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Borceux, Francis (B-UCL-IRM); **Campanini, Federico** (I-PADV);
Gran, Marino (B-UCL-IRM)
The stable category of preorders in a pretopos II: the universal property. (English summary)
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A. Facchini and C. A. Finocchiaro [Ann. Mat. Pura Appl. (4) **199** (2020), no. 3, 1073–1089; MR4102802] observed that, in the category, PreOrd, of pre-ordered sets, there is a natural pretorsion theory, $(\mathsf{Eq}, \mathsf{ParOrd})$, where Eq is the *torsion* subcategory of equivalence relations and ParOrd is the *torsion-free* subcategory of partial orders. Writing $\mathcal{Z}$ for $\mathsf{Eq} \cap \mathsf{ParOrd}$, and calling a morphism $\mathcal{Z}$*-trivial* if it factors via an object in $\mathcal{Z}$, this just amounts to the observations that any morphism from an equivalence relation to a partial order is $\mathcal{Z}$-trivial, whilst any preorder fits in a canonical $\mathcal{Z}$-trivial exact sequence, analogous to the torsion subobject/torsion-free quotient exact sequence in a classical torsion-theoretic setting.

Part I [F. Borceux, F. Campanini and M. Gran, J. Pure Appl. Algebra **226** (2022), no. 9, Paper No. 106997; MR4403640] of this series established that, whenever $\mathbb{C}$ is a coherent category [P. T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 1*, Oxford Logic Guides, 43, Oxford Univ. Press, New York, 2002; MR1953060], it is possible to give a purely categorical construction of the stable category $\mathsf{Stab}(\mathbb{C})$ of the category $\mathsf{PreOrd}(\mathbb{C})$ of internal preorders in $\mathbb{C}$. Furthermore, when $\mathbb{C}$ is a pretopos, the canonical functor

$$\Sigma\colon \mathsf{PreOrd}\,(\mathbb{C}) \to \mathsf{Stab}\,(\mathbb{C})$$

preserves coproducts, sending short $\mathcal{Z}$-exact sequences in $\mathsf{PreOrd}(\mathbb{C})$ to short exact sequences in the pointed category $\mathsf{Stab}(\mathbb{C})$ [Part I, op. cit. (Theorem 7.14)].

This paper, as the second of the series, aims to establish the universal property of the stable category $\mathsf{Stab}(\mathbb{C})$, demonstrating:

Theorem 1. The canonical functor

$$\Sigma\colon \mathsf{PreOrd}\,(\mathbb{C}) \to \mathsf{Stab}\,(\mathbb{C})$$

is universal among all finite coproduct preserving torsion theory functors

$$G\colon \mathsf{PreOrd}\,(\mathbb{C}) \to \mathbb{X}$$

where $\mathsf{PreOrd}(\mathbb{C})$ is rigged out in the pretorsion theory $(\mathsf{Eq}(\mathbb{C}), \mathsf{ParOrd}(\mathbb{C}))$, and $\mathbb{X}$ is a pointed category with coproducts rigged out in a torsion theory $(\mathcal{T}, \mathcal{F})$.

{Editor's comment: This review also appears in zbMATH as Zbl 07605314.}

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.