 0 \cdots (3)$ for $\tau \geq t > 1$. Hence, from (A 4.9) we obtain $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (A 4.13). ∎

Let us define

$$\texttt{S}_{20}\boxed{\circledS\blacktriangle \mid \circledast\|} = \left\{\begin{array}{l}\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1)\quad \boxed{\circledS\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \texttt{Conduct}_{\tau \geq t > 1\blacktriangle}, \\ (2)\quad \boxed{\circledast\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \texttt{Conduct}_{\tau \geq t > 1\blacktriangle}.\end{array}\right\}$$

□ **Tom A 4.4** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta a \geq b$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
(c) *Let $\beta a < b$.*
  1. *Let $\beta = 1$.*
    i. *Let $a - s \leq a^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
    ii. *Let $a - s > a^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
    i. *Let $b > 0$ ($\kappa > 0$). Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
    ii. *Let $b = 0$ ($\kappa = 0$).*
      1. *Let $\beta a - s \leq a^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta a - s > a^\star$. Then* $\boxed{\circledS\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
    iii. *Let $b < 0$ ($\kappa < 0$).*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\texttt{S}_{20}$(p.287) $\boxed{\circ\blacktriangle \mid *\|}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$. First see (A 4.14(p.286))

(a) Since $x_K \geq (\lambda\beta a - s)/\delta = \beta a - s = V_1$ due to Lemma A 2.3(p.276) (j2) and (6.5.5(p.31)), we have $K(V_1) \geq 0$ due to Lemma A 2.3(p.276) (j1), hence $V_2 - V_1 \geq 0$ from (A 4.12), i.e., $V_1 \leq V_2$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.6(p.31)) and Lemma A 2.3(p.276) (e) we have $V_t \leq \max\{K(V_t)+V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Consider a sufficiently large $M > 0$ such that $\beta a - s \leq M$ and $b \leq M$, hence $V_1 \leq M$. Suppose $V_{t-1} \leq M$. Then, from Lemma A 2.3(p.276) (e) and (A 2.5 (2) (p.275)) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\} \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \leq M$ for $t > 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.6(p.31)) we have $V = \max\{K(V) + V, \beta V\}$, hence $0 = \max\{K(V), -(1-\beta)\beta V\}$. Thus, since $K(V) \leq 0$, we have $V \geq x_K$ from Lemma A 2.3(p.276) (j1).

(b) Let $\beta a \geq b \cdots (1)$. Then $x_L \leq \beta a - s = V_1$ from Lemma A 2.5(p.277) (c1) with $\lambda = 1$ and $\delta = 1$, hence $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ due to Corollary A 2.1(p.276) (a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Hence, from (A 4.11(p.286)) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$, hence $t^*_\tau = 1$ for $\tau > 1$ due to Preference Rule 7.2.1(p.35), i.e., $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau > 1$.

(c) Let $\beta a < b$.

(c1) Let $\beta = 1 \cdots (2)$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" in the lemma. Then $(\lambda\beta a - s)/\delta = a - s \cdots (3)$ due to (2) and (A 4.14 (2) (p.286)). In addition, we have $x_L = x_K \cdots (4)$ from Lemma A 2.4(p.277) (b), we have $K(x_L) = K(x_K) = 0 \cdots (5)$.

(c1i) Let $a - s \leq a^\star$. Then $x_L = x_K = a - s = V_1$ from (4), Lemma A 2.3(p.276) (j2), (3), and (6.5.5(p.31)). Accordingly, since $x_L \leq V_{t-1}$ for $t > 1$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 1$ due to Lemma A 2.2(p.275) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau > 1$.

(c1ii) Let $a - s > a^\star$. Then $x_L = x_K > a - s = V_1 > a^\star$ from (4), Lemma A 2.3(p.276) (j2), and (6.5.5(p.31)), hence $a^\star < V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} < x_L$, hence $L(V_{t-1}) > 0$ from Lemma A 2.2(p.275) (e1). Then, from (A 4.10), Lemma A 2.3(p.276) (g), and (4) we have $V_t < K(x_L) + x_L = K(x_K) + x_L = x_L$. Accordingly, by induction $V_{t-1} < x_L$ for $t > 1$, hence, $L(V_{t-1}) > 0$ for $t > 1$ from Lemma A 2.2(p.275) (e1). Thus, for the same reason as in the proof of Tom A 4.3(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ ($s > 0$).

(c2i) Let $b > 0$ ⟪$\kappa > 0$⟫ . Then $x_L > x_K > 0 \cdots$ **(6)** from Lemma A 2.4(p.277) (c ⟪d⟫). Now, since $x_K \geq \beta a - s$ due to Lemma A 2.3(p.276) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_K \geq V_1$ from (6.5.5(p.31)). Suppose $x_K \geq V_{t-1}$. Then, from Lemma A 2.3(p.276) (e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (6), thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.276) (a). Hence, for the same reason as in the proof of `Tom` A 4.3(b) we have ⓢ `dOITs`$_\tau\langle\tau\rangle$$_\blacktriangle$ for $\tau > 1$ and `conduct`$_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2ii) Let $b = 0$ ⟪$\kappa = 0$⟫ . Then $x_L = x_K \cdots$ **(7)** from Lemma A 2.4(p.277) (c ⟪d⟫).

(c2ii1) Let $\beta a - s \leq a^\star$. Then, $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.276) (j2) and (6.5.5(p.31)). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (7), thus $L(V_{t-1}) = L(x_L) = 0$. Then, from (A 4.10) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (7). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.11), hence, for the same reason as in the proof of (b) we obtain ● `dOITd`$_\tau\langle 1\rangle$$_{||}$ for $\tau > 1$.

(c2ii2) Let $\beta a - s > a^\star$. Then, since $V_1 > a^\star$, we have $V_{t-1} > a^\star$ for $t > 1$ due to (a). In addition, we have $x_K > \beta a - s = V_1$ from Lemma A 2.3(p.276) (j2) and (6.5.5(p.31)). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (7). Then, since $L(V_{t-1}) > 0$ due to Corollary A 2.1(p.276) (a), from (A 4.10) and Lemma A 2.3(p.276) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, thus $x_L > V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.276) (a), for the same reason as in the proof of `Tom` A 4.3(b) we have ⓢ `dOITs`$_\tau\langle\tau\rangle$$_\blacktriangle$ for $\tau > 1$ and `Conduct`$_{\tau\blacktriangle}$ for $\tau \geq t > 1$.

(c2iii) Let $b < 0$ ⟪$\kappa < 0$⟫ . Then $x_L < x_K \cdots$ **(8)** from Lemma A 2.4(p.277) (c ⟪d⟫).

(c2iii1) Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. First, let $\beta a - s \leq a^\star$. Then, since $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.276) (j2), we have $x_L < V_1$ from (8), hence $x_L \leq V_1$. Next, let $s_{\mathcal{L}} \leq s$. Then, since $x_L \leq \beta a - s$ due to Lemma A 2.5(p.277) (d), we have $x_L \leq V_1$ and (6.5.5(p.31)). Accordingly, whether $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, so $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary A 2.1(p.276) (a), for the same reason as in the proof of (b) we obtain ● `dOITd`$_\tau\langle 1\rangle$$_{||}$ for $\tau > 1$.

(c2iii2) Let $\beta a - s > a^\star \cdots$ **(9)** and $s < s_{\mathcal{L}}$. Then, from (8) and Lemma A 2.5(p.277) (b) we have $x_K > x_L > \beta a - s = V_1 \cdots$ **(10)**, hence $K(V_1) > 0 \cdots$ **(11)** from Lemma A 2.3(p.276) (j1). In addition, since $V_1 > a^\star$ due to (9), we have $V_{t-1} > a^\star$ for $t > 0$ from (a). Now, from (A 4.12) and (11) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from (6.5.6(p.31)) and Lemma A 2.3(p.276) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (10). Assume that $V_{t-1} \leq x_L$ for *all* $t > 1$, hence $V \leq x_L$. Now, from (8) and $V \geq x_K$ in (a) we have the contradiction of $V \geq x_K > x_L \geq V$. Hence, it is impossible that $V_{t-1} \leq x_L$ for all $t > 1$, implying that there exists $t^\bullet > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet - 1} < x_L \leq V_{t^\bullet} < V_{t^\bullet + 1} < V_{t^\bullet + 2} < \cdots, \tag{A 4.15}$$

from which

$$V_{t-1} < x_L, \quad t^\bullet \geq t > 1, \qquad x_L \leq V_{t-1}, \quad t > t^\bullet. \tag{A 4.16}$$

Therefore, from Corollary A 2.1(p.276) (a) we have $L(V_{t-1}) > 0 \cdots$ **(12)** for $t^\bullet \geq t > 1$ and $L(V_{t-1}) \leq 0 \cdots$ **(13)** for $t > t^\bullet$.

1. Let $t^\bullet \geq \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots$ **(14)** for $\tau \geq t > 1$ from (12), for the same reason as in the proof of `Tom` A 4.3(b) we have ⓢ `dOITs`$_\tau\langle\tau\rangle$$_\blacktriangle$ for $t^\bullet \geq \tau > 1$ and `Conduct`$_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\mathtt{S}_{20}$(p.287) (1) is true.
2. Let $\tau > t^\bullet$. Firstly, let $\tau \geq t > t^\bullet$. Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet$ from (A 4.11), thus

   $$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet} V_{t^\bullet} \cdots \textbf{(15)}.$$

   Next, let $t^\bullet \geq t > 1$. Then, from (12) and (A 4.9) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t^\bullet \geq t > 1$, hence

   $$V_{t^\bullet} > \beta V_{t^\bullet - 1} > \beta^2 V_{t^\bullet - 2} > \cdots > \beta^{t^\bullet - 1} V_1 \cdots \textbf{(16)}.$$

   From (15) and (16) we have

   $$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet} V_{t^\bullet} > \beta^{\tau - t^\bullet + 1} V_{t^\bullet - 1} > \beta^{\tau - t^\bullet + 2} V_{t^\bullet - 2} > \cdots > \beta^{\tau - 1} V_1,$$

   hence we obtain $t^*_\tau = t^\bullet$ for $\tau > t^\bullet$ due to Preference Rule 7.2.1(p.35), i.e., ⊛ `ndOIT`$_\tau\langle t^\bullet\rangle$$_{||}$ for $\tau > t^\bullet$. In addition, we have `Conduct`$_{t\blacktriangle}$ for $t^\bullet \geq t > 1$ due to (12) and (A 4.13). Hence $\mathtt{S}_{20}$(p.287) (2) is true. ∎

A 4.3 $\mathscr{A}\{\tilde{\mathtt{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

Since $\tilde{K}(x) + (1 - \beta)x = \tilde{L}(x)$ due to (5.1.34(p.19)) and (5.1.33(p.19)), from (6.5.8(p.31)) we have

$$V_t - \beta V_{t-1} = \min\{\tilde{L}(V_{t-1}), 0\} \leq 0, \quad t > 1. \tag{A 4.17}$$

Accordingly:

1. If $\tilde{L}(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = \tilde{L}(V_{t-1})$, hence

   $$V_t = \tilde{L}(V_{t-1}) + \beta V_{t-1} = \tilde{K}(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.18}$$

2. If $\tilde{L}(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

   $$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.19}$$

Now, from (6.5.8(p.31)) with $t = 2$ we have

$$V_2 - V_1 = \min\{\tilde{K}(V_1), -(1-\beta)V_1\}. \quad \text{(A 4.20)}$$

Finally, from (A 4.17) we see that

$$\tilde{L}(V_{t-1}) < (>)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_t). \quad \text{(A 4.21)}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.14) (b)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.14})), \qquad \delta = 1 \quad (\text{see } (9.2.1(\text{p.42}))). \quad \text{(A 4.22)}$$

(A 4.23)

□ **Tom A 4.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$. □

Proof Let $\beta = 1$ and $s = 0$. Then, from (5.1.34(p.19)) we have $\tilde{K}(x) = \tilde{T}(x) \leq 0 \cdots$ **(1)** for any x due to Lemma 13.6.1(p.89) (g), hence from (6.5.8(p.31)) and (1) we have $V_t = \min\{\tilde{T}(V_{t-1}) + V_{t-1}, V_{t-1}\} = \tilde{T}(V_{t-1}) + V_{t-1} \cdots$ **(2)** for $t > 1$.

(a) Since $V_2 = \tilde{T}(V_1) + V_1$, we have $V_2 \leq V_1$ due to (1). Suppose $V_{t-1} \geq V_t$. Then, from Lemma A 3.7(p.281) (d) we have $V_t \geq \tilde{T}(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$.

(b) Since $V_1 = b$ from (6.5.7(p.31)), we have $V_1 > a$. Suppose $V_{t-1} > a$. Then, noting $b^\star > b > a$ due to Lemma A 3.7(p.281) (n), from (2) we have $V_t > \tilde{T}(a) + a = a$ due to Lemma A 3.7(p.281) (l,g). Accordingly, by induction $V_{t-1} > a$ for $t > 1$, hence $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Lemma A 3.8(p.282) (d), thus $\tilde{L}(V_{t-1}) < 0 \cdots$ **(3)** for $\tau \geq t > 1$. Hence, from (A 4.17) we obtain $V_t - \beta V_{t-1} < 0$ for $\tau \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly $V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-1}V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (A 4.21). ∎

Let us define

$$\texttt{S}_{21}\boxed{ⓢ\blacktriangle}\boxed{⊛\|} = \left\{ \begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1) \quad \boxed{ⓢ\ \texttt{dOITs}_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle}_\blacktriangle \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}, \\ (2) \quad \boxed{⊛\ \texttt{ndOIT}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle}_\| \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}. \end{array} \right\}$$

□ **Tom A 4.6** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \leq a$. Then .*
(c) *Let $\beta b > a$.*
  1. *Let $\beta = 1$.*
    i. *Let $b + s \geq b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
    ii. *Let $b + s < b^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
    i. *Let $a < 0$ ($\kappa < 0$). Then* $\boxed{ⓢ\ \texttt{dOITs}_{\tau>1}\langle\tau\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
    ii. *Let $a = 0$ ($\kappa = 0$).*
      1. *Let $\beta b + s \geq b^\star$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta b + s < b^\star$. Then* $\boxed{ⓢ\ \texttt{dOITs}_\tau\langle\tau > 1\rangle}_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
    iii. *Let $a > 0$ ($\kappa > 0$).*
      1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{L}} \leq s$. Then* $\boxed{\bullet\ \texttt{dOITd}_{\tau>1}\langle 1\rangle}_\|$.
      2. *Let $\beta b + s < b^\star$ and $s < s_{\tilde{L}}$. Then* $\texttt{S}_{21}\boxed{\circ\blacktriangle}\boxed{*\triangle}$ *is true.* □

Proof Let $\beta < 1$ or $s > 0$. First note (A 4.22 (1,2) (p.289)).

(a) Since $x_{\tilde{K}} \leq (\beta b + s)/\delta = \beta b + s = V_1$ due to Lemma A 3.9(p.282) (j2) and (6.5.7(p.31)), we have $\tilde{K}(V_1) \leq 0$ due to Lemma A 3.9(p.282) (j1), hence $V_2 - V_1 \leq 0$ from the right side of (A 4.20), i.e., $V_1 \geq V_2$. Suppose $V_{t-1} \geq V_t$. Then, from (6.5.8(p.31)) and Lemma A 3.9(p.282) (e) we have $V_t \geq \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \geq V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta b + s \geq M$ and $a \geq M$, hence $V_1 \geq M$. Suppose $V_{t-1} \geq M$. Then, from Lemma A 3.9(p.282) (e) and (A 3.5 (2) (p.282)) we have $V_t \geq \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \geq \min\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \geq M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.8(p.31)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \geq 0$, we have $V \leq x_{\tilde{K}}$ from Lemma A 3.9(p.282) (j1).

(b) Let $\beta b \leq a \cdots$ **(1)**. Then $x_{\tilde{L}} \geq \beta b + s = V_1$ from Lemma A 3.11(p.283) (c1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ due to Corollary A 3.2(a), we have $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > 1$. Hence, from (A 4.19) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau > 1$ due to Preference Rule 7.2.1(p.35), i.e., $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_\|$ for $\tau > 1$.

(c) Let $\beta b > a$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" in the lemma. Then, we see that $(\lambda\beta b + s)/\delta = b + s \cdots$ **(3)** due to (2) and (A 4.22) and that $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(4)** from Lemma A 3.10(p.283) (b), hence $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots$ **(5)**.

(c1i) Let $b + s \geq b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} = b + s = V_1$ from (4), Lemma A 3.9(p.282) (j2, (3), and (6.5.7(p.31))). Accordingly, since $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ due to Lemma A 3.8(p.282) (e1). Hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c1ii) Let $b + s < b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} < b + s = V_1 < b^\star$ from (4), Lemma A 3.9(p.282) (j2), and (6.5.7(p.31))), hence $b^\star > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Lemma A 3.8(p.282) (e1). Then, from (A 4.18), Lemma A 3.9(p.282) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Lemma A 3.8(p.282) (e1). Thus, for the same reason as in the proof of Tom A 4.5(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c2i) Let $a < 0$ ⟪$\tilde{\kappa} < 0$⟫. Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots$ **(6)** from Lemma A 3.10(p.283) (c ⟪d⟫). Now, since $x_{\tilde{K}} \leq \beta b + s$ due to Lemma A 3.9(p.282) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \leq V_1$ from (6.5.7(p.31))). Suppose $x_{\tilde{K}} \leq V_{t-1}$. Then, from Lemma A 3.9(e) we have $V_t \geq \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \geq x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.282) (a). Hence, for the same reason as in the proof of Tom A 4.5(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$, and $\texttt{CONDUCT}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2ii) Let $a = 0$ ⟪$\tilde{\kappa} = 0$⟫. Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(7)** from Lemma A 3.10(p.283) (c ⟪d⟫).

(c2ii1) Let $\beta b + s \geq b^\star$. Then, $x_{\tilde{K}} = \beta b + s = V_1$ from Lemma A 3.9(p.282) (j2) and (6.5.7(p.31))). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.18) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.19), hence, for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2ii2) Let $\beta b + s < b^\star$. Then, since $V_1 < b^\star$ from (6.5.7(p.31))), we have $V_{t-1} < b^\star$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta b + s = V_1$ from Lemma A 3.9(p.282) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Corollary A 3.2(p.282) (a), from (A 4.18) and Lemma A 3.9(p.282) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.282) (a), for the same reason as in the proof of Tom A 4.5(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2iii) Let $a > 0$ ⟪$\tilde{\kappa} > 0$⟫. Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots$ **(8)** from Lemma A 3.10(c ⟪d⟫).

(c2iii1) Let $\beta b + s \geq b^\star$ or $s_{\tilde{L}} \leq s$. Firstly, let $\beta b + s \geq b^\star$. Then, since $x_{\tilde{K}} = \beta b - s = V_1$ from Lemma A 3.9(p.282) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \geq V_1$. Next, let $s_{\tilde{L}} \leq s$. Then, since $x_{\tilde{L}} \geq \beta b + s$ due to Lemma A 3.11(p.283) (d), we have $x_{\tilde{L}} \geq V_1$. Accordingly, whether $\beta b + s \geq b$ or $s_{\tilde{L}} \leq s$, we have $x_{\tilde{L}} \geq V_1$, thus $x_{\tilde{L}} \geq V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \geq 0$ for $t > 1$ from Corollary A 3.2(p.282) (a), for the same reason as in the proof of (b) we obtain $\boxed{\bullet\ \texttt{dOITd}_\tau\langle 1\rangle}_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta b + s < b^\star \cdots$ **(9)** and $s < s_{\tilde{L}}$. Then, from (8) and Lemma A 3.11(p.283) (d) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta b + s = V_1 \cdots$ **(10)**, hence $\tilde{K}(V_1) < 0 \cdots$ **(11)** from Lemma A 3.9(p.282) (j1). In addition, since $V_1 < b^\star$ due to (9), we have $V_{t-1} < b^\star$ for $t > 0$ from (a). Now, from (A 4.20) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from Lemma A 3.9(p.282) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10). Assume that $V_{t-1} \geq x_{\tilde{L}}$ for *all* $t > 1$, hence $V \geq x_{\tilde{L}}$ due to (a). Then, from (8) and $V \leq x_{\tilde{K}}$ in (a) we have the contradiction of $V \leq x_{\tilde{K}} < x_{\tilde{L}} \leq V$. Hence, it is impossible that $V_{t-1} \geq x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t^\bullet > 1$ such that

$$V_1 > V_2 > \cdots > V_{t^\bullet-1} > x_{\tilde{L}} \geq V_{t^\bullet} > V_{t^\bullet+1} > V_{t^\bullet+2} > \cdots, \tag{A 4.24}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t^\bullet \geq t > 1, \qquad x_{\tilde{L}} \geq V_{t-1}, \quad t > t^\bullet. \tag{A 4.25}$$

Therefore, from Corollary A 3.2(p.282) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots$ **(12)** for $t^\bullet \geq t > 1$ and $\tilde{L}(V_{t-1}) \geq 0 \cdots$ **(13)** for $t > t^\bullet$.

1. Let $t^\bullet \geq \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots$ **(14)** for $\tau \geq t > 1$ from (12), for the same reason as in the proof of Tom A 4.5(b) we have $\boxed{\circledS\ \texttt{dOITs}_\tau\langle\tau\rangle}_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\texttt{S}_{21}$(p.289) (1) is true.
2. Let $\tau > t^\bullet$. Firstly, let $\tau \geq t > t^\bullet$. Then, since $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet$ from (13), we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > t^\bullet$ from (A 4.17), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet$, thus

   $$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet} V_{t^\bullet} \cdots \textbf{(15)}.$$

   Next, let $t^\bullet \geq t > 1$. Then, from (12) and (A 4.17) we have $V_t - \beta V_{t-1} < 0$ for $t^\bullet \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t^\bullet \geq t > 1$, hence

   $$V_{t^\bullet} < \beta V_{t^\bullet-1} < \beta^2 V_{t^\bullet-2} < \cdots < \beta^{t^\bullet-1} V_1 \cdots \textbf{(16)}.$$

   From (15) and (16) we have

   $$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet} V_{t^\bullet} < \beta^{\tau-t^\bullet+1} V_{t^\bullet-1} < \beta^{\tau-t^\bullet+2} V_{t^\bullet-2} < \cdots < \beta^{\tau-1} V_1,$$

   hence we obtain $t^*_\tau = t^\bullet$ for $\tau > t^\bullet$ due to Preference Rule 7.2.1(p.35), i.e., $\boxed{\circledast\ \texttt{ndOIT}_\tau\langle t^\bullet\rangle}$ for $\tau > t^\bullet$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet \geq t > 1$ due to (12) and (A 4.21). Hence $\texttt{S}_{21}$(2) is true. ∎

A 5 Optimal Initiating Time in Markovian Decision Processes

This section defines the optimal initiating time OIT in Markovian decision processes (MDP) [23,Howard,1960][0528] and discusses its implications.

A 5.1 Standard Definition of Markovian Decision Processes

To begin with, let us provide the most standard definition of Markovian decision process with a finite horizon [23,Howard][0528] [39, Ross][0535] .

A 5.1.1 Maximization MDP

Consider a finite number of points in time equally spaced on the time axis (see Figure 2.1.1(p.7)). At each time $t \geq 0$ the process is observed to be in a *state* $i \in \mathcal{I}$, called the *state space*. At each time $t \geq 0$ in a state i an *action* $x \in \mathcal{A}(i)$, called the *action space*. If an action $x \in \mathcal{A}(i)$ is taken at a given time t in state $i \in \mathcal{I}$, then a *reward* $r(i, x)$ can be obtained and the present state i changes into $j \in \mathcal{I}$ at the next time $t-1$ with a known probability $p(j|i, x)$. Then let us call the process the *maximization* MDP for short. By $v_t(i)$ let us denote the maximum of the total expected present discounted reward gained over a given planning horizon starting from a time t in state i. Then the following system of equations holds:

$$v_t(i) \ = \ \max_{x \in \mathcal{A}(i)} \big\{ r(i, x) + \beta \textstyle\sum_{j \in \mathcal{I}} p(j|i, x) v_{t-1}(j) \big\}, \quad i \in \mathcal{I}, \quad t > 0, \tag{A 5.1}$$

where $v_0(i)$ is a value specified for an reason inherent for the process; in many cases, $v_0(i) = \max_{x \in \mathcal{A}(i)} r(i, x)$.

A 5.1.2 Minimization MDP

This is the inverse of the maximization MDP where if an action $x \in \mathcal{A}(i)$ is taken at a given time $t \geq 0$ in state $i \in \mathcal{I}$, a *cost* $r(i, x)$ must be paid. Then let us call the process the *minimization* MDP for short. By $v_t(i)$ let us denote the minimum of the total expected present discounted cost incurred over a given planning horizon from starting a time t in state i. Then the following system of equations holds:

$$v_t(i) \ = \ \min_{x \in \mathcal{A}(i)} \big\{ r(i, x) + \beta \textstyle\sum_{j \in \mathcal{I}} p(j|i, x) v_{t-1}(j) \big\}, \quad i \in \mathcal{I}, \quad t > 0, \tag{A 5.2}$$

where $v_0(i)$ is a value specified for a reason inherent for the process; in many cases, $v_0(i) = \min_{x \in \mathcal{A}(i)} r(i, x)$.

A 5.2 Optimal Initiating Time

A 5.2.1 Initiating State i°

Markovian decision processes is what is already completed as a *mathematical theory*. However, in Concept 1(p.9) we pointed out that an *activity of human being* regarding a decision process is first triggered only when its existence is *recognized*; in other words, without its recognition, human being does not take on any behavior involved with the decision process at all. Below let us consider only iiA-case (see A2bii(p.9)), i.e., whenever it reaches the starting time τ, you have the option whether to immediately initiate the process or to postpone its initiation. Furthermore, let us postulate that whenever the process is initiated at a time t, it is assumed to be always in a specified state $i^\circ \in \mathscr{I}$, called the *initiating state*. Then let us define

$$V_t \overset{\text{def}}{=} v_t(i^\circ), \quad t \geq 0. \tag{A 5.3}$$

A 5.2.2 Some Examples

By using some examples, below let us show that the monotonicity of the original sequence $V_{[\tau]}$ is not always inherited to the β-adjusted sequence $V_{\beta[\tau]}$ (see Section 7.2.4.3(p.35)). Below let:

original sequence $V_{[\tau]}$ $\rightarrow$

β-adjusted sequence $V_{\beta[\tau]}$ $\rightarrow$

optimal initiating time t^*_τ $\rightarrow$ $\bullet$

Example **1.5.1 (maximization MDP)** Suppose the original sequence $V_{[\tau]}$ is strictly increasing in t with

$$V_\tau > V_{\tau-1} > V_{\tau-2} > \cdots > V_0 > 0.$$

Then, as seen in Figure A 5.1 below, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0 > 0$; i.e., the monotonicity of the original sequence $V_{[\tau]}$ is <u>inherited</u> to the β-adjusted sequence $V_{\beta[\tau]}$ where the optimal initiating time $t^*_{16} = 16$ (ⓢ). □

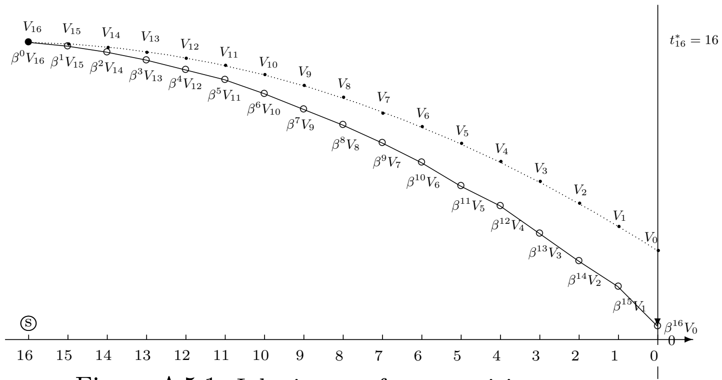


Figure A 5.1: Inheritance of monotonicity

Example **1.5.2 (maximization MDP)** Suppose the original sequence $V_{[\tau]}$ is strictly increasing in t with

$$V_\tau > \beta V_{\tau-1} > V_{\tau-2} > \cdots > V_{\tau-t'} > 0 > V_{\tau-t'-1} > \cdots > V_0.$$

Then, as seen in Figure A 5.2 below, the monotonicity of the original sequence $V_{[\tau]}$ is <u>inherited</u> to its β-adjusted sequence $V_{\beta[\tau]}$; however, in this example, the optimal initiating time does not change, i.e., $t^*_{16} = 16$ (ⓢ). □

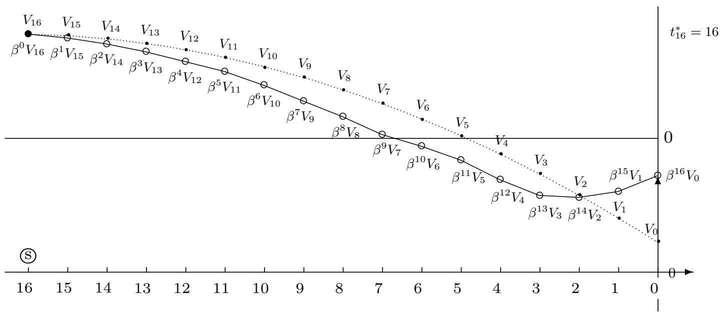


Figure A 5.2: Inheritance of monotonicity

Example **1.5.3 (maximization MDP)** Suppose V_t is strictly decreasing in t where

$$0 < V_\tau < \beta V_{\tau-1} < V_{\tau-2} < \cdots < V_0.$$

Then, as seen in Figure A 5.3 below, the monotonicity of the original sequence $V_{[\tau]}$ collapses in its β-adjusted sequence $V_{\beta[\tau]}$ where $t_1^* = 6$ (⊛). □

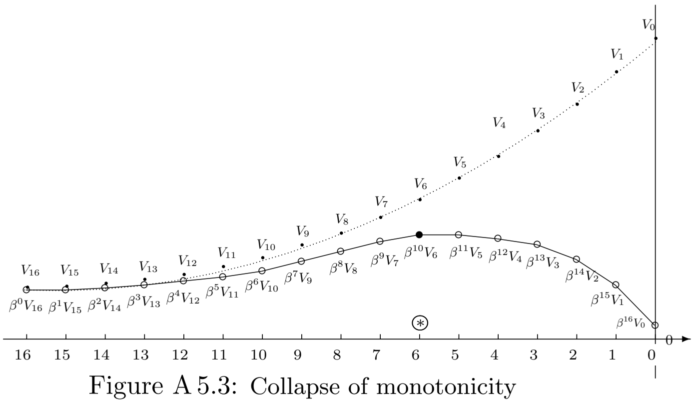


Figure A 5.3: Collapse of monotonicity

Example **1.5.4 (minimization MDP)** Suppose V_t is strictly decreasing in t with

$$0 < V_\tau < \beta V_{\tau-1} < \cdots < V_{\tau-t'} < 0 < V_{\tau-t'-1} < \cdots < V_0.$$

Then, as seen in Figure A 5.4 below, the monotonicity in the original sequence $V_{[\tau]}$ is inherited to its β-adjusted sequence $V_{\beta[\tau]}$ where $t_{16}^* = 16$ (Ⓢ). □

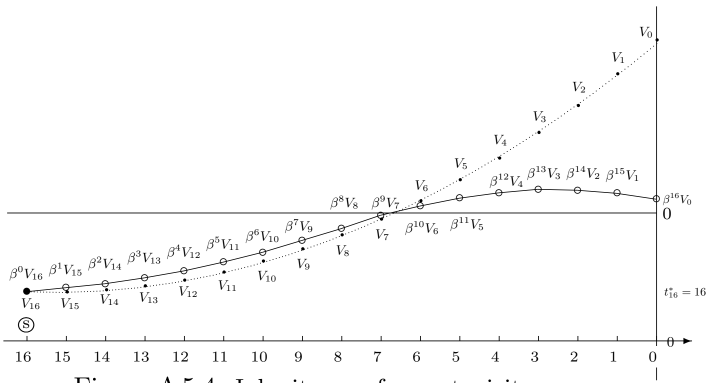


Figure A 5.4: Inheritance of monotonicity

A 6 Numerical Calculations

A 6.1 Numerical Example and Numerical Experiment

In general, a numerical calculation is to calculate a given expression by substitute numerical values for constants, parameters, and variables which the expression have. In the paper we attempt to conduct numerical calculations from the following two viewpoints. One is to reconfirm the results which have *already proved*, the other to confirm expectations which it is *hard to theoretically prove*; Let us call the former the *numerical example* and the latter the *numerical experiment*. Throughout the paper, in numerical calculations we use the uniform distribution function (see (A 7.5(p.296)) and (A 7.6)).

A 6.2 Calculation of Solutions x_K, x_L, and $s_{\mathcal{L}}$

In the numerical calculation of a given The most basic function used in numerical calculations is T-Function defined by (5.1.1(p.17))).model, very often it is required to calculate the solutions x_K, x_L, and $s_{\mathcal{L}}$ (see Section 5.2(p.19)). The following lemma is used for the calculations.

Lemma A 6.1 (x_K, x_L, $s_{\mathcal{L}}$)

(a) $\min\{a, (\lambda\beta\mu - s)/\delta\} \le x_K \le \max\{b, 0\}$.
(b) $\min\{a, (\lambda\beta\mu - s)/\lambda\} \le x_L \le b$.
(c) $0 \le s_{\mathcal{L}} \le \lambda\beta\mu - \min\{a, 0\}$. □

Proof (a) First, let $x \le a \cdots$ **(1)**. Then, from (9.2.4 (1) (p.42)) we have $K(x) = \lambda\beta\mu - s - \delta x = \delta((\lambda\beta\mu - s)/\delta - x)$, hence $K(x) \ge 0$ for $x \le (\lambda\beta\mu - s)/\delta$. From this and (1) we have $K(x) \ge 0$ for $x \le \min\{a, (\lambda\beta\mu - s)/\delta\}$, hence $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) \ge 0$.

1. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\delta\} < x_K \cdots$ **(2)** due to Corollary 9.2.2(p.44) (a).
2. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) = 0$.
   - If $\beta = 1$ and $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} \ge x_K$ due to Lemma 9.2.2(i). Since $\min\{a, (\lambda\beta\mu - s)/\delta\} \le a < b = x_K$, hence $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$.

- If $\beta < 1$ or $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$ due to Lemma 9.2.2(j1).

Accordingly $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K \cdots$ **(3)** whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$".

Thus $\min\{a, (\lambda\beta\mu - s)/\delta\} \le x_K \cdots$ **(4)** from (2) and (3).

Next, let $b \le x \cdots$ **(5)**. Then, from (9.2.5 (2) (p.42)) we have $K(x) \le 0$ for $0 \le x$. From this and (5) we have $K(x) \le 0$ for $\max\{b, 0\} \le x$, hence $0 \ge K(\max\{b, 0\})$. Accordingly, due to Corollary 9.2.2(p.44) (a) we have $x_K \le \max\{b, 0\}$. From this and (4) the assertion becomes true.

(b) First, let $x \le a \cdots$ **(6)**. Then, from (9.2.3 (1) (p.42)) we have $L(x) = \lambda\beta\mu - s - \lambda\beta x = \lambda\beta((\lambda\beta\mu - s)/\lambda\beta - x)$, hence $L(x) \ge 0$ for $x \le (\lambda\beta\mu - s)/\lambda\beta$. From this and (6) we have $L(x) \ge 0$ for $x \le \min\{a, (\lambda\beta\mu - s)/\lambda\beta\}$, hence $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) \ge 0$.

1. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} < x_L \cdots$ **(7)** due to Corollary 9.2.1(a).
2. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) = 0$.
   - If $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \ge x_L$ due to Lemma 9.2.1(d). Since $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\beta\} \le a < b = x_L$, hence $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$.
   - If $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$ due to Lemma 9.2.1(e1).

   Accordingly, $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L \cdots$ **(8)** whether $s = 0$ or $s > 0$.

Thus $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \le x_L \cdots$ **(9)** from (7) and (8).

Next, let $b \le x \cdots$ **(10)**. Then, from (5.1.3) and Lemma 9.1.1(g) we have $L(x) = -s$, hence $0 \ge L(b)$. Accordingly, due to Corollary 9.2.1(p.43) (a) we have $x_L \le b$. From this and (9) the assertion becomes true.

(c) From (5.1.5(p.17)) and (5.1.3) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) \ge 0 \cdots$ **(11)** due to Lemma 9.1.1(g). For a sufficiently large $s > 0$ such that $\lambda\beta\mu - s \le a$ and $\lambda\beta\mu - s \le 0 \cdots$ **(12)**, hence $s \ge \lambda\beta\mu - a$ and $s \ge \lambda\beta\mu$, so $s \ge \max\{\lambda\beta\mu - a, \lambda\beta\mu\} = \lambda\beta\mu + \max\{-a, 0\} = \lambda\beta\mu - \min\{a, 0\} \cdots$ **(13)**. Then, from (5.1.5(p.17)) and Lemma 9.1.1(f) we have $\mathcal{L}(s) = \lambda\beta T(\lambda\beta\mu - s) - s = \lambda\beta(\mu - \lambda\beta\mu + s) - s = \lambda\beta\mu - \lambda\beta(\lambda\beta\mu - s) - s = (1 - \lambda\beta)(\lambda\beta\mu - s)$. Hence, due to (12) we have $\mathcal{L}(s) \le 0$ for $s \ge \lambda\beta\mu - \min\{a, 0\}$, so $\mathcal{L}(\lambda\beta\mu - \min\{a, 0\}) \le 0$. From this and (11) we have $\mathcal{L}(0) \ge 0 \ge \mathcal{L}(\lambda\beta\mu - \min\{a, 0\})$, hence due to Lemma 9.2.4(p.44) (a) we have $0 \le s_{\mathcal{L}} \le \lambda\beta\mu - \min\{a, 0\}$. ∎

A 6.3 Calculation of Solutions $x_{\tilde{K}}$, $x_{\tilde{L}}$, and $s_{\tilde{\mathcal{L}}}$

The following lemma is used to find the solutions $x_{\tilde{K}}$, $x_{\tilde{L}}$, and $s_{\tilde{\mathcal{L}}}$ by the numerical calculation.

Lemma A 6.2 ($x_{\tilde{K}}$, $x_{\tilde{L}}$, $s_{\tilde{\mathcal{L}}}$ **)**

(a) $\max\{b, (\lambda\beta\mu + s)/\delta\} \ge x_{\tilde{K}} \ge \min\{a, 0\}$.
(b) $\max\{b, (\lambda\beta\mu + s)/\lambda\beta\} \ge x_{\tilde{L}} \ge a$.
(c) $0 \le s_{\tilde{\mathcal{L}}} \le -\lambda\beta\mu + \max\{b, 0\}$. □

Proof First, applying the operation $\mathcal{R}$ to Lemma A 6.1(p.293) leads to

(a) $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\delta \le -\hat{x}_K \le \max\{-\hat{b}, 0\}$.
(b) $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\lambda\}\beta \le -\hat{x}_L \le -\hat{b}$.
(c) $0 \le s_{\mathcal{L}} \le -\lambda\beta\hat{\mu} - \min\{-\hat{a}, 0\}$.

Secondly, rearranging the above produces

(a) $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta \le -\hat{x}_K \le -\min\{\hat{b}, 0\}$.
(b) $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \le -\hat{x}_L \le -\hat{b}$.
(c) $0 \le s_{\mathcal{L}} \le -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$.

Thirdly, rearranging the above yields

(a) $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta \ge \hat{x}_K \ge \min\{\hat{b}, 0\}$.
(b) $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \ge \hat{x}_L \ge \hat{b}$.
(c) $0 \le s_{\mathcal{L}} \le -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$.

Next, applying the operation $\mathcal{C}_{\mathbb{R}}$ to the above yields

(a) $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\delta \ge \check{\hat{x}}_K \ge \min\{\check{a}, 0\}$.
(b) $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\lambda\}\beta \ge \check{\hat{x}}_L \ge \check{a}$.
(c) $0 \le s_{\check{\tilde{\mathcal{L}}}} \le -\lambda\beta\check{\mu} + \max\{\check{b}, 0\}$.

Finally, applying the operation $\mathcal{I}_{\mathbb{R}}$ to the above leads to the three inequalities in the lemma . ∎

A 7 Others

A 7.1 Formulas

Proposition A 7.1 For given sets X and Y we have

$$\{\check{F} \mid F \in X \cap Y\} = \{\check{F} \mid F \in X\} \cap \{\check{F} \mid F \in Y\}. \tag{A 7.1}$$

$$\{\check{F} \mid F \in X \cup Y\} = \{\check{F} \mid F \in X\} \cup \{\check{F} \mid F \in Y\}. \ \square \tag{A 7.2}$$

• *Proof of* (A 7.1) First, consider a $\check{F}$ defined by (11.1.2(p.55)) where $F \in \mathscr{F}$ and let the $\check{F} \in \{\check{F} \mid F \in X \cap Y\} \cdots (*)$. Then, for the F corresponding to the $\check{F}$ we have $F \in X \cap Y$ or equivalently $F \in X$ and $F \in Y$. Hence, for the $\check{F}$ corresponding to the F, clearly $\check{F} \in \{\check{F} \mid F \in X\}$ and $\check{F} \in \{\check{F} \mid F \in Y\}$, thus the $\check{F} \in \{\check{F} \mid F \in X\} \cap \{\check{F} \mid F \in Y\} \cdots (**)$. Accordingly $(*) \subseteq (**)$. Next, consider a $\check{F}$ and let $\check{F} \in \{\check{F} \mid F \in X\} \cap \{\check{F} \mid F \in Y\} \cdots (**)$. Then, we have $\check{F} \in \{\check{F} \mid F \in X\}$ and $(\cap)$ $\check{F} \in \{\check{F} \mid F \in Y\}$, hence, for F corresponding to the $\check{F}$ we have $F \in X$ and $(\cap)$ $F \in Y$ or equivalently $F \in X \cap Y$. Thus, for the $\check{F}$ corresponding to the F we have $\check{F} \in \{\check{F} \mid F \in X \cap Y\} \cdots (*)$, so that $(**) \subseteq (*)$. Accordingly, we have $(*) = (**)$.

• *Proof of* (A 7.2) Almost the same as the proof of (A 7.1) only except that the symbol $\cap$ changes into $\cup$. ∎

Proposition A 7.2 For given sets X_k, $k \in K = \{1, 2, \cdots, n\}$, we have

$$\{\check{F} \mid F \in \cap_{k \in K} X_k\} = \cap_{k \in K} \{\check{F} \mid F \in X_k\}. \tag{A 7.3}$$

$$\{\check{F} \mid F \in \cup_{k \in K} X_k\} = \cup_{k \in K} \{\check{F} \mid F \in X_k\}. \ \square \tag{A 7.4}$$

Proof Evident from Proposition A 7.1. ∎

A 7.2 Monotonicity of Solution

Proposition A 7.3 For the solution x_t of a given equation $g_t(x) = 0$ we have:

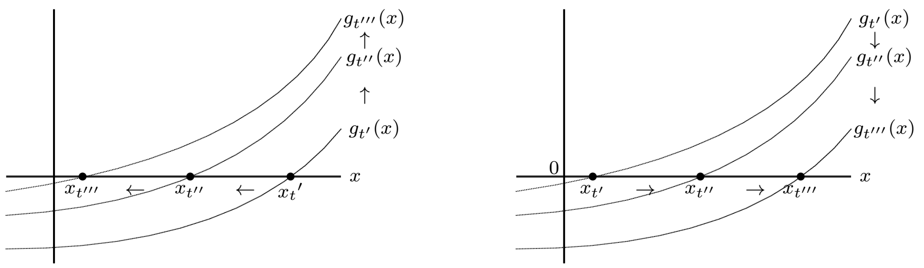
[a] Let $g_t(x)$ is nondecreasing in x for all t.

1. If $g_t(x)$ is nondecreasing in t, then x_t is nonincreasing in t.
2. If $g_t(x)$ is nonincreasing in t, then x_t is nondecreasing in t.

[b] Let $g_t(x)$ is nonincreasing in x for all t.

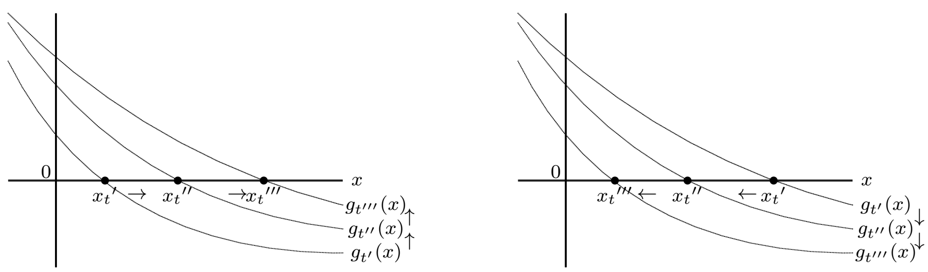
1. If $g_t(x)$ is nondecreasing in t, then x_t is nondecreasing in t.
2. If $g_t(x)$ is nonincreasing in t, then x_t is nonincreasing in t. $\square$

Proof Evident from figures below:



Case that $g_t(x)$ is nondecreasing in t ([a1]) Case that $g_t(x)$ is nonincreasing in t ([a2])

Figure A 7.1: Case that $g_t(x)$ is nondecreasing in x ([a])



Case that $g_t(x)$ is nondecreasing in t ([b1]) Case that $g_t(x)$ is nonincreasing in t ([b2])

Figure A 7.2: Case that $g_t(x)$ is nonincreasing in x ([b])

A 7.3 Uniform Probability Density Function

For given a and b such as $-\infty < a < b < \infty$ let consider the uniform probability density function:

$$f(x) = \begin{cases} 0, & x < a, \\ 1/(b-a), & a \le x \le b, \\ 0, & b < x, \end{cases} \tag{A 7.5}$$

where the expectation is $\mu = 0.5(a+b)$. Then, noting Lemma 9.1.1(p.41) (f,g), we have:

$$T(x) = \begin{cases} 0.5(a+b) - x, & x < a, & \cdots (1), \\ 0.5(b-x)^2/(b-a), & a \le x \le b, & \cdots (2), \\ 0, & b < x, & \cdots (3). \end{cases} \tag{A 7.6}$$

A 7.4 Graphs of $T_{\mathbb{R}}(x)$

From Lemma 9.1.1(p.41) (b,f,g) it can be immediately seen that $T_{\mathbb{R}}(x)$ is depicted as in (I) of the graph below. Similarly, from Lemma 9.2.2(p.43) (b, (9.2.4 (1) (p.42)), and (9.2.5 (2)) it can be immediately seen that $K_{\mathbb{R}}(x)$ is depicted as in (II) of the graph below.

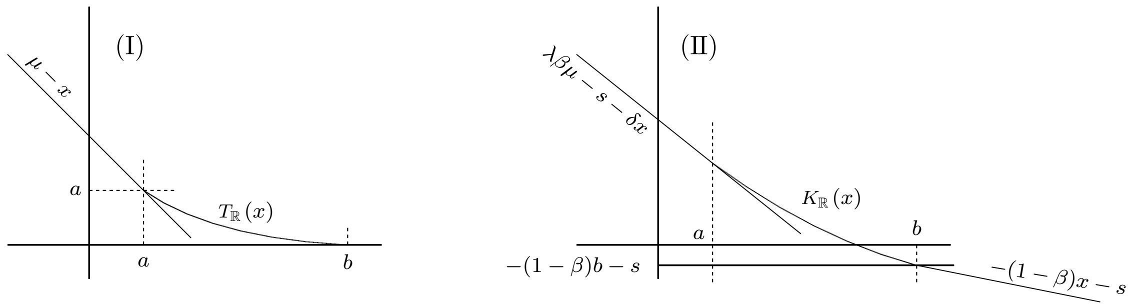


Figure A 7.3: Graph of $\boldsymbol{T_{\mathbb{P}}(x)}$ and $\boldsymbol{K_{\mathbb{P}}(x)}$

A 7.5 Graph of $T_{\mathbb{P}}(x)$

From Lemma 12.2.1(p.77) (b,f,g) it can be immediately seen that $T_{\mathbb{P}}(x)$ can be depicted as in the graph below.

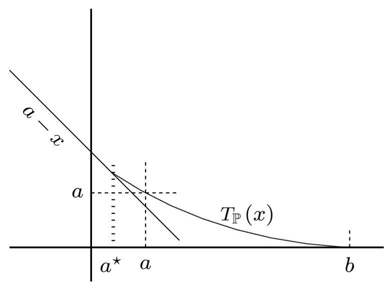


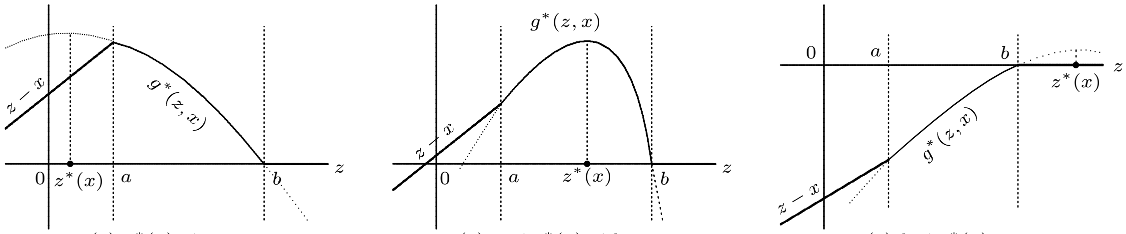
Figure A 7.4: Graph of $\boldsymbol{T_{\mathbb{P}}(x)}$

Below let us consider the uniform distribution function. Then $p(z) = 1$ for $z \le a$ from (5.1.28 (1) (p.18)), $p(z) = \int_z^b 1/(b-a)d\xi = (b-z)/(b-a)$ for $a \le z \le b$, and $p(z) = 0$ for $b \le z$ from (5.1.29 (2) (p.18)), hence

$$T(z,x) \overset{\text{def}}{=} p(z)(z-x) = \begin{cases} z - x, & z \le a & \cdots \textbf{(1)} \\ (b-z)(z-x)/(b-a), & a \le z \le b & \cdots \textbf{(2)} \\ 0, & b \le z & \cdots \textbf{(3)} \end{cases}$$

where $T(x) = \max_z T(z,x) = T(z(x),x) \cdots \textbf{(4)}$. Let us here define $g^*(z,x) = (b-z)(z-x)/(b-a)$ for any x and z on $(-\infty,\infty)$. By $z^*(x)$ let us denote z attaining the maximum of $g^*(z,x)$ for a given x, then clearly $z^*(x) = (b+x)/2 \cdots \textbf{(5)}$. Note that $g^*(z,x)$ can be depicted as one of the three possible *smooth* curves in Figure A 7.5 below, depending on a value that $z^*(x)$ takes on, i.e., $z^*(x) \le a$, $a \le z^*(x) \le b$, and $b \le z^*(x)$. Accordingly, it follows that $T(x)$ can be depicted as one of the three possible *broken* curves, each of which has the line of the angle 45° on $z \le a$ and the horizontal line on $b \le z$.

Figure A 7.5: Graph of $g^*(z,x)$



(1) $z^*(x) \le a$ (2) $a \le z^*(x) \le b$ (3) $b \le z^*(x)$

From (5) and Figure A 7.5 we can immediately know that

1. Let $z^*(x) \le a$, i.e., $(b+x)/2 \le a$, hence $x \le 2a-b$. Then $z(x) = a \cdots$ **(6)**, hence $T(x) = T(a,x) = a-x \cdots$ **(7)** on $(-\infty, 2a-b]$ from (1).
2. Let $a < z^*(x) \le b$, i.e., $a < (b+x)/2 \le b$, hence $2a-b < x \le b$. Then $z(x) = z^*(x) = (b+x)/2 > a \cdots$ **(8)** on $(2a-b, b]$, hence $T(x) = T(z^*(x), x) = g^*(z^*(x), x) = (b - z^*(x))(z^*(x) - x)/(b-a) = (b-x)^2/4(b-a)$. Now, we have $m(x) \stackrel{\text{def}}{=} T(x) - a + x = ((b-x)^2 - 4(b-a)(a-x))/4(b-a)$. Then $m'(x) = (x - 2a + b)/2(b-a) > 0$. Accordingly $m(x)$ is strictly increasing on $2a - b < x \le b$. In addition to the fact, since it can be easily confirmed that $m(2a-b) = 0$, it follows that $m(x) > 0$ on $2a - b < x \le b$, hence $T(x) - a + x > 0$ on $2a - b < x \le b$, so $T(x) > a - x \cdots$ **(9)** on $(2a-b, b]$.
3. Let $b \le z^*(x)$, i.e., $b \le (b+x)/2$, hence $b \le x$. Then $z(x) = b > a \cdots$ **(10)** on $[b, \infty)$, hence $T(x) = T(b,x) = 0$ from (3), hence $T(x) = 0 \ge b - x > a - x \cdots$ **(11)** on $[b, \infty)$.

Collecting up (7), (9), and (11), we have

$$T(x) \begin{cases} = a - x, & x \le 2a - b, \\ > a - x, & 2a - b < x \le b, \\ > a - x, & b \le x. \end{cases}$$

Similarly, collecting up (6), (8), and (10), we have

$$z(x) \begin{cases} = a, & x \le 2a - b, \\ > a, & 2a - b < x \le b, \\ > a, & b \le x. \end{cases}$$

Accordingly, from (5.1.26(p.18)) and (5.1.27) we immediately obtain

$$a^\star = 2a - b \cdots (1) \quad \text{and} \quad x^\star = 2a - b \cdots (2). \qquad \text{(A 7.7)}$$

Example **1.7.1** (Discontinuity of $z(x)$) $z(x)$ is not always continuous in $x = x^\star$; in fact we can demonstrate a numerical example in which $z(x)$ is not continuous in $x = x^\star$. For example let us consider $F(w)$ with $f(w)$ such that $f(w) \approx 0.05701$ on $[0.1, 0.599]$, $f(w)$ is a triangle on $[0.599, 0.7]$ with its maximum at $w = 0.6$, and $f(w) \approx 0.06982$ on $[0.7, 3.0]$. Then we have $z(x) \approx 0.599$ for $x \le 0.48568$ and $z(x) \approx 1.7$ for $x < 0.48568$, i.e., $z(x)$ is discontinuous at $x = 0.48568$. □

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