Bourke, John; **Lack, Stephen**

Accessible ∞-cosmoi. (English) Zbl 07654860

J. Pure Appl. Algebra 227, No. 5, Article ID 107255, 28 p. (2023)

The theory of ∞-categories has experienced an explosion of interest in recent years, multiple approaches having emerged to multiple definitions of $(\infty, 1)$-category or multiple models in the usual parlance, prominent examples of which include quasicategories, complete Segal spaces, and Segal categories. Over a number of years, Riehl and Verity have been unfurling a model-independent approach under the name of ∞-*cosmos*, and their theory has now attained a high level of power and sophistication [*E. Riehl* and *D. Verity*, Elements of ∞-category theory. Cambridge: Cambridge University Press (2022; Zbl 1492.18001)].

In a joint work [[Adv. Math. 412, Article ID 108812, 52 p. (2023; Zbl 1502.18014)]] with *L. Vokřínek*, the authors established a very general homotopical adjoint functor theorem for enriched categories as a generalization of Freyd's celebrated General Adjoint Functor Theorem (GAFT), the authors' motivation lying in the study of ∞-cosmoi, which are in fact certain simplicially enriched categories. In the case of ordinary categories, the solution set condition appearing in the GAFT is to be simplified by use of the theory of *accessible* categories [*J. Adámek* and *J. Rosický*, Locally presentable and accessible categories. Cambridge: Cambridge University Press (1994; Zbl 0795.18007); *M. Makkai* and *R. Paré*, Accessible categories: The foundations of categorical model theory. Providence, RI: American Mathematical Society (1989; Zbl 0703.03042)].

This paper introduces a notion of accessibility for ∞-cosmoi stronger than that of the earlier paper because of the good stability properties it enjoys. A precursor to the present work is [*J. Bourke*, J. Pure Appl. Algebra 225, No. 3, Article ID 106519, 43 p. (2021; Zbl 1451.18015)] dealing with 2-categories of categorical structures rather than ∞-cosmoi of ∞-categorical structures.

The synopsis of the paper goes as follows.

§2 is a brief review of the necessary background on ∞-cosmoi and accessible categories.

§3 introduces the main concept of accessible ∞-cosmos, showing that these include the basic examples arising from suitable simplicially enriched model categories.

§4 studies a first raft of closure properties of accessible ∞-cosmoi, including ∞-cosmoi∞-cosmoi of isofibrations, slices and duals of ∞-cosmoi as well as pullbacks of cosmological embeddings.

§5 is the technical heart of the paper, showing that for an accessible ∞-cosmos $\mathcal{K}$, the ∞-cosmos $\boldsymbol{Rari}(\mathcal{K})$ of *left adjoint left inverses* in $\mathcal{K}$ is also accessible.

§6 establishes the corresponding fact about *trivial fibrations* in $\mathcal{K}$ with further results on equivalences.

§7 deduces all remaining closure properties for accessible ∞-cosmoi by exploiting the results of the previous three sections.

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MSC:

18N60 $(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories
18C35 Accessible and locally presentable categories
18D20 Enriched categories (over closed or monoidal categories)
18N40 Homotopical algebra, Quillen model categories, derivators

Full Text: DOI arXiv

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