Rogers, Morgan
Toposes of topological monoid actions. (English) Zbl 07641774
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The author ["Toposes of discrete monoid actions", Preprint, arXiv:1905.10277] investigated properties of presheaf toposes of the form $\mathrm{PSh}(M)$ for a monoid M. For a topological group (G, τ), the category $\mathrm{Cont}(G, \tau)$ of continuous G-actions on discrete topological spaces is a Grothendieck topos. This paper studies the category of continuous actions of topological monoids.

The synopsis of the paper goes as follows.

§1 exhibits the necessary data to establish that the forgetful functor from the category $\mathrm{Cont}(M, \tau)$ of continuous actions of a monoid with respect to an arbitrary topology τ to the topos $\mathrm{PSh}(M)$ is left exact and comonadic (Proposition 1.4). The adjoint is to be expressed using either clopen subsets of (M, τ) or open relations. The existence of the adjunction of guarantees that $\mathrm{Cont}(M, \tau)$ is an elementary topos, so that the forgetful functor is the inverse image of a hyperconnected geometric morphism. Theoretical results from *M. Rogers* [Theory Appl. Categ. 37, 1017–1079 (2021; Zbl 1473.18011)] on supercompactly generated toposes are recalled in §1.2, being applied in §1.3 to conclude that any topos of the form $\mathrm{Cont}(M, \tau)$ is moreover a supercompactly generated Grothendieck topos, which gives rise to an intuitive Morita-equivalence result in terms of the category of continuous M-sets (Corollary 1.22). It is finally shown in §1.4 that the above characterization of toposes of the form $\mathrm{Cont}(M, \tau)$ is not yet complete.

§2 examines the question of how much is recoverable about a topology τ on a monoid M from the hyperconnected morphism
$$\mathrm{PSh}(M) \to \mathrm{Cont}(M, \tau)$$

§3 considers the canonical surjective point of $\mathrm{Cont}(M, \tau)$, which is the ccomposite of the canonical essential surjective point of $\mathrm{PSh}(M)$ and the hyperconnected morphism obtained in §1. The author aims to characterize this class of toposes in terms of the existence of a point of this form, just as *O. Caramello* [Adv. Math. 291, 646–695 (2016; Zbl 1401.18007)] did for topological groups.

§4 shows that continuous semigroup homomorphisms between topological monoids induce geometric morphisms between the corresponding toposes of continuous actions, while it was shown in [the author, "Toposes of discrete monoid actions", Preprint, arXiv:1905.10277] that semigroup homomorphisms correspond to essentially geometric morphisms between toposes of discrete monoid actions. When a geometric morphism g is induced by a continuous semigroup homomorphism ϕ between complete monoids, it is shown that the surjection-inclusion factorization of g is canonically represented by the factorization of ϕ into a monoid homomorphism followed by an inclusion of a subsemigroup (Theorem 4.15). It is also shown that the hyperconnected-localic factorization of g can be identified with the dense-closed factorization of ϕ (Theorem 4.19).

§5 summarizes the unresolved problems the author has encountered along the way, suggesting some future directions this research might proceed.

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MSC:

18B25 Topoi
18B40 Groupoids, semigroupoids, semigroups, groups (viewed as categories)
18D25 Actions of a monoidal category, tensorial strength

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