Bravo, Daniel; Gillespie, James; Pérez, Marco A.

Locally type FP_n and n-coherent categories. (English) Zbl 07672037

Appl. Categ. Struct. 31, No. 2, Paper No. 16, 21 p. (2023)

The first and third author [J. Pure Appl. Algebra 221, No. 6, 1249–1267 (2017; Zbl 1362.18019)] described a nice interplay between modules of type FP_n and n-coherent rings in terms of closure properties. The principal objective in this paper is to present and study the concept of n-coherent categories as a general framework for the study of finiteness conditions of objects, based mainly in the proposal of the concept of locally type FP_n categories and n-coherent objects, as generalizations of locally f-initely generated and locally finitely presented categories, and of noetherian and coherent objects [*B. Stenström*, Rings of quotients. An introduction to methods of ring theory. Berlin-Heidelberg-New York: Springer-Verlag (1975; Zbl 0296.16001; *J. Stovicek*, "On purity and applications to coderived and singularity categories", Preprint, arXiv:1412.1615]. The main result is Theorem 5.5, where several characterizations of n-coherent categories are given. One of these characterizations is given in terms of the existence of a hereditary small cotorsion theory generated by the class of objects of type FP_n. Theorem 5.5 also generalizes the results in [*D. Bravo* and *M. A. Pérez*, J. Pure Appl. Algebra 221, No. 6, 1249–1267 (2017; Zbl 1362.18019)] concerning modules of type FP_n, FP_n-injective modules and n-coherent rings to the more general context of Grothendieck categories.

The synopsis of the paper goes as follows.

- §2 is concerned with categorical and homological preliminaries.
- §3 presents the concept of objects of type FP_n in a Grothendieck category, studying several closure properties along with some alternative descriptions under some extra assumption in the ground category. The authors also define locally type FP_n categories as a formal setting for the existence of objects of type FP_n.
- §4 investigates injectivity relative to objects of type FP_n. The authors define the class $\mathcal{FP}_n$-Inj of FP_n-injective objects, showing that this class is the right half of a complete cotorsion pair $\left({}^{\perp_1}(\mathcal{FP}_n\text{-Inj}), \mathcal{FP}_n\text{-Inj}\right)$ cogenerated by a set in any locally type FP_n category.
- §5 is devoted to n-coherent categories. One of the principal results in this section is that the previous cotorsion pair is hereditary iff the ground category is n-coherent. Another important result holding in n-coherent categories is that $\mathcal{FP}_n$-Inj is a covering class.
- §6 defines the Gorenstein FP_n-injective objects, constructing model structures such that they form the class of fibrant objects.

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MSC:

18C35 Accessible and locally presentable categories
18A25 Functor categories, comma categories
18E10 Abelian categories, Grothendieck categories
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
18G15 Ext and Tor, generalizations, Künneth formula (category-theoretic aspects)
18G25 Relative homological algebra, projective classes (category-theoretic aspects)
18N40 Homotopical algebra, Quillen model categories, derivators

Keywords:

objects of type FP_n; FP_n-injective objects; locally type FP_n categories; n-coherent categories

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