Takens-type reconstruction theorems of one-sided dynamical systems

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TAKENS-TYPE RECONSTRUCTION THEOREMS OF ONE-SIDED DYNAMICAL SYSTEMS

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Abstract. The reconstruction theorem deals with dynamical systems that are given by a map $T : X \to X$ of a compact metric space X together with an observable $f : X \to \mathbb{R}$ from X to the real line $\mathbb{R}$. In 1981, by use of Whitney's embedding theorem, Takens proved that if $T : M \to M$ is a (two-sided) diffeomorphism on a compact smooth manifold M with $\dim M = d$, for generic (T, f) there is a bijection between elements $x \in M$ and corresponding sequence $(fT^j(x))_{j=0}^{2d}$, and moreover, in 2002 Takens proved a generalized version for endomorphisms.

In natural sciences and physical engineering, there has been an increase in the importance of fractal sets and more complicated spaces, and also in mathematics, many topological and dynamical properties and stochastic analysis of such spaces have been studied. In the present paper, by use of some topological methods we extend the Takens' reconstruction theorems of compact smooth manifolds to reconstruction theorems of "non-invertible" dynamical systems for a large class of compact metric spaces, which contains PL-manifolds, manifolds with branched structures and some fractal sets, e.g. Menger manifolds, Sierpiński carpet and Sierpiński gasket and dendrites, etc.

1. Introduction

Throughout this paper, all spaces are separable metric spaces and maps are continuous functions. Let $\mathbb{N}$ be the set of all nonnegative integers, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathbb{Z}$ be the set of all integers and $\mathbb{R}$ the real line.

A map $h : X \to Y$ is an *embedding* if $h : X \to h(X)$ is a homeomorphism. A pair (X, T) is called a *one-sided dynamical system* (abbreviated as *dynamical system*) if X is a separable metric space and $T : X \to X$ is any map. Moreover, if $T : X \to X$ is a homeomorphism, i.e., invertible, then (X, T) is called a *two-sided dynamical system.* Also if $T : X \to X$ is not a homeomorphism, (X, T) called a *non-invertible dynamical system.*

Reconstruction of dynamical systems from a scalar time series is a topic that has been extensively studied. The theoretical basis for methods of recovering dynamical systems on compact manifolds from one-dimensional

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data was studied by Takens [Tak81, Tak02]. The embedding theorem of Takens forms a bridge between the theory of (two-sided) nonlinear dynamical systems on smooth manifolds and the analysis of experimental time series. In 1981, Takens [Tak81], by use of Whitney's embedding theorem, proved that under some conditions of (two-sided) diffeomorphisms on a manifold, the dynamical system can be reconstructed from the observations made with generic functions.

Theorem 1.1. (Takens' reconstruction theorem for diffeomorphisms [Tak81] and [Noa91]) *Suppose that M is a compact smooth manifold of dimension d. Let $D^r(M,M)$ be the space of all C^r-diffeomorphisms on M and $C^r(M,\mathbb{R})$ the set of all C^r-functions ($r \geq 1$) from M to $\mathbb{R}$. If E is the set of all pairs $(T,f) \in D^r(M,M) \times C^r(M,\mathbb{R})$ such that the delay observation map $I_{T,f}^{(0,1,2,..,2d)} : M \to \mathbb{R}^{2d+1}$ defined by*

$$x \mapsto (fT^j(x))_{j=0}^{2d}$$

is an embedding, then E is open and dense in $D^r(M,M) \times C^r(M,\mathbb{R})$.

Moreover, in 2002 Takens [Tak02], extended his theorem for endomorphisms on compact smooth manifolds as follows.

Theorem 1.2. (Takens' reconstruction theorem for endomorphisms [Tak02]) *Suppose that M is a compact smooth manifold of dimension d. Then there is an open dense subset $\mathcal{U} \subset \mathrm{End}^1(M,M) \times C^1(M,\mathbb{R})$, where $\mathrm{End}^1(M,M)$ denotes the space of all C^1-endomorphisms on M, such that, whenever $(T,f) \in \mathcal{U}$, there is a map $\pi : I_{T,f}^{(0,1,..,2d)}(M) \to M$ with $\pi \cdot I_{T,f}^{(0,1,..,2d)} = T^{2d}$.*

Embeddings of two-sided dynamical systems in the two-sided shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$ have been studied by many authors (e.g. see [AAM18, Coo15, Gut15, Gut16, GQS18, GT14, Jaw74, Lin99, LW00, Ner91, SYC91, Tak81]).

In general, one-sided dynamical systems are more diverse than two-sided dynamical systems. In fact we know that two-sided (invertible) dynamical systems of the unit interval $I = [0,1]$ are very simple, but one-side (non-invertible) dynamical systems of I are very complicated and diverse, and so now many researchers are trying to clarify them.

In [Kat20], we studied embeddings of non-invertible dynamical systems in the one-sided shift $(\mathbb{R}^{\mathbb{N}}, \sigma)$. In this paper, by use of the topological methods introduced in the paper [Kat20], we extend the above Takens' reconstruction theorems of differential dynamical systems on compact manifolds to theorems of "non-invertible" dynamical systems for a large class of compact metric spaces. The main results of this paper are Theorem 5.4 and Theorem 6.9.

In this paper, we do not assume injectivity of T and so the proofs of our results cannot any longer rely on the embedding theorems of Whitney and Menger-Nöbeling [Eng95]. Instead, an essential role is played by the notion defined in Definition 2.1.

2. Definitions and notations

For a space X, $\dim X$ means the topological (covering) dimension of X (e.g. see [Eng95], [HW41] and [Nag65]). Let X be compact metric space and Y a space with a complete metric d_Y. Let $C(X,Y)$ denote the space consisting of all maps $f : X \to Y$. We equip $C(X,Y)$ with the metric d defined by

$$d(f,g) = \sup_{x\in X} d_Y(f(x), g(x)).$$

Recall that $C(X,Y)$ is a complete metric space and hence Baire's category theorem holds in $C(X,Y)$.

A map $g : X \to Y$ of separable metric spaces is *n-dimensional* ($n = 0,1,2,...$) if $\dim g^{-1}(y) \leq n$ for each $y \in Y$. Note that a closed map $g : X \to Y$ is 0-dimensional if and only if for any 0-dimensional subset D of Y, $\dim g^{-1}(D) \leq 0$ (see [Eng95, Hurewic's theorem (1.12.4)]). A map $T : X \to X$ is *doubly 0-dimensional* if for each closed set $A \subset X$ of dimension 0, one has $\dim T^{-1}(A) \leq 0$ and $\dim T(A) = 0$.

If K is a subset of a space X, then $\mathrm{cl}(K)$, $\mathrm{bd}(K)$ and $\mathrm{int}(K)$ denote the closure, the boundary and the interior of K in X, respectively. A subset A of a space X is an F_σ-set of X if A is a countable union of closed subsets of X. Also, a subset B of X is a G_δ-set of X if B is an intersection of countably many open subsets of X.

An indexed family $(C_s)_{s\in S}$ of subsets of a set X will by abuse of notation also be denoted by $\{C_s\}_{s\in S}$ or $\{C_s : s \in S\}$. Hence if $\mathcal{C} = \{C_s\}_{s\in S}$ is such a family then its members C_s and C_t will be considered as different whenever $s \neq t$. We then put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x(\mathcal{C}) : x \in X\}, \text{ where } \mathrm{ord}_x(\mathcal{C}) = |\{s \in S|\ x \in C_s\}|.$$

Note that $\mathrm{ord}(\mathcal{C})$ so defined is by 1 larger than it would be according to the usual definition, as e.g. in [Eng95, (1.6.6) Definition].

Modifying the definition of TSP in [Kat20], we define the notion of (k,η) trajectory-separation property for $k \in \mathbb{N}$ and $\eta > 0$ which is very important in this paper.

Definition 2.1. *Let $T : X \to X$ be a map of a compact metric space X with $\dim X = d < \infty$ and let $k \in \mathbb{N}, \eta > 0$. Then T has the (k,η) trajectory-separation property ((k,η)-TSP for short) provided that there is a closed set H of X such that*
(1) $X \setminus H$ is a union of finitely many disjoint open sets of diameter at most η, and
(2) $\mathrm{ord}\{T^{-p}(H)\}_{p=0}^{k} \leq d$.

3. RECONSTRUCTION SPACES OF DYNAMICAL SYSTEMS

For a space K, we consider the (one-sided) shift $\sigma : K^{\mathbb{N}} \to K^{\mathbb{N}}$ which is defined by

$$\sigma(x_0, x_1, x_2, x_3 ....) = (x_1, x_2, x_3 ....),\ x_i \in K.$$

Let (X, T) and (X', T') be dynamical systems. If $h : X \to X'$ is a map such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow T & & \downarrow T' \\ X & \xrightarrow{h} & X' \end{array}$$

is commutative, then we say that $h : (X, T) \to (X', T')$ is a *morphism* of dynamical systems.

In this paper, we need the following definition from [Kat20].

Definition 3.1. *Let $T : X \to X$ be a map of a compact metric space X.*
(a) Given a set $S \subset \mathbb{N}$ and a map $f : X \to \mathbb{R}$, the map $(fT^j)_{j\in S} : X \to \mathbb{R}^S$ will be denoted by $I^S_{T,f}$. We call this map the delay observation map at times $j \in S$. Note that $I_{T,f} := I^{\mathbb{N}}_{T,f} : (X, T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$ is a morphism of dynamical systems. We call $I_{T,f}$ the infinite delay observation map for (T, f).
(b) We say that I^S_f is a trajectory-embedding if $I^S_f(x) \neq I^S_f(y)$ whenever $T^j(x) \neq T^j(y)$ for all $j \in S$.

Remark 1. (1) In the statement of Theorem 1.2, the existence of such a map $\pi : I^{(0,1,..,2d)}_{T,f}(M) \to M$ is equivalent to that $I^{(0,1,...,2d)}_{T,f}$ is a trajectory-embedding (see (2) of Proposition 4.1).
(2) In the statement (*b*) of Definition 3.1, for the case where $T : X \to X$ is injective, I^S_f is an embedding if and only if I^S_f is a trajectory-embedding.

Let (X, T) be a dynamical system of a compact metric space X. For $n \geq 1$, let $P_n(T)$ be the set of all periodic points of T with period $\leq n$ and $P(T)$ the set of all periodic points of T, i.e.

$$P_n(T) = \{x \in X|\ \text{there is an } i \text{ such that } 1 \leq i \leq n \text{ and } T^i(x) = x\}$$

$$\text{and } P(T) = \bigcup_{n\geq 1} P_n(T).$$

Two points x and y of X are *trajectory-separated* for T if $T^j(x) \neq T^j(y)$ for $j \in \mathbb{N}$. A morphism $h : (X, T) \to (X', T')$ is a *trajectory-monomorphism* if $h(x), h(y)$ are trajectory-separated for T', whenever $x, y \in X$ are trajectory-separated for T.

Let $x, y \in X$ and let $o_T(x) = (T^i(x))_{i\in\mathbb{N}}$ and $o_T(y) = (T^i(y))_{i\in\mathbb{N}}$ be orbits of x and y for T respectively. We say that the orbit $o_T(x)$ is *eventually equivalent* to the orbit $o_T(y)$ if the orbits will be equal in the future, i.e., there exists an $n \in \mathbb{N}$ such that $T^i(x) = T^i(y)$ for each $i \geq n$. In this case, we wright $o_T(x) \sim_e o_T(y)$. We see that this relation is an equivalence relation. So we have the equivalence class

$$[o_T(x)] = \{o_T(y) |\ o_T(x) \sim_e o_T(y)\}$$

containing $o_T(x)$ and we put

$$[O(T)] = \{[o_T(x)] |\ x \in X\}.$$

Note that if $T : X \to X$ is injective, the function $o : X \to [O(T)]$ defined by $x \mapsto [o_T(x)]$ is bijective, i.e., $o : X \cong [O(T)]$. Also, note that if $h : (X,T) \to (X',T')$ is a morphism of dynamical systems, then h induces the function $h : [O(T)] \to [O(T')]$ defined by $h([o_T(x)]) = [o_{T'}(h(x))]$ for $x \in X$. A morphism $h : (X,T) \to (X',T')$ of dynamical systems is a *trajectory-isomorphism* if h induces the bijection $h : [O(T)] \cong [O(T')]$.

Proposition 3.2. *Suppose that a morphism $h : (X,T) \to (X',T')$ is a trajectory-monomorphism and h is surjective, i.e., $h(X) = X'$. Then h is a trajectory-isomorphism:*

$$h : [O(T)] \cong [O(T')]$$

Proof. Since h is a trajectory-monomorphism, h induces an injective function from $[O(T)]$ to $[O(T')]$. Also h induces a surjective function from $[O(T)]$ onto $[O(T')]$, because that h is a surjective function. □

We need the definition of topological entropy and here we give the definition by Bowen [Bow78]. Let $T : X \to X$ be any map of a compact metric space X. A subset E of X is *(n,ϵ)-separated* if for any $x, y \in E$ with $x \neq y$, there is a $j \in \mathbb{N}$ such that $0 \leq j < n$ and $d(T^j(x), T^j(y)) \geq \epsilon$. If K is any nonempty closed subset of X, $s_n(\epsilon; K)$ denotes the largest cardinality of any set $E \subset K$ which is (n,ϵ)-separated. Also we define

$$s(\epsilon; K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon; K),$$
$$h(T; K) = \lim_{\epsilon\to 0} s(\epsilon; K).$$

It is well known that the topological entropy $h(T)$ of T is equal to $h(T; X)$ (see [Bow78]).

Let (X,T) and (Y,S) be one-sided dynamical systems of compact metric spaces. The *inverse limit* of T is the space

$$\varprojlim(X,T) = \{(x_i)_{i=0}^{\infty} \in X^{\mathbb{N}} \mid T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\}$$

which has the topology inherited as a subspace of the product space $X^{\mathbb{N}}$. If $h : (X,T) \to (Y,S)$ is a morphism of dynamical systems, then the map

$$\varprojlim h : \varprojlim(X,T) \to \varprojlim(Y,S)$$

is defied by $\varprojlim h((x_i)_i) = (h(x_i))_i$ for $(x_i)_i \in \varprojlim(X,T)$. Note that if T is a homeomorphism, then $X \cong \varprojlim(X,T)$.

Now, we will introduce the notion of *reconstruction space* of dynamical systems which is the main theme of this paper.

Definition 3.3. *A compact metric space X is a reconstruction space of dynamical systems if there exists a G_δ-dense set E of $C(X,X) \times C(X,\mathbb{R})$ such that for $(T,f) \in E$, the infinite delay observation map*

$$I_{T,f} := I_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions (1) and (2):
(1) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$, *where* $\sigma_{T,f} = \sigma | I_{T,f}(X)$, *and*
(2) $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism.*

$$\begin{array}{ccccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} & & \downarrow \sigma \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \end{array}$$

Remark 2. In Definition 3.3, (1) implies that one can understand the structure of orbits of (X,T) from the analysis of time series $(I_{T,f}(X), \sigma_{T,f})$, and (2) implies that $(I_{T,f}(X), \sigma_{T,f})$ reflects topological and dynamical properties of (X,T). In fact, let $\mathcal{P}$ be any dynamical property such that (X,T) has $\mathcal{P}$ if and only if $(\varprojlim(X,T), \varprojlim T)$ has $\mathcal{P}$; e.g. as such a property $\mathcal{P}$ minimal, topological transitive, topological mixing, sensitive, etc. Then (X,T) has $\mathcal{P}$ whenever $(I_{T,f}(X), \sigma_{T,f})$ has $\mathcal{P}$ because that there is the following commutative diagram of homeomorphisms:

$$\begin{array}{ccc} \varprojlim(X,T) & \xrightarrow{\varprojlim I_{T,f}} & \varprojlim(I_{T,f}(X), \sigma_{T,f}) \\ \downarrow \varprojlim T & & \downarrow \varprojlim \sigma_{T,f} \\ \varprojlim(X,T) & \xrightarrow{\varprojlim I_{T,f}} & \varprojlim(I_{T,f}(X), \sigma_{T,f}) \end{array}$$

We show that many compact metric spaces (e.g. PL-manifolds, manifolds with branched structures, Menger manifolds, Sierpiński carpet, Sierpiński gasket and many fractal sets) are reconstruction spaces of dynamical systems. Our result means that almost all dynamical systems (X,T) on a reconstruction space X can be reconstructed from (observation) maps $f : X \to \mathbb{R}$ in the sense of 'eventually equivalent orbits and inverse limits', and so it forms a bridge between the theory of nonlinear one-sided dynamical systems and nonlinear time series analysis.

4. Trajectory-embeddings in $(\mathbb{R}^{\mathbb{N}}, \sigma)$

In this section, we study some fundamental properties of trajectory-embeddings.

Proposition 4.1. *Let (X,T) be a dynamical system, $f : X \to \mathbb{R}$ a map and $k \in \mathbb{N}$. Suppose that $I_{T,f}^{(0,1,..,k)} : X \to \mathbb{R}^{k+1}$ is a trajectory-embedding. Then the following* (1)-(4) *hold.*

(1) *There is the unique map* $\sigma_{T,f}^{(0,1,..,k)} : I_{T,f}^{(0,1,..,k)}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ *such that the diagram*

$$\begin{array}{ccc} X & \overset{I_{T,f}^{(0,1,..,k)}}{\longrightarrow} & I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,..,k)} \\ X & \overset{I_{T,f}^{(0,1,..,k)}}{\longrightarrow} & I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1}. \end{array}$$

is commutative. In other words, the map $\sigma_{T,f}^{(0,1,..,k)}$ *defined by*

$$(fT^i(x))_{i=0}^k) \mapsto (fT^i(x))_{i=1}^{k+1})\ (x \in X)$$

is well-defined. And the morphism

$$I_{T,f}^{(0,1,..,k)} : (X, T) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$$

is a trajectory-isomorphism. In particular, $I_{T,f} := I_{T,f}^{\mathbb{N}} : (X, T) \to (R^{\mathbb{N}}, \sigma)$ *is a trajectory-monomorphism.*

(2) *There is a map* $\pi : I_{T,f}^{(0,1,..,k)}(X) \to X$ *such that*

$$\pi \cdot I_{T,f}^{(0,1,..,k)} = T^k \text{ and } I_{T,f}^{(0,1,..,k)} \cdot \pi = (\sigma_{T,f}^{(0,1,..,k)})^k$$

and so the map $\varprojlim I_{T,f}^{(0,1,..,k)} : \varprojlim(X, T) \to \varprojlim(I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ *is a homeomorphism.*

(3) *Let* $p_{(0,1,..,k)} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{k+1}$ *be the projection defined by* $(x_i)_{i\in\mathbb{N}} \mapsto (x_i)_{i=0}^k$. *Then* $p_{(0,1,..,k)} : (I_{T,f}(X), \sigma_{T,f}) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ *is an isomorphism of dynamical systems, i.e.,* $p_{(0,1,..,k)} : I_{T,f}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ *is a homeomorphism.*

(4) $h(T) = h(\sigma_{T,f}) = h(\sigma_{T,f}^{(0,1,..,k)})$.

Proof. Recall that $I_{T,f}^{(0,1,..,k)}(x) = (fT^i(x))_{i=0}^k$ for $x \in X$. Since $I_{T,f}^{(0,1,..,k)}$ is a trajectory-embedding, the following claim $(\star)$ holds:

$(\star)$ If $x, y \in X$ with $(fT^i(x))_{i=0}^k = (fT^i(y))_{i=0}^k$, then $T^k(x) = T^k(y)$.

We prove (1). Let $x, y \in X$ with $(fT^i(x))_{i=0}^k = (fT^i(y))_{i=0}^k$. By $(\star)$, $T^{k+1}(x) = T^{k+1}(y)$ and so $fT^{k+1}(x) = fT^{k+1}(y)$. This implies that

$$(fT^i(x))_{i=1}^{k+1} = (fT^i(y))_{i=1}^{k+1}.$$

Thus $\sigma_{T,f}^{(0,1,..,k)}$ is well-defined. Also, we see that the morphism

$$I_{T,f}^{(0,1,..,k)} : (X, T) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$$

is a trajectory-isomorphism.

We prove (2). We show that there is a map $\pi : I_{T,f}^{(0,1,..,k)}(X) \to X$ such that $\pi \cdot I_{T,f}^{(0,1,..,k)} = T^k$. Let $x, y \in X$ such that $I_{T,f}^{(0,1,..,k)}(x) = I_{T,f}^{(0,1,..,k)}(y)$. By $(\star)$, $T^k(x) = T^k(y)$. So the function $\pi : I_{T,f}^{(0,1,..,k)}(X) \to X$ defined by

$\pi(z) = T^k(x)$ $(z = I_{T,f}^{(0,1,..,k)}(x) \in I_{T,f}^{(0,1,..,k)}(X))$ is well-defined. Then
(i) $\pi \cdot I_{T,f}^{(0,1,..,k)} = T^k$.
And so we have

$$I_{T,f}^{(0,1,..,k)} \cdot \pi \cdot I_{T,f}^{(0,1,..,k)} = I_{T,f}^{(0,1,..,k)} \cdot T^k = (\sigma_{T,f}^{(0,1,..,k)})^k \cdot I_{T,f}^{(0,1,..,k)}.$$

Since $I_{T,f}^{(0,1,..,k)} : X \to I_{T,f}^{(0,1,..,k)}(X)$ is an onto map, we see that
(ii) $I_{T,f}^{(0,1,..,k)} \cdot \pi = (\sigma_{T,f}^{(0,1,..,k)})^k$.
By use of (i) and (ii), we see that

$$\varprojlim I_{T,f}^{(0,1,..,k)} : \varprojlim(X,T) \to \varprojlim(I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$$

is a homeomorphism and $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ is also a homeomorphism.

We prove (3). Note that $p_{(0,1,..,k)}(I_{T,f}(X)) = I_{T,f}^{(0,1,..,k)}(X)$. Suppose that $I_{T,f}^{(0,1,..,k)}(x) = I_{T,f}^{(0,1,..,k)}(y)$ $(x, y \in X)$. By $(\star)$, we see that $T^k(x) = T^k(y)$ and so

$$I_{T,f}(x) = (fT^i(x))_{i\in\mathbb{N}} = (fT^i(y))_{i\in\mathbb{N}} = I_{T,f}(y).$$

This implies that $p_{(0,1,..,k)} : I_{T,f}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ is a homeomorphism.

We prove (4). Since the diagram

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}^{(0,1,..,k)}} & Z = I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,..,k)} \\ X & \xrightarrow{I_{T,f}^{(0,1,..,k)}} & Z = I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1}. \end{array}$$

is commutative, by Bowen's theorem (e.g. see [MS93, Theorem 7.1]) we have

$$h(\sigma_{T,f}^{(0,1,..,k)})) \leq h(T) \leq h(\sigma_{T,f}^{(0,1,..,k)}) + \sup\{h(T; (I_{T,f}^{(0,1,..,k)})^{-1}(z))| \ z \in Z\}.$$

Let $z = I_{T,f}^{(0,1,..,k)}(x)$ $(x \in X)$. By $(\star)$, we see that

$$T^k((I_{T,f}^{(0,1,..,k)})^{-1}(z)) = \{T^k(x)\}$$

is a one point set, and so $h(T; (I_{T,f}^{(0,1,..,k)})^{-1}(z)) = 0$. Hence $h(T) = h(\sigma_{T,f}^{(0,1,..,k)})$. By (3), $h(T) = h(\sigma_{T,f}^{(0,1,..,k)}) = h(\sigma_{T,f})$.

□

By Proposition 4.1 and [Kat20, Theorem 3.1], we have the following result.

Theorem 4.2. *Let X be a compact metric space with $\dim X = d < \infty$ and let $T : X \to X$ be a doubly 0-dimensional map with $\dim P(T) \leq 0$. Then there is a dense G_δ-set D of $C(X, \mathbb{R})$ such that for all $f \in D$,*

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:
(a) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,

(b) $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism,*
(c) $h(T) = h(\sigma_{T,f})$ *and*
(d) if $x, y \in X$ *are trajectory-separated for* T*, then*

$$|\{i \in \mathbb{N}|\ I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

5. Reconstruction theorem in the one-sided shift $(\mathbb{R}^{\mathbb{N}}, \sigma)$

For a compact metric space (Y,d), 2^Y denotes the space whose elements are nonempty closed subsets of Y and the space 2^Y has the Hausdorf metric d_H. Note that $(2^Y, d_H)$ is a compact metric space. Let Z be a space and let $\varphi : Z \to 2^Y \cup \{\emptyset\}$ be a set-valued function, where we consider that the empty set $\emptyset$ is an isolated point of the space $2^Y \cup \{\emptyset\}$. Then $\varphi : Z \to 2^Y \cup \{\emptyset\}$ is *upper semi-continuous* if for any $z \in Z$ and any open neighborhood V of $\varphi(z)$ in Y, there is an open neighborhood U of z in Z such that $\varphi(z') \subset V$ for any $z' \in U$.

Let (X,T) be any one-sided dynamical system. A point $x \in X$ is a *chain recurrent point* of T if for any $\epsilon > 0$ there is a finite sequence $x = x_0, x_1, \cdots, x_m = x$ $(m \geq 1)$ of points of X such that $d(T(x_i), x_{i+1}) < \epsilon$ for each $i = 0, 1, \cdots, m-1$. Let $CR(T)$ be the set of all chain recurrent points of T. Note that $P(T) \subset CR(T)$, $CR(T)$ is a nonempty closed subset of X and the set-valued function

$$CR : C(X,X) \to 2^X,\ T \mapsto CR(T)$$

is upper semi-continuous (see [BF85]).

We will define a class 0-$\mathcal{DCR}$ of compact metric spaces.

Definition 5.1. *Let 0-*$\mathcal{DCR}$ *be the class of all compact metric spaces* X *such that* X *satisfies the following two conditions:*
(0-$\mathcal{D}$) *The set of doubly 0-dimensional maps* $T : X \to X$ *is dense in* $C(X,X)$*.*
(0-$\mathcal{CR}$) *The set of maps* $T : X \to X$ *with* $\dim CR(T) = 0$ *is dense in* $C(X,X)$*.*

Remark 3. Note that for a compact metric space X, both the set of 0-dimensional maps $T : X \to X$ and the set of maps $T : X \to X$ with $\dim CR(T) = 0$ are G_δ-sets of $C(X,X)$ (e.g. see [KOU16]). So we see that if X belongs to 0-$\mathcal{DCR}$, then the set of all maps $T : X \to X$ such that T is a 0-dimensional map with $\dim CR(T) = 0$ is a dense G_δ-set of $C(X,X)$.

Let A be a (nonempty) closed subset of a compact metric space X. Here we consider the following condition: $D(A) < \eta$ if A can be decomposed into finitely many mutually disjoint closed sets A_i with $\mathrm{diam}(A_i) < \eta$ for each i, i.e., $A = \bigcup_i A_i$, $\mathrm{diam}(A_i) < \eta$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. Note that $\dim A = 0$ if and only if $D(A) < \eta$ for each $\eta > 0$.

Modyfying the proof of [KM20, Lemma 3.11], we have the following.

Lemma 5.2. (c.f. [KM20, Lemma 3.11]) *Let $\eta > 0$ and $k \in \mathbb{N}$. Suppose that $T : X \to X$ is a doubly 0-dimensional map of a compact metric space X such that $\dim X = d < \infty$ and $D(\mathrm{cl}[\bigcup_{p=0}^{4k} T^{-p}(P(T))]) < \eta$. Then T has (k, η)-TSP.*

Proof. Since $D(\mathrm{cl}[\bigcup_{p=0}^{4k} T^{-p}(P(T))]) < \eta$, there is an open cover

$$\mathcal{C} = \{C_i \mid 1 \leq i \leq M\}$$

of X such that
(a) $\mathrm{diam}(C_i) < \eta$ for each $1 \leq i \leq M$, and
(b) $\mathrm{bd}(C_i) \cap (\mathrm{cl}[\bigcup_{p=0}^{4k} T^{-p}(P(T))]) = \emptyset$ for each $1 \leq i \leq M$.

Put $K = \bigcup_{i=1}^{M} \mathrm{bd}(C_i)$. By (b) there is an open neighborhood K' of K in X such that for any point $z \in K'$, $T^t(z) \cap T^{t'}(z) = \emptyset$ for $-2k \leq t < t' \leq 2k$.

By modyfying the proof of [KM20, Lemma 3.11], we see that there is an open cover $\mathcal{C}' = \{C_i' \mid 1 \leq i \leq M\}$ of X such that
(1) $C_i' \subset C_i$ for each $1 \leq i \leq M$, and
(2) $\mathrm{ord}\{f^{-p}(\mathrm{bd}(C_i')) \mid 1 \leq i \leq M, p = 0, 1, ..., k\} \leq d$, and
(3) $\mathrm{bd}(C_i') \cap (\mathrm{cl}[\bigcup_{p=0}^{4k} T^{-p}(P(T))]) = \emptyset$ for each $1 \leq i \leq M$.

Put $c_1' = \mathrm{cl}(C_1')$, $c_i' = \mathrm{cl}(\mathrm{int}[(C_i') \setminus (\bigcup_{j<i} C_j')])$ for $2 \leq i \leq M$. We define

$$H = \bigcup_{i=1}^{M} \mathrm{bd}(c_i') \text{ and } U_i = \mathrm{int}(c_i') \ (i = 1, 2, .., M).$$

Then H satisfies the desired conditions of (k, η)-TSP. □

Lemma 5.3. (A version of Borsuk's homotopy extension theorem, c.f. [Bor67, (8.1)Theorem] and [Mil01, Theorem 4.1.3]) *Let X be a compact metric space and M a closed subset of X, and let maps $f', g' : M \to \mathbb{R}$ satisfy $d(f', g') < \epsilon$. If $g : X \to \mathbb{R}$ is an extension of g', then f' has an extension $f : X \to \mathbb{R}$ such that $d(f, g) < \epsilon$.*

Let X be any compact metric space. For each $\alpha > 0$ and $S \subset \mathbb{N}$ a set of cardinarity $2d+1$, let $E(\alpha; S)$ be the subset of $C(X, X) \times C(X, \mathbb{R})$ consisting of all pairs (T, f) such that $I_{T,f}^S : X \to \mathbb{R}^S$ is an α *trajectory-embedding* (i.e., $I_{T,f}^S(x) \neq I_{T,f}^S(y)$ whenever $x, y \in X$ with $d(T^j(x), T^j(y)) \geq \alpha$ for all $j \in S$).

The main theorem of this paper is the following.

Main Theorem 5.4. (Reconstruction theorem of dynamical systems) *Let X be a compact metric space with $\dim X = d < \infty$. Suppose that X belongs to the class 0-$\mathcal{DCR}$.Then the following assertions $(1) - (3)$ hold.*

(1) (α trajectory-embedding) *Let $\alpha > 0$ and $S \subset \mathbb{N}$ a set of cardinarity $2d + 1$. Then the set $E(\alpha; S)$ is a dense open set of $C(X, X) \times C(X, \mathbb{R})$.*

(2) (Trajectory-embedding) *There exists a G_δ-dense set E of $C(X, X) \times C(X, \mathbb{R})$ such that if $(T, f) \in E$, for any $S \subset \mathbb{N}$ of cardinality $2d + 1$*

$$I_{T,f}^S : X \to \mathbb{R}^S$$

is a trajectory-embedding.

(3) (Infinite delay observation) *If E is the set as in the above (2), then for any $(T,f) \in E$,*

$$I_{T,f} = T^{\mathbb{N}}_{T,f} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:

(a) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,

(b $\varprojlim I_{T,f} : \varprojlim (X,T) \to \varprojlim (I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism,*

(c) $h(T) = h(\sigma_{T,f})$ *and*

(d) *if $x, y \in X$ are trajectory-separated for T, then*

$$|\{i \in \mathbb{N} | \; I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

In particular, X is a reconstruction space of dynamical systems.

$$\begin{array}{ccccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} & & \downarrow \sigma \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \end{array}$$

Proof. We prove (1). Let $\alpha > 0$ and $S \subset \mathbb{N}$ of cardinality $2d+1$. For each $T \in C(X,X)$, we put

$$L(T : \alpha, S) = \{(x,y) \in X \times X | \; d(T^j(x), T^j(y)) \geq \alpha \text{ for } j \in S\} \subset X \times X.$$

Recall the set

$$E(\alpha; S) = \{(T,f) \in C(X,X) \times C(X,\mathbb{R}) | I^S_{T,f}(x) \neq I^S_{T,f}(y) \text{ for } (x,y) \in L(T : \alpha, S)\}.$$

We will show that $E(\alpha; S)$ is an open subset of $C(X,X) \times C(X,\mathbb{R})$. Let $(T,f) \in E(\alpha; S)$. Since $L(T : \alpha, S)$ is compact, we can choose a neighborhood K of $L(T : \alpha, S)$ in $X \times X$ and $\epsilon > 0$ such that for any $(x,y) \in K$,

$$d(I^S_{T,f}(x), I^S_{T,f}(y)) \geq 2\epsilon.$$

Note that if $L(T : \alpha, S) = \emptyset$, we choose K as an empty set $\emptyset$. Since the set-valued function

$$L(\alpha, S) : C(X,X) \to 2^{X \times X} \cup \{\emptyset\}, T \mapsto L(T : \alpha, S)$$

is an upper semi-continuous set-valued function, we can choose a neighborhood $U(T)$ of T in $C(X,X)$ and a neighborhood $V(f)$ of $f \in C(X,\mathbb{R})$ such that if $(T', f') \in U(T) \times V(f)$ then $L(T' : \alpha, S) \subset K$ and for $(x,y) \in K$,

$$d(I^S_{T',f'}(x), I^S_{T',f'}(y)) \geq \epsilon.$$

Since $L(T' : \alpha, S) \subset K$, we see that $I^S_{T',f'}(x) \neq I^S_{T',f'}(y)$ for $(x,y) \in L(T' : \alpha, S)$. Then $(T', f') \in E(\alpha, S)$ and so $U(T) \times V(f) \subset E(\alpha; S)$. Hence $E(\alpha, S)$ is an open set of $C(X,X) \times C(X,\mathbb{R})$.

Next, we will show that $E(\alpha; S)$ is dense in $C(X,X) \times C(X,\mathbb{R})$. Let $(T,f) \in C(X,X) \times C(X,\mathbb{R})$ and $\epsilon > 0$. Since $f : X \to \mathbb{R}$ is uniformly continuous, there is a sufficiently small positive number $\eta > 0$ such that $\eta <$

α and if $x, y \in X$ with $d(x,y) < \eta$, then $d(f(x), f(y)) < \epsilon$. Let $k = \max S$. By Remark 3, we can choose $T_1 \in C(X,X)$ such that $d(T, T_1) < \epsilon/2$ and T_1 is a 0-dimensional map with $\dim CR(T_1) = 0$. Since

$$\dim(\bigcup_{p=0}^{4k} T_1^{-p}(CR(T_1))) = 0,$$

we choose a closed neighborhood W of $\bigcup_{p=0}^{4k} T_1^{-p}(CR(T_1))$ in X such that $D(W) < \eta$. Since the set function $CR : C(X,X) \to 2^X$ is upper semi-continuous and X satisfies the condition 0-$\mathcal{D}$ of Definition 5.1, we can choose a doubly 0-dimensional map $T_2 \in C(X,X)$ such that $d(T_1, T_2) < \epsilon/2$ and $\bigcup_{p=0}^{4k} T_2^{-p}(CR(T_2)) \subset W$. Then $D(\bigcup_{p=0}^{4k} T_2^{-p}(CR(T_2))) < \eta$ and so

$$D(\mathrm{cl}[\bigcup_{p=0}^{4k} T_2^{-p}(P(T_2))]) < \eta.$$

By Lemma 5.2, we see that T_2 has (k,η)-TSP. Hence there is a closed set H of X such that
(1) $X \setminus H$ is a union of finitely many disjoint open sets U_i $(i = 1, 2, ..., m)$ of diameter at most η, and
(2) $\mathrm{ord}\{T_2^{-p}(H)\}_{p=0}^{k} \le d$.

We choose a small open neighborhood G of H in M such that
(2′) $\mathrm{ord}\{T_2^{-j}(G)\}_{j=0}^{k} \le d$.
Then we may assume that $X \setminus \mathrm{cl}(G)$ is a union of disjoint open sets $V_i (i = 1, 2, ..., m)$ such that $\mathrm{cl}(V_i) \subset U_i$. Note that $\mathrm{cl}(V_i) \cap \mathrm{cl}(\mathrm{V}_j) = \emptyset (i \neq j)$. For each i, take a point t_i which belongs to a sufficiently small neighborhood of $f(\mathrm{cl}(\mathrm{V}_i))$ in $\mathbb{R}$ such that $t_i \neq t_j$ if $i \neq j$. We define a map

$$g' : \bigcup_{i=1}^{m} \mathrm{cl}(V_i) \to \mathbb{R}$$

by $g'(\mathrm{cl}(V_i)) = t_i$. Then by Lemma 5.3, we have an extension $g : X \to \mathbb{R}$ of g' with $d(g, f) < \epsilon$. We will prove $(T_2, g) \in E(\alpha; S)$. Let $(x, y) \in L(T_2 : \alpha, S)$. By (2'),

$$|\{j \in S |\ T_2^j(x) \in G\}| \le d$$

and

$$|\{j \in S |\ T_2^j(y) \in G\}| \le d.$$

Since $|S| = 2d + 1$, we can find some $j \in S$ such that $T_2^j(x), T_2^j(y) \in X \setminus G$. Since $d(T_2^j(x), T_2^j(y)) \ge \alpha$ and $\mathrm{diam}(\mathrm{cl}(V_i)) < \eta < \alpha$ for each $i = 1, 2, .., m$, there are n, n' such that $n \neq n'$ and $T_2^j(x) \in \mathrm{cl}(V_n)$ and $T_2^j(y) \in \mathrm{cl}(V_{n'})$. Then $gT_2^j(x) = t_n \neq t_{n'} = gT_2^j(y)$. This implies $I_{T_2,g}^S(x) \neq I_{T_2,g}^S(y)$ and hence

$$(T_2, g) \in E(\alpha; S).$$

Note that $d(T, T_2) < \epsilon$ and $d(f, g) < \epsilon$. So we see that $E(\alpha; S)$ is a dense open set of $C(X, X) \times C(X, \mathbb{R})$.

We will prove (2). Let J be the set of all set $S \subset \mathbb{N}$ of cardinality $(2d+1)$. Note that J is a countable set. We define

$$E = \bigcap \{E(1/n; S) |\ S \in J \text{ and } n \in \mathbb{N} \setminus \{0\}\}.$$

Then we see that E is a desired dense G_δ-set in $C(X, X) \times C(X, \mathbb{R})$.

Finally we will prove (3). Let $(T, f) \in E$. Note that if $k = 2d$, then (T, f) satisfies the conditions of Proposition 4.1. Hence

$$I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})],\ h(T) = h(\sigma_{T,f})$$

and $\varprojlim I_{T,f} : \varprojlim(X, T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ is a homeomorphism.

We prove (d). Let $x, y \in X$ be trajectory-separated points for T. Suppose, on the contrary, that

$$|\{i \in \mathbb{N}|\ I_{T,f}(x)_i = I_{T,f}(y)_i\}| > 2d.$$

Then we can choose a set $S' \subset \{i \in \mathbb{N}|\ I_{T,f}(x)_i = I_{T,f}(y)_i\}$ with $|S'| = 2d+1$. This is a contradiction to the fact that $I^{S'}_{T,f}$ is a trajectory-embedding.

This completes the proof.

□

6. The class 0-$\mathcal{DCR}$

In this section, we consider the following general problem.

Problem 6.1. *What kinds of compact metric spaces belong to the class 0-$\mathcal{DCR}$?*

We will show that PL-manifolds, some branched manifolds and some fractal sets, e.g. Menger manifolds, Sierpiński carpet, Sierpiński gasket and dendrites, belong to the class 0-$\mathcal{DCR}$.

In [KOU16] Krupski, Omiljanowski and Ungeheuer defined the class 0-$\mathcal{CR}$ which is the family of all compact metric spaces X such that the set $CR(T)$ is 0-dimensional for a generic map $T \in C(X, X)$. They proved the following result.

Theorem 6.2. ([KOU16, Theorem 5.1]) *If X is a (compact) polyhedron, then $X \in$ 0-$\mathcal{CR}$. Moreover, if X is a compact metric space that admits an ϵ-retraction $r_\epsilon : X \to P$ onto a polyhedron $P \subset X$ for each $\epsilon > 0$ (i.e., $d(r_\epsilon, id_X) < \epsilon$ and $r_\epsilon|P = id_P$), then $X \in$ 0-$\mathcal{CR}$.*

Now, we consider the family 0-$\mathcal{D}$ of all compact metric spaces X such that the set of all doubly 0-dimensional maps on X is dense in $C(X, X)$. A map $T : X \to X$ is said to be a *piecewise embedding* if there is a countable family $\{F_i\}_{i \in \mathbb{N}}$ of closed subsets of X such that $X = \bigcup_{i \in \mathbb{N}} F_i$ and $T|F_i : F_i \to X$ is injective for each $i \in \mathbb{N}$. Note that if a map $T : X \to X$ is a piecewise embedding, then T is doubly 0-dimensional because that $\dim T^{-1}(x)$ is a countable set for each $x \in X$ and

$$\dim T(A) = \max\{\dim T(A \cap F_i) \mid i \in \mathbb{N}\} \leq 0$$

for any 0-dimensional closed set A of X (see the countable sum theorem for dimension [Eng95, Theorem 3.1.8]).

A (compact) d-dimensional polyhedron P ($d \geq 1$) is called a *manifold with branched structures* if $P = \bigcup_{j \in J} M_j \cup M$, where
(1) $\{M_j\}_{j \in J}$ ($|J| < \infty$) is a finite family of mutually disjoint sub-polyhedra M_j of P such that for each $j \in J$,

$$M_j = N_j \cup_{\varphi_\alpha} \bigcup \{N_{j,\alpha} | \alpha \in J_j\},$$

where J_j is a finite set, N_j and $N_{j,\alpha}$ ($\alpha \in J_j$) are d-dimensional PL-manifolds with boundaries, and M_j is obtained from N_j by attaching $N_{j,\alpha}$ ($\alpha \in J_j$) via locally embedding maps $\varphi_\alpha : N'_{j,\alpha} \to \partial N_j$ from a $(d-1)$-dimensional (compact) submanifold $N'_{j,\alpha}$ of $\partial N_{j,\alpha}$ into ∂N_j, i.e., M_j is the quotient space of the topological sum $N_j \amalg_{\alpha \in J_j} N_{j,\alpha}$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in N'_{j,\alpha} \subset \partial N_{j,\alpha}$ and the quotient map is denoted by $q_j : N_j \amalg_{\alpha \in J_j} N_{j,\alpha} \to M_j$ $(= N_j \cup \bigcup \{q_j(N_{j,\alpha}) | \ \alpha \in J_j\})$,
(2) M is a close set of P with

$$M \cap \bigcup_{j \in J} N_j = \emptyset,$$

(3) $P \setminus \bigcup_j N_j$ ($\supset M$) is a d-dimensional (non-compact) manifold.

Remark 4. All PL-manifolds are manifolds with branched structures. The associated template of the well-know Lorenz attractor is a manifold with branched structures [GL02].

Let K be a simplicial complex and let $K^{(m)}$ be the m-skeleton of K, i.e., the set of all simplexes of K whose dimension are $\leq m$. For a vertex v of $K^{(0)}$, let $\mathrm{St}(v, K)$ be the closed star of v, i.e., $\mathrm{St}(v, K) = \bigcup \{\sigma \in K | \ v \in \sigma\}$. Also let βK denote the barycentric subdivision of K. Let $\Delta =< p_0, p_1, \cdots, p_n >$ be a fixed n-simplex. Let $\sigma =< v_0, v_1, ..., v_n >$ be an n-simplexes and let F be the set of all sequence $v = s_0, s_1, \cdots, s_n = \sigma$ of faces of σ such that s_{i-1} is a face of s_i and $\dim s_{i-1} + 1 = \dim s_i$ $(= i)$ for $i = 1, 2, ..., n$. Then $|F| = (n+1)!$ and

$$\sigma = \bigcup \{< b(s_0), b(s_1), \cdots, b(s_n) > | \ (s_0, s_1, \cdots, s_n) \in F\},$$

where $b(s_i)$ is the barycenter of s_i. Consider the *folding map (at barycenters)* $f_\sigma : |\beta\sigma| \to \Delta$ which is the simplicial map defined by $f_\sigma(b(s_i)) = p_i$ for each $i = 0, 1, 2, ..., n$. Note that f_σ is a piecewise embedding.

Proposition 6.3. *Let P be a manifold with branched structures with* $\dim P \geq 1$. *Then the set of all piecewise embedding maps $T : P \to P$ is dense in*

$C(P,P)$. In particular, P belongs to 0-$\mathcal{DCR}$. Hence P is a reconstruction space of dynamical systems.

Proof. Let $\dim P = d \geq 1$. Since P is a polyhedron, by Theorem 6.2, P belongs to the class 0-$\mathcal{CR}$.

We will show that P belongs to the class 0-$\mathcal{D}$. Let $T \in C(P,P)$ and $\epsilon > 0$. We choose a simplicial complex K of P such that mesh(K) is sufficiently small, i.e., mesh$(K) < \epsilon/2$. Take a simplicial approximation $T_1 : P = |L| \to |K|$ of T such that $d(T, T_1) < \epsilon/2$, where L is a subdivision of K.

By modifying T_1, we will construct a map $T_1' : |\beta L| \to P$ such that for each d-simplex s of βL, $T_1'|s : s \to P$ is an embedding and $d(T_1, T_1') < \epsilon$. We consider the following abstract simplicial complex $\tilde{K}$ which contains the simplicial complex K as follows: For each $0 \leq k \leq d$, let

$$A_k = \{(a_0, a_1, .., a_k) \in \mathbb{N}^{k+1} \mid d = k + \Sigma_{i=0}^k a_i\}.$$

For each k-simplex $\sigma =< v_0, v_1, ..., v_k >$ $(k \leq d)$ of K and each $(a_0, a_1, .., a_k) \in A_k$, we consider the abstract d-simplex

$$< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >$$

$$=< p_{(v_0,0)}, p_{(v_0,1)}, .., p_{(v_0,a_0)}, p_{(v_1,0)}, p_{(v_1,1)}, .., p_{(v_1,a_1)}, \cdots, p_{(v_k,0)}, p_{(v_k,1)} .., p_{(v_k,a_k)} >$$

where we assume $v_i = p_{(v_i,0)} \in K^{(0)}$ $(i = 0, 1, ..., k)$. In particular,

$$< v_0, v_1, ..., v_d; (0, 0, .., 0) >=< p_{(v_0,0)}, p_{(v_1,0)}, .., p_{(v_d,0)} >=< v_0, v_1, ..., v_d >\in K^{(d)}$$

and

$$< v; d >=< p_{(v,0)}, p_{(v,1)}, ..., p_{(v,d)} >$$

for each vertex $v \in K^{(0)}$, where $v = p_{(v,0)}$. We define the abstract simplicial complex $\tilde{K}$ as follows:

$$\tilde{K} = K \bigcup \{s \mid s \text{ is a face of } < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >, 0 \leq k \leq d-1,$$
$$(a_0, a_1, .., a_k) \in A_k, \text{and} < v_0, v_1, ..., v_k >\in K^{(k)} \setminus K^{(k-1)}\}.$$

For each $0 \leq k \leq d-1$, we put

$$H_k = \bigcup \{< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) > \mid < v_0, v_1, ..., v_k >\in K^{(k)} \setminus K^{(k-1)},$$
$$(a_0, a_1, .., a_k) \in A_k\}.$$

We will construct a retraction $r : |\tilde{K}| \to |K|$ such that

$$r| < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >$$

is injective. Recall that $P = |K|$ is a manifold with branched structures. So we assume that

$$P = \bigcup_{j \in J} M_j \cup M, M_j = N_j \cup_{\varphi_\alpha} \bigcup \{N_{j,\alpha} | \alpha \in J_j\},$$

$N_{j,\alpha}', \varphi_\alpha$ and q_j are defined as before.

By induction on k $(0 \le k \le d-1)$, we construct $h_k : \bigcup_{i=0}^{k} H_i \to |K|$. First, for the case $k=0$ we will construct a map $h_0 : H_0 \to |K|$ as follows. Let $v \in K^{(0)}$.

If $v \notin \bigcup \{N'_{j,\alpha} \mid j \in J, \alpha \in J_j\}$, we choose an embedding

$$h_0 :< v; d > (=< p_{(v,0)}, p_{(v,1)}, ..., p_{(v,d)} >) \to P \setminus \bigcup \{N'_{j,\alpha} \mid j \in J, \alpha \in J_j\}$$

with $h_0(v) = v$, because that $P \setminus \bigcup \{N'_{j,\alpha} | \ j \in J, \alpha \in J_j\}$ is a d-dimensional (non-compact) manifold.

If $v \in N'_{j,\alpha}$ for some $j \in J$ and $\alpha \in J_j$, we choose an embedding

$$h_0 :< v; d > \to N_j$$

with $h_0(v) = v$, because that N_j is a d-dimensional manifold. So we have a map $h_0 : H_0 \to |K|$.

Now we assume that $h_{k-1} : \bigcup_{i=0}^{k-1} H_i \to |K|$ have been constructed. Let $< v_0, v_1, ..., v_k >$ be a k-simplex of K.

If $< v_0, v_1, ..., v_k > \subset P \setminus \bigcup \{N'_{j,\alpha} | \ j \in J, \alpha \in J_j\}$, then we can choose an embedding

$$h_k :< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) > \to P \setminus \bigcup \{N'_{j,\alpha} | \ j \in J, \alpha \in J_j\}$$

satisfying the following conditions (a) and (b):
(a) $h_k| < v_0, v_1, ..., v_k >= id$ and
(b)

$$h_k | H_{k-1} \cap < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >=$$

$$h_{k-1} | H_{k-1} \cap < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) > .$$

If $< v_0, v_1, ..., v_k > \subset N_j$ for some $j \in J$ and

$$< v_0, v_1, ..., v_k > \cap \bigcup \{N'_{j,\alpha} | \ j \in J, \alpha \in J_j\} \neq \emptyset,$$

then we choose an embedding $h_k :< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) > \to N_j$ satisfying (a) and (b) as above.

If $< v_0, v_1, ..., v_k > \subset N_{j,\alpha}$ and

$$< v_0, v_1, ..., v_k > \cap N'_{j,\alpha} \neq \emptyset \neq < v_0, v_1, ..., v_k > \setminus N'_{j,\alpha},$$

then we choose an embedding

$$h_k :< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) > \to N_j \cup N_{j,\alpha}$$

satisfying (a) and (b) as above, because that as the assumption of the case $k-1$, we can assume that

$$h_{k-1}(H_{k-1} \cap \partial < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >)$$

is contained in a d-dimensional manifold in $N_j \cup N_{j,\alpha}$. By induction on k, we obtain h_{d-1}. By use of h_{d-1} we have a retraction $r : |\tilde{K}| \to |K|$ such that $r| < v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >$ is injective.

Next, we will define a PL-map $\varphi : |L| \to |\tilde{K}|$ which is a piecewise embedding. For each d-simplex σ of L, we consider the simplex

$$T_1(\sigma) =< v_0, v_1, ..., v_k >\in K \ (k \le d).$$

For each vertex v_i $(i = 0, 1, .., k)$ of $T_1(\sigma)$, we consider the face

$$T^{-1}(v_i) \cap \sigma =< w_{(i,0)}, w_{(i,1)}, .., w_{(i,a_i)} >= \sigma_{v_i}$$

of σ. Note that $d = k + \Sigma_{i=0}^{k} a_i$ and

$$\sigma =< w_{(v_0,0)}, .., w_{(v_0,a_0)}, w_{(v_1,0)}, .., w_{(v_1,a_1)}, \cdots , w_{(v_k,0)}, .., w_{(v_k,a_k)} >$$
$$\equiv \sigma_{v_0} * \sigma_{v_1} * \cdots * \sigma_{v_k}.$$

We put

$$\beta\sigma_{v_0} * \beta\sigma_{v_1} * \cdots * \beta\sigma_{v_k} = \{\tau_0 * \tau_1 \cdots * \tau_k | \ \tau_i \in \beta\sigma_i, \dim \tau_i = a_i\}.$$

Then $\beta\sigma_{v_0} * \beta\sigma_{v_1} * \cdots * \beta\sigma_{v_k}$ gives a subdivision of σ. Consider the (abstract) d-simplex

$$\Delta_\sigma =< v_0, v_1, ..., v_k; (a_0, a_1, .., a_k) >$$
$$=< p_{(v_0,0)}, .., p_{(v_0,a_0)}, p_{(v_1,0)}, .., p_{(v_1,a_1)}, \cdots , p_{(v_k,0)}, .., p_{(v_k,a_k)} >$$

of $\tilde{K}$ and consider the folding map

$$f_{\sigma_{v_i}} : |\beta\sigma_{v_i}| \to \Delta_{v_i} =< p_{(v_i,0)}, .., p_{(v_i,a_i)} > \ (\in \tilde{K})$$

$(0 \le i \le k)$ defined as before.

For each d-simplex σ of L, we have a map

$$\varphi_\sigma = f_{\sigma_{v_0}} * f_{\sigma_{v_1}} \cdots * f_{\sigma_{v_k}} :$$
$$\sigma = |\beta\sigma_{v_0} * \beta\sigma_{v_1} * \cdots * \beta\sigma_{v_k}| \to \Delta_{v_0} * \Delta_{v_1} * \cdots * \Delta_{v_k} = \Delta_\sigma \in \tilde{K}.$$

Note that if $\dim T_1(\sigma) = d$, $\varphi_\sigma = T_1|\sigma$.

By use of φ_σ $(\sigma \in L^{(d)} \setminus L^{(d-1)})$, we have a desired PL map $\varphi : |L| \to |\tilde{K}|$ which is a piecewise embedding. Finally, we put $T_1' = r\varphi : P \to P$. Then T_1' is a piecewise embedding. Also by the constraction of r, we may assume that $d(T_1, T_1') < \epsilon/2$. This means that P satisfies the condition (0-$\mathcal{D}$). This completes the proof. □

Many dynamical properties of Cantor sets have been studied. Here we consider dynamical properties of higher dimensional fractal sets.

For $0 \le k < n$, we will construct a space L_k^n in an n-simplex $M_0 =< v_0, v_1, ..., v_n >$ by Lefshetz's method (see [Chi96, p.129] and [Lef31]). We define a sequence $\{(M_i, L_i)\}_{i\in\mathbb{N}}$ of compact n-dimensional polyhedra M_i with triangulations L_i inductively as follows. Let M_0 be the n-simplex $< v_0, v_1, ..., v_n >$ with the standard simplicial complex structure L_0. Suppose (M_i, L_i) has been defined. Let

$$M_{i+1} = \bigcup\{\mathrm{S}t(v, \beta^2(L_i)) \mid v \text{ is a vertex of } \ \beta(L_i^{(k)})\}$$

and

$$L_{i+1} = \beta^2 L_i | M_{i+1}.$$

Note that M_{i+1} may be regarded as a regular neighborhood of the k-skeleton of L_i. Then $\{M_i\}_{i\in\mathbb{N}}$ is a decreasing sequence and we obtained a compact metric space

$$L_k^n = \bigcap_{i\in\mathbb{N}} M_i.$$

Note that L_0^1 is a Cantor set and L_d^{2d+1} $(= \mu^d)$ is called the d-dimensional *Menger compactum*. Also L_1^2 is called the *Sierpiński carpet*. A space X is a d-dimensional *Menger manifold* if X is compact and each point x of X has a neighborhood W of x in X such that W is homeomorphic to the d-dimensional Menger compactum μ^d (see [Bes88] for geometric properties of μ^d).

Also the Sierpiński gasket can be constructed from an equilateral triangle by repeated removal of (open) triangular subsets: Start with an equilateral triangle. Subdivide it into four smaller congruent equilateral triangles and remove the central (open) triangle. Repeat this step with each of the remaining smaller triangles infinitely. So we have a sequence $\{X_i\}_{i\in\mathbb{N}}$ of 2-dimensional polyhedra in the plane and the intersection $X = \bigcap_{i\in\mathbb{N}} X_i$ is called the *Sierpiński gasket*.

A compact connected metric space (=continuum) X is said to be a *dendrite* if X is a 1-dimensional locally connected continuum which contains no simple closed curve.

Proposition 6.4. *Let M be a d-dimensional Menger manifold. Then M belongs to 0-$\mathcal{DCR}$ and hence M is a reconstruction space. More precisely, there exists a G_δ-dense set E' of $C(M,M)\times C(M,\mathbb{R})$ such that if $(T,f)\in E'$, then for any $S\subset\mathbb{N}$ of cardinality $2d+1$, $I_{T,f}^S : M\to\mathbb{R}^S$ is an embedding and so*

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (M,T)\to(\mathbb{R}^{\mathbb{N}},\sigma)$$

is an embedding.

Proof. By [Bes88, Definition 1.2.1 and Corollary 5.2.2], for each $\epsilon>0$, M admits an ϵ-retraction $r_\epsilon : M\to P$ onto a d-dimensional polyhedron $P\subset M$. Hence by Theorem 6.2, M belongs to 0-$\mathcal{CR}$. Also it is well-known that the set $e(M,M)$ of all embeddings $T: M\to M$ is a G_δ dense set of $C(M,M)$ (see [Bes88, Theorem 2.3.8]). Hence M belongs to 0-$\mathcal{DCR}$. By use of the fact that $e(M,M)$ is a G_δ dense set of $C(M,M)$ and by modifying the proof of Theorem 5.4, we can complete the latter part of the proof. □

We will show that the Sierpiński carpet belongs to 0-$\mathcal{DCR}$. In [Why58, p.323], Whyburn proved that the Sierpiński carpet is homeomorphic to any S-curve X, i.e., X is a plane locally connected 1-dimensional continuum whose complement in the plane consists of countably many components with frontiers being mutually disjoint simple closed curves $\{S_i\}_{i\in\mathbb{N}}$. Such simple closed curves $\{S_i\}_{i\in\mathbb{N}}$ are called the *rational circles* of the S-curve X. The union of all these circles $\{S_i\}_{i\in\mathbb{N}}$ is called the *rational part* of X, and the remainder $X \setminus (\bigcup_{i \geq 0} S_i)$ is called the *irrational part* of X. Moreover, Whyburn ([Why58]) proved that if K_1, K_2 are S-curves and C_1, C_2 are frontiers of components of complements of K_1, K_2 in the plane $\mathbb{R}^2$ respectively, then each homeomorphism of C_1 onto C_2 can be extended to a homeomorphism of K_1 onto K_2. We need the following lemma.

Lemma 6.5. *Let X be an S-curve in the plane $\mathbb{R}^2$ and let $\{S_i\}_{i \in \mathbb{N}}$ be rational circles of X, and S_0 the frontier of the unbounded component of $\mathbb{R}^2 \setminus X$. Let B_k $(k \geq 1)$ be the disk in $\mathbb{R}^2$ with $\partial B_k = S_k$. If $p : \mathbb{R}^2 \to H$ is the decomposition map of $\mathbb{R}^2$ obtained by identifying the sets $B_1, B_2, ...$ to single points respectively, then the decomposition space H is homeomorphic to $\mathbb{R}^2$, $p(X) = D$ is a disk in the plane H with $\partial D = p(S_0)$, and the set $\{p(S_i) \mid i = 1, 2, ..\}$ is a countable set in $D \setminus \partial D$. Moreover, for a point x of $X \setminus S_0$, x is in the irrational part of X if and only if $p^{-1}(p(x)) = \{x\}$.*

Proof. By the Moore's theorem [Kur68, p.380], we see that H is homeomorphic to $\mathbb{R}^2$ and $p(X) = D$ is a disk. Note that the set $\{p(S_j) | j \geq 1\}$ is a countable set in the disk D. □

Proposition 6.6. *Let $X = L_1^2 \subset \mathbb{R}^2$ be the Sierpiński carpet. Then X belongs to 0-$\mathcal{DCR}$.*

Proof. Let $\epsilon > 0$. Recall the Lefshetz's construction of L_1^2 as before. We see that M_{i+1} is regarded as a regular neighborhood of the 1-skeleton of L_i. So we can easily see that X admits an ϵ-retraction $r_\epsilon : X \to |L_i^{(1)}|$ for a sufficiently large $i \in \mathbb{N}$. Hence X belongs to 0-$\mathcal{CR}$.

We will show that X belongs to the class 0-$\mathcal{D}$. Let $T \in C(X, X)$ and $\epsilon > 0$. Let $M_0 = \Delta_2$ be a 2-simplex in the plane $\mathbb{R}^2$ with the standard simplicial complex structure L_0. We have the sequence (M_i, L_i) defined as before, i.e.,

$$M_{i+1} = \bigcup \{\mathrm{St}(v, \beta^2(L_i)) \mid v \text{ is a vertex of } \beta(L_i^{(1)})\}$$

and

$$L_{i+1} = \beta^2 L_i | M_{i+1}.$$

Then $X = L_1^2 = \bigcap_{i \in \mathbb{N}} M_i$. Note that $\mathrm{St}(v, \beta^2(L_i))$ is a disk in $\mathbb{R}^2$ and

$$\mathrm{St}(v, \beta^2(L_i)) \cap X$$

is an S-curve. Choose a sufficiently large natural number i_0 so that

$$\mathrm{diam}\ \mathrm{St}(v, \beta^2(L_{i_0})) < \epsilon$$

for each vertex v of $\beta(L_{i_0}^{(1)})$. Put $D_v = \mathrm{St}(v, \beta^2(L_{i_0}))$.

Let $\{S_k\}_{k \geq 1}$ be the family of rational circles S_k of the S-curve X such that $S_k \subset \mathrm{int}_{\mathbb{R}^2} M_{i_0+1}$ and let B_k be the disk with $\partial B_k = S_k$ for each $k \geq 1$. Let $p : \mathbb{R}^2 \to H$ be the decomposition map of $\mathbb{R}^2$ obtained by identifying the sets $B_1, B_2, ...$ to single points respectively. Then $p(D_v)$ is a disk in the plane H. So we have a family

$$\{p(D_v) \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)})\}$$

of disks in H such that

$$p(M_{i_0+1}) = \bigcup \{p(D_v) \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)})\}$$

and

$$\mathrm{ord}(\{p(D_v) \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)})\}) \leq 2.$$

Since $p(S_k) = \{s_k\}$ is a one point set and so the set $Z = \{s_k \mid k \geq 1\}$ is a countable set in $p(M_{i_0+1})$, for each $v \in \beta(L_{i_0}^{(1)})$ we have a disk E_v in $p(M_{i_0+1})$ ($\subset H$) such that
(1) $\partial E_v \cap Z = \emptyset$,
(2) $p(D_v) \subset \mathrm{int}_{p(M_{i_0+1})} E_v$,
(3) diam $p^{-1}(E_v) < \epsilon$ and
(4) $\mathrm{ord}\{E_v \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)})\} \leq 2$.

If necessary, by use of homeomorphism of $\mathbb{R}^2$ we may assume that $H = \mathbb{R}^2$ and each E_v is a convex set in H. Put $D'_v = p^{-1}(E_v) \cap X$. Since $D_v \subset \mathrm{int}_X D'_v$, the family

$$\{\mathrm{int}_X D'_v \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)})\}$$

is an open cover of X.

Choose a large natural number $j_0 \geq i_0$ such that for each 2-simplex σ of L_{j_0}, there is a vertex v of $\beta(L_{i_0}^{(1)})$ such that $T(\sigma \cap X) \subset \mathrm{int}_X D'_v$. For each $w \in L_{j_0}^{(0)}$, we put

$$V(w) = \{v \mid v \text{ is a vertex of } \beta(L_{i_0}^{(1)}) \text{ and } T(w) \in D'_v\}.$$

Note that $1 \leq |V(w)| \leq 2$. Since Z is a countable set in $H = \mathbb{R}^2$ and by use of usual general position arguments in the plane, we see that for any $w \in L_{j_0}^{(0)}$, take a point $\tilde{w}$ of the irrational part of $\bigcap\{D'_v \mid v \in V(w)\}$ such that
(5) if $w, w' \in L_{j_0}^{(0)}$ and $w \neq w'$, then $\tilde{w} \neq \tilde{w}'$,
(6) the set $\{p(\tilde{w}) \mid w \in L_{j_0}^{(0)}\}$ is in general position of the plane $H = \mathbb{R}^2$ and the segment $[p(\tilde{w}), p(\tilde{w}')]$ in $\mathbb{R}^2$ contains no point of Z.

Let σ be any 2-simplex in L_{j_0} and $\sigma =< w_0, w_1, w_2 >$. Consider the 2-simplex $\tilde{\sigma} =< p(\tilde{w}_0), p(\tilde{w}_1), p(\tilde{w}_2 >$ in $H = \mathbb{R}^2$. Then we have a natural homeomorphism $h_\sigma : \partial\sigma \to p^{-1}(\partial\tilde{\sigma})$ with $h_\sigma(w_i) = \tilde{w}_i$. We may assume that if σ_1, σ_2 are 2-simplexes in L_{j_0} with $\partial\sigma_1 \cap \partial\sigma_2 \neq \emptyset$, then

$$h_{\sigma_1} | \partial\sigma_1 \cap \partial\sigma_2 = h_{\sigma_2} | \partial\sigma_1 \cap \partial\sigma_2.$$

Since $\sigma \cap X$ and $p^{-1}(\tilde{\sigma}) \cap X$ are S-curves, by Whyburn theorem as above there is a homeomorphism $\varphi_\sigma : \sigma \cap X \to p^{-1}(\tilde{\sigma}) \cap X$ which is an extension of h_σ. By use of φ_σ ($\sigma \in L_{j_0}^{(2)} \setminus L_{j_0}^{(1)}$), we have a desired piecewise embedding $T' : X \to X$ with $d(T, T') < \epsilon$. Hence X belongs to 0-$\mathcal{D}$. □

Proposition 6.7. *Let X be the Sierpiński gasket. Then X belongs to 0-$\mathcal{DCR}$.*

Proof. For each $\epsilon > 0$, X admits an ϵ-retraction $r_\epsilon : X \to P$ $(= P_\epsilon)$ onto a subgraph P of X and so X belongs to 0-$\mathcal{CR}$. We will show that X belongs to the class 0-$\mathcal{D}$. Let $T \in C(X, X)$ and $\epsilon > 0$. Since T is uniform continuous, we choose a sufficiently small positive number $0 < \delta < \epsilon$ such that $d(T, Tr_\delta) < \epsilon$, where r_δ is a δ-retraction. Note that X is a countable union of segments J_n $(n \in \mathbb{N})$ in $\mathbb{R}^2$ and also we can choose such a retraction r_δ such that $r_\delta | J_n$ is injective, and hence it is a doubly 0-dimensional map. Consider the map $r_\delta T|P : P \to P$. Since P is a graph and hence it is a 1-dimensional manifold with branched Published online: 18 December 2020structures, we have a piecewise embedding map $g : P \to P$ such that $d(g, r_\delta T|P) < \epsilon$. Then

$$d(T, gr_\delta) \leq d(T, Tr_\delta) + d(Tr_\delta, r_\delta Tr_\delta) + d(r_\delta Tr_\delta, gr_\delta) < 3\epsilon$$

and gr_δ is a doubly 0-dimensional map. Hence X belongs to the class 0-$\mathcal{DCR}$. □

Proposition 6.8. *Let X be any dendrite. Then X belongs to 0-$\mathcal{DCR}$.*

Proof. Since X is a dendrite, we see that for each $\epsilon > 0$, X admits an ϵ-retraction $r_\epsilon : X \to P$ onto a subtree P of X. Hence X belongs to 0-$\mathcal{CR}$ (see also [KOU16]).

We will show that X belongs to the class 0-$\mathcal{D}$. Note that X is a countable union of arcs J_n $(n \in \mathbb{N})$ and we can choose such a retraction r_ϵ such that $r_\epsilon | J_n$ is injective and hence it is a doubly 0-dimensional map. By the same arguments as the proof of Proposition 6.7, we see that X belongs to 0-$\mathcal{D}$. □

Finally, we obtain the following consequence.

Theorem 6.9. *Let X be one of the following spaces: PL-manifold, manifold with branched structures, Menger manifold, Sierpiński carpet, Sierpiński gasket and dendrite. Then X is a reconstruction space of dynamical systems.*

7. Application: Reconstructions of one-sided dynamical systems from nonlinear time series analysis

There have been attempts to reconstruct dynamical models directly from data, and nonlinear methods for the analysis of time series data have been extensively investigated. This research is an inverse problem to the numerical analysis of dynamical systems model, in that it seeks to identify models that fit data.

Time-delay embedding is well-known for nonlinear time series analysis, and it is used in several research fields such as physics, meteorology, informatics, neuroscience and so on. In laboratories, experimentalists are striving to find principles of phenomenons from a lot of data and they use delay embedding for reconstructing the dynamical systems from experimental time series. For smooth dynamical systems on manifolds, the celebrated Takens' reconstruction theorem ensures validity of the delay embedding analysis.

Takens' theorem means that many dynamics theoretically can be reconstructed by the delay coordinate system, more precisely almost all (two-sided) dynamical systems can be reconstructed from observation maps (see Takens [Tak81, Tak02] and Sauer,Yorke and Casdagli [SYC91]). So Takens' theorem is the basis for nonlinear time series analysis and form a bridge between the theory of nonlinear differential dynamical systems on smooth manifolds and nonlinear time series analysis.

However, unfortunately the systems may not to be two-sided and moreover, they may not be systems on manifolds. Recently we freqently encounter a situation where we have to study dynamical systems of spaces that cannot have differential structure. In natural sciences and physical engineering, there has been an increase in importance of fractal sets and more complicated spaces, and also in mathematics, the dynamical properties and stochastic analysis of such spaces have been studied by many authors. Our reconstruction theorem theoretically ensures validity of the delay embedding analysis for (topological) dynamical systems on such complicated compact metric spaces, i.e., almost all one-sided dynamical systems (X,T) of spaces X belonging to 0-$\mathcal{DCR}$ can be reconstructed from observation maps $f : X \to \mathbb{R}$ in the sense of "trajectory embedding", i.e., the delay observation map

$$I_{T,f}^{(0,1,2,\cdots,k)} : (X,T) \to (I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,\cdots k)})$$

is a trajectory-embedding for a natural number $k \geq 2\dim X$, and so the dynamical system

$$(I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$$

may reflect many dynamical and topological properties of the original dynamical system (X,T). Especially,

$$I_{T,f} : [O(T)] \cong [O(\sigma_{T,f}^{(0,1,2,\cdots,k)})]$$

and

$$\varprojlim I_{T,f} : \varprojlim(X,T) \cong \varprojlim(I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$$

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}^{(0,1,2,\cdots,k)}} & I_{T,f}^{(0,1,2,\cdots,k)}(X) \subset \mathbb{R}^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,2,\cdots,k)} \\ X & \xrightarrow{I_{T,f}^{(0,1,2,\cdots,k)}} & I_{T,f}^{(0,1,2,\cdots,k)}(X) \subset \mathbb{R}^{k+1}. \end{array}$$

In laboratories, experimentalists may understand how the system (X,T) will go in the future in the sense of orbital classification from the analysis of experimental time series and they understand the geometric properties of (X,T) by use of the inverse limit space $\varprojlim(\mathrm{I}_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$. More precisely, for $x,y \in X$, if one can find a time $n \in \mathbb{N}$ such that

$$|\{i \in \mathbb{N}|\ fT^i(x) = fT^i(y), 0 \leq i \leq n\}| = 2\dim X + 1,$$

then $T^j(x) = T^j(y)$ for $j \geq n$ and hence $[o_T(x)] = [o_T(y)]$.

For more general case where a d-dimensional compact metric space X does not belong to 0-$\mathcal{DCR}$ and (X, T) is any one-sided dynamical system, we have an extension (μ^d, T') of (X, T), where μ^d is the d-dimensional Menger compactum containing X and $T' : \mu^d \to \mu^d$ is an extension of T (see [Bes88]). By Proposition 6.4, there is a possibility to be able to investigate the approximate properties of the dynamical system (X, T) by use of time-delay embedding of the dynamical system (μ^d, T').

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