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Sheaf representation of monoidal categories. (English) Zbl 07658620
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This paper aims to prove that any monoidal category embeds into a product of easier ones, showing that nice enough monoidal categories are always equivalent to a dependent product of easy ones. Exactly speaking, the principal results of this paper are the following theorem and its corollary.

Theorem. Any small monoidal category with universal finite joins of central idempotents is monoidally equivalent to the category of global sections of a sheaf of $\vee$-local categories. Any small monoidal category with universal joins of central idempotents forming a spatial frame is monoidally equivalent to the category of global sections of a sheaf of $\vee$-local categories.

Corollary. Any small stiff monoidal category monoidally embeds into a category of global sections of a sheaf of $\vee$-local categories, and into a product of $\vee$-local monoidal categories.

This subsumes sheaf representation theorem for toposes [*S. Awodey*, Outst. Contrib. Log. 20, 39–57 (2021; Zbl 07440903); *J. Lambek* and *I. Moerdijk*, Stud. Logic Found. Math. 110, 275–295 (1982; Zbl 0511.03028); *J. Lambek* and *P. J. Scott*, Introduction to higher order categorical logic. Cambridge University Press (1986; Zbl 0596.03002); *S. Mac Lane* and *I. Moerdijk*, Sheaves in geometry and logic: a first introduction to topos theory. New York etc.: Springer-Verlag (1992; Zbl 0822.18001)], where the lattice of central idempotents corresponds to that of elements of the subobject classifier. This improves on the authors' earlier work [*P. Enrique Moliner* et al., J. Pure Appl. Algebra 224, No. 10, Article ID 106378, 35 p. (2020; Zbl 1445.18010)], which focused on the special case of central idempotents called subunits. From a logical viewpoint, it extends the sheaf representation of categorical models of higher-order intuitionistic logic [*M. Anel* and *G. Catren*, in: New spaces in mathematics. Formal and conceptual reflections. Cambridge: Cambridge University Press. 1–27 (2021; Zbl 07425814); *J. Lambek*, "On the sheaf of possible worlds", McGill University, Department of Mathematics and Statistics. 18 p. (1989; https://www.ioc.ee/~matt/papers/lambek_on-the-sheaf-of-possible-worlds.pdf); *J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)] to categorical models of multiplicative linear logic.

The synopsis of the paper goes as follows.

- §2 defines central idempotents, giving examples.
- §3 addresses basic properties of central idempotents making the category nice.
- §4 constructs the topological space on which the structure sheaf is based.
- §5 details the presheaf structure.
- §6 establishes the sheaf condition.
- §7 investigates the stacks.
- §8 shows that they are local.
- §9 extends to the case of infinitary joins.
- §10 deepens the sheaf representation by demonstrating that it preserves Booleaness, having limits, closedness, compactness, having a trace, and satisfying the external axiom of choice.
- §11 works out several examples to which sheaf representations for toposes do not apply directly.
- §12 settles functoriality of the main construction.
- §13 establishes the Corollary above.
- §14 discusses several open questions.
- Appendix A compares central idempotents with the special case of subunits [loc. cit., Zbl 1445.18010].

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18Bxx Special categories
03Gxx Algebraic logic

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