Barwick, Clark; Schommer-Pries, Christopher
On the unicity of the theory of higher categories. (English) Zbl 07397080
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Any model for the theory of (∞, n)-categories must form an ∞-category $\mathcal{C}$ containing the *gaunt n-categories*. The principal result of this paper goes as follows.

Theorem. The moduli space $\mathrm{Thy}_{(\infty,n)}$ of theories of (∞, n)-categories is a $B\left(\mathbb{Z}/2\right)^n$.

Any theory of (∞, n)-categories has internal Hom, so that it is canonically enriched in itself. *D. Gepner* and *R. Haugseng* [Adv. Math. 279, 575–716 (2015; Zbl 1342.18009); Algebr. Geom. Topol. 15, No. 4, 1931–1982 (2015; Zbl 1327.18015)] showed that categories enriched in (∞, n)-categories are a model of $(\infty, n+1)$-categories, so that the unicity theorem for the ∞-category of (∞, n)-categories implies that of the $(\infty, n+1)$-category of (∞, n)-categories.

C. Simpson ["Some properties of the theory of n-categories", Preprint, arXiv:math/0110273] conjectured a similar unicity result for the theory of n-categories, suggesting ten axioms extremely different from those here. *B. Toën* [K-Theory 34, No. 3, 233–263 (2005; Zbl 1083.18003)] established a unicity theorem of the kind above for the theory of $(\infty, 1)$-categories. Toën's axioms and the authors' ones speficy the same homotopy theory.

The paper multifurcates into three parts.

The first part is concerned with aspects of strict n-category theory, most particularly including the theory of gaunt n-categories.

The second part is concerned with the axiomatization, showing that $\mathrm{Thy}_{(\infty,n)}$ is nonempty by explicitly constructing a theory of (∞, n)-categories that abides by the axioms. It is then shown that $\mathrm{Thy}_{(\infty,n)}$ is connected. The space of autoequivalences of the model of (∞, n)-categories is computed.

The third part establishes that most of the purported models of (∞, n)-categories in the literature are observant of the axioms in this paper. These include:

1. Charles Rezk's complete Segal Θ_n-spaces [*C. Rezk*, Geom. Topol. 14, No. 4, 2301–2304 (2010; Zbl 1203.18016); Geom. Topol. 14, No. 1, 521–571 (2010; Zbl 1203.18015)],
2. the n-fold complete Segal spaces of the first-named author [https://repository.upenn.edu/dissertations/AAI3165639/],
3. André Hirschowitz and Simpson's Segal n-categories [*B. Bollobás* and *D. B. West*, "A note on generalized chromatic number and generalized girth", Preprint, arXiv:math/98070449],
4. the n-relative categories of the first-named author and *D. M. Kan*, Indag. Math., New Ser. 23, No. 1–2, 42–68 (2012; Zbl 1245.18006); Homology Homotopy Appl. 15, No. 2, 281–300 (2013; Zbl 1291.18009)],
5. categories enriched in any internal model category whose underlying homotopy theory is a homotopy theory of (∞, n)-categories,
6. when $n = 1$, Boardman and Vogt's quasicategories [*J. M. Boardman* and *R. M. Vogt*, Homotopy invariant algebraic structures on topological spaces. Springer, Cham (1973; Zbl 0285.55012)],
7. when $n = 1$, Lurie's marked simplicial sets [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)], and
8. when $n = 2$, Lurie's scaled simplicial sets [*J. Lurie*, "(Infinity,2)-categories and the Goodwillie calculus I", Preprint, arXiv:0905.0462].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18N60 $(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories

18N65 (∞, n)-categories and (∞, ∞)-categories

Full Text: DOI arXiv

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